MA 508 Homework 3

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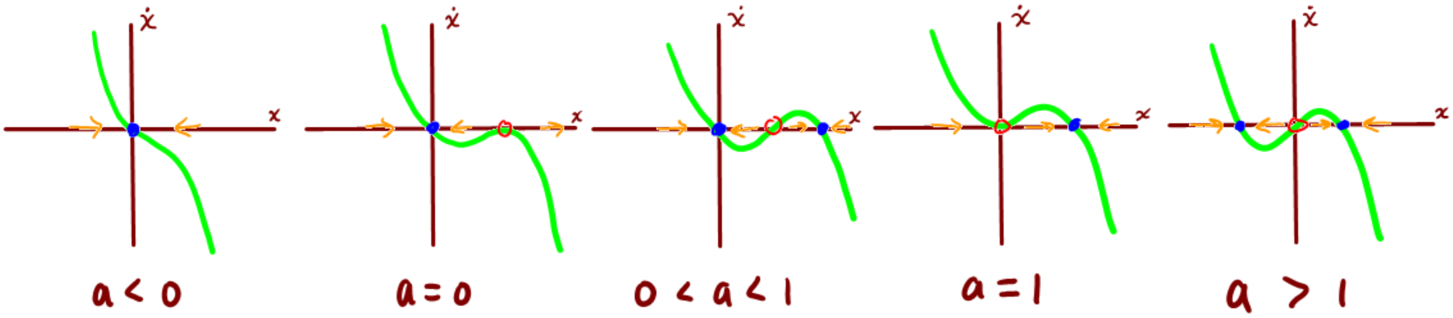
1. Consider the equation

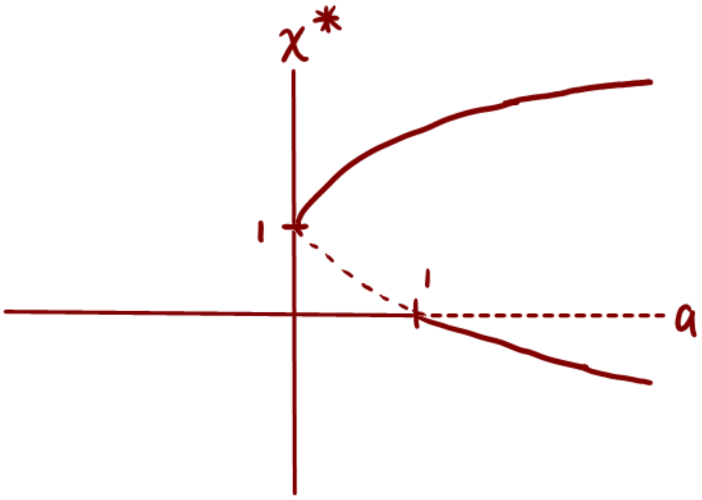
$$\dot{x} = ax - x(1-x)^2 \tag{1}$$

a) Draw a bifurcation diagram for this equation as a varies.
b) In the neighborhood of all bifurcation(s), if they exist, transform Eq. 1 into the normal form.

a)



Bifurcation Diagram



b) Near $a=0, x=1$, $x(t) = 1+\varepsilon(t)$, $a = 0+\delta$

$$\dot{\varepsilon} = f(a,x) = f(0,1) + \frac{\partial f}{\partial x}\Big|_{(0,1)}\varepsilon + \frac{\partial f}{\partial a}\Big|_{(0,1)}\delta + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{(0,1)}\varepsilon^2$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial a^2}\Big|_{(0,1)}\delta^2 + \frac{\partial^2 f}{\partial x \partial a}\Big|_{(0,1)}\varepsilon\delta + \dots$$
$$= 0 + (a-1+4x-3x^2)\Big|_{(0,1)}\varepsilon + x\Big|_{(0,1)}\delta +$$
$$+ \frac{1}{2}(4-6x)\Big|_{(0,1)}\varepsilon^2 + \frac{1}{2}\cdot 0\Big|_{(0,1)}\delta^2 + 1\Big|_{(0,1)}\varepsilon\delta + \dots$$
$$= 0 + 0 + \delta - \varepsilon^2 + 0 + \varepsilon\delta + \dots$$
$$= -\varepsilon^2 + \varepsilon\delta + \delta + \dots$$
$$\approx -\varepsilon^2 + \varepsilon\delta + \delta.$$

