MA 503 : Lebesgue Measure and Integration

Dane Johnson

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Chapter 5 : Differentiation and Integration

1 Differentiation of Monotone Functions

Definition Let $\mathcal{J}$ be a collection of intervals. We say that $\mathcal{J}$ **covers** E **in the sense of Vitali** if for each ϵ and any $x \in E$, there is an interval $I \in \mathcal{J}$ such that $x \in I$ and $l(I) < \epsilon$. The intervals may be open, half-open, or closed, but we do not allow degenerate intervals consisting of only one point.

Lemma 1 (Vitali) Let E be a set of finite outer measure and $\mathcal{J}$ a collection of intervals that covers E in the sense of Vitali. Then given $\epsilon > 0$ there is a finite disjoint collection $\{I_1, ..., I_N\} \subset \mathcal{J}$ such that

$$m^* \left[E \backslash \bigcup_{n=1}^{N} I_n \right] < \epsilon \ .$$

Proof: The proof will assume that each interval in $\mathcal{J}$ is closed.
Let O be a set of finite measure containing E. Since $\mathcal{J}$ is a Vitali covering of E, assume without loss of generality that for each $I \in \mathcal{J}$, $I \subset O$. If $I \not\subset O$, then since $E \subset O$ it must be that for any point $x \in I$ such that $x \notin O$, $x \notin E$. So we can redefine I so that $I \subset O$ without losing coverage of any point in E. Let I_1 be any interval in $\mathcal{J}$ and assume $I_1, ..., I_n$ have been chosen. Let k_n be the supremum of the lengths of the intervals of $\mathcal{J}$ that do not intersect with any of $I_1, ..., I_n$. Since each interval is contained in O, $k_n \leq m(O) < \infty$. .....

Definition The **derivates** of f at x are:

$$D^+ f(x) = \lim_{h \to 0^+} \sup \frac{f(x+h) - f(x)}{h}$$

$$D^- f(x) = \lim_{h \to 0^+} \sup \frac{f(x) - f(x-h)}{h}$$

$$D_+ f(x) = \lim_{h \to 0^+} \inf \frac{f(x+h) - f(x)}{h}$$

$$D_- f(x) = \lim_{h \to 0^+} \inf \frac{f(x) - f(x-h)}{h}$$

Since $\inf \frac{f(x+h)-f(x)}{h} \leq \sup \frac{f(x+h)-f(x)}{h}$ for each $h > 0$, $D_+ f(x) \leq D^+ f(x)$. Similarly, $D_- f(x) \leq D^- f(x)$. If $D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \neq \pm\infty$, we say that f is **differentiable** at x and define $f'(x)$ to be the common value of the derivates at x. If $D^+ f(x) = D_+ f(x)$, f has a **right-hand derivative** at x and define $f'(x+)$ as the common value. Similarly, f has a **left-hand derivative** at x if $D^- f(x) = D_- f(x)$, with $f'(x-)$ defined as the common value.

Proposition 2 If f is continuous on $[a, b]$ and one of its derivates (say D^+) is everywhere nonnegative on (a, b), then $f(x) \leq f(y)$ for $x \leq y$, $x, y \in [a, b]$.

Proof: Suppose $x, y \in [a, b]$ with $x \leq y$.