MA 508
Homework 1
Due 9/8/2021

Group: Dane Johnson (writer)
Timothy Day
Wesley Lo
Kayleigh Campbell

1. Consider the following chemical reaction, where one chemical (A) turns into a different chemical (B) and vice versa. Suppose that the total amount of chemical is constant, that is $A(t) + B(t) = C$, where C is a positive constant. This reaction can be represented schematically in the following way:

$$A \underset{k^-}{\overset{k^+}{\rightleftharpoons}} B$$

where the two positive constants k^+ and k^- are called rate constants.

The following differential equation describes how A changes with time

$$\frac{dA}{dt} = -k^+ A + k^- B \tag{1}$$

Recall that, in addition to this differential equation, we also have the conservation constraint $A(t)+B(t) = C$.

a) Solve for $A(t)$, given $A(0) = A_0$, with A_0 being a positive constant such that $A_0 < C$.
b) Use Matlab to check your answer for a few choices of A_0, C, k^+, and k^-. (I have provided code that will assist you).

a) $\frac{dA}{dt} = -k^+ A + k^-(C - A)$

$\frac{dA}{dt} = -k^+ A - k^- A + Ck^-$

$\frac{dA}{dt} = Ck^- - (k^+ + k^-)A$

$\frac{dA}{dt} = \gamma - KA$, $\gamma := Ck^-$, $K = k^+ + k^-$

$\int \frac{1}{KA - \gamma}\, dA = -t + d$, $d \in \mathbb{R}$

$\frac{1}{K} \ln|KA - \gamma| = -t + d$

$\ln|KA - \gamma| = -Kt + d$

$KA - \gamma = de^{-Kt}$, $d := \pm e^d$ (since d will be determined using the I.C., we account for the absolute value this way).

$A(t) = \gamma/K + \frac{d}{K} e^{-Kt}$

$A(t) = \frac{Ck^-}{k^+ + k^-} + \frac{d}{k^+ + k^-} e^{-(k^+ + k^-)t}$

$A_0 = A(0) = \frac{Ck^-}{k^+ + k^-} + \frac{d}{k^+ + k^-} \longrightarrow d = A_0(k^+ + k^-) - Ck^-$

$$\boxed{A(t) = \frac{Ck^-}{k^+ + k^-} + \left(A_0 - \frac{Ck^-}{k^+ + k^-}\right) e^{-(k^+ + k^-)t}}$$

b) See Matlab files / plots

2. The position of an object moving in 1D ($x(t)$) on a damped, linear spring obeys the following differential equation

$$m\ddot{x} = -b\dot{x} - kx \tag{2}$$

where m, b, and k are positive constant representing the mass of the object, the damping coefficient and the stiffness of the spring, respectively.

a) Solve for $x(t)$, given $x(0) = x_0$, and $\dot{x}(0) = v_0$.
b) Use Matlab to check your answer for a few choices of x_0, v_0, m, b, and k. (I have provided code that will assist you).

a) $m\ddot{x} + b\dot{x} + Kx = 0$

$mr^2 + br + K = 0$ (characteristic Eqn for $x(t) = e^{rt}$)

$$r = \frac{-b \pm \sqrt{b^2 - 4mK}}{2m}$$

Case 1: $b^2 - 4mk > 0$ (overdamped)

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$\boxed{r_1 = \frac{-b + \sqrt{b^2 - 4mK}}{2m}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4mK}}{2m}}$$

$x_0 = x(0) = c_1 + c_2$

$v_0 = \dot{x}(0) = r_1 c_1 + r_2 c_2$

$c_1 = x_0 - c_2$

$v_0 = r_1(x_0 - c_2) + r_2 c_2$

$v_0 = r_1 x_0 - r_1 c_2 + r_2 c_2$

$v_0 = r_1 x_0 + c_2(r_2 - r_1)$

$$c_2 = \frac{v_0 - r_1 x_0}{r_2 - r_1}$$

$$c_1 = \frac{r_2 x_0 - v_0}{r_2 - r_1}$$

$$\boxed{x(t) = \frac{r_2 x_0 - v_0}{r_2 - r_1} e^{r_1 t} + \frac{v_0 - r_1 x_0}{r_2 - r_1} e^{r_2 t}}$$

Note that $r_2 - r_1 = -\sqrt{b^2 - 4mk}/m$ does not make this expression any more simple, so we leave our result in terms of r_1 and r_2.