Near $a=1, x=0$, $x(t) = 0+\varepsilon(t)$, $a = 1+\delta$

$$\dot{\varepsilon} = f(a,x) = f(1,0) + \frac{\partial f}{\partial x}\Big|_{(1,0)}\varepsilon + \frac{\partial f}{\partial a}\Big|_{(1,0)}\delta + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{(1,0)}\varepsilon^2$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial a^2}\Big|_{(1,0)}\delta^2 + \frac{\partial^2 f}{\partial x \partial a}\Big|_{(1,0)}\varepsilon\delta + \dots$$
$$= 0 + (a-1+4x-3x^2)\Big|_{(1,0)}\varepsilon + x\Big|_{(1,0)}\delta +$$
$$+ \frac{1}{2}(4-6x)\Big|_{(1,0)}\varepsilon^2 + \frac{1}{2}\cdot 0\Big|_{(1,0)}\delta^2 + 1\Big|_{(1,0)}\varepsilon\delta + \dots$$
$$= 0 + 0\varepsilon + 0\delta + 2\varepsilon^2 + \varepsilon\delta + \dots$$
$$\approx 2\varepsilon^2 + \varepsilon\delta$$

2. Consider the following bifurcation diagram, showing fixed points (x^*) as a function of parameter, p, for an equation of the form $\dot{x} = f(x)$. Note that there are three different parameter values indicated, p_1, p_2, and p_3.

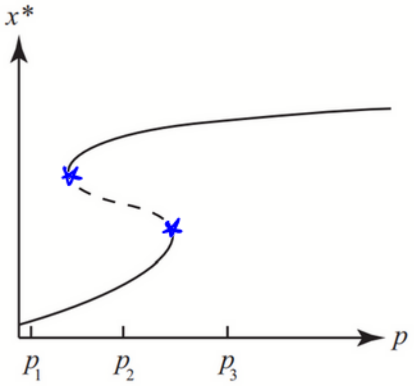
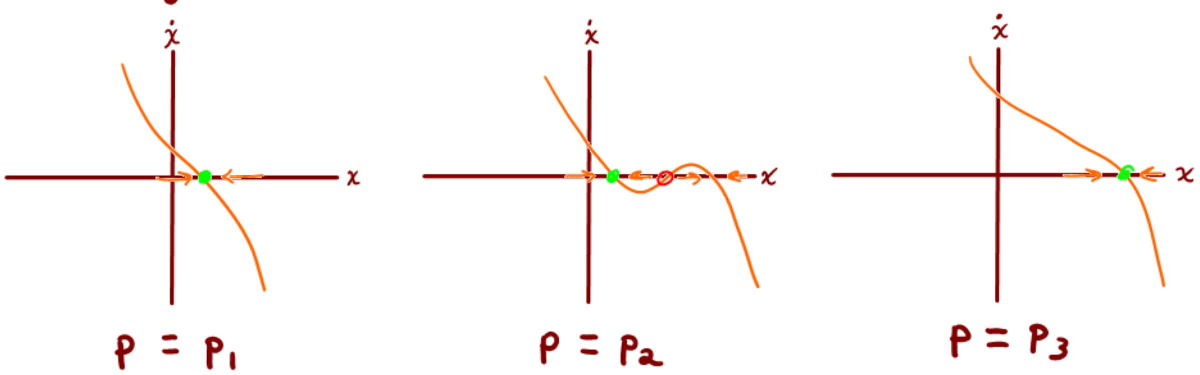


Figure 1: Bifurcation diagram; stable fixed points are drawn as a solid line, unstable as a dashed line.

a) Label and identify all bifurcations in the figure.
b) Draw phase portraits consistent with the bifurcation curve at each of the parameter values, p_1, p_2 and p_3. (You should draw one plot for each parameter value, a total of three phase portraits).
c) Can this system exhibit hysteresis (according to the definition used in class)?
d) How would your answer to c) change if the stability in Fig. 1 were flipped (i.e., each stable fixed point were unstable, and each unstable fixed point were stable)? Explain.

a) Labelled by: ✱

b)



c) Yes. The system is bistable and has bifurcations.

If the system starts at an unstable fixed point a perturbation would cause a jump to one of the stable fixed point branches. We would have to raise or lower p through a bifurcation point in order to return to the original state. This lack of reversibility as a parameter is varied is hysterisis.

d) No. The definition given in class requires bistability. If we reversed the stable / unstable fixed points there would be at most 1 stable fixed point p* given a particular value of p.

