MA 503 : Homework 4+

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Problem 40

Let F be a closed set of real numbers and f a real valued function which is defined and continuous on F. Show there is a function $g : \mathbb{R} \to \mathbb{R}$ such that g is continuous and $f(x) = g(x)$ for each $x \in F$.

If $F = \emptyset$, use $g(x) \equiv 0$. Then g is continuous and the requirement that $f(x) = g(x)$ for each $x \in F$ is vacuously true.

If $F = \mathbb{R}$, then set $g \equiv f$. Since f is continuous g is also continuous on $\mathbb{R}$ and $g(x) = f(x)$ for each $x \in F = \mathbb{R}$.

If $F \subsetneq \mathbb{R}$ then F^c is a nonempty open set of real numbers. Using Proposition 8, F^c is the union of a countable collection of disjoint open sets, not all empty. As in the proof of this proposition, for each $y \in F^c$, there is an interval $I_y = (a_y, b_y) \subset F^c$ with $a_y = \inf\{a \in \mathbb{R} : (a, y) \subset F^c\}$ and $b_y = \sup\{b \in \mathbb{R} : (y, b) \subset F^c\}$. The disjoint union $F^c = \cup_{y \in F^c} I_y$ is countable and so we may write $F^c = \cup_{i=1}^{\infty}(a_i, b_i)$ for real numbers $a_i, b_i,\ i \in \mathbb{N}$. Relabeling the intervals if necessary, assume $a_1 < b_1 < a_2 < b_2 < a_3 < ...$ (if it is possible to write $F^c = \cup_{i=1}^{n}(a_i, b_i)$ then assume $a_1 < b_1 < .... < a_n < b_n$). We will handle the case that one or two of these intervals is an infinite interval soon since this will alter the linear equation we use to define g if this occurs. For each interval (a_i, b_i), sup $(a_i, b_i) = b_i \in F$ and inf $(a_i, b_i) = a_i \in F$. For each (a_i, b_i) in the disjoint union $F^c = \cup_{i=1}^{\infty}(a_i, b_i)$ (or $F^c = \cup_{i=1}^{n}(a_i, b_i)$ if it is possible) set

$$g(x) = \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i) + f(a_i) \quad x \in (a_i, b_i)\ .$$

Then for $x \in F$, set $g(x) = f(x)$. If $(a_i, b_i) = (-\infty, b_i)$ for some interval, set $g(x) = f(b_i)$ for all $x \in (-\infty, b_i)$. If $(a_i, b_i) = (a_i, \infty)$, set $g(x) = f(a_i)$ for all $x \in (a_i, \infty)$. If the intervals have been ordered and any interval contained within another interval has been deleted from the collection then $(-\infty, b_i) = (a_1, b_1)$ and $(a_n, b_n) = (a_n, \infty)$ if infinite intervals exist in the collection. Considering these cases, where we still consider F^c as the disjoint union $F^c = \cup_i(a_i, b_i)$,

$$g(x) = \begin{cases} f(x) & x \in F \\ \frac{f(b_i)-f(a_i)}{b_i-a_i}(x - a_i) + f(a_i) & x \in (a_i, b_i) \text{ and } a_i, b_i \text{ finite} \\ f(b_i) & x \in (a_i, b_i) \text{ and } a_i = -\infty \\ f(a_i) & x \in (a_i, b_i) \text{ and } b_i = \infty \end{cases}$$

Since g is linear on each open interval in $F^c = \cup_i(a_i, b_i)$, g is continuous on F^c. For each endpoint b_i, since g is continuous on F^c, $f = g$ on F and $\lim_{x \to b_i^-} g(x) = f(b_i) = g(b_i)$, g is continuous at each interval endpoint b_i. Similarly, g is continuous at each a_i. Since $f = g$ on F and f is continuous on F, g is therefore continuous on all of F. Since g is continuous for any $x \in F^c \cup F$, g is continuous on $\mathbb{R}$.

Homework 4+: Problem 1

Let $S \subset \mathbb{R}$ and suppose S is not closed. Show that there exists a function $f : S \to \mathbb{R}$ such that f is continuous on S but there does not exist an extension $g : \mathbb{R} \to \mathbb{R}$ of f to $\mathbb{R}$ such that g is continuous.

