

## MA 528 Exercise Set 2

Dane Johnson

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### Chapter 4 Probabilities on a Finite or Countable Space

#### Exercise 4.1 (Poisson Approximation to the Binomial)

Let  $P$  be a binomial probability with probability of success  $p$  and number of trials  $n$ . Let  $\lambda = pn$ . . Show that

$$P(k \text{ successes}) = \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left\{ \frac{n}{n} \frac{n-1}{n} \dots \frac{n-k+1}{n} \right\} \left(1 - \frac{\lambda}{n}\right)^{-k}.$$

Let  $n \rightarrow \infty$  and let  $p$  change so that  $\lambda$  remains constant. Conclude that for small  $p$  and large  $n$ ,

$$P(k \text{ successes}) \approx \frac{\lambda^k}{k!} e^{-\lambda}, \quad \text{where } \lambda = pn.$$

Answer:

$$\begin{aligned} P(X = k) &= \binom{n}{k} p^k (1-p)^{n-k} \\ &= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} \frac{n(n-1)\dots(n-k+1)}{n^k} \\ &= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} \frac{n}{n} \frac{n-1}{n} \dots \frac{n-k+1}{n} \\ &= \frac{\lambda^k}{k!} \left(1 + \left(-\frac{\lambda}{n}\right)\right)^n \left\{ \frac{n}{n} \frac{n-1}{n} \dots \frac{n-k+1}{n} \right\} \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &\approx \frac{\lambda^k}{k!} e^{-n\lambda/n} \left\{ \frac{n}{n} \frac{n-1}{n} \dots \frac{n-k+1}{n} \right\} \left(1 - \frac{\lambda}{n}\right)^{-k} \quad ((1+a)^n \approx e^{na} \text{ for small } a)^* \\ &= \frac{\lambda^k}{k!} e^{-\lambda} \left\{ \frac{n}{n} \frac{n-1}{n} \dots \frac{n-k+1}{n} \right\} \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &\rightarrow \frac{\lambda^k}{k!} e^{-\lambda} \cdot 1 \cdot 1^{-k} = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

\*The approximation  $(1+a) \approx e^{na}$  for 'small  $a$ ' warrants discussion.

$$\begin{aligned}
\frac{1}{1+a} &= 1 - a + a^2 - a^3 + \dots, \quad |a| < 1 \\
\ln(1+a) &= a - \frac{a^2}{2} + \frac{a^3}{3} - \dots \\
1+a &= e^a e^{-a^2/2} e^{a^3/3} \dots \\
(1+a)^n &= e^{na} e^{-na^2/2} e^{na^3/3} \dots
\end{aligned}$$

If  $na \ll 1$ ,  $(1+a)^n \approx e^0 = 1$ . In our calculation,  $n(-\lambda/n) = \lambda$  does not satisfy this criterion. If  $na^2 \ll 1$ ,  $(1+a)^n \approx e^{na}$ . In our calculation  $n(-\lambda/n)^2 = \lambda/n \ll 1$  for large  $n$  assuming that we let  $p$  change so that  $\lambda$  remains constant as  $n \rightarrow \infty$ .

### Exercise 4.2 (Poisson Approximation to the Binomial continued)

In the setting of Exercise 4.1, let  $p_k = P(\{k\})$  and  $q_k = 1 - p_k$ . Show that the  $q_k$  are the probabilities of singletons for a Binomial distribution  $B(1-p, n)$ . Deduce a Poisson approximation of the Binomial when  $n$  is large and  $p$  is close to 1.