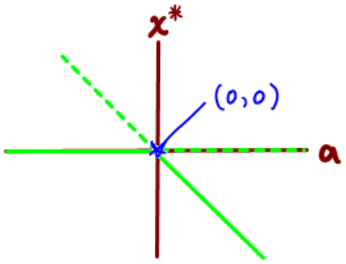
3. Here is the equation for an imperfect transcritical bifurcation

$$\dot{x} = ax + x^2 + \varepsilon \tag{2}$$

a) Sketch bifurcation diagrams for 1) $\varepsilon = 0$, 2) $\varepsilon > 0$, 3) $\varepsilon < 0$. You may assume ε is small in the latter two cases. On each of the three bifurcation diagrams, indicate stable fixed points with a solid line, unstable fixed points with a dashed line, and label all bifurcations.

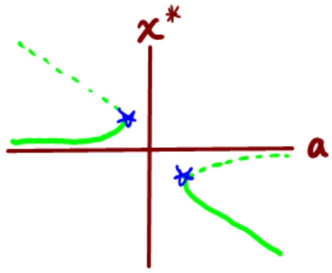
b) Sketch a stability diagram (Recall that a stability diagram will have a and ε as axes, and will indicated regions where there are differing numbers of fixed points).

a) i. $\varepsilon = 0$



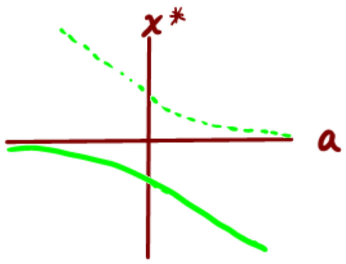
Transcritical bifurcation at $a = 0$

ii. $\varepsilon > 0$



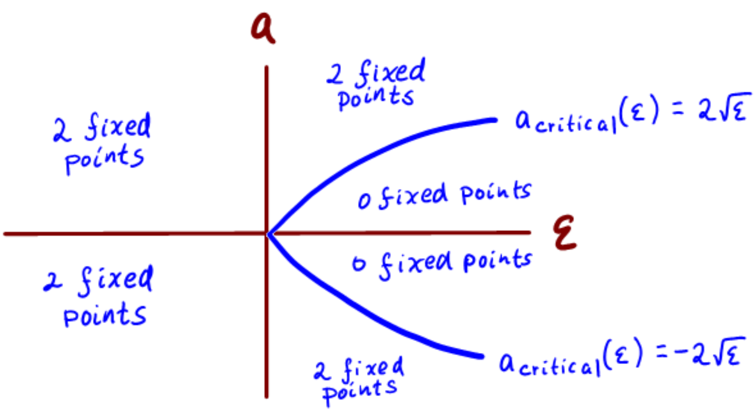
Saddle node bifurcations at $a = \pm 2\sqrt{\varepsilon}$

iii. $\varepsilon < 0$



No bifurcation points

b)



$$0 = x^2 + ax + \varepsilon$$

$$x^* = \frac{-a \pm \sqrt{a^2 - 4\varepsilon}}{2}$$

Two solutions for $\varepsilon > 0$.

If $\varepsilon > 0$, distinct real solutions for $a^2 - 4\varepsilon > 0 \rightarrow |a| > 2\sqrt{\varepsilon}$ while there are 0 real solutions for $|a| < 2\sqrt{\varepsilon}$.

Alternatively solve: $\begin{cases} 0 = \frac{d}{dx}(x^2 + ax) \\ -\varepsilon = x^2 + ax \end{cases}$

$$0 = 2x + a \rightarrow x = -a/2$$

$$-\varepsilon = a^2/4 - a^2/2 \rightarrow a = \pm 2\sqrt{\varepsilon}$$

Worksheet 3

1. $$\dot{x} = -\frac{k}{b}\left(\sqrt{x^2+h^2} - \ell_0\right)\frac{x}{\sqrt{x^2+h^2}} = \frac{k}{b}\left(\ell_0 \frac{x}{\sqrt{x^2+h^2}} - x\right)$$

a. $\tau = t/T$ $\rightarrow$ set $T = b/k$

Note: using τ instead of $\hat{t}$ for convenience (easier to write).

