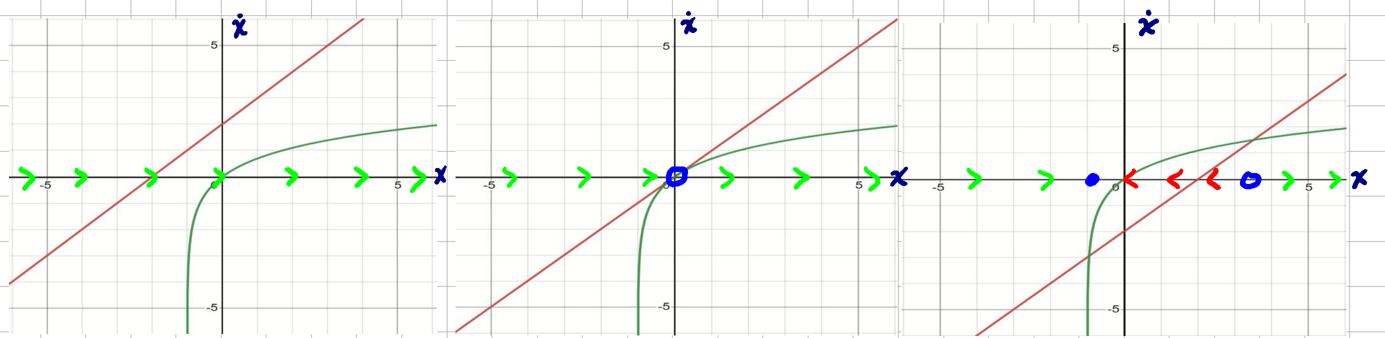
1. (Problem 3.1.3 in Strogatz) Consider the following non-dimensional equation

$$\dot{x} = r + x - \ln(1+x)$$

a) Sketch all qualitatively different phase portraits – remember to label fixed points, indicate their stability, and indicate the flow direction on the horizontal axis. [Note: "phase portrait" is a generic term for what I've been calling the phase line in class].
b) Show that a saddle-node bifurcation occurs at some critical value of r.
c) Sketch the bifurcation diagram.

Legend:
— $y = r + x$
— $y = \ln(1+x)$
•, o x^*

a)



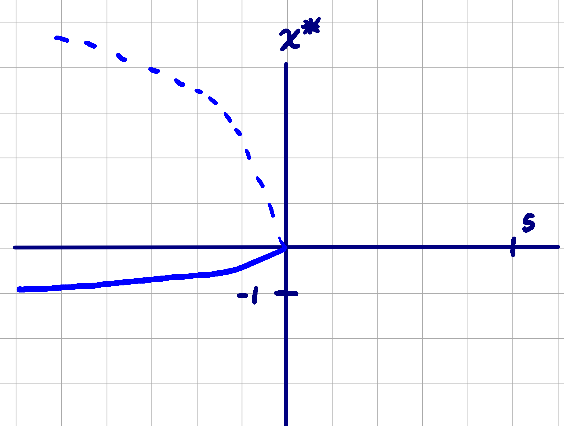
$r > 0$ No fixed points

$r = 0$ One semistable (unstable) fixed point $x^* = 0$

$r < 0$ One stable fixed point $x^* < 0$ and one unstable fixed point $x^* > 0$.

b) For $r > 0$, $r + x - \ln(1+x) > 0$ for all x, so there are no fixed points. For $r = 0$, $r + x - \ln(1+x) = 0$ when $x = 0$. This fixed point is unstable. For $r < 0$, there are two fixed points. One of these fixed points $x^* < 0$ is stable. The other fixed point $x^* > 0$ is unstable. So we transition from no fixed points to a pair of fixed points with opposite stability as r is varied. This describes a saddle node bifurcation.

c)



stable x^*: —
unstable x^*: - - -

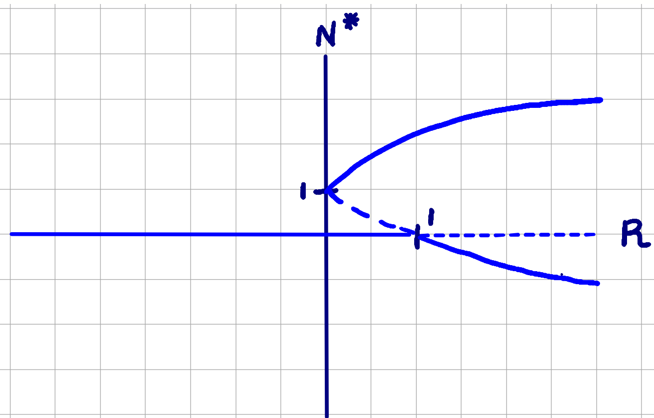
The stable fixed point approaches -1 as $r \to -\infty$ and the unstable fixed point approaches ∞ as $r \to -\infty$.