Answer: For  $X \sim B(n, p)$  and  $Y \sim B(n, 1-p)$ ,

$$\begin{aligned}
P(\{k\}) &= P(X = k) \\
&= \binom{n}{k} p^k (1-p)^{n-k} \\
&= \binom{n}{n-k} p^k (1-p)^{n-k} \\
&= \binom{n}{j} p^{n-j} (1-p)^j, \quad j = n-k \\
&= \binom{n}{j} (1-p)^j p^{n-j} \\
&= \binom{n}{j} q^j (1-q)^{n-j} \\
&= P(Y = j) = P(\{j\})
\end{aligned}$$

$$\lambda = pn \implies n - \lambda = n(1-p) = nq.$$

$$P(Y = j) \approx \frac{(n-\lambda)^j}{j!} e^{\lambda-n} \quad \text{for large } n$$

### Exercise 4.3

We consider the setting of the hypergeometric distribution, except that we have  $m$  colors and  $N_i$  balls of color  $i$ . Set  $N = N_1 + \dots + N_m$ , and call  $X_i$  the number of balls of color  $i$  drawn among  $n$  balls. Of course  $X_1 + \dots + X_m = n$ . Show that

$$P(X_1 = x_1, \dots, X_m = x_m) = \begin{cases} \frac{\binom{N_1}{x_1} \dots \binom{N_m}{x_m}}{\binom{N}{n}} & \text{if } x_1 + \dots + x_m = n \\ 0 & \text{otherwise} \end{cases}$$

Answer: Consider an outcome to be a subset (containing  $n$  elements) of the set  $\{1, 2, \dots, N\}$  of all  $N = N_1 + \dots + N_m$  balls (which can be assumed to be numbered from 1 to  $N$ ). That is,  $\Omega$  is the family of all subsets of  $\{1, \dots, N\}$  with  $n$  points and  $\#(\Omega) = \binom{N}{n}$ .

Consider the case where  $P$  is the uniform probability on  $\Omega$ . The set  $\{X_1 = x_1, \dots, X_m = x_m\} = X^{-1}(\{(x_1, \dots, x_m)\})$  contains  $\binom{N_1}{x_1} \binom{N_2}{x_2} \dots \binom{N_m}{x_m}$  points for  $x_1 + x_2 + \dots + x_m = n$ . If  $x_1 + \dots + x_m \neq n$ ,  $\{X_1 = x_1, \dots, X_m = x_m\} = \emptyset$  so  $\#(X_1 = x_1, \dots, X_m = x_m) = 0$ . Therefore,

$$P(X_1 = x_1, \dots, X_m = x_m) = \frac{\#(X_1 = x_1, \dots, X_m = x_m)}{\#(\Omega)} = \begin{cases} \frac{\binom{N_1}{x_1} \binom{N_2}{x_2} \dots \binom{N_m}{x_m}}{\binom{N}{n}} & \text{if } x_1 + \dots + x_m = n \\ 0 & \text{otherwise} \end{cases}$$

## Chapter 5 Random Variables on a Countable Space

### Exercise 5.1

Let  $g : [0, \infty) \rightarrow [0, \infty)$  be strictly increasing and nonnegative. Show that

$$P(\{|X| \geq a\}) \leq \frac{E\{g(|X|)\}}{g(a)}, \quad a > 0.$$

Answer: For  $a > 0$ ,  $\{|X| \geq a\} = \{\omega : |X(\omega)| \geq a\} = \{\omega : g(|X(\omega)|) \geq g(a)\} = \{g(|X|) \geq g(a)\}$  since  $g$  is strictly increasing and nonnegative. Also  $|X|$  is random variable since  $X$  is a random variable. Since  $g$  is strictly increasing and has domain  $[0, \infty)$ , the only possible  $x$  such that  $g(x) = 0$  is  $x = 0$ . We must have  $g(a) > 0$  because  $a > 0$ . By Theorem 5.1,

$$P(\{|X| \geq a\}) = P(\{g(|X|) \geq g(a)\}) \leq \frac{E\{g(|X|)\}}{g(a)}.$$

### Exercise 5.6

Let  $X$  be Binomial  $B(p, n)$ . For what value of  $j$  is  $P(X = j)$  the greatest?

