MA 514 Homework 3

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Exercise 6.1

If P is an orthogonal projector, then $I - 2P$ is unitary. Prove this algebraically, and given a geometric interpretation.

Answer: Suppose P is an orthogonal projector. Since P is a projector, $P^2 = P$ by definition. By Theorem 6.1, P is orthogonal if and only if $P = P^*$. To show that $I - 2P$ is unitary, we need to show $(I-2P)^*(I-2P) = (I-2P)(I-2P)^* = I$.

$$\begin{aligned}(I-2P)^*(I-2P) &= (I^* - 2P^*)(I-2P) = (I-2P)(I-2P) \\ &= I^2 - 4P + 4P^2 = I - 4P + 4P = I\end{aligned}$$

$$\begin{aligned}(I-2P)(I-2P)^* &= (I-2P)(I^* - 2P^*) = (I-2P)(I-2P) \\ &= I^2 - 4P + 4P^2 = I - 4P + 4P = I\end{aligned}$$

Exercise 6.2

Let E be the $m \times m$ matrix that extracts the "even part" of an m-vector: $Ex = (x + Fx)/2$, where F is the $m \times m$ matrix that flips $(x_1, ..., x_m)^*$ to $(x_m, ..., x_1)^*$. Is E is an orthogonal projector, an oblique projector, or not a projector at all? What are the entries of E?

Answer: First note that for any $x = (x_1, ..., x_m)^* \in \mathbb{C}^m$, we have $F^2 x = F(Fx) = F(F(x_1, ..., x_m)^*) = F(x_m, ..., x_1)^* = (x_1, ..., x_m)^*$. This shows that $F^2 = I$. To see if E is a projector, we check whether $E^2 = E$ to meet the definition.

$$E^2 = \frac{I+F}{2}\frac{I+F}{2} = \frac{I^2+2F+F^2}{4} = \frac{I+2F+I}{4} = \frac{I+F}{2} = E \ .$$

Therefore, we know at least that E is a projector. To see if E is orthogonal, we check whether $E^* = E$ (according to Theorem 6.1). First note that for F to perform the transformation as the problem statement describes, it must be that F is the $m \times m$ matrix with ones on the antidiagonal and zeros elsewhere. That is,

$$F = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 0 & 0 & ... & 1 & 0 \\ : & : & ... & : & : \\ 0 & 1 & ... & 0 & 0 \\ 1 & 0 & ... & 0 & 0 \end{pmatrix} \quad \text{so that} \quad F(x_1, ..., x_m)^* = (x_m, ..., x_1)^* \ .$$

But then since all elements of F are real and the elements on the antidiagonal of F are all the same, $F^* = F$. Using this fact,

$$E^* = \left(\frac{I+F}{2}\right)^* = \frac{I^*+F^*}{2} = \frac{I+F}{2} = E \ .$$

Therefore, we see that E is an orthogonal projector. Since we defined an *oblique projector* to be a projector that is not an orthogonal projector, we conclude that E is not an oblique projector.

The entries of E are such that there are 1's along the main diagonal and 1's along the antidiagonal. If E is $n \times n$ for n an odd integer, however, these 1's intersect in the very middle entry of the matrix and so in this case there is a 2 in the middlemost entry of E.

6.4

Consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

(a) What is the orthogonal projector P onto range(A), and what is the image under P of the vector $(1, 2, 3)^*$.

Answer: It is immediate that the columns of A span range(A) and since the columns of A are linearly independent, the reasoning on page 46 applies so that we can use equation (6.13) to find P. Then,

$$\begin{aligned}
P &= A(A^*A)^{-1}A^* \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} .
\end{aligned}$$

$$P(1,2,3)^* = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} .$$

(b) What is the orthogonal projector P onto range(B), and what is the image under P of the vector $(1,2,3)^*$.

Answer: Just as before, since the columns of B are a basis for range(B) we may apply equation (6.13). Then,

$$\begin{aligned}
P &= B(B^*B)^{-1}B^* \\
&= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{pmatrix} .
\end{aligned}$$

$$P(1,2,3)^* = \begin{pmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} .$$

Exercise 7.1

Consider again the matrices A and B of Exercise 6.4.

(a) Using any method you like, determine (on paper) a reduced QR factorization $A = \hat{Q}\hat{R}$ and a full QR factorization $A = QR$.

Answer: Denote the columns of A by a_1 and a_2.

- $$q_1 = \frac{a_1}{r_{11}} = \frac{a_1}{||a_1||_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

- $$q_2 = \frac{a_2 - r_{12}q_1}{r_{22}}, \quad r_{12} = q_1^* a_2, \quad r_{22} = ||a_2 - r_{12}q_1||_2$$

$$r_{12} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$r_{22} = ||a_2 - 0q_1||_2 = 1$$

$$q_2 = \frac{a_2 - 0q_1}{r_{22}} = a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- $$A = \hat{Q}\hat{R} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

To get a full QR factorization we need to add a third column q_3 to $\hat{Q}$ such that $\{q_1, q_2, q_3\}$ are orthonormal and then add a row of zeros to $\hat{R}$. By inspection, $q_3 = \frac{1}{\sqrt{2}}(1, 0, -1)^*$ will work. Therefore, a full QR factorization is:

$$A = QR = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(b) Again using any method you like, determine reduced and full QR factorizations $B = \hat{Q}\hat{R}$ and $B = QR$.

Answer: Denote the columns of B by b_1 and b_2. Since $b_1 = a_1$, q_1 and r_{11} are the same as in part (a). Next,

$$q_2 = \frac{b_2 - r_{12}q_1}{r_{22}}, \quad r_{12} = q_1^* b_2, \quad r_{22} = ||b_2 - r_{12}q_1||_2$$

$$r_{12} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2}$$

$$b_2 - r_{12}q_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \implies r_{22} = \sqrt{3}$$

$$q_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$$

So we have the reduced QR factorization of B given by:

$$B = \hat{Q}\hat{R} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}$$

To find a full QR factorization, we find that any multiple of $(-1, 2, 1)^*$ will be orthogonal to q_1 and q_2. So normalizing this vector gives an option for q_3 so that:

$$B = QR = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}.$$