MA 502 Homework 1

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Exercise 1

Consider the space $V = \{v = (v_1, \ldots, v_n) \in \mathbb{R}^n \mid v = \nabla f(0)$ for some function $f \in C^1(\mathbb{R}^n)$ defined in a neighborhood of the origin$\}$

(1) Prove that V, equipped with the usual vector sum and scalar multiplication operations, is a vector space.

To prove this statement, we verify that all the vector space axioms hold. (Note: I realize that much effort could be saved if we first just notice that $V \subseteq \mathbb{R}^n$ and then verify that V is closed under vector addition and multiplication by a scalar but perhaps this was not the intent of the exercise and I also didn't notice this until after checking all vector space axioms manually).

Proof

Let $u, v, w \in V$. By definition of the set V there exist functions $f, g, h : \mathbb{R}^n \to \mathbb{R}$ with each function continuously differentiable and defined in a neighborhood of the origin such that $u = \nabla f(0), v = \nabla g(0)$, and $w = \nabla h(0)$.

VS 1

$$\begin{aligned}
(u+v)+w &= (\nabla f(0)+\nabla g(0))+\nabla h(0)\\
&= [(\partial_1 f(0),...,\partial_n f(0))+(\partial_1 g(0),...,\partial_n g(0))]+(\partial_1 h(0),...,\partial_n h(0))\\
&= [(\partial_1 f(0)+\partial_1 g(0),...,\partial_n f(0)+\partial_n g(0))]+(\partial_1 h(0),...,\partial_n h(0))\\
&= (\partial_1 f(0)+\partial_1 g(0),...,\partial_n f(0)+\partial_n g(0))+(\partial_1 h(0),...,\partial_n h(0))\\
&= (\partial_1 f(0)+\partial_1 g(0)+\partial_1 h(0),...,\partial_n f(0)+\partial_n g(0)+\partial_n h(0))\\
&= (\partial_1 f(0)+(\partial_1 g(0)+\partial_1 h(0)),...,\partial_n f(0)+(\partial_n g(0)+\partial_n h(0)))\\
&= (\partial_1 f(0),...,\partial_n f(0))+(\partial_1 g(0)+\partial_1 h(0),...,\partial_n g(0)+\partial_n h(0))\\
&= (\partial_1 f(0),...,\partial_n f(0))+[(\partial_1 g(0),...,\partial_n g(0))+(\partial_1 h(0),...,\partial_n h(0))]\\
&= \nabla f(0)+(\nabla g(0)+\nabla h(0))\\
&= u+(v+w)\,.
\end{aligned}$$

VS 2

Let $i:\mathbb{R}^n\to\mathbb{R}$, $i(x_1,...,x_n)=0$. Then $\nabla i=(0,....,0)$ and in particular $\nabla i(0)=(0,...,0)$. Define the vector $O=(0,...,0)\in\mathbb{R}^n$. Then it is the case that $O=\nabla i(0)$, so that $O\in V$. We claim that O is the additive identity of V. To see this note that

$$\begin{aligned}
u+O &= (\partial_1 f(0),...,\partial_n f(0))+(0,...,0)\\
&= (\partial_1 f(0)+0,...,\partial_n f(0)+0)\\
&= (0+\partial_1 f(0),...,0+\partial_n f(0))=O+u
\end{aligned}$$

$$O+u=(0+\partial_1 f(0),...,0+\partial_n f(0))=(\partial_1 f(0),...,\partial_n f(0))=u\,.$$

Therefore $O\in V$ and since u was arbitrary $O+u=u+O=u$ for any choice of $u\in V$.

VS 3

Given $u=\nabla f(0)$, let $-u=-\nabla f(0)$. Then it is the case that $-u\in V$ and we have

$$u+(-u)=\nabla f(0)-\nabla f(0)=(\partial_1 f(0)-\partial_1 f(0),...,\partial_n f(0)-\partial_n f(0))=(0,...,0)=O\,.$$

VS 4

Given $u, v \in V$, with $u = \nabla f(0), v = \nabla g(0)$, we have

$$\begin{aligned}
u + v &= (\partial_1 f(0), ..., \partial_n f(0)) + (\partial_1 g(0), ..., \partial_n g(0)) \\
&= (\partial_1 f(0) + \partial_1 g(0), ..., \partial_n f(0) + \partial_n g(0)) \\
&= (\partial_1 g(0) + \partial_1 f(0), ..., \partial_n g(0) + \partial_n f(0)) \\
&= v + u\,.
\end{aligned}$$

