**MA502 Fall 2019 – Homework 10. Due December 10th, 2019**

**Write down detailed proofs of every statement you make**

1. Let $A$ be a real $n \times n$ matrix with an eigenvalue $\lambda$ having algebraic multiplicity $n$. Prove that for any $t$ real one has

$$e^{tA} = e^{\lambda t}\left(I + (A - \lambda I)t + ... + \frac{(A - \lambda I)^{n-1}}{(n-1)!}t^{n-1}\right)$$

2. Let $A$ denote the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

- Find an orthogonal matrix $O$ such that $O^T A O$ is diagonal
- Compute the matrix $e^A$.

3. Consider the vector space of polynomials with real coefficients and with inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(t)g(t)(1 - t^2)dt.$$

Apply the Graham-Schmidt process to find an orthonormal basis, with respect to this inner product, for the subspace generated by $\{\frac{\sqrt{3}}{2}, \frac{\sqrt{15}}{2}t, t^2\}$.

4. Let $A$ be a real $n \times n$ matrix. Define $\langle x, y\rangle := \sum_{i,j=1}^{n} a_{ij}x_i y_j$. Find necessary and sufficient conditions on $A$ for this operation to be a inner product on $\mathbb{R}^3$.

5. Show that the system $Ax = b$ has no solution and find the least square solution of the problem $Ax \approx b$ with

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$