MA 503 : Homework 11

Dane Johnson

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Proposition 18 Let f be an extended real-valued function whose domain is measurable. The following statements are equivalent:

i. For each real number α the set $\{x : f(x) > \alpha\}$ is measurable.
ii. For each real number α the set $\{x : f(x) \geq \alpha\}$ is measurable.
iii. For each real number α the set $\{x : f(x) < \alpha\}$ is measurable.
iv. For each real number α the set $\{x : f(x) \leq \alpha\}$ is measurable.
These statements imply
v. For each real number α the set $\{x : f(x) = \alpha\}$ is measurable.

Definition A function $f : D \to \overline{\mathbb{R}}$ is said to be (Lebesgue) measurable if $D \subset \mathbb{R}$ is measurable and f satisfies one of statements (i)-(iv) in Proposition 18.

Proposition 19 Let c be a constant and f and g two measurable real-valued functions on the same domain D (which must be measurable by the definition above). Then the functions $f + c$, cf, $f + g$, $g - f$, and fg are also measurable.

Problem 21

a. Let E and D be measurable sets and f a function with domain $E \cup D$. Show that f is measurable if and only if its restrictions to D and E are measurable.

Suppose that f is measurable and let $\alpha \in \mathbb{R}$ be given. Then the set $\{x \in E \cup D : f(x) < \alpha\}$ is measurable (the notation needs to be more explicit here so we can consider whether $x \in D$ or $x \in E$). Consider that

$$\{x \in D : f|_D(x) < \alpha\} = \{x \in D : f(x) < \alpha\} = \{x \in E \cup D : f(x) < \alpha\} \cap D \,.$$

Then $\{x \in D : f(x) < \alpha\}$ is measurable as the intersection of two measurable sets. Since α was arbitrary, conclude that $\{x \in D : f|_D(x) < \alpha\}$ is measurable for any α. Conclude that the restriction of f to D, $f|_D$, is a measurable function. By swapping the positions of D and E, the same reasoning shows that the restriction of f to E, $f|_E$, is a measurable function as well.

Suppose that $f|_D$ and $f|_E$ are measurable functions and let $\alpha \in \mathbb{R}$ be given. Since E and D are measurable, the domain $E \cup D$ of f is measurable and

$$\begin{aligned} \{x \in E \cup D : f(x) < \alpha\} &= \{x \in E : f(x) < \alpha\} \cup \{x \in D : f(x) < \alpha\} \\ &= \{x \in E : f|_E(x) < \alpha\} \cup \{x \in D : f|_D(x) < \alpha\} \,. \end{aligned}$$

Since $\{x \in E : f|_E(x) < \alpha\}$ and $\{x \in D : f|_D(x) < \alpha\}$ are measurable sets, so is their union. Since α was arbitrary, $\{x \in E \cup D : f(x) < \alpha\}$ is measurable for each α. Therefore, f is a measurable function.

b. Let f be a function with a measurable domain D and let

$$g(x) = \begin{cases} f(x) & x \in D \\ 0 & x \notin D \end{cases} \,.$$

Show that f is measurable if and only if g is measurable.

The domain of g has not been specified in the problem. If the domain of g is not measurable, then by definition it is impossible for g to be measurable and this problem cannot be completed. So we need to assume that the domain E of g is measurable. If $E = D$ then the result is immediate. If $E \subsetneq D$, then it would seem the definition of g does not make sense and also it would not be possible to show that the measurability of g implies the measurability of f. So most likely we are to assume that $E \supsetneq D$. Since the reasoning is similar for any such measurable E, just assume that the domain of g is $\mathbb{R}$ (which is measurable in what follows. While the work below is still correct if $D = \mathbb{R}$, the result would again be immediate in this case and so this is meant to handle a measurable $D \subsetneq \mathbb{R}$.