$$\dot{x} = \frac{1}{T}\frac{dx}{d\tau}$$

$$\frac{1}{T}\frac{dx}{d\tau} = \frac{k}{b}\left(\ell_0 \frac{x}{\sqrt{x^2+h^2}} - x\right)$$

$$\frac{dx}{d\tau} = \ell_0 \frac{x}{\sqrt{x^2+h^2}} - x \longrightarrow$$

set $\hat{x} = x/\ell_0$

$$\frac{d\hat{x}}{d\tau} = \frac{d\hat{x}}{dx}\frac{dx}{d\tau} = \frac{1}{\ell_0}\frac{dx}{d\tau}$$

$$\ell_0 \frac{d\hat{x}}{d\tau} = \frac{\ell_0^2 \hat{x}}{\sqrt{\ell_0^2\hat{x}^2+h^2}} - \ell_0\hat{x}$$

$$\frac{d\hat{x}}{d\tau} = \frac{\ell_0 \hat{x}}{\ell_0\sqrt{\hat{x}^2+\alpha^2}} - \hat{x}, \quad \alpha = h/\ell_0$$

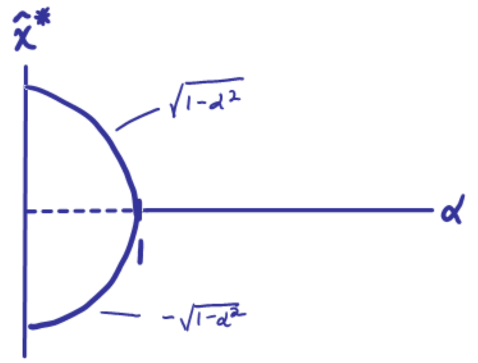
$$\boxed{\frac{d\hat{x}}{d\tau} = \frac{\hat{x}}{\sqrt{\hat{x}^2+\alpha^2}} - \hat{x} \qquad \hat{x} = \frac{1}{\ell_0}x, \ \tau = kt/b, \ \alpha = h/\ell_0}$$

b. Let $\dot{\hat{x}} := \frac{d\hat{x}}{d\tau}$. Set $\dot{\hat{x}} = 0$.

$$\hat{x}\left(1/\sqrt{\hat{x}^2+\alpha^2} - 1\right) = 0$$

$$\hat{x}^* = 0, \pm\sqrt{1-\alpha^2}, \quad \alpha < 1 \quad (\text{By assumption } \alpha = h/\ell_0 > 0)$$

Bifurcation Diagram



c. Near the bifurcation point $\alpha = 1$, $\hat{x} = 0$:

$$f(\hat{x}) = \frac{\hat{x}}{\phi} - \hat{x}, \quad \phi(\alpha, \hat{x}) = \sqrt{\hat{x}^2 + \alpha^2}, \quad c = (1, 0), \quad \phi(c) = \sqrt{0^2 + 1^2} = 1$$

$$\left.\frac{\partial f}{\partial \hat{x}}\right|_c = \left.\left(\frac{1}{\phi} - \frac{\hat{x}^2}{\phi^3} - 1\right)\right|_c = 0 \qquad \left.\frac{\partial f}{\partial \alpha}\right|_c = -\left.\frac{x\alpha}{\phi^3}\right|_c = 0$$

$$\left.\frac{\partial^2 f}{\partial \hat{x}^2}\right|_c = -\left.\frac{3\hat{x}\alpha^2}{\phi^5}\right|_c = 0 \qquad \left.\frac{\partial^2 f}{\partial \alpha^2}\right|_c = -\left.\frac{\hat{x}(\hat{x}^2 - 2\alpha^2)}{\phi^5}\right|_c = 0$$

$$\left.\frac{\partial^3 f}{\partial \hat{x}^3}\right|_c = -\left.\frac{3\alpha^2(\alpha^2 - 4\hat{x}^2)}{\phi^7}\right|_c = -3 \qquad \left.\frac{\partial^3 f}{\partial \alpha^3}\right|_c = \left.\frac{9\hat{x}^3\alpha - 6\hat{x}\alpha^3}{\phi^7}\right|_c = 0$$

$$\left.\frac{\partial^2 f}{\partial x \partial \alpha}\right|_c = \left.\frac{\partial^2 f}{\partial \alpha \partial x}\right|_c = -\left.\frac{\alpha(\alpha^2 - 2\hat{x}^2)}{\phi^5}\right|_c = -1$$

$$\left.\frac{\partial^3 f}{\partial \alpha^2 \partial x}\right|_c = \left.\frac{2\hat{x}^4 - 11\hat{x}^2\alpha^2 + 2\alpha^4}{\phi^7}\right|_c = 2$$