VS 5

Let c be a scalar (we assume the field in this exercise is the field of real numbers). Then

$$\begin{aligned}
c[u + v] &= c[(\partial_1 f(0) + \partial_1 g(0), ..., \partial_n f(0) + \partial_n g(0))] \\
&= (c\partial_1 f(0) + c\partial_1 g(0), ..., c\partial_n f(0) + c\partial_n g(0)) \\
&= (c\partial_1 f(0), ..., c\partial_n f(0)) + (c\partial_1 g(0), ..., c\partial_n g(0)) \\
&= c(\partial_1 f(0), ..., \partial_n f(0)) + c(\partial_1 g(0), ..., \partial_n g(0)) \\
&= cu + cv\,.
\end{aligned}$$

VS 6

If a and b are two numbers, then

$$\begin{aligned}
(a + b)u &= ((a + b)\partial_1 f(0), ..., (a + b)\partial_n f(0)) \\
&= (a\partial_1 f(0) + b\partial_1 f(0), ..., a\partial_n f(0) + b\partial_n f(0)) \\
&= (a\partial_1 f(0), ..., a\partial_n f(0)) + (b\partial_1 f(0), ..., b\partial_n f(0)) \\
&= a(\partial_1 f(0), ..., \partial_n f(0)) + b(\partial_1 f(0), ..., \partial_n f(0)) \\
&= au + bu\,.
\end{aligned}$$

VS 7

If a and b are two numbers, then

$$\begin{aligned}
(ab)u &= ((ab)\partial_1 f(0), ..., (ab)\partial_n f(0)) \\
&= (a(b\partial_1 f(0)), ..., a(b\partial_n f(0))) \\
&= a(b\partial_1 f(0), ..., b\partial_n f(0)) \\
&= a(bu)\,.
\end{aligned}$$

VS 8

Using the arbitrary element $u \in V$, we have

$$1u = 1(\partial_1 f(0), ..., \partial_n f(0)) = (1\partial_1 f(0), ..., 1\partial_n f(0)) = (\partial_1 f(0), ..., \partial_n f(0)) = u$$

We conclude that the space V is a vector space.

(2) Prove that $V = \mathbb{R}^n$.

Let $v \in V$. There exists a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $v = \nabla f(0) = (\frac{\partial f}{\partial x_1}(0), ..., \frac{\partial f}{\partial x_n}(0)) = (\partial_1 f(0), ..., \partial_n f(0))$. Since f maps from $\mathbb{R}^n$ to $\mathbb{R}$, it is also the case that $\frac{\partial f}{\partial x_i} : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, ..., n$. But this means that when evaluating the partial derivatives we have $\frac{\partial f}{\partial x_i}(0) \in \mathbb{R}$ for $i = 1, 2, .., n$ so that $v = (\frac{\partial f}{\partial x_1}(0), ..., \frac{\partial f}{\partial x_n}(0)) = (\partial_1 f(0), ..., \partial_n f(0)) \in \mathbb{R}^n$. Therefore, $V \subseteq \mathbb{R}^n$.

Let $P = (c_1, c_2, ..., c_n) \in \mathbb{R}^n$. Consider the function $g : \mathbb{R}^n \to \mathbb{R}$ given by the rule $g(x_1, x_2, ..., x_n) = c_1x_1 + c_2x_2 + ... + c_nx_n$. Then we have $\nabla g = (c_1, ..., c_n)$, which means $\nabla g(0) = (c_1, ..., c_n)$ as well. If we define $w = \nabla g(0)$ we see that since g is defined in any neighborhood of zero and is continuously differentialable that $P = (c_1, ..., c_n) = w \in V$. Therefore $\mathbb{R}^n \subseteq V$.

Since $V \subseteq \mathbb{R}^n$ and $\mathbb{R}^n \subseteq V$ we conclude that $V = \mathbb{R}^n$.

Exercise 2

Let $\mathbb{V}$ be a vector space over the field K. If $\mathbb{X}$ and $\mathbb{Y}$ are subspaces of $\mathbb{V}$, then the intersection $\mathbb{X} \cap \mathbb{Y}$ is also a subspace of $\mathbb{V}$.

