MA 503 Homework 1

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Exercise 1

Let $x, y \in \mathbb{R}^d$. Assume that $x \neq 0$. Set $e = \frac{x}{||x||}$, $P(y) = \langle y, e\rangle e$ and $z = y - P(y)$.

(i) Show that $\langle z, P(y)\rangle = 0$.

$$\begin{aligned}
\langle z, P(y)\rangle &= \langle y - P(y), P(y)\rangle \\
&= \langle y, P(y)\rangle + \langle -P(y), P(y)\rangle \\
&= \langle y, P(y)\rangle - \langle P(y), P(y)\rangle \\
&= \langle y, \langle y, e\rangle e\rangle - \langle \langle y, e\rangle e, \langle y, e\rangle e\rangle \\
&= \langle y, e\rangle\langle y, e\rangle - \langle y, e\rangle\langle y, e\rangle\langle e, e\rangle \\
&= \langle y, \frac{x}{||x||}\rangle\langle y, \frac{x}{||x||}\rangle - \langle y, \frac{x}{||x||}\rangle\langle y, \frac{x}{||x||}\rangle\langle \frac{x}{||x||}, \frac{x}{||x||}\rangle \\
&= \left(\frac{1}{||x||}\right)^2 \langle y, x\rangle\langle y, x\rangle - \left(\frac{1}{||x||}\right)^2 \langle y, x\rangle\langle y, x\rangle \left(\frac{1}{||x||}\right)^2 \langle x, x\rangle \\
&= \left(\frac{1}{||x||}\right)^2 \langle y, x\rangle\langle y, x\rangle - \left(\frac{1}{||x||}\right)^2 \langle y, x\rangle\langle y, x\rangle \left(\frac{1}{||x||}\right)^2 (||x||)^2 \\
&= \left(\frac{1}{||x||}\right)^2 \langle y, x\rangle\langle y, x\rangle - \left(\frac{1}{||x||}\right)^2 \langle y, x\rangle\langle y, x\rangle \\
&= 0\ .
\end{aligned}$$

(ii) Show that $||P(y)|| \leq ||y||$.

$$
\begin{aligned}
||P(y)||^2 &= \langle P(y), P(y)\rangle \\
&= \langle \langle y, e\rangle e, \langle y, e\rangle e\rangle \\
&= \langle y, e\rangle^2 \langle e, e\rangle \\
&= \langle y, \frac{x}{||x||}\rangle^2 \langle \frac{x}{||x||}, \frac{x}{||x||}\rangle \\
&= \langle y, \frac{x}{||x||}\rangle^2 \langle x, x\rangle \left(\frac{1}{||x||}\right)^2 \\
&= \langle y, \frac{x}{||x||}\rangle^2 (||x||)^2 \left(\frac{1}{||x||}\right)^2 \\
&= \langle y, \frac{x}{||x||}\rangle^2 \\
&\leq \langle y, \frac{y}{||y||}\rangle^2 \\
&= \left(\frac{1}{||y||}\right)^2 \langle y, y\rangle^2 \\
&= \left(\frac{1}{||y||}\right)^2 (||y||^2)^2 \\
&= ||y||^2 \ .
\end{aligned}
$$

Since $||y||, ||P(y)|| \geq 0$ and $||P(y)||^2 \leq ||y||^2$ we conclude $||P(y)|| \leq ||y||$.

(iii) Infer the Cauchy - Schwarz Inequality. Here this takes the form: $|\langle x, y\rangle| \leq ||x|| \, ||y||$.

$$
\begin{aligned}
||x|| \, ||y|| &\geq ||x|| \, ||P(y)|| \\
&= ||x|| \, ||\langle y, e\rangle e|| \\
&= |\langle y, e\rangle| \, ||x|| \, ||e|| \\
&= |\langle y, \frac{x}{||x||}\rangle| \, ||x|| \, || \left(\frac{x}{||x||}\right) || \\
&= \frac{1}{||x||} |\langle y, x\rangle| \, ||x|| \, \frac{1}{||x||} \, ||x|| \\
&= |\langle y, x\rangle| \ .
\end{aligned}
$$

Exercise 2

For $x = (x_1, ..., x_d) \in \mathbb{R}^d$, set

$$||x||_1 = \sum_{i=1}^{d} |x_i| \;, \quad ||x||_\infty = \max\{|x_i| \;:\; i = 1, ..., d\} \;.$$

Show that ρ_1 and ρ_∞ define two distances on $\mathbb{R}^d$, where for $x, y \in \mathbb{R}^d$ we define

$$\rho_1(x, y) = ||x - y||_1 \;, \qquad \rho_\infty(x, y) = ||x - y||_\infty \;.$$

Let $x, y, z \in \mathbb{R}^d$. First we show that ρ_1 is a distance on $\mathbb{R}^d$ by verifying that the properties of ρ_1 satisfy the definition of a distance function.