Answer:

For  $k \geq 1$ ,  $P(X = k - 1) > 0$  and

$$\frac{P(X = k)}{P(X = k - 1)} = \frac{\binom{n}{k} p^k (1 - p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1 - p)^{n-k+1}} = \frac{n!}{k!(n-k)!} \frac{(k-1)!(n-k+1)!}{n!} \frac{p}{1-p} = \frac{n-k+1}{k} \frac{p}{1-p}.$$

We have  $P(X = k) \geq P(X = k - 1)$  for  $P(X = k)/P(X = k - 1) \geq 1$ .

$$\frac{n-k+1}{k} \frac{p}{1-p} \geq 1 \iff (n+1)p \geq k.$$

Let  $j = \lceil (n+1)p \rceil$ , where  $\lceil x \rceil$  is the floor function. For any  $k < j$ ,  $k \leq j - 1$  so  $P(X = j) \geq P(X = k)$ . For any  $k > j$ ,  $P(X = j) < P(X = k)$ . Conclude that  $P(X = j)$  is greatest for  $j = \lceil (n+1)p \rceil$ .

### Exercise 5.7

Let  $X$  be Binomial  $B(p, n)$ . Find the probability  $X$  is even.

Answer: If  $p = 0$  then  $X$ , there will always be 0 successes so the probability that  $X$  is even is 1. If  $p = 1$ , then  $X = n$  and so the probability that  $X$  is even is 1 if  $n$  is even and 0 if  $n$  is odd.

Let  $P_n$  be the probability that  $X \sim B(p, n)$ ,  $p \in (0, 1)$  is an even number. To calculate the probability  $P_{n+1}$  consider for  $X \sim B(p, n+1)$  whether there are an even number of successes or an odd number of successes from  $n$  trials. Then  $P_{n+1} = (1-p)P_n + p(1-P_n) = p + (1-2p)P_n$ . Motivated by the idea that  $P_n$  approaches  $1/2$  as  $n$  approaches  $+\infty$ , let  $x_n = P_n - \frac{1}{2}$ .

$$x_{n+1} = P_{n+1} - \frac{1}{2} = p + (1-2p)(x_n + \frac{1}{2}) + \frac{1}{2} = (1-2p)x_n.$$

Since the probability that  $X$  is even for  $X \sim B(0, p)$  is 1,  $x_0 = P_0 - \frac{1}{2} = \frac{1}{2}$ . Thus  $(x_n)$  is a geometric sequence with common ratio  $(1-2p)$  so we write  $x_n = \frac{1}{2}(1-2p)^n$ . For  $0 < p < 1$ ,  $|1-2p| < 1$  so  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ . This means that the probability that  $X \sim B(n, p)$  is even is  $P_n = \frac{1}{2} + x_n = \frac{1}{2}(1 + (1-2p)^n)$ . Checking  $P_n$  for  $p = 0$  and  $p = 1$  shows that this formula holds for any  $p \in [0, 1]$ .

### Exercise 5.9

Let  $X$  be Poisson ( $\lambda$ ). What value of  $j$  maximizes  $P(X = j)$ ?

Answer: Consider that  $P(X = k) \geq P(X = k-1)$  for  $P(X = k)/P(X = k-1) \geq 1$ .

$$\frac{P(X = k)}{P(X = k-1)} = \frac{\frac{\lambda^k}{k!} e^{-\lambda}}{\frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}} = \frac{\lambda}{k}.$$

This shows that  $P(X = k) \geq P(X = k-1)$  for  $k \leq \lambda$ . Let  $j = [\lambda]$ , where  $[x]$  is the floor function. By the reasoning given in Exercise 5.6,  $j$  maximizes  $P(X = j)$ .

### Exercise 5.12

Let  $X$  be Binomial  $B(p, n)$ . Show that for  $\lambda > 0$  and  $\epsilon > 0$ ,

$$P(X - np > n\epsilon) \leq E\{\exp(\lambda(X - np - n\epsilon))\}.$$

Answer: Using the fact that the exponential function is strictly increasing and nonnegative as well as Markov's Inequality

$$P(X - np - n\epsilon \geq 0) = P(\exp(X - np - n\epsilon) \geq 1) = P(|\exp(X - np - n\epsilon)| \geq 1) \leq E\{\exp(X - np - n\epsilon)\}.$$

Now where does  $\lambda$  appear? I must have missed this before.

