Worksheet 2

1. $0 = \zeta\dot{\theta} + K\theta - mgl\sin\theta \longrightarrow \frac{d\hat{\theta}}{d\tau} = -\beta\hat{\theta} + \sin\hat{\theta}$

Let $\tau = t/T$, T to be chosen later.

$$\frac{d\theta}{dt} = \frac{d\theta}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\theta}{d\tau}$$

Substitute into $0 = \zeta\dot{\theta} + K\theta - mgl\sin\theta$

$$0 = \zeta/T \frac{d\theta}{d\tau} + K\theta - mgl\sin\theta$$

$$\zeta/T \frac{d\theta}{d\tau} = -K\theta + mgl\sin\theta$$

$$\zeta/(mglT)\frac{d\theta}{d\tau} = -\frac{K}{mgl}\theta + \sin\theta$$

$$\zeta/(mglT)\frac{d\theta}{d\tau} = -\beta\theta + \sin\theta, \quad \beta := \frac{K}{mgl}$$

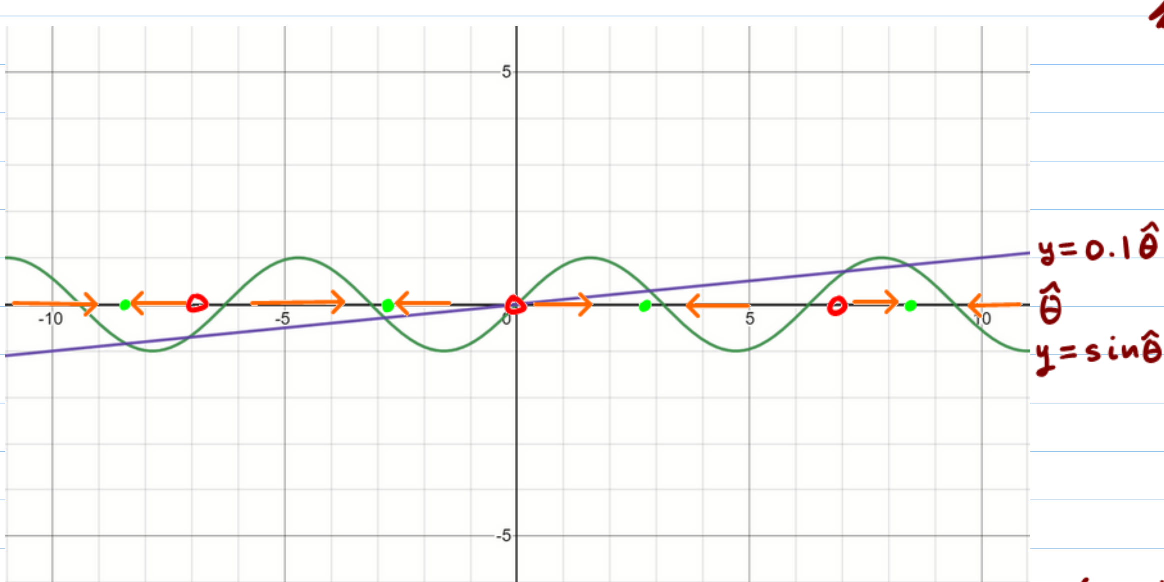
$$\zeta\beta/(KT)\frac{d\theta}{d\tau} = -\beta\theta + \sin\theta$$

$$\frac{d\theta}{d\tau} = -\beta\theta + \sin\theta, \quad T := \zeta\beta/K = \frac{\zeta}{mgl}$$

Then $\tau = t/T = tmgl/\zeta$. To match the worksheet, $\hat{t} = \tau$ and $\hat{\theta}(\hat{t}) := \theta(t/T)$.

$$\boxed{\frac{d\hat{\theta}}{d\hat{t}} = -\beta\hat{\theta} + \sin\hat{\theta}, \quad \beta = \frac{K}{mgl}, \quad \hat{t} = tmgl/\zeta}$$

