MA 503 : Homework 14

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Proposition 8 If f and g are nonnegative measurable functions, then:

(i) $\int_E cf = c\int_E f, \quad c > 0$.

(ii) $\int_E (f+g) = \int_E f + \int_E g$.

(iii) If $f \leq g$ a.e., then $\int_E f \leq \int_E g$.

Problem 3 Let f be a nonnegative measurable function. Show that $\int f = 0$ implies $f = 0$ a.e.

For $n \in \mathbb{N}$, let $E_n = \{x : f(x) > 1/n\}$. If $x \in E_n$, then $f(x) > 1/n = 1/n\chi_{E_n}(x)$. If $x \notin E_n$, then $1/n\chi_{E_n}(x) = 0 \leq f(x) \leq 1/n$. So $f(x) \geq \chi_{E_n}$ for all x. By proposition 8 (iii),

$$0 = \int f \geq \int \frac{1}{n}\chi_{E_n} = \frac{1}{n}m(E_n) \implies 0 = m(E_n)\ .$$

Since $f \geq 0$, $\{x : f(x) \neq 0\} = \{x : f(x) > 0\}$. If $f(x) > 0$, then there is an $n \in \mathbb{N}$ such that $f(x) > 1/n$ by the axiom of Archimedes. Conversely if $f(x) > 1/n$ for some $n \in \mathbb{N}$ then $f(x) > 0$. Then $\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty}\{x : f(x) > 1/n\} = \bigcup_{n=1}^{\infty} E_n$ from which it follows

$$m\left(\{x : f(x) \neq 0\}\right) = m\left(\{x : f(x) \neq 0\}\right) = m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} 0 = 0\ .$$

Conclude that $f = 0$ a.e.

By the way, the converse is also true. If $f = 0$ a.e. then the set $E = \{x : f(x) > 0\} = \{x : f(x) \neq 0\}$ has measure zero. Let $f_n = n\chi_E$. If $x \in E$, then for each $k \in \mathbb{N}$, $\inf_{n\geq k} n\chi_E = k$ so that $\lim_{k\to\infty}\inf_{n\geq k} f_n(x) = \infty \geq f(x)$. If $x \notin E$, then $f(x) = 0$ and $\inf_{n\geq k} n\chi_E = 0$. So for all x, $f(x) \leq \liminf f_n(x)$. Since $f \geq 0$, $\int f \geq \int 0 = 0$ by proposition 8 (iii). Also using proposition 8 (iii) along with Fatou's Lemma,

$$0 \leq \int f \leq \int \liminf f_n \leq \liminf \int f_n = \liminf\left(\frac{1}{n}m(E)\right) = \liminf 0 = 0 \implies \int f = 0\ .$$