MA 503 : Homework 9

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Proposition 8 (Chapter 2) Every open set of real numbers is the union of a countable collection of disjoint open intervals.

Proposition 5 Given any set A and $\epsilon > 0$, there is an open set $\mathcal{O}$ such that $A \subset \mathcal{O}$ and $m^*(O) \leq m^*(A) + \epsilon$. There is a $G \in G_\delta$ such that $A \subset G$ and $m^*(A) = m^*(G)$.

Theorem 10 The collection $\mathfrak{M}$ is a σ-algebra. Moreover, every set with outer measure zero is measurable.

Theorem 12 Every Borel set is measurable. In particular each open set and each closed set is measurable.

Proposition 14 Let (E_i) be a sequence of decreasing measurable sets, that is, a sequence with $E_{n+1} \subset E_n$ for each $n \in \mathbb{N}$. Let $m(E_1) < \infty$. Then,

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)\ .$$

Proposition 15 Let E be a given set. The following five statements are equivalent.

i. E is measurable.
ii. Given $\epsilon > 0$ there is an open set $O \supset E$ such that $m^*(O \backslash E) < \epsilon$.
iii. Given $\epsilon > 0$ there is a closed set $F \subset E$ such that $m^*(E \backslash F) < \epsilon$.
iv. There is a $G \in G_\delta$ with $E \subset O$ such that $m^*(G \backslash E) = 0$.
v. There is an $F \in F_\sigma$ with $F \subset E$ such that $m^*(E \backslash F) = 0$.

If $m^*(E) < \infty$, the above statements are equivalent to:

vi. Given $\epsilon > 0$, there is a finite union U of open intervals such that $m^*(U \triangle E) < \epsilon$.

Problem 13 Prove Proposition 15.

(a) Assume for part (a) that $m^*(E) < \infty$.

(i) $\implies$ (ii)
Suppose E is measurable and let $\epsilon > 0$. By Proposition 5, since $E \subset \mathbb{R}$, there is an open set $O \supset E$ such that $m^*(O) \leq m^*(E) + \epsilon$. As we saw in the proof of this proposition, the inequality can be made strict if $m^*(E) < \infty$ (or just start with $m^*(O) \leq m^*(E) + \epsilon/2$). Then since E is measurable,

$$\begin{aligned} m^*(O) &= m^*(O \cap E) + m^*(O \cap E^c) \\ &= m^*(E) + m^*(O \backslash E) \\ \implies m^*(O \backslash E) &= m^*(O) - m^*(E) \\ &< m^*(E) + \epsilon - m^*(E) \\ &= \epsilon\ . \end{aligned}$$

However, there appears to be no reason to use the assumption that E is measurable. Since $E \subset O$, we can write O as a disjoint union $O = (O \backslash E) \cup E$ so that $m^*(O) < m^*(E) + \epsilon \implies m^*(O \backslash E) =$

$m^*(O) - m^*(E) < \epsilon$. Note that $m^*(O) < \infty$ as $m^*(O) < m^*(E) + \epsilon < \infty$.

(ii) $\implies$ (vi)
Assume (ii) holds and let $\epsilon > 0$. There exists an open set $O \supset E$ such that $m^*(O \backslash E) < \epsilon/2$. By Proposition 8 of Chapter 2, there is a disjoint collection of open intervals $\{I_n\}$ such that $O = \biguplus_{n=1}^{\infty} I_n$ (If the collection $\{I_n\}$ is actually finite, with say $\biguplus_{n=1}^{p} I_n = O$, set $I_n = \emptyset$ for $n > p$). Since $O \supset E$, $O = (O \backslash E) \biguplus E$ is a disjoint union and $m^*(O) = m^*(O \backslash E) + m^*(E) < \epsilon + m^*(E)$. This implies that $m^*(O) < \infty$.

$$\infty > m^* \left(\biguplus I_n \right) = \sum_{n=1}^{\infty} m^*(I_n) \ .$$

Since the series converges to a finite value, the nonnegative sequence of partial sums converges to 0. This implies that there is an N such that $\sum_{n=N}^{\infty} m^*(I_n) < \epsilon/2$. Let $U = \biguplus_{n=1}^{N} I_n$. From $E \subset O$ and $U \subset O$ it follows that

$$m^*(E \backslash U) \leq m^*(O \backslash U) = m^*(O) - m^*(U) = \sum_{n=1}^{\infty} m^*(I_n) - \sum_{n=1}^{N} m^*(I_n) = \sum_{n=N}^{\infty} m^*(I_n) < \epsilon/2 \ .$$

