# MA 508 Homework 4

1. $\dot{x} = 1 + y - e^{-x}$
   $\dot{y} = x^3 - y$

a) Find the fixed point(s) by setting $\dot{x} = \dot{y} = 0$.

$$0 = \dot{x}(x^*, y^*) = \dot{y}(x^*, y^*) \Rightarrow \begin{cases} y = x^3 \\ 0 = 1 + x^3 - e^{-x} \end{cases} \Rightarrow \boxed{(x^*, y^*) = (0,0)}$$

Evaluate the Jacobian matrix at the fixed point(s). Use the eigenvalues to classify the fixed point(s).

$$J(x,y) = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} e^{-x} & 1 \\ 3x^2 & -1 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \qquad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -1 \end{matrix}$$

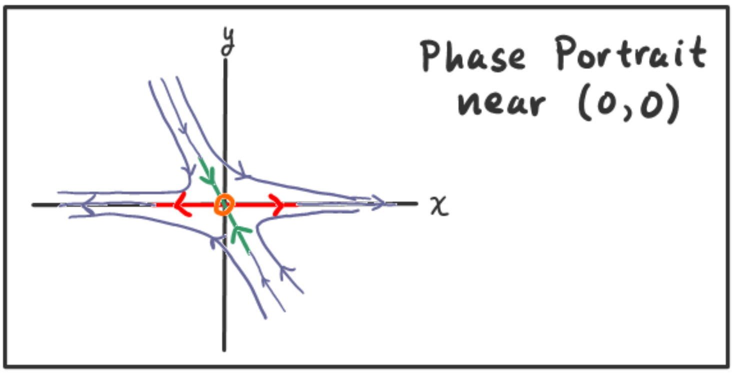
Both eigenvalues are real. $\lambda_1 = 1 > 0$ and $\lambda_2 = -1 < 0$ means

$\boxed{(x^*, y^*) = (0,0) \text{ is a saddle point}}$

b) Sketch a local phase portrait near each fixed point.

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$

$$\left(\begin{array}{cc|c} 1+1 & 1 & 0 \\ 3 & -1+1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$ is an eigenvector corresponding to $\lambda_2 = -1$



2. $\dot{x} = xy - 1$
$\dot{y} = x - y^3$

a) $0 = \dot{x} = \dot{y} \Rightarrow \begin{cases} x = y^3 \\ 1 = y^4 \end{cases} \Rightarrow (x^*, y^*) = (1,1), (-1,-1)$

$$J(x,y) = \begin{pmatrix} y & x \\ 1 & -3y^2 \end{pmatrix} \qquad \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$$

$$J(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \qquad 0 = \lambda^2 + 2\lambda - 4 \Rightarrow \begin{matrix} \lambda_1 = -1 + \sqrt{5} > 0 \\ \lambda_2 = -1 - \sqrt{5} < 0 \end{matrix}$$

$\boxed{(x^*, y^*) = (1,1) \text{ is a saddle point}}$

$$\left(\begin{array}{cc|c} 1-\lambda_1 & 1 & 0 \\ 1 & -3-\lambda_1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2-\sqrt{5} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_1 = \begin{pmatrix} 1 \\ \sqrt{5}-2 \end{pmatrix}$$ is an eigenvalue corresponding to $\lambda_1 = -1 + \sqrt{5}$.

$$\left(\begin{array}{cc|c} 1-\lambda_2 & 1 & 0 \\ 1 & -3-\lambda_2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2+\sqrt{5} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_2 = \begin{pmatrix} 1 \\ -\sqrt{5}-2 \end{pmatrix}$$ is an eigenvalue corresponding to $\lambda_2 = -1 - \sqrt{5}$.

$$J(-1,-1) = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} \qquad \begin{aligned} 0 &= \lambda^2 + 4\lambda + 4 \\ 0 &= (\lambda + 2)^2 \\ \lambda_1 &= \lambda_2 = -2 \end{aligned}$$

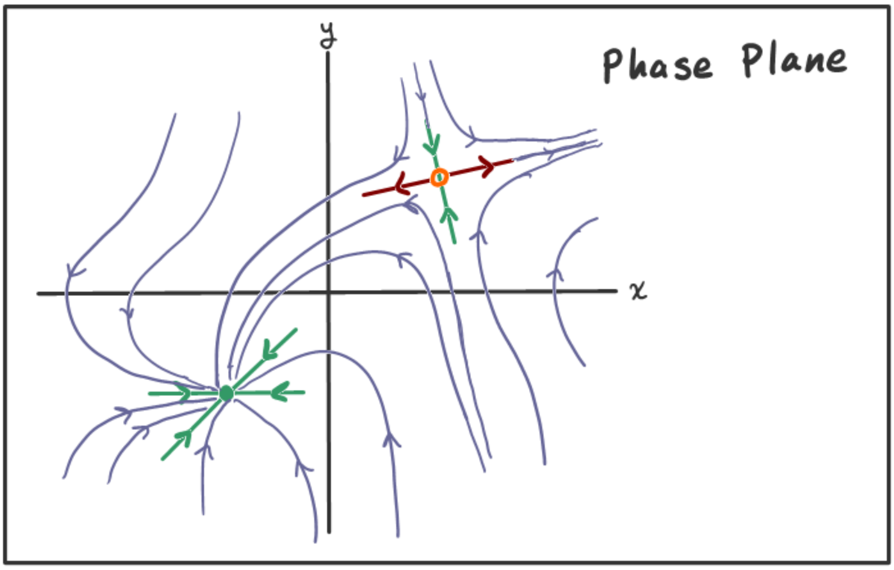
$\boxed{(x^*, y^*) = (-1,-1) \text{ is a stable node}}$

$$\left(\begin{array}{cc|c} -1-(-2) & -1 & 0 \\ 1 & -3-(-2) & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right) \quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\vec{x}(t) \approx c_1 e^{-2t} v_1 + c_2 e^{-2t}(t v_1 + v_2)$ near $(x^*, y^*) = (-1,-1)$

b)



3. $\dot{x} = x(3-x-y)$ $\qquad 3x - x^2 - xy$

$\dot{y} = y(2-x-y)$ $\qquad 2y - xy - y^2$

a) $\dot{x} = \dot{y} = 0 \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix} \quad \begin{matrix} x=0 \\ y=2 \end{matrix} \quad \begin{matrix} y=0 \\ x=3 \end{matrix} \quad (x^*, y^*) = (0,0), (0,2), (3,0)$

$$J(x,y) = \begin{pmatrix} 3-2x-y & -x \\ -y & 2-2y-x \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \quad \lambda_1 = 3,\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \lambda_2 = 2,\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\boxed{(x^*, y^*) = (0,0) \text{ is an unstable node}}$

$$J(0,2) = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix} \quad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -2 \end{matrix}$$

$\boxed{(x^*, y^*) = (0,2) \text{ is a saddle point}}$

$$\left(\begin{array}{cc|c} 1-1 & 0 & 0 \\ -2 & -2-1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 3 & 0 \end{array}\right) \Rightarrow v_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1+2 & 0 & 0 \\ -2 & -2+2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$J(3,0) = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix} \quad \begin{matrix} \lambda_1 = -3 \\ \lambda_2 = -1 \end{matrix}$$

$\boxed{(x^*, y^*) = (3,0) \text{ is a stable node}}$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} -3+1 & -3 & 0 \\ 0 & -1+1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

b)