MA 503 : Lebesgue Measure and Integration

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December 2, 2020

Chapter 6 : The Classical Banach Spaces

1 The L^p Spaces

Definition Let p be a positive real number. A measurable function defined on $[0,1]$ is said to belong to the space $L^p([0,1])$ if $\int_0^1 |f|^p < \infty$. For a function $f \in L^p([0,1])$ (abbreviate $f \in L^p$),

$$||f|| = ||f||_p := \left(\int_0^1 |f|^p \right)^{1/p} .$$

Remarks

1. $f \in L^p([0,1])$ if and only if f is integrable on $[0,1]$.

Proof: If $f \in L^1([0,1])$, $\int_0^1 |f| < \infty$. If either $\int_0^1 f^+ = \infty$ or $\int_0^1 f^- = \infty$, then the fact that $0 \leq f^+, f^- \leq f^+ + f^-$ would give the contradiction $\infty = \int_0^1 (f^+ + f^-) = \int_0^1 |f| < \infty$. So both f^+ and f^- are integrable on $[0,1]$ and thus f is integrable on $[0,1]$. Conversely, if f is integrable on $[0,1]$, $\int_0^1 f^+, \int_0^1 f^- < \infty$. This means $\int_0^1 |f| < \infty$ and $f \in L^p([0,1])$.

2. If $f \in L^p$ and $\alpha \in \mathbb{R}$, then $\int_0^1 |\alpha f|^p = |\alpha|^p \int_0^1 |f|^p < \infty \implies \alpha f \in L^p$.

3. If one accepts the use of the inequality $|f+g|^p \leq 2^p(|f|^p + |g|^p)$, it follows that if $f, g \in L^p$, then $f + g \in L^p$. (Note: Other sources, like Wikipedia, use the stricter 2^{p-1} in this inequality. Using 2^p is either an error or intentional to handle certain cases I'm not aware of).

Proof:

$$\int_0^1 |f+g|^p \leq 2^p \int_0^1 (|f|^p + |g|^p) = 2^p \left(\int_0^1 |f|^p + \int_0^1 |g|^p \right) < \infty .$$

4. It follows from 2 and 3 that if $f, g \in L^p$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in L^p$. Therefore, for each p, L^p is a vector space (or linear space), where the vectors are real valued functions defined on $[0,1]$ and the scalar field is $\mathbb{R}$.

5. $||f|| = 0$ if and only if $f = 0$ almost everywhere.

Proof: $||f|| = 0 \implies 0 = \int_0^1 |f|^p$. Since f is measurable, $|f|$ is measurable. The product of measurable p measurable functions $|f||f|...|f| = |f|^p$ is also measurable. Therefore $|f|^p$ is a non-negative measurable function for which $\int_0^1 |f|^p = 0$. By Problem 4.3, $|f|^p = 0$ a.e. and so $|f| = 0$ a.e., which implies that $f = 0$ a.e. Conversely, if $f = 0$ a.e then $|f|^p = 0$ a.e. and so again using Problem 4.3, $\int_0^1 |f|^p = 0$, which means $(\int_0^1 |f|^p)^{1/p} = 0$. So all of the steps were reversible.

6. If $\alpha \in \mathbb{R}$, $||\alpha f||_p = \left(\int_0^1 |\alpha f|^p \right)^{1/p} = |\alpha| \left(\int_0^1 |f|^p \right)^{1/p} = |\alpha| ||f||_p$.

7. In the next section the Minkowski inequality will show that $||f+g||_p \leq ||f||_p + ||g||_p$ if $p \geq 1$, so starting now assume that $||\cdot|| = ||\cdot||_p$. A linear space (or vector space) is said to be a normed linear space if we have assigned a nonnegative real number $||f||$ to each f such that $||\alpha f|| = |\alpha| ||f||$, $||f+g|| \leq ||f|| + ||g||$ and $||f|| = 0 \iff f \equiv 0$. By 5, we see that the L^p spaces fail this last condition. To ameliorate this, consider two measurable functions equivalent if they are equal almost

everywhere. Then a function that is zero almost everywhere will be considered equivalent to the zero function. So if the elements of an L^p space are considered as equivalence classes of functions then L^p can be treated as a normed linear space. But in practice what will be done is to treat the elements of L^p as functions as we originally introduced the concept and then just not distinguish between equivalent functions.

