MA 503 Test 1 Dane Johnson

1. Show that if $f: \mathbb{R} \to \mathbb{R}$ satisfies $\forall \alpha \in \mathbb{Q}$, $\{x : f(x) > \alpha\}$ is measurable, then f is measurable

Answer:

The domain of f, $\mathbb{R}$, is a measurable set ✓.

Let $\beta \in \mathbb{R}$. For each $n \in \mathbb{N}$, $\exists$ is a rational number r_n s.t. $\beta - \frac{1}{n} < r_n < \beta$

For each n, the set $\{x : f(x) > r_n\}$ is measurable by hypothesis. Since $\mathcal{M}$ is a σ-algebra, the countable intersection $\bigcap_{n \in \mathbb{N}} \{x : f(x) > r_n\}$ is measurable. So,

$\{x : f(x) \geq \beta\} = \bigcap_{n \in \mathbb{N}} \{x : f(x) > r_n\}$ is measurable. Proof of set equality:

$y \in \{x : f(x) \geq \beta\}$ implies $f(y) \geq \beta > r_n \; \forall n$, so $y \in \bigcap_{n \in \mathbb{N}} \{x : f(x) > r_n\}$

$y \in \bigcap_{n \in \mathbb{N}} \{x : f(x) > r_n\}$ implies $f(y) > r_n > \beta - \frac{1}{n} \; \forall n$, so $f(y) + \frac{1}{n} > \beta \; \forall n$, implies $f(y) \geq \beta$, $y \in \{x : f(x) \geq \beta\}$

Since β was arbitrary, $\{x : f(x) \geq \beta\}$ is measurable for any β. Conclude by Proposition 18 and defn of a measurable fn. that f is measurable.

2. Show that if $S \subset C \subset \mathbb{R}$ and C is closed, then $\bar{S} \subset C$.

<u>Answer:</u> Suppose, for contradiction, that $\bar{S} \not\subset C$. Then $\exists\, x \in \bar{S}$ such that $x \notin C$. So $x \in C^c$. Since C is closed, C^c is open. So $\exists\, \varepsilon > 0$ s.t. if $|x-y| < \varepsilon$, then $y \in C^c$.

But since x is in the closure of S, for this same ε, there must exist an element $z \in S$ such that $|x-z| < \varepsilon$. This shows that $z \in S$ and $z \in C^c$, so $z \in S$ and $z \notin C$. This contradicts the assumption that S is a subset of C. This contradiction arose from supposing that $\bar{S} \not\subset C$, so conclude that $\bar{S} \subset C$.

3. Show that if $f : \mathbb{R} \to [0, \infty)$ is continuous and $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0$, then f has a maximum.

Answer: Since $\lim_{x \to -\infty} f(x) = 0$, $\exists\ a \in \mathbb{R}$ s.t. $0 \leq f(x) < 1$ for all $x < a$. Since $\lim_{x \to +\infty} f(x) = 0$, $\exists\ y \in \mathbb{R}$ s.t. $0 \leq f(x) < 1$ for all $x > y$. Pick $b \geq y$ so that $b > a$. Then f is continuous on the closed and bounded interval $[a, b] \subset \mathbb{R}$. Therefore f attains a maximum on $[a, b]$. Let $\alpha = \max_{x \in [a,b]} f(x)$.

If $\alpha \geq 1$, then since $f(x) < 1\ \forall x \notin [a, b]$, conclude $\alpha = \max_{x \in \mathbb{R}} f(x)$. Otherwise it may be the case that if $\alpha < 1$, f attains its maximum on $(-\infty, a)$ and/or (b, ∞). So suppose $\alpha < 1$. The set $\{f(x) : x \in (-\infty, a)\}$ is a nonempty set bdd above by 1. Let $M_1 = \sup\{f(x) : x \in (-\infty, a)\} = \max\{f(x) : x \in (-\infty, a)\}$ (the supremum is the max b/c f is continuous). Similarly, let $M_2 = \sup\{f(x) : x \in (b, \infty)\} = \max\{f(x) : x \in (b, \infty)\}$. We know $M_1, M_2 < 1$. Take $M = \max\{M_1, M_2\}$. If $M \geq \alpha$, conclude $M = \max_{x \in \mathbb{R}} f(x)$. If $M \leq \alpha$, conclude $\alpha = \max_{x \in \mathbb{R}} f(x)$.