MA 502 Homework 8

Dane Johnson

November 21, 2019

1

Let A be an $n \times n$ matrix with the single eigenvalue $\alpha \in \mathbb{C}$. Set $d_i = \dim(\mathrm{Ker}(A - \alpha I)^i)$ and let $d_0 = 0$.

-

Suppose $n = 4, d_1 = 2, d_2 = 4$. To find the Jordan Canonical Form of A, we note the facts:

There are $d_1 - d_0 = 2$ blocks of size 1×1 or larger.
There are $d_2 - d_1 = 2$ blocks of size 2×2 or larger.

We cannot have a block of size 3×3 or larger since the assumption that $n = 4$ would then require that we cannot have two blocks of size 2×2 or larger. Therefore, the JCF must have two blocks of size 2×2. We conclude that the JCF of A is:

$$J = \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & \alpha \end{pmatrix}.$$

-

Suppose $n = 6, d_1 = 3, d_2 = 5, d_3 = 6$. Then in the JCF of A it must be the case that:

There are $d_1 - d_0 = 3$ blocks of size 1×1 or larger.
There are $d_2 - d_1 = 2$ blocks of size 2×2 or larger.

There is $d_3 - d_2 = 1$ block of size 1×1 or larger.

We cannot have a block of size 4×4 or larger since the requirements that we have at least two blocks of size 2×2 or larger and three blocks in total would require us to exceed the dimensions of a 6×6 matrix. So the maximum size of a block must be 3×3. With one block of size 3×3 the remaining blocks must be of sizes 2×2 and 1×1. Therefore,

$$J = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 0 & \alpha \end{pmatrix}.$$

Suppose $n = 5$ and that A has two eigenvalues. For the eigenvalue $\alpha = 0$, we have $d_1 = 2, d_2 = 3, d_3 = 4$. For the eigenvalue $\alpha = 1$ we have $d_1 = 1$. This means:

For $\alpha = 0$, there are 2 blocks of size 1×1 or larger
There is 1 block of size 2×2 or larger
There is 1 block of size 3×3 or larger

For $\alpha = 1$, there is 1 block of size 1×1 or larger.

For $\alpha = 0$, we need one block of size 3×3 or larger and two blocks of size 1×1 or larger. If we include a block of size 2×2 it will be impossible to include a block for $\alpha = 1$. Then we must have one block of size 1×1 for $\alpha = 1$, one block of size 1×1 for $\alpha = 0$, and one block of size 3×3 for $\alpha = 0$. Up to the ordering of blocks, the JCF of A is:

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2

Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} .$$

Note that since A is upper triangular, the eigenvalues of A lie along the diagonal. So we have that $\lambda = 0$ is the only eigenvalue of A with algebraic multiplicity 3. Putting $\lambda = 0$ in $A - \lambda I$, we have $A - \lambda I = A - 0I = A$. Therefore $\text{Null}(A - \lambda I) = \text{Null}(A) = \{k(1, 0, 0) \mid k \in \mathbb{C}\}$. Therefore we have only one independent eigenvector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

Next we see that

$$A^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

And so $d_2 = \text{Null}((A - 0I)^2) = \text{Null}(A^2) = 2$ and similarly $d_3 = \text{Null}(A^3) = 3$. Therefore in the JCF of A we have that there is $d_1 - d_0 = 1$ block of size 1×1 or larger, $d_2 - d_1 = 1$ block of size 2×2 or larger, and $d_3 - d_2 = 1$ block of size 3×3 or larger. But since A is size 3×3 we conclude from this information that there must just be one block in the JCF of size 3×3 and

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

3

Suppose $A : V \to V$ is a linear transformation with eigenvalues $\{\lambda_1, ..., \lambda_n\}$ and that $f(t)$ is any polynomial. Consider the eigenvalue $\lambda \in \{\lambda_1, ..., \lambda_n\}$ with corresponding eigenvector v. Note that since A is linear and $A(v) = \lambda v$, then $A^2(v) = A(A(v)) = A(\lambda v) = \lambda A(v) = \lambda^2 v$, which shows that λ^2 is an eigenvalue of the composition A^2. Applying this reasoning inductively we have that λ^m is an eigenvalue of the transformation A^m stil corresponding to the eigenvector v. Also, the linearity of A^m for any positive nonnegative integer m means that $(kA^m)(v) = k(A^m(v)) = k\lambda^m v$. Therefore, if λ^m is an eigenvalue of A^m then $k\lambda^m$ is an eigenvalue of kA. Since f is a polynomial,

f is of the form $f(t) = a_0 + a_1 t + ... + a_m t^m$. Then, letting $I : V \to V$ denote the identity transformation,

$$\begin{aligned} [f(A)](v) &= (a_0 I + a_1 A + ... + a_m A^m)(v) && (1) \\ &= a_0 I(v) + a_1 A(v) + ... a_m A^m(v) && (2) \\ &= a_0 \lambda v + a_1 \lambda v + ... + a_m \lambda^m v && (3) \\ &= (a_0 \lambda + a_1 \lambda + ... a_m \lambda^m) v && (4) \\ &= f(\lambda) v \, . && (5) \end{aligned}$$

This shows that for each $\lambda_i \in \{\lambda_1, ..., \lambda_n\}$, $f(\lambda_i)$ is an eigenvalue of the transformation of $f(A)$. Also, $f(A)$ can have no eigenvalues not included in this set since $f(A)$ is constructed using compositions of A with itself (and scaling), which means that the set of eigenvalues of $f(A)$ must be a subset of the set of eigenvalues of A.

4

Suppose A is a square matrix with zero determinant. Certainly we know by the Cayley - Hamilton Theorem that since the characteristic polynomial χ_A has the property that $\chi_A(A) = 0$, then $A\chi_A = A(0) = 0$. But we can show further that the characteristic polynomial is not the only nonzero polynomial, p, for which $Ap(A)$ holds. We write

$$\chi_A(\lambda) = a_0 + a_1 \lambda + ... + a_n \lambda^n \, .$$

Then we have

$$0 = \chi_A(0) = a_0 \implies \chi_A(\lambda) = \lambda(a_1 + \lambda a_2 + ... + \lambda^{n-1} a_n) \, .$$

The since $\chi_A(A) = 0$, we see that

$$0 = \chi_A(A) = A(a_1 + a_2 A + ... + a_n A^{n-1}) \, .$$

Taking $p(t) = a_1 + a_2 t + ... + a_n t^{n-1}$ we conclude that $Ap(A) = 0$.

5

To find 4×4 matrices with minimal polynomials of degree $1, 2, 3,$ and 4, one strategy is to first pick specific polynomials of each degree and find matrices for which each polynomial is the minimal polynomial. Set $p_1(x) =$

x, which is degree 1. To find the matrix A_1 for which p_1 is the minimal polynomial, just take $A_1 = 0 \in \mathbb{R}^{4\times 4}$. Next consider the polynomial $p_2(x) = x^2$. If we require that A_2 have minimal polynomial $p_2(x)$, we may take

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies p_2(A) = A_2^2 = 0 \ .$$

Take $p_3(x) = x^3$. Then p_3 is the minimal polynomial of

$$A_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies A_3^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , p_3(A) = A_3^3 = 0 \ .$$

Take $p_4(x) = x^4$. Then p_4 is the minimal polynomial of

$$A_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies$$

$$A_4^2 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , A_4^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , p_4(A) = A^4 = 0 \ .$$