MA502 Fall 2019 – Homework 6. Due October 31st, 2019

Write down detailed proofs of every statement you make

1. A 4×4 matrix A has eigenvalues $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 2, \lambda_4 = -1$.
   - Is A invertible? Why or why not?
   - Is A diagonalizable? Why or why not?
   - Find the the characteristic polynomial, the trace and determinant of A.

2. Find a relation between the eigenvalues of a non-singular matrix A and those of its inverse A^{-1}

3. Find a relation between the eigenvalues of a matrix A and those of its square $A^2 = AA$

4. For the following either find an example or prove that such example cannot exist:
   (a) A 4×4 matrix with eigenvalues $\lambda_1 = 1$ with algebraic multiplicity 2 and geometric multiplicity 1; $\lambda_2 = 2$ with algebraic multiplicity 1 and geometric multiplicity 1 and $\lambda_3 = 3$ with algebraic multiplicity 1 and geometric multiplicity 1.
   (b) A 4×4 matrix with eigenvalues $\lambda_1 = 1$ with algebraic multiplicity 1 and geometric multiplicity 2; $\lambda_2 = 2$ with algebraic multiplicity 2 and geometric multiplicity 1 and $\lambda_3 = 3$ with algebraic multiplicity 1 and geometric multiplicity 1.
   (c) A 4×4 matrix with eigenvalues $\lambda_1 = 1$ with algebraic multiplicity 2 and geometric multiplicity 1; $\lambda_2 = 2$ with algebraic multiplicity 2 and geometric multiplicity 1 and $\lambda_3 = 3$ with algebraic multiplicity 1 and geometric multiplicity 1.
   (d) A 4×4 matrix with one eigenvalue $\lambda_1 = \pi$ with algebraic multiplicity 4 and geometric multiplicity 1;

5. Construct a 3×3 matrix A with eigenvalues π, π^2, π^3 and corresponding eigenvectors $(1, 0, 1)$, $(1, 1, 0)$, $(0, 0, 1)$. Is such matrix unique?

MA 502 Homework 6

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1

Assume that A is a 4×4 matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 2$, and $\lambda_4 = -1$.

Claim: A is not invertible.

Proof: Since $\lambda_2 = 0$ is an eigenvalue of A, then

$$0 = \det(A - \lambda_2 I) = \det(A - 0I) = \det(A) \ .$$

This shows that $\det(A) = 0$, so we conclude that A is not invertible.

Claim: A is diagonalizable.

Proof: Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T(x) = Ax$. We know that T has four distinct eigenvalues since for each eigenvalue λ of the matrix A, with corresponding eigenvector v, $T(v) = Av = \lambda v$. Since T has four distinct eigenvalues we can find four distinct eigenvectors. Therefore, T is a diagonalizable transformation, so A must be a diagonalizable matrix.

The characteristic polynomial of A is $p(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda + 1)$ since for this polynomial, $p(\lambda) = 0$ precisely when $\lambda \in \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and the degree of p does not exceed 4. The trace of A, $\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2$. Since we know that A is not invertible, $\det(A) = 0$. It is also the case that since the determinant of A is the product of the eigenvalues of A, we could also verify this statement by calculating $\det(A) = \lambda_1\lambda_2\lambda_3\lambda_4 = 0$.

2

Assume that A is a nonsingular matrix and that λ is an eigenvalue of A. We know that $\lambda \neq 0$ since if $\lambda = 0$ is an eigenvalue of A, then A is singular by the previous exercise. Suppose that v is an eigenvector corresponding to the eigenvalue λ. Then,

$$\begin{aligned} Av &= \lambda v \\ A^{-1}Av &= A^{-1}\lambda v \\ v &= \lambda A^{-1} v \\ \frac{1}{\lambda} v &= A^{-1} v \ . \end{aligned}$$

Since each of these steps is reversible, this shows that λ is and eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

3

Suppose that λ is an eigenvalue of the matrix A. Then for an eigenvector v corresponding to λ,

$$\begin{aligned} &Av = \lambda v \implies \\ &A^2 v = A\lambda v = \lambda A v = \lambda^2 v \ . \end{aligned}$$

This shows that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

Suppose that λ^2 is an eigenvalue of A^2 and that v. This means that

$$0 = \det(A^2 - \lambda^2 I) = \det[(A - \lambda I)(A + \lambda I)] = \det(A - \lambda I)\det(A + \lambda I) \ .$$

So we may conclude that if λ^2 is an eigenvalue of A^2, then λ is an eigenvalue of A or $-\lambda$ is an eigenvalue of A. This is the strongest statement we can make without further assumptions - it is not necessarily the case that both λ and $-\lambda$ must be eigenvalues of A. To see this consider:

$$A^2 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \ .$$

Then 4 is an eigenvalue of A^2 but -2 is not an eigenvalue of A.

4

(a) The matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

has eigenvalue $\lambda_1 = 1$ with algebraic multiplicity 2 and geometric multiplicity 1, eigenvalue $\lambda_2 = 2$ with algebraic and geometric multiplicities both of 1, and eigenvalue λ_3 with algebraic and geometric multiplicities 1 as well.

(b) It is not possible to construct a matrix such that the geometric multiplicity of an eigenvalue exceeds the algebraic multiplicity of the same eigenvalue. Here we are given that the eigenvalue $\lambda_1 = 1$ should have algebraic multiplicity 1 and geometric multiplicity 2. No matrix can satisfy this requirement.

(c) It is not possible to construct a 4×4 matrix such that the sum of the algebraic multiplicities of the eigenvalues of the matrix exceed 4. A 4×4 matrix can be used as a linear transformation with a 4-dimensional domain. The sum of the algebraic multiplicities of the eigenvalues must not be larger than the dimension of the domain of such a transformation. Here we require that the algebraic multiplicities of the eigenvalues sum to 5. This is not possible.

(d) The matrix

$$D = \begin{pmatrix} \pi & \pi & \pi & \pi \\ 0 & \pi & \pi & \pi \\ 0 & 0 & \pi & \pi \\ 0 & 0 & 0 & \pi \end{pmatrix}$$

has eigenvalue $\lambda = \pi$, where λ has algebraic multiplicity 4 and geometric multiplicity 1.

5

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi^2 & 0 \\ 0 & 0 & \pi^3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \pi & \pi^2 - \pi & 0 \\ 0 & \pi^2 & 0 \\ \pi - \pi^3 & \pi^3 - \pi & \pi^3 \end{pmatrix}.$$

Then we have

$$A\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}\pi\\0\\\pi\end{pmatrix} = \pi\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$A\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}\pi^2\\\pi^2\\0\end{pmatrix} = \pi^2\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$A\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\pi^3\end{pmatrix} = \pi^3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

as desired. Note that $(1,0,1),(1,1,0),(0,0,1)$ form a basis for $\mathbb{R}^3$ since they form a collection of three linearly independent vectors in $\mathbb{R}^3$. The transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$, $T_A(x) = Ax$ is uniquely determined by specifying its values on a basis for $\mathbb{R}^3$. So the corresponding matrix A is the only 3×3 matrix with the given eigenvalue/eigenvector pairs.