1. $f_n, f : E \to [0,\infty)$ are msble, $\boxed{m(E) < \infty}$ and $f_n \to f$ uniformly. show $\int f_n \to \int f$

Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, $\exists N$ s.t. $\forall n \geq N$ and $\forall x \in E$, $|f_n(x) - f(x)| < \varepsilon / m(E)$. then for all $n \geq N$

$$\left| \int_E f_n - \int_E f \right| = \left| \int_E (f_n - f) \right|$$

$$\leq \int_E |f_n - f|$$

$$< \int_E \varepsilon / m(E)$$

$$= \frac{\varepsilon\, m(E)}{m(E)}$$

$$= \varepsilon.$$

(Note that $f_n, f \geq 0$ on E)

$(m(E) < \infty)$

Since $\varepsilon > 0$ was arbitrary conclude that $\lim \int f_n = \int f$.

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2. Suppose $f \in L^1([0,1])$ with $|f| \leq 1$ a.e.
Find $\lim_{n\to\infty} \int_0^1 |f|^n$.

Let $A = \{x \in [0,1] : |f(x)| = 1\}$

$B = \{x \in [0,1] : |f(x)| < 1\}$

$c = \{x \in [0,1] : |f(x)| > 1\}$

Then $m(c) = 0$. since $|f| \leq 1$ a.e.

$$\int_{[0,1]} |f|^n = \int_A |f|^n + \int_B |f|^n + \int_C |f|^n$$
$$= \int_A |f|^n + \int_B |f|^n$$
$$= \int_A 1^n + \int_B |f|^n$$
$$= m(A) + \int_B |f|^n$$

~~If~~ $\cancel{\sup B < 1, \ 0 \leq \lim_{n\to\infty} \int_{[0,1]} |f|^n = m(A) + \lim_{n\to\infty} \int_B |\sup B|^n = m(A).}$

~~If~~ $\cancel{\sup B = 1,}$

$$\cancel{\int_B |f|^n = \int_{\{x : |f(x)| \leq \sup B - \varepsilon\}} |f|^n + \int_{\{x : \sup B - \varepsilon < |f(x)| < 1\}} |f|^n}$$

$$\lim_{n\to\infty} \int_{[0,1]} |f|^n = \lim_{n\to\infty} \left(m(A) + \int_B |f|^n \right)$$
$$= m(A) + \lim_{n\to\infty} \int_B |f|^n$$
$$= m(A) + 0$$
$$= m(A)$$

$\lim_{n\to\infty} \int_B |f|^n = 0$?

Let $h_n = |f|^n \leq 1 =: g$

Then $|h_n| \leq g$ and g integrable on B. We have $h_n \to 0$ everywhere on B. By LCT,

$$0 = \int_B 0 = \int_B \lim |f|^n = \lim \int_B |f|^n .$$

Therefore, the conclusion

$\lim \int_{[0,1]} |f|^n = m(A)$ follows.

3. Let $f, g \in L^2(\mathbb{R})$. Set $f_n(x) := \frac{1}{n} f(x+n)$, $n \in \mathbb{N}$.

Show that $\int f_n g \to 0$

$$0 \le \int_{\mathbb{R}} |f_n(x) g(x)|$$

$$= \int_{\mathbb{R}} \left| \frac{1}{n} f(x+n) g(x) \right|$$

$$= \frac{1}{n} \int_{\mathbb{R}} | f(x+n) g(x) |$$

$$\le \frac{1}{n} \|f\|_2 \|g\|_2$$

$$\|f\|_2 = \left[\left(\int_{\mathbb{R}} |f(x)|^2 \right)^{1/2} = \left(\int_{\mathbb{R}} |f(x+n)|^2 \right)^{1/2} \right]$$

(Hölder's Inequality with $p = q = 2$)

Similarly,

$$0 = \lim_{n\to\infty} 0 \le \lim_{n\to\infty} \int_{\mathbb{R}} |f_n g| \le \lim_{n\to\infty} \frac{1}{n} \|f\|_2 \|g\|_2 = 0$$

which means $\lim_{n\to\infty} \int_{\mathbb{R}} |f_n g| = 0$.

This implies that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n g = 0$.