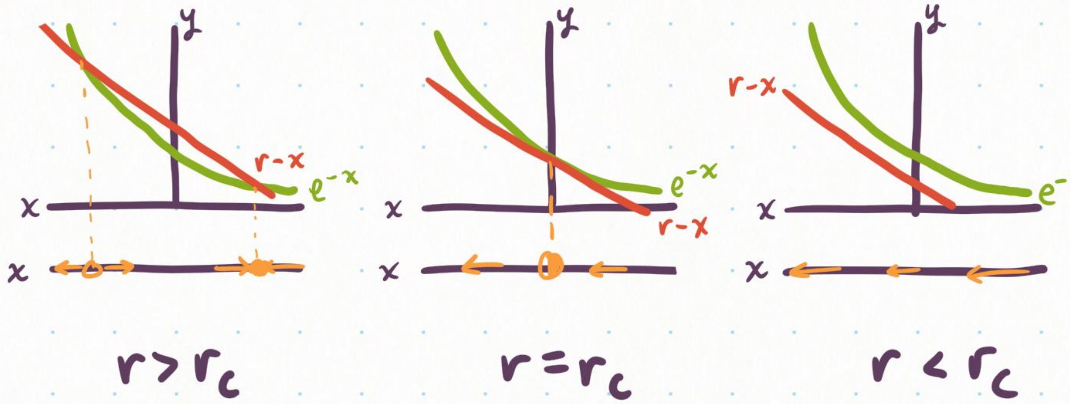
Chapter 3 Bifurcations

§ 3.1 Saddle Node Bifurcation

Fixed points of a system created or destroyed by varying a parameter. Normal Form: $\dot{x} = r + x^2$

Ex 3.1.2 Show that $\dot{x} = r - x - e^{-x}$ undergoes a saddle node bifurcation as r is varied, and find the critical value $r = r_c$ at the bifurcation point.

Sketch the qualitatively different phase portraits for varying r by plotting $r - x$ and e^{-x} on the same set of axes.



To determine r_c, notice by the second phase portrait that at $r = r_c$ and $x = x^*$ we have

$$\frac{d}{dx}(r - x) = \frac{d}{dx} e^{-x} \quad \text{and} \quad r - x = e^{-x}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$-1 = -e^{-x} \qquad\qquad r - 0 = e^{-0}$$

$$x = 0 \longrightarrow \qquad\qquad r = 1$$

$$\therefore r_c = 1$$

§ 3.2 Transcritical Bifurcation

Fixed points of a system change stability as a parameter is varied. Normal Form: $\dot{x} = rx - x^2$

Ex 3.2.1 Show that $\dot{x} = x(1-x^2) - a(1-e^{-bx})$ undergoes a transcritical bifurcation at $x=0$ when the parameters a and b satisfy a certain equation, to be determined.

Note $x=0$ is a fixed point for any a, b values. For small x, we use the Maclaurin Expansion

$$e^{-bx} = 1 - bx + \tfrac{1}{2}b^2x^2 + \mathcal{O}(x^3)$$

to get:

$$\dot{x} = x - x^3 - a\left(bx - \tfrac{1}{2}b^2x^2 + \mathcal{O}(x^3)\right)$$
$$\dot{x} = (1-ab)x + \tfrac{1}{2}ab^2x^2 + \mathcal{O}(x^3)$$

Comparing this to the normal form we see that the system undergoes a transcritical bifurcation when $ab=1$. The nonzero fixed point can be approximated:

$$0 \approx (1-ab) + \tfrac{1}{2}ab^2x^* \longrightarrow x^* = \frac{2(ab-1)}{ab^2}$$

This approximation assumes x^* is near 0 and thus when $ab \approx 1$.

Ex 3.2.2 Analyze $\dot{x} = r\ln x + x - 1$ near $x=1$ and undergoes a transcritical bifurcation at a certain value of r. Find variables X, R s.t. the system reduces approximately to the normal form $\dot{X} \approx RX - X^2$ near the bifurcation.

Note $x=1$ is a fixed point for any r. Let $u = x-1$ so $u \approx 0$ when $x \approx 1$ and $\dot{u} = \dot{x}$.

$$
\begin{aligned}
\dot{u} = \dot{x} &= r\ln(u+1) + u \\
&= r\left(u - \tfrac{1}{2}u^2 + \mathcal{O}(u^3)\right) + u \\
&= (1+r)u - \tfrac{1}{2}ru^2 + \mathcal{O}(u^3)
\end{aligned}
$$

Thus there is a transcritical bifurcation at $r_c = -1$.