2. The following non-dimensional equation models population growth

$$\dot{N} = RN - N(1-N)^2 \tag{1}$$

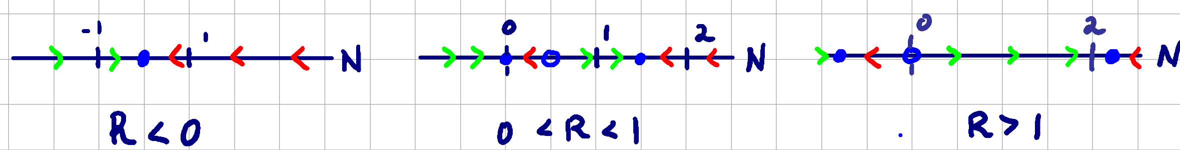
a) Draw a bifurcation diagram for this equation as R varies.
b) At each bifurcation, the system's qualitative dynamics change, so that the phase portrait differs from one side of the bifurcation to the other. Identify regions with similar dynamics and sketch a phase portrait for each region. [Note: "phase portrait" is a generic term for what I've been calling the phase line in class].
c) Pick one region and sketch several trajectories, $N(t)$, for several different initial conditions. On your plot of $N(t)$, indicate the fixed points.
d) Use Matlab to check your answer to part c. (You don't need to turn this in – but it's a good idea to use Matlab to check your work when you can).

a)



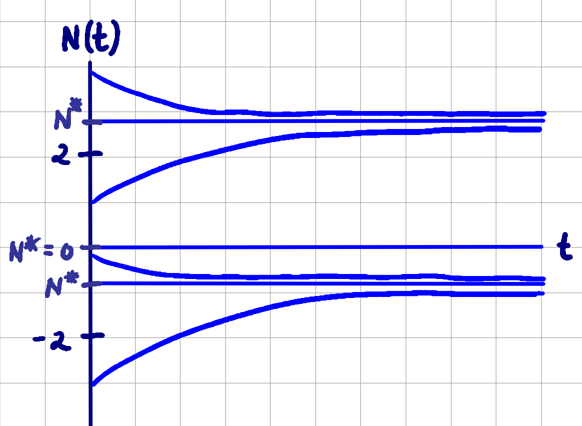
Fixed points:
—— (stable)
- - - - (unstable)

b)



• : stable fixed point   o : unstable fixed point

c)



Trajectories $N(t)$ for several initial conditions. Picked the region $R > 1$ (using $R = 2$ in particular for this sketch). Three fixed points $N^* > 2$ (stable), $N^* = 0$ (unstable), $N^* < 0$ (stable) is typical of this region ($R > 1$).

d) Use Matlab to check part c.

3. Consider the equation

$$\dot{x} = x^{1/3} \tag{2}$$

a) Using Matlab, perform the following two simulations:
i. Simulate this equation, running time backwards from $t = 1.52$ to $t = 0$, starting from $x(t = 1.52) = 1$.
ii. Simulate this equation, running time forwards from $t = 0$ to $t = 1.52$, starting from $x(t = 0) = 0$.
b) If you've done the simulations correctly (or at least in the same way that I did), then the trajectories cross. In class, I claimed that trajectories cannot cross. Explain this apparent contradiction.
c) Calculate the exact solution for the simulation in part i. Is this solution unique? How does it differ from the computed solution? Explain the source of any differences.
d) Calculate the exact solution for the simulation in part ii. Is this solution unique? How does it differ from the computed solution? Explain the source of any differences.

