MA502 Fall 2019 – Homework 5. Due October 10th, 2019

1. Let $X = C([0,1])$ denote the space of continuous functions defined in the unit interval. Prove that the map $T(g) = \int_0^1 g(x)dx$ is in X^*.

2. Consider a basis of $\mathbb{R}^3$ composed of the vectors

$$(1,0,-1),\ (1,1,1) \text{ and } (2,2,0)$$

find its dual basis.

3. Prove that the determinant, interpreted as a transformation

$$D : \mathbb{R}^{n^2} \to \mathbb{R} \ \text{ with } D(A) = determinant(A)$$

is linear in each of the rows. That is, if a row R of the matrix A is given by $R = \alpha R_1 + \beta \mathbb{R}_2$ with $R_1, R_2 \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, then

$$D(A) = \alpha D(A_1) + \beta D(A_2)$$

where A_i is the matrix constructed by taking A and replacing row R with tow R_i. This property is denoted as *the determinant is a multi-linear transformation row by row.*

4. Prove that the determinant map $D : \mathbb{R}^{n^2} \to \mathbb{R}$ defined above is *alternating*, i.e. if rows R_i and R_j in a matrix

$$A = \begin{pmatrix} R_1 \\ \dots \\ R_i, \\ \dots \\ R_j \\ \dots \\ R_n \end{pmatrix} \text{ are exchanged to obtain a new matrix } \tilde{A} = \begin{pmatrix} R_1 \\ \dots \\ R_j, \\ \dots \\ R_i \\ \dots \\ R_n \end{pmatrix}$$

then $D(A) = -D(\tilde{A})$.

5. Prove that for 2×2 matrices the determinant is the only map $D : \mathbb{R}^4 \to \mathbb{R}$ that is both multilinear as a function of the 2 rows and alternating, and that takes the value $D(I) = 1$ at the identity. The proof can be

done directly, using multilinearity and the alternating property. Just write any row in the matrix as a sum of vectors in the canonical basis.

Note This result, a characterization of the determinant, holds in any dimensions and can be used as an alternative (and equivalent) definition of the determinant.