Integral Equations

Fredholm

2^{nd} kind : $\phi(x) = F(x) + \lambda\int_a^b K(x,y)\phi(y)\,dy \qquad a \le x \le b$

1^{st} kind : $F(x) = \int_a^b K(x,y)\phi(y)\,dy \qquad a \le x \le b$

Volterra

2^{nd} kind : $\phi(x) = F(x) + \lambda\int_0^x K(x,y)\phi(y)\,dy \qquad K(x,y) = 0$ for $y > x$

1^{st} kind : $F(x) = \int_0^x K(x,y)\phi(y)\,dy \qquad K(x,y) = 0$ for $y > x$

$K(x,y)$: kernel $\qquad \phi(x)$: unknown function

Homogeneous if $F(x) \equiv 0$. Inhomogeneous otherwise.

<u>Example</u> Inhomogeneous Fredholm integral equation of the second kind

$\phi(x) = x + \lambda\int_{-1}^1 xy\phi(y)\,dy \qquad -1 \le x \le 1$

$\phi(x) = x + \lambda x A \qquad A = \int_{-1}^1 y\phi(y)\,dy$ (constant)

$\phi(x) = (1+\lambda A)x \qquad A = \int_{-1}^1 (1+\lambda A)y^2\,dy = (1+\lambda A)\left.\frac{y^3}{3}\right|_{-1}^1 = \frac{2}{3}(1+\lambda A)$

$\phi(x) = \left(1 + \frac{2\lambda}{3-2\lambda}\right)x \qquad A(1 - \tfrac{2}{3}\lambda) = \tfrac{2}{3} \longrightarrow A = \frac{2}{3-2\lambda}$

If $\lambda = 3/2$, no such A exists

$\phi(x) = \frac{3}{3-2\lambda}x$

$\phi(x) = (1 - \frac{2}{3}\lambda)^{-1}x, \quad \lambda \in \mathbb{R} \quad \lambda \ne 3/2$

$$\begin{cases} \lambda = 3/2 & \text{no soln } \phi(x) \\ \lambda \ne 3/2 & \text{unique soln } \phi(x) \end{cases}$$

<u>Example</u> Homogeneous Fredholm integral equation of the second kind

$\phi(x) = \lambda\int_{-1}^1 xy\phi(y)\,dy \qquad -1 \le x \le 1$

$\phi(x) = \lambda A x \qquad A = \int_{-1}^1 y\phi(y)\,dy = \int_{-1}^1 \lambda A y^2\,dy = \frac{2}{3}\lambda A$

$A(1 - \tfrac{3}{2}\lambda) = 0 \rightarrow \lambda = 3/2$ for nontrivial soln

$(A = 0 \rightarrow \phi \equiv 0)$

$\therefore$ If $\lambda = 3/2$
$\phi(x) = \alpha x$, α arbitrary

$$\begin{cases} \lambda \ne 3/2 & \text{trivial soln } \phi(x) \\ \lambda = 3/2 & \text{infinitely many soln's } \phi(x) \end{cases}$$

Example Inhomogeneous Fredholm integral equation of the second kind

$\phi(x) = 1 + \lambda \int_{-1}^{1} xy\phi(y)\,dy \qquad -1 \le x \le 1$

$\phi(x) = 1 + A\lambda x \qquad A = \int_{-1}^{1} y\phi(y)\,dy = \int_{-1}^{1} (y + A\lambda y^2)\,dy = \frac{2}{3}A\lambda \rightarrow A = 0 \text{ or } \lambda = 3/2$

If $A = 0$, unique solution $\phi(x) = 1$ for any λ.

If $A \neq 0$, we have $\lambda = 3/2$ and $\phi(x) = 1 + \frac{3}{2}Ax$, $A \neq 0$ arbitrary
$\phi(x) = 1 + \alpha x$, $\alpha \neq 0$ arbitrary

Example Fredholm integral equation of the first kind

Case A

$1 = \int_0^1 xy\phi(y)\,dy \quad 0 \le x \le 1$

$1/x = \int_0^1 y\phi(y)\,dy$

The left side varies while the right side is constant

$\therefore$ No solution $\phi(x)$ exists

Case B

$x = \int_0^1 xy\phi(y)\,dy \quad 0 \le x \le 1$

$1 = \int_0^1 y\phi(y)\,dy$

Any function Ψ satisfying $\int_0^1 y\Psi(y)\,dy \neq 0$ produces a solution:

$\phi(x) = \left[\int_0^1 y\Psi(y)\,dy\right]^{-1} \Psi(x)$

$\therefore$ Infinitely many soln's $\phi(x)$.

Example Volterra integral equation of the second kind

$\phi(x) = ax + \lambda x \int_0^x \phi(y)\,dy \qquad$ (Note $\phi(x) \to ax + \mathcal{O}(x^3)$ as $x \to 0$)

$\frac{\phi(x)}{x} = a + \lambda \int_0^x \phi(y)\,dy$

$\frac{d}{dx}\left\{\phi(x)/x\right\} = \lambda\phi(x)$

$du/dx = \lambda x u \qquad u(x) := \phi(x)/x$

$\int \frac{1}{u}\,du = \int \lambda x\,dx$

$\phi(x) = Axe^{\lambda x^2/2} \left(= Ax\left(1 + \frac{\lambda x^2}{2} + \frac{1}{2}\left(\frac{\lambda x^2}{2}\right)^2 + \ldots\right) \rightarrow Ax + \mathcal{O}(x^3) \text{ as } x \to 0\right)$

$\therefore A = a$
$\phi(x) = axe^{\lambda x^2/2}, \quad \lambda \in \mathbb{R}$