MA 503 : Homework 18

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Problem 10 Let (f_n) be a sequence of functions in L^∞. Prove that (f_n) converges to f in L^∞ if and only if there is a set E of measure zero such that f_n converges to f uniformly on E^c.

Suppose that (f_n) converges to f in $L^\infty([0,1])$ and let $\epsilon > 0$. There is an $N \in \mathbb{N}$ such that $||f - f_n||_\infty < \epsilon$ for all $n \geq N$. But since $|f(x) - f_n(x)| \leq ||f - f_n||_\infty$ for almost all x, this means there is a set E of measure zero such that $|f(x) - f_n(x)| \leq ||f - f_n||_\infty < \epsilon$ for all $x \in E^c$ and for all $n \geq N$. Therefore, (f_n) converges to f uniformly on E^c.

Suppose there is a set E of measure zero such that (f_n) converges to f uniformly on E^c. Let $\epsilon > 0$. There is an N such that for all $n \geq N$ and all $x \in E^c$, $|f(x) - f_n(x)| < \epsilon$. That is, for $n \geq N$, $|f(x) - f_n(x)| < \epsilon$ almost everywhere so $\epsilon \in \{M : |f(x) - f_n(x)| < M \text{ a.e. }\}$. Then $||f - f_n||_\infty = \inf\{M : |f(x) - f_n(x)| \text{ a.e. }\} \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, conclude that $||f - f_n|| \to 0$ as $n \to \infty$.

Problem 11 Prove that L^∞ is complete.

Suppose (f_k) is a Cauchy sequence in $L^\infty([0,1])$. Then for each $n \in \mathbb{N}$, there is an N such that $||f_k - f_j||_\infty < 1/n$ for all $k, j \geq N$. Then since $|f_k(x) - f_j(x)| \leq ||f_k - f_j||_\infty$ for almost all x, there is a set $E_{k,j,n}$ of measure zero such that

$$|f_k(x) - f_j(x)| < 1/n \quad \forall x \in E^c_{k,j,n} \ .$$

Let $E = \bigcup_{k,j,n} E_{k,j,n}$ so that $m(E) = 0$ and for each x in E, the sequence $(f_k(x))$ is a real Cauchy sequence and so convergent in $\mathbb{R}$. Define the function f (actually equivalence class of functions equal a.e.) pointwise by $f(x) = \lim_{k\to\infty} f_k(x)$ for each x in N^c. Since $m(E) = 0$, $f(x)$ can be defined arbitrarily for $x \in E$. Then for each n there is an N such that for all $j \geq N$ and all $x \in E^c$,

$$|f(x) - f_j(x)| = \lim_{k\to\infty} |f_k(x) - f_j(x)| \leq \lim_{k\to\infty} 1/n = 1/n \ .$$

This shows that (f_j) is a sequence of functions in $L^\infty([0,1])$ that converges uniformly to f outside a set of measure zero. By problem 10, (f_j) converges to f in $L^\infty([0,1])$.

Problem 13 Let $C = C([0,1])$ be the space of continuous functions on $[0,1]$ and define $||f|| = \max |f(x)|$. Show that C is a Banach space.

Let (f_n) be Cauchy in $C([0,1])$ under the given norm. Note that for each $x \in [0,1]$ the sequence $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$. So we define the function $f : [0,1] \to \mathbb{R}$ pointwise as $f(x) = \lim_{n\to\infty} f_n(x)$. To show that (f_n) converges to f under the given norm, let $\epsilon > 0$ and take N such that for all $m, n \geq N$, $||f_n - f_m|| < \epsilon$. But then for any $x \in [0,1]$ and $m \geq N$,

$$|f(x) - f_m(x)| = \lim_{n\to\infty} |f_n(x) - f_m(x)| \leq \lim_{n\to\infty} ||f_n - f_m|| \leq \epsilon \ .$$

This shows that the sequence (f_n) of continuous functions converges uniformly to on the compact set $[0,1]$ f and therefore $f \in C([0,1])$. Also, $||f - f_m|| = \lim_{n\to\infty} ||f_n - f_m|| \leq \epsilon$ so that (f_n) converges to f under the given norm. Alternatively, to show continuity, we know that since each function in the sequence (f_n) is continuous and $[0,1]$ is a compact set, each function in the sequence is uniformly continuous on $[0,1]$. Let $\epsilon > 0$ and take N such that for $n \geq N$, $||f - f_n|| < \epsilon/3$ and $\delta > 0$ so that $|f_n(x) - f_n(y)| < \epsilon/3$ whenever $|x - y| < \delta$.

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \ .$$