1. $\ddot{x} = -x - \beta\dot{x} + \frac{x}{\sqrt{x^2+\alpha^2}}$

a) $y = \dot{x}$ $\quad \dot{x} = f(x,y) = y$

$\dot{y} = \ddot{x}$ $\quad \dot{y} = g(x,y) = -x - \beta y + \frac{x}{\sqrt{x^2+\alpha^2}}$

$$\begin{matrix} f(x^*, y^*) = 0 \\ g(x^*, y^*) = 0 \end{matrix} \longrightarrow (x^*, y^*) = (0,0), (\pm\sqrt{1-\alpha^2}, 0)$$

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1+\phi & -\beta \end{pmatrix} \qquad \phi = \frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+\alpha^2}} = \frac{\alpha^2}{(x^2+\alpha^2)^{3/2}}$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1+1/\alpha & -\beta \end{pmatrix}, \quad \lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4 + 4/\alpha}}{2}$$

If $\alpha < 1$, $\sqrt{\beta^2-4+4/\alpha} > \beta$ so this is a saddle node.

If $\alpha = 1$, $\lambda_1 = 0$, $\lambda_2 = -\beta \longrightarrow$ line of fixed points (ignore this).

If $\alpha > 1$,

If $\beta^2 - 4 + 4/\alpha > 0$, this is a stable node (both $\lambda \in \mathbb{R}$, $\lambda < 0$)

If $\beta^2 - 4 + 4/\alpha < 0$, this is a stable spiral

If $\beta^2 - 4 + 4/\alpha = 0$, this is degenerate node / star node (ignore)

$$J(\pm\sqrt{1-\alpha^2}, 0) = \begin{pmatrix} 0 & 1 \\ \alpha^2-1 & -\beta \end{pmatrix}, \quad \lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4 + 4\alpha^2}}{2}$$

If $\alpha > 1$, $1 - \alpha^2 < 0 \rightarrow$ these are not actually fixed points.

If $\alpha = 1$, $\Delta = 0 \longrightarrow$ line of fixed points (ignore this).

If $\alpha < 1$, $\Delta > 0$

If $\beta^2 - 4 + 4\alpha^2 < 0$, these points are stable nodes

If $\beta^2 - 4 + 4\alpha^2 > 0$, these points are stable spirals

If $\beta^2 - 4 + 4\alpha^2 = 0$, these points are degenerate nodes / stars. (ignore)

b) Let $\alpha = 2$, $\beta = \sqrt{11}$

At $(x^*, y^*) = (0,0)$

$$\lambda = \frac{-\sqrt{11} \pm \sqrt{11 - 4 + 4/2}}{2} = \frac{-\sqrt{11} \pm 3}{2}$$

$\lambda_1 = (-\sqrt{11} + 3)/2$, $\lambda_2 = (-\sqrt{11} - \sqrt{3})/2$

Note that $\lambda_1, \lambda_2 < 0$ confirming the local stability of this fixed point.

Here are some linearized equations:

$u = x - x^* = x - 0 = x$, $\dot{u} = \dot{x}$

$v = y - y^* = y - 0 = y$, $\dot{v} = \dot{y}$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J(0,0)\begin{pmatrix} u \\ v \end{pmatrix} + \text{quadratic terms}$$

$$\approx J(0,0)\begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 + 1/2 & -\sqrt{11} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1/2 & -\sqrt{11} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} y \\ -x/2 - \sqrt{11}\,y \end{pmatrix}$$

That is, near $(0,0)$, $\dot{x} \approx y$ (actually $\dot{x} = y$ by defn) and $\dot{y} \approx -x/2 - \sqrt{11}\,y$. Then since $\dot{x} = y$, $\ddot{x} = \dot{y}$,

$$\boxed{\ddot{x} \approx -\tfrac{1}{2}x - \sqrt{11}\,\dot{x}.}$$

c) Comparing to $\ddot{x} = -(k/m)x - (b/m)\dot{x}$, $\boxed{k/m = 1/2 = 1/\alpha}$ and $\boxed{b/m = \sqrt{11} = \beta.}$

d) ✓

2. $\ddot{x} = -\beta\dot{x} - \alpha x + \sin(x)$

a) $y = \dot{x}$ $\quad\quad \dot{x} = f(x,y) = y$

$\dot{y} = \ddot{x}$ $\quad\quad \dot{y} = g(x,y) = -\beta y - \alpha x + \sin x$

$$\begin{matrix} f(x^*, y^*) = 0 \\ g(x^*, y^*) = 0 \end{matrix} \rightarrow \begin{matrix} y^* = 0 \\ \alpha x^* = \sin x^* \end{matrix} \rightarrow \boxed{(x^*, y^*) = (0,0)}$$

we only consider this fixed point (if $\alpha < 1$ there are more fixed points but whether or not $\alpha < 1$ we still only discuss $(0,0)$ as mentioned in class).

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -\alpha + \cos x & -\beta \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1-\alpha & -\beta \end{pmatrix} \quad\quad 0 = \lambda^2 + \beta\lambda + \alpha - 1$$

$$\lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha + 4}}{2}$$

If $\alpha > 1$, $\beta^2 - 4\alpha + 4 < \beta^2$. In this case both λ are real and negative so long as $\beta^2 - 4\alpha + 4 > 0$. The fixed point is a stable node. If $\beta^2 - 4\alpha + 4 < 0$ the fixed point is instead a stable spiral.

If $\alpha < 1$, $\beta^2 - 4\alpha + 4 > \beta^2$. In this case both λ are real but of opposite sign. The fixed point is a saddle points. Since $\alpha > 0$, we cannot make $\beta^2 - 4\alpha + 4$ arbitrarily large — so you couldn't make $(0,0)$ an unstable node (unstable in the sense that both $\lambda_1 > 0$, $\lambda_2 > 0$).

If $\alpha = 1$, $\lambda_1 = 0$ and $\lambda_2 = -\beta < 0$. I recall that we were told to not consider cases like this.

b) Choose $\beta = \sqrt{20}$, $\alpha = 2$. Then

$$\lambda = \frac{-\sqrt{20} \pm \sqrt{20 - 8 + 4}}{2} = \frac{-2\sqrt{5} \pm 4}{2} \rightarrow \begin{matrix} \lambda_1 = -\sqrt{5} + 2 < 0 \\ \lambda_2 = -\sqrt{5} - 2 < 0 \end{matrix}$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -2\sqrt{5} \end{pmatrix} \rightarrow \boxed{\begin{matrix} \text{Linearized Equations}: & \dot{x} = y \\ & \ddot{x} = \dot{y} = -x - 2\sqrt{5}y = -x - 2\sqrt{5}\dot{x} \end{matrix}}$$

c) $-x - 2\sqrt{5}\dot{x} = -K/m\, x - b/m\, \dot{x} \Rightarrow \boxed{K/m = 1 = \alpha/2 \, , \, b/m = 2\sqrt{5} = \beta}$

d) ✓