MA 503 : Homework 12

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Problem 20

(Part 1) Show that the sum of simple functions is a simple function and the product of simple functions is a simple function.

Let ϕ_1 and ϕ_2 be simple functions both defined on the measurable set $E \subset \mathbb{R}$ (by definition, a simple function must be measurable which means its domain must be measurable) with:

$$\phi_1(x) = \sum_{n=1}^{N} \alpha_n \chi_{A_n}, \quad \text{where } A_n = \{x : \phi_1(x) = \alpha_n\}$$
$$\phi_2(x) = \sum_{m=1}^{M} \beta_m \chi_{B_m}, \quad \text{where } B_m = \{x : \phi_2(x) = \beta_m\}$$

Since ϕ_1 and ϕ_2 are measurable functions, the sum $\phi_1 + \phi_2$ is measurable. To show that $\phi_1 + \phi_2$ is simple, we need to show that $\phi_1 + \phi_2$ assumes only a finite number of values. Adding sets and relabeling if necessary, we can assume that $E = \cupdot A_m$ and $E = \cupdot$. This allows us to write:

$$\begin{aligned}
\phi_1 + \phi_2 &= \sum_{m=1}^{M} \sum_{n=1}^{N} (\alpha_n + \beta_m)\chi_{A_n \cap B_m} \\
&= \sum_{m=1}^{M} [(\alpha_1 + \beta_m)\chi_{A_1 \cap B_m} + ... + (\alpha_N + \beta_m)\chi_{A_N \cap B_m}] \\
&= [(\alpha_1 + \beta_1)\chi_{A_1 \cap B_1} + ... + (\alpha_N + \beta_1)\chi_{A_N \cap B_1}] \\
&+ [(\alpha_1 + \beta_2)\chi_{A_1 \cap B_2} + ... + (\alpha_N + \beta_2)\chi_{A_N \cap B_2}] \\
&\vdots \\
&+ [(\alpha_1 + \beta_M)\chi_{A_1 \cap B_M} + ... + (\alpha_N + \beta_M)\chi_{A_N \cap B_M}]
\end{aligned}$$

$$\begin{aligned}
&:= [\gamma_1 \chi_{C_1} + ... + \gamma_N \chi_{C_N}] \\
&+ [\gamma_{N+1}\chi_{C_{N+1}} + ... + \gamma_{2N}\chi_{C_{2N}}] \\
&\vdots \\
&+ [\gamma_{(M-1)N+1}\chi_{C_{(M-1)N+1}} + ... + \gamma_{MN}\chi_{C_{MN}}] \\
&= \sum_{i=1}^{MN} \gamma_i \chi_{C_i} \ ,
\end{aligned}$$

where we used the fact that for any pair n, m, $\alpha_n + \beta_m$ is a real number and $A_n \cap B_m$ is a measurable set so that we can we can assign a coefficient γ_i and a characteristic function χ_{C_i} to each term in the finite sum. Also, since the A_n and B_m are disjoint, $(A_n \cap B_m) \cap (A_{n'} \cap B_{m'}) = \emptyset$ whenever $(n, m) \neq (n', m')$. This shows that $\phi_1 + \phi_2$ can be written in the form of a simple function and so takes on only a finite number of values. Conclude that $\phi_1+\phi_2$ is simple. By very similar reasoning, using coefficients of the form $\gamma_i = \alpha_n\beta_m$ instead of $\gamma_i = \alpha_n + \beta_m$ for the possible pairs (n, m),

$$\phi_1\phi_2 = \sum_{m=1}^{M}\sum_{n=1}^{N} \alpha_n\beta_m\chi_{A_n\cap B_m} = \sum_{i=1}^{MN} \gamma_i\chi_{C_i} \ ,$$

where the C_i are disjoint measurable sets with $\cup C_i = E$. Conclude that $phi_1\phi_2$ is a measurable function (as the product of measurable functions) that assumes only a finite number of values and is therefore a simple function.

(Part 2) Let A and B be sets of real numbers and χ_A and χ_B corresponding characteristic functions. Show that the sum $\chi_A + \chi_B$ and the product $\chi_A\chi_B$ are simple functions and that,

$$\begin{aligned}
\chi_{A\cap B} &= \chi_A\chi_B \\
\chi_{A\cup B} &= \chi_A + \chi_B - \chi_A\chi_B \\
\chi_{A^c} &= 1 - \chi_A
\end{aligned}$$

We may as well assume that $\chi_A, \chi_B, \chi_{A\cap B}, \chi_{A\cup B}, \chi_{\overline{A}} : D \subset \mathbb{R} \to \{0, 1\}$; that is, all characteristic functions mentioned have the same domain D where $A, B \subset D$. The reasoning would look the same for any choice of $D \subset \mathbb{R}$. Whether or not D is measurable is not important for the identities we are proving here, only for whether the functions are measurable or not. Let $x \in D$.

