MA 503 : Homework 4

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We define continuity using the usual ϵ, δ definition of continuity at a point and say that a function is continuous on a set A if it is continuous at every point in A.

Remark (Assigned as HW Problem) Suppose $f : E \to \mathbb{R}$ and let $A \subset E$. To say that f is continuous on A means that f is continuous at every point $x \in A$. That is, if $x \in A$, then given $\epsilon > 0$ there is a $\delta > 0$ such that if $y \in E$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. The function $f|_A : A \to \mathbb{R}$ is a function defined on A with $f|_A(x) = f(x)$ for every $x \in A \subset E$. To say that $f|_A$ is continuous means that $f|_A$ is continuous on its domain. That is, at every point $x \in A$. This means that for every $x \in A$, given $\epsilon > 0$ there is a $\delta > 0$ such that if $y \in A$ with $|x - y| < \delta$ then $|\ f|_A(x) - f|_A(y)\ | < \epsilon$. The statements:

(i) f is continuous on $A \subset E$,
(ii) $f|_A$ is continuous,

are identical except that for (ii), since $f|_A : A \to \mathbb{R}$, we added the requirement that $y \in A$ while for (i) we allow $y \in E$. Then (i) $\implies$ (ii). Suppose (i) holds and let $x \in A$ be arbitrary. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that for $y \in E$ with $|x - y| < \delta$ it follows that $|f(x) - f(y)| < \epsilon$. Then for any $y \in A$ with $|x - y| < \delta$, $y \in E$ since $A \subset E$ and $|\ f|_A(x) - f|_A(y)\ | = |f(x) - f(y)| < \epsilon$. Since $x \in A$ was arbitrary conclude that if f is continuous on A, then $f|_A$ is continuous.

Statement (i), however, does not necessarily follow from statement (ii). Let us show this with a counterexample. Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = 1$ if $x \in \mathbb{R}\backslash\mathbb{Q}$, $f(x) = 0$ if $x \in \mathbb{Q}$. Then $f|_{\mathbb{Q}}$ is continuous but f is not continuous on $\mathbb{Q} \subset \mathbb{R}$. Let $x \in \mathbb{Q}$ be arbitrary. Let $\epsilon > 0$ and take $\delta = 1$ (any choice of $\delta > 0$ will work). If $y \in \mathbb{Q}$ with $|x - y| < 1$ then $|\ f|_{\mathbb{Q}}(x) - f|_{\mathbb{Q}}(y)\ | = |0 - 0| = 0 < \epsilon$. This shows that $f|_{\mathbb{Q}}$ is continuous. To show that f is not continuous on $\mathbb{Q}$, we need to show that f is discontinuous at at least one rational number. In fact it is the case that f is discontinuous at every rational number for this example. Let $q \in \mathbb{Q}$ and let $\epsilon = 1/27$. Then no matter our choice of $\delta > 0$ there is an irrational number $y \in (q - \delta, q + \delta)$. Note that there is no requirement that $y \in \mathbb{Q}$ when considering whether f is continuous at the point q since $f : \mathbb{R} \to \mathbb{R}$. Then $|q - y| < \delta$ but $|f(q) - f(y)| = 1 > 1/27$. This means given any rational number q, there is an $\epsilon > 0$ such that there does not exist $\delta > 0$ such that $|q - y| < \delta \implies |f(q) - f(y)| < \epsilon$. Therefore, f is not continuous at any point $r \in \mathbb{Q}$ (again we only needed this at one point in $\mathbb{Q}$) and so f is not continuous on $\mathbb{Q}$.

Problem 40
Let F be a closed set of real numbers and f a real valued function which is defined and continuous on F. Show there is a function $g : \mathbb{R} \to \mathbb{R}$ such that g is continuous and $f(x) = g(x)$ for each $x \in F$.

If $F = \emptyset$, use $g(x) \equiv 0$. Then g is continuous and the requirement that $f(x) = g(x)$ for each $x \in F$ is vacuously true.

