Chapter 2 Flows on the Line

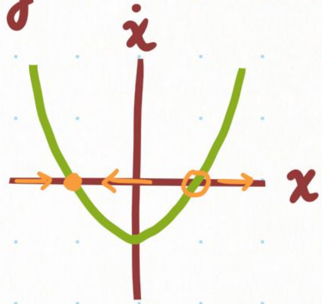
§ 2.2 Fixed Points and Stability

Ex 2.2.1 Find all fixed points of $\dot{x} = f(x) = x^2 - 1$ and classify their stability.

The fixed points are x^* s.t. $0 = f(x^*)$.

$$0 = (x^*)^2 - 1$$
$$x^* = \pm 1$$

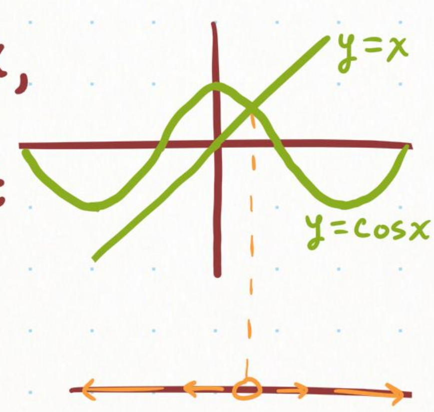
Their stability can be analyzed graphically:



The flow is to the right where $x^2 - 1 > 0$ and to the left where $x^2 - 1 < 0$. This means $x^* = -1$ is stable and $x^* = 1$ is unstable.

Ex 2.2.3 Sketch the phase portrait for $\dot{x} = x - \cos x$.

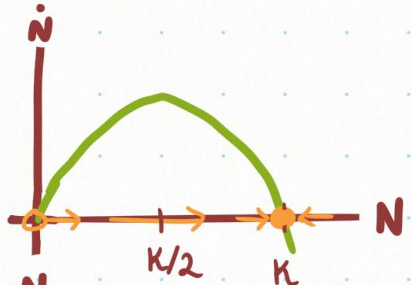
Instead of graphing $x - \cos x$, graph x, $\cos x$ on the same axes. The flow is to the right where $x > \cos x$ and to the left where $x < \cos x$. This shows there is one fixed point x^* where $x^* = \cos x^*$ and this fixed point is unstable.



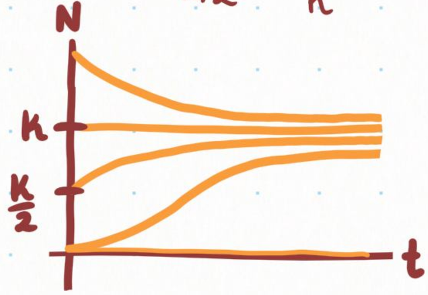
§ 2.3 Population Growth

Let $N(t)$ be the number of organisms at time $t \geq 0$. Assuming the per capita growth rate $\dot{N}/N$ decreases linearly with N leads to the logistic equation

$$\dot{N} = rN(1 - N/K) \qquad r: \text{growth rate}$$
$$t \geq 0 \qquad K: \text{carrying capacity}$$
$$N \geq 0$$



Unstable fixed point $N^* = 0$, Stable fixed point $N^* = K$, Maximum positive flow at $N = K/2$, $\dot{N} = rK/4$.



If $N_0 = N(0) > K$, $N(t)$ decreases asymptotically to K. If $N_0 < K$, $N(t)$ increases asymptotically to K. Note that for $0 < N_0 < K/2$, $N(t)$ has an inflection point when $N(t) = K/2$. If $N_0 = 0$ or $N_0 = K$, $N(t)$ is constant (at equilibrium).

§ 2.4 Linear Stability Analysis

Let x^* be a fixed point and $\eta(t) = x(t) - x^*$ a small perturbation away from x^*.

$$\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta) = f(x^*) + \eta f'(x^*) + \mathcal{O}(\eta^2)$$

Since $f(x^*) = 0$, $\dot{\eta} = \eta f'(x^*) + \mathcal{O}(\eta^2) \approx \eta f'(x^*)$ for $f'(x^*) \neq 0$. This linearization shows that the perturbation η grows exponentially if $f(x^*)>0$ and decays exponentially if $f(x^*)<0$. So $f(x^*)>0$ means x is unstable and $f(x^*)<0$ means x^* is stable.

If $f(x^*)=0$, the $\mathcal{O}(\eta^2)$ terms are not negligible and linear stability analysis fails. Consider $\dot{x}=-x^3$, $\dot{x}=x^3$, $\dot{x}=x^2$, $\dot{x}=0$, which all have $f'(x^*)=0$.

§ 2.5 Existence and Uniqueness

Existence & Uniqueness Theorem: Consider the initial value problem

$$\dot{x}=f(x), \quad x(0)=x_0$$

Suppose $f(x)$ and $f'(x)$ are continuous on an open interval R of the x-axis and $x_0 \in R$. Then the IVP has a unique solution $x(t)$ on some interval $(-\tau,\tau)$ about the origin.

Ex 2.5.2 Discuss the existence and uniqueness of solutions to $\dot{x}=1+x^2$, $x(0)=x_0$. Do solutions exist for all time?

$f(x)=1+x^2$ and $f'(x)=2x$ are continuous on any open interval R. Unique solutions exist for any x_0, but may not exist for all t. For example, if $x_0=0$, $x(t)=\tan t$ on $-\pi/2<t<\pi/2$.

$$\int \frac{1}{1+x^2}\,dx=\int dt$$

$$\arctan x = t + C$$

$$\arctan x_0 = C$$

$$x(t)=\tan(t+\arctan x_0)$$

§ 2.7 Potentials

For a first order system $\dot{x} = f(x)$, we define the potential $V(x)$ by $f(x) = -dV/dx$.

Since $x = x(t)$,

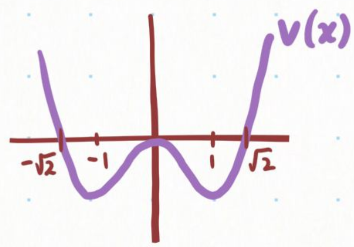
$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx} f(x) = -\left(\frac{dV}{dx}\right)^2 \le 0$$

Thus V decreases along any trajectory $x(t)$ to lower potential. Note $dV/dx = 0$ iff $\dot{x} = 0$, so the equilibria of V occur at fixed points x^*.

The local minima of V correspond to stable fixed points and the local maxima correspond to unstable fixed points.

Ex 2.7.2 Graph the potential for the system $\dot{x} = x - x^3$ and identify all equilibrium points.

$$\frac{dV}{dx} = x^3 - x \longrightarrow V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 \text{ (set } + C = 0\text{)}.$$



The critical points of $V(x)$ are $x^3 - x = 0 \rightarrow x = 0, \pm 1$. $V(x)$ has minima at $x = \pm 1$ with $V(\pm 1) = -1/4$. The stable fixed points of $\dot{x} = x - x^3$ are $x^* = \pm 1$ and the unstable fixed point of $\dot{x} = x - x^3$ is $x^* = 0$.

$V(x)$ shown here is called a double-welled potential and $\dot{x} = x - x^3$ is bistable.