MA502 Fall 2019 – Homework 5. Due October 10th, 2019

1. Let $X = C([0,1])$ denote the space of continuous functions defined in the unit interval. Prove that the map $T(g) = \int_0^1 g(x)dx$ is in X^*.

2. Consider a basis of $\mathbb{R}^3$ composed of the vectors

$$(1,0,-1),\ (1,1,1) \text{ and } (2,2,0)$$

find its dual basis.

3. Prove that the determinant, interpreted as a transformation

$$D : \mathbb{R}^{n^2} \to \mathbb{R} \ \text{ with } D(A) = determinant(A)$$

is linear in each of the rows. That is, if a row R of the matrix A is given by $R = \alpha R_1 + \beta \mathbb{R}_2$ with $R_1, R_2 \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, then

$$D(A) = \alpha D(A_1) + \beta D(A_2)$$

where A_i is the matrix constructed by taking A and replacing row R with tow R_i. This property is denoted as *the determinant is a multilinear transformation row by row.*

4. Prove that the determinant map $D : \mathbb{R}^{n^2} \to \mathbb{R}$ defined above is *alternating*, i.e. if rows R_i and R_j in a matrix

$$A = \begin{pmatrix} R_1 \\ ... \\ R_i, \\ ... \\ R_j \\ ... \\ R_n \end{pmatrix} \text{ are exchanged to obtain a new matrix } \tilde{A} = \begin{pmatrix} R_1 \\ ... \\ R_j, \\ ... \\ R_i \\ ... \\ R_n \end{pmatrix}$$

then $D(A) = -D(\tilde{A})$.

5. Prove that for 2×2 matrices the determinant is the only map $D : \mathbb{R}^4 \to \mathbb{R}$ that is both multilinear as a function of the 2 rows and alternating, and that takes the value $D(I) = 1$ at the identity. The proof can be

done directly, using multilinearity and the alternating property. Just write any row in the matrix as a sum of vectors in the canonical basis.

Note This result, a characterization of the determinant, holds in any dimensions and can be used as an alternative (and equivalent) definition of the determinant.

MA 502 Homework 5

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1

Let $X = C([0,1])$ denote the space of continuous maps defined on the unit interval. We will prove that the map $T(g) = \int_0^1 g(x)\,dx$ is in $X^* = \{L : C([0,1]) \to \mathbb{R} \mid L \text{ is a linear map }\}$.

First, since a function that is continuous on $[0,1]$ is integrable on $[0,1]$, we are able to define $T(g)$ for all $g \in X$. Also, $\int_0^1 g(x)\,dx \in \mathbb{R}$ for all $g \in X$. Therefore, the definition $T : C([0,1]) \to \mathbb{R}$, $T(g) = \int_0^1 g(x)\,dx$ makes sense. To prove that T is linear, let $a \in \mathbb{R}$ and $g, h \in X = C([0,1])$. Then $T(ag) = \int_0^1 ag(x)\,dx = a\int_0^1 g(x)\,dx = aT(g)$. Also, $T(g+h) = \int_0^1 [g(x)+h(x)]\,dx = \int_0^1 g(x)\,dx + \int_0^1 h(x)\,dx = T(g) + T(h)$. We conclude that $T : X \to \mathbb{R}$ is a linear map and that so $T \in X^*$.

2

Consider the following basis for $\mathbb{R}^3$:

$$\{(1,0,-1),(1,1,1),(2,2,0)\}\ .$$

We will find the corresponding dual basis.

For a basis $\{b_1, ..., b_n\}$ of a vector space V, the dual basis $\{\beta_1, ...\beta_n\}$ consists of linear functions satisfying

$$\beta_i(b_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Here this means $\beta_1, \beta_2, \beta_3 : \mathbb{R}^3 \to \mathbb{R}$ satisfying

$$\beta_1(b_1) = 1,\ \beta_1(b_2) = 0,\ \beta_1(b_3) = 0.$$

$$\beta_2(b_1) = 0,\ \beta_2(b_2) = 1,\ \beta_2(b_3) = 0.$$

$$\beta_3(b_1) = 0,\ \beta_3(b_2) = 0,\ \beta_3(b_3) = 1.$$

Let $[\beta_i]$ denote the matrix representation of β_i with respect to the standard basis (as a 1x3 matrix). Based on our requirements for the properties of the members of the dual basis, it must be that:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} [\beta_1] \\ [\beta_2] \\ [\beta_3] \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \implies \begin{pmatrix} [\beta_1] \\ [\beta_2] \\ [\beta_3] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -1/2 & 1 & -1/2 \end{pmatrix}.$$

From this we see that the dual basis, using $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ using coordinates with respect to the standard basis, is

$$\{\beta_1(x) = x_1 - x_2,\ \beta_2(x) = x_1 - x_2 + x_3,\ \beta_3(x) = -\frac{x_1}{2} + x_2 - \frac{x_3}{2}\}.$$

It can be verified that the requirement

$$\beta_i(b_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

is in fact satisfied by the dual basis we have found.

3

Prove that the determinant, interpreted as the transformation

$$D : \mathbb{R}^{n^2} \to \mathbb{R}, \quad D(A) = \text{determinant}(A)$$

is linear in each of its rows.

