MA502 Fall 2019 – Homework 1. Due August 29th, 2019

1. Consider the space V of all vectors

$$\{v = (v_1, ..., v_n) \in \mathbb{R}^n \text{ such that } v = \nabla f(0) = (\partial_1 f(0), \partial_2 f(0), ..., \partial_n f(0))$$
$$\text{for some } C^1 \text{ function } f \text{ defined in a neighborhood of the origin}\}.$$

   (1) Prove that V, equipped with the usual operations of vector sum and multiplication by a scalar is a vector space. (2) Prove that $V = \mathbb{R}^n$.

2. Show that if $\mathbb{X}$ and $\mathbb{Y}$ are subspaces of a vector space $\mathbb{V}$, then $\mathbb{X} \cap \mathbb{Y}$ is also a subspace of $\mathbb{V}$.

3. Consider

$$\mathbb{X} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \right\}, \tag{1}$$

$$\mathbb{Y} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid b_1 x_1 + b_2 x_2 + b_3 x_3 = 0 \right\}, \tag{2}$$

   where the a_i's and b_i's are given real numbers.

   (1) Prove that $\mathbb{X}$ and $\mathbb{Y}$ are vector spaces.

   (2) Describe $\mathbb{X} \cap \mathbb{Y}$ in geometric terms, considering all possible choices of the coefficients. Is $\mathbb{X} \cap \mathbb{Y}$ a vector space?

4. Which of the following are subspaces of the given vector spaces? Justify rigorously your answers. (1) $\{x \in \mathbb{R}^n : \ Ax = 0\} \subseteq \mathbb{R}^n$, where A is a given $m \times n$ matrix.

   (2) $\{p \in \mathbb{P} : \ p(x) = p(-x) \text{ for all } x \in \mathbb{R}\} \subseteq \mathbb{P}$, where $\mathbb{P}$ is the set of all polynomials with real coefficients.

   (3) $\{p \in \mathbb{P} : \ p \text{ has degree less or equal than } n\} \subseteq \mathbb{P}$.

   (4) $\{f \in C[0,1] : \ f(1) = 2f(0)\} \subseteq C[0,1]$, where $C[0,1]$ is the set of all continuous functions on $[0,1]$.

   (5) The unit sphere in $\mathbb{R}^n$.

5. In the following, determine the dimension of each subspace and find a basis for it.

(1) $\left\{ x = (x_1, x_2) \mid \; x_1 + x_3 = 0 \right\} \subseteq \mathbb{R} \times \mathbb{R}$.

(2) The set of all $n \times n$ square matrices with real coefficients that are equal to their transpose.

(3)$\{p \in \mathbb{P}_2 : \; p(0) = 0\} \subseteq \mathbb{P}_2$, where $\mathbb{P}_2$ is the set of all polynomials of degree ≤ 2.