To write the system in normal form, we need the quadratic term to have coefficient -1. Let $u = av$ for some $a \in \mathbb{R}$

$$
\begin{aligned}
a\dot{v} = \dot{u} &= (1+r)av - \tfrac{1}{2}ra^2v^2 + \mathcal{O}(v^3) \\
\dot{v} &= (1+r)v - \tfrac{1}{2}rav^2 + \mathcal{O}(v^3)
\end{aligned}
$$

Set $a = 2/r$ so that $\dot{v} = (1+r)v - v^2 + \mathcal{O}(v^3)$
Let $X = v$ and $R = 1+r$ so that

$$
\dot{X} = \dot{v} = RX - X^2 + \mathcal{O}(X^3) \approx RX - X^2
$$

In terms of the original variables r and x,

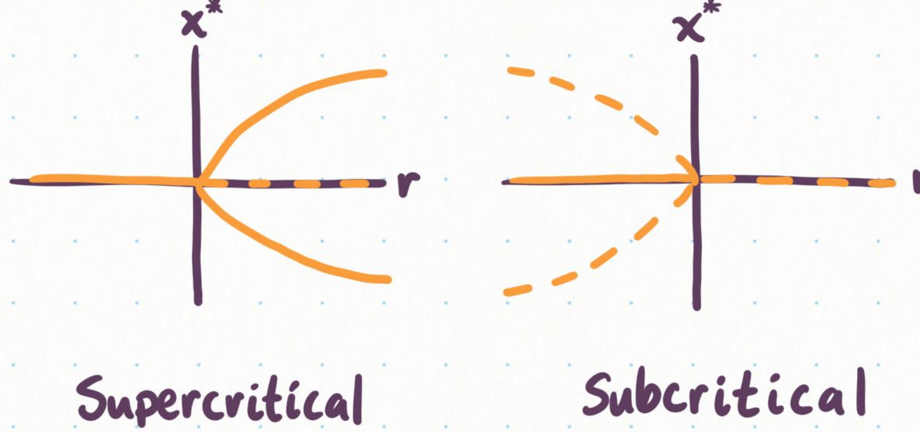
$$
\begin{aligned}
X &= v = u/a = ru/2 = r(x-1)/2 \\
R &= 1+r
\end{aligned}
$$

§ 3.4 Pitchfork Bifurcation

The system transitions from one fixed point to three. There are two cases:

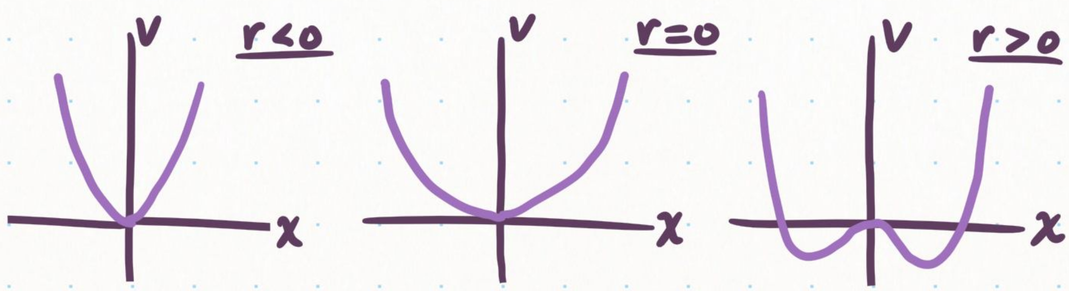
- Supercritical Case
  Normal form $\dot{x} = rx - x^3$
  For $r<0$, one stable equilibrium at $x=0$.
  For $r>0$, one unstable equilibrium at $x=0$ and two stable equilibria at $x=\pm\sqrt{r}$.

- Subcritical Case
  Normal form $\dot{x} = rx + x^3$
  For $r>0$, one unstable equilibrium at $x=0$.
  For $r<0$, one stable equilibrium at $x=0$ and two unstable equilibria at $x=\pm\sqrt{-r}$.