$$\left.\frac{\partial^2 f}{\partial x^2 \partial \alpha}\right|_c = \left.\frac{9\hat{x}\alpha^3 - 6\hat{x}^3\alpha}{\phi^7}\right|_c = 0$$

$$\begin{aligned}
\dot{\varepsilon} = f(\alpha, x) &= f(c) + \left.\frac{\partial f}{\partial \hat{x}}\right|_c \varepsilon + \left.\frac{\partial f}{\partial \alpha}\right|_c \delta + \frac{1}{2}\left.\frac{\partial^2 f}{\partial \hat{x}^2}\right|_c \varepsilon^2 + \frac{1}{2}\left.\frac{\partial^2 f}{\partial \alpha^2}\right|_c \delta^2 \\
&\quad + \left.\frac{\partial^2 f}{\partial x \partial \alpha}\right|_c \varepsilon\delta + \frac{1}{6}\left.\frac{\partial^3 f}{\partial \hat{x}^3}\right|_c \varepsilon^3 + \frac{1}{2}\left.\frac{\partial^3 f}{\partial \hat{x}^2 \partial \alpha}\right|_c \varepsilon^2\delta + \frac{1}{2}\left.\frac{\partial^3 f}{\partial \hat{x} \partial \alpha^2}\right|_c \varepsilon\delta^2 \\
&\quad + \frac{1}{6}\left.\frac{\partial^3 f}{\partial \alpha^3}\right|_c \delta^3 + \dots \\
&= 0 + 0 \cdot \varepsilon + 0\delta + \frac{1}{2} \cdot 0 \cdot \varepsilon^2 + \frac{1}{2} \cdot 0 \cdot \delta^2 - \varepsilon\delta \\
&\quad + \frac{1}{6}(-3)\varepsilon^3 + \frac{1}{2} \cdot 0 \cdot \varepsilon^2\delta + \frac{1}{2}(2)\varepsilon\delta^2 + \frac{1}{6} \cdot 0 \cdot \delta^3 \\
&= \boxed{-\frac{1}{2}\varepsilon^3 - \varepsilon\delta} \quad \left(\neq -\frac{2}{3}\varepsilon^3 + \varepsilon\delta\right)
\end{aligned}$$

Notes on the board in class showed $\frac{\partial f}{\partial \hat{x}} = \frac{1}{\phi} - \frac{2\hat{x}^2}{\phi^3} - 1$. This doesn't seem correct and could account for the difference.

Compare $\dot{\varepsilon} = -\frac{1}{2}\varepsilon^3 - \varepsilon\delta$ to the normal form for a pitchfork bifurcation $\dot{x} = ax - x^3$.

2. $\dot{x} = -\frac{k}{b}\left(\sqrt{x^2+h^2} - \ell_0\right)\frac{x}{\sqrt{x^2+h^2}} - \frac{mg}{b}$

a) Using $\hat{x} = \frac{1}{\ell_0}x$, $\tau = kt/b$, $\alpha = h/\ell_0$

$$\frac{d\hat{x}}{d\tau} = \frac{d\hat{x}}{dx}\frac{dx}{d\tau} = \frac{1}{\ell_0}\frac{dx}{d\tau} = \frac{1}{\ell_0}\frac{dx}{dt}\frac{dt}{d\tau} = \frac{b}{\ell_0 k}\dot{x} \Rightarrow \dot{x} = \frac{k\ell_0}{b}\frac{d\hat{x}}{d\tau}$$

$$\frac{k\ell_0}{b}\frac{d\hat{x}}{d\tau} = -\frac{k}{b}\left(\sqrt{\ell_0^2\hat{x}^2+\ell_0^2\alpha^2} - \ell_0\right)\frac{\ell_0\hat{x}}{\sqrt{\ell_0^2\hat{x}^2+\ell_0^2\alpha^2}} - \frac{mg}{b}$$

$$\ell_0\frac{d\hat{x}}{d\tau} = -\left(\ell_0\sqrt{\hat{x}^2+\alpha^2} - \ell_0\right)\frac{\hat{x}}{\sqrt{\hat{x}^2+\alpha^2}} - \frac{mg}{k}$$

$$\frac{d\hat{x}}{d\tau} = -\left(\sqrt{\hat{x}^2+\alpha^2} - 1\right)\frac{\hat{x}}{\sqrt{\hat{x}^2+\alpha^2}} - \frac{mg}{\ell_0 k}$$