If $F = \mathbb{R}$, then set $g \equiv f$. Since f is continuous g is also continuous on $\mathbb{R}$ and $g(x) = f(x)$ for each $x \in F = \mathbb{R}$.

If $F \subsetneq \mathbb{R}$ then F^c is a nonempty open set of real numbers. Using Proposition 8, F^c is the union of a countable collection of disjoint open sets, not all empty. As in the proof of this proposition, for each $y \in F^c$, there is an interval $I_y = (a_y, b_y) \subset F^c$ with $a_y = \inf\{a \in \mathbb{R} : (a, y) \subset F^c\}$ and $b_y = \sup\{b \in \mathbb{R} : (y, b) \subset F^c\}$. The disjoint union $F^c = \cup_{y \in F^c} I_y$ is countable and so we may write $F^c = \cup_{i=1}^{\infty}(a_i, b_i)$ for real numbers $a_i, b_i,\ i \in \mathbb{N}$. Relabeling the intervals if necessary, assume $a_1 < b_1 < a_2 < b_2 < a_3 < ...$ (if it is possible to write $F^c = \cup_{i=1}^{n}(a_i, b_i)$ then assume $a_1 < b_1 < .... < a_n < b_n$). We will handle the case that one or two of these intervals is an infinite interval soon since this will alter the linear equation we use to define g if this occurs. For each interval (a_i, b_i), sup $(a_i, b_i) = b_i \in F$ and inf $(a_i, b_i) = a_i \in F$. For each (a_i, b_i) in the disjoint union $F^c = \cup_{i=1}^{\infty}(a_i, b_i)$ (or $F^c = \cup_{i=1}^{n}(a_i, b_i)$ if it is possible) set

$$g(x) = \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i) + f(a_i) \quad x \in (a_i, b_i)\ .$$

Then for $x \in F$, set $g(x) = f(x)$. If $(a_i, b_i) = (-\infty, b_i)$ for some interval, set $g(x) = f(b_i)$ for all $x \in (-\infty, b_i)$. If $(a_i, b_i) = (a_i, \infty)$, set $g(x) = f(a_i)$ for all $x \in (a_i, \infty)$. If the intervals have been ordered and any interval contained within another interval has been deleted from the collection then $(-\infty, b_i) = (a_1, b_1)$ and $(a_n, b_n) = (a_n, \infty)$ if infinite intervals exist in the collection. Considering these cases, where we still consider F^c as the disjoint union $F^c = \cup_i(a_i, b_i)$,

$$g(x) = \begin{cases} f(x) & x \in F \\ \frac{f(b_i)-f(a_i)}{b_i-a_i}(x - a_i) + f(a_i) & x \in (a_i, b_i) \text{ and } a_i, b_i \text{ finite} \\ f(b_i) & x \in (a_i, b_i) \text{ and } a_i = -\infty \\ f(a_i) & x \in (a_i, b_i) \text{ and } b_i = \infty \end{cases}$$

Since g is linear on each open interval in $F^c = \cup_i(a_i, b_i)$, g is continuous on F^c. For each endpoint b_i, since g is continuous on F^c, $f = g$ on F and $\lim_{x \to b_i^-} g(x) = f(b_i) = g(b_i)$, g is continuous at each interval endpoint b_i. Similarly, g is continuous at each a_i. Since $f = g$ on F and f is continuous on F, g is therefore continuous on all of F. Since g is continuous for any $x \in F^c \cup F$, g is continuous on $\mathbb{R}$.

Problem 42 Let (f_n) be a sequence of functions defined on a set E. Prove that if (f_n) converges uniformly to f on E, then f is continuous on E.

Let $\epsilon > 0$. Since (f_n) converges to f uniformly, there is an $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon/3$ for all $z \in E$ and $n \geq N$. In particular,

$$|f_N(z) - f(z)| < \epsilon \text{ for all } z \in E \quad (1)\ .$$

Since f_N is continuous, there is a $\delta > 0$ such that if $y \in E$ with $|x - y| < \delta$,

$$|f_N(x) - f_N(y)| < \epsilon/3 \quad (2)\ .$$

Let $x \in E$. Then for any $y \in E$ with $|x - y| < \delta$,

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \underbrace{\frac{\epsilon}{3}}_{\text{By (1)}} + \underbrace{\frac{\epsilon}{3}}_{\text{By (2)}} + \underbrace{\frac{\epsilon}{3}}_{\text{By (1)}} \\ &= \epsilon\ . \end{aligned}$$

Since $x \in E$ was arbitrary f is continuous on E.

