MA 528 Exercise Set 3

Dane Johnson

September 27, 2021

Chapter 6 Construction of a Probability Measure

Chapter 7 Construction of a Probability Measure on $\mathbb{R}$

Exercise 7.1

Let $(A_n)_{n\geq 1}$ be any sequence of pairwise disjoint events and P a probability. Show that $\lim_{n\to\infty} P(A_n) = 0$.

Answer: Since $P(A_i) \geq 0$ for each i the sequence $s_n = \sum_{i=1}^n P(A_i) = P(\cup_{i=1}^n A_i) \leq 1$ is a nonnegative sequence bounded that is bounded above 1 and is therefore convergent. This means the series $\lim_{n\to\infty} s_n = \sum_{i=1}^\infty P(A_i)$ is a convergent series, which implies $\lim_{n\to\infty} P(A_n) = 0$.

Exercise 7.6

Show that the maximum of the Lognormal density occurs at $x = e^\mu e^{-\sigma^2}$.

Answer: The Lognormal distribution with parameters μ, σ^2 $(-\infty < \mu < \infty,\ 0 < \sigma^2 < \infty)$ is

$$f(x) = \begin{cases} \frac{1}{x} g_{\mu,\sigma^2}(\log x) & x > 0 \\ 0 & x \leq 0 \end{cases},$$

where $g_{\mu,\sigma^2}(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-\mu)^2/(2\sigma^2)}$ is the normal distribution with parameters μ, σ^2. Fix the parameters μ, σ^2 and let $g(z) = g_{\mu,\sigma^2}(z)$, $\alpha = 1/(\sqrt{2\pi}\sigma)$. Since $g > 0$, $f(x) > 0$ for $x > 0$. So the maximum of f on $\mathbb{R}$ is the maximum of f on $(0, \infty)$.

$$\begin{aligned} 0 = f'(x) &= -\frac{1}{x^2} g(\log x) + \frac{1}{x^2} g'(\log x) \\ g(\log x) &= g'(\log x) \\ \alpha e^{-(\log x - \mu)^2/(2\sigma^2)} &= \alpha e^{-(z-\mu)^2/(2\sigma^2)} \frac{\mu - \log x}{\sigma^2} \\ 1 &= \frac{\mu - \log x}{\sigma^2} \\ \log x &= \mu - \sigma^2 \\ x &= e^\mu e^{-\sigma^2} \end{aligned}$$

Since $g > 0$, $\lim_{z\to\pm\infty} g(z) = 0$, and that there is only on critical point of f, conclude that this value of x must maximize $f(x)$.

Exercise 7.11

Let $P(A) = \int_{-\infty}^{\infty} 1_A(x)f(x)dx$ for a nonnegative function f with $\int_{-\infty}^{\infty} f(x)dx = 1$. Let $A = \{x_0\}$, a singleton (that is, the set A consists of one single point on the real line). Show that A is a Borel set and also a null set (that is, $P(A) = 0$).

Answer: We can write A as

$$(x_0 - 1, x_0] \cap [x_0, x_0 + 1)$$

$$(x_0 - 1, x_0] = \cap_{n=1}^{\infty}(x_0 - 1, x_0 + 1/n), \quad [x_0, x_0 + 1) = \cap_{n=1}^{\infty}(x_0 - 1/n, x_0)$$

This shows that A can be written as the intersection of two Borel sets, which must also be a Borel set and

$$P(A) = \int_{-\infty}^{\infty} 1_{\{x_0\}}(x)f(x)dx = \int_{x_0}^{x_0} f(x)dx = 0.$$

Exercise 7.14

Let $(A_i)_{i\geq 1}$ be a sequence of null sets. Show that $B = \cup_{i=1}^{\infty} A_i$ is also a null set.

Answer: For each A_i, there is a $B_i \in \mathcal{A}$ (the σ-algebra on Ω) such that $A_i \subset B_i$ and $P(B_i) = 0$. Let $B' = \cup_{i=1}^{\infty} B_i$. Then $B \subset B'$ and $0 \leq P(B') \leq \sum_{i=1}^{\infty} P(B_i) = 0$. This shows that $P(B') = 0$ and therefore B is a null set for P.

Exercise 7.15

Let X be a r.v. defined on a countable Probability space. Suppose $E\{|X|\} = 0$. Show that $X = 0$ except possibly on a null set. Is it possible to conclude, in general, that $X = 0$ everywhere (i.e., for all ω)?

Answer: If $0 = E\{|X|\} = \sum_{\omega\in\Omega} |X(\omega)|P(\{\omega\})$ implies that for each ω either $|X(\omega)| = 0$ of $P(\{\omega\}) = 0$. If $|X(\omega)| = 0$ for all ω we are done. Otherwise let B be the union of all singletons $\{\omega\}$ such that $P(\{\omega\}) = 0$. Since Ω is countable, this is either a finite or countably infinite union. By the previous exercise, B is also a null set. It is not possible to conclude that $X = 0$ everywhere.

Exercise 7.16

Let F be a distribution function. Show that in general F can have an infinite number of jump discontinuities, but that there can be at most countably many.

Answer: Since F is right continuous the set of discontinuities of F is the set $D = \{x : F(x-) \neq F(x+)\} = \{x : F(x-) < F(x)\}$. For each $x \in D$, select one $q_x \in \mathbb{Q}$ such that $F(x-) < q_x < F(x)$. If $x, y \in D$ with $x \neq y$ we may assume $x < y$ without loss of generality. Since F is nondecreasing, $q_x < F(x) \leq F(y-) < q_y$ so $q_x \neq q_y$. This means $r : D \to \mathbb{Q}$ defined by $r(x) = q_x$ is injective. Since $\mathbb{Q}$ is countable, $r(D) \subseteq \mathbb{Q}$ must also be countable. But then since r is injective conclude that D must be countable.

