MA508 – Worksheet 2

An overdamped pendulum on a torsional spring obeys the following differential equation

$$0 = \zeta\dot{\theta} + \kappa\theta - mg\ell\sin(\theta)$$

where $\theta(t)$ is the angle of the pendulum (with $\theta = 0$ being straight up), ζ is the torsional damping coefficient, κ is the torsional spring constant, m is the mass of the pendulum, g is the gravity constant and ℓ is the length of the pendulum.

The equation can be non-dimensionalized to

$$\frac{d\hat{\theta}}{d\hat{t}} = -\beta\hat{\theta} + \sin\hat{\theta}$$

1. In the non-dimensionalized form, what are $\hat{t}$ and β in terms of the dimensional variables and parameters $(t, \theta, \zeta, \kappa, m, g, \ell)$?

2a) Sketch a phase portrait for the non-dimensionalized equation, for the case $\beta = 0.1$ [Note: "phase portrait" is a generic term for what I've been calling the phase line in class]. In your diagram:
i. Indicate stable fixed points with a filled circle and unstable ones with a hollow circle.
ii., Indicate flow directions with arrows on the horizontal axis.

2b) For this case, sketch $\theta(t)$ when $\theta(0)$ is a small positive number (the pendulum is initially pointing almost straight up).

3) Sketch a bifurcation diagram for the non-dimensionalized equation. In your diagram:
i. Indicate stable fixed points with a solid line and unstable fixed points with a dashed line
ii. Show your calculations for how you determined the fixed points
iii. Explain how you determined stability and/or show your calculations
iv. Clearly indicate any bifurcation(s) (if they exist)
v. Clearly identify and label any saddle-node bifurcation(s)

4) If you've done part 3 correctly, you found a bifurcation at $\beta = 1$, $\theta = 0$. This is a new kind of bifurcation, called a transcritical bifurcation. By doing a Taylor expansion about this point, show that transcritical bifurcations (including this one) have the normal form $\dot{x} = ax - x^2$.

5) Suppose you have a pendulum whose stiffness, κ, can be tuned. You perform a series of experiments, where the pendulum starts nearly vertical and then is released. For the first experiment, the spring is very weak ($\kappa \ll 1$) and you make it stronger for each subsequent experiment. Explain what would happen.