Suppose that f is measurable and let $\alpha \in \mathbb{R}$ be given.

$$\begin{aligned}\{x : g(x) < \alpha\} &= \{x \in D : g(x) < \alpha\} \cup \{x \notin D : g(x) < \alpha\}\\ &= \{x \in D : f(x) < \alpha\} \cup \{x \in D^c : 0 < \alpha\} .\end{aligned}$$

Since f is measurable, the set $\{x \in D : f(x) < \alpha\}$ is always measurable, so it remains for us to see if $D^c \cap \{x : g(x) < \alpha\} = \{x \in D^c : g(x) < \alpha\} = \{x \in D^c : 0 < \alpha\}$ is measurable. Either $0 < \alpha$ or $0 \geq \alpha$. If $0 < \alpha$, $\{x \in D^c : 0 < \alpha\} = D^c$. If $0 \geq \alpha$, $\{x \in D^c : 0 < \alpha\} = \emptyset$. Since D^c and $\emptyset$ are measurable sets, $\{x \in D^c : 0 < \alpha\}$ in either case. Therefore, $\{x : g(x) < \alpha\}$ is measurable as the union of two measurable sets.

Suppose that g is measurable (again with the assumptions mentioned in the first paragraph about the domain of g and f). Let α be given. The set $\{x \in \mathbb{R} : g(x) < \alpha\}$ is measurable and D is measurable.

$$\{x \in D : f(x) < \alpha\} = \{x \in D : g(x) < \alpha\} = \{x \in \mathbb{R} : g(x) < \alpha\} \cap D .$$

Then $\{x \in D : f(x) < \alpha\}$ is measurable as the intersection of two measurable sets. Since α was arbitrary, $\{x \in D : f(x) < \alpha\}$ is measurable for any α. Conclude that f is a measurable function.

Problem 22

a. Let $f : D \to \overline{\mathbb{R}}$ where D is a measurable set. Let $D_1 = \{x : f(x) = \infty\}$ and $D_2 = \{x : f(x) = -\infty\}$. Show that f is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D \backslash (D_1 \cup D_2)$ is measurable.

Suppose that f is measurable and let $f^\dagger$ denote the restriction of f to $D\backslash(D_1 \cup D_2)$. Since $\{x : f(x) > n\}$ and $\{x : f(x) < -n\}$ are measurable sets for each $n \in \mathbb{N}$,

$$\begin{aligned}D_1 = \{x : f(x) = \infty\} = \cap_n \{x : f(x) > n\} \in \mathfrak{M},\\ D_2 = \{x : f(x) = -\infty\} = \cap_n \{x : f(x) < -n\} \in \mathfrak{M} .\end{aligned}$$

This implies that $D \cap D_1^c \cap D_2^c = D\backslash(D_1 \cup D_2) \in \mathfrak{M}$ as well. Let $\alpha \in \mathbb{R}$. Since $f^\dagger$ is only defined for $x \in D\backslash(D_1 \cup D_2)$,

$$\{x : f^\dagger(x) < \alpha\} = \{x : f(x) < \alpha\} \cap (D_1 \cup D_2)^c .$$

This shows that $\{x : f^\dagger(x) < \alpha\}$ is measurable as the intersection of measurable sets. Since α was arbitrary, $\{x : f^\dagger(x) < \alpha\}$ is measurable for each α and since the domain of $f^\dagger$ is measurable, conclude that $f^\dagger$ is a measurable function.
Suppose that $f\dagger$ is measurable and that D_1 and D_2 are measurable sets. Let $\alpha \in \mathbb{R}$. The set $\{x : f^\dagger(x) < \alpha\}$ is measurable and so

$$\begin{aligned}\{x \in D : f(x) < \alpha\} &= \{x \in D\backslash(D_1 \cup D_2) : f(x) < \alpha\} \cup \{x \in D : f(x) = -\infty\}\\ &= \{x : f^\dagger(x) < \alpha\} \cup D_2 \in \mathfrak{M} .\end{aligned}$$

Since α was arbitrary and D is measurable, conclude that f is measurable.

b. Prove that the product of two measurable extended real-values functions is measurable.