Case 2: $b^2 - 4mK < 0$ (underdamped)

$r = -b/2m \pm \sqrt{4mk-b^2}\, i$

$$x(t) = e^{\alpha t}(c_1 \cos\omega t + c_2 \sin\omega t)$$

$$\boxed{\alpha := -b/(2m)\ ,\ \omega := \sqrt{4mK-b^2}/(2m)}$$

$x_0 = x(0) = c_1$

$v_0 = \dot{x}(0) = c_1 + \omega c_2 = x_0 + \omega c_2 \longrightarrow \frac{v_0 - x_0}{\omega} = c_2$

$$\boxed{x(t) = e^{\alpha t}\left(x_0 \cos\omega t + \frac{v_0 - x_0}{\omega}\sin\omega t\right)}$$

Case 3: $b^2 - 4mK = 0$ (critically damped)

$r_1 = r_2 = -b/(2m)$

The repeated root case (critical damping) is omitted (by email instruction).

b) see Matlab files / plots

3. The following equation describes the velocity, $v(t)$, of a relatively large object falling through a relatively inviscid medium (e.g., a baseball falling through the air)

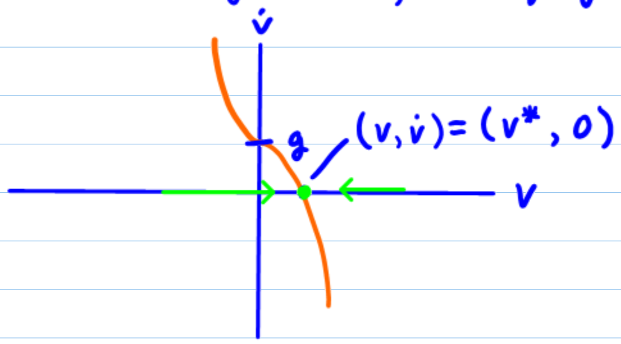
$$m\dot{v} = -cv|v| + mg \tag{3}$$

where m, c and g are positive constants representing the mass of the object, the drag of the medium, and the pull of gravity. a) draw a plot of $\dot{v}$ vs. v. Label any equilibrium point(s) and indicate the stability of each. On the horizontal axis, indicate the flow direction.
b) without solving the equation, sketch $v(t)$ as a function of t for several different initial conditions.
c) solve the equation for $v(t)$, given $v(0) = 0$. (It will simplify your life to assume that $v \geq 0$ to get rid of the absolute value sign. Once you have a solution, you can determine whether this is a reasonable assumption).
d) Use Matlab to check your solution. I have not provided code, but you should be able to modify the code for problem 1.

a) $\dot{v} = -cv|v|/m + g = \begin{cases} -cv^2/m + g, & v \geq 0 \\ cv^2/m + g, & v < 0 \end{cases}$

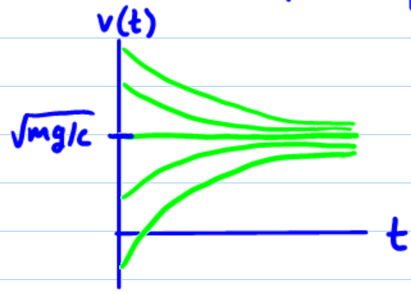
$0 = \dot{v} = -cv|v|/m + g \rightarrow v|v| = mg/c$

If $v < 0$, $-v^2 = mg/c$. But $mg/c > 0$ and $-v^2 < 0$. So there are no $v^* < 0$. If $v \geq 0$, $v^* = \sqrt{mg/c}$ (equilibrium point)



The fixed point $\boxed{v^* = \sqrt{mg/c} \text{ is a stable equilibrium}}$.

b)



c) Assume $v \geq 0$.