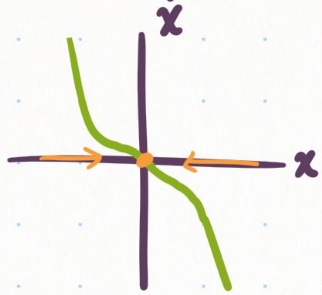
Ex 3.4.2 Plot the potential $V(x)$ for the system $\dot{x} = rx - x^3$ for $r<0$, $r=0$, and $r>0$.

$$\frac{dV}{dx} = x^3 - rx \longrightarrow V(x) = \tfrac{1}{4}x^4 - \tfrac{1}{2}rx^2$$
$$0 = V(x) = \tfrac{1}{2}x^2\left(\tfrac{1}{2}x^2 - r\right) \longrightarrow x = 0, \pm\sqrt{2r} \quad (r>0)$$

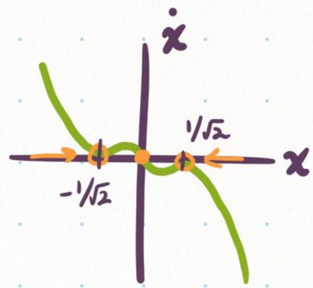


$\dot{x} = rx + x^3 - x^5$

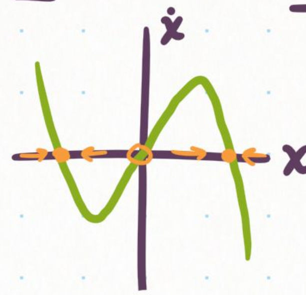
$0 = \dot{x} = x(r + x^3 - x^5) \longrightarrow x^* = 0, \pm \left[\frac{1 \pm (1+4r)^{1/2}}{2} \right]^{1/2}$



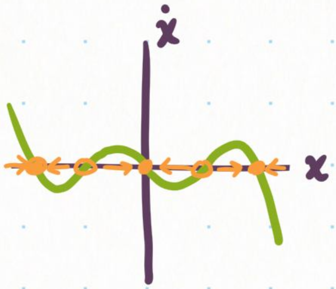
$r < -1/4$



$r = r_s = -1/4.$

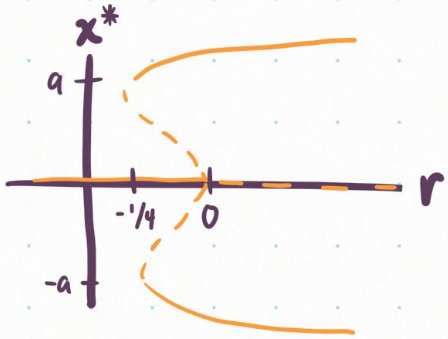


$r > 0$



$-1/4 < r < 0$

For $r < -1/4$, one stable fixed point at $x = 0$.
For $-1/4 < r < 0$, five fixed points.
For $r > 0$, three fixed points.

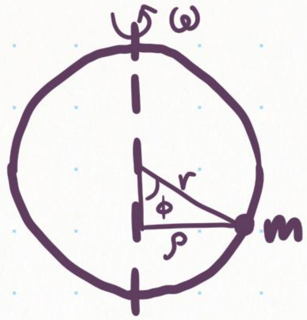


Bifurcation Diagram

If $-1/4 < r < 0$, note that $\lim_{t \to \infty} x(t)$ depends on $x_0 \geq a$, $x_0 \leq -a$, $|x_0| < a$.

If a system starts at $x^* = 0$ and $r < r_s$, this state remains stable as r is increased until $r = 0$. Then any perturbation pushes the system to one of the stable large-amplitude branches. But then the system is stable to small perturbations as r decreases until $r < r_s$. This lack of reversibility as a parameter is varied (here across $r = 0$) is called <u>hysteresis</u>.

