Homework groups:

You will complete each of twelve homework assignment as part of a three- or four-person group. Group members are assigned randomly from the class and will remain the same for the duration of the quarter. Each group turns in one homework, and each *participating* group member receives the same grade on the assignment. One member of the group is responsible for writing the homework (**the writer**), and this writer rotates for every assignment.

Homework groups work best if: Each member of the homework group finishes (or honestly attempts) the homework independently. At some appointed time, well before the due date, the group meets and everyone compares answers. Any discrepancies are discussed until a consensus is achieved. The writer notes the group consensus and makes sure she or he understands how to do the problem. After the meeting, but before class, the writer neatly and clearly writes the homework according to the Homework guidelines.

Homework groups don't work if: One or more of the members skips meetings; each group member does not honestly attempt the homework prior to the meeting; a consensus in not reached for each assigned problem. *If a group member does not adequately participate in the homework, write a note on the homework and alert your PLA. That person will not receive credit.*

Homework guidelines for writers:

(Adapted from the website of Professor Andy Ruina). To get full credit, please do these things on each homework.

1. As a group writer, upload your homework **as a single PDF** on the day it is due. Homework is available via Canvas Wednesday, and is due the following week by 5pm Eastern Time (unless stated otherwise). At my discretion, late homework may or may not be accepted for reduced credit.

2. On the first page of your homework, please do the following. On the top left corner, please put the course information, homework number and date, e.g.:

   MA 508
HW 2
Due September 15, 2021.

   On the top right corner, please put your group number, the names of your group members, with the writer at the top and clearly indicated. Also indicate any non-participating group members, e.g.:

   Group 3
Jaromir Jagr (writer)
Sarah Jessica Parker
Michelle Wie
James Van der Beek (did not participate)

3. **CITE YOUR HELP.** At the top of each problem, clearly acknowledge all help you got from faculty, students or any other source (with exceptions for lecture and the text, which need not be cited). You could write, for example: "Mary Jones pointed out to me that I had forgotten to divide by three in problem 2," or "Nadia Chow showed me how to do problem 3 from start to finish," or "I copied this solution word for word from Jane Lewenstein" or "I found a problem just like this one, number 9, at

cheatonyourhomework.com, and copied it," etc. You will not lose credit for getting and citing such help. Don't violate academic integrity rules: be clear about which parts of your presentation you did not do on your own. Violations of this policy are violations of the WPI Code of Academic Conduct.

4. Your work should be laid out neatly enough to be read by someone who does not know how to do the problem. For most jobs, it is not sufficient to know how to do a problem, you must convince others that you know how to do it. Your job on the homework is to practice this. **Box your answers.**

DUE: September 15, 2021. Your answers must be uploaded on Canvas, as a single PDF, by 5pm Eastern Time.

This homework covers 1. Existence/uniqueness; 2. Non-dimensionalization/scaling; 3. Bifurcation diagrams; and 4. Saddle-node bifurcations.

These topic are covered in §2.4-3.1 in Strogatz.

For your bifurcation diagrams:
i. Indicate stable fixed points with a solid line and unstable fixed points with a dashed line
ii. Show your calculations for how you determined the fixed points
iii. Explain how you determined stability and/or show your calculations
iv. Clearly indicate any bifurcation(s) (if they exist)
v. Clearly identify and label bifurcation(s) (saddle-node, transcritical, pitchfork, if they exist)

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1. (Problem 3.1.3 in Strogatz) Consider the following non-dimensional equation

$$\dot{x} = r + x - \ln(1 + x)$$

a) Sketch all qualitatively different phase portraits – remember to label fixed points, indicate their stability, and indicate the flow direction on the horizontal axis. [Note: "phase portrait" is a generic term for what I've been calling the phase line in class].
b) Show that a saddle-node bifurcation occurs at some critical value of r.
c) Sketch the bifurcation diagram.

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2. The following non-dimensional equation models population growth

$$\dot{N} = RN - N(1 - N)^2 \tag{1}$$

a) Draw a bifurcation diagram for this equation as R varies.
b) At each bifurcation, the system's qualitative dynamics change, so that the phase portrait differs from one side of the bifurcation to the other. Identify regions with similar dynamics and sketch a phase portrait for each region. [Note: "phase portrait" is a generic term for what I've been calling the phase line in class].
c) Pick one region and sketch several trajectories, $N(t)$, for several different initial conditions. On your plot of $N(t)$, indicate the fixed points.
d) Use Matlab to check your answer to part c. (You don't need to turn this in – but it's a good idea to use Matlab to check your work when you can).

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3. Consider the equation

$$\dot{x} = x^{1/3} \tag{2}$$

a) Using Matlab, perform the following two simulations:
i. Simulate this equation, running time backwards from $t = 1.52$ to $t = 0$, starting from $x(t = 1.52) = 1$.
ii. Simulate this equation, running time forwards from $t = 0$ to $t = 1.52$, starting from $x(t = 0) = 0$.
b) If you've done the simulations correctly (or at least in the same way that I did), then the trajectories cross. In class, I claimed that trajectories cannot cross. Explain this apparent contradiction.
c) Calculate the exact solution for the simulation in part i. Is this solution unique? How does it differ from the computed solution? Explain the source of any differences.

d) Calculate the exact solution for the simulation in part ii. Is this solution unique? How does it differ from the computed solution? Explain the source of any differences.

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4. In class (and on Homework 1), we've been discussing the non-dimensional form of an equation modeling the production of a protein:

$$\frac{d\hat{p}}{d\hat{t}} = -\hat{p} + \alpha \frac{\hat{p}^2}{\hat{p}^2 + 1^2}$$

a) Suppose $\alpha = 4$. Linearize about each fixed point to determine stability. Which one is the most stable (i.e., where do small perturbations decay the fastest)?
b) For how big an initial perturbation is the linear approximation good? Use Matlab to explore this. To do so, you'll need to define what a "good" approximation is (e.g., error less than 1%, 0.1%), and then use Matlab to calculate the difference between the "exact" solution, generated by Matlab, and the linear approximation.

5. Turn in a completed version of worksheet 2, which you worked on during class on September 8.