MA 503 : Homework 13

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Problem 2.40
Let F be a closed set of real numbers and f a real valued function which is defined and continuous on F. Show there is a function $g : \mathbb{R} \to \mathbb{R}$ such that g is continuous and $f(x) = g(x)$ for each $x \in F$.

Proposition 15 Let E be a given set. The following five statements are equivalent.

i. E is measurable.
ii. Given $\epsilon > 0$ there is an open set $O \supset E$ such that $m^*(O \backslash E) < \epsilon$.
iii. Given $\epsilon > 0$ there is a closed set $F \subset E$ such that $m^*(E \backslash F) < \epsilon$.
iv. There is a $G \in G_\delta$ with $E \subset O$ such that $m^*(G \backslash E) = 0$.
v. There is an $F \in F_\sigma$ with $F \subset E$ such that $m^*(E \backslash F) = 0$.

If $m^*(E) < \infty$, the above statements are equivalent to:

vi. Given $\epsilon > 0$, there is a finite union U of open intervals such that $m^*(U \triangle E) < \epsilon$.

Proposition 22 Let f be a measurable function defined on an interval $[a, b]$, and assume that f takes on the values $\pm\infty$ only on a set of measure zero. Then given ϵ, we can find a step function g and a continuous function h such that

$$|f - g| < \epsilon \text{ and } |f - h| < \epsilon$$

except on a set of measure less than ϵ; i.e., $m(\{x : |f(x) - g(x)| \geq \epsilon\}) < \epsilon$ and $m(\{x : |f(x) - g(x)| \geq \epsilon\}) < \epsilon$. If in addition, $m \leq f \leq M$, then we may choose the functions g and h such that $m \leq g, h \leq M$.

Proposition 24 Let E be a measurable set of finite measure, and (f_n) a sequence of measurable functions that converge to a real-valued function f almost everywhere on E (the set B of points such that f_n does not converge to f pointwise is such that $m(B) = 0$ and $f_n \to f$ pointwise on $E \backslash B$). Then given $\epsilon > 0$ and $\delta > 0$, there is a set $A \subset E$ with $m(A) < \delta$, and an N such that for all $x \notin A$ and for all $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon \ .$$

Problem 30 (Proving **Egoroff's Theorem**) If (f_n) is a sequence of measurable functions that converge to a real-valued function f almost everywhere on a measurable set E with $m(E) < \infty$, then given $\eta > 0$, there is a subset $A \subset E$ with $m(A) < \eta$ such that $f_n \to f$ uniformly on $E \backslash A$.

Proof: Let $\eta > 0$. For every $n \in \mathbb{N}$, there exists by Proposition 24 a measurable set $A_n \subset E$ such that $m(A_n) < \delta_n := 2^{-n}\eta$ and an N_n such that for all $k \geq N_n$ and all $x \in E \backslash A_n$, $|f_k(x) - f(x)| < \epsilon_n := 1/n$. Since each $A_n \subset E$ is measurable, $\cup_{n=1}^\infty A_n$ is measurable and $\cup_{n=1}^\infty A_n \subset E$. Let $A := \cup_{n=1}^\infty A_n$. By Proposition 13 (subadditivity),

$$m(A) \leq \sum_{n=1}^\infty m(A_n) < \sum_{n=1}^\infty 2^{-n}\eta = \eta \ .$$

So A is a subset of E such that $m(A) < \eta$. If we can show that f_n converges uniformly to f on $E\backslash A$ then we will have proven Egoroff's Theorem. Now let $\epsilon > 0$ be given. Then there is an m such that $A_m \subset E$ and a corresponding N_m such that for all $k \geq N_m$ and $x \in E\backslash A_m$, $|f_k(x) - f(x)| < 1/m < \epsilon$. But then for $x \in E\backslash A$, $x \in (\cup_{n=1}^{\infty} A_n)^c = \cap_{n=1}^{\infty} A_n^c$. In particular, $x \in E\backslash A_m$. Therefore, given $\epsilon > 0$, there is a positive integer N_m such that for all $k \geq N_m$ and $x \in E\backslash A$, $|f_k(x) - f(x)| < 1/m < \epsilon$. Since ϵ was arbitrary, conclude that there is a measurable set $A \subset E$ with $m(A) < \eta$ such that $f_k \to f$ uniformly on $E\backslash A$.

Problem 31 (Proving **Lusin's Theorem**) Let f be a measurable real-valued function on $[a, b]$. Given $\delta > 0$, there is a continuous function ϕ on $[a, b]$ such that $m(\{x : f(x) \neq \phi(x)\}) < \delta$.

Proof: Since f is real-valued, $m(\{x : f(x) = \pm\infty\}) = m(\emptyset) = 0$. For each $k \in \mathbb{N}$, there exists by Proposition 22 a continuous function h_k such that $|f - h_k| < 1/k$ except on a set of measure less than $1/k$, i.e. $m(\{x \in [a, b] : |f(x) - h_k(x)| \geq 1/k\}) < 1/k$. Then the sequence (h_k) of continuous (and therefore measurable) functions converges to f almost everywhere on $[a, b]$. By Egoroff's Theorem, there is a set $A \subset [a, b]$ with $m(A) < \delta/2$ such that $h_k \to f$ uniformly on $[a, b]\backslash A$. By Proposition 15 (iii) (more specifically (i) $\iff$ (iii) and the fact that $[a, b]\backslash A$ is a measurable set), there exists a closed set $F \subset ([a, b]\backslash A)$ such that $m^*(([a, b]\backslash A)\backslash F) < \delta/2$. Since $F \subset [a, b]\backslash A$, $h_k \to f$ uniformly on F. Since the convergence of (h_k) to f is uniform on F and each h_k is continuous, f must be continuous on F. By Problem 2.40, there exists a continuous function ϕ defined on $(-\infty, \infty)$ (and so ϕ is defined and continuous on $[a, b]$ - we may as well assume $\phi : [a, b] \to \mathbb{R}$) such that $\phi(x) = f(x)$ for all $x \in F$. Therefore, if $x \in [a, b]$ and $f(x) \neq \phi(x)$, it must be the case that $x \notin F$. So $x \in ([a, b]\backslash A)\backslash F$ or $x \in A$. That is,

$$m(\{x \in [a, b] : f(x) \neq \phi(x)\}) = m\left[(([a, b]\backslash A)\backslash F) \cup A\right] \leq m\left(([a, b]\backslash A)\backslash F\right) + m(A) < \delta/2 + \delta/2 = \delta\ .$$

If instead f is measurable real-valued function on $(-\infty, \infty)$, consider that given $\delta > 0$, there is a continuous function ϕ_n on $[n, n+1]$ for each $n \in \mathbb{Z}$ such that $m(\{x : f(x) \neq \phi_n(x)\}) < \delta/2^{|n|}$ by the above. Let ϕ be the function defined piecewise on $\mathbb{R}$ as $\phi(x) = \phi_n(x)$ for $x \in [n, n+1]$. Then,

$$m(\{x : f(x) \neq \phi(x)\}) = \sum_{n=0}^{\infty} \delta/2^n + \sum_{n=-1}^{-\infty} \delta/2^{-n} = 3\delta\ .$$