MA 503 : Homework 16

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1. Suppose (f_n) is a sequence of nonnegative functions integrable on E such that $m(E) < \infty$ and $f_n \to f$ a.e. on E. Show that if for all $\epsilon > 0$ there is a $\delta > 0$ such that $\int_A f_n < \epsilon$ for all $n \in \mathbb{N}$ whenever $m(A) < \delta$, then

$$\int_E f_n \to \int_E f \ .$$

Since each f_n is integrable, each f_n must be measurable. The sequence (f_n) meets the conditions of Fatou's Lemma and since the f_n are nonnegative, f is nonnegative almost everywhere. Since the f_n are integrable, $\int_E f_n < \infty$ for each n. This means:

$$\int_E |f| = \int_E f \leq \liminf \int_E f_n < \infty \ .$$

Next show that $\limsup \int_E f_n \leq \int_E f$. Let $\epsilon > 0$. There is a $\delta > 0$ such that $\int_A f_n < \epsilon$ for all n whenever $m(A) < \delta$. In particular, the set $A = \{x \in E : f_n(x) \not\to f(x)\}$ has measure zero, so...

2. Suppose (f_n) is a sequence of nonnegative measurable functions with $f_n \to f$ a.e. on $\mathbb{R}$ and $\int_E f_n \to \int_E f$ for all bounded measurable sets E. Suppose for all $\epsilon > 0$ there is an $M > 0$ such that $\int_{[-M,M]^c} f_n < \epsilon$ for all $n \in \mathbb{N}$. Then $\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f$. Note/Hint? $f_n = \chi_{[n,n+1]}, f_n \to 0$.

Again by Fatou's Lemma,

$$\int_E f \leq \liminf \int_E f_n \ .$$

Next show that $\limsup \int_E f_n \leq \int_E f$. Let $\epsilon > 0$. There is an $M > 0$ such that:

$$\int_{\mathbb{R}} f_n = \int_{[-M,M]} f_n + \int_{[-M,M]^c} f_n < \int_{[-M,M]} f_n + \epsilon \ .$$

$$\limsup \int_{\mathbb{R}} f \leq \limsup \int_{[-M,M]} f_n + \epsilon$$

$$\limsup \int_{\mathbb{R}} f \leq \int_{[-M,M]} f + \epsilon \leq \int_{\mathbb{R}} f?$$