Since $U \subset O$, $U \backslash E \subset O \backslash E$ and $m^*(U \backslash E) \leq m^*(O \backslash E) < \epsilon/2$. For the finite union of open intervals U,

$$m^* \left(U \bigtriangleup E \right) = m^* \left((U \backslash E) \cup (E \backslash U) \right) \leq m^*(U \backslash E) + m^*(E \backslash U) < \epsilon/2 + \epsilon/2 = \epsilon \ .$$

(vi) $\implies$ (ii)
Assume (vi) holds and let $\epsilon > 0$. By Proposition 5, since E is a set of real numbers there is an open set O such that $m^*(O) < m^*(E) + \epsilon < \infty$ (again the inequality can always be made strict so long as $m^*(E) < \infty$). Then $m^*(O \backslash E) = m^*(O) - m^*(E) < \epsilon$. There appears to be no need to use (vi). If we want to explicitly involve (vi), consider that we can show (iv) $\implies$ (i) and use (i) $\implies$ (ii) to conclude that (iv) $\implies$ (i). The strategy used in this case is then better described as (i) $\implies$ (ii) $\implies$ (vi) $\implies$ (i).

Assume (iv) holds and let $A \subset \mathbb{R}$ and $\epsilon > 0$. As we have seen, it is always the case that $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$. There is a finite union of open intervals U such that $m^*(U \bigtriangleup E) = m^* \left[(E \cap U^c) \cup (U \cap E^c) \right] < \epsilon/2$. As U is the union of open sets, U is open. By Theorem 12, U is measurable.

$$\begin{aligned}
&m^*(A \cap E) + m^*(A \cap E^c) \\
&= [m^*(A \cap E \cap U) + m^*(A \cap E \cap U^c)] + [m^*(A \cap E^c \cap U) + m^*(A \cap E^c \cap U^c)] \\
&\leq [m^*(A \cap U) + m^*(A \cap E \cap U^c)] + [m^*(A \cap E^c \cap U) + m^*(A \cap E^c \cap U^c)] \\
&\leq [m^*(A \cap U) + m^*(E \cap U^c)] + [m^*(A \cap E^c \cap U) + m^*(A \cap E^c \cap U^c)] \\
&\leq [m^*(A \cap U) + m^*(E \cap U^c)] + [m^*(E^c \cap U) + m^*(A \cap E^c \cap U^c)] \\
&\leq [m^*(A \cap U) + m^*(E \cap U^c)] + [m^*(E^c \cap U) + m^*(A \cap U^c)] \\
&= [m^*(A \cap U) + m^*(A \cap U^c)] + [m^*(E^c \cap U) + m^*(E \cap U^c)] \\
&= m^*(A) + m^*(E^c \cap U) + m^*(E \cap U^c) \\
&= m^*(A) + m^*(U \cap E^c) + m^*(E \cap U^c) \\
&< m^*(A) + \epsilon \\
&\implies m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A) \quad \text{as } \epsilon \text{ was arbitrary.}
\end{aligned}$$

The last inequality above follows from:

$$\begin{aligned}
m^*(E \cap U^c) &\leq m^* \left[(E \cap U^c) \cup (U \cap E^c) \right] < \epsilon/2 \\
m^*(U \cap E^c) &\leq m^* \left[(E \cap U^c) \cup (U \cap E^c) \right] < \epsilon/2
\end{aligned}$$

$$m^*(E \cap U^c) + m^*(U \cap E^c) < \epsilon \, .$$

Therefore, (iv) implies that E is measurable. This (i) and so (ii) follows.

(b) Let E be a given set.

(i) $\implies$ (ii)
Suppose E is measurable. By part (a), if $m^*(E) < \infty$ then (i) $\implies$ (ii). If instead $m^*(E) = \infty$, then E cannot be bounded. If $E \subset [-M, M]$ for some $M \in \mathbb{R}$ then the contradiction $\infty = m^*(E) \leq 2M$ follows. Define $E_k = \{e \in E : |e| \leq k\}$ for each $k \in \mathbb{N}$. Then (E_k) is an increasing sequence of sets. Each E_k is a bounded set of real numbers (or possibly empty) with $m^*(E_k) < \infty$. For each E_k there is by Proposition 5 an open set $O_k \supset E_k$ with $m^*(O_k) < m^*(E_k) + \epsilon/2^k < \infty \implies m(O_k \backslash E_k) < \epsilon/2^k$. Set $O = \bigcup_{k=1}^{\infty} O_k$. Since $E \subset \mathbb{R}$, for any element $e \in E$ there is a k such that $|e| \leq k$ and $e \in E_n \subset O_n \subset O$ for all $n \geq k$. So $E_k \uparrow E$ and $E \subset O$.