Definition Let L^∞ denote the space of bounded measurable functions on $[0,1]$ (or rather all measurable functions bounded except possibly on a subset of measure zero considering remark 7). Again identify functions which are equivalent. Then L^∞ is a linear space (quick mental proof).

$$||f||_\infty := \text{ess }\sup |f(t)| \ ,$$

where $\text{ess }\sup f(t)$ is the infimum of $\sup g(t)$ as g ranges over all functions which are equal to f almost everywhere. Said another way, the essential supremum of f is the smallest number M such that the set $\{x \in [0,1] : f(x) > M\}$ has measure zero, or :

$$\begin{aligned} \text{ess }\sup f(t) &= \inf\{M : m(\{t : f(t) > M\}) = 0\} \ , \\ \text{ess }\sup |f(t)| &= \inf\{M : m(\{t : |f(t)| > M\}) = 0\} \ . \end{aligned}$$

L^∞ is a normed linear space under $||\cdot||_\infty$.

Lemma Eggcorn If A and B are sets of real numbers bounded below, then $\inf(A+B) = \inf(A) + \inf(B)$.

Proof: Let $a+b \in A+B$ with $a \in A$ and $b \in B$. Since $\inf(A) \le a$ and $\inf(B) \le b$, $\inf(A)+\inf(B) \le a+b$. So $\inf(A)+\inf(B)$ is a lower bound of $A+B$ and $\inf(A)+\inf(B) \le \inf(A+B)$. For each $\epsilon > 0$ there is exist $a \in A$ and $b \in B$ such that $a < \inf(A)+\epsilon/2$ and $b < \inf(B)+\epsilon/2$ so that $\inf(A+B) \le a+b < \inf(A)+\inf(B)+\epsilon$. Then since $\inf(A+B) < \inf(A)+\inf(B)+\epsilon$ for every $\epsilon > 0$, $\inf(A+B) \le \inf(A)+\inf(B)$.

Problem 1 Show that $||f+g||_\infty \le ||f||_\infty + ||g||_\infty$.

$$\begin{aligned} A &:= \{P : m(\{t : |f(t)+g(t)| > P\}) = 0\} = \{P : |f+g| \le P \text{ a.e.}\} \\ B &:= \{M : m(\{t : |f(t)| > M\}) = 0\} = \{M : |f| \le M \text{ a.e.}\} \\ C &:= \{N : m(\{t : |g(t)| > N\}) = 0\} = \{N : |g| \le N \text{ a.e.}\} \end{aligned}$$

$$B+C := \{M+N : m(\{t : |f(t)| > M\}) = 0, m(\{t : |g(t)| > N\}) = 0\} = \{M+N : |f| \le M \text{ a.e.}, |g| \le N \text{ a.e.}\} \ .$$

Let $M+N \in B+C$ such that $M \in B$ and $N \in C$. Then $|f| \le M$ a.e. and $|g| \le N$ a.e. which means that $|f|+|g| \le M+N$ a.e. But then $|f+g| \le |f|+|g| \le M+N$ a.e. so that $M+N \in A$. Therefore $B+C \subset A$ and $\inf A \le \inf(B+C)$. Since B and C are sets of real numbers bounded each bounded below by 0 (because $|f|, |g| \ge 0$), $\inf(B+C) = \inf B + \inf C$ by Lemma Eggcorn. That is,

$$\begin{aligned} ||f+g||_\infty &= \inf\{P : m(\{t : |f(t)+g(t)| > P\}) = 0\} \\ &= \inf A \\ &\le \inf B + \inf C \\ &= \inf\{M : m(\{t : |f(t)| > M\}) = 0\} + \inf\{N : m(\{t : |g(t)| > N\}) = 0\} \\ &= ||f||_\infty + ||g||_\infty \ . \end{aligned}$$

We can also stick with the provided definition of essential supremum and follow a similar but messier argument. If $M \in B$ and $N \in C$, then $m(\{t : |f(t)| > M\}) = 0$ and $m(\{t : |f(t)| > N\}) = 0$.

