MA 502 Homework 10

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1

$$
\begin{align}
e^{tA} &= e^{tA-\lambda tI+\lambda tI} \tag{1}\\
&= e^{\lambda tI}e^{(A-\lambda I)t} \tag{2}\\
&= e^{\lambda tI}\left(I+(A-\lambda I)t+(A-\lambda I)^2\frac{t^2}{2}+(A-\lambda I)^3\frac{t^3}{3!}+...\right) \tag{3}\\
&= e^{\lambda tI}\left(I+(A-\lambda I)t+(A-\lambda I)^2\frac{t^2}{2}+...(A-\lambda I)^{n-1}\frac{t^{n-1}}{(n-1)!}\right) \tag{4}\\
&= e^{\lambda t}\left(I+(A-\lambda I)t+(A-\lambda I)^2\frac{t^2}{2}+...(A-\lambda I)^{n-1}\frac{t^{n-1}}{(n-1)!}\right) \tag{5}
\end{align}
$$

Note that to get from the infinite sum in line (3) to the finite sum in line (4) we used the fact that for the characteristic polynomial of A, $p_A(x) = (x-\lambda)^n$, we have $p_A(A) = (A-\lambda I)^n = 0$ by Cayley-Hamilton. Therefore $(A-\lambda I)^m = 0$ for any $m \geq n$. Also to arrive at line (5) from line (4) we used

$$e^{\lambda tI} = I + \lambda tI + \frac{\lambda^2 t^2}{2}I^2 + ... = I(1+\lambda t+\frac{\lambda^2t^2}{2}+....) = I(e^{\lambda t})$$

We can then distribute the identity into the sum on line (5) without any change.

2

Consider the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

• First we fine an orthogonal matrix O such that $O^T AO$ is diagonal. We will find eigenvalues and corresponding eigenvectors that are orthonormal.

$$0 = p_A(\lambda) = (1-\lambda)^2 - 9 \implies \lambda = -2, 4\,.$$

Putting $\lambda = -2$ into $A - \lambda I = 0$ gives

$$A - \lambda I = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \hat{v} = 0.$$

Then any multiple of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector. We take the unit vector $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Similarly, for $\lambda = 4$, we have

$$A - \lambda I = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \hat{v} = 0,$$

which means that any multiple of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector. We take the unit vector $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Note that v_1 and v_2 are already orthogonal vectors. Set

$$O = \begin{pmatrix} v_2 & v_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = O^T.$$

Then $O^T O = OO^T = I$, which means that O is an orthogonal matrix (and also that O^T is orthogonal). But we require that $O^T AO$ is diagonal. So we multiply to get

$$O^T AO = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$

In fact the eigenvalues of A lie on the diagonal of $O^T AO$.

• To compute e^A, rewrite A as $A = ODO^T$. Then,

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \ldots.$$

$$= I + ODO^T + \frac{(ODO^T)^2}{2!} + \frac{(ODO^T)^3}{3!} + \frac{(ODO^T)^4}{4!} + ...$$
$$= I + ODO^T + \frac{OD^2O^T}{2!} + \frac{OD^3O^T}{3!} + \frac{OD^4O^T}{4!} + ...$$

3

Define the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)(1-t^2)\,dt$$

on the vector space of polynomials with real coefficients. We find an orthonormal basis $\{\phi_0, \phi_1, \phi_2\}$ for the subspace generated by $\{\frac{\sqrt{3}}{2}, \frac{\sqrt{15}}{2}t, t^2\}$.

Take $\psi_0(t) = \frac{\sqrt{3}}{2}$. To normalize ψ_0, we set

$$\phi_0(t) = \frac{\psi_0}{(\langle\psi_0, \psi_0\rangle)^{1/2}} = \frac{\frac{\sqrt{3}}{2}}{\sqrt{\int_{-1}^{1}(\frac{\sqrt{3}}{2})^2(1-t^2)dt}} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}.$$

Next set

$$\psi_1(t) = \frac{\sqrt{15}}{2}t - \langle\frac{\sqrt{15}}{2}t, \phi_0(t)\rangle\phi_0 = \frac{\sqrt{15}}{2}t - \left(\int_{-1}^{1} \frac{\sqrt{15}}{2}t\frac{\sqrt{3}}{2}(1-t^2)\,dt\right)\frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{15}}{2}t - 0\frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{2}t.$$

Then to find a normalized vector take

$$\phi_1(t) = \frac{\psi_1}{(\langle\psi_1, \psi_1\rangle)^{1/2}} = \frac{\frac{\sqrt{15}}{2}t}{\left(\int_{-1}^{1}(\frac{\sqrt{15}}{2}t)^2(1-t^2)dt\right)^{1/2}} = \frac{\frac{\sqrt{15}}{2}t}{1} = \frac{\sqrt{15}}{2}t\,.$$