§ 3.5 Overdamped Bead on a Rotating Hoop



$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$$

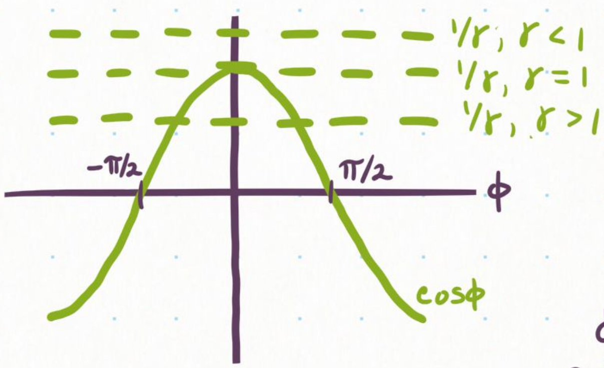
$$-\pi < \phi \leq \pi$$

If we find the appropriate conditions to omit $\ddot{\phi}$, we can analyze the first order system:

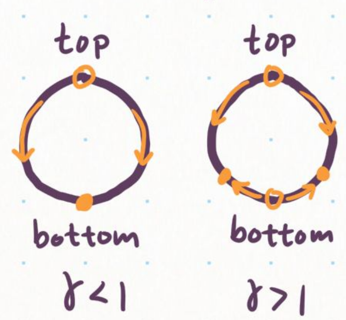
$$b\dot{\phi} = mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right)$$

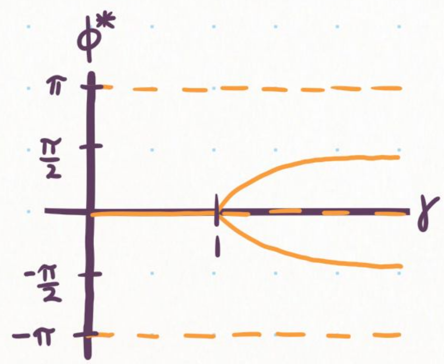
Fixed points at $\phi^* = 0, \pi$ $(\sin\phi = 0)$ for any m, r, ω^2. If $r\omega^2/g > 1$, $\exists\, \phi$ s.t. $0 = r\omega^2\cos\phi/g - 1$. This means that if the hoop spins fast enough (ω^2 large enough), $\exists$ two additional fixed points:

$$\phi = \pm\cos^{-1}(g/r\omega^2) = \pm\cos^{-1}(1/\gamma),$$
$$\gamma := r\omega^2/g$$



Fixed points on both sides of $\phi = 0$ if $\gamma > 1$ and these $\phi^* \to \pm\pi/2$ as $\gamma \to \infty$





Supercritical Pitchfork Bifurcation at $\gamma_c = 1$.

Dimensional Analysis and Scaling

Define dimensionless time $\tau = t/T$, where T is characteristic time scale (chosen later).

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}$$

$$\ddot{\phi} = \frac{d}{dt}\dot{\phi} = \frac{d}{dt}\frac{1}{T}\frac{d\phi}{d\tau} = \frac{1}{T}\frac{d^2\phi}{d\tau^2}\frac{d\tau}{dt} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$$

Substitute into the governing equation

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$$

$$\frac{mr}{T^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{T}\frac{d\phi}{d\tau} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$$

This a balance of forces. Divide by mg to yield the dimensionless equation

$$\frac{r}{gT^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{mgT}\frac{d\phi}{d\tau} - \sin\phi + \frac{r\omega^2}{g}\sin\phi\cos\phi$$

Want the LHS to be neglible and the terms on the RHS to be comparable in size. Since all derivatives are $O(1)$ and $\sin\phi \approx O(1)$,

$$-\frac{b}{mgT} \approx O(1) \quad \text{and} \quad \frac{r}{gT^2} \ll 1$$

Take $T = -\frac{b}{mg}$ so $\frac{rm^2g}{b^2} \ll 1$ or $rm^2g \ll b^2$.
That is, very strong damping or very small mass.
Introduce the dimensionless group $\varepsilon = m^2rg/b^2$:

$$\varepsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\sin\phi\cos\phi$$

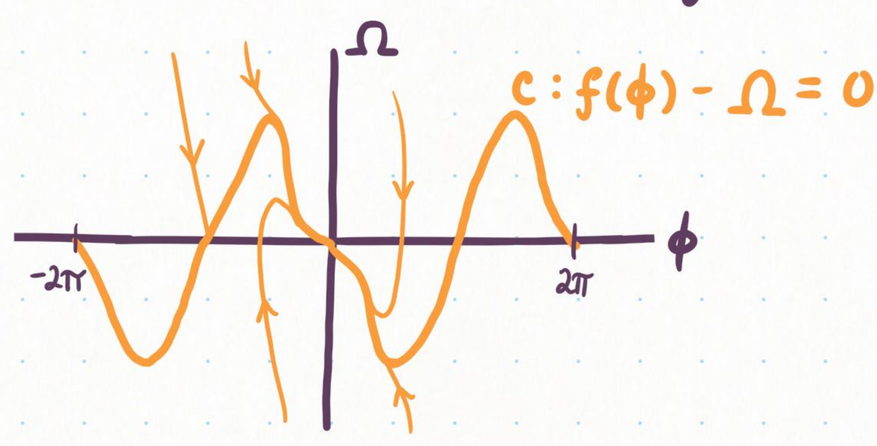
In the overdamped limit $\varepsilon \to 0$ this reduces to
$d\phi/d\tau = f(\phi) = \sin\phi(\gamma\cos\phi - 1)$