Let $f, g : D \to \overline{\mathbb{R}}$, where D is a measurable set on which both f and g and thus fg can be defined. Assume that both f and g are measurable. By part (a), the sets $\{x : f(x) = \infty\}$, $\{x : f(x) = -\infty\}$, $\{x : g(x) = \infty\}$, and $\{x : g(x) = -\infty\}$ are measurable. Also the sets $\{x : f(x) < 0\}$, $\{x : f(x) > 0\}$, $\{x : g(x) < 0\}$, and $\{x : g(x) > 0\}$ are measurable since f and g are measurable (and Proposition 18). By repeatedly using the fact that the σ-algebra $\mathfrak{M}$ is closed under complement and intersection and the conventions from section 2.3 for multiplication in $\overline{\mathbb{R}}$, the set

$$\begin{aligned}
D_1 &:= \{x : (fg)(x) = \infty\} \\
&= [\{x : f(x) = \infty\} \cap \{x : g(x) > 0\}] \\
&\cup [\{x : f(x) = -\infty\} \cap \{x : g(x) < 0\}] \\
&\cup [\{x : g(x) = \infty\} \cap \{x : f(x) > 0\}] \\
&\cup [\{x : g(x) = -\infty\} \cap \{x : f(x) < 0\}] \\
&\cup [\{x : f(x) = \infty\} \cap \{x : g(x) = \infty\}] \\
&\cup [\{x : f(x) = -\infty\} \cap \{x : g(x) = -\infty\}] \\
&\in \mathfrak{M}\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
D_2 &:= \{x : (fg)(x) = -\infty\} \\
&= [\{x : f(x) = \infty\} \cap \{x : g(x) < 0\}] \\
&\cup [\{x : f(x) = -\infty\} \cap \{x : g(x) > 0\}] \\
&\cup [\{x : g(x) = \infty\} \cap \{x : f(x) < 0\}] \\
&\cup [\{x : g(x) = -\infty\} \cap \{x : f(x) > 0\}] \\
&\cup [\{x : f(x) = \infty\} \cap \{x : g(x) = -\infty\}] \\
&\cup [\{x : f(x) = \infty\} \cap \{x : g(x) = -\infty\}] \\
&\in \mathfrak{M}\,.
\end{aligned}$$

This implies $D_1 \cup D_2$ and $D \backslash (D_1 \cup D_2)$ are measurable. By Problem 21 (a), the restriction of f to $D \backslash (D_1 \cup D_2)$ is measurable. Similarly, the restriction of g to $D \backslash (D_1 \cup D_2)$ is measurable. Moreover, these restrictions are measurable real-valued functions and so by Proposition 19, the restriction of fg to $D \backslash (D_1 \cup D_2)$ is measurable. By part (a) of this problem, since D_1 and D_2 are measurable and the restriction of fg to $D \backslash (D_1 \cup D_2)$ is measurable, we conclude that fg is measurable.

c. If f and g are measurable extended real-valued functions, and $\alpha \in \mathbb{R}$ is fixed, prove that

$$(f+g)(x) := \begin{cases} \alpha & f(x) = \infty, \quad g(x) = -\infty \\ \alpha & f(x) = -\infty, \quad g(x) = \infty \\ f(x) + g(x) & \qquad \text{otherwise} \end{cases}$$

is measurable.

Let $f, g : D \to \overline{\mathbb{R}}$. By part (a) the sets $\{x : f(x) = \infty\}$, $\{x : f(x) = -\infty\}$, $\{x : g(x) = \infty\}$, and $\{x : g(x) = -\infty\}$ are measurable. This implies $\{x : -\infty < g(x) < \infty\} = D \backslash (\{x : g(x) = \infty\} \cup \{x : g(x) = -\infty\})$ and similarly $\{x : -\infty < f(x) < \infty\}$ are measurable.

$$\begin{aligned}
D_1 &:= \{x : (f+g)(x) = \infty\} \\
&= [\{x : f(x) = \infty\} \cap \{x : -\infty < g(x) < \infty\}] \\
&\cup [\{x : -\infty < f(x) < \infty\} \cap \{x : g(x) = \infty\}] \\
&\cup [\{x : f(x) = \infty\} \cap \{x : g(x) = \infty\}]
\end{aligned}$$

$$\begin{aligned}
D_2 &:= \{x : (f+g)(x) = -\infty\} \\
&= [\{x : f(x) = -\infty\} \cap \{x : -\infty < g(x) < \infty\}] \\
&\cup [\{x : -\infty < f(x) < \infty\} \cap \{x : g(x) = -\infty\}] \\
&\cup [\{x : f(x) = -\infty\} \cap \{x : g(x) = -\infty\}]
\end{aligned}$$