$$\begin{aligned}
m^*(O \backslash E) &= m^*\left((\bigcup_{k=1}^{\infty} O_k) \cap E) \right) \\
&= m^*\left(\bigcup_{k=1}^{\infty} (O_k \cap E) \right) \\
&\leq \sum_{k=1}^{\infty} m^*(O_k \cap E) \\
&\leq \sum_{k=1}^{\infty} m^*(O_k \cap E_k) \quad (E_k \subset E \implies O_k \cap E \subset O_k \cap E_k) \\
&< \sum_{k=1}^{\infty} \epsilon 2^{-k} \\
&= \epsilon \, .
\end{aligned}$$

Conclude that O is an open set with $O \supset E$ and $m^*(O \cap E) < \epsilon$.

(ii) $\implies$ (iv)
For any $k \in \mathbb{N}$, there is an open set $O_k \supset E$ such that $m^*(O_k \backslash E) < 1/k$. Let $G = \cap_{k=1}^{\infty} O_k$. For any $N \in \mathbb{N}$,

$$\begin{aligned}
m^*(G \backslash E) &= m^*\left(\left(\bigcap_{k=1}^{\infty} O_k \right) \cap E \right) \\
&\leq m^*\left(\left(\bigcap_{k=1}^{N} O_k \right) \cap E \right) \quad (\cap_{k=1}^{\infty} O_k \subset \cap_{k=1}^{N} O_k) \\
&\leq m^*(O_N \backslash E) \quad (\cap_{k=1}^{N} O_k \subset O_N) \\
&< 1/N \\
\implies m^*(G \backslash E) &= \lim_{N\to\infty} m^*(G \backslash E) \leq \lim_{N\to\infty} 1/N = 0 \, .
\end{aligned}$$

Since $m^*(G \backslash E) \geq 0$ by definition of m^*, conclude that for the countable intersection of open sets $G \in G_\delta$, $m^*(G \backslash E) = 0$.

(iv) $\implies$ (i)
Let $A \subset \mathbb{R}$ and by (iv) let $G \in G_\delta$ such that $m^*(G \backslash E) = 0$. Since $\mathfrak{M}$ is a σ-algebra by Theorem 10, $G = \bigcap_{k=1}^{\infty} G_k = \left(\bigcup_{k=1}^{\infty} G_k^c \right)^c \in \mathfrak{M}$ using σ-algebra properties and the fact that each open set $G_k \in \mathfrak{M}$. As noted several times in this section, $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$ whether or not E is measurable. For the reverse inequality,

$$
\begin{aligned}
m^*(A\cap E)+m^*(A\cap E^c) &= m^*(A\cap E)+m^*(A\cap E^c)\cap G)+m^*((A\cap E^c)\cap G^c)\\
&= m^*(A\cap E)+m^*(A\cap(E^c\cap G))+m^*(A\cap(E^c\cap G^c))\\
&= m^*(A\cap E)+m^*(A\cap(G\backslash E))+m^*(A\cap G^c)\quad (E\subset G\implies G^c\subset E^c)\\
&\leq m^*(A\cap E)+m^*(G\backslash E)+m^*(A\cap G^c)\quad (A\cap(G\backslash E)\subset G\backslash E)\\
&= m^*(A\cap E)+0+m^*(A\cap G^c)\\
&\leq m^*(A\cap G)+m^*(A\cap G^c)\quad (A\cap E\subset A\cap G)\\
&= m^*(A)\quad (G \text{ is measurable})\ .
\end{aligned}
$$

Therefore $m^*(A\cap E)+m^*(A\cap E^c)\leq m^*(A)$, from which we conclude that E is measurable.

(c) Let E be a given set.

(i) $\implies$ (iii)
Assume that E is measurable. Then E^c is also measurable. By part (b), if (i) holds then (ii) holds as well so there is an open set $O\supset E^c$ such that $m^*(O\backslash E^c)<\epsilon$. But this means $\epsilon>m^*(O\cap(E^c)^c)=m^*(O\cap E)$. Since $E^c\subset O$, $O^c\subset E$. This gives a closed set, O^c, such that $O^c\subset E$ and $m^*(E\backslash O^c)=m^*(E\cap(O^c)^c)=m^*(E\cap O)<\epsilon$.