Next set

$$\begin{aligned}
\psi_2(t) &= t^2 - \langle t^2, \phi_0\rangle\phi_0 - \langle t^2, \phi_1\rangle\phi_1 \\
&= t^2 - \left(\int_{-1}^{1} t^2\frac{\sqrt{3}}{2}(1-t^2)\,dt\right)\frac{\sqrt{3}}{2} - \left(\int_{-1}^{1} t^2(\frac{\sqrt{15}}{2}t)(1-t^2)\,dt\right)\frac{\sqrt{15}}{2}t \\
&= t^2 - \frac{2}{5\sqrt{3}}\frac{\sqrt{3}}{2} - 0\frac{\sqrt{15}}{2}t \\
&= t^2 - \frac{1}{5}\,.
\end{aligned}$$

Finally, we set

$$\phi_2(t) = \frac{\psi_2}{(\langle \psi_2, \psi_1 \rangle)^{1/2}} = \frac{t^2 - \frac{1}{5}}{\left(\int_{-1}^{1} (t^2 - \frac{1}{5})^2 (1 - t^2)\, dt \right)^{1/2}} = \frac{t^2 - \frac{1}{5}}{\frac{4\sqrt{42}}{105}} = \frac{105t^2 - 21}{4\sqrt{42}} .$$

4

For an $n \times n$ matrix A define $\langle x, y \rangle = \sum_{i,j=1}^{n} a_{ij} x_i y_j$. To determine under what conditions $\langle x, y \rangle$ defines an inner product on $\mathbb{R}^3$, we will need $n = 3$ for the definition to make sense. First note that if we write this sum in terms of matrix algebra we have

$$\langle x, y \rangle \equiv x^T A y = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where we use the symbol '$\equiv$' only to regard the fact that the matrix product above results in a 1×1 real matrix while $\sum_{i,j=1}^{n} a_{ij} x_i y_j$ is a real number. Since the set of 1×1 real matrices is isomorphic to the set of real numbers we can readily pass back and forth between the matrix algebra representation of $\langle x, y \rangle$ and $\sum_{i,j=1}^{n} a_{ij} x_i y_j$. We consider the definition of an inner product on $\mathbb{R}^3$ in order to find information about what properties A must have.

Let $x, y, z \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

1)

To have inner product it must be that $\langle x + y, z \rangle >= \langle x, z \rangle + \langle y, z \rangle$. However, we see that

$$\langle x + y, z \rangle = (x + y)^T A z = (x^T + y^T) A z = x^T A z + y^T A z = \langle x, z \rangle + \langle y, z \rangle$$

for any $A \in \mathbb{R}^{3 \times 3}$, so we gain no information.

2)

Another requirement for an inner product is that $\langle cx, y \rangle = c \langle x, y \rangle$. However, we see that

$$\langle cx, y\rangle = (cx)^T Ay = cx^T Ay = c\langle x, y\rangle$$

for any A and again gain no helpful information.

3) To have an inner product we require $\langle x, y\rangle = \langle y, x\rangle$. If A is symmetric, then $A = A^T$ and remembering that since $x^T Ay$ is a 1×1 matrix, it is also symmetric we have

$$\langle x, y\rangle = x^T Ay = (x^T Ay)^T = (x^T A^T y)^T = y^T Ax = \langle y, x\rangle \,.$$

Also, if it is the case that $\langle x, y = \sum_{i,j=1}^{n} a_{ij} x_i y_j = \sum_{i,j=1}^{n} a_{ij} y_i x_j = \langle y, x\rangle$ then this means the sum is the same even if i and j are reversed. That is, $a_{ij} = a_{ji}$ so that A must be symmetric. Therefore, it is a necessary and sufficient condition that A is symmetric.

4) We require that $\langle x, x\rangle \geq 0$ for any x and $\langle x, x\rangle = 0$ if and only if $x = 0$. Here this meanst that $x^T Ax \geq 0$ for any $x \in \mathbb{R}^3$ and $x^T Ax = 0$ if and only if $x = 0$. But this is the definition of positive definiteness of A (since we have already established that A must be symmetric it is acceptable to discuss positive definiteness). Since definition are bidirectional, the final requirement on $\langle \cdot, \cdot\rangle$ is satisfied if and only if A is a positive definite matrix.

The conclusion is that $\langle x, y\rangle = \sum_{i,j=1}^{n} a_{ij} x_i y_j$ defines an inner product on $\mathbb{R}^3$ if and only if A is a symmetric matrix that is also positive definite.

5

The system

$$Ax = \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{pmatrix} b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

has no solution. To see this, we use Gaussian elimination on an augmented matrix:

$$\begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The second and third row of this augment matrix shows that if $x = (x_1, x_2)$ is a solution to $Ax = b$, then $x_2 = 0$ and also that $0x_2 = 1$. But this would imply that $0 = 1$. Therefore, no $x \in \mathbb{R}^2$ can satisfy the system.

We instead seek an approximate solution $A\hat{x} \approx b$ using least squares approximation.

The system $A^T A\hat{x} = A^T b$ will have a unique solution $\hat{x}$ and $\hat{x}$ minimizes $||Ax - b||^2$. We have,

$$A^T A\hat{x} = A^T b$$

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{pmatrix} \hat{x} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \hat{x} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$\hat{x} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}.$$

Our approximation gives

$$A\hat{x} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/3 \\ -2/3 \end{pmatrix}.$$