$$
\begin{aligned}
&m(\{t : |f(t) + g(t)| > M + N\}) \\
&\le m(\{t : |f(t)| + |g(t)| > M + N\}) \quad \text{(from the triangle inequality)} \\
&= m[(\{t : |f(t)| > M\} \cap \{t : |g(t)| > N\}) \cup \\
&(\{t : |g(t)| \le N\} \cap \{t : |f(t)| > M + N - |g(t)|\}) \cup (\{t : |f(t)| \le M\} \cap \{t : |g(t)| > M + N - |f(t)|\})] \\
&\le m(\{t : |f(t)| > M\} \cap \{t : |g(t)| > N\}) + m(\{t : |g(t)| \le N\} \cap \{t : |f(t)| > M + N - |g(t)|\}) \\
&+ m(\{t : |f(t)| \le M\} \cap \{t : |g(t)| > M + N - |f(t)|\}) \\
&\le m(\{t : |f(t)| > M\}) + m(\{t : |f(t)| > M\}) + m(\{t : |g(t)| > N\}) \\
&= 0\ .
\end{aligned}
$$

This shows that if $M + N \in B + C$ with $M \in B$ and $N \in C$, then $M + N \in A$. Therefore, $B + C \subset A$ and so $\inf(A) \le \inf(B + C) = \inf(B) + \inf(C)$. That is, $||f + g||_\infty \le ||f||_\infty + ||g||_\infty$.

Problem 2 Let f be a bounded measurable function on $[0, 1]$. Prove that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

We have $|f| \le ||f||_\infty$ a.e. and so

$$
\begin{aligned}
||f||_p &= \left(\int_0^1 |f|^p\right)^{1/p} \\
&\le \left(\int_0^1 ||f||_\infty^p\right)^{1/p} \\
&= (||f||_\infty^p m([0,1]))^{1/p} \\
&= ||f||_\infty \\
\implies \lim_{p\to\infty} ||f||_p &\le |||f||_\infty\ .
\end{aligned}
$$

For each $p \in \mathbb{N}$, the set $B_p = \{x : |f(x)| > ||f||_\infty - 1/p\}$ has positive measure since if $m(B_p) = 0$, we have $||f||_\infty - 1/p \in \{M : m(\{x : |f(x)| > M\}) = 0\}$ and $||f||_\infty - 1/p < ||f||_\infty = \inf\{M : m(\{x : |f(x)| > M\}) = 0\}$, which is a contradiction.

$$
\begin{aligned}
\left(\int_{B_p} |f|^p\right)^{1/p} &\ge \left(\int_{B_p} |\ ||f||_\infty - 1/p\ |^p\right)^{1/p} \\
&= |\ ||f||_\infty - 1/p\ | \,(m(B_p))^{1/p}
\end{aligned}
$$

As $p \to \infty$, $||f||_\infty - 1/p \to ||f||_\infty$, so $B_p \to [0, 1]$. Then,

$$
\lim_{p\to\infty} ||f||_p \ge \lim_{p\to\infty} |\ ||f||_\infty - 1/p\ | \,(m(B_p))^{1/p} = ||f||_\infty m([0,1]) = ||f||_\infty\ .
$$

Problem 3 Prove that $||f + g||_1 \le ||f||_1 + ||g||_1$.

Suppose $f, g \in L^1([0, 1])$. For each $x \in [0, 1]$, $|f + g| \le |f| + |g|$, so

$$
||f + g||_1 = \int_0^1 |f + g| \le \int_0^1 (|f| + |g|) = \int_0^1 |f| + \int_0^1 |g| = ||f||_1 + ||g||_1\ .
$$

Problem 4 Show that if $f \in L^1$ and $g \in L^\infty$,

$$
\int |fg| \le ||f||_1 \cdot ||g||_\infty\ .
$$

We have $|g| \le ||g||_\infty$ almost everywhere so $|fg| = |f||g| \le |f|||g||_\infty$ almost everywhere. We have defined L^p spaces in this section on the interval $[0, 1]$, so

$$
\int |fg| = \int_0^1 |fg| = \int_0^1 |f||g| \le \int_0^1 |f|||g||_\infty = ||g||_\infty \int_0^1 |f| = ||g||_\infty \int |f| = ||g||_\infty ||f||_1\ .
$$

2 The Minkowski and Hölder Inequalities

1. The Minkowski Inequality If $f, g \in L^p$ with $1 \le p \le \infty$, then $f + g \in L^p$ and

$$||f+g||_p \le ||f||_p + ||g||_p \ .$$

If $1 < p < \infty$, then inequality can only hold if there are nonnegative constants α and β such that $\beta f = \alpha g$.