(iii) $\implies$ (v)
For each $n\in\mathbb{N}$, there is by (iii) a closed set $F_n\subset E$ such that $m^*(E\backslash F_n)<1/n$. Let $F=\bigcup_{n=1}^{\infty}F_n$. Then $F\in F_\sigma$ is also a closed set and for any $N\in\mathbb{N}$,

$$
\begin{aligned}
m^*(E\backslash F)=m^*\left[E\cap\left(\bigcup_{n=1}^{\infty}F_n\right)^c\right] &= m^*\left[E\cap\left(\bigcap_{n=1}^{\infty}F_n^c\right)\right]\\
&= m^*\left[\bigcap_{n=1}^{\infty}(E\cap F_n^c)\right]\\
&\leq m^*\left[\bigcap_{n=1}^{N}(E\cap F_n^c)\right]\quad \left(\bigcap_{n=1}^{\infty}F_n^c\subset\bigcap_{n=1}^{N}F_n^c\right)\\
&\leq m^*(E\cap F_n^c)\quad \left(\bigcap_{n=1}^{N}F_n^c\subset F_N^c\right)\\
&= m^*(E\backslash F_N)<1/N\implies m^*(E\backslash F)\leq 0\ .
\end{aligned}
$$

Since $m^*(E\backslash F)\geq 0$ by definition of m^*, conclude that $F\in F_\sigma$ satisfies the conditions of (v).

(v) $\implies$ (i)
Let $F\in F_\sigma$ such that $F\subset E$ and $m^*(E\backslash F)=0$. As the union of closed (and therefore measurable) sets, F is measurable. Let $A\subset\mathbb{R}$. Then $m^*(A)\leq m^*(A\cap E)+m^*(A\cap E^c)$. To show the reverse inequality,

$$
\begin{aligned}
m^*(A\cap E)+m^*(A\cap E^c) &= m^*((A\cap E)\cap F)+m^*((A\cap E)\cap F^c)+m^*(A\cap E^c)\\
&= m^*(A\cap(E\cap F))+m^*(A\cap(E\backslash F))+m^*(A\cap E^c)\\
&= m^*(A\cap F)+m^*(A\cap(E\backslash F))+m^*(A\cap E^c)\\
&\leq m^*(A\cap F)+m^*(E\backslash F)+m^*(A\cap E^c)\\
&= m^*(A\cap F)+0+m^*(A\cap E^c)\\
&\leq m^*(A\cap F)+m^*(A\cap F^c)\\
&= m^*(A)
\end{aligned}
$$

Therefore, $m^*(A)=m^*(A\cap E)+m^*(A\cap E^c)$, which shows that E is measurable.

Definition A **ternary expansion** of $x \in [0,1]$ is a sequence (a_n) with $0 \le a_n < 3$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \, .$$

Definition The **Cantor Ternary Set** C consists of all those real numbers $[0,1]$ that have a ternary expansion (a_n) for which a_n is never 1 (if x has two ternary expansions, we put x in C if one of the expansions has no term equal to 1). The set C is closed and obtained by first removing $(1/3, 2/3)$ from $[0,1]$, then removing $(1/9, 2/9)$ from $[0, 1/3]$ and $(7/9, 8/9)$ from $[2/3, 1]$, and so on. Using the definition of ternary expansion,

$$C = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n \in \{0,1,2\} \right\}$$

Problem 14

(a) Prove that the Cantor Ternary Set has measure 0.

Define,

$$\begin{aligned}
E_1 &= [0/3, 1/3] \cup [2/3, 3/3] \\
E_2 &= [0/9, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 9/9] \\
E_3 &= [0/27, 1/27] \cup [2/27, 3/27] \cup [6/27, 7/27] \cup [8/27, 9/27] \cup [18/27, 19/27] \cup [20/27, 21/27] \\
&\quad \cup [24/27, 25/27] \cup [26/27, 27/27] \\
&\vdots
\end{aligned}$$

Continue such that E_{n+1} is obtained by removing the open interval making up the middle third of each closed interval in the union forming E_n. Then $E_{n+1} \subset E_n$ for all n and each E_n is measurable as a union of closed intervals (which are measurable by Theorem 12). For each n, E_n is the union of 2^n closed intervals, each of length $(1/3)^n$. For each $n \ge 2$, E_n is obtained by removing 2^{n-1} open intervals each of length $(1/3)^n$ from E_{n-1}.

$$\begin{aligned}
m^*(E_n) = m(E_n) &= m([0,1]) - \sum_{k=1}^{n} \frac{2^{k-1}}{3^k} \\
&= 1 - \sum_{k=1}^{n} \frac{2^{k-1}}{3^k} \\
&= 1 - \sum_{k=1}^{n} \frac{1}{2} \left(\frac{2}{3}\right)^k \\
&= 1 - \left(1 - \left(\frac{2}{3}\right)^n\right) \\
&= \left(\frac{2}{3}\right)^n \, .
\end{aligned}$$

The sequence of measurable sets (E_n) is decreasing with $m(E_1) = 2/3 < \infty$. By Proposition 14:

$$m(C) = m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0 \, .$$