Phase Plane Analysis

Let $\Omega = \phi' \equiv d\phi/d\tau$. The previous governing equation becomes $\varepsilon\Omega' = f(\phi) - \Omega$, which yields the vector field

$$\phi = \Omega'$$
$$\Omega' = 1/\varepsilon\,(f(\phi) - \Omega)$$

Interpret the vector (ϕ', Ω') at the point (ϕ, Ω) as the velocity of a phase fluid flowing steadily on the plane. Let C be the curve defined by $f(\phi) = \Omega$. As $\varepsilon \to 0$, all trajectories follow C after a rapid initial transient. During this it's incorrect to ignore $\varepsilon\, d^2\phi/d\tau^2$ but afterward the second order system does behave like a first order system. For example, suppose a phase point lies $O(1)$ distance below C so $\Omega < f(\phi)$. Since $f(\phi) - \Omega \approx O(1)$, $\Omega' \approx O(1/\varepsilon) >> 1$. The phase points zaps upward to where $f(\phi) - \Omega \approx O(\varepsilon)$ and evolves according to $\Omega \approx f(\phi)$.

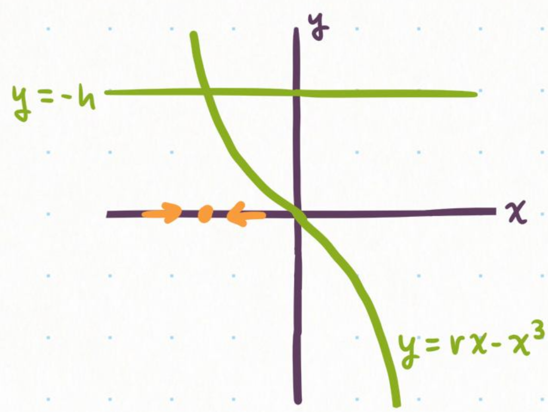


This problem illustrates a concept. In some limit of interest (like damping), the highest order term drops out of the governing equation. Such a limit is called singular.

§ 3.6 Imperfect Bifurcations & Catastrophes

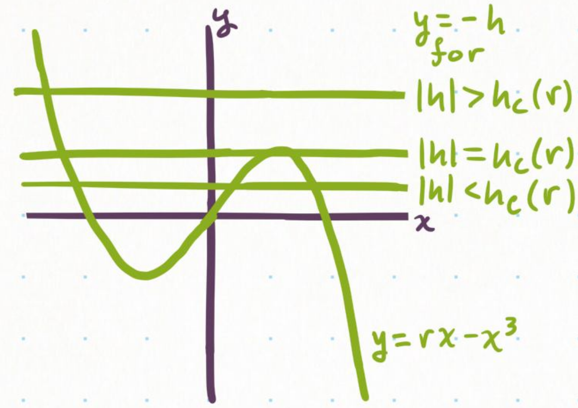
$\dot{x} = h + rx - x^3$, h: imperfection parameter

To find the fixed points, fix r and plot $y = rx - x^3$, $y = -h$ on the same axes.



$r \le 0$

one stable x^*



$r > 0$

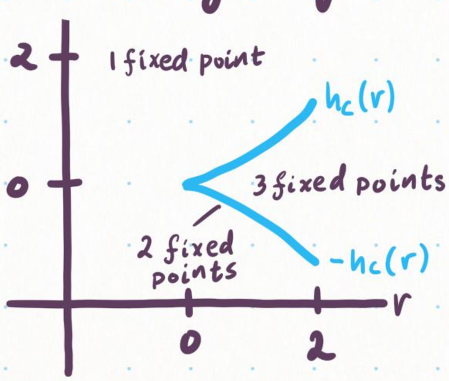
one, two, or three x^* depending on $|h|$