Proof: The case when $p = \infty$ is problem 1 of the previous section. If $||f|| = 0$, then $f = 0$ a.e. and so $f + g = g$ a.e. By the convention mentioned in remark 7 of the previous section, $||f + g|| = ||g|| = 0 + ||g|| = ||f|| + ||g||$. The case is similar if $||g|| = 0$. Otherwise, assume $1 \le p < \infty$ and $||f|| = \alpha \ne 0$, $||g|| = \beta \ne 0$. Let f_0 and g_0 be functions such that $|f| = \alpha f_0$ and $|g| = \beta g_0$ so that

$$||f_0|| = \left(\int_0^1 |f|^p/|\alpha|^p\right)^{1/p} = \left(\int_0^1 |f|^p\right)^{1/p} \frac{1}{|\alpha|} = \frac{||f||}{|\alpha|} = 1 \ .$$

$$||g_0|| = \left(\int_0^1 |g|^p/|\beta|^p\right)^{1/p} = \left(\int_0^1 |g|^p\right)^{1/p} \frac{1}{|\beta|} = \frac{||g||}{|\beta|} = 1 \ .$$

$$\lambda := \frac{\alpha}{\alpha+\beta}, \quad 1-\lambda = \frac{\alpha+\beta}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta} \ .$$

For all $x \in [0,1]$,

$$\begin{aligned}(|f+g|)^p &\le (|f|+|g|)^p \\ &= (\alpha f_0 + \beta g_0)^p \\ &= (\alpha+\beta)^p \left(\frac{\alpha}{\alpha+\beta} f_0 + \frac{\beta}{\alpha+\beta} g_0\right)^p \\ &= (\alpha+\beta)^p \left(\lambda f_0 + (1-\lambda) g_0\right)^p \\ &\le (\alpha+\beta)^p \left(\lambda (f_0)^p + (1-\lambda)(g_0)^p\right)\end{aligned}$$

The last inequality used the convexity of the function $\varphi(t) = t^p$ on $[0, \infty)$ for $1 \le p < \infty$ meaning $\varphi(\lambda f_0 + (1-\lambda) g_0) = \lambda\varphi(f_0) + (1-\lambda)\varphi(g_0)$. If $1 < p < \infty$ the inequality is strict unless $f_0(x) = g_0(x)$ and sgn $f(x) =$ sgn $g(x)$ (I don't think I'll prove this). Integrating the inequality established above,

$$\begin{aligned}||f+g||^p &\le (\alpha+\beta)^p \left(\lambda ||f_0||^p + (1-\lambda)||g_0||^p\right) \\ &= (\alpha+\beta)^p \left(\lambda + (1-\lambda)\right) \\ &= (\alpha+\beta)^p \\ &= (||f|| + ||g||)^p \ . \\ \therefore ||f+g|| &\le ||f|| + ||g|| \ .\end{aligned}$$

If $1 < p < \infty$ the inequality is strict unless $f_0 = g_0$ a.e. and sgn $f =$ sgn g a.e. If this occurs then $\alpha f_0 = f$ whenever $\beta g_0 = g$ (a.e.) and $-\alpha f_0 = f$ whenever $-\beta g_0 = g$ (a.e) so that $f = \alpha f_0 = \alpha g_0 = \alpha g/\beta \implies \beta f = \alpha g$.

2. Minkowski Inequality for $0 < p < 1$ Let f and g be two nonnegative functions which belong to the space L^p with $0 < p < 1$. Then,

$$||f+g|| \ge ||f|| + ||g|| \ .$$

Lemma 3 Let $1 \le p < \infty$. Then for a, b, t nonnegative we have

$$(a+bt)^p \ge a^p + ptba^{p-1} \ .$$

Proof: For $\varphi(t) = (a+tb)^p - a^p - ptba^{p-1}$, $\varphi(0) = 0$ and

$$\varphi'(t) = pb(a+tb)^{p-1} - pba^{p-1} = pb[(a+tb)^{p-1} - a^{p-1}] \geq 0 \; .$$

So $\varphi(t)$ is nonnegative and increasing for $t > 0$.

Hölder Inequality If p and q are nonnegative extended real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1 \; ,$$

and $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and

$$\int |fg| \leq ||f||_p ||g||_q \; .$$

Equality holds if and only if for some constants α and β, not both zero, $\alpha|f|^p = \beta|g|^p$ a.e.

