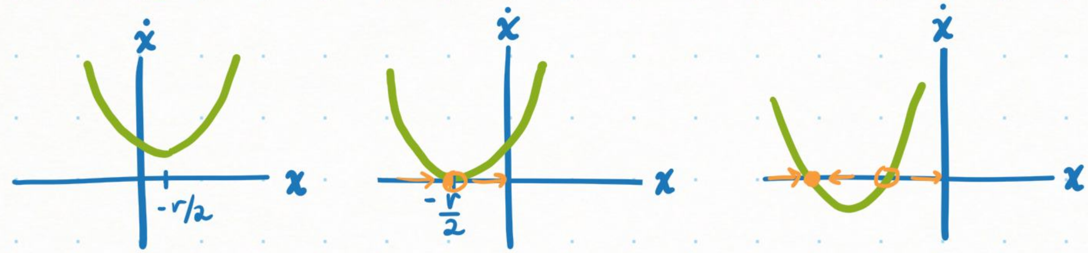
3.1.1

$\dot{x} = 1 + rx + x^2$

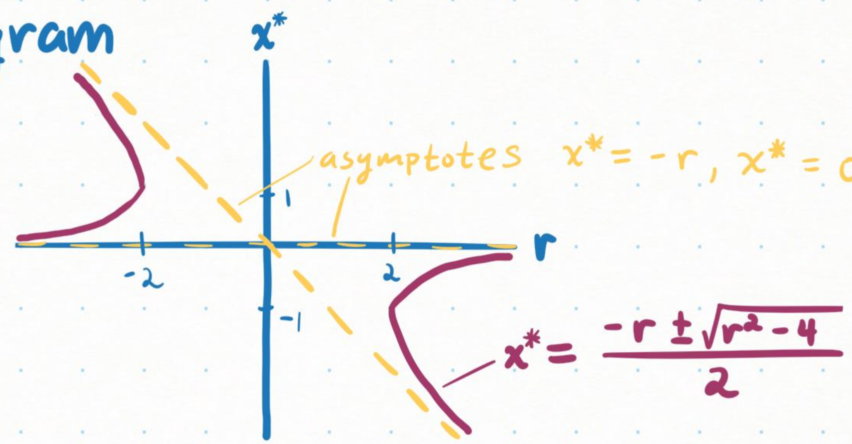
$x^* = \frac{-r \pm \sqrt{r^2 - 4}}{2}$ for $|r| > 2$, $x^* = -\frac{r}{2} = -1$ if $|r| = 2$. No fixed points for $|r| < 2$. There are saddle node bifurcations at $r_c = \pm 2$.



The 3 qualitatively different cases.

$\lvert r \rvert < 2$	$\lvert r \rvert = 2$	$\lvert r \rvert > 2$
No x^*	1 semistable x^*	1 stable x_1^* 1 unstable $x_2^* > x_1^*$

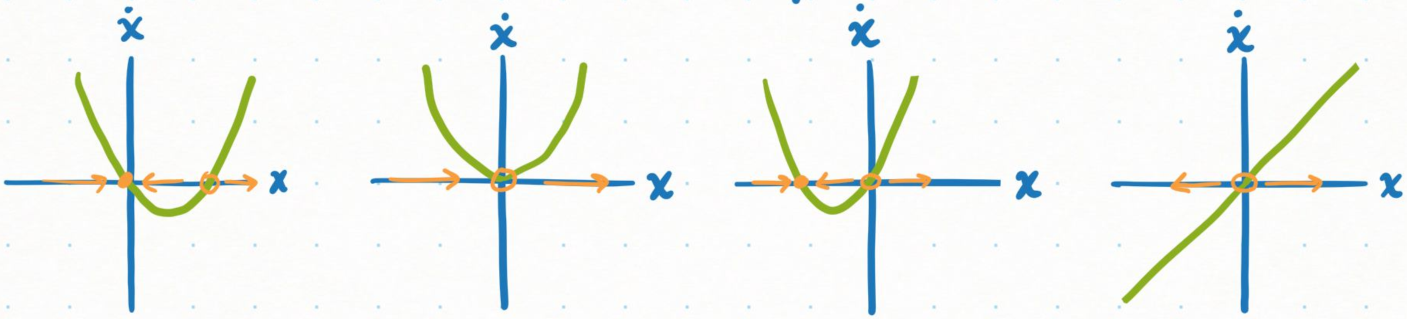
Bifurcation Diagram



Since the number of fixed points changes as r is varied, the bifurcations at $r_c = \pm 2$ are saddle node bifurcation.

