MA502 Fall 2019 – Practice Problems Set and Bonus points (Homework 11 and 12)
You can work this set of problems in groups, just list every member of the group below
Due on the last day of class of B term

Group:

Write down detailed proofs of every statement you make

1. Let A be a $n \times n$ diagonal matrix with eigenvalues $\lambda_i \in \mathbb{R}$. Find sufficient and necessary conditions for the eigenvalues so that the function

$$v \to \left(\sum_{i,j=1}^{n} a_{ij} v_i v_j \right)^{\frac{1}{2}}$$

is a norm in $\mathbb{R}^n$.

2. For $p \geq 1$, consider the $p-$norm

$$||(x,y)||_p = \left(|x|^p + |y|^p \right)^{\frac{1}{p}}$$

Sketch the shape of the unit ball, as well as balls of very large and very small radii as the parameter p varies among all integers and converges to ∞.

3. The ODE $x'' + dx' + kx = 0$ is a simple model of a damped mechanical oscillator with unit mass, in which x is the displacement from equilibrium, $d > 0$ is the drag coefficient, and $k > 0$ is the "spring constant."

   - write the ODE as a system of first order ODE of the form $z' = Az$ where A is a 2×2 matrix.
   - Find a change of basis that diagonalizes A
   - Solve the diagonal system and use this to find a solution for the original system

4. Let A be a real $n \times n$ matrix with an eigenvalue λ having algebraic multiplicity n. Prove that for any t real one has

$$e^{tA} = e^{\lambda t} \left(I + (A - \lambda I)t + ... + \frac{(A - \lambda I)^{n-1}}{(n-1)!} t^{n-1} \right)$$

5. Let A denote the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

- Find an orthogonal matrix O such that $O^T AO$ is diagonal
- Compute the matrix e^A.

6. Consider the vector space of polynomials with real coefficients and with inner product
$$\langle f, g\rangle := \int_{-1}^{1} f(t)g(t)(1-t^2)dt.$$
Apply the Graham-Schmidt process to find an orthonormal basis, with respect to this inner product, for the subspace generated by $\{\frac{\sqrt{3}}{2}, \frac{\sqrt{15}}{2}t, t^2\}$.

7. Let A be a real $n \times n$ matrix. Define $\langle x, y\rangle := \sum_{i,j=1}^{n} a_{ij}x_i y_j$. Find necessary and sufficient conditions on A for this operation to be a inner product on $\mathbb{R}^3$.

8. Show that the system $Ax = b$ has no solution and find the least square solution of the problem $Ax \approx b$ with
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

9. Let A be a $n \times n$ matrix. Prove that there exists a $n \times n$ matrix B such that $AB = 0$ and $rank(A) + rank(B) = n$.

10. Prove that any square matrix can be obtained as a limit of matrices which have all distinct eigenvalues (i.e. algebraic multiplicity one).

11. Prove that if A is a non-singular square matrix, then there exists a polynomial $f(t)$ such that $A^{-1} = f(A)$.