a) ✓

b) The existence and uniqueness theorem states that for the IVP $\dot{x} = f(x)$, $x(0) = x_0$, that if $f(x)$ and $f'(x)$ are continuous on an open interval R of the x-axis and $x_0 \in R$, the IVP has a solution $x(t)$ on some interval $(-\tau, \tau)$ about $t = 0$ and that solution is unique. Here $f(x) = x^{1/3}$ is continuous everywhere but since $f'(x) = \frac{1}{3}x^{-2/3}$ is not defined at $x = 0$, f' cannot be continuous at $x = 0$. The theorem cannot be applied to guarantee uniqueness in any $(-\tau, \tau)$. Although $x^* = 0$ is a fixed point, it is very unstable since the slope of $f(x)$ becomes infinite as $x \to 0$. While $x(t) \equiv 0$ is one solution with $x(0) = 0$, so is $x(t) = \left(\frac{2}{3}t\right)^{3/2}$. The solution with $x(0) = 1$ also doesn't even have the same IC.

c) $\dot{x} = x^{1/3}$, $x(0) = 1$

$\int x^{-1/3}\, dx = \int dt$

$\frac{3}{2}x^{2/3} = t + C$

$\frac{3}{2}1^{2/3} = 1.52 + C$

$-0.02 = C$

$x(t) = \left[\frac{2}{3}(t - .02)\right]^{3/2}$

Since there exists a neighborhood R of $x = 1$ (take any $R = (a, b)$ with $a > 0$ and $b > 1$) for which $f(x)$ and $f'(x)$ are both continuous, we know there must exist some interval $(-\tau, \tau)$ containing $t = 0$ on which this solution is unique.

Does this differ from the computed solution. If so, why?
Need to double check but I saw a small difference between this and what ode45 gave.

d) $\dot{x} = x^{1/3}$, $x(0) = 0$

$x(t) \equiv 0$ is a solution (the solution ode45 gives in Matlab). Also,

$\int x^{-1/3}\, dx = \int dt$

$\frac{3}{2}x^{2/3} = t + C$

$\frac{3}{2}0^{2/3} = 0 + C$

$x(t) = \left(\frac{2}{3}t\right)^{3/2}.$

Does this differ from the computed solution. If so, why?

4. In class (and on Homework 1), we've been discussing the non-dimensional form of an equation modeling the production of a protein:

$$\frac{d\hat{p}}{d\hat{t}} = -\hat{p} + \alpha \frac{\hat{p}^2}{\hat{p}^2 + 1^2}$$

a) Suppose $\alpha = 4$. Linearize about each fixed point to determine stability. Which one is the most stable (i.e., where do small perturbations decay the fastest)?
b) For how big an initial perturbation is the linear approximation good? Use Matlab to explore this. To do so, you'll need to define what a "good" approximation is (e.g., error less than 1%, 0.1%), and then use Matlab to calculate the difference between the "exact" solution, generated by Matlab, and the linear approximation.

a) $0 = -\hat{p} + 4\frac{\hat{p}^2}{\hat{p}^2+1}$

$\hat{p}(\hat{p}^2+1) = 4\hat{p}^2$

$\hat{p}(\hat{p}^2 - 4\hat{p} + 1) = 0$

$\hat{p}^* = 0, \ \hat{p}^* = 2 \pm \sqrt{3}$

Linearization about a fixed point x^*:
Let $\eta = x(t) - x^*$ be a small perturbation away from x^*. Then

$$\dot{\eta} = \dot{x} = f(x^* + \eta) = \eta f'(x^*) + \mathcal{O}(\eta^2) \approx \eta f'(x^*)$$

for $f'(x^*) \neq 0$. Perturbations grow/decay exponentially if $f'(x^*) > 0$, $f'(x^*) < 0$ respectively.

$$f'(\hat{p}) = -1 + 4\frac{2\hat{p}(\hat{p}^2+1) - \hat{p}^2(2\hat{p})}{(\hat{p}^2+1)^2} = -1 + \frac{8\hat{p}}{(\hat{p}^2+1)^2}$$

$f'(0) = -1, \ f'(2+\sqrt{3}) = -\frac{\sqrt{3}}{2} \approx -0.866, \ f'(2-\sqrt{3}) = \frac{\sqrt{3}}{2} \approx 0.866$.