Either $x \in A \cap B$ or $x \notin A \cap B$. If $x \in A \cap B$ then $\chi_{A\cap B}(x) = 1$, $\chi_A(x) = 1$, and $\chi_B(x) = 1$ so $\chi_{A\cap B}(x) = 1 = 1 \cdot 1 = \chi_A(x)\chi_B(x)$. If $x \notin A \cap B$ then $\chi_{A\cap B}(x) = 0$ and $\chi_A(x) = 0$ or $\chi_B(x) = 0$ (or both of course). Then $\chi_{A\cap B}(x) = 0 = \chi_A\chi_B$. Since x was arbitrary, this shows that $\chi_{A\cap B} \equiv \chi_A\chi_B$ on D.

Exactly one of these four possibilities must hold: (i) $x \in A \cap B$, (ii) $x \in A\backslash B$, (iii) $x \in B\backslash A$, or (iv) $x \notin A \cup B$. We check that the equality $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$ is satisfied in each case.

$$\begin{aligned}
&\text{(i) } \chi_{A\cup B}(x) = \chi_A(x) = \chi_B(x) = \chi_{A\cap B}(x) = 1 \\
&\quad \implies \chi_{A\cup B}(x) = 1 = 1 + 1 - 1 = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x) \\
&\text{(ii) } \chi_{A\cup B}(x) = \chi_A(x) = 1, \chi_B(x) = \chi_{A\cap B}(x) = 0 \\
&\quad \implies \chi_{A\cup B} = 1 = 1 + 0 - 0 = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x) \\
&\text{(iii) } \chi_{A\cup B}(x) = \chi_B(x) = 1, \chi_A(x) = \chi_{A\cap B}(x) = 0 \\
&\quad \implies \chi_{A\cup B} = 1 = 0 + 1 - 0 = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x) \\
&\text{(iv) } \chi_{A\cup B}(x) = \chi_A(x) = \chi_B(x) = \chi_{A\cap B}(x) = 0 \\
&\quad \implies \chi_{A\cup B}(x) = 0 = 0 + 0 - 0 = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)
\end{aligned}$$

Either $x \in A$ or $x \in A^c$. If $x \in A$, then $\chi_{A^c}(x) = 0$ and $\chi_A = 1$ so $\chi_{A^c} = 0 = 1 - 1 = 1 - \chi_A$. If $x \in A^c$, then $\chi_{A^c} = 1$ and $\chi_A = 0$ so $\chi_{A^c} = 1 = 1-0 = 1-\chi_A$. Conclude that $\chi_{A^c} \equiv 1-\chi_A$ on D.

Proposition 14 Let (E_i) be a sequence of decreasing measurable sets, that is, a sequence with $E_{n+1} \subset E_n$ for each $n \in \mathbb{N}$. Let $m(E_1) < \infty$. Then,

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n) \ .$$

Proposition 15 Let E be a given set. The following five statements are equivalent.

i. E is measurable.
ii. Given $\epsilon > 0$ there is an open set $O \supset E$ such that $m^*(O\backslash E) < \epsilon$.
iii. Given $\epsilon > 0$ there is a closed set $F \subset E$ such that $m^*(E\backslash F) < \epsilon$.
iv. There is a $G \in G_\delta$ with $E \subset O$ such that $m^*(G\backslash E) = 0$.

v. There is an $F \in F_\sigma$ with $F \subset E$ such that $m^*(E \backslash F) = 0$.

If $m^*(E) < \infty$, the above statements are equivalent to:

vi. Given $\epsilon > 0$, there is a finite union U of open intervals such that $m^*(U \triangle E) < \epsilon$.

Proposition 22 Let f be a measurable function defined on an interval $[a, b]$, and assume that f takes on the values $\pm\infty$ only on a set of measure zero. Then given ϵ, we can find a step function g and a continuous function h such that

$$|f - g| < \epsilon \text{ and } |f - h| < \epsilon$$

except on a set of measure less than ϵ; i.e., $m(\{x : |f(x) - g(x)| \geq \epsilon\}) < \epsilon$ and $m(\{x : |f(x) - g(x)| \geq \epsilon\}) < \epsilon$. If in addition, $m \leq f \leq M$, then we may choose the functions g and h such that $m \leq g, h \leq M$.