Proof: If $p = 1$ and $q = \infty$, then with some abuse of mathematical rigor (division by ∞ was not defined in chapter 2) $1/p + 1/q = 1 + 0 = 1$. Suppose $f \in L^1$ and $g \in L^\infty$. Then $|g| \leq ||g||_\infty$ a.e. and $|fg| \leq |f| \cdot ||g||_\infty$ a.e. so that by Proposition 4.15 (iii),

$$\int |fg| \leq \int (|f| \cdot ||g||_\infty) = \left(\int |f| \right) (||g||_\infty) = ||f||_1 ||g||_\infty \; .$$

Otherwise assume $1 < p < \infty$, which forces $1 < q < \infty$ as well.

Since $|fg| = |f||g|$, we can consider the case that $f, g \geq 0$, replacing f and g with $|f|$ and $|g|$ respectively if necessary.

$$\frac{1}{p} + \frac{1}{q} = 1 \iff \frac{q}{p} + 1 = q \iff q + p = pq \iff q = pq - p \iff \frac{q}{p} = q - 1$$

Set

$$h(x) = g(x)^{q-1} = g(x)^{q/p} \implies g(x) = h(x)^{p-1} = h(x)^{p/q} \; .$$

For nonnegative t, $h(x)^p + ptf(x)h(x)^{p-1} \leq (h(x) + tf(x))^p$ by Lemma 3, so

$$ptf(x)g(x) = ptf(x)h(x)^{p-1} \leq (h(x) + tf(x))^p - h(x)^p \; .$$

$$pt \int fg \leq \int |h + tf|^p - \int |h|^p = ||h + tf||^p - ||h||^p \leq (||h|| + t||f||)^p - ||h||^p \; .$$

$$\frac{d}{dt}\left[pt \int fg \right] \leq \frac{d}{dt} \left[(||h|| + t||f||)^p - ||h||^p \right]$$

$$p \int fg \leq p \, (||h|| + t||f||)^{p-1} \, ||f||$$

This holds for any nonnegative t. In particular, for $t = 0$,

$$p \int fg \leq p||h||^{p-1}||f|| = p||g|| \cdot ||f|| = p||f||_p \cdot ||g||_p \; .$$

$$\therefore \int |fg| \leq ||f||_p \cdot ||g||_p \; .$$

Problem 5

a. Prove the Minkowski inequality for $0 < p < 1$.

b. Show that if $f \in L^p$, $g \in L^p$, then $f + g \in L^p$ even for $0 < p < 1$. Hint: $||f + g||^p \leq 2^p \left(||f||^p + ||g||^p \right)$.

3 Convergence and Completeness

Definition A sequence (f_n) in a normed linear spaced is said to converge to an element f in the space if given $\epsilon > 0$, there is an N such that for all $n \geq N$ we have $||f - f_n|| < \epsilon$. If f_n converges to f, we write $f = \lim f_n$ of $f_n \to f$.

Note that $f_n \to f$ if $||f - f_n|| \to 0$. Convergence in the space L^p, $1 \leq p < \infty$, is referred to as **convergence in the mean of order** p. A sequence of functions (f_n) is said to converge to f in the mean of order p if each f_n belongs to L^p and $||f - f_n||_p \to 0$. Convergence in L^∞ is nearly uniform convergence.

Definition A normed linear space is **complete** if for every Cauchy sequence (f_n) in the space there is an element f in the space such that $f_n \to f$. A complete normed linear space is called a **Banach space**.

A series (f_n) is **summable** to s if s is in the space and $||s - \sum_{i=1}^n f_i|| \to 0$. In this case, write $s = \sum_{i=1}^\infty f_i$. The series (f_n) is **absolutely summable** if $\sum_{n=1}^\infty ||f_n|| < \infty$.

Proposition 5 A normed linear space X is complete if and only if every absolutely summable series is summable.

Proof: Let X be complete and (f_n) an absolutely summable series of elements of X. Since $\sum ||f_n|| = M < \infty$, for every $\epsilon > 0$ there is an N such that $\sum_{n=N}^\infty ||f_n||$. Let $s_n = \sum_{i=1}^n f_n$. Then for $n \geq m \geq N$,

$$||s_n - s_m|| = ||\sum_{i=m+1}^n f_i|| \leq \sum_{i=m+1}^n ||f_i|| \leq \sum_{i=m+1}^\infty ||f_i|| \leq \sum_{i=N}^\infty ||f_i|| < \epsilon \ .$$

This shows that (s_n) is a Cauchy sequence and since X is complete s_n converges to some element $s \in X$.

