# MA 503 : Homework 17

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**Lemma Eggcorn** If $A$ and $B$ are sets of real numbers bounded below, then $\inf(A+B) = \inf(A)+\inf(B)$.

Proof: Let $a+b \in A+B$ with $a \in A$ and $b \in B$. Since $\inf(A) \leq a$ and $\inf(B) \leq b$, $\inf(A)+\inf(B) \leq a+b$. So $\inf(A)+\inf(B)$ is a lower bound of $A+B$ and $\inf(A)+\inf(B) \leq \inf(A+B)$. For each $\epsilon > 0$ there is exist $a \in A$ and $b \in B$ such that $a < \inf(A)+\epsilon/2$ and $b < \inf(B)+\epsilon/2$ so that $\inf(A+B) \leq a+b < \inf(A)+\inf(B)+\epsilon$. Then since $\inf(A+B) < \inf(A)+\inf(B)+\epsilon$ for every $\epsilon > 0$, $\inf(A+B) \leq \inf(A)+\inf(B)$.

**Problem 1** Show that $||f+g||_\infty \leq ||f||_\infty + ||g||_\infty$.

$$\begin{aligned}
A &:= \{P : m(\{t : |f(t)+g(t)| > P\}) = 0\} = \{P : |f+g| \leq P \text{ a.e.}\} \\
B &:= \{M : m(\{t : |f(t)| > M\}) = 0\} = \{M : |f| \leq M \text{ a.e.}\} \\
C &:= \{N : m(\{t : |g(t)| > N\}) = 0\} = \{N : |g| \leq N \text{ a.e.}\}
\end{aligned}$$

$$B+C := \{M+N : m(\{t : |f(t)| > M\}) = 0, m(\{t : |g(t)| > N\}) = 0\} = \{M+N : |f| \leq M \text{ a.e.}, |g| \leq N \text{ a.e.}\}\ .$$

Let $M+N \in B+C$ such that $M \in B$ and $N \in C$. Then $|f| \leq M$ a.e. and $|g| \leq N$ a.e. which means that $|f|+|g| \leq M+N$ a.e. But then $|f+g| \leq |f|+|g| \leq M+N$ a.e. so that $M+N \in A$. Therefore $B+C \subset A$ and $\inf A \leq \inf(B+C)$. Since $B$ and $C$ are sets of real numbers bounded each bounded below by 0 (because $|f|, |g| \geq 0$), $\inf(B+C) = \inf B + \inf C$ by Lemma Eggcorn. That is,

$$\begin{aligned}
||f+g||_\infty &= \inf\{P : m(\{t : |f(t)+g(t)| > P\}) = 0\} \\
&= \inf A \\
&\leq \inf B + \inf C \\
&= \inf\{M : m(\{t : |f(t)| > M\}) = 0\} + \inf\{N : m(\{t : |g(t)| > N\}) = 0\} \\
&= ||f||_\infty + ||g||_\infty\ .
\end{aligned}$$

We can also stick with the provided definition of essential supremum and follow a similar but messier argument. If $M \in B$ and $N \in C$, then $m(\{t : |f(t)| > M\}) = 0$ and $m(\{t : |f(t)| > N\}) = 0$.

$$\begin{aligned}
&m(\{t : |f(t)+g(t)| > M+N\}) \\
&\leq m(\{t : |f(t)|+|g(t)| > M+N\}) \quad \text{(from the triangle inequality)} \\
&= m[(\{t : |f(t)| > M\} \cap \{t : |g(t)| > N\}) \cup \\
&(\{t : |g(t)| \leq N\} \cap \{t : |f(t)| > M+N-|g(t)|\}) \cup (\{t : |f(t)| \leq M\} \cap \{t : |g(t)| > M+N-|f(t)|\})] \\
&\leq m(\{t : |f(t)| > M\} \cap \{t : |g(t)| > N\}) + m(\{t : |g(t)| \leq N\} \cap \{t : |f(t)| > M+N-|g(t)|\}) \\
&+ m(\{t : |f(t)| \leq M\} \cap \{t : |g(t)| > M+N-|f(t)|\}) \\
&\leq m(\{t : |f(t)| > M\}) + m(\{t : |f(t)| > M\}) + m(\{t : |g(t)| > N\}) \\
&= 0\ .
\end{aligned}$$

This shows that if $M+N \in B+C$ with $M \in B$ and $N \in C$, then $M+N \in A$. Therefore, $B+C \subset A$ and so $\inf(A) \leq \inf(B+C) = \inf(B)+\inf(C)$. That is, $||f+g||_\infty \leq ||f||_\infty + ||g||_\infty$.

**Problem 2** Let $f$ be a bounded measurable function on $[0,1]$. Prove that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

We have $|f| \le ||f||_\infty$ a.e. and so

$$\begin{aligned} ||f||_p &= \left(\int_0^1 |f|^p\right)^{1/p} \\ &\le \left(\int_0^1 ||f||_\infty^p\right)^{1/p} \\ &= \left(||f||_\infty^p m([0,1])\right)^{1/p} \\ &= ||f||_\infty \\ \implies \lim_{p\to\infty} ||f||_p &\le |||f||_\infty\ . \end{aligned}$$

For each $p \in \mathbb{N}$, the set $B_p = \{x : |f(x)| > ||f||_\infty - 1/p\}$ has positive measure since if $m(B_p) = 0$, we have $||f||_\infty - 1/p \in \{M : m(\{x : |f(x)| > M\}) = 0\}$ and $||f||_\infty - 1/p < ||f||_\infty = \inf\{M : m(\{x : |f(x)| > M\}) = 0\}$, which is a contradiction.

$$\begin{aligned} \left(\int_{B_p} |f|^p\right)^{1/p} &\ge \left(\int_{B_p} |\ ||f||_\infty - 1/p\ |^p\right)^{1/p} \\ &= |\ ||f||_\infty - 1/p\ | \left(m(B_p)\right)^{1/p} \end{aligned}$$

As $p \to \infty$, $||f||_\infty - 1/p \to ||f||_\infty$, so $B_p \to [0,1]$. Then,

$$\lim_{p\to\infty} ||f||_p \ge \lim_{p\to\infty} |\ ||f||_\infty - 1/p\ | \left(m(B_p)\right)^{1/p} = ||f||_\infty m([0,1]) = ||f||_\infty\ .$$

**Problem 3** Prove that $||f+g||_1 \le ||f||_1 + ||g||_1$.

Suppose $f, g \in L^1([0,1])$. For each $x \in [0,1]$, $|f+g| \le |f| + |g|$, so

$$||f+g||_1 = \int_0^1 |f+g| \le \int_0^1 (|f| + |g|) = \int_0^1 |f| + \int_0^1 |g| = ||f||_1 + ||g||_1\ .$$

**Problem 4** Show that if $f \in L^1$ and $g \in L^\infty$,

$$\int |fg| \le ||f||_1 \cdot ||g||_\infty\ .$$

We have $|g| \le ||g||_\infty$ almost everywhere so $|fg| = |f||g| \le |f|||g||_\infty$ almost everywhere. We have defined $L^p$ spaces in this section on the interval $[0,1]$, so

$$\int |fg| = \int_0^1 |fg| = \int_0^1 |f||g| \le \int_0^1 |f|||g||_\infty = ||g||_\infty \int_0^1 |f| = ||g||_\infty \int |f| = ||g||_\infty ||f||_1\ .$$