Linear stability analysis shows that $\hat{p}^* = 2-\sqrt{3}$ is unstable, $\hat{p}^* = 2+\sqrt{3}$ is stable, and $\hat{p}^* = 0$ is stable. Since $f'(0) < f'(2+\sqrt{3})$, $\hat{p}^* = 0$ is more stable than $\hat{p}^* = 2+\sqrt{3}$ in the sense that small perturbations decay faster. Both perturbations decay exponentially but the exponent is more negative for perturbations near 0.

b) Linearized equation: $\frac{d\hat{p}}{d\hat{t}} \approx f(\hat{p}^*) + f'(\hat{p}^*)(\hat{p} - \hat{p}^*) = f'(\hat{p}^*)(\hat{p} - \hat{p}^*)$ for a fixed point $\hat{p}^*$.

For $\hat{p}^* = 0$, $\frac{d\hat{p}}{d\hat{t}} \approx -(\hat{p} - \hat{p}^*) = -\hat{p}$

For $\hat{p}^* = 2+\sqrt{3}$, $\frac{d\hat{p}}{d\hat{t}} \approx -\frac{\sqrt{3}}{2}(\hat{p} - 2 - \sqrt{3})$

For $\hat{p}^* = 2-\sqrt{3}$, $\frac{d\hat{p}}{d\hat{t}} \approx \frac{\sqrt{3}}{2}(\hat{p} - 2 + \sqrt{3})$

Needs Matlab comparison still.

Worksheet 2

1. $0 = \zeta\dot{\theta} + K\theta - mgl\sin\theta \longrightarrow \frac{d\hat{\theta}}{d\tau} = -\beta\hat{\theta} + \sin\hat{\theta}$

Let $\tau = t/T$, T to be chosen later.

$$\frac{d\theta}{dt} = \frac{d\theta}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\theta}{d\tau}$$

Substitute into $0 = \zeta\dot{\theta} + K\theta - mgl\sin\theta$

$$0 = \zeta/T \frac{d\theta}{d\tau} + K\theta - mgl\sin\theta$$

$$\zeta/T \frac{d\theta}{d\tau} = -K\theta + mgl\sin\theta$$

$$\zeta/(mglT)\frac{d\theta}{d\tau} = -\frac{K}{mgl}\theta + \sin\theta$$

$$\zeta/(mglT)\frac{d\theta}{d\tau} = -\beta\theta + \sin\theta \quad , \quad \beta := \frac{K}{mgl}$$

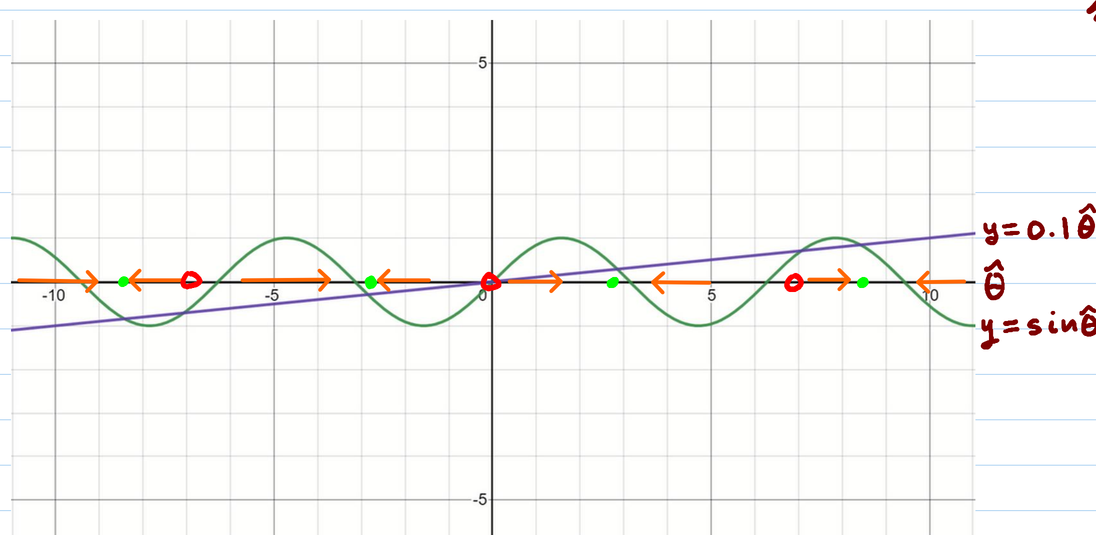
$$\zeta\beta/(KT)\frac{d\theta}{d\tau} = -\beta\theta + \sin\theta$$

$$\frac{d\theta}{d\tau} = -\beta\theta + \sin\theta \quad , \quad T := \zeta\beta/K = \frac{\zeta}{mgl}$$