Let (f_n) be a Cauchy sequence in X. For each integer k there is an integer n_k such that $||f_n - f_m|| < 2^{-k}$ for all n and m greater than n_k. Choose the n_k's so that $n_{k+1} > n_k \geq k$. Then $(f_{n_k})_{k=1}^\infty$ is a subsequence of (f_n). Set $g_1 = f_{n_1}$ and $g_k = f_{n_k} - f_{n_{k-1}}$ for $k > 1$ to obtain the sequence (g_k). Then,

$$s_j = \sum_{k=1}^j g_k = f_{n_1} + (f_{n_2} - f_{n_1}) + ... + (f_{n_j} - f_{n_{j-1}}) = f_{n_j} \ ,$$
$$||g_k|| = ||f_{n_k} - f_{n_{k-1}}|| < 2^{-k} < 2^{-k+1}, \quad k > 1 \ ,$$
$$\sum ||g_k|| \leq ||g_1|| + \sum 2^{-k+1} = ||g_1|| + 1 < \infty \ .$$

This shows that the series (g_k) is absolutely summable and by the hypothesis therefore summable. That is, there is an element $f \in X$ such that $f = \lim_{j\to\infty} \sum_{k=1}^j g_k = \lim_{j\to\infty} f_{n_j}$.

Since (f_n) is Cauchy, given $\epsilon > 0$ there is an N such that $||f_n - f_m|| < \epsilon/2$ for all n and m larger than N. Since $f_{n_k} \to f$, there is a K such that for all $k \geq K$, $||f_{n_k} - f|| < \epsilon/2$. Then there is a k such that $k \geq K$ and $n_k \geq N$. For $n > N$,

$$||f_n - f|| \leq ||f_n - f_{n_k}|| + ||f_{n_k} - f|| < \epsilon/2 + \epsilon/2 = \epsilon \ .$$

Therefore for $n > N$, $||f_n - f|| < \epsilon$, and so $f_n \to f$.

Theorem 6 (Riesz-Fischer) The L^p spaces are complete.

Proof: Suppose $p = \infty$ and let (f_n) be a Cauchy sequence in L^∞ and suppose $f_n \to f$. We want to show that $f \in L^\infty$. There is an N such that for all $m, n \geq N$, $||f_n - f_m|| < 1$ and $||f - f_n|| < 1$. Then $||f|| = ||f - f_n + f_n|| \leq ||f - f_n|| + ||f_n|| \leq 1 + ||f_n|| < \infty$ a.e. since (f_n) is bounded a.e. on $[0, 1]$.

Assume $1 \le p < \infty$. We will prove that every absolutely summable series in L^p is summable in L^p to some element in L^p and then apply proposition 5.

Let (f_n) be a sequence in L^p with $\sum_{n=1}^{\infty} ||f_n||_p = M < \infty$ and for each n define g_n by $g_n(x) = \sum_{k=1}^{n} |f_k(x)|$. From the Minkowski inequality,

$$||g_n|| = || \sum_{k=1}^{n} |f_k| \, || \le \sum_{k=1}^{n} ||f_k|| \le \sum_{k=1}^{\infty} ||f_k|| = M \implies \int |g_n|^p \le \int |M|^p = M^p \ .$$

For each x, $(g_n(x))$ is an increasing sequence of (extended) real numbers and so must converge to an extended real number $g(x)$. The function g defined in this way is measurable and since $g_n \ge 0$, $\inf_{n \ge k} \int g_n^p \le M^p$ for each k, and $\lim_{k\to\infty} M^p = M^p$, by Fatou's Lemma,

$$\int g^p \le \liminf \int g_n^p \le M^p \ .$$

Hence g^p is integrable which implies that $g(x)$ is finite for almost all x. For each x such that $g(x)$ is finite the series $\sum_{k=1}^{\infty} f_k(x)$ is an absolutely summable series of real numbers and so must be summable to a real number $s(x)$. Set $s(x) = 0$ for x such that $g(x) = \infty$. This defines a function s that is almost everywhere the limit of the partial sums $s_n = \sum_{k=1}^{n} f_k$. Hence s is measurable. Since $|s_n(x)|$ for each n, $|s(x)| \le g(x)$ and $\int |s|^p \le \int g^p \le M^p < \infty$ so $s \in L^p$.