3.2.3 $\dot{x} = x - rx(1-x)$

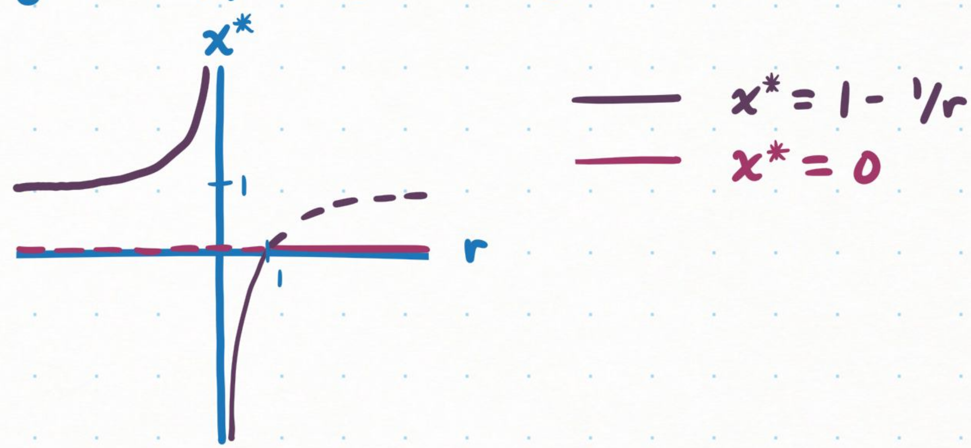
$$0 = x(1 - r(1-x)) \longrightarrow x^* = 0, \ \frac{r-1}{r} \ (r \neq 0)$$



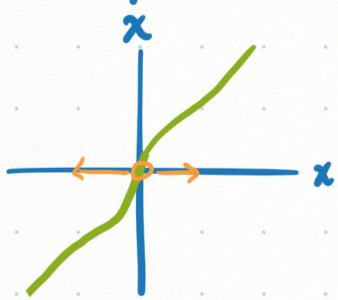
$r > 1$ | $r = 1$ | $r < 1, r \neq 0$ | $r = 0$

There is a transcritical bifurcation at $r_c = 1$. The fixed point $x^* = 0$ is stable for $r > 1$ and unstable for $r \leq 1$ (semistable for $r = 1$). The fixed point $x = (r-1)/r$ (when it exists and is distinct from 0) is stable for $r < 1$ and unstable for $r > 1$.

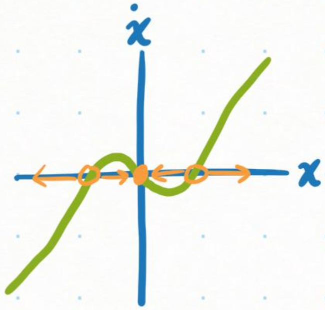
Bifurcation Diagram



3.4.4 $\dot{x} = x + \frac{rx}{1+x^2}$ $\qquad 0 = x\left(1+\frac{r}{1+x^2}\right) \longrightarrow x^* = 0, \pm\sqrt{-1-r} \quad (r<-1)$



$r \geq -1$



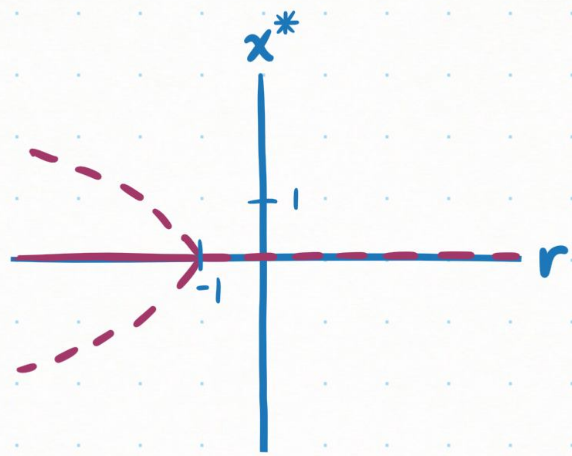
$r < -1$

For $r \geq -1$ there is one unstable fixed point $x^* = 0$.

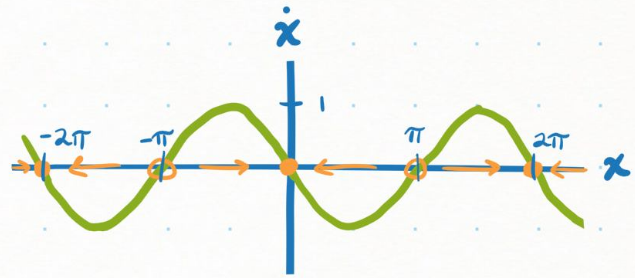
For $r<-1$ there is one stable fixed point $x = 0$ and two unstable fixed points $x^* = \pm\sqrt{-1-r}$.

Since the fixed point at zero becomes stable as two unstable fixed points appear as r is varied, there is a subcritical pitchfork bifurcation at $r_c = -1$.