Then $\tau = t/T = tmgl/\zeta$. To match the worksheet, $\hat{t} = \tau$ and $\hat{\theta}(\hat{t}) := \theta(t/T)$.

$$\boxed{\frac{d\hat{\theta}}{d\hat{t}} = -\beta\hat{\theta} + \sin\hat{\theta} \quad , \quad \beta = \frac{K}{mgl} \quad , \quad \hat{t} = tmgl/\zeta}$$

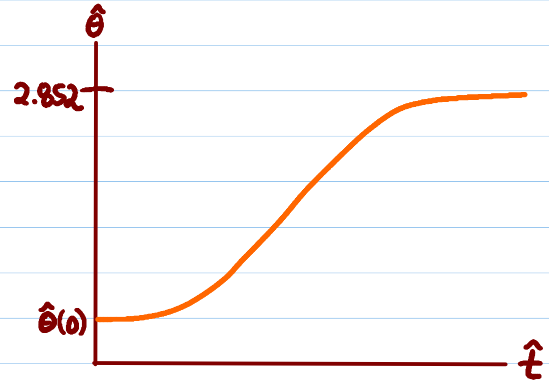
2. a) $\beta = 0.1$, $\frac{d\hat{\theta}}{d\hat{t}} = -0.1\hat{\theta} + \sin\hat{\theta}$



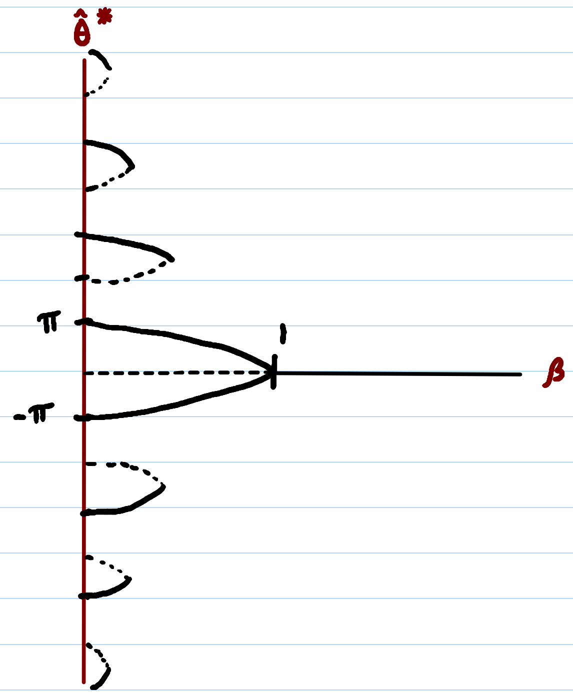
Fixed points at $\hat{\theta}^* \approx -8.423$, -7.068, -2.852, 0 (exact),
stable, unstable, stable, unstable

2.852, 7.068, 8.423
stable, unstable, stable

b)



3.



4. $\dot{\theta} = -\beta\theta + \sin\theta$ (θ for $\hat{\theta}$ for convenience)

$\dot{\theta} \approx -\beta\theta + \theta$ for θ near 0.

$\dot{\theta} \approx (1-\beta)\theta$

$\theta(t) = \varepsilon(t)$, $\beta = 1-\delta$ ($\delta = 1-\beta$)

$\dot{\varepsilon} = \dot{\theta} = \delta\varepsilon - \frac{1}{3!}\varepsilon^3 + \mathcal{O}(\varepsilon^5)$

5. For $K \ll 1$ and fixed m, ℓ we have $\beta = \frac{K}{mg\ell} \approx 0$. Then, $\dot{\theta} \approx \sin\theta$ and $\theta^* = 0$ (vertical starting position) is unstable. As K increases, this position becomes "less unstable" in the sense that f' evaluated at 0 becomes less positive meaning perturbations grow at a slower (although still exponential rate). Once $K > mg\ell$, the vertical starting position ($\theta^* = 0$) becomes a stable fixed point: once the spring is stiff enough, the system returns to this position under small perturbations.