

MA 502 Homework 5

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1

Let $X = C([0, 1])$ denote the space of continuous maps defined on the unit interval. We will prove that the map $T(g) = \int_0^1 g(x) dx$ is in $X^* = \{L : C([0, 1]) \rightarrow \mathbb{R} \mid L \text{ is a linear map}\}$.

First, since a function that is continuous on $[0, 1]$ is integrable on $[0, 1]$, we are able to define $T(g)$ for all $g \in X$. Also, $\int_0^1 g(x) dx \in \mathbb{R}$ for all $g \in X$. Therefore, the definition $T : C([0, 1]) \rightarrow \mathbb{R}$, $T(g) = \int_0^1 g(x) dx$ makes sense. To prove that T is linear, let $a \in \mathbb{R}$ and $g, h \in X = C([0, 1])$. Then $T(ag) = \int_0^1 ag(x) dx = a \int_0^1 g(x) dx = aT(g)$. Also, $T(g + h) = \int_0^1 [g(x) + h(x)] dx = \int_0^1 g(x) dx + \int_0^1 h(x) dx = T(g) + T(h)$. We conclude that $T : X \rightarrow \mathbb{R}$ is a linear map and that so $T \in X^*$.

2

Consider the following basis for \mathbb{R}^3 :

$$\{(1, 0, -1), (1, 1, 1), (2, 2, 0)\}.$$

We will find the corresponding dual basis.

For a basis $\{b_1, \dots, b_n\}$ of a vector space V , the dual basis $\{\beta_1, \dots, \beta_n\}$ consists of linear functions satisfying

$$\beta_i(b_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Here this means $\beta_1, \beta_2, \beta_3 : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfying

$$\beta_1(b_1) = 1, \beta_1(b_2) = 0, \beta_1(b_3) = 0.$$

$$\beta_2(b_1) = 0, \beta_2(b_2) = 1, \beta_2(b_3) = 0.$$

$$\beta_3(b_1) = 0, \beta_3(b_2) = 0, \beta_3(b_3) = 1.$$

Let $[\beta_i]$ denote the matrix representation of β_i with respect to the standard basis (as a 1×3 matrix). Based on our requirements for the properties of the members of the dual basis, it must be that:

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} [\beta_1] \\ [\beta_2] \\ [\beta_3] \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \implies \begin{pmatrix} [\beta_1] \\ [\beta_2] \\ [\beta_3] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -1/2 & 1 & -1/2 \end{pmatrix}. \end{aligned}$$

From this we see that the dual basis, using $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ using coordinates with respect to the standard basis, is

$$\{\beta_1(x) = x_1 - x_2, \beta_2(x) = x_1 - x_2 + x_3, \beta_3(x) = -\frac{x_1}{2} + x_2 - \frac{x_3}{2}\}.$$

It can be verified that the requirement

$$\beta_i(b_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

is in fact satisfied by the dual basis we have found.

3

Prove that the determinant, interpreted as the transformation

$$D : \mathbb{R}^{n^2} \rightarrow \mathbb{R}, \quad D(A) = \text{determinant}(A)$$

is linear in each of its rows.

Let R be the r^{th} row of the $n \times n$ matrix A , and suppose that $R = \alpha R_1 + \beta R_2$ where $R_1, R_2 \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. We define the matrix A_i as the matrix where row R (which is the r^{th} row of matrix A) is replaced by

R_i . We need to introduce notation to denote the elements of R, R_1 and R_2 .
We let

$$R = (a_{r1} \ a_{r2} \ \dots \ a_{rn}) \quad R_1 = (b_{r1} \ b_{r2} \ \dots \ b_{rn}) \quad R_2 = (c_{r1} \ c_{r2} \ \dots \ c_{rn}) .$$

By hypothesis, $a_{ri} = \alpha b_{ri} + \beta c_{ri}$ for $i = 1, \dots, n$. Using the permutation definition of the determinant of a matrix we have

