MA 503 : Homework 8

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Definition A set E is said to be **(Lebesgue) measurable** if for each set $A \subset \mathbb{R}$ we have using (Lebesgue) outer measure, m^*, that $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$.

Let $D_1 = A \cap E$, $D_2 = A \cap E^c$, $D_n = \emptyset$ for $n \geq 3$. Then $A = (A \cap E) \cup (A \cap E^c) = \bigcup D_n$. By Proposition 2,

$$m^*(A) = m^*\left((A \cap E) \cup (A \cap E^c)\right) = m^*\left(\bigcup D_n\right) \leq \sum m^*(D_n) = m^*(A \cap E) + m^*(A \cap E^c)\ .$$

Since we always have $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$, we see that E is measurable if and only if $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$ for each set A. Since the definition is symmetric, E is measurable if and only if E^c is measurable. For any A, $m^*(A \cap \emptyset) + m^*(A \cap \mathbb{R}) = m^*(\emptyset) + m^*(A) = m^*(A)$, which shows that both $\emptyset$ and $\mathbb{R}$ are measurable.

Lemma 9 Let A be any set and $E_1, ..., E_n$ a finite sequence of disjoint meaurable sets. Then

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i)\ .$$

Theorem 10 The collection $\mathfrak{M}$ is a σ-algebra. Moreover, every set with outer measure zero is measurable.

Definition If E is a measurable set, we define the Lebesgue measure of E, $m(E)$, as the outer measure of E. That is $m : \mathfrak{M} \to [0, \infty]$, $m(E) = m^*(E)$ is the set function obtained by restricting the set function m^* to the family $\mathfrak{M}$ of measurable sets.

Proposition 13 Let (E_i) be a sequence of measurable sets. Then,

$$m\left(\bigcup E_i\right) \leq \sum m(E_i)\ .$$

If the E_i are pairwise disjoint,

$$m(\left(\bigcup E_i\right) = \sum m(E_i)\ .$$

Problem 10

Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)\ .$$

First if $m(E_1) = \infty$ or $m(E_2) = \infty$, then $E_1, E_2 \subset E_1 \cup E_2$ implies that $m(E_1 \cup E_2) = \infty$ as well. In this case the equality holds since both sides are ∞. Otherwise assume that both $m(E_1)$ and $m(E_2)$ are finite.

Since E_1 and E_2 are measurable and $\mathfrak{M}$ is a σ-algebra, the sets $E_1 \cup E_2$, $E_1 \cap E_2^c$, $E_2 \cap E_1^c$, and $E_1 \cap E_2$ are all also measurable. Then for each set mentioned, $m^* = m$. Since $m(E_1), m(E_2) < \infty$, each of these sets mentioned also has finite measure so that addition and subtraction are meaningful in the equations below (no expressions like $\infty - \infty$ arise). All sets mentioned are contained in either E_1 or E_2 or both except $E_1 \cup E_2$. If $m(E_1 \cup E_2) = \infty$, then for every cover $\{I_n\}$ of $E_1 \cup E_2$ by open intervals, $\sum l(I_n) = \infty$. However, since there exists a collection $\{I_j\}$ and a collection $\{I_k\}$ such that $E_1 \subset \bigcup I_j$, $E_2 \subset \bigcup I_k$ and $\sum l(I_j) < \infty$, $\sum l(I_k) < \infty$, we have that $\{I_n\} = \{I_k\} \cup \{I_j\}$ is a collection of open intervals such that $E_1 \cup E_2 \subset \bigcup I_n$ and $\sum l(I_n) = \sum I_j + \sum I_k < \infty$. This is a contradiction, so $m(E_1 \cup E_2) < \infty$. Also by Lemma 9, setting $A = \mathbb{R}$, we have that if $C_1, ..., C_n$ are disjoint measurable sets then $m(C_1 \cup ... \cup C_n) = m^*(C_1 \cup ... \cup C_n) = \sum_{i=1}^n m^*(C_i) = \sum_{i=1}^n m(C_i)$ (or use Proposition 13 with $C_i = \emptyset$ for $i > n$).

$$(1) \quad m(E_1 \cup E_2) = m(E_1 \cap E_2^c) + m(E_2 \cap E_1^c) + m(E_1 \cap E_2)$$

$$(2) \quad m(E_1) = m(E_1 \cap E_2^c) + m(E_1 \cap E_2)$$

$$(3) \quad m(E_2) = m(E_2 \cap E_1^c) + m(E_1 \cap E_2)$$

Add equations (2) and (3) to get (4)

$$(4) \quad m(E_1) + m(E_2) = m(E_1 \cap E_2^c) + m(E_2 \cap E_1^c) + 2m(E_1 \cap E_2)$$

$$(5) \quad m(E_1) + m(E_2) - m(E_1 \cap E_2) = m(E_1 \cap E_2^c) + m(E_2 \cap E_1^c) + m(E_1 \cap E_2)$$

Compare equation (5) to equation (1)

$$(6) \quad m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$$

$$(7) \quad m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2) \; .$$

Proposition 2 Let A_n be a countable collection of sets of real numbers.

$$m^*\left(\bigcup A_n\right) \leq \sum m^*(A_n) \; .$$

Problem 12

Let (E_i) be a sequence of disjoint measurable sets and $A \subset \mathbb{R}$.

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i) \; .$$

Proof:

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = m^*\left(\bigcup_{i=1}^{\infty}(A \cap E_i)\right) \leq \sum_{i=1}^{\infty} m^*(A \cap E_i) \quad \text{(by Proposition 2)} \; .$$

Since $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i$ for every n, $A \cap \bigcup_{i=1}^{\infty} \supset A \cap \bigcup_{i=1}^{n} E_i$ for every n.

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i),$$

where the equality comes from Lemma 9. Since the left hand side is independent of n, we have

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \lim_{n\to\infty} \sum_{i=1}^{n} m^*(A \cap E_i) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

This means $m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i)$ as well.