Chapter 2: Flows on the Line

2.1.1 $\dot{x} = \sin x$

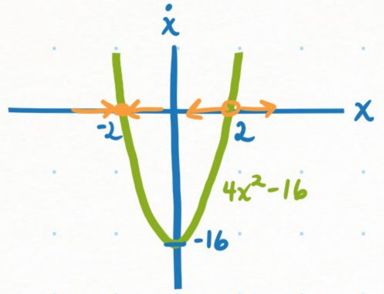
The fixed points of the flow are the points x^* s.t. $\dot{x} = 0$.
$\dot{x} = 0$ when $x^* = n\pi$, $n \in \mathbb{Z}$.

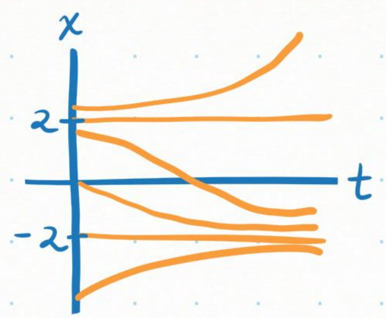
2.1.3

a) $\ddot{x} = \frac{d}{dt}\dot{x} = \dot{x}\cos x = \sin x \cos x = \frac{1}{2}\sin 2x$.

b) The flow has maximum positive acceleration at the points x that maximize $\ddot{x} = \frac{1}{2}\sin 2x$. That is, $x = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$.

2.2.1 $\dot{x} = 4x^2 - 16$





Fixed points:
$x^* = 2$ is unstable
$x^* = -2$ is stable

2.2.10

a) $\dot{x} \equiv 0$ has a fixed point at every real number.

b) $\dot{x} = \sin(\pi x)$. Since $\sin(\pi x) = 0$ iff $\pi x = n\pi$ for $n \in \mathbb{Z}$, $x^* = n \in \mathbb{Z}$.

c) Not possible for $\dot{x} = f(x)$ with f smooth. Suppose $x_1 < x_2 < x_3$ are three fixed points. In order for all three points to be stable, $\dot{x}$ must change sign five times. This gives five fixed points. Since this contradicts the assumption of only three fixed points, at least one of x_1, x_2, x_3 must not be stable. Between any two stable fixed points, there must be an unstable fixed point.

d) $\dot{x} \equiv 1$ has no fixed points since $\dot{x}$ is always nonzero.

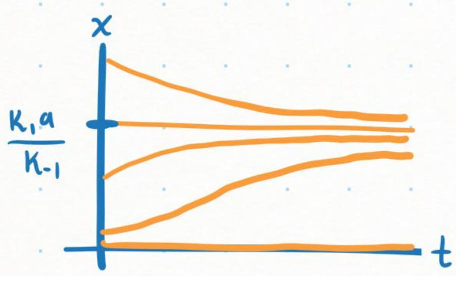
e) $\dot{x} = \prod_{k=1}^{100} (x-k)$ has exactly 100 fixed points at $x^* = 1, 2, \ldots, 100$.

2.3.2

$\dot{x} = k_1 a x - k_{-1} x^2$, rate constants $k_1, k_{-1} > 0$, $a > 0$

a) $0 = \dot{x} = x(k_1 a - k_{-1} x) \longrightarrow x^* = 0,\ k_1 a / k_{-1}$.
$k_1 a x - k_{-1} x^2$ is a parabola opening down with $k_1 a / k_{-1} > 0$.
$\therefore\ x^* = 0$ is unstable and $x^* = k_1 a / k_{-1}$ is stable.

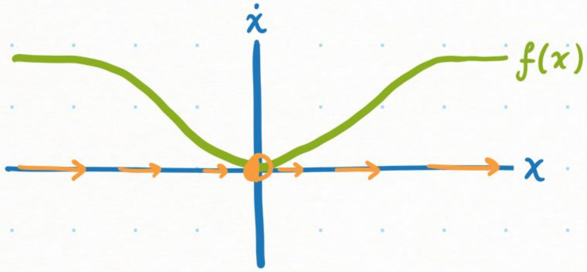
b)



2.4.5 $\dot{x} = f(x) = 1 - e^{-x^2}$

$x^* = 0$

$f'(x) = 2xe^{-x^2}$, $f'(x^*) = 0$ linear stability analysis fails.