Exercise 7.17

Suppose a distribution function F is given by

$$F(x) = \frac{1}{4}1_{[0,\infty)} + \frac{1}{2}1_{[1,\infty)} + \frac{1}{4}1_{[2,\infty)}.$$

Let P be given by $P((-\infty, x]) = F(x)$. Then find the probabilities of the following events:

a) $A = \left(-\frac{1}{2}, \frac{1}{2}\right)$

b) $B = \left(-\frac{1}{2}, \frac{3}{2}\right)$

c) $C = \left(\frac{2}{3}, \frac{5}{2}\right)$

d) $D = [0, 2)$

e) $E = (3, \infty)$.

Answer:

a) $P(A) = F\left(\frac{1}{2}-\right) - F\left(-\frac{1}{2}\right) = \frac{1}{4} - 0 = \frac{1}{4}$.

b) $P(B) = F\left(\frac{3}{2}-\right) - F\left(-\frac{1}{2}\right) = \frac{3}{4} - 0 = \frac{3}{4}$.

c) $P(C) = F\left(\frac{5}{2}-\right) - F\left(-\frac{2}{3}\right) = 1 - \frac{1}{4} = \frac{3}{4}$.

d) $P(D) = F(2-) - F(0-) = \frac{3}{4} - 0 = \frac{3}{4}$.

e) $P(E) = (\lim_{x\to\infty} F(x)) - F(3) = 1 - 1 = 0$.

Exercise 7.18

Suppose a function F is given by

$$F(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} 1_{[\frac{1}{i},\infty)}.$$

Show that F is a distribution on $\mathbb{R}$. Let us define P by $P((-\infty, x]) = F(x)$. Find the probabilities of the following events:

a) $A = [1, \infty)$

b) $B = \left[\frac{1}{10}, \infty\right)$

c) $C = \{0\}$

d) $D = \left[0, \frac{1}{2}\right)$

e) $E = (-\infty, 0)$

f) $G = (0, \infty)$

Answer: $F(x) = 0$ for $-\infty < x \leq 0$ and $F(x) = 1$ for $1 \leq x < \infty$. For $0 < x < 1$, F has the form of a step function with infinitely many steps of decreasing length and altitude. We have $F(x) = \frac{1}{n}$ on each interval $[1/n, 1/(n-1)$ for $n = 2, 3, \ldots$. That is $F(x) = 1/2$ on $1/2 \leq x < 1$, $F(x) = 1/3$ on $1/3 \leq x < 1/2$, and so on. Using this information we can show that F is a distribution.

1. Since $1_{[1/i,\infty)}(x) = 0$ for all i whenever $x \leq 0$, we have $F(x) = 0$ for any $x \leq 0$. Then $\lim_{x\to-\infty} F(x) = 0$. Since $1_{[1/i,\infty)}(x) = 1$ for all i whenever $x \geq 1$, we have $F(x) = \sum_{i=1}^{\infty} 1/2^i = 1$ for any $x \geq 1$. Then $\lim_{x\to\infty} F(x) = 1$.

2. Suppose $x < y$. Since $1_{[1/i,\infty)}(x) \leq 1_{[1/i,\infty)}(y)$ for each $i = 1, 2, \ldots$,

$$F(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} 1_{[1/i,\infty)}(x) \leq \sum_{i=1}^{\infty} \frac{1}{2^i} 1_{[1/i,\infty)}(y) = F(x).$$

3. Since $F(y) = 1$ for all $y \geq 1$, we have $\lim_{y \downarrow x} F(y) = \lim_{y \downarrow x} 1 = 1 = F(x)$ for all $x \geq 1$. Since $F(y) = 0$ for all $y \leq 0$ we have $\lim_{y \downarrow x} F(x) = 0 = F(x)$ for all $x < 0$. This is because if $y \downarrow x$ for $x < 0$ the sequence of y values approaching x must eventually be negative. For $0 < x < 1$ consider that x must be in the interval $[1/n, 1/(n-1))$ for some $n \geq 2$. Since $F(x)$ is constant on each such interval (with $F(x) = \sum_{i=n}^{\infty} \frac{1}{2^i}$), there must exist an interval $(x, x+\epsilon)$ on which $F(y) = F(x)$ for all $y \in (x, x+\epsilon)$. Thus $\lim_{y \downarrow x} F(y) = F(x)$. Finally consider $x = 0$. We have $F(0) = 0$ but $F(y) > 0$ for all $y > 0$. However, by the previous observation we see that for $y \in (0, 1)$ there exists some positive integer n such that $y \in [1/n, 1/(n-1))$ so that $F(y) = \sum_{i=n}^{\infty} \frac{1}{2^i}$. As $y \downarrow 0$, $n \to \infty$. We sum an increasingly smaller tail of the geometric series, whose terms tend to 0. Thus $F(y) \downarrow 0 = F(x)$ and $F(x)$ is right continuous at $x = 0$.

a) $P(A) = (\lim_{x \to \infty} F(x)) - F(1-) = 1 - \frac{1}{2} = \frac{1}{2}$.

b) $P(B) = 1 - F\left(\frac{1}{10}-\right) = 1 - \frac{1}{11} = \frac{10}{11}$.

c) $P(C) = P([0, 0]) = F(0) - F(0-) = 0 - 0 = 0$.

d) $P(D) = F\left(\frac{1}{2}-\right) - F(0-) = \frac{1}{3} - 0 = \frac{1}{3}$

e) $P(E) = F(0-) - \lim_{x \to -\infty} F(x) = 0 - 0 = 0$.

f) $P(G) = (\lim_{x \to \infty} F(x)) - F(0) = 1 - 0 = 1$.