# MA 503 : Homework 1

Dane Johnson

September 11, 2020

## Chapter 2 : The Real Number System

**C. Completeness Axiom** Every nonempty set $S$ of real numbers which has an upper bound has a least upper bound.

**1. Proposition** Let $L$ and $U$ be nonempty subsets of $\mathbb{R}$ such that $\mathbb{R} = L \cup U$ and such that for each $l \in L$ and for each $u \in U$, $l < u$. Then either $L$ has a greatest element of $U$ has a least element.

**Problem 3**
Prove Proposition 1 using Axiom C.

Proof: The statement that either $L$ has a greatest element or $U$ has a least element is equivalent to the statement that if $U$ does not have a least element then $L$ must have a greatest element. Suppose $U$ does not have a least element and let $u \in U$ be arbitrary. Since $l < u$ for all $l \in L$, $u$ is an upper bound of $L$ so $L$ has a least upper bound, which we denote sup $L$. Since $u$ was arbitary and sup $L$ is the least upper bound of $L$, we have sup $L \leq u$ for all $u \in U$. Therefore, sup $L \leq$ inf $U$. Suppose that sup $L$ is not an element of $L$. Then since $\mathbb{R} = L \cup U$, it follows that sup $L$ is an element of $U$. Since inf $U \leq u$ for every element $u \in U$, this means inf $U \leq$ sup $L$. Thus inf $U =$ sup $L$. This means that sup $L$ is an element of $U$, less than or equal to any element of $U$. That is, sup $L$ is the least element of $U$. This contradicts the assumption that $U$ does not have a least element. So it must be the case that sup $L$ is an element of $L$. Since sup $L$ is greater than or equal to any element of $L$, sup $L$ is the greatest element of $L$.