Homework 3 (Due: Wednesday, 2/5)

This assignment is due on **Wednesday, February 5**, by 11:59 PM. Your assignment should be well-organized, typed (or neatly written and scanned) and saved as a .pdf for submission on Canvas. You must show all of your work to receive full credit. For problems requiring the use of MATLAB code, remember to also submit your .m-files on Canvas as a part of your completed assignment. Your code should be appropriately commented to receive full credit.

Problems

$\boxed{1}$ (30 points) The goal of this problem is to explore the use of the truncated singular value decomposition (TSVD) in solving linear discrete inverse problems. Consider the Phillips' test problem, which is a discretization of a Fredholm integral equation of the first kind; for the problem description, see the 'Regularization Tools' manual (by P. C. Hansen, link posted on Canvas). The true solution, operator matrix, and noiseless data can be generated using the .m-file `phillips.m` (also posted on Canvas).

(a) Use `phillips.m` with $n = 64$ to generate the true solution $\mathbf{x}_{\text{true}}$, operator matrix G, and noiseless data $\mathbf{y}_{\text{true}}$. Report the condition number of G. Generate noisy data $\mathbf{y}$ by adding a normally distributed perturbation $\mathbf{e}$ with zero mean and standard deviation 10^{-4}, so that $||\mathbf{e}||_2 \approx 8 \times 10^{-4}$. Make figures plotting the true solution $\mathbf{x}_{\text{true}}$ and the data (show $\mathbf{y}_{\text{true}}$ and $\mathbf{y}$ in the same plot).

(b) Compute the singular value decomposition of G using MATLAB's `svd` function. Plot the singular values on a `semilogy` plot, and discuss the range of the spectrum (i.e. report the largest and smallest singular value). How many nonzero singular values does G have; i.e. what is p in the compact SVD for this problem?

(c) Make a `semilogy` plot of the coefficients $|\mathbf{u}_i^\mathsf{T}\mathbf{y}|$, singular values σ_i, and ratios $|\mathbf{u}_i^\mathsf{T}\mathbf{y}|/\sigma_i$ vs. the index i for $i = 1, \ldots, n$. Does the Discrete Picard Condition hold for $\mathbf{y}$ here? Discuss your findings.

(d) Compute the generalized inverse solution using the compact SVD matrices to compute the pseudoinverse, and plot your estimated solution compared to the true solution. Does the generalized inverse solution give a reasonable result for this problem? Discuss.

(e) Consider the TSVD with truncation parameter k. Generate an "L-curve" using a `loglog` plot comparing the 2-norm of the TSVD solution to the 2-norm of the residual for different values of k between $k = 2$ and $k = 62$. From this plot, what is the "optimal" value of k? Use the "optimal" k to compute the TSVD solution, and plot your estimated solution compared to the true solution. Does the TSVD give a reasonable result? Discuss.

(f) Repeat this problem increasing the level of noise added in part (a) (i.e. increase the standard deviation of the perturbation) and discuss how your results change. In particular, how does the "optimal" k value differ when the noise in the data increases?

Note: For any of the above problems for which you use MATLAB to help you solve, you must submit your code/.m-files as part of your work. Your code must run in order to receive full credit. If you include any plots, make sure that each has a title, axis labels, and readable font size, and include the final version of your plots as well as the code used to generate them.