MA 528 Exercise Set 2

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Chapter 4 Probabilities on a Finite or Countable Space

Exercise 4.1 (Poisson Approximation to the Binomial)

Let P be a binomial probability with probability of success p and number of trials n. Let $\lambda = pn$. . Show that

$$P(k \text{ successes}) = \frac{\lambda^k}{k!}(1-\frac{\lambda}{n})^n \left\{\frac{n}{n}\frac{n-1}{n}\dots\frac{n-k+1}{n}\right\}(1-\frac{\lambda}{n})^{-k}.$$

Let $n \to \infty$ and let p change so that λ remains constant. Conclude that for small p and large n,

$$P(k \text{ successes}) \approx \frac{\lambda^k}{k!}e^{-\lambda}, \quad \text{where } \lambda = pn.$$

Answer:

$$\begin{aligned}
P(X=k) &= \binom{n}{k}p^k(1-p)^{n-k} \\
&= \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}\frac{n(n-1)\dots(n-k+1)}{n^k} \\
&= \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}\frac{n}{n}\frac{n-1}{n}\dots\frac{n-k+1}{n} \\
&= \frac{\lambda^k}{k!}\left(1+\left(-\frac{\lambda}{n}\right)\right)^n\left\{\frac{n}{n}\frac{n-1}{n}\dots\frac{n-k+1}{n}\right\}\left(1-\frac{\lambda}{n}\right)^{-k} \\
&\approx \frac{\lambda^k}{k!}e^{-n\lambda/n}\left\{\frac{n}{n}\frac{n-1}{n}\dots\frac{n-k+1}{n}\right\}\left(1-\frac{\lambda}{n}\right)^{-k} \qquad ((1+a)^n \approx e^{na} \text{ for small } a)^* \\
&= \frac{\lambda^k}{k!}e^{-\lambda}\left\{\frac{n}{n}\frac{n-1}{n}\dots\frac{n-k+1}{n}\right\}\left(1-\frac{\lambda}{n}\right)^{-k} \\
&\to \frac{\lambda^k}{k!}e^{-\lambda}\cdot 1\cdot 1^{-k} = \frac{\lambda^k}{k!}e^{-\lambda} \quad \text{as} \quad n \to \infty.
\end{aligned}$$

*The approximation $(1+a) \approx e^{na}$ for 'small a' warrants discussion.

$$
\begin{aligned}
\frac{1}{1+a} &= 1 - a + a^2 - a^3 + \dots, \quad |a| < 1 \\
\ln(1+a) &= a - \frac{a^2}{2} + \frac{a^3}{3} - \dots \\
1+a &= e^a e^{-a^2/2} e^{a^3/3} \dots \\
(1+a)^n &= e^{na} e^{-na^2/2} e^{na^3/3} \dots
\end{aligned}
$$

If $na << 1$, $(1+a)^n \approx e^0 = 1$. In our calculation, $n(-\lambda/n) = \lambda$ does not satisfy this criterion. If $na^2 << 1$, $(1+a)^n \approx e^{na}$. In our calculation $n(-\lambda/n)^2 = \lambda/n << 1$ for large n assuming that we let p change so that λ remains constant as $n \to \infty$.

Exercise 4.2 (Poisson Approximation to the Binomial continued)

In the setting of Exercise 4.1, let $p_k = P(\{k\})$ and $q_k = 1 - p_k$. Show that the q_k are the probabilities of singletons for a Binomial distribution $B(1-p, n)$. Deduce a Poisson approximation of the Binomial when n is large and p is close to 1.

Answer: For $X \sim B(n, p)$ and $Y \sim B(n, 1-p)$,

$$
\begin{aligned}
P(\{k\}) &= P(X = k) \\
&= \binom{n}{k} p^k (1-p)^{n-k} \\
&= \binom{n}{n-k} p^k (1-p)^{n-k} \\
&= \binom{n}{j} p^{n-j} (1-p)^j, \quad j = n-k \\
&= \binom{n}{j} (1-p)^j p^{n-j} \\
&= \binom{n}{j} q^j (1-q)^{n-j} \\
&= P(Y = j) = P(\{j\})
\end{aligned}
$$

$\lambda = pn \implies n - \lambda = n(1-p) = nq$.

$$
P(Y = j) \approx \frac{(n-\lambda)^j}{j!} e^{\lambda - n} \quad \text{for large } n
$$

Exercise 4.3

We consider the setting of the hypergeometric distribution, except that we have m colors and N_i balls of color i. Set $N = N_1 + \dots + N_m$, and call X_i the number of balls of color i drawn among n balls. Of course $X_1 + \dots + X_m = n$. Show that

$$
P(X_1 = x_1, \dots X_m = x_m) = \begin{cases} \frac{\binom{N_1}{x_1} \dots \binom{N_m}{x_m}}{\binom{N}{n}} & \text{if } x_1 + \dots x_m = n \\ 0 & \text{otherwise} \end{cases}
$$

Answer: Consider an outcome to be a subset (containing n elements) of the set $\{1, 2, \dots, N\}$ of all $N = N_1 + \dots N_m$ balls (which can be assumed to be numbered from 1 to N). That is, Ω is the family of all subsets of $\{1, \dots, N\}$ with n points and $\#(\Omega) = \binom{N}{n}$.

