Homework 10 (Due: Wednesday, 4/22)

This assignment is due on **Wednesday, April 22**, by 11:59 PM. Your assignment should be well-organized, typed (or neatly written and scanned) and saved as a .pdf for submission on Canvas. You must show all of your work to receive full credit. For problems requiring the use of MATLAB code, remember to also submit your .m-files on Canvas as a part of your completed assignment. Your code should be appropriately commented to receive full credit.

Problems

$\boxed{1}$ (20 points) This problem highlights the difference between the maximum a posteriori (MAP) and conditional mean (CM) Bayesian point estimators. Consider a random variable $X : S \to \mathbb{R}$, and assume the posterior density is given by

$$\pi_{\text{post}}(x) = \frac{\alpha}{\sigma_0}\phi\left(\frac{x}{\sigma_0}\right) + \frac{1-\alpha}{\sigma_1}\phi\left(\frac{x-1}{\sigma_1}\right)$$

where $0 < \alpha < 1$, $\sigma_0, \sigma_1 > 0$, and $\phi(x)$ is the Gaussian distribution

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

as in Kaipio and Somersalo (2005) Ch. 3, Example 1.

(a) Using their respective definitions, show that

$$x_{\text{MAP}} = \begin{cases} 0\ , & \text{if } \dfrac{\alpha}{\sigma_0} > \dfrac{1-\alpha}{\sigma_1} \\ 1\ , & \text{if } \dfrac{\alpha}{\sigma_0} < \dfrac{1-\alpha}{\sigma_1} \end{cases}$$

and

$$x_{\text{CM}} = 1 - \alpha\ .$$

Hint: If you come across some complicated integration in computing x_{CM}, stop and think about which useful definitions (or densities, actually) could help!

(b) Make figures in MATLAB that show where x_{MAP} and x_{CM} lie with respect to the posterior $\pi_{\text{post}}(x)$ for the following cases:

- $\alpha = 0.5$, $\sigma_0 = 0.08$, $\sigma_1 = 0.04$
- $\alpha = 0.01$, $\sigma_0 = 0.001$, $\sigma_1 = 0.1$

Does the MAP estimate or CM estimate give a better representation of the underlying posterior density? Explain your findings for each case.

If the conditional covariance

$$\sigma^2 = \int_{-\infty}^{\infty} (x - x_{\text{CM}})^2 \pi_{\text{post}}(x) dx$$

is given by

$$\sigma^2 = \alpha\sigma_0^2 + (1-\alpha)(\sigma_1^2 + 1) - (1-\alpha)^2$$

how does this spread estimator change in each of the above cases? Comment on your results.

$\boxed{2}$ (10 points) Given the random variables $X : S \to \mathbb{R}^n$ and $Y, E : S \to \mathbb{R}^m$ and the forward operator $G : \mathbb{R}^n \to \mathbb{R}^m$, consider the multiplicative noise model

$$Y = E. * G(X)$$

where $.*$ denotes component-wise multiplication. One way to construct a likelihood function corresponding to this noise model is to transform it into an additive noise model in the natural log space, assuming that E is *log-normally distributed*, i.e., that the natural log of the noise follows a normal distribution $\mathcal{N}(0, \sigma^2 \mathsf{I}_m)$ with variance σ^2.

Write the likelihood function for $\log Y$ conditioned on $X = x$ when σ^2 is both known and unknown.

Note: For any of the above problems for which you use MATLAB to help you solve, you must submit your code/.m-files as part of your work. Your code must run in order to receive full credit. If you include any plots, make sure that each has a title, axis labels, and readable font size, and include the final version of your plots as well as the code used to generate them.

ALSO: Please note that this is your last official homework assignment for MA 590! 😄 Your homework for the remainder of the course is to focus on your final project report and presentation – more details on this to follow next week!