Proposition 19 (Intermediate Value Theorem) Let f be a continuous real valued function on $[a, b]$ and suppose that $f(a) \leq \gamma \leq f(b)$ [or $f(b) \leq \gamma \leq f(b)$]; then there is a a point $c \in [a, b]$ such that $f(c) = \gamma$.

Problem 45 Prove Proposition 19.

Proof: Let $C = \{x \in [a, b] : f(x) < \gamma\}$. Since $a \in [a, b]$ and $f(a) < \gamma$, $C \neq \emptyset$ and since $C \subset [a, b]$, C is bounded. Let $c = \sup C$. If $f(c) > \gamma$, then since f is continuous and $f(c) - \gamma > 0$, there is a $\delta > 0$ such that $f(y) > \gamma$ for all $y \in (c - \delta, c + \delta)$. But then we cannot have any point $x \in C$ with $x \in (c - \delta, c + \delta)$ since otherwise $f(x) > \gamma$. But this contradicts the assumption that c is the supremum of $C = \{x \in [a,b] : f(x) < \gamma\}$. So it must be the case that $f(c) \le \gamma$. If $f(c) < \gamma$, then since f is continuous and $0 < \gamma - f(c)$ there is a $\delta > 0$ such that $f(y) < \gamma$ for all$y \in (c-\delta, c+\delta)$. But then there is an $x \in (c, c+\delta)$ such that $f(x) < \gamma$, which contradictions the assumption that c is an upper bound of the set $C = \{x \in [a,b] : f(x) < \gamma\}$. So it cannot be the case that $f(c) < \gamma$ either. Therefore, conclude that $f(c) = \gamma$. Since c is a cluster point of C and $x \in [a,b]$ for all $x \in C$, the fact that $[a,b]$ is closed implies $c \in [a,b]$ as well.

Could you read this alternative proof? I don't think it's correct since $x_n \in f^{-1}((\gamma - 1/n, \gamma + 1/n))$ may be empty. But I wonder if something similar to this might be possible.
Proof: If $\gamma = f(a)$ or $\gamma = f(b)$ then since $a \in [a,b]$ and $b \in [a,b]$ the conclusion holds. So suppose $f(a) < \gamma < f(b)$. Then $(f(a), f(b))$ is an open set and $\gamma \in (f(a), f(b))$. There is an $\epsilon > 0$ such that $(\gamma - \epsilon, \gamma + \epsilon) \subset (f(a), f(b))$. Then for any $n \in \mathbb{N}$ with $1/n < \epsilon$ it follows that $(\gamma - 1/n, \gamma + 1/n) \subset (f(a), f(b))$. Since f is continuous, $f^{-1}((\gamma - 1/n, \gamma + 1/n)) \subset (a,b)$ is open by Proposition 18 (while Proposition 18 is stated with a function defined on the real line, we saw in Problem 40 that it is possible to find a continuous extension of f to the real line that matches f on the compact set $[a,b]$ so we will continue to just use f in this proof instead of the extension g). Then for each n with $n < 1/\epsilon$ take $x_n \in f^{-1}((\gamma - 1/n, \gamma + 1/n))$ to construct the sequence (x_n). Since $x_n \in [a,b]$ for all n and $[a,b]$ is a closed and bounded set, there is a subsequence (x_{n_k}) of (x_n) such that $\lim x_{n_k} = c \in [a,b]$. Then $\lim f(x_{n_k}) = f(c) = \gamma$ since f is continuous. We know that $f(c) = \gamma$ since $f(x_n) \in (\gamma - 1/n, \gamma + 1/n) \to \{\gamma\}$ as $n \to \infty$ and $n_k \ge n$ for all n.