$$|s_n(x) - s(x)|^p = |s_n(x) + (-s(x))|^p \le 2^p(|s_n(x) + |-s(x)|)^p \le 2^p(g(x) + g(x))^p = 2^{p+1}[g(x)]^p \ .$$

Since $2^{p+1}g^p$ is integrable and $|s_n(x) - s(x)|$ converges to 0 almost everywhere, $\int |s_n - s|$ is integrable and

$$||s_n - s||^p = \int |s_n - s|^p \to 0$$

by the dominated convergence theorem. Then $||s - \sum_{k=1}^{n} f_k|| = ||s - s_n|| \to 0$. Therefore (f_n) is summable to $s \in L^p$. Conclude that L^p is complete by proposition 5.

Problem 10 Let (f_n) be a sequence of functions in L^∞. Prove that (f_n) converges to f in L^∞ if and only if there is a set E of measure zero such that f_n converges to f uniformly on E^c.

Suppose that (f_n) converges to f in $L^\infty([0,1])$ and let $\epsilon > 0$. There is an $N \in \mathbb{N}$ such that $||f - f_n||_\infty < \epsilon$ for all $n \ge N$. But since $|f(x) - f_n(x)| \le ||f - f_n||_\infty$ for almost all x, this means there is a set E of measure zero such that $|f(x) - f_n(x)| \le ||f - f_n||_\infty < \epsilon$ for all $x \in E^c$ and for all $n \ge N$. Therefore, (f_n) converges to f uniformly on E^c.

Suppose there is a set E of measure zero such that (f_n) converges to f uniformly on E^c. Let $\epsilon > 0$. There is an N such that for all $n \ge N$ and all $x \in E^c$, $|f(x) - f_n(x)| < \epsilon$. That is, for $n \ge N$, $|f(x) - f_n(x)| < \epsilon$ almost everywhere so $\epsilon \in \{M : |f(x) - f_n(x)| < M \text{ a.e. }\}$. Then $||f - f_n||_\infty = \inf\{M : |f(x) - f_n(x)| \text{ a.e. }\} \le \epsilon$. Since $\epsilon > 0$ was arbitrary, conclude that $||f - f_n|| \to 0$ as $n \to \infty$.

Problem 11 Prove that L^∞ is complete.

Suppose (f_k) is a Cauchy sequence in $L^\infty([0,1])$. Then for each $n \in \mathbb{N}$, there is an N such that $||f_k - f_j|| < 1/n$ for all $n \ge N$. Then since $|f_k(x) - f_j(x)| \le ||f_k - f_j||$ for almost all x, there is a set $E_{k,j,n}$ of measure zero such that

$$|f_k(x) - f_j(x)| < 1/n \quad \forall x \in E^c_{k,j,n} \ .$$

Let $E = \bigcup_{k,j,n} E_{k,j,n}$ so that $m(E) = 0$ and for each x in E, the sequence $(f_k(x))$ is a real Cauchy sequence and so convergent in $\mathbb{R}$. Define the function f (actually equivalence class of functions equal a.e.) pointwise by $f(x) = \lim_{k\to\infty} f_k(x)$ for each x in N^c. Since $m(E) = 0$, $f(x)$ can be defined arbitrarily for $x \in E$. Then for each n there is an N such that for all $j \ge N$ and all $x \in E^c$,

$$|f(x) - f_j(x)| = \lim_{k\to\infty} |f_k(x) - f_j(x)| \le \lim_{k\to\infty} 1/n = 1/n \ .$$

This shows that (f_j) is a sequence of functions in L^∞ that converges uniformly to f outside a set of measure zero. By problem 10, (f_j) converges to f in L^∞.

Problem 13 Let $C = C([0,1])$ be the space of continuous functions on $[0,1]$ and define $||f|| = \max |f(x)|$. Show that C is a Banach space.

