# MA 502 Homework 3

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September 19, 2019

## Exercise 1

Given two bases $\mathbb{B}_1 = \{v_1, ..., v_n\}$ and $\mathbb{B}_2 = \{w_1, ..., w_n\}$ for the vector space $V$ and a linear transformation $L : V \to V$, prove that

$$[L]_{\mathbb{B}_2 \to \mathbb{B}_1}[a]_{\mathbb{B}_2} = [\mathbb{B}_2 \to \mathbb{B}_1][L]_{\mathbb{B}_1 \to \mathbb{B}_2}[a]_{\mathbb{B}_1} \ .$$

First we have that

$$[L]_{\mathbb{B}_1 \to \mathbb{B}_2}[a]_{\mathbb{B}_2} = [La]_{\mathbb{B}_1} \ .$$

Next we have

$$\begin{aligned}
[\mathbb{B}_2 \to \mathbb{B}_1][L]_{\mathbb{B}_1 \to \mathbb{B}_2}[a]_{\mathbb{B}_1} &= [\mathbb{B}_2 \to \mathbb{B}_1][La]_{\mathbb{B}_2} \\
&= [I]_{\mathbb{B}_2 \to \mathbb{B}_1}[La]_{\mathbb{B}_2} \\
&= [I(La)]_{\mathbb{B}_1} = [La]_{\mathbb{B}_1} \ .
\end{aligned}$$

Therefore, we conclude that

$$[L]_{\mathbb{B}_2 \to \mathbb{B}_1}[a]_{\mathbb{B}_2} = [\mathbb{B}_2 \to \mathbb{B}_1][L]_{\mathbb{B}_1 \to \mathbb{B}_2}[a]_{\mathbb{B}_1} \ .$$

## Exercise 2

Consider the linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ represented in canonical coordinates by the matrix

$$[L]_{\mathcal{C} \to \mathcal{C}} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ 2 & 1 & 3 \end{pmatrix} \ .$$

First we find the null space and range of $L$ using the transformation's matrix representation (since the matrix representation will provide all the information we need about the transformation to find these spaces).

The null space consists of vectors from $\mathbb{R}^3$ that will satisfy:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

For computation we use row reduction on the corresponding augmented matrix:

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 0 & 4 & 0 \\ 2 & 1 & 3 & 0 \end{pmatrix} &\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ -2 & 0 & -2 & 0 \\ 2 & 1 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ -2 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Translating the results of our matrix computation to the properties of $L$, we see that if $(x_1, x_2, x_3) \in \mathcal{N}(L)$, then $x_1 = -x_3$ and $x_2 = -x_3$, where we may choose $x_3$ at will. Thus the null space of $L$ is the set

$$\mathcal{N}(L) = \{c(-1, -1, 1) \in \mathbb{R}^3 , c \in \mathbb{R}\} .$$

To find the range of $L$, we first note that the columns of the matrix are dependent. In particular,

$$\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} .$$

Therefore the column space of the matrix is the same as the column space of the first two columns of the matrix. Since the column space of the matrix is a matrix representation of the range of the linear transformation $L$, we see that the range of $L$ is the set:

$$\{c_1(1, 4, 2) + c_2(2, 0, 1) \mid c_1, c_2 \in \mathbb{R}\} .$$

Consider the linear system $Lv = (1,2,0)$. To see if a solution exists, we use matrix computations:

$$\begin{pmatrix}1 & 2 & 3 & 1\\ 4 & 0 & 4 & 2\\ 2 & 1 & 3 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3 & 1\\ 0 & -8 & -8 & -2\\ 0 & -3 & -3 & 0\end{pmatrix}$$
$$\sim \begin{pmatrix}1 & 2 & 3 & 1\\ 0 & 0 & 0 & -2\\ 0 & 1 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & -2\end{pmatrix}.$$

In order for $v = (v_1, v_2, v_3)$ to satisfy the linear system, this row reduced system tells us that $0v_3 = -2$, so that $0 = -2$. Since this cannot be, we conclude that there is no solution to the system $Lv = (1,2,0)$.