Since S is not closed, $\overline{S}\backslash S \neq \emptyset$. Let $c \in \overline{S}\backslash S$. That is, c is a limit point of the set S but $c \notin S$. The function $j : S \to \mathbb{R}$, $j(x) = x - c$ is continuous on S and $j(x) \neq 0$ for all $x \in S$ since $c \notin S$. The function $h : j(S) \to \mathbb{R}$, $h(y) = 1/y$ is also continuous (and by the above, $y \neq 0$ for any $y \in j(S)$). Therefore the composition $f = h \circ j : S \to \mathbb{R}$, $f(x) = 1/(x-c)$ is continuous on S as the composition of continuous functions. Then either $\lim_{x\to c^-} f(x) = \infty$ or $\lim_{x\to c^-} f(x)$ does not exist. Similarly, $\lim_{x\to c^+} f(x) = -\infty$ or $\lim_{x\to c^+} f(x)$ does not exist. In any case, $\lim_{x\to c} f(x) \notin \mathbb{R}$. If g is any extension of f, since $g : \mathbb{R} \to \mathbb{R}$, we must define g at c and must have $g(c) \in \mathbb{R}$. By definition of continuity at a point (this is at least one of the common ways to define continuity in introductory real analysis), for g to be continuous at c, we must have for $y \in \mathbb{R}$, $\lim_{y\to c} g(y) = g(c)$. Since g is an extension of f, $g(x) = f(x)$ for $x \in S$, so we must also have for $x \in S \subset \mathbb{R}$ that $\lim_{x\to c} g(x) = \lim_{x\to c} f(x)$. Therefore, we may choose between two options: either define g for all real numbers (so that g is indeed an extension to $\mathbb{R}$) but with g discontinuous at c or choose not to define $g(c)$ so that g may still be continuous (but then g cannot be an extension of f to $\mathbb{R}$). Conclude that it is impossible to find a continuous extension, g, of f to $\mathbb{R}$.

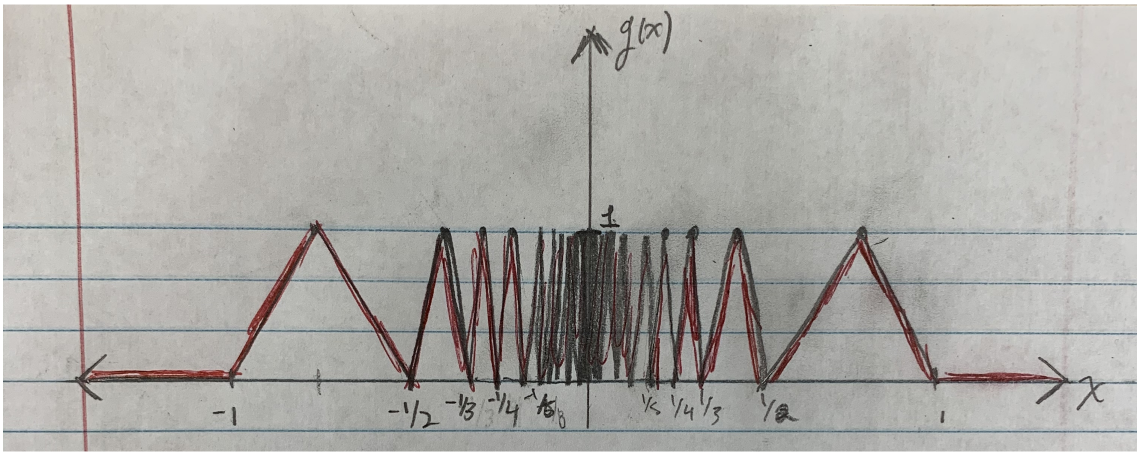
Homework 4+: Problem 2

Show there exists a closed set F, a continuous function $f : F \to \mathbb{R}$, an extension $g : \mathbb{R} \to \mathbb{R}$ (that is, $g|_F = f$) such that F^c is the $\bigcup\{I_n\}$, where $\{I_n\}$ is a collection of disjoint open intervals, $g|_{\overline{I_n}}$ is continuous for all n, but g is not continuous.

Consider $F = \{-1/n : n \in \mathbb{N}\} \cup \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Then $F^c = (-\infty, -1) \cup \{(-1/n, -1/(n+1)) : n \in \mathbb{N}\} \cup \{(1/(n+1), 1/n) : n \in \mathbb{N}\} \cup (1, \infty)$ is open as the union of open sets. This means F is closed. Labeling $I_1 = (-\infty, -1)$, $I_2 = (1, \infty)$, $I_3 = (-1, -1/2)$, $I_4 = (1/2, 1)$, $I_5 = (-1/2, -1/3)$, $I_6 = (1/3, 1/2)$, and continuing in this way we have $I_k = \left(-\frac{1}{(k-1)/2}, -\frac{1}{(k-1)/2+1}\right)$ for odd $k \geq 3$ and $I_k = \left(\frac{1}{k/2}, \frac{1}{k/2-1}\right)$ for even $k \geq 4$. Then $F^c = \bigcup\{I_n\}$ is a countable union of disjoint open intervals.