1) $\underline{\rho_1(x, y) \geq 0 \text{ and } \rho_1(x, y) = 0 \text{ iff } x = y}$

$$\rho_1(x, y) = \sum_{i=1}^{d} |x_i - y_i| \geq \sum_{i=1}^{d} 0 = 0 \;.$$

Therefore the function satisfies the nonnegativity property.

Suppose that $x = y$. This means that $x_i = y_i$ for $i = 1, ..., d$, which implies

$$\rho_1(x, y) = \sum_{i=1}^{d} |x_i - y_i| = \sum_{i=1}^{d} 0 = 0 \;.$$

Suppose that $\rho_1(x, y) = 0$. Then since $|x_i - y_i| \geq 0$ for $i = 1, ..., d$, we have

$$0 = \rho_1(x, y) = \sum_{i=1}^{d} |x_i - y_i| \implies |x_i - y_i| = 0 \quad \forall i \in \{1, ..., d\} \;.$$

Since $|x_i - y_i| = 0$, it follows that $x_i - y_i = 0$, so that $x_i = y_i$, for $i = 1, ..., d$. Therefore $x = y$.

2) $\underline{\rho_1(x, y) = \rho_1(y, x)}$

Using the fact that $|a-b| = |b-a|$ for any real numbers a, b it follows that

$$\rho_1(x,y) = \sum_{i=1}^{d} |x_i - y_i| = \sum_{i=1}^{d} |y_i - x_i| = \rho_1(y,x) \ .$$

3) $\underline{\rho_1(x,z) \leq \rho_1(x,y) + \rho_1(y,z)}$

Note: We will use the triangle inequality for real numbers in one of the steps below without proof, which should be a reasonable assumption.

$$\begin{aligned}
\rho_1(x,z) &= \sum_{i=1}^{d} |x_i - z_i| \\
&= \sum_{i=1}^{d} |x_i - y_i + y_i - z_i| \\
&\leq \sum_{i=1}^{d} (|x_i - y_i| + |y_i - z_i|) \\
&= \sum_{i=1}^{d} |x_i - y_i| + \sum_{i=1}^{d} |y_i - z_i| \\
&= \rho_1(x,y) + \rho_1(y,z) \ .
\end{aligned}$$

Therefore we conclude that ρ_1 is a distance function on $\mathbb{R}^d$.

Next we show that ρ_∞ is a distance function on $\mathbb{R}^d$ using the same process as for ρ_1. Let $x, y, z \in \mathbb{R}^d$.

1) $\underline{\rho_\infty(x,y) \geq 0 \text{ and } \rho_\infty(x,y) = 0 \text{ iff } x = y}$

Since $|a| \geq 0$ for any $a \in \mathbb{R}$, it follows that

$$\rho_\infty(x,y) = \max\{|x_i - y_i|, ..., |x_d - y_d|\} \geq \min\{|x_i - y_i|, ..., |x_d - y_d|\} \geq 0 \ .$$

Suppose $x = y$ so that $x_i = y_i$ for $i = 1, ..., d$. Then $|x_i - y_i| = 0$. This implies that

$$\rho_\infty(x,y) = \max\{|x_i - y_i| \ : \ i = 1, ..., d\} = \max\{0, ..., 0\} = 0 \ .$$

Suppose that $\rho_\infty(x,y) = \max\{|x_i - y_i|, ..., |x_d - y_d|\} = 0$. This implies that $|x_i - y_i| \leq 0$ for $i = 1, ..., d$. But since it must also be the case that $|x_i - y_i| \geq 0$, we conclude that $|x_i - y_i| = 0$. So $x_i = y_i$ for $i = 1, ..., d$ and therefore $x = y$.