Definition If A is any set, we define the characteristic function χ_A of the set A as

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} .$$

The function A is measurable if and only if A is measurable.

Remark By this definition, the existence of a nonmeasurable set implies the existance of a nonmeasurable function.

Definition A real-valued function ϕ is called simple if it is measurable and assumes only a finite number of values. If ϕ is simple and has the values $\alpha_1, ..., \alpha_n$ then $\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where $A_i = \{x : \phi(x) = \alpha_i\}$. The sum, product, and difference of two simple functions are simple.

Problem 23 Prove Proposition 22 by establishing the following lemmas:

(a) Given a measurable function f on $[a, b]$ that takes on the values $\pm\infty$ only on a set of measure zero, and given $\epsilon > 0$, there is an M such that $|f| \leq M$ except on a set of measure less than $\epsilon/3$.

Let $E_n = \{x : |f(x)| > n\}$ for each $n \in \mathbb{N}$. Since f is measurable, $|f|$ is measurable and therefore each set E_n is measurable. Also, $E_{n+1} \subset E_n$ for each n and $m(E_1) \leq m([a, b]) < \infty$. By Proposition 14,

$$\lim_{n\to\infty} m(E_n) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = m(\{x : f(x) = \pm\infty\}) = 0 .$$

It follows that given $\epsilon > 0$, there is an M such that for all $n \geq M$, $m(E_n) < \epsilon/3$. In particular, $m(E_M) < \epsilon/3$ and $E_M^c = \{x : |f(x)| \leq M\}$. That is, $|f| \leq M$ except on the set E_M, which is of measure $\epsilon/3$.

(b) Let f be a measurable function on $[a, b]$. Given $\epsilon > 0$ and $M \geq 0$ there is a simple function ϕ such that $|f(x) - \phi(x)| < \epsilon$ except where $|f(x)| \geq M$. If $m \leq f \leq M$, then we may take ϕ so that $m \leq \phi \leq M$.

Let $\epsilon > 0$. Since $M < \infty$, there is a $k \in \mathbb{N}$ such that $k\epsilon \geq M$ and $-k\epsilon \leq M$. Assume k to be the smallest such integer.

$$\begin{aligned}
A_1 &= f^{-1}([0,\epsilon)),\\
A_2 &= f^{-1}([\epsilon,2\epsilon)),\\
A_3 &= f^{-1}([2\epsilon,3\epsilon)),\\
&\vdots\\
A_k &= f^{-1}([(k-1)\epsilon,k\epsilon))
\end{aligned}$$

$$\begin{aligned}
A_{-1} &= f^{-1}([-\epsilon,0)),\\
A_{-2} &= f^{-1}([-2\epsilon,-\epsilon))\\
&\vdots\\
A_{-k} &= f^{-1}([-k\epsilon,-(k-1)\epsilon))
\end{aligned}$$

The A_i are disjoint and for $I := \{-k,...,-1,1,...,k\}$, $\cup_{i\in I}A_n \supset [-M,M]$. So for each $x \in [a,b]$ such that $|f(x)| \leq M$, $f(x)$ lies within exactly one of the A_i. Let $\alpha_i = i\epsilon - \epsilon/2$ for $i = 1,...,k$ and $\alpha_i = -i\epsilon + \epsilon/2$ for $i = -1,...,-k$. That is, α_i is the midpoint of the half open interval used in the definition of the set A_i. Define $\phi(x) = \sum_{i\in I}\alpha_i\chi_{A_i}(x)$ for each $x \in [a,b]$ such that $|f(x)| \leq M$. Then since each of the A_i are measurable (this follows from the fact that f is measurable), χ_{A_i} is measurable for each i and so ϕ is measurable. Since ϕ is measurable and assumes only finitely many values, ϕ is a simple function. Let $x \in [a,b]$ such that $|f(x)| \leq M$. Then $x \in A_i$ for some $i \in I$ so $|f(x) - \phi(x)| = |f(x) - \alpha_i| \leq \epsilon/2 < \epsilon$.