Since $\mathfrak{M}$ is a σ-algebra, D_1 and D_2 are measurable. By part (a), if we can show that the restriction of f to $D\backslash(D_1 \cup D_2)$, $h(x) := (f+g)|_{D\backslash(D_1\cup D_2)}(x)$ is measurable then we can conclude the extended real-valued function $f+g : D \to \overline{\mathbb{R}}$ is measurable. Let $E := D\backslash(D_1 \cup D_2)$. Let $\beta \in \mathbb{R}$ be arbitrary. With Proposition 18 (iii) in mind, we want to show that the set $\{x \in E : h(x) := (f+g)(x) < \beta\}$ is measurable. Since α is fixed, consider whether $\alpha < \beta$ or $\alpha \geq \beta$. Let $F = E \cap \{x : -\infty < f(x) < \infty\} \cap \{x : -\infty < g(x) < \infty\} \in \mathfrak{M}$. For $x \in F$, $f(x)$ and $g(x)$ are both finite so that $h(x) = (f+g)(x)$ is a measurable function by Proposition 19. So the set $\{x \in F : h(x) < \beta\}$ is measurable.

$$\begin{aligned}
\{x \in E : h(x) < \beta\} &= \{x \in F : h(x) < \beta\} \\
&\cup [\{x \in E : f(x) = \infty\} \cap \{x \in E : g(x) = -\infty\}] \\
&\cup [\{x : f(x) = -\infty\} \cap \{x : g(x) = \infty\}] \in \mathfrak{M}, \quad \alpha < \beta \\
\{x \in E : h(x) < \beta\} &= \{x \in F : h(x) < \beta\} \in \mathfrak{M}, \quad \alpha \geq \beta\,.
\end{aligned}$$

Note that the sets above of a form like $\{x \in E : f(x) = \infty\} = \{x \in D : f(x) = \infty\} \cap D_1^c \cap D_2^c$ are indeed measurable and that it is necessary to mention these cases as $D_1 \cup D_2$ does not include all possible instances where f and g are infinite. Since β was arbitrary, conclude that the restriction of $f+g$ to $D\backslash(D_1 \cup D_2)$ is measurable and so by part (a), the extended real valued function $f+g$ is measurable.

(d) Let $f, g : D \to \overline{\mathbb{R}}$ be measurable extended real-valued functions such that f and g are each finite almost everywhere. Show that $f+g$ is measurable no matter how it is defined at points where it is of the form $\infty - \infty$ (and presumably $-\infty + \infty$).

$$\begin{aligned}
&C_1 := \{x : f(x) = \infty\} \cup \{x : f(x) = -\infty\} \\
&C_1 := \{x : g(x) = \infty\} \cup \{x : g(x) = -\infty\} \\
&m(C_1) = m(C_2) = 0 \text{ by hypothesis.} \\
&0 \leq m(C_1 \cup C_2) \leq m(C_1) + m(C_2) = 0 \implies m(C_1 \cup C_2) = 0 \\
&B := [\{x : f(x) = \infty\} \cap \{x : g(x) = -\infty\}] \cup [\{x : f(x) = -\infty\} \cap \{x : g(x) = \infty\}]
\end{aligned}$$

The set B is the set of points at which $f+g$ is of the form $\infty - \infty$ or $-\infty + \infty$. Let $f+g$ be defined arbitrarily at points in B. To see that $B \subset C_1 \cup C_2$, let $y \in B$. If $y \in \{x : f(x) = \infty\} \cap \{x : g(x) = -\infty\}$, then $f(y) = \infty$ and $g(y) = -\infty$. So $y \in C_1$ and $y \in C_2$ and $y \in C_1 \cap C_2 \subset C_1 \cup C_2$. If $y \in \{x : f(x) = -\infty\} \cap \{x : g(x) = \infty\}$ it follows similarly that $y \in C_1 \cup C_2$. This implies that $m(B) = 0$. Define $h : D \to \overline{\mathbb{R}}$,

$$h(x) = \begin{cases} (f+g)(x) & x \in B^c \\ 27 & x \in B \end{cases}.$$

Then h is measurable by part (c) and the set of points at which $h \neq f+g$ has measure zero. That is, h is a measurable function and $h = f+g$ almost everywhere. By Proposition 21, conclude that $f+g$ is measurable.