Let (f_n) be Cauchy in $C([0,1])$ under the given norm. Note that for each $x \in [0,1]$ the sequence $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$. So we define the function $f : [0,1] \to \mathbb{R}$ pointwise as $f(x) = \lim_{n\to\infty} f_n(x)$. To show that (f_n) converges to f under the given norm, let $\epsilon > 0$ and take N such that for all $m, n \geq N$, $||f_n - f_m|| < \epsilon$. But then for any $x \in [0,1]$ and $m \geq N$,

$$|f(x) - f_m(x)| = \lim_{n\to\infty} |f_n(x) - f_m(x)| \leq \lim_{n\to\infty} ||f_n - f_m|| \leq \epsilon \, .$$

This shows that the sequence (f_n) of continuous functions on $[0,1]$ converges uniformly to f on $[0,1]$ and therefore $f \in C([0,1])$ and also that $||f - f_m|| = \lim_{n\to\infty} ||f_n - f_m|| \leq \epsilon$ so that (f_n) converges to f under the given norm. Alternatively, to show continuity, we know that since each function in the sequence (f_n) is continuous and $[0,1]$ is a compact set, each function in the sequence is uniformly continuous on $[0,1]$. Let $\epsilon > 0$ and take N such that for $n \geq N$, $||f - f_n|| < \epsilon/3$ and $\delta > 0$ so that $|f_n(x) - f_n(y)| < \epsilon/3$ whenever $|x - y| < \delta$.

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \, .$$

4 Approximation in L^p

In this section we establish versions of Littlewood's second principle which says that for every function $f \in L^p$, $1 \leq p < \infty$, f is 'nearly' a step function and 'nearly' continuous. That is, given f and $\epsilon > 0$, there is a step function φ and a continuous function ψ with $||f - \varphi||_p < \epsilon$ and $||f - \psi||_p < \epsilon$.

If $\Delta = \{\xi_0, ..., \xi_n\}$ is a subdivision, $0 = \xi_0 < \xi_1 < ... < \xi_n = 1$, of $[0,1]$, define the step function φ_Δ to be constant on each interval $[\xi_k, \xi_{k+1})$ and equal to the average of f over that interval. We will show that $||f - \varphi_\Delta|| \to 0$ as the length δ of the largest subinterval of Δ goes to zero.

Lemma 7 Given $f \in L^p$, $1 \leq p < \infty$, and $\epsilon > 0$, there is a bounded measurable function f_M with $|f_M| \leq M$ and $||f - f_M|| < \epsilon$.

Proof:

$$f_N = \begin{cases} N & N \leq f(x) \\ f(x) & -N \leq f(x) \leq N \\ -N & f(x) \leq -N \end{cases}$$

Then $|f_N| \leq N$ and (f_n) converges to f almost everywhere (since $f \in L^p$, $|f| < \infty$ almost everywhere otherwise $||f||_p \not< \infty$) so $|f - f_N|^p \to 0$ almost everywhere. Since $|f - f_N|^p \leq |f|^p$, and $|f|^p$ is integrable,

$$||f - f_N||^p = \int |f - f_N|^p \to 0 \qquad \text{as } N \to \infty \, .$$

This implies that $||f - f_N|| \to 0$ so that given $\epsilon > 0$ there is an M such that $||f - f_M|| < \epsilon$.

Proposition 8 Given $f \in L^p$, $1 \leq p < \infty$ and $\epsilon > 0$, there is a step function φ and a continuous function ψ such that $||f - \varphi||_p < \epsilon$ and $||f - \psi||_p < \epsilon$.

Proof: By Lemma 7 we can find a bounded function f_M such that $||f - f_M|| < \epsilon/2$. By Proposition 3.22, we can find a step function φ such that $|f_M - \varphi| < \epsilon/4$ except on a set E of measure less than $\delta = (\epsilon/(4M))^p$.

$$
\begin{aligned}
||f_M - \varphi||^p &= \int_0^1 |f_M - \varphi|^p \\
&= \int_{[0,1]\setminus E} |f_M - \varphi|^p + \int_E |f_M - \varphi|^p \\
&< \frac{\epsilon^p}{4^p} + \frac{M^p \epsilon^p}{4^p M^p} \\
&= \frac{\epsilon^p}{4^p} + \frac{\epsilon^p}{4^p} \\
&= \frac{2\epsilon^p}{4^p} \\
&\leq \frac{\epsilon^p}{2^p} \qquad \left[\frac{2\epsilon^p}{4^p} \leq \frac{\epsilon^p}{2^p} \iff \left(\frac{2}{4}\right)^p \leq \frac{1}{2} \iff p \geq 1 \right]
\end{aligned}
$$

Consequently, $||f_M - \varphi|| < \epsilon/2$. By the Minkowski inequality,

$$||f - \varphi|| \leq ||f - f_M|| + ||f_M - \varphi|| < \epsilon \ .$$

It turns out that any step function can be approximated in L^p by a continuous function, which will lead to the existence of ψ.