The critical value $h_c(r)$ is the case where $y = -h$ is tangent to the local maximum or local minimum.

$$\frac{d}{dx}\left[rx - x^3\right] = r - 3x^2 = 0 \longrightarrow x = \pm\sqrt{r/3}$$

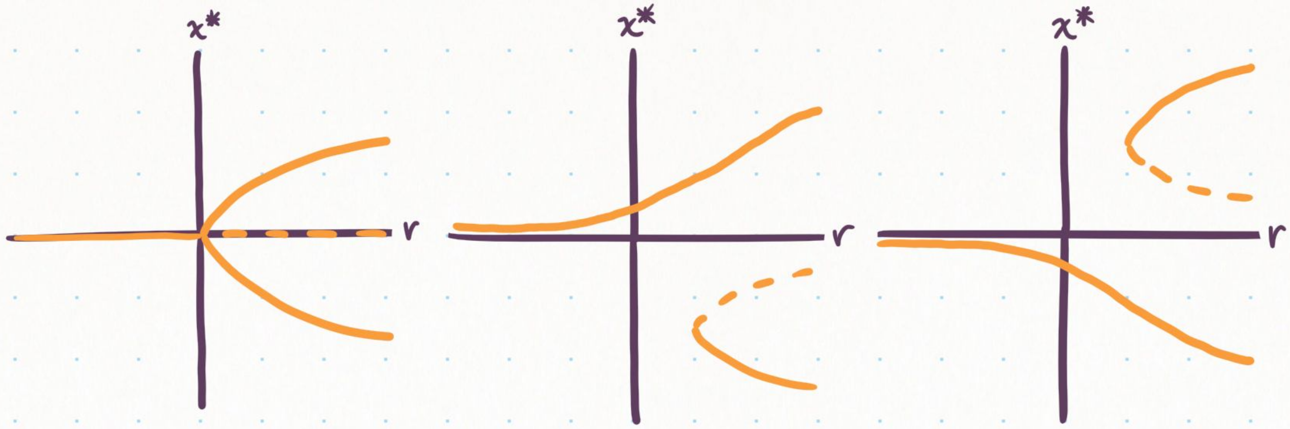
$$\pm h_c(r) = \mp(rx - x^3) = \pm \frac{2r}{3}\sqrt{r/3}$$

Stability Diagram



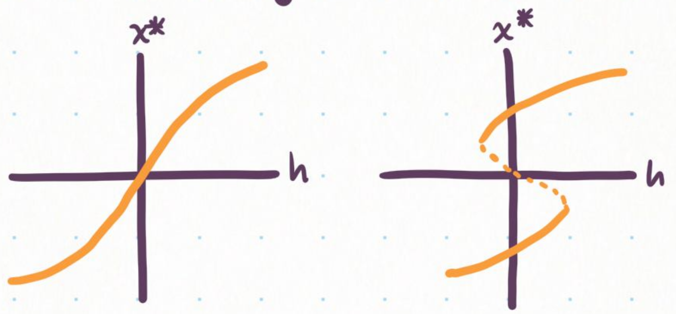
The bifurcation curves $h = \pm h_c(r)$ meet tangentially at the cusp point $(r, h) = (0, 0)$.

To show bifurcation diagrams of x vs. r, hold h constant. If $h = 0$, we have a pitchfork bifurcation.



h=0 h<0 h>0

The h<0 and h>0 cases are not really qualitatively different but separated to help with understanding. Using the graphs from the previous page to plot x^* vs. h,



$r \leq 0$ $r > 0$

For $r \leq 0$, x^* is at the intersection of the monotone decreasing $y = rx - x^3$ and $y = -h$. As h grows $y = -h$ shifts downward so x^* increases. As h becomes more negative, $y = -h$ shifts upward so x^* decreases. For $r > 0$, $y = rx - x^3$ has two 'humps' near the origin. For large $|h|$, $y = -h$ does not intersect with these and so there is only one x^*. For smaller $|h|$ ($|h| < h_c$), there are three x^*. The orientation of this curve follows the same reasoning as for $r \leq 0$ (as $h \to \infty$, $y = -h \to -\infty$, so $x^* \to \infty$, etc.).

Note that these graphs are a $90°$ clockwise rotation of the graphs on the previous page.