If $m \leq f \leq M$ use a similar approach. Find $k \in \mathbb{N}$ such that $m + k\epsilon \leq M$ but for which $m + (k+1)\epsilon > M$.

$$\begin{aligned}
A_1 &= f^{-1}([m,m+\epsilon))\\
A_2 &= f^{-1}([m+\epsilon,m+2\epsilon))\\
&\vdots\\
A_k &= f^{-1}([m+(k-1)\epsilon,m+k\epsilon))\\
A_{k+1} &= f^{-1}([m+k\epsilon,M]) \quad (\epsilon \geq M-(m+k\epsilon) \geq 0)
\end{aligned}$$

Define $\alpha_1 = m+\epsilon/2$, $\alpha_2 = m+(3/2)\epsilon$,...,$\alpha_k = m+(k-1/2)\epsilon$, $\alpha_{k+1} = (m+k\epsilon+M)/2$. That is, take α_i to be the midpoint of the interval used to define A_i. Let $\phi = \sum_{i=1}^{k+1}\alpha_i\chi_{A_i}$. Then ϕ is a simple function. For each $x \in [a,b]$, $x \in A_i$ for exactly one of the disjoint A_i and so $|\phi(x) - f(x)| < \epsilon$ and $m \leq \phi \leq M$.

(c) Given a simple function ϕ defined on $[a,b]$, there is a step function g defined on $[a,b]$ such that $g(x) = \phi(x)$ except on a set of measure $\epsilon/3$. If $m \leq \phi(x) \leq M$, then we may take g so that $m \leq g \leq M$.

Let $\phi : [a,b] \to \{\alpha_1,...,\alpha_n\}$, $\phi(x) = \sum_{i=1}^{n}\alpha_i\chi_{A_i}(x)$. Each A_i is measurable and $m^*(A_i) = m(A_i) \leq m([a,b]) < \infty$. By Proposition 15 (ii) there is an open set O_i' such that $m^*(O_i'\backslash A_i) < \epsilon/(6n)$. Let $O_i = O_i' \cap [a,b]$ for which we still have $A_i \subset O_i$ (since $A_i \subset [a,b]$ and $A_i \subset O_i'$) and $m^*(O_i\backslash A_i) < \epsilon/(6n)$. The open set O_i can be written as a countable union of disjoint open intervals, $O_i = \cup_{n\in\mathbb{N}}I_n$.

$$m^*(O_i) = m^*\left(\bigcup I_n\right) = m\left(\bigcup I_n\right) = \sum_{n=1}^{\infty} m(I_n) = \lim_{N\to\infty} \sum_{n=1}^{N} m(I_n)$$

$$\implies \exists N \text{ s.t. } \sum_{N+1}^{\infty} m(I_n) < \epsilon/(6n) \quad \text{(the terms of a convergent series tend to zero)} .$$

Define $U_i := \cup_{n=1}^{N} I_n$

$$m^*(O_i \backslash U_i) = m^*(O_i \cap U_i^c) = m^*(\cup_{n=N+1}^{\infty} I_n) = m(\cup_{n=N+1}^{\infty} I_n) = \sum_{n=N+1}^{\infty} m(I_n) < \epsilon/(6n)$$

Repeat this process for each A_i to produce a U_i. Since each U_i is a union of open intervals, we can define the step function $g = \sum_{i=1}^{n} \alpha_i \chi_{U_i}$. Since the U_i are disjoint, for each $x \in [a,b]$, $x \in U_i$ for at most one U_i. Then $g(x) = \alpha_i = \phi(x)$ for $x \in A_i \cap U_i$. So for each i we have $\phi(x) = \alpha_i = g(x)$ except on $U_i \triangle A_i$ and:

$$\begin{aligned}
m(U_i \triangle A_i) &= m((U_i \backslash A_i) \cup (A_i \backslash U_i)) \\
&\leq m(U_i \backslash A_i) + m(A_i \backslash U_i) \\
&= m^*(U_i \backslash A_i) + m^*(A_i \backslash U_i) \\
&\leq m^*(O_i \backslash A_i) + m^*(O_i \backslash U_i) \\
&< \epsilon/(6n) + \epsilon/(6n) = \epsilon/(3n) .
\end{aligned}$$

In total, $\phi(x) = g(x)$ except on a set of measure $n\epsilon/(3n) = \epsilon/3$.

If $m \leq \phi(x) \leq M$, define g just as before except that instead of using χ_{U_i}, use

$$\chi'_{U_i} = \begin{cases} 1 & x \in U_i \\ m & x \notin U_i \end{cases} .$$

Then whenever $\phi(x) = g(x)$, it must be that $m \leq g(x) \leq M$ and whenever $\phi(x) \neq g(x)$ we still have either $m \leq g(x) = \alpha_i \leq M$ for some α_i or $g(x) = m \leq M$. This change accounts for the possibility that $m > 0$ which would allow $g(x) = 0 < m$ to occur.