Proof: To prove that $\mathbb{X} \cap \mathbb{Y}$ is a subspace of $\mathbb{V}$, it suffices to show that $\mathbb{X} \cap \mathbb{Y}$ is a subset of $\mathbb{V}$ that is closed under scalar multiplication and vector addition.

Since $\mathbb{X} \cap \mathbb{Y} \subseteq \mathbb{X}$ and $\mathbb{X} \subseteq \mathbb{V}$, we have $\mathbb{X} \cap \mathbb{Y} \subseteq \mathbb{V}$.

Let $u, v \in \mathbb{X} \cap \mathbb{Y}$. Then $u, v \in \mathbb{X}$. Since $\mathbb{X}$ is a subspace of $\mathbb{V}$, $\mathbb{X}$ is itself a vector space, so that it must be the case that $u + v \in \mathbb{X}$. Similarly, we know that $u, v \in \mathbb{Y}$ and that $\mathbb{Y}$ is also a vector space. Thus, $u + v \in \mathbb{Y}$. We have therefore shown that $u + v \in \mathbb{X} \cap \mathbb{Y}$.

Let $k \in K$ and $w \in \mathbb{X} \cap \mathbb{Y}$. Since $w \in \mathbb{X}$ and $\mathbb{X}$ is a vector space, we may conclude that $kw \in \mathbb{X}$ (vector spaces are closed under scalar multiplication). Similarly, since $w \in \mathbb{Y}$ and $\mathbb{Y}$ is a vector space we have that $kw \in \mathbb{Y}$. Therefore $kw \in \mathbb{X} \cap \mathbb{Y}$.

Since $\mathbb{X} \cap \mathbb{Y}$ is a subset of $\mathbb{V}$ and is closed under vector addition and scalar multiplication, we conclude that $\mathbb{X} \cap \mathbb{Y}$ is a subspace of $\mathbb{V}$.

Note: The textbook mentions that we must show that $\mathbb{X} \cap \mathbb{Y}$ contains the zero vector of the space $\mathbb{V}$ but this actually follows from the fact that $\mathbb{X} \cap \mathbb{Y}$ is closed under scalar multiplication (use the identity element of the field K as the scalar).

Exercise 3

Consider

$$\mathbb{X} = \{x = (x_1, x_2, x_3) \mid \; a_1 x_1 + a_2 x_2 + a_3 x_3 = 0\}$$

$$\mathbb{Y} = \{x = (x_1, x_2, x_3) \mid \; b_1 x_1 + b_2 x_2 + b_3 x_3 = 0\}$$

$$\text{where } a_i, b_i \in \mathbb{R} \text{ for } i = 1, 2, 3 \, .$$

(1)

To prove that $\mathbb{X}$ and $\mathbb{Y}$ are vector spaces we just prove the case for $\mathbb{X}$ since the proof for $\mathbb{Y}$ is identical if we replace the $'a'$ coefficients with $'b'$ coefficients.

Instead of verifying each vector space axiom, we instead first note that $\mathbb{X} \subseteq \mathbb{R}^3$. Since $\mathbb{R}^3$ is already known to be a vector space, it suffices to show that $\mathbb{X}$ is closed under vector addition and scalar multiplication (with $\mathbb{X}$ inheriting the usual operations for the vector space $\mathbb{R}^3$).

Let $x, y \in \mathbb{X}$ with $x = (a_1 x_1, a_2 x_2, a_3 x_3)$ and $y = (a_1 y_1, a_2 y_2, a_3 y_3)$ and let $k \in \mathbb{R}$. We have

$$x + y = (a_1x_1 + a_1y_1, a_2x_2 + a_2y_2, a_3x_3 + a_3y_3)\ .$$

To see that $x + y \in \mathbb{X}$, note that

$$a_1x_1+a_1y_1+a_2x_2+a_2y_2+a_3x_3+a_3y_3 = a_1x_1+a_2x_2+a_3x_3+a_1y_1+a_2y_2+a_3y_3 = 0+0 = 0\ .$$

Consider next $kx = (kc_1x_1, kc_2x_2, kc_3x_3)$. Since $kc_1x_1 + kc_2x_2 + kc_3x_3 = k(c_1x_1 + c_2x_2 + c_3x_3) = k(0) = 0$, we know $kx \in \mathbb{X}$.