§ 3.7 Insect Outbreak

$$\dot{N} = RN(1 - N/K) - p(N)$$

$$p(N) = \frac{BN^2}{A^2 + N^2} \quad A, B > 0$$

$N(t)$: Budworm Population
R : Growth Rate
K : Carrying Capacity
$p(N)$: predation rate

Dimensionless Formulation

Let $x = N/A$, $dx/dt = \dot{N}/A$

$$\frac{1}{B}\dot{N} = \frac{RN}{B}(1 - N/K) - \frac{N^2}{A^2+N^2}$$

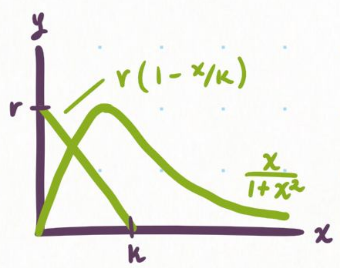
$$\frac{A}{B}\frac{dx}{dt} = \frac{RAx}{B}(1 - Ax/K) - \frac{x^2}{1+x^2}$$

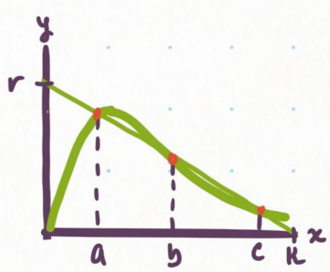
Let $\tau = Bt/A$, $r = RA/B$, $k = K/A$

$$\frac{dx}{d\tau} = rx(1 - x/k) - \frac{x^2}{1+x^2}$$

Analysis of Fixed Points

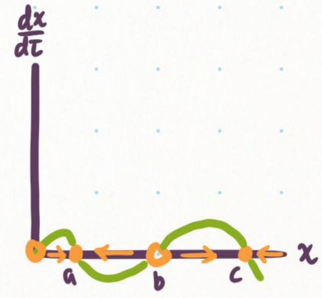
The dimensionless equation has $x^* = 0$ as a fixed point, which is unstable. The other fixed points satisfy $r(1 - x/k) = x/(1+x^2)$





1, 2, or 3 fixed points depending on r, k
Saddle node bifurcation(s) at tangency.

Vector field for three $x^* > 0$
a : refuge
b : threshold
c : outbreak



Since $x^* = 0$ is unstable and stability alternates along the x-axis, $x^* = a, c$ are stable & $x^* = b$ is unstable.

Calculating the Bifurcation Curves

The condition for a saddle node bifurcation is that $r(1-x/K)$ intersects $x/(1+x^2)$ tangentially.

$$r(1-x/K) = \frac{x}{1+x^2} \quad \text{and}$$

$$-\frac{r}{K} = \frac{d}{dx}\left[r(1-x/K)\right] = \frac{d}{dx}\left[\frac{x}{1+x^2}\right] = \frac{1-x^2}{(1+x^2)^2}$$

$$\longrightarrow \quad r - rx/K = \frac{x}{1+x^2}$$

$$r = \frac{x}{1+x^2} - \frac{1-x^2}{(1+x^2)^2}x$$

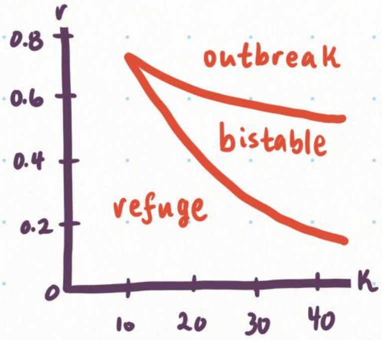
$$r = \frac{x(1+x^2)}{(1+x^2)^2} - \frac{x(1-x^2)}{(1+x^2)^2}$$

$$\left[r(x) = \frac{2x^3}{(1+x^2)^2}\right]$$

$$K = r\frac{(1+x^2)^2}{x^2-1}$$

$$\left[K(x) = \frac{2x^3}{x^2-1}\right]$$

The condition $K>0$ implies the restriction $x>1$. Plotting the parametric $(K(x), r(x))$ for $x>1$ shows the bifurcation curves.



The refuge level $x^* = a$ is the only stable state for low r. The outbreak level $x^* = c$ is the only stable state for high r. For (K, r) in the bistable region both $x^* = a$ and $x^* = c$. In this case $x \longrightarrow a$ if $0 < x_0 < b$ and $x \rightarrow c$ if $x_0 > b$.