MA 503 : Homework 7

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Problem 9 Show that if E is a measurable set that each translate $E + y$, $y \in \mathbb{R}$, of E is also measurable.

Lemma 9A Let $D \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then $(D + x)^c = D^c + x$.

Proof: Let $a \in (D + x)^c$. Then $a \notin D + x$ and so $a \neq d + x$ for any $d \in D$. This means that $a - x \neq d$ for any $d \in D$ and so $a - x \notin D$. Thus $a - x \in D^c$ and so $a \in D^c + x$.

Let $a \in D^c + x$. Then $a = \tilde{d} + x$ for some $\tilde{d} \in D^c$. Thus $a - x = \tilde{d} \in D^c$. Then $a - x \neq d$ for any $d \in D$ which means also $a \neq d + x$ for any $d \in D$. Then it cannot be the case that $a \in D + x$, so $a \in (D + x)^c$.

Lemma 9B Let $C, D \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then $(C \cap D) - x = (C - x) \cap (D - x)$.

Proof: Let $a \in (C \cap D) - x$. Then $a = b - x$ for some $b \in C \cap D$ and so $a = b - x$ for some b such that $b \in C$ and $b \in D$. Thus $a = b - x \in C - x$ and $a = b - x \in D - x$. Therefore, $a \in (C - x) \cap (D - x)$.

Let $a \in (C - x) \cap (D - x)$. Then $a \in C - x$ and $a \in D - x$. Therefore, $a = c - x$ for some $c \in C$ and $a = d - x$ for some $d \in D$. But since $c - x = d - x$, $c = d$ and so $c = d \in C \cap D$. Thus $a = c - x$ for some $c \in C \cap D$ and therefore $a \in (C \cap D) - x$.

Now let E be a measurable set, $A \subset \mathbb{R}$, and $y \in \mathbb{R}$.

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\begin{aligned}
m^*[A \cap (E + y)] + m^*[A \cap (E + y)^c] &= m^*[A \cap (E + y)] + m^*[A \cap (E^c + y)] \quad \text{(Lemma 9A)} \\
&= m^*[(A \cap (E + y)) - y] + m^*[(A \cap (E^c + y) - y] \quad \text{(Problem 7)} \\
&= m^*[(A - y) \cap (E + y - y)] + m^*[(A - y) \cap (E^c + y - y)] \quad \text{(Lemma 9B)} \\
&= m^*[(A - y) \cap (E + 0)] + m^*[(A - y) \cap (E^c + 0)] \\
&= m^*[(A - y) \cap E] + m^*[(A - y) \cap E^c] \\
&= m^*(A - y) \quad \text{(Since } E \text{ is measurable)} \\
&= m^*(A) \quad \text{(Problem 7)}\ .
\end{aligned}
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Since A and y were arbitrary, we have shown that for any translate $E + y$ of E, that $m^*(A) = m^*(A \cap (E + y)) + m^*(A \cap (E + y)^c)$ for any set A. Therefore, $E + y$ is measurable.

Alternative Proof
Let $\{I_n\}$ be any cover of E by open intervals. Then $\{I_n + y\}$ is a cover of $(E + y)$ by open intervals since if $x \in E + y$ then $x - y \in E$ so $x - y \in I_n$ for some interval in $\{I_n\}$. Thus $x \in I_n + y$. Also, for any interval $I_n = (a_n, b_n)$, $l(I_n) = b_n - a_n = (b_n + y) - (a_n + y) = l(I_n + y)$. Since $A \cap E \subset E$ and $A \cap (E + y) \subset E + y$, any cover $\{I_n\}$ of E by open intervals will be a cover of $A \cap E$ and since the corresponding cover $\{I_n + y\}$ contains $E + y$, $A \cap (E + y) \subset \bigcup (I_n + y)$. This means that the values of the sums in the set $\{\sum l(I_n) : (A \cap E) \subset \bigcup I_n\}$ are the same as the values of the sums in the set $\{\sum l(I_n + y) : (A \cap (E + y)) \subset \bigcup (I_n + y)\}$. Therefore, $m^*(A \cap E) = m^*(A \cap (E + y))$. Similarly, $m^*(A \cap E^c) = m^*(A \cap (E + y)^c) = m^*(A \cap (E^c + y))$ (Lemma 9A). By Problem 7, m^* is translation invariant and since E is measurable we have:

$$m^*(A \cap (E+y)) + m^*(A \cap (E+y)^c) = m^*(A \cap E) + m^*(A \cap E^c) = m^*(A) \ .$$

Since A and y were arbitrary this shows that if E is measurable, $E+y$ is measurable.