Consider the case where P is the uniform probability on Ω. The set $\{X_1 = x_1, \dots, X_m = x_m\} = X^{-1}(\{(x_1, \dots, x_m\})$ contains $\binom{N_1}{x_1}\binom{N_2}{x_2}\dots\binom{N_m}{m}$ points for $x_1 + x_2 + \dots + x_m = n$. If $x_1 + \dots + x_m \neq n$, $\{X_1 = x_1, \dots X_m = x_m\} = \emptyset$ so $\#(X_1 = x_1, \dots X_m = x_m) = 0$. Therefore,

$$P(X_1 = x_1, \dots, X_m = x_m) = \frac{\#(X_1 = x_1, \dots, X_m = x_m)}{\#(\Omega)} = \begin{cases} \frac{\binom{N_1}{x_1}\binom{N_2}{x_2}\dots\binom{N_m}{x_m}}{\binom{N}{n}} & \text{if } x_1 + \dots + x_m = n \\ 0 & \text{otherwise} \end{cases}$$

Chapter 5 Random Variables on a Countable Space

Exercise 5.1

Let $g : [0, \infty) \to [0, \infty)$ be strictly increasing and nonnegative. Show that

$$P(\{|X| \geq a\}) \leq \frac{E\{g(|X|)\}}{g(a)}, \quad a > 0.$$

Answer: For $a > 0$, $\{|X| \geq a\} = \{\omega : |X(\omega)| \geq a\} = \{\omega : g(|X(\omega)| \geq g(a)\} = \{g(|X|) \geq g(a)\}$ since g is strictly increasing and nonnegative. Also $|X|$ is random variable since X is a random variable. Since g is strictly increasing and has domain $[0, \infty)$, the only possible x such that $g(x) = 0$ is $x = 0$. We must have $g(a) > 0$ because $a > 0$. By Theorem 5.1,

$$P(\{|X| \geq a\}) = P(\{g(|X|) \geq g(a)\}) \leq \frac{E\{g(|X|)\}}{g(a)}.$$

Exercise 5.6

Let X be Binomial $B(p, n)$. For what value of j is $P(X = j)$ the greatest?

Answer:
For $k \geq 1$, $P(X = k - 1) > 0$ and

$$\frac{P(X = k)}{P(X = k - 1)} = \frac{\binom{n}{k}p^k(1-p)^{n-k}}{\binom{n}{k-1}p^{k-1}(1-p)^{n-k+1}} = \frac{n!}{k!(n-k)!}\frac{(k-1)!(n-k+1)!}{n!}\frac{p}{1-p} = \frac{n-k+1}{k}\frac{p}{1-p}.$$

We have $P(X = k) \geq P(X = k - 1)$ for $P(X = k)/P(X = k - 1) \geq 1$.

$$\frac{n-k+1}{k}\frac{p}{1-p} \geq 1 \iff (n+1)p \geq k.$$

Let $j = [(n + 1)p]$, where $[x]$ is the floor function. For any $k < j$, $k \leq j - 1$ so $P(X = j) \geq P(X = k)$. For any $k > j$, $P(X = j) < P(X = k)$. Conclude that $P(X = j)$ is greatest for $j = [(n + 1)p]$.

Exercise 5.7

Let X be Binomial $B(p, n)$. Find the probability X is even.

Answer: If $p = 0$ then X, there will always be 0 successes so the probability that X is even is 1. If $p = 1$, then $X = n$ and so the probability that X is even is 1 if n is even and 0 if n is odd.

Let P_n be the probability that $X \sim B(p, n), p \in (0, 1)$ is an even number. To calculate the probability P_{n+1} consider for $X \sim B(p, n + 1)$ whether there are an even number of successes or an odd number of successes from n trials. Then $P_{n+1} = (1-p)P_n + p(1-P_n) = p + (1-2p)P_n$. Motivated by the idea that P_n approaches 1/2 as n approaches $+\infty$, let $x_n = P_n - \frac{1}{2}$.

$$x_{n+1} = P_{n+1} - \frac{1}{2} = p + (1 - 2p)(x_n + \frac{1}{2}) + \frac{1}{2} = (1 - 2p)x_n.$$

Since the probability that X is even for $X \sim B(0, p)$ is 1, $x_0 = P_0 - \frac{1}{2} = \frac{1}{2}$. Thus (x_n) is a geometric sequence with common ratio $(1 - 2p)$ so we write $x_n = \frac{1}{2}(1 - 2p)^n$. For $0 < p < 1$, $|1 - 2p| < 1$ so $x_n \to 0$ as $n \to \infty$. This means that the probability that $X \sim B(n, p)$ is even is $P_n = \frac{1}{2} + x_n = \frac{1}{2}(1 + (1 - 2p)^n)$. Checking P_n for $p = 0$ and $p = 1$ shows that this formula holds for any $p \in [0, 1]$.