Since $\mathbb{X}$ is closed under scalar multiplication and vector addition, $\mathbb{X}$ is a subspace of $\mathbb{R}^3$ and therefore a vector space.

(2)

There are several geometric possibilities for $\mathbb{X} \cap \mathbb{Y}$ considering that although we know a_1, a_2, a_3, b_1, b_2, and b_3 are real numbers we have not specified exactly which real numbers they are.

In the case that both $\mathbb{X}$ and $\mathbb{Y}$ are planes, then by their definition they must both pass through the origin. If $a_i = b_i$ for each i, then these two planes are actually the same plane so that the intersection is also a plane. If the two planes are distinct then the intersection is a line in $\mathbb{R}^3$.

It may also be the case that $a_1 = a_2 = a_3 = 0$ (or similarly with the b_i's). If this is the case, then the geometric representation of $\mathbb{X}$ (or similarly $\mathbb{Y}$) is not a plane but in fact all of $\mathbb{R}^3$. If it is the case that $\mathbb{X}$ is all of $\mathbb{R}^3$ but $\mathbb{Y}$ is a plane then the intersection will be the plane that represents $\mathbb{Y}$. If it is the case that $a_i = b_i = 0$ for each i, then $\mathbb{X} = \mathbb{Y} = \mathbb{R}^3$ so that $\mathbb{X} \cap \mathbb{Y} = \mathbb{R}^3$.

We conclude that there are three possible geometric representations of $\mathbb{X} \cap \mathbb{Y}$: a line passing through the origin, a plane passing through the origin, or all of $\mathbb{R}^3$.

Yes $\mathbb{X} \cap \mathbb{Y}$ is a vector space. In part (1) of this exercise, we showed that both $\mathbb{X}$ and $\mathbb{Y}$ are both subspaces of the vector space $\mathbb{R}^3$. By the result of Exercise 2, we may conclude that $\mathbb{X} \cap \mathbb{Y}$ is also a subspace of $\mathbb{R}^3$ and therefore a vector space. We may also appeal to the geometric discussion we just provided: a line, a plane, or all of $\mathbb{R}^3$ are all subspaces of $\mathbb{R}^3$. Note that although the empty set $\emptyset$ is also a subspace of $\mathbb{R}^3$, it is never the case that $\mathbb{X} \cap \mathbb{Y} = \emptyset$ since the origin is an element of both sets no matter our choice of coefficients.

Exercise 4

(1)

The set $X = \{x \in \mathbb{R}^n \mid Ax = 0\}$, where A is a given $m \times n$ matrix, **is** a subspace of the vector space $\mathbb{R}^n$.

Let $x, y \in X$, so that $Ax = Ay = 0$. Then we have $A(x+y) = Ax+Ay = 0 + 0 = 0$, which shows that $x + y \in X$.

Let $c \in \mathbb{R}$. We have $A(cx) = c(Ax) = c(0) = 0$, which shows that $cx \in X$. Thus X is a subspace of $\mathbb{R}^n$.

(2)

The set $X = \{p \in \mathbb{P} \mid p(x) = p(-x) \text{ for all } x \in \mathbb{R}\}$ **is** is a subspace of the vector space of all polynomials with real coefficients $\mathbb{P}$.

Let $p, q \in X$. Then $(p + q)(x) = p(x) + q(x) = p(-x) + q(-x) = (p + q)(-x)$. Thus $p + q \in X$.

Let $k \in \mathbb{R}$. Then $kp(x) = kp(-x)$, which shows that if $p \in X$ it is also the case that $kp \in X$. Thus X is a subspace of $\mathbb{P}$.

(3)

The set $X = \{p \in \mathbb{P} \mid p \text{ has degree less than or equal to } n\}$ **is** a subspace of $\mathbb{P}$.

Let $p, q \in X$. Then p and q are polynomials of degree at most n. The result of summing of two polynomials of degree at most n (to do this perform the real number sums of the corresponding coefficients of each polynomial as usual) must also be a polynomial of degree at most n. Therefore, $p+q \in X$.

Let $k \in \mathbb{R}$. Then since the degree of $p \in X$ is at most n, the polynomial kp is of degree at most n (note that if $k = 0$, then kp is the zero polynomial, but this is still of degree less than n no matter which of the common conventions we use for defining the degree of the zero polynomial - but it is impossible for kp to be of higher degree than p). Thus $kp \in X$.