$$\begin{aligned} \alpha D(A_1) + \beta D(A_2) &= \alpha \sum_{\pi} \sigma(\pi) a_{1\pi(1)} \dots a_{(r-1)\pi(r-1)} b_{r\pi(r)} \dots a_{n\pi(n)} \\ &\quad + \beta \sum_{\pi} \sigma(\pi) a_{1\pi(1)} \dots a_{(r-1)\pi(r-1)} c_{r\pi(r)} \dots a_{n\pi(n)} \\ &= \sum_{\pi} \sigma(\pi) a_{1\pi(1)} \dots a_{(r-1)\pi(r-1)} \alpha b_{r\pi(r)} \dots a_{n\pi(n)} \\ &\quad + \sum_{\pi} \sigma(\pi) a_{1\pi(1)} \dots a_{(r-1)\pi(r-1)} \beta c_{r\pi(r)} \dots a_{n\pi(n)} \\ &= \sum_{\pi} \sigma(\pi) (a_{1\pi(1)} \dots a_{(r-1)\pi(r-1)} a_{(r+1)\pi(r+1)} \dots a_{n\pi(n)}) (\alpha b_{r\pi(r)} + \beta c_{r\pi(r)}) \\ &= \sum_{\pi} \sigma(\pi) (a_{1\pi(1)} \dots a_{(r-1)\pi(r-1)} a_{(r+1)\pi(r+1)} \dots a_{n\pi(n)}) (a_{r\pi(r)}) \\ &= \sum_{\pi} \sigma(\pi) a_{1\pi(1)} \dots a_{(r-1)\pi(r-1)} a_{r\pi(r)} a_{(r+1)\pi(r+1)} \dots a_{n\pi(n)} \\ &= D(A) \end{aligned}$$

4

We prove that the determinant map, $D : \mathbb{R}^{n^2} \rightarrow \mathbb{R}$, $D(A) = \text{determinant}(A)$ is alternating. That is, if we exchange two rows of a matrix A , R_i and R_j with $j \neq i$ to get the matrix \tilde{A} , then $D(A) = -D(\tilde{A})$.

Although it is not necessary to write the multiplications in ascending order in the definition of the determinant, we assume without loss of generality that $i < j$ for organizational convenience. Using the definition of the

determinant of A ,

$$D(\tilde{A}) = \sum_{\pi} \sigma(\pi) a_{1\pi(1)} \dots a_{ji\pi(i)} \dots a_{i\pi(j)} \dots a_{n\pi(n)} \quad (1)$$

$$= \sum_{\pi} \sigma(\pi) a_{1\pi(1)} \dots a_{i\pi(j)} \dots a_{j\pi(i)} \dots a_{n\pi(n)} \quad (2)$$

$$= - \sum_{\pi} \sigma(\pi) a_{1\pi(1)} \dots a_{i\pi(i)} \dots a_{j\pi(j)} \dots a_{n\pi(n)} \quad (3)$$

$$= -D(A). \quad (4)$$

We arrived at line 3 from line 2 of this calculation using the fact that the number of permutations changed by 1. If a permutation parity is even, changing the permutation count by 1 would give an odd permutation parity and reverse if a permutation parity is odd. Since $\sigma(\pi) = 1$ for even parity and $\sigma(\pi) = -1$ for odd parity, we switch the sign.

5

For 2×2 matrices, we prove that the determinant map is the only map $D : \mathbb{R}^{n^2} \rightarrow \mathbb{R}$ that is multilinear as a function of 2 rows, alternating, and for which $D(I) = 1$.

Consider the determinant of the arbitrary matrix A :

$$\begin{aligned} D(A) &= D\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\ &= D\left(\begin{pmatrix} ae_1^T & be_2^T \\ c & d \end{pmatrix}\right) \\ &= D\left(\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\right) + D\left(\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}\right) \quad \text{by multilinearity} \\ &= aD\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right) + bD\left(\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}\right) \quad \text{by multilinearity} \\ &= aD\left(\begin{pmatrix} 1 & 0 \\ ce_1^T & de_2^T \end{pmatrix}\right) + bD\left(\begin{pmatrix} 0 & 1 \\ ce_1^T & de_2^T \end{pmatrix}\right) \\ &= a \left[D\left(\begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}\right) + D\left(\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\right) \right] + b \left[D\left(\begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\right) + D\left(\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix}\right) \right] \quad \text{multilinearity} \\ &= a \left[cD\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) + dD\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \right] + b \left[cD\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + dD\left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) \right] \quad \text{multilinearity} \end{aligned}$$

$$\begin{aligned}
&= a \left[c(0) + dD\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \right] + b \left[cD\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + d(0) \right] \quad (*) \\
&= a \left[dD\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \right] + b \left[-cD\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \right] \quad \text{by the alternating property} \\
&= ad - bc \quad \text{by } D(I) = 1 .
\end{aligned}$$

In particular, if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{then} \quad D\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (1)(1) - (0)(0) = 1 .$$

(*) By the alternating property of the determinant map and the fact that the rows are identical in each case,

$$\begin{aligned}
D\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) &= -D\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) \implies D\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) = 0 \\
D\left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) &= -D\left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) \implies D\left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) = 0
\end{aligned}$$