$m\dot{v} = mg - cv^2$

$\dot{v} = g - c/m\, v^2$

$\frac{1}{g - \frac{c}{m}v^2}\dot{v} = 1$

$\frac{1}{g}\int \frac{1}{1 - \frac{c}{mg}v^2}\, dv = t + d, \; d \in \mathbb{R}$

$\frac{1}{g}\int \frac{1}{1 - u^2}\sqrt{\frac{mg}{c}}\, du = t + d$

$\sqrt{\frac{m}{gc}}\, \text{arctanh}\left(\sqrt{\frac{c}{mg}}\, v\right) = t + d$ *
(for $v < \sqrt{mg/c}$)

* Found this integral formula at:
https://en.wikipedia.org/wiki/Inverse_hyperbolic_functions#atanh

$\int \frac{1}{1-w^2}\, dw = \text{arctanh}\, w, \quad |w| < 1$

Apply I.C. $v(0) = 0$
Since $\text{arctanh}(0) = 0$ this gives $d = 0$.

$\text{arctanh}\left(\sqrt{\frac{c}{mg}}\, v\right) = \sqrt{\frac{gc}{m}}\, t$

$\boxed{v(t) = \sqrt{\frac{mg}{c}}\, \tanh\left(\sqrt{gc/m}\; t\right)}$

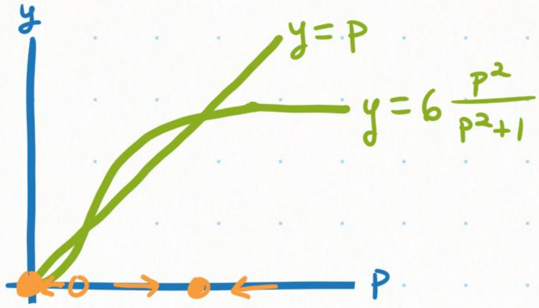
Since $v(t) > 0$ for $t > 0$ it is reasonable to choose one direction for positive velocity and assume $v \geq 0$.

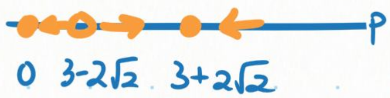
d) See Matlab files / plots

4. $\underline{1}\ \dot{p} = -p + A\frac{p^2}{p^2+1}$, $p(t) \geq 0$, $A > 0$ (Worksheet 1)

a) $A = 6$

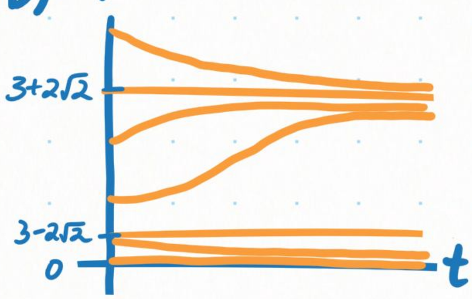
$p^* = 0,\ 3 \pm 2\sqrt{2}$





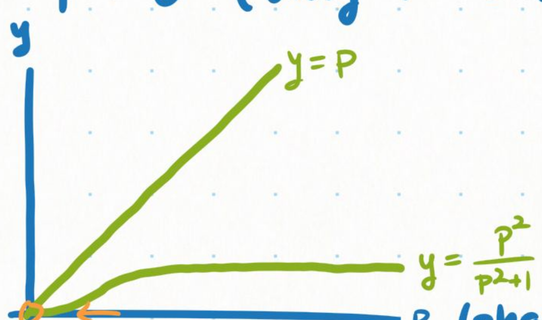
(phase line)

b)



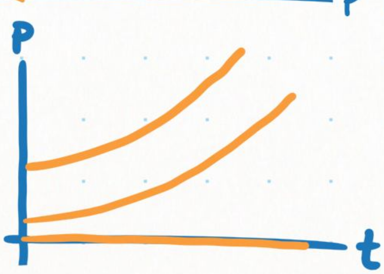
c) $A = 1$

$p^* = 0$ (only 1 real fixed point)



(phase line)

d)



e) Choose $A=6$ since the system has nonzero steady states for $A=6$ but not $A=1$. You must also start with $p(0) \geq 3-2\sqrt{2}$.

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$\boxed{\text{In order to exhibit the behavior seen in the video, there must be 3 real } p^*.}$

$$0 = -p + A\frac{p^2}{p^2+1}$$

$$0 = p(p^2 - Ap + 1)$$

$$p^* = 0,\ \frac{A \pm \sqrt{A^2-4}}{2}$$

$\boxed{\text{There are 3 real } p^* \text{ if } A^2 > 4 \rightarrow A > 2.}$