MA508 – Worksheet 4

A damped linear oscillator is a classical mechanical system. One typically analyzes it to death in math, physics and engineering courses. Its importance lies in the fact that, near equilibrium, many systems behave like a damped linear oscillator. Here, you'll see how it works.

Here are three differential equations that govern non-linear oscillators of one sort or another.

1. A mass on a wire (like you saw last week, but here it is not overdamped, so it obeys a second-order equation)

$$m\ddot{x} = -b\dot{x} - k\left(\sqrt{x^2+h^2} - \ell_0\right)\frac{x}{\sqrt{x^2+h^2}} \tag{1}$$

A non-dimensional form of this equation is (note that this should be in terms of $\hat{x} = x/X$ and $\hat{t} = t/T$ to relate to the previous equation)

$$\ddot{x} = -x - \beta\dot{x} + \frac{x}{\sqrt{x^2+\alpha^2}} \tag{2}$$

2. A pendulum on a torsional spring (like you saw two weeks ago, but here it is not overdamped, so it obeys a second-order equation)

$$-m\ell^2\ddot{\theta} = \zeta\dot{\theta} + \kappa\theta - mg\ell\sin(\theta) \tag{3}$$

A non-dimensional form of this equation is (note that this should be in terms of $x = \theta$ and $\hat{t} = t/T$ to relate to the previous equation)

$$\ddot{x} = -\beta\dot{x} - \alpha x + \sin(x) \tag{4}$$

3. Duffing's oscillator (a model for a slender metal beam interacting with two magnets, which we will likely revisit), in non-dimensional form

$$\ddot{x} = -\dot{x} + \beta x - \alpha x^3 \tag{5}$$

a) Find the fixed point(s) of each oscillator and classify them (i.e., stable node, unstable node, saddle, stable spiral, unstable spiral, etc.). Note that, in ALL CASES, $\beta > 0$ and $\alpha > 0$.

b) For each oscillator, choose a fixed point that is stable in some parameter regime and write linearized equations.

c) Compare your linearization to that of a linear oscillator ($\ddot{x} = -(k/m)x - (b/m)\dot{x}$) and determine the effective spring constant, k/m, and effective damping constant, b/m, for each system.

d) Use Matlab to check your work. Pick value of α and β and run some simulations of the three non-linear oscillators. Compare these with the predictions of the linear system you found in part c, which can be solved analytically (you did this on HW 1).