2) $\underline{\rho_\infty(x,y) = \rho_\infty(y,x)}$

$$\rho_\infty(x,y) = \max\{|x_i - y_i| \ : \ i = 1, ...d\} = \max\{|y_i - x_i| \ : \ i = 1, ...d\} = \rho_\infty(y,x) \ .$$

3) $\underline{\rho_\infty(x,z) \leq \rho_\infty(x,y) + \rho_\infty(y,z)}$

$$\begin{aligned}
\rho_\infty(x,z) &= \max\{|x_i - z_i\} \ : \ i = 1, ..., d\} \\
&= \max\{|x_i - y_i + y_i - z_i\} \ : \ i = 1, ..., d\} \\
&\leq \max\{|x_i - y_i| \ : \ i = 1, ..., d\} + \max\{|y_i - z_i| \ : \ i = 1, ..., d\} \quad (*) \\
&= \rho_\infty(x,y) + \rho_\infty(y,z) \ .
\end{aligned}$$

(*) Note that we this step is justified by the following reasoning: For each i, we have $|x_i - y_i + y_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$. By splitting into two 'max' sets, we certainly include this case but also allow for cases $|x_i - y_i| + |y_j - z_j|$ with $i \neq j$. Thus we are only adding flexibility to our sum, allowing for potentially larger sums than if the subscripts must match while simultaneously covering all matching subscript cases. The more restrictive scenario must lead to a value no larger than the more flexible scenario.

Therefore we conclude that ρ_∞ is a distance function on $\mathbb{R}^d$.

Exercise 3

Show that for $a, b, c \geq 0$, if $a \leq b + c$ then $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$.

First we will establish some inequalities to use in the proof:

$$a, b, c \geq 0 \, , \; a \leq b + c \implies a + 1 \leq b + c + 1 \implies \frac{1}{b+c+1} \leq \frac{1}{a+1}$$
$$\implies -\frac{1}{b+c+1} \geq -\frac{1}{a+1} \ .$$

$$b, c \geq 0 \implies b+1 \leq b+c+1 \implies \frac{1}{b+c+1} \leq \frac{1}{b+1} \ .$$

Similarly we have $\frac{1}{b+c+1} \leq \frac{1}{c+1}$. Now we may begin the proof:

$$\begin{aligned}
\frac{a}{1+a} &= \frac{1+a}{1+a} - \frac{1}{1+a} \\
&= 1 - \frac{1}{1+a} \\
&\leq 1 - \frac{1}{b+c+1} \\
&= \frac{b+c+1}{b+c+1} - \frac{1}{b+c+1} \\
&= \frac{b}{b+c+1} + \frac{c}{b+c+1} \\
&\leq \frac{b}{1+b} + \frac{c}{1+c} \ .
\end{aligned}$$

Let (X, ρ) be a metric space and $x, y \in X$. We define

$$d(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$$

and claim that d is a distance function. In what follows we use the assumption that ρ itself is distance function and therefore satisfies all the necessary properties of a distance function to establish properties for d.

1) $\underline{d(x,y) \geq 0 \text{ and } d(x,y) = 0 \text{ iff } x = y}$

Since $\rho(x,y), \rho(x,y)+1 \geq 0$, we have

$$d(x,y) = \rho(x,y)\frac{1}{1+\rho(x,y)} = \frac{\rho(x,y)}{1+\rho(x,y)} \geq 0 \ .$$

Suppose $x = y$. Then $\rho(x,y) = 0$ so that

$$d(x,y) = \frac{0}{1+0} = 0 \ .$$

Suppose $0 = d(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$. Then $\rho(x,y) = 0(1+\rho(x,y)) = 0$, which implies that $x = y$ by the assumption that ρ is a distance function.

2) $\underline{d(x,y) = d(y,x)}$

Since $\rho(x,y) = \rho(y,x)$, we have

$$d(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)} = \frac{\rho(y,x)}{1+\rho(y,x)} = d(y,x) \ .$$

3) $\underline{d(x,z) \leq d(x,y) + d(y,z)}$

Note that since ρ is a distance, we know that $\rho(x,z), \rho(x,y), \rho(y,z) \geq 0$ and $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. Then the inequality from problem 9.6 of the Royden text we established above applies, and we see that

$$d(x,z) = \frac{\rho(x,y)}{1+\rho(x,y)} \leq \frac{\rho(x,y)}{1+\rho(x,y)} + \frac{\rho(y,z)}{1+\rho(y,z)} = d(x,y) + d(y,z) \ .$$

We conclude that d is a distance on $\mathbb{R}^d$.

Exercise 4

Let (x_n) be a convergent sequence in the metric space (X, ρ). Then the limit of (x_n) is unique.

<u>Proof</u> Suppose the limit of the sequence (x_n) is not unique. Then there are at least two distinct limits, say L_1 and L_2, with $L_1 \neq L_2$.

