MA502 Fall 2019 – Homework 4. Due October 26th, 2019

1. Consider the set of all $n \times n$ real matrices. This set is has a vector space structure, as we have seen in class. Prove that

$$S = \{A \in \mathbb{R}^{n\times n} | A^T = -A\}$$

that is the set of all skew symmetric matrices, is a subspace. Here, we have denoted by A^T the transpose of A, that is the matrix $\{a^T_{ij}\} = A^T$ defined by $a^T_{ij} = a_{ji}$.

2. Consider $T : \mathbb{P}_3 \to \mathbb{P}_2$ defined by differentiation, i.e., by $T(p) = p' \in \mathbb{P}_2$ for $p \in \mathbb{P}_3$. Find the range and the Null space for T.

3. Let A be a $n \times n$ matrix with real coefficients and let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ denote the linear operator defined by

$$T_A x = A \cdot x,$$

for every $x \in \mathbb{R}^n$. Prove that $R(T_A)$ is equal to the span of the columns of A.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator and for every $k \in \mathbb{N}$ set T^k to denote the composition of T with itself k times.

   (i) Show that for every $k \in \mathbb{N}$ one has $R(T^{k+1}) \subset R(T^k)$.

   (ii) Show that there exists a positive integer m such that for all $k \geq m$ one has $R(T^k) = R(T^{k+1})$.

5. Let A and B be two square, $n \times n$ matrices. Prove that if $AB = 0$ (as matrix products), then

$$R(T_A) + R(T_B) \leq n,$$

where we have denoted by T_A and T_B the linear operators associated to the matrices A and B respectively.