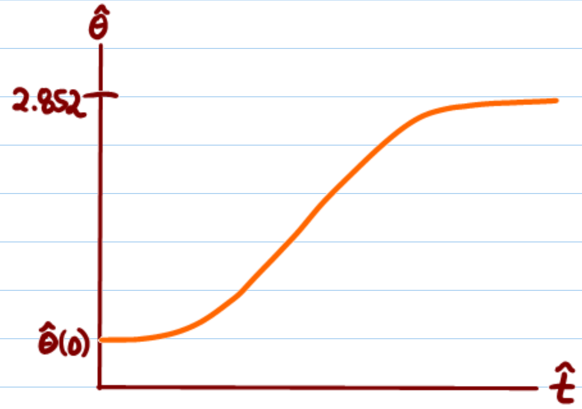
2. a) $\beta = 0.1$, $\frac{d\hat{\theta}}{d\hat{t}} = -0.1\hat{\theta} + \sin\hat{\theta}$ [redacted]



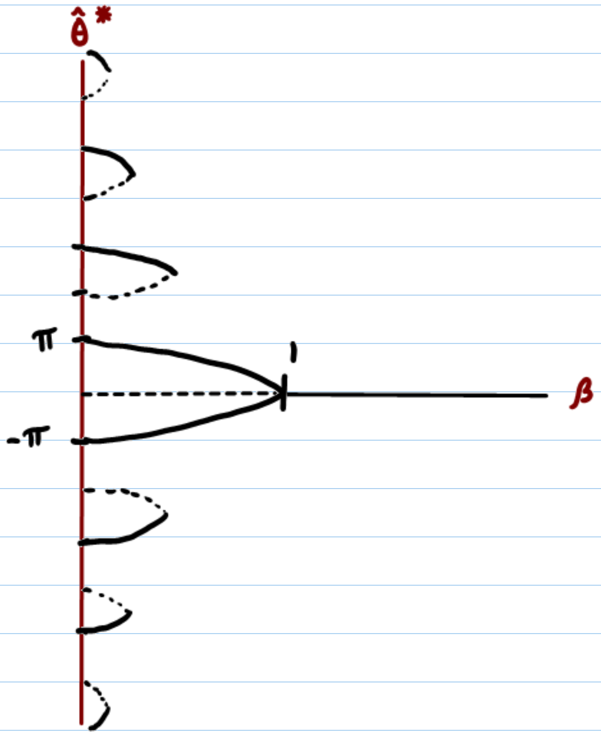
Fixed points at $\hat{\theta}^* \approx -8.423$, -7.068, -2.852, 0 (exact),
stable, unstable, stable, unstable

2.852, 7.068, 8.423
stable, unstable, stable

b)



3.



4. $\dot{\theta} = -\beta\theta + \sin\theta$ (θ for $\hat{\theta}$ for convenience)

$\dot{\theta} \approx -\beta\theta + \theta$ for θ near 0.

$\dot{\theta} \approx (1-\beta)\theta$

$\theta(t) = \varepsilon(t)$, $\beta = 1-\delta$ $\quad (\delta = 1-\beta)$

$\dot{\varepsilon} = \dot{\theta} = \delta\varepsilon - \frac{1}{3!}\varepsilon^3 + \mathcal{O}(\varepsilon^5)$

5. For $K \ll 1$ and fixed m, l we have $\beta = \frac{K}{mgl} \approx 0$. Then, $\dot{\theta} \approx \sin\theta$ and $\theta^* = 0$ (vertical starting position) is unstable. As K increases, this position becomes "less unstable" in the sense that f' evaluated at 0 becomes less positive meaning perturbations grow at a slower (although still exponential rate). Once $K > mgl$, the vertical starting position ($\theta^* = 0$) becomes a stable fixed point: once the spring is stiff enough, the system returns to this position under small perturbations.