Consider the linear system $Lv = (6,8,6)$. We use row reduction to determine whether solutions to this system exist:

$$\begin{pmatrix}1 & 2 & 3 & 6\\ 4 & 0 & 4 & 8\\ 2 & 1 & 3 & 6\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3 & 1\\ 0 & -8 & -8 & -16\\ 0 & -3 & -3 & -6\end{pmatrix}$$
$$\sim \begin{pmatrix}1 & 2 & 3 & 1\\ 0 & 1 & 1 & 2\\ 0 & -1 & -1 & -2\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3 & 6\\ 0 & 1 & 1 & 2\\ 0 & 0 & 0 & 0\end{pmatrix}$$
$$\sim \begin{pmatrix}1 & 0 & 1 & 2\\ 0 & 1 & 1 & 2\\ 0 & 0 & 0 & 0\end{pmatrix}$$

From this we have that if $v = (v_1, v_2, v_3)$ is a solution to the system it must be the case that $v_1 = 2 - v_3$ and $v_2 = 2 - v_3$. This shows that any vector of the form

$$v = \begin{pmatrix}2\\ 2\\ 0\end{pmatrix} + c\begin{pmatrix}-1\\ -1\\ 1\end{pmatrix} \quad c \in \mathbb{R}.$$

Since we may choose $c$ freely, there are infinitely many solutions to the system. For one specific solution, set $c = 0$ in the above to get $L(2,2,0) = (6,8,6)$.

## Exercise 3

Consider the operator $T : \mathbb{P} \to \mathbb{P}$, $T(p) = \int p(x)\,dx$, where $\mathbb{P}$ is the space of all polynomials. We find the null space and range of $T$.

Note that if $p(x)$ is not the zero polynomial, then $\int p(x)\,dx \neq 0$ and that if $p(x)$ is the zero polynomial, then $\int p(x)\,dx = 0$. So the null space of $T$ includes the zero polynomial but no other polynomial.

In order for $\mathbb{P}$ to indeed be a vector space, we must work under the agreement that the constant of integration is 0.

The range of $T$ is the set of all polynomials in $\mathbb{P}$ with a zero constant term. That is, $p(x)$ of the form $p(x) = a_1x + ... + a_nx^n$ for some natural number $n$. To see why, first we see that if $p(x)$ is of the form $p(x) = a_0 + a_1x + ... + a_nx^n$ with $a_0 \neq 0$, then there does not exist any polyomial $q(x) \in \mathbb{P}$ such that $T(q) = \int q(x)\,dx = p(x)$ since integration of a polynomial cannot result in any constant terms unless the constant of integration is nonzero, and we assumed that the constant of integration is zero. Next, if $p(x) = a_1x + ... + a_nx^n \in \mathbb{P}$, set $q(x) = a_1 + 2a_2x + ... + na_nx^{n-1}$. Then $T(q) = \int(a_1 + 2a_2x + ...na_nx^{n-1})\,dx = p(x)$.

## Exercise 4

Let $T : \mathbb{R}^3 \to \mathbb{R}^{100}$ be a linear transformation. We show that $T$ cannot be a surjective transformation.

In class we proved that if $T : X \to Y$ is a linear transformation between vectors spaces $X$ and $Y$ and if dim $Y >$ dim $X$, then $T$ is not surjective. Here we see that dim $\mathbb{R}^3 = 3$ and dim $\mathbb{R}^{100} = 100$. Since dim $\mathbb{R}^{100} >$ dim $\mathbb{R}^3$, we conclude that $T$ is not surjective.

## Exercise 5

Let $T : \mathbb{R}^{100} \to \mathbb{R}^3$ be a linear transformation. We show that $T$ cannot be an injective transformation.

In class we proved that if $T : X \to Y$ is a linear transformation between vectors spaces $X$ and $Y$ and if dim $Y <$ dim $X$, then $T$ is not injective.

Here we see that dim $\mathbb{R}^{100} = 100$ and dim $\mathbb{R}^3 = 3$. Since dim $\mathbb{R}^3 <$ dim $\mathbb{R}^{100}$, we conclude that $T$ is not injective.

It is possible for such a map to be onto. Take $T : \mathbb{R}^{100} \to \mathbb{R}^3$, $T((x_1, x_2, x_3, ..., x_{100})) = (x_1, x_2, x_3)$. Then for any element of $\mathbb{R}^3$, we see that any vector from $\mathbb{R}^{100}$ for which the first three components are the same as the target vector will suffice (while $x_4, x_5, ..., x_{100}$ may be taken arbitrarily).