MA 502 Homework 7

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Let A be an $n \times n$ matrix and $J = \{f(t) \in \mathbb{K}[t] \mid f(A) = 0\}$. To show that J is an ideal, we check that the zero polynomial is in J, that if $f, g \in J$ then $f + g \in J$, and that if $f \in J$ and $g \in K[t]$ then $gf \in J$.

Let $\mathcal{O}$ denote the zero polynomial, then $\mathcal{O}(A) = 0$, so $\mathcal{O} \in J$.

Suppose $f, g \in J$. Then $f(A) = 0$ and $g(A) = 0$, so it follows that $(f + g)(A) = f(A) + g(A) = 0$. Thus, $f + g \in J$.

Let $g \in K[t]$ and $f \in J$. Then $(gf)(A) = g(A)f(A) = g(A)\,0 = 0$.

Let $p \in K[t]$ denote the characteristic polynomial of A. Then p is of degree n and by the Cayley Hamilton Theorem, $p(A) = 0$. Therefore $p \in J$ and since J is an ideal, we also have that $p^2 \in J$. Then p^2 is a polynomial of degree n^2 and it is the case that $p^2(A) = p(A)p(A) = 0$ as well.

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Let $\mathrm{ad}(A)$ denote the classical adjoint of the $n \times n$ matrix A, where $\mathrm{ad}(A)$ $=(\mathrm{co}(A))^T$. That is, the adjoint is the transpose of the cofactor matrix of A. Let $(A\mathrm{ad}(A))_{ij}$ denote the element in the i^{th} row and j^{th} column of the product $A\mathrm{ad}(A)$, a_{ij} the element of A in row i and column j, and b_{ij} the element of $\mathrm{ad}(A)$ in row i and column j. Then considering the j^{th} column,

$$(A\mathrm{ad}(A))_{ij} = \sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n} (-1)^k \mathrm{det} a_{ik} \mathrm{det}(A_{jk}) \ .$$

When $i = j$ this computes the determinant of A but when $i \neq j$, this computes the determinant of a matrix with a repeated row, which must have a 0 determinant. So we conclude that $(A\text{ad}(A))_{ij}$ =det(A) when $i = j$ and $(A\text{ad}(A))_{ij} = 0$ when $i \neq j$. Therefore $A\text{ad}(A)$ =det$(A)I$.

Next we note that the cofactor matrix of A^T is the transpose of the cofactor matrix of A. That is, $\text{co}(A^T) = (\text{co}(A))^T$. Also, recall that det$(A)$ =det(A^T). Then since

$$\text{ad}(A)A = (\text{co}(A))^T A = (A^T \text{co}(A))^T = (A^T (\text{co}(A^T))^T)^T,$$

we apply the reasoning above to see that $(\text{ad}(A)A)^T = A^T(\text{co}(A^T))^T = \det(A^T)I = \det(A)I$. But since $(\text{ad}(A)A)^T = \det(A)I$, which is a diagonal matrix, we conclude that $\text{ad}(A)A = \det(A)I$ as well.

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Let A be an upper triangular $n \times n$ matrix.

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If we let A_{ij} denote the element of A in the i^{th} row and j^{th} column, then $A_{ij} = 0$ for $i > j$. First we prove that the product of two upper triangular matrices is upper triangular.

Let B be an $n \times n$ upper triangular matrix as well. We define B_{ij} and $(AB)_{ij}$ similarly to A_{ij}. Note that if $i = 1$, then $1 = i > j$ cannot occur for any column number j and since the definition of upper triangular requires only that $(AB)_{ij} = 0$ for $i > j$, we assume $i > 1$ in what follows.

The element in the i^{th} row and j^{th} column of AB is found by:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{i-1} A_{ik}B_{kj} + \sum_{k=i}^{n} A_{ik}B_{kj}.$$

Inspecting the summation from $k = 1$ to $k = i - 1$, we see that since $k < i$, $A_{ik} = 0$ in each term so that $sum_{k=1}^{i-1} A_{ik}B_{kj} = 0$. This means that

$$(AB)_{ij} = \sum_{k=i}^{n} A_{ik}B_{kj}.$$

Note that since $k \geq i$, then if it is the case that $i > j$, then $k > j$ so that $B_{kj} = 0$. Therefore, we may conclude that $(AB)_{ij} = 0$ whenever it is the case that $i > j$, which means by definition of an upper triangular matrix that AB is upper triangular.

Using the result above, we see that since A^2 is the product of two upper triangular $n \times n$ matrices, A^2 is also upper triangular. Then since A and A^2 are upper triangular, $A^3 = A^2 A$ is upper triangular. Applying the reasoning inductively we conclude that $A^k = A^{k-1}A$ is upper triangular for all positive powers k.