Therefore X is subspace of $\mathbb{P}$.

(4)

The set $X = \{f \in C[0, 1] \mid f(1) = 2f(0)\}$ **is** a subspace of $C[0, 1]$, where $C[0, 1]$ is the set of all continuous functions on $[0, 1]$.

Let $f, g \in X$, which means that f and g are continuous and that $f(1) = 2f(0), g(1) = 2g(0)$. Then we have

$$(f+g)(1) = f(1) + g(1) = 2f(0) + 2g(0) = 2(f+g)(0) \ .$$

This shows that $f + g \in X$

Let $k \in \mathbb{R}$. Then for $f \in X$, since $f(1) = 2f(0)$ it follows that $kf(1) = k[2f(0)] = 2kf(0)$. Thus $kf \in X$ as well.

Therefore X is a subspace of $C[0,1]$.

(5)

The unit sphere in $\mathbb{R}^n$ **is not** a subspace of $\mathbb{R}^n$. Since the claim that the unit sphere in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ in the general case (that is, for any choice of n), it suffices to give a counterexample for $n = 2$. This counterexample is amenable to generalization if desired.

Let $(x_1, y_1), (x_2, y_2) \in U$, where U is the unit sphere in $\mathbb{R}^2$. This means $x_1^2 + y_1^2 = 1$ and $x_2^2 + y_2^2 = 1$. We show that U is not closed under vector addition. We define vector addition in the usual way, giving

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \ .$$

However, we see that

$$(x_1+x_2)^2+(y_1+y_2)^2 = x_1^2+y_1^2+x_2^2+y_2^2 2x_1x_2+2y_1y_2 = 2+2x_1x_2+2y_1y_2 \ .$$

For $(x_1, y_1) + (x_2, y_2) \in U$, we would need $2 + 2x_1x_2 + 2y_1y_2 = 1$. To see why this does not hold generally, take $(x_1, y_1) = (1, 0)$ and $(x_2, y_2) = (0, 1)$. Then $(1, 0) + (0, 1) = (1, 1) \notin U$.

Exercise 5

(1)

Let $X = \{(x_1, x_2) \mid x_1 + x_2 = 0\} \subseteq \mathbb{R} \times \mathbb{R}$. Then X is a subspace of dimension 1. The element $(1, -1) \in X$ is a basis for X. The zero vector of X is the element $(0, 0)$. The only solution to the equation $c(1, -1) = (0, 0)$ is the trivial case that $c = 0$. Thus the set $\{(1, -1)\}$ is then clearly a linearly independent set. To show that $(1, -1)$ generates X, let $(x_1, x_2) \in X$. Then we know that $x_1 + x_2 = 0 \implies -x_1 = x_2$, so that we may write

$(x_1, x_2) = (x_1, -x_1)$. Since $x_1 \in \mathbb{R}$, which also happens to be our scalar field in this example, we take $x_1(1, -1) = (x_1, -x_1) = (x_1, x_2)$. Since this vector was an arbitrary element of X we have shown that $(1, -1)$ generates X and may conclude that X is a subspace of dimension 1 with the element $(1, -1)$ as a basis for the subspace.

(2)

Let M be the set of all $n \times n$ symmetric matrices with real entries with proposed field also the set $\mathbb{R}$. Then M is a subspace of the vector space of all $n \times n$ real matrices. The dimension of M is 6 and a basis for the subspace is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

Since it is not required by the exercise to rigorously prove these statements and the proof that this is a basis will be very tedious, the proof is omitted.

(3)

Let $Q = \{p \in \mathbb{P}_2 \mid p(0) = 0\} \subseteq \mathbb{P}_2$. Note that $p \in \mathbb{P}_2$ must be of the form $p(x) = k_2x^2 + k_1x + k_0$. If $p \in Q$ as well we require $0 = p(0) = k_0$. This shows that any $q \in Q$ must be of the form $q(x) = k_2x^2 + k_1x$. Then we see that Q is a subspace of $\mathbb{P}_2$ of dimension 2 where we may use the set $\{x, x^2\}$ as a basis. This set is linearly independent and generates Q so we may conclude that it is indeed a basis for the subspace.