Let R be the r^{th} row of the $n \times n$ matrix A, and suppose that $R = \alpha R_1 + \beta R_2$ where $R_1, R_2 \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. We define the matrix A_i as the matrix where row R (which is the r^{th} row of matrix A) is replaced by

R_i. We need to introduce notation to denote the elements of R, R_1 and R_2. We let

$$R = (a_{r1} \quad a_{r2} \quad ... \quad a_{rn}) \quad R_1 = (b_{r1} \quad b_{r2} \quad ... \quad b_{rn}) \quad R_2 = (c_{r1} \quad c_{r2} \quad ... \quad c_{rn})\ .$$

By hypothesis, $a_{ri} = \alpha b_{ri} + \beta c_{ri}$ for $i = 1, ..., n$. Using the permutation definition of the determinant of a matrix we have

$$\begin{aligned}
\alpha D(A_1) + \beta D(A_2) &= \alpha \sum_{\pi} \sigma(\pi) a_{1\pi(1)} ... a_{(r-1)\pi(r-1)} b_{r\pi(r)} ... a_{n\pi(n)} \\
&+ \beta \sum_{\pi} \sigma(\pi) a_{1\pi(1)} ... a_{(r-1)\pi(r-1)} c_{r\pi(r)} ... a_{n\pi(n)} \\
&= \sum_{\pi} \sigma(\pi) a_{1\pi(1)} ... a_{(r-1)\pi(r-1)} \alpha b_{r\pi(r)} ... a_{n\pi(n)} \\
&+ \sum_{\pi} \sigma(\pi) a_{1\pi(1)} ... a_{(r-1)\pi(r-1)} \beta c_{r\pi(r)} ... a_{n\pi(n)} \\
&= \sum_{\pi} \sigma(\pi) (a_{1\pi(1)} ... a_{(r-1)\pi(r-1)} a_{(r+1)\pi(r+1)} ... a_{n\pi(n)}) (\alpha b_{r\pi(r)} + \beta c_{r\pi(r)}) \\
&= \sum_{\pi} \sigma(\pi) (a_{1\pi(1)} ... a_{(r-1)\pi(r-1)} a_{(r+1)\pi(r+1)} ... a_{n\pi(n)}) (a_{r\pi(r)}) \\
&= \sum_{\pi} \sigma(\pi) a_{1\pi(1)} ... a_{(r-1)\pi(r-1)} a_{r\pi(r)} a_{(r+1)\pi(r+1)} ... a_{n\pi(n)} \\
&= D(A)
\end{aligned}$$

4

We prove that the determinant map, $D : \mathbb{R}^{n^2} \to \mathbb{R}$, $D(A) = \text{determinant}(A)$ is alternating. That is, if we exchange two rows of a matrix A, R_i and R_j with $j \neq i$ to get the matrix $\tilde{A}$, then $D(A) = -D(\tilde{A})$.

Although it is not necessary to write the multiplications in ascending order in the definition of the determinant, we assume without loss of generality that $i < j$ for organizational convenience. Using the definition of the determinant of A,

$$D(\tilde{A}) = \sum_{\pi} \sigma(\pi) a_{1\pi(1)} ... a_{ji\pi(i)} ... a_{i\pi(j)} ... a_{n\pi(n)} \tag{1}$$
$$= \sum_{\pi} \sigma(\pi) a_{1\pi(1)} ... a_{i\pi(j)} ... a_{j\pi(i)} ... a_{n\pi(n)} \tag{2}$$
$$= -\sum_{\pi} \sigma(\pi) a_{1\pi(1)} ... a_{i\pi(i)} ... a_{j\pi(j)} ... a_{n\pi(n)} \tag{3}$$
$$= -D(A). \tag{4}$$

We arrived at line 3 from line 2 of this calculation using the fact that the number of permutations changed by 1. If a permutation parity is even, changing the permutation count by 1 would give an odd permutation parity and reverse if a permutation parity is odd. Since $\sigma(\pi) = 1$ for even parity and $\sigma(\pi) = -1$ for odd parity, we switch the sign.

5

For 2×2 matrices, we prove that the determinant map is the only map $D : \mathbb{R}^{n^2} \to \mathbb{R}$ that is multilinear as a function of 2 rows, alternating, and for which $D(I) = 1$.

Consider the determinant of the arbitrary matrix A:

$$\begin{aligned}
D(A) &= D(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \\
&= D(\begin{pmatrix} ae_1^T & + & be_2^T \\ c & & d \end{pmatrix}) \\
&= D(\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}) + D(\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}) \quad \text{by multilinearity} \\
&= aD(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}) + bD(\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}) \quad \text{by multinearity} \\
&= aD(\begin{pmatrix} 1 & & 0 \\ ce_1^T & + & de_2^T \end{pmatrix}) + bD(\begin{pmatrix} 0 & & 1 \\ ce_1^T & + & de_2^T \end{pmatrix}) \\
&= a\left[D(\begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}) + D(\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix})\right] + b\left[D(\begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}) + D(\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix})\right] \quad \text{multilinearity} \\
&= a\left[cD(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}) + dD(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})\right] + b\left[cD(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) + dD(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix})\right] \quad \text{multilinearity}
\end{aligned}$$

$$= a\left[c(0) + dD(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})\right] + b\left[cD(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}) + d(0)\right] \quad (*)$$
$$= a\left[dD(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})\right] + b\left[-cD(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})\right] \quad \text{by the alternating property}$$
$$= ad - bc \quad \text{by } D(I) = 1\,.$$

In particular, if

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{then} \quad D(\begin{pmatrix}a & b\\ c & d\end{pmatrix}) = (1)(1) - (0)(0) = 1\,.$$

(*) By the alternating property of the determinant map and the fact that the rows are identical in each case,

$$D(\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}) = -D(\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}) \implies D(\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}) = 0$$

$$D(\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}) = -D(\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}) \implies D(\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}) = 0$$