MA 514 Homework 2

Dane Johnson

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Exercise 4.1

(a) Determine the SVD of the matrix:

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} .$$

We want a decomposition of the form $A = U\Sigma V^*$, where U and V are unitary matrices and Σ a diagonal matrix containing the square roots of the singular values of A. We then know that

$$AA^* = U\Sigma V^* V \Sigma^* U^* = U\Sigma\Sigma^* U^* \quad A^*A = V\Sigma^* U^* U \Sigma V^* = V\Sigma^*\Sigma V^* .$$

In particular, $U\Sigma\Sigma^*U^*$ and $V\Sigma^*\Sigma V^*$ are a diagonalizations of A^*A (note that $\Sigma\Sigma^*$ and $\Sigma^*\Sigma$ are diagonal matrices. Therefore we can determine Σ and V by determining the eigenvalues of A^*A and finding corresponding eigenvectors.

$$A^*A = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$$

Since A^*A is diagonal, the eigenvalues are the diagonal elements. The eigenvalues of A^*A are the squares of the singular values of A. Thus, $\sigma_1^2 = 9, \sigma_2^2 = 4$ and next we find corresponding normalized eigenvectors to use a columns of V (so that V will be unitary). In this case, since A^*A is diagonal, we can find eigenvectors by inspection as $v_1 = e_1$ and $v_2 = e_2$. So we have

$$\Sigma = \begin{pmatrix} \sqrt{9} & 0 \\ 0 & \sqrt{4} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Next, use the requirement that $AV = U\Sigma$ to find U.

$$U\Sigma = AV = A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

Since Σ scales the first column of U by $\sqrt{\sigma_1} = 3$ and scales the second column by $\sqrt{\sigma_2} = 2$, so we have

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \implies U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We conclude that an SVD factorization of A is given by

$$A = U\Sigma V^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise 4.2

Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating A ninety degree clockwise on paper. We prove that A and B have the same singular values.

Let A be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

Consider A^T (we do not consider A^* in case this is different from A^T):

$$A^T = \begin{pmatrix} a_{11} & a_{21} & ... & a_{m1} \\ a_{12} & a_{22} & ... & a_{m2} \\ ... & ... & ... & ... \\ a_{1n} & a_{2n} & ... & a_{mn} \end{pmatrix}$$

We can reverse the ordering of the columns of A^T by multiplying by the $m \times m$ matrix P that has 1s on the antidiagonal and 0s elsewhere:

$$A^T P = \begin{pmatrix} a_{11} & a_{21} & ... & a_{m1} \\ a_{12} & a_{22} & ... & a_{m2} \\ ... & ... & ... & ... \\ a_{1n} & a_{2n} & ... & a_{mn} \end{pmatrix} \begin{pmatrix} 0 & ... & 0 & 1 \\ 0 & ... & 1 & 0 \\ ... & ... & ... & ... \\ 1 & 0 & ... & 0 \end{pmatrix} = \begin{pmatrix} a_{m1} & ... & a_{21} & a_{11} \\ a_{m2} & ... & a_{22} & a_{12} \\ ... & ... & ... & ... \\ a_{mn} & ... & a_{2n} & a_{1n} \end{pmatrix} = B$$

Therefore, we have found the matrix equation that relates the matrix A and B from the given description of how B is obtained from A. Next, since A has an SVD $A = U\Sigma V^*$, we see that $A^T = (V^*)^T\Sigma^T U^T = (V^*)^T\Sigma U^T$. This shows that A has the same singular values as A^T (even if the left and right singular vectors may change). Also, since P is an orthogonal matrix, we have $(A^TP)(A^TP)^* = A^TPP^*(A^T)^* = A^T(A^T)^*$. This implies that the singular values of A^TP are the same as A^T (since these are the square roots of the eigenvalues of $A^T(A^T)^* = (A^TP)(A^TP)^*$. But since $A^TP = B$, we have shown that the singular values of $A^TP = B$ are the same as A. Thus, the singular values of A are the same as the singular values of B, as desired.

Exercise 4.3

See the MATLAB script for this exercise.

Exercise 5.3

Consider the matrix

$$A = \begin{pmatrix} -2 & 11 \\ -10 & 5 \end{pmatrix}.$$

(a) We will determine a real SVD of A of the form $A = U\Sigma V^T$ such that one has the minimal number of minus signs in U and V. Using $AA^* = AA^T = U\Sigma\Sigma^*U^* = U\Sigma^2U^T$, we have that $U\Sigma U^T$ is a diagonalization of AA^T. So we will find the eigenvalues, σ_1^2 and σ_2^2 of Σ^2 and corresponding normalized eigenvectors to form U.

$$\begin{aligned} 0 &= \det(AA^T - \sigma^2 I) \\ &= \det\left(\begin{pmatrix} 125-\sigma & 75 \\ 75 & 125-\sigma^2 \end{pmatrix}\right) \\ &= (125-\sigma^2)^2 - 75^2 \\ &\implies \sigma_1^2 = 200, \quad \sigma_2^2 = 50 \end{aligned}$$

Next find an eigenvector for $\sigma_1^2 = 200$.

$$\begin{pmatrix} 125-200 & 75 & 0 \\ 75 & 125-200 & 0 \end{pmatrix} = \begin{pmatrix} -75 & 75 & 0 \\ 75 & -75 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Next find an eigenvector for $\sigma_2^2 = 50$.

$$\begin{pmatrix} 125-50 & 75 & 0 \\ 75 & 125-50 & 0 \end{pmatrix} = \begin{pmatrix} 75 & 75 & 0 \\ 75 & 75 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

We take $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \sqrt{200} & 0 \\ 0 & \sqrt{50} \end{pmatrix} = \begin{pmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{pmatrix}$.

Then since $A = U\Sigma V^T$, $U^T A = \Sigma V^T$ so that $A^T U \Sigma^{-1} = V$. We can use this to find V:

$$\begin{aligned}
V &= A^T U \Sigma^{-1} \\
&= \begin{pmatrix} -2 & -10 \\ 11 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} -2 & -10 \\ 11 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{20} & \frac{1}{10} \\ \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \\
&= \begin{pmatrix} -12/20 & 8/10 \\ 16/20 & 6/10 \end{pmatrix} \\
&= \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}.
\end{aligned}$$

Therefore we have a factorization $A = U\Sigma V^T$ with:

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{pmatrix}, \quad V = \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}.$$

(b) The singular values of A are $\sigma_1 = 10\sqrt{2}$ and $\sigma_2 = 5\sqrt{2}$.

The left singular vectors of A are:

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

The right singular vectors of A are:

$$v_1 = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix} .$$