### Exercise 5.13

Let  $X_n$  be Binomial  $B(p, n)$  with  $p > 0$  fixed. Show that for any fixed  $b > 0$ ,  $P(X_n \leq b)$  tends to 0.

Partial Answer:

$$\begin{aligned} P(X_n \leq b) &= P(X_n \leq [b]) \\ &= P(X = 0) + \dots + P(X = [b]) \\ &= \sum_{k=0}^{[b]} \binom{n}{k} p^k (1-p)^{n-k} \end{aligned}$$

If I could show that this expression tends to 0 this would give the result. But I forgot to ask for help on this one.

**Exercise 5.16**

Let  $X$  be Geometric. Show that for  $i, j > 0$ ,

$$P(X > i + j | X > i) = P(X > j).$$

Answer:

$$\begin{aligned}
 P(X > i + j | X > i) &= \frac{P(\{X > i + j\} \cap \{X > i\})}{P(X > i)} \\
 &= \frac{P(X > i + j)}{P(X > i)} \\
 &= \frac{P(X = i + j + 1) + P(X = i + j + 2) + \dots}{P(X = i + 1) + \dots + P(X = i + j + 1) + \dots} \\
 &= \frac{p(1 - p)^{i+j+1} + p(1 - p)^{i+j+2} + \dots}{p(1 - p)^{i+1} + p(1 - p)^{i+2} + \dots + p(1 - p)^{i+j+1} + \dots} \\
 &= \frac{(1 - p)^j + (1 - p)^{j+1} + \dots}{1 + (1 - p) + \dots + (1 - p)^j + \dots} \\
 &= \frac{(1 - p)^j (1 + (1 - p) + \dots)}{1 + (1 - p) + \dots} \\
 &= (1 - p)^j \\
 &= p(1 - p)^j \frac{1}{1 - (1 - p)} \\
 &= p(1 - p)^j (1 + (1 - p) + (1 - p)^2 + \dots) \\
 &= p(1 - p)^{j+1} + p(1 - p)^{j+2} + \dots \\
 &= P(X = j + 1) + P(X = j + 2) + \dots \\
 &= P(X > j).
 \end{aligned}$$

**Exercise 5.17**

Let  $X$  be Geometric ( $p$ ). Show

$$E \left\{ \frac{1}{1 + X} \right\} = \log \left( p^{\frac{p}{p-1}} \right).$$

Answer:

$$\begin{aligned}
E\left\{\frac{1}{1+X}\right\} &= \sum_{k=0}^{\infty} \frac{1}{1+k} P(X=k) \\
&= \sum_{k=0}^{\infty} \frac{1}{1+k} p(1-p)^k \\
&= \sum_{j=1}^{\infty} \frac{1}{j} p(1-p)^{j-1} \\
&= \frac{p}{1-p} \sum_{j=1}^{\infty} \frac{(1-p)^j}{j} \\
&= -\frac{p}{1-p} \log(1 - (1-p)) \\
&= \frac{p}{p-1} \log(p) \\
&= \log\left(p^{\frac{p}{p-1}}\right)
\end{aligned}$$

### Exercise 5.19

Show that for a sequence of events  $(A_n)_{n \geq 1}$ ,

$$E\left\{\sum_{n=1}^{\infty} 1_{A_n}\right\} = \sum_{n=1}^{\infty} P(A_n).$$

Answer: Using properties (i) and (v) of  $\mathcal{L}^1$  given on page 28,

$$E\left\{\sum_{n=1}^{\infty} 1_{A_n}\right\} = \sum_{n=1}^{\infty} E\{1_{A_n}\} = \sum_{n=1}^{\infty} P(A_n).$$

### Exercise 5.20

Suppose  $X$  takes all its values in  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ . Show that

$$E\{X\} = \sum_{n=0}^{\infty} P(X > n).$$

Answer:

$$\begin{aligned}
\sum_{n=0}^{\infty} P(X > n) &= \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} P(X=k) \\
&= \sum_{n=1}^{\infty} n P(X=n) \\
&= \sum_{n=0}^{\infty} n P(X=n) \\
&= E\{X\}.
\end{aligned}$$