Let $\epsilon \in \mathbb{R}$ with $\epsilon > 0$. Since (x_n) converges to L there exists some $N_1 \in \mathbb{N}$ such that for every $n > N_1$ it is the case that $\rho(x_n, L_1) < \frac{\epsilon}{2}$. Since (x_n) converges to L_2, there exists some $N_2 \in \mathbb{N}$ such that for ever $n > N_2$ it is the case that $\rho(x_n, L_2) < \frac{\epsilon}{2}$. Take $N = \max\{N_1, N_2\}$. It follows that whenever $n > N$, both inequalities $\rho(x_n, L_1) < \frac{\epsilon}{2}$ and $\rho(x_n, L_2) < \frac{\epsilon}{2}$ must hold. Letting $n > N$, we obtain

$$\rho(L_1, L_2) \leq \rho(L_1, x_n) + \rho(x_n, L_2) = \rho(x_n, L_1) + \rho(x_n, L_2) < \epsilon = \epsilon + 0 \ .$$

Since ϵ was arbitrary we have shown that $\rho(L_1, L_2) < \epsilon + 0$ holds for any choice of ϵ. This implies that $\rho(L_1, L_2) \leq 0$. But since $\rho(L_1, L_2) \geq 0$ we conclude that $\rho(L_1, L_2) = 0$, which means that $L_1 = L_2$. This contradicts the assumption that $L_1 \neq L_2$. Therefore, we conclude that the limit of the sequence (x_n) must be unique.

Exercise 5

Let (X, ρ) be a metric space, where X is a finite set. Then any subset $\mathcal{O}$ of X is both open and closed.

Proof Since X is a finite set, we may enumerate the set as $X = \{x_1, x_2, ..., x_n\}$, where $n \in \mathbb{N}$. Let $\mathcal{O} \subseteq X$ and write $\mathcal{O} = \{x_1, ..., x_m\}$, with $0 \leq m \leq n$. If it is the case that $\mathcal{O} = \emptyset$ then we are done, since $\emptyset$ is both open and closed. So assume $\mathcal{O} \neq \emptyset$.

Let $x_i \in \mathcal{O}$. Since $\mathcal{O}$ is a finite set, there exists a finite number of distances between x_i and any other point in $\mathcal{O}$. For any $x_j \in \mathcal{O}$, with $j \neq i$ we denote $\rho(x_i, x_j) = r_{ij} > 0$. The set $R = \{r_{ij} \ : \ j \in \{1, ..., m\} - \{i\}\}$ is a finite set of positive numbers. Take $r = \min R$ and note that $r > 0$. It follows that the open ball

$$B\left(x_i, \frac{r}{2}\right) = \{x \in \mathcal{O} \ : \ \rho(x, x_i) < \frac{r}{2}\} = \emptyset \, .$$

This shows that $B\left(x_i, \frac{r}{2}\right) = \emptyset \subset \mathcal{O}$. Since $x_i \in \{x_1, ..., x_m\}$ was arbitrary, we conclude by definition that $\mathcal{O}$ is open in X.

To show that $\mathcal{O}$ is closed in X, consider the complement of $\mathcal{O}$, denoted $\mathcal{O}^C$. Since $\mathcal{O}^C$ is itself a subset of X, we know from our work above that $\mathcal{O}^C$ is open in X (we must assume that the proof above is correct, but if not it should still be reasonable to move forward with the remainder of this proof). Since we have established in lecture that $\mathcal{O}^C$ is open iff $(\mathbb{O}^C)^C = \mathcal{O}$ is closed, we may immediately conclude that $\mathcal{O}$ is closed in X. Therefore, any subset of X is both open and closed.

Exercise 6

Let (X, ρ) be a metric space and let A be a subset of X. Then V is an open subset of A iff there is an open subset W of X such that $V = A \cap W$.

Proof Suppose V is an open subset of A. It follows that V is then also an open subset of X. Take $W = V$. Since V is a subset of A, $A \cap W = A \cap V = V$. Thus we have found an open subset W of X such that V is the intersection of A and W.

Suppose V is a subset of A and that there exists an open subset W of X such that $V = A \cap W$. We need to show that V is open in A. Let $v \in V = A \cap W$. Since $v \in W$, there exists an open ball $B(v,r)$ centered at v contained in W. Define $B_V(v,r) = B(v,r) \cap A$. Then $B_V(v,r)$ an open subset of A and an open subset W, which means that $B_V(v,r)$ is an open subset of $A \cap W = V$. Generalizing this reasoning to all points in V we see that $V = \cup_{v \in V} B_V(v,r)$ is the union of open balls contained in A. Thus we conclude V must be open in A.