By graphical analysis, $x^* = 0$ is a semistable fixed point (stable for $x_0 \leq 0$ and unstable for $x_0 > 0$).

2.4.6 $\dot{x} = f(x) = \ln x$

$x^* = 1$

$f'(x) = 1/x$, $f'(x^*) = 1 > 0 \implies x^* = 1$ is unstable.

2.5.1 $\dot{x} = -x^c$, $x \geqslant 0$, $c \in \mathbb{R}$

a) Since we only consider $x \geqslant 0$, $x = 0$ is a stable fixed point for any $c > 0$ because $-0^c = 0$ and $-x^c < 0$ for $x > 0$. For $c \leq 0$, $x = 0$ is not a fixed point since $-x^c \neq 0$ for any x.

b) For $c > 0$, $\int -x^{-c}\,dx = \int dt \rightarrow \frac{1}{c-1} x^{1-c} = t + K$, $K \in \mathbb{R}$. Suppose $x(t_1) = 1$ and $x(t_0) = 0$. The time for a particle to travel from $x = 1$ to $x = 0$ is $t_0 - t_1 = 0 - \frac{1}{c-1} 1^{1-c} = (1-c)^{-1}$.

2.6.2 $x(t) = x(t+T),\ T>0$ and $x(t) \neq x(t+s)\ \forall\ 0<s<T,\quad \dot{x} = f(x)$

$$\int_t^{t+T} f(x)\frac{dx}{dt}\,dt = \int_t^{t+T} (\dot{x})^2\,dt > 0$$ assuming $\dot{x} \not\equiv 0$. Yet also

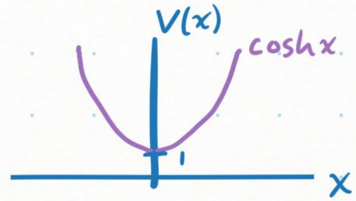
$$\int_t^{t+T} f(x)\frac{dx}{dt}\,dt = \int_{x(t)}^{x(t+T)} f(x)\,dt = 0$$ since $x(t) = x(t+T)$.

$\therefore$ $x(t)$ cannot be periodic.

[Substitution $u = \varphi(t)$:
$$\int_a^b g(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} g(u)\,du$$]

2.7.5 $\dot{x} = f(x) = -\sinh x$

$$f(x) = -\frac{dV}{dx} \rightarrow V(x) = \cosh x$$



$x^* = 0$ is a stable fixed point V has a local minimum at 0 (pg 31).

2.8.1 The slope is constant along horizontal lines because the slope, $\dot{x}$, is an autonomous equation – the value of $\dot{x}$ is determined by x and is time independent.

2.8.7

Exact $x(t_1) \equiv x(t_0 + \Delta t)$ Euler estimate $x_1 = x_0 + \Delta t\, f(x_0)$

a) $x(t_1) = x(t_0 + \Delta t) = x(t_0) + \dot{x}(t_0)\Delta t + \frac{1}{2}\ddot{x}(t_0)(\Delta t)^2 + \mathcal{O}((\Delta t)^3)$

$= x_0 + \Delta t\, f(x_0) + \frac{1}{2}(\Delta t)^2 f'(x_0) f(x_0) + \mathcal{O}((\Delta t)^3)$

using $\ddot{x}(t) = \frac{d}{dt}\dot{x}(t) = \frac{d}{dt} f(x(t)) = f'(x(t))\,\dot{x}(t) = f'(x) f(x)$

b) $|x(t_1) - x_1| = \left|\frac{1}{2}(\Delta t)^2 f'(x_0) f(x_0) + \mathcal{O}((\Delta t)^3)\right| \sim C((\Delta t)^2)$, $C = \frac{1}{2}|f'(x_0) f(x_0)|$