It simplifies the notation to write for an arbitrary finite interval I_n, $n \geq 3$ from the collection $\{I_n\}$ as $I_n = (a_n, b_n)$. For each such interval we will define g to make a sawtooth with base of length $b_n - a_n$ and height 1. The specific values of a_n and b_n determine how steep each sawtooth is. Define $f : F \to \mathbb{R}$ by $f(x) = 0$ for all $x \in F$. Define $g : \mathbb{R} \to \mathbb{R}$ as

$$\begin{cases} 0 & x \in F \\ 0 & x \in I_1 = (-\infty, -1) \\ 0 & x \in I_2 = (1, \infty) \\ \frac{2}{b_n - a_n}(x - a_n) & x \in (a_n, a_n + \frac{b_n - a_n}{2}] \\ 1 - \frac{2}{b_n - a_n}\left(x - (a_n + \frac{b_n - a_n}{2})\right) & x \in [a_n + \frac{b_n - a_n}{2}, b_n) \end{cases}$$



Then g is defined for all real numbers and $g(x) = f(x)$ for $x \in F$ (that is, $g|_F = f$). So g is an extension of f to $\mathbb{R}$. Also, g is continuous on I_1 and I_2 as g is identically 0 on these open intervals.

For each finite open interval $I_n = (a_n, b_n)$, g is an increasing linear function on $(a_n, a_n + \frac{b_n - a_n}{2})$ and a decreasing linear function on $(a_n + \frac{b_n - a_n}{2}, b_n)$. Also, at the midpoint of each (a_n, b_n), $a_n + \frac{b_n - a_n}{2}$,

$$1 - \frac{2}{b_n - a_n}\left(a_n + \frac{b_n - a_n}{2} - \left(a_n + \frac{b_n - a_n}{2}\right)\right) = 1 - \frac{2}{b_n - a_n}(0) = 1$$
$$\frac{2}{b_n - a_n}\left(a_n + \frac{b_n - a_n}{2} - a_n\right) = \frac{2}{b_n - a_n}\frac{b_n - a_n}{2} = 1 \ .$$

So indeed the two lines used to define g within any $I_n = (a_n, b_n)$ agree at a maximum of 1 at the midpoint of the interval. Therefore g is continuous on each I_n. Also, at each endpoint a_n and each endpoint b_n, we have

$$\lim_{x \to a_n^+} g(x) = \lim_{x \to a_n^+} \frac{2}{b_n - a_n}(x - a_n) = 0$$
$$\lim_{x \to b_n^-} g(x) = \lim_{x \to b_n^-} 1 - \frac{2}{b_n - a_n}\left(x - (a_n + \frac{b_n - a_n}{2})\right) = 0 \ .$$

Since each $a_n, b_n \in F$ and $g(a_n) = g(b_n) = 0$, this shows that g is continuous at each endpoint (recall also that $g \equiv 0$ on $(-\infty, -1)$ and $(1, \infty)$). Therefore, g is continuous on $\overline{I_n}$ for each interval I_n (this would only be an issue if we need to define g on $\overline{R}$ for the case of $\overline{I_1}$, $\overline{I_2}$ but this cannot be necessary since we are asked to find $g : \mathbb{R} \to \mathbb{R}$). As we proved in Homework 4, "g is continuous on $\overline{I_n} \implies g|_{\overline{I_n}}$ is continuous".

However, g is not continuous at 0. Since $0 \in F$, $g(0) = f(0) = 0$. Consider $\epsilon = 1 > 0$ and let $\delta > 0$ be arbitrary. Then there is a $k \in \mathbb{N}$ such that k is even and $k > 2/\delta + 2 \implies 0 < 1/(k/2) < 1/(k/2 - 1) < \delta$. Thus we have an interval $I_k = \left(\frac{1}{k/2}, \frac{1}{k/2-1}\right) \subset (0, \delta)$ (we have $I_k = I_n = (a_n, b_n)$ for some n but now it is more convenient to revert to the original notation). But then for the midpoint of this interval, $\alpha_k = 2/k + \frac{1}{2}\left(\frac{1}{k/2-1} - \frac{1}{k/2}\right)$, $g(\alpha_k) = 1$ using our definition of g. Therefore, there is an $\epsilon > 0$ (specifically $\epsilon = 1$) such that for every $\delta > 0$, we can find an x (α_k) such that $|x - 0| < \delta$ but $|g(x) - g(0)| = |1 - 0| = 1 \geq \epsilon$. Therefore g is not continuous at 0.