(d) Given a step function g defined on $[a,b]$ there is a continuous function h defined on $[a,b]$ such that $g(x) = h(x)$ except on a set of measure $\epsilon/3$. If $m \leq g \leq M$ we can take h such that $m \leq h \leq M$.

Let g be a step function defined on $[a,b]$. Then g can be written in the form $g = \sum_{i=1}^{m} \alpha_i \chi_{I_i}$ where the I_i are intervals. The intervals can be taken so that they are disjoint and $\cup_{i=1}^{m} I_i = [a,b]$. Also, if it is the case that I_j and I_k are consecutive intervals and $\alpha_j = \alpha_k$ then we can collapse I_j and I_k into a single interval $I_l = I_j \cup I_k$ on which $g(x) = \alpha_j = \alpha_k =: \alpha_l$. Relabelling if necessary, assume a is the left endpoint of the interval I_1 and b the right endpoint of the last interval, I_m, used in the definition of g. We will use a construction that does not depend on whether the endpoints of any particular interval are open or closed (to include a and b we need at least two closed endpoints). For purely notational convenience, therefore, we will write most of the intervals as if they were all open - but it should be understood that these intervals may not take this form. Write

$$\begin{aligned}
I_1 &= [p_0, p_1) = [a, p_1) \\
I_2 &= (p_1, p_2) \\
&\vdots \\
I_n &= (p_{n-1}, p_n] = (p_{n-1}, b] .
\end{aligned}$$

The function g is discontinuous at the $n-1$ points $p_1, ..., p_{n-1}$. We define h to be equal to g except at intervals of the size $\epsilon/[3(n-1)]$ around each of these points of discontinuity (if $n = 1$, g is constant on $[a,b]$ and so already continuous itself). Let $\Delta x = \epsilon/[3(n-1)]$.

$$h(x) = \begin{cases} \alpha_1 & a \leq x \leq p_1 - \Delta x/2 \\ \frac{\alpha_2 - \alpha_1}{\Delta x}[x - (p_1 - \Delta x/2)] + \alpha_1 & p_1 - \Delta x/2 < x < p_1 + \Delta x/2 \\ \alpha_2 & p_1 + \Delta x/2 \leq x \leq p_2 - \Delta x/2 \\ \frac{\alpha_3 - \alpha_2}{\Delta x}[x - (p_2 - \Delta x/2)] + \alpha_2 & p_2 - \Delta x/2 < x < p_2 + \Delta x/2 \\ \vdots & \\ \frac{\alpha_n - \alpha_{n-1}}{\Delta x}[x - (p_{n-1} - \Delta x/2)] + \alpha_{n-1} & p_{n-1} - \Delta x/2 < x < p_{n-1} + \Delta x/2 \\ \alpha_n & p_{n-1} + \delta x/2 \leq x \leq b \end{cases}$$

The result is that h is constant and agrees with g except near the 'jumps' of g, where h is then defined to be a linear function connecting each of the constant portions in order to satisfy continuity. Then h disagrees with g on $n-1$ intervals each of length $\epsilon/[3(n-1)]$. That is, h is a continuous function for which $h(x) = g(x)$ except on a set of measure $\epsilon/3$. Using this construction of h, it follows that if $m \leq g \leq M$, then $m \leq h \leq M$ as well.

Conclusion Let f be a measurable function defined on $[a, b]$ and assume that f takes on the values $\pm\infty$ only on a set of measure zero. Then given $\epsilon > 0$, there is an M such that $|f| \leq M$ except on a set A of measure less than $\epsilon/3$ by (a). By (b) there is a simple function ϕ such that $|f - \phi| < \epsilon$ except where $|f| \geq M$. By (c) there is a step function g such that $\phi = g$ except on a set C of measure less than $\epsilon/3$. So $|f-g| = |f-\phi| < \epsilon$ except possibly on $A \cup C$ where $m(A \cup C) < 2\epsilon/3 < \epsilon$. By (d) there is a continuous function h such that $g = h$ except on a set D of measure less than $\epsilon/3$. So $|f-h| = |f-g| = |f-\phi| < \epsilon$ except possibly on $A \cup C \cup D$ with $m(A \cup C \cup D) < 3\epsilon/3 = \epsilon$. The results are only improved if $m \leq f \leq M$ since in this case we can find ϕ such that $|f - \phi| < \epsilon$ over a more inclusive set.