$$\frac{d\hat{x}}{d\tau} = -\left(\sqrt{\hat{x}^2+\alpha^2} - 1\right)\frac{\hat{x}}{\sqrt{\hat{x}^2+\alpha^2}} - \beta, \quad \beta = \frac{mg}{\ell_0 k}$$

$$\boxed{\begin{gathered}\frac{d\hat{x}}{d\tau} = -\hat{x} + \frac{\hat{x}}{\sqrt{\hat{x}^2+\alpha^2}} - \beta \\ \hat{x} = \frac{1}{\ell_0}x, \quad \tau = kt/b, \quad \alpha = h/\ell_0, \quad \beta = \frac{mg}{\ell_0 k}\end{gathered}}$$

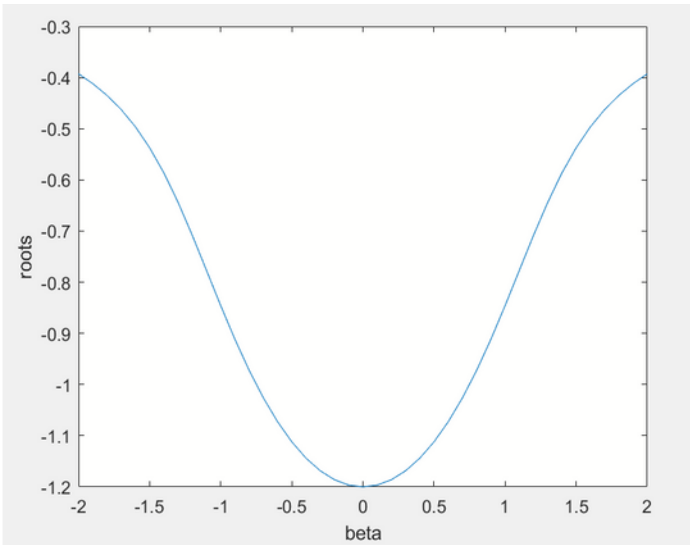
b) For $\beta = 0$, $\hat{x}^* = 0, \sqrt{1-\alpha^2}, -\sqrt{1-\alpha^2}$

For $\beta = 0.1, 0.2$ you can't solve for x^* as a function of α. This is what our group tried but it doesn't seem accurate:

```
y = 1;
i = 1;
list = [];

for a = -2:0.1:2
    y = fzero(@(x) x/(sqrt(x^2 + a^2))-x-0.2,y);
    list(i) = y;
    i = i+1;
end

figure(1);
clf;
plot([-2:0.1:2],list)
xlabel('beta');
ylabel('roots')
```



c) maybe we could cover matlab in class in more detail?

$$\dot{x} = \frac{x}{\sqrt{x^2+\alpha^2}} - x - \beta = 0$$

$$0 = \frac{\alpha^2}{(x_c^2+\alpha^2)^{3/2}} - 1 \qquad (f'(x, \alpha, \beta) = 0)$$

$$(x_c^2+\alpha^2)^{3/2} = \alpha^2$$

$$x_c^2 + \alpha^2 = \alpha^{4/3} \longrightarrow x_c = \alpha\sqrt{\alpha^{2/3}-1}$$

$$\beta = \frac{\alpha\sqrt{\alpha^{2/3}-1}}{\sqrt{\alpha^2(\alpha^{2/3}-1)+\alpha^2}} - \alpha\sqrt{\alpha^{2/3}-1}$$

$$= \frac{\alpha\sqrt{\alpha^{2/3}-1}}{\sqrt{\cancel{\alpha^2}(\alpha^{2/3}\cancel{-1}\cancel{+1})}} - \alpha\sqrt{\alpha^{2/3}-1} = \frac{\sqrt{\alpha^{2/3}-1}}{\alpha^{1/3}} - \alpha\sqrt{\alpha^{2/3}-1}$$

$$= \sqrt{\alpha^{2/3}-1}\,(\alpha^{-1/3}-\alpha)$$

```matlab
y = 1;
i = 1;
list = [];

for a = -2:0.1:2
    y = fsolve(@(x) ...
        (x*sqrt(x^(2/3)-1))/(sqrt(x^2*(x^(2/3)-1))+x^2)-(x*sqrt(x^(2/3)-1)),y);
    list(i) = y;
    i = i+1;
end

figure(1);
clf;
plot([-2:0.1:2],list)
xlabel('alpha')
ylabel('beta')
```