MA 503 : Homework 19

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Proposition 14 Let f be a nonnegative function which is integrable over a set E. Then given ϵ, there is a δ such that for every set $A \subset E$ with $m(A) < \delta$ we have

$$\int_A f < \epsilon \, .$$

Proof: If f is bounded with $|f| \leq M$, then to get $\int_A f < \epsilon$, require that $m(A) < \delta = \epsilon/M$ so that $\int_A f \leq \int_A M = Mm(A) < M\epsilon/M = \epsilon$. So consider the case that f is not bounded but still f is integrable over E, meaning that $0 \leq \int_E f < \infty$ is bounded. Define

$$f_n(x) = \begin{cases} f(x) & f(x) \leq n \\ n & f(x) > n \end{cases} .$$

Each f_n is bounded and nonnegative as $0 \leq f_n(x) \leq n$. If $f(x) < \infty$, then there is an N such that $f(x) \leq N$. Then $f_n(x) = f(x)$ for all $n \geq N$ and so $f_n(x) \to f(x)$. If $f(x) = \infty$ (measurable functions by definition may be extended real valued functions so it is necessary to consider this - it's unclear whether $\int_E f < \infty$ guarantees that $f(x)$ is finite for all $x \in E$, only that $f(x) = \infty$ on a set of measure zero), then $f_n(x) = n$ for all n. But then $f_n(x) = n \to \infty = f(x)$ as well. So $f_n \to f$ pointwise on E. Since $f(x)$ is measurable and $g(x) = n$ is measurable for each n, each f_n is measurable. For each x, $f_n(x)$ is increasing since either $f_n(x) = n$ for all n or $f_n(x) = n$ for finitely many n and then constantly $f_n(x) = f(x)$ for all $n \geq N$ for some N. We have established that (f_n) is an increasing sequence of nonnegative measurable functions that $f = \lim f_n$ pointwise (and thereby satisfies the weaker condition of a.e. convergence). That each f_n is integrable follows from Proposition 8 (iii). By Theorem 10 (Monotone Convergence Theorem),

$$\int_E f = \lim \int f_n \, .$$

Then given $\epsilon > 0$, there is an N such that $\int_E f_n > \int_E f - \epsilon/2$ and so $\int_E (f - f_n) < \epsilon/2$ for all $n \geq N$. In particular this holds for N specifically. Then for any $A \subset E$ with $m(A) < \delta = \epsilon/2N$.

$$\begin{aligned}
\int_A f = \int_A [(f - f_N) + f_N] &= \int_A (f - f_N) + \int_A f_N \\
&\leq \int_E (f - f_N) + \int_A N \\
&< \frac{\epsilon}{2} + \int_A N \\
&= \frac{\epsilon}{2} + Nm(A) \\
&< \frac{\epsilon}{2} + \frac{N\epsilon}{2N} \\
&= \epsilon \, .
\end{aligned}$$

Problem 24 Use proposition 14 to prove directly that if $f_n \to f$ in measure and if there is an integrable function g such that for all n we have $|f_n| \leq g$, then $\int |f_n - f| \to 0$.

By proposition 18, there is a subsequence (f_{n_k}) of (f_n) that converges to f almost everywhere. Since $|f_{n_k}| \leq g$ for each k, it follows that $|f| \leq g$ almost everywhere. Therefore $\int |f| \leq \int g$. Since

$|f_n| \leq g$ for each n, $\int |f_n| \leq \int g$ for each n. Then $\int |f_n - f| \leq \int |f_n| + |f| \leq 2\int g < \infty$.

Let $\epsilon > 0$. Since g is integrable, there is an $M \in \mathbb{N}$ such that $\int_{[-M,M]^c} g < \epsilon/4$. Given $\epsilon/4$, there exists by proposition 14 a $\delta > 0$ such that $\int_A |f_n - f| < \epsilon/4$ if $m(A) < \delta$. Let $\delta^* = \min\{\delta, \epsilon/(8M)\}$. Since (f_n) converges to f in measure, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $m(\{x : |f_n(x) - f(x)| \geq \delta^*\}) < \delta^* \leq \delta$. Let $A_n = \{x : |f_n(x) - f(x)| \geq \delta^*\}$. By proposition 14, $\int_{A_n} |f_n(x) - f(x)| < \epsilon/4$.