We cannot prove that A^k is upper triangular for $k < 0$ without further assuming that A is invertible. We do not know whether A is invertible here.

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Since A is an upper triangular matrix, its eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ are the n elements on the diagonal of A (not necessarily distinct). Since f is a polynomial, $f(A)$ is the result of exponentiating A, multiplying by elements $k \in \mathbb{K}$ and adding matrices. That is, if f is a degree m polynomial then $f(A) = k_0 I + k_1 A^1 + ... + k_m A^m$. Since A^p is upper triangular for all nonnegative integers p, the result of multiplying an upper triangular matrix by a scalar is upper triangular, and the sum of two upper triangular matrices is upper triangular, we conclude that $f(A)$ must be an upper triangular matrix. Next we note that if λ is an eigenvalue of A with corresponding eigenvalue v, then since $Av = \lambda v$ we have

$$kA^p v = k\lambda A^{p-1} v = k\lambda A^{p-2} Av = k\lambda^2 A^{p-2} v = ... = k\lambda^p v \quad k \in \mathbb{K}\,, p \in \mathbb{N}\,.$$

This shows that $k\lambda^p$ is an eigenvalue of kA^p. Then for $f(A) = k_0 I + k_1 A + ... + k_m A^m$, if λ is an eigenvalue of A with corresponding eigenvalue v, then

$$f(A)v = k_0 I + k_1 A + ... + k_m A^m = k_0 \lambda^0 v + k_1 \lambda v + ... + k_m \lambda^m v = (k_0 + k_1 \lambda + k_m \lambda^m) v\,.$$

This implies that the eigenvalues of $f(A)$ are $f(\lambda_i)$, where each λ_i is one of the n diagonal elements of A.

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Let A be a nonsingular matrix and let λ be an eigenvalue of A. Then for an eigenvector v corresponding to the eigenvalue λ we have $Av = \lambda v$. Since A is invertible, we have $\lambda \neq 0$ and

$$\begin{aligned} A^{-1}Av &= A^{-1}\lambda v \\ v &= \lambda A^{-1}v \\ \frac{1}{\lambda}v &= A^{-1}v \; . \end{aligned}$$

Therefore, if λ is an eigenvalue of A then λ^{-1} is an eigenvalue of A^{-1}.

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Suppose A is a 3×3 upper triangular matrix with eigenvalues $-1, 0, 1$. Since 0 is an eigenvalue of A, A is not invertible. Consider however, the matrix $A^3 - 3A^2 + I$.

Using our results above, the eigenvalues of $A^3 - 3A^2 + I$ are

$$1^3 - 3(1)^2 + 1 = 1 - 3 + 1 = -1$$

$$0^3 - 3(0)^2 + 1 = 1$$

$$(-1)^3 - 3(-1)^2 + 1 = -3 \; .$$

Note that since none of the eigenvalues of $A^3 - 3A^2 + I$ are 0, then the matrix $A^3 - 3A^2 + I$ is invertible.

Therefore the eigenvalues of $(A^3 - 3A^2 + I)^{-1}$ are $-1, 1$ and $-1/3$.

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Let A be an $n \times n$ matrix with eigenvalues $1, 2$, and 3 with corresponding eigenvalues v_1, v_2, and v_3. In general for an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector v,

$$Av = \lambda v \implies A^{100}v = \lambda A^{99}v = \lambda^2 A^{98}v = ... = \lambda^{100}v \; .$$

Therefore λ^{100} is an eigenvalue of A^{100}.

So we conclude that $1, 2^{100}$, and 3^{100} are eigenvalues of A^{100}. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the transformation given by $T(x) = Ax$. Then there exists a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}\to\mathcal{B}}$ is an upper triangular matrix. The eigenvalues of the transformation T are the same as the matrix A since A is one matrix representation of the transformation T and the eigenvalues of T do not depend on the choice of basis used in the representation of T. By the previous exercise, the eigenvalues of the matrix $([T]_{\mathcal{B}\to\mathcal{B}})^{100}$ are found by applying the polynomial $f(\lambda) = \lambda^{100}$ to each eigenvalue of $[T]_{\mathcal{B}\to\mathcal{B}}$. Thus the eigenvalues of $([T]_{\mathcal{B}\to\mathcal{B}})^{100}$ are just $1, 2^{100}$, and 3^{100}. Since these must be the same set of eigenvalues as the matrix A^{100}, we conclude that A^{100} does not have any eigenvalues aside from $1, 2^{100}$, and 3^{100}.