Trifurcation Diagram



<u>3.4.11</u> $\dot{x} = rx - \sin x$

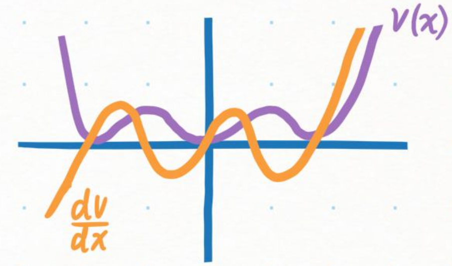


a) $r = 0 \rightarrow \dot{x} = -\sin x$, $x^* = n\pi$, $n \in \mathbb{Z}$.
$x^* = n\pi$ is stable for even n and unstable for odd n.

b) For $r > 1$, we have $rx = \sin x$ at $x = 0$ but since $\frac{d}{dx} rx = r > 1$ and $|{}^{d}/_{dx} \sin x| = |\cos x| \leq 1$, $rx \neq \sin x$ for any $x \neq 0$. This means the only fixed point is $x^* = 0$. Since $\dot{x} > 0$ for $x > 0$ and $\dot{x} < 0$ for $x < 0$, $x^* = 0$ is unstable.

c) By graphical analysis, there is a subcritical pitchfork bifurcation at $r_c = 1$ and then infinitely many saddle node bifurcations as $r \rightarrow 0$ into the set of fixed points described in part a.

3.4.15 $\dot{x} = f(x) = rx + x^3 - x^5, \quad f(x) = -\frac{dV}{dx}$



$$\frac{dV}{dx} = x^5 - x^3 - rx$$
$$V(x) = \tfrac{1}{6}x^6 - \tfrac{1}{4}x^4 - \tfrac{r}{2}x^2 \quad (+c = 0 \text{ by convention})$$
$$0 = V(x) = x^2\left(\tfrac{1}{6}x^4 - \tfrac{1}{4}x^2 - \tfrac{r}{2}\right)$$

implies $V(x)$ has a double root at $x=0$. Since the coefficient of x^6 is positive, $V(x)$ opens upward. So $x=0$ is a well of $V(x)$ and $V(0)=0$. This means the remaining roots of $V(x)$ are double roots and wells of $V(x)$. Thus the discrimant must be 0 in:

$$x^2 = \frac{1/4 \pm \sqrt{1/16 + r/3}}{1/3}$$

That is, $0 = 1/16 + r/3 \longrightarrow r_c = -3/16$.

3.5.7 $\dot{N} = rN(1 - N/K), \quad N(0) = N_0$

$$x(0) = \tfrac{1}{K}N(0) = \tfrac{1}{K}N_0$$

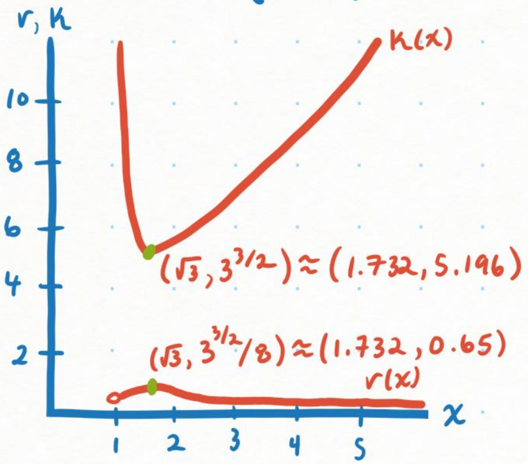
a) $\dot{N}$: organisms / time unit $\rightarrow$ r : (time unit)$^{-1}$, N_0, K : organisms

b) Let $x = N/K$, $x_0 = N_0/K$, $\tau = rt$. Each of x, x_0, τ are dimensionless by part a.

$$\left.\begin{aligned} K\frac{dx}{dt} &= \dot{N} \\ \frac{dx}{d\tau} &= \frac{dx}{dt}\frac{dt}{d\tau} = \frac{1}{r}\frac{dx}{dt} \end{aligned}\right\} \longrightarrow \begin{aligned} K\frac{dx}{dt} &= rKx(1-x), \quad x(0) = N(0)/K \\ rK\frac{dx}{d\tau} &= rKx(1-x), \quad x(0) = N_0/K \\ \frac{dx}{d\tau} &= x(1-x), \quad x(0) = x_0 \end{aligned}$$

3.7.2 $r(x) = \dfrac{2x^3}{(1+x^2)^2}$, $K(x) = \dfrac{2x^3}{x^2-1}$, $x > 1$

a)



$\lim_{x\to\infty} K(x) = \lim_{x\to 1^+} K(x) = \infty$

$\lim_{x\to\infty} r(x) = 0$, $\lim_{x\to 1} r(x) = 1/2$

b) $0 = \dfrac{d}{dx}K(x) = \dfrac{6x^2(x^2-1) - 2x^3(2x)}{(x^2-1)^2} \rightarrow x = 0, \pm\sqrt{3}$

$0 = \dfrac{d}{dx}r(x) = \dfrac{6x^2(1+x^2)^2 - 2x^3(4x(1+x^2))}{(1+x^2)^4} \rightarrow x = 0, \pm\sqrt{3}$

The cusp point is the point where both $r(x)$ and $K(x)$ reverse their direction in the K, r plane. This corresponds to the value of x where $K(x), r(x)$ have a local minimum / local maximum respectively for $x > 1$. Thus $x = \sqrt{3}$, $K(\sqrt{3}) = 3^{3/2}$, $r(x) = 3^{3/2}/8$.