MA502 Fall 2015 – Homework 7. Due November 12th, 2019

Write down detailed proofs of every statement you make

1. Let A be a $n \times n$ matrix and let J be the set of polynomials $f(t) \in \mathbb{K}(t)$ such that $f(A) = 0$. Prove that J is an ideal. Can you point out a specific polynomial of degree n and one of degree n^2 in J?

2. For any $n \times n$ matrix define the cofactor matrix coA to be the $n \times n$ matrix whose (i, j) entry is $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained from A deleting the $i-$th row and $j-$th columns. Let the classical adjoint matrix $ad(A)$ (also called adjugate or adjunct) be defined as the transpose of the cofactor matrix. Prove that $Aad(A) = ad(A)A = \det(A)I$.

3. Let A be an upper triangular $n \times n$ matrix.
   - Prove that all powers A^k are upper triangular.
   - Derive a formula for the eigenvalues of $f(A)$ when $f \in \mathbb{K}(t)$ is a polynomial.
   - Find a relation between the eigenvalues of a non-singular matrix A and those of its inverse A^{-1}
   - Using the property above, find the eigenvalues and the characteristic polynomial of
   $$(A^3 - 3A^2 + I)^{-1}$$
   where A is an upper triangular 3×3 matrix with eigenvalues $1, 0, -1$. (As part of the problem you will need to check that $A^3 - 3A^2 + I$ is indeed invertible even if A is clearly not so)

4. If A is a square matrix with eigenvalues $1, 2, 3$ find the eigenvalues of A^{100}. Provide a detailed proof of your answer (note we are not assuming that A is 3×3).