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\begin{aligned}
\int |f_n - f| &= \int_{[-M,M]^c} |f_n - f| + \int_{[-M,M]} |f_n - f| \\
&\leq \int_{[-M,M]^c} 2g + \int_{[-M,M]} |f_n - f| \\
&< \frac{2\epsilon}{4} + \int_{[-M,M]} |f_n - f| \\
&= \frac{\epsilon}{2} + \int_{[-M,M]\cap A_n} |f_n - f| + \int_{[-M,M]\cap A_n^c} |f_n - f| \\
&\leq \frac{\epsilon}{2} + \int_{A_n} |f_n - f| + \int_{[-M,M]\cap A_n^c} |f_n - f| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \int_{[-M,M]\cap A_n^c} |f_n - f| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \int_{[-M,M]\cap A_n^c} \delta^* \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{4} + m([-M,M]\cap A_n^c)\delta^* \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + m([-M,M])\delta^* \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{4} + 2M\delta^* \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{2M\epsilon}{8M} \\
&= \epsilon\ .
\end{aligned}
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Problem 25 A sequence f_n is said to be Cauchy in measure if given $\epsilon > 0$ there is an N such that for all $m, n \geq N$ we have

$$m\left(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}\right) < \epsilon\ .$$

Show that if (f_n) is a Cauchy sequence in measure, then there is a function f to which the sequence (f_n) converges in measure.

I had trouble figuring out how to work with the hint so did something probably much more complicated instead.

Since (f_n) is Cauchy, for each $v \in \mathbb{N}$, there is an n_v such that for all $k, j \geq n_v$ we have $m(\{x : |f_k(x) - f_j(x)| \geq 2^{-v}\}) < 2^{-v}$. In particular, $m(\{x : |f_{n_v} - f_{n_{v+1}}(x)| \geq 2^{-v}\}) < 2^{-v}$ and we can choose each n_{v+1} such that $n_{v+1} > n_v$. Let $E_v = \{x : |f_{n_{v+1}}(x) - f_{n_v}(x)| \geq 2^{-v}\}$ and $F_v = \bigcup_{j=v}^{\infty} E_j$ so that F_v is measurable and $m(F_v) < 2^{-(v-1)}$. If $i \geq j \geq v$ and $x \notin F_v$, then

$$|f_{n_i}(x) - f_{n_j}(x)| \leq |f_{n_i}(x) - f_{n_{i-1}}(x)| + ... + |f_{n_{j+1}}(x) - f_{n_j}(x)| \leq \frac{1}{2^{i-1}} + ... + \frac{1}{2^j} < \frac{1}{2^{j-1}}\ .$$

Let $F = \bigcap_{v=1}^{\infty} F_v$ so that F is measurable and $m(F) = 0$. Then (f_{n_j}) converges outside of F. Define,

$$f(x) = \begin{cases} \lim f_{n_j}(x) & x \notin F \\ 0 & x \in F \end{cases}$$

so that (f_{n_j}) converges a.e. to the measurable real valued function f. As $i \to \infty$, we have that if $j \geq v$ and $x \notin F_v$, then

$$|f(x) - f_{n_j}(x)| \leq \frac{1}{2^{j-1}} \leq \frac{1}{2^{k-1}} \ .$$

This shows that (f_{n_j}) converges uniformly to f on the complement of each set F_v. Let α, ϵ be positive real numbers and take v large enough such that $m(F_v) < 2^{-(v-1)} < \min\{\alpha, \epsilon\}$. If $j \geq v$, we have

$$\{x : |f(x) - f_{n_j}(x)| \geq \alpha\} \subseteq \{x : |f(x) - f_{n_j}(x)| > 2^{-(v-1)}\} \subseteq F_v \ .$$

But then $m\left(x : |f(x) - f_{n_j}(x)| \geq \alpha\}\right) \leq m(F_v) < \epsilon$ for all $j \geq v$ so that (f_{n_j}) converges in measure to f.

We have seen that there is a subsequence (f_{n_v}) of (f_n) which converges in measure to f and next need to show that the entire sequence converges in measure to f. Since,

$$|f(x) - f_n(x)| \leq |f(x) - f_{n_v}(x)| + |f_{n_v}(x) - f_n(x)| \ ,$$

it follows that,

$$\{x : |f(x) - f_n(x)| \geq \alpha\} \subseteq \{x : |f(x) - f_{n_v}(x) \geq \frac{\alpha}{2}\} \bigcup \{x : |f_{n_v}(x) - f_n(x)| \geq \frac{\alpha}{2}\} \ .$$

The convergence in measure of (f_n) to f follows from this relation above and since (f_{n_v}) converges to f in measure and (f_n) converges in measure to f_{n_v} (I think this is how it would work but not totally sure).

This function f is a.e. unique. Suppose that (f_n) converges to f and g in measure.

$$|f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)|$$

$$\{x : |f(x) - g(x)| \geq \alpha\} \subseteq \{x : |f(x) - f_n(x)| \geq \frac{\alpha}{2}\} \bigcup \{x : |f_n(x) - g(x)| \geq \frac{\alpha}{2}\} \ .$$

Then $m(\{x : |f(x) - g(x)| \geq \alpha\}) = 0$ for all $\alpha > 0$. Then taking $\alpha = 1/n$ for each $n \in \mathbb{N}$, conclude that $f = g$ except on a set of measure zero.