Exercise 5.9

Let X be Poisson (λ). What value of j maximizes $P(X = j)$?

Answer: Consider that $P(X = k) \geq P(X = k - 1)$ for $P(X = k)/P(X = k - 1) \geq 1$.

$$\frac{P(X = k)}{P(X = k - 1)} = \frac{\frac{\lambda^k}{k!}e^{-\lambda}}{\frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}} = \frac{\lambda}{k}.$$

This shows that $P(X = k) \geq P(X = k - 1)$ for $k \leq \lambda$. Let $j = [\lambda]$, where $[x]$ is the floor function. By the reasoning given in Exercise 5.6, j maximizes $P(X = j)$.

Exercise 5.12

Let X be Binomial $B(p, n)$. Show that for $\lambda > 0$ and $\epsilon > 0$,

$$P(X - np > n\epsilon) \leq E\{\exp(\lambda(X - np - n\epsilon))\}.$$

Partial Answer:

$$P(X - np > n\epsilon) \leq P(|X - np| \geq n\epsilon) \leq \frac{E\{\exp(\lambda|X - np|)\}}{\exp(\lambda n\epsilon)} = E\{\exp(\lambda(|X - np| - n\epsilon\}$$

Exercise 5.13

Let X_n be Binomial $B(p, n)$ with $p > 0$ fixed. Show that for any fixed $b > 0$, $P(X_n \leq b)$ tends to 0.

Exercise 5.16

Let X be Geometric. Show that for $i, j > 0$,

$$P(X > i + j|X > i) = P(X > j).$$

Answer:

$$
\begin{aligned}
P(X > i+j | X > i) &= \frac{P(\{X > i+j\} \cap \{X > i\})}{P(X > i)} \\
&= \frac{P(X > i+j)}{P(X > i)} \\
&= \frac{P(X = i+j+1) + P(X = i+j+2) + \dots}{P(X = i+1) + \dots + P(X = i+j+1) + \dots} \\
&= \frac{p(1-p)^{i+j+1} + p(1-p)^{i+j+2} + \dots}{p(1-p)^{i+1} + p(1-p)^{i+2} + \dots + p(1-p)^{i+j+1} + \dots} \\
&= \frac{(1-p)^j + (1-p)^{j+1} + \dots}{1 + (1-p) + \dots + (1-p)^j + \dots} \\
&= \frac{(1-p)^j (1 + (1-p) + \dots}{1 + (1-p) + \dots} \\
&= (1-p)^j \\
&= p(1-p)^j \frac{1}{1-(1-p)} \\
&= p(1-p)^j (1 + (1-p) + (1-p)^2 + \dots) \\
&= p(1-p)^{j+1} + p(1-p)^{j+2} + \dots \\
&= P(X = j+1) + P(X = j+2) + \dots \\
&= P(X > j).
\end{aligned}
$$

Exercise 5.17

Let X be Geometric (p). Show

$$
E\left\{\frac{1}{1+X}\right\} = \log\left((1-p)^{\frac{p}{p-1}}\right).
$$

Partial Answer:

$$
\begin{aligned}
E\left\{\frac{1}{1+X}\right\} &= \sum_{k=0}^{\infty} \frac{1}{1+k} P\left(X = k\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{1+k} p(1-p)^k \\
&= \sum_{j=1}^{\infty} \frac{1}{j} p(1-p)^{j-1} \\
&= \frac{p}{1-p} \sum_{j=1}^{\infty} \frac{(1-p)^j}{j} \\
&= -\frac{p}{1-p} \log(1-(1-p)) \\
&= \frac{p}{p-1} \log(p) \\
&= \log\left(p^{\frac{p}{p-1}}\right)
\end{aligned}
$$

Exercise 5.19

Show that for a sequence of events $(A_n)_{n\geq 1}$,

$$E\left\{\sum_{n=1}^{\infty} 1_{A_n}\right\} = \sum_{n=1}^{\infty} P(A_n).$$

Answer: Using properties (i) and (v) of $\mathcal{L}^1$ given on page 28,

$$E\left\{\sum_{n=1}^{\infty} 1_{A_n}\right\} = \sum_{n=1}^{\infty} E\{1_{A_n}\} = \sum_{n=1}^{\infty} P(A_n)\ .$$

Exercise 5.20

Suppose X takes all its values in $\mathbb{N} = \{0, 1, 2, 3, ...\}$. Show that

$$E\{X\} = \sum_{n=0}^{\infty} P(X > n).$$