MA 590 Homework 8

Dane Johnson

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Exercise 1

Part a

Prove that $\mathbb{E}[(X - \bar{x})(Y - \bar{y})] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

$$\begin{aligned}
\text{Cov}(X,Y) &\equiv \mathbb{E}[(X - \bar{x})(Y - \bar{y})] &(1)\\
&= \mathbb{E}[XY - \bar{y}X - \bar{x}Y + \bar{x}\bar{y})] &(2)\\
&= \mathbb{E}[XY] + \mathbb{E}[-\bar{y}X] + \mathbb{E}[-\bar{x}Y] + \mathbb{E}[\bar{x}\bar{y}] &(3)\\
&= \mathbb{E}[XY] - \bar{y}\mathbb{E}[X] - \bar{x}\mathbb{E}[Y] + \bar{x}\bar{y} &(4)\\
&= \mathbb{E}[XY] - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y} &(5)\\
&= \mathbb{E}[XY] - \bar{x}\bar{y} &(6)\\
&= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\ . &(7)
\end{aligned}$$

(1) Definition of $\text{Cov}(X,Y)$. (2) Distributive property. (3) Linearity of expectation. (4) Linearity of expectation. (5) Definition of $\bar{x}, \bar{y}$. (6) Simplification. (7) Definition of $\bar{x}, \bar{y}$.

Part b

Prove that if X and Y are independent random variables, then X and Y are uncorrelated (that if the correlation between X and Y is denoted by r, then $r = 0$).

Suppose X and Y are independent. Then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. Using part a, this means that $\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] = 0$. Therefore, $r \equiv \text{Cov}(X,Y)/(\sigma_X\sigma_Y) = 0/(\sigma_X\sigma_Y) = 0$, showing that X and Y are uncorrelated.

Part c

Prove that for $s \in \mathbb{R}$ and X a random variable, that $\mathrm{Var}(sX) = s^2\mathrm{Var}(X)$.

For a random variable Z, $\mathrm{Var}(Z) \equiv \mathbb{E}[(Z - \bar{z})^2]$, where $\bar{z} = \mathbb{E}[Z]$. In the case of sX we have $\mathbb{E}[sX] = s\mathbb{E}[X] = s\bar{x}$ by linearity and:

$$\begin{aligned}
\mathrm{Var}(sX) &= \mathbb{E}[(sX - s\bar{x})^2] && (8)\\
&= \mathbb{E}[s^2(X - \bar{x})^2] && (9)\\
&= s^2\mathbb{E}[(X - \bar{x})^2] && (10)\\
&= s^2\mathrm{Var}(X) \; . && (11)
\end{aligned}$$

(8) Definition of variance. (9) Factoring. (10) Linearity of expectation. (11) Definition of variance.

Part d

Prove that $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)$.
First note that for a random variable Z:

$$\begin{aligned}
\mathrm{Var}(Z) \equiv \mathbb{E}[(Z - \bar{z})^2] &= \mathbb{E}[Z^2 - 2\bar{z}Z + (\bar{z})^2]\\
&= \mathbb{E}[Z^2] - 2(\bar{z})^2 + (\bar{z})^2\\
&= \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 \; .
\end{aligned}$$

Let X, Y be random variables. The variance of the random variable $X + Y$ is:

$$\begin{aligned}
\mathrm{Var}(X + Y) &= \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2\\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y]^2 - (\mathbb{E}[X])^2 - 2\mathbb{E}[X]\mathbb{E}[Y] - (\mathbb{E}[Y])^2\\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 + \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right)\\
&= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right)\\
&= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y) \; .
\end{aligned}$$

Exercise 2

Consider the random variable $A = X\mathbf{e}_1 + Y\mathbf{e}_2$, where $X, Y \sim \mathcal{N}(0, \sigma^2)$. If we define $R = ||A||_2$, then since no matter what the dimension of A only the first two components of A are nonzero (by our definition of A), $R = \sqrt{X^2 + Y^2}$. Since X and Y are continuous random variables, it follows that R is also a continuous random variable. Then the cumulative distribution function of R, which we denote $F_R(t)$ is given by:

$$F_R(t) \equiv P(R \leq t) = P(\sqrt{X^2 + Y^2} \leq t) = \int_{-t}^{t} \int_{-\sqrt{t^2-x^2}}^{\sqrt{t^2-x^2}} f_{X,Y}(x, y)\, dy\, dx \ ,$$

where $f_{X,Y}$ is the joint probability density function of X and Y. If we assume that X and Y are independent (and so by the above iid) random variables, $f_{X,Y}(x, y) = f_X(x) f_Y(y)$. The pdf of a normal random variable Z with mean zero and standard deviation σ is $f_Z(z) = 1/(\sqrt{2\pi}\sigma)\exp(-z^2/(2\sigma^2))$.Then,

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-y^2}{2\sigma^2}} = \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}} \ .$$

Converting the double integral above to polar coordinates we have

$$\begin{aligned}
F_R(t) &= \int_0^{2\pi} \int_0^t \frac{1}{2\pi\sigma^2} e^{\frac{-r^2}{2\sigma^2}}\, r\, dr d\theta \\
&= \frac{1}{2\pi\sigma^2} \int_0^{2\pi} \int_0^t r e^{\frac{-r^2}{2\sigma^2}}\, dr d\theta \\
&= \frac{1}{2\pi\sigma^2} \int_0^{2\pi} d\theta \int_0^{t^2} \frac{1}{2} e^{\frac{-u}{2\sigma^2}}\, du \\
&= \frac{1}{\sigma^2} \int_0^{t^2} \frac{1}{2} e^{\frac{-u}{2\sigma^2}}\, du \\
&= \frac{1}{\sigma^2} \frac{1}{2} \frac{2\sigma^2}{-1} \left(e^{\frac{-u}{2\sigma^2}} \right) |_0^{t^2} \\
&= -\left(e^{\frac{-t^2}{2\sigma^2}} - 1 \right) \\
&= 1 - e^{\frac{-t^2}{2\sigma^2}} \ .
\end{aligned}$$

We note the restriction $t \geq 0$ since $t = \sqrt{X^2 + Y^2} \geq 0$. The probability density function of R, $f_R(t)$, is the derivative of $F_R(t)$. Therefore, the probability density function of the Rayleigh distribution is:

$$f_R(t) = \frac{d}{dt}\ F_R(t) = -\frac{-2t}{2\sigma^2} e^{\frac{-t^2}{2\sigma^2}} = \frac{t}{\sigma^2} e^{\frac{-t^2}{2\sigma^2}}\ , \quad t \geq 0\ .$$