Chapter 4 Problem Set 1

§ 4.1 Distributions and the Delta Function

4.1.1 Show that $\lim_{K\to\infty}\int_{-\infty}^{\infty} S_K(x)\,\phi(x)\,dx = \phi(0)$ for any $\phi \in C^1(\mathbb{R})$ for which $\phi(x)$ is bounded on $-\infty < x < \infty$.

(a) $$S_K(x) = \begin{cases} 1/2 + \sum_{j=1}^{K} \cos j\pi x \,, & |x| < 1 \\ 0 \,, & |x| \geq 1 \end{cases}$$

$$\int_{-\infty}^{\infty} S_K(x)\,dx = \int_{-1}^{1}\left[1/2 + \sum_{j=1}^{K} \cos j\pi x \right] = 1 \quad \forall K \in \{1,2,3,\dots\}$$

$$\int_{-\infty}^{\infty} S_K(x)\,\phi(x)\,dx = \int_{-1}^{1} \left(1/2 + \sum_{j=1}^{K} \cos j\pi x\right)\phi(x)\,dx$$
$$= \int_{-1}^{1} \tfrac{1}{2}\phi(x)\,dx + \int_{-1}^{1} \phi(x) \sum_{j=1}^{K} \cos j\pi x\,dx$$

$$\int_{-\infty}^{\infty} S_K(x)\,\phi(x)\,dx = \int_{-1}^{1}\left[1/2 + \sum_{j=1}^{K} \cos j\pi x\right]\phi(x)\,dx$$
$$= \left[x/2 + \sum_{j=1}^{K} 1/j\pi \sin j\pi x \right]\phi(x)\Big|_{-1}^{1}$$
$$- \int_{-1}^{1}\left[x/2 + \sum_{j=1}^{\infty} 1/j\pi \sin j\pi x\right]\phi'(x)\,dx$$
$$= \tfrac{1}{2}\phi(1) + \tfrac{1}{2}\phi(-1) - \int_{-1}^{1}\left[x/2 + \sum_{j=1}^{\infty} 1/j\pi \sin j\pi x\right]\phi'(x)\,dx$$

(b) $$S_K(x) = \begin{cases} -4K^2|x| + 2K \,, & |x| \leq 1/2K \\ 0 \,, & |x| \geq 1/2K \end{cases}$$

$$\int_{-\infty}^{\infty} S_K(x)\,dx = 2\int_{0}^{1/2K}\left[-4K^2 x + 2K\right]dx = 2\left[-2K^2x^2 + 2Kx\right]\Big|_0^{1/2K} = 2\left[-\tfrac{2K^2}{4K^2} + \tfrac{2K}{2K}\right] = 1$$

$$\int_{-\infty}^{\infty} S_K(x)\,\phi(x)\,dx = \int_{-1/2K}^{1/2K}\left[2K - 4K^2|x|\right]\phi(x)\,dx$$
$$= \int_{-1/2K}^{0}\left[2K + 4K^2 x\right]\phi(x)\,dx + \int_{0}^{1/2K}\left[2K - 4K^2x\right]\phi(x)\,dx$$
$$= \left[2Kx + 2K^2x^2\right]\phi(x)\Big|_{-1/2K}^{0} - \int_{-1/2K}^{0}\left[2Kx + 2K^2x^2\right]\phi'(x)\,dx$$
$$+ \left[2Kx - 2K^2x^2\right]\phi(x)\Big|_0^{1/2K} - \int_0^{1/2K}\left[2Kx - 2K^2x^2\right]\phi'(x)\,dx$$
$$= \tfrac{1}{2}\left[\phi(1/2K) + \phi(-1/2K)\right] - 2K\int_{-1/2K}^{1/2K} x\,\phi'(x) - 2K^2\int_{-1/2K}^{0} 2K^2x^2\phi'(x)\,dx$$
$$+ \frac{1}{2K}\int_0^{1/2K} x^2\,\phi'(x)\,dx$$

4.1.2 Show $1+2\sum_{n=1}^{\infty}\cos 2n\pi x = \sum_{k=-\infty}^{\infty}\delta(x-k)$ in the sense of distributions.

This means $\int_{-\infty}^{\infty}\left(1+2\sum_{n=1}^{\infty}\cos n\pi x\right)\phi(x)\,dx = \int_{-\infty}^{\infty}\phi(x)\sum_{k=-\infty}^{\infty}\delta(x-k)\,dx$ for any $\phi\in C^{\infty}(\mathbb{R})$ with compact support. That is, ϕ can be differentiated infinitely many times and $\phi(x)=0$ outside of some compact set, called the support of ϕ.

$$= \begin{matrix}\cos 2n\pi m\\ \cos(-2n\pi m)\end{matrix}$$

$$\int\left(\phi+2\phi\sum\cos 2n\pi x\right)dx =$$

$m\in\mathbb{N}$
$$\int_{-m}^{m}\phi\sum\cos 2n\pi x\,dx = \overset{0}{\cancel{\phi\sum\frac{1}{2n\pi}\sin 2n\pi x\Big|_{-m}^{m}}} - \int_{-m}^{m}\phi'\sum\frac{1}{2n\pi}\sin 2n\pi x\,dx$$
$$= -\left(\phi'\sum\frac{-1}{(2n\pi)^2}\cos 2n\pi x\Big|_{-m}^{m} + \int_{-\infty}^{\infty}\phi''\sum\frac{1}{(2n\pi)^2}\cos 2n\pi x\,dx\right)$$
$$\overset{?}{=} \phi'\sum\frac{2}{(2n\pi)^2} + \phi''\sum\frac{2}{(2n\pi)^3} + \dots$$

$$\int_{-\infty}^{\infty}\phi\sum_{-\infty}^{\infty}\delta(x-k)\,dx = \int_{-m}^{m}\phi\sum_{-m}^{m}\delta(x-k)\,dx = \sum_{-m}^{m}\phi(k)$$

$m\in\mathbb{Z}$
$B\notin\mathbb{Z}$

$$\int_{-\infty}^{\infty}\phi\sum\delta(x-k)$$
$$=\int_{-B}^{B}\phi\sum\delta(x-k)$$
$$=\int_{-m}^{m}\phi\,\delta(x-k)$$
$$=\sum_{-m}^{m}\phi(k)$$
$$=\sum_{-m-1}^{m+1}\phi(k)$$
as $\phi(m+1)=\phi(-m-1)=0$

$N=m+1$

$$\int_{-N}^{N}\phi\left(1+2\sum_{n=1}^{\infty}\cos 2n\pi x\right)dx$$
$$=\phi\left(x+2\sum_{n=1}^{\infty}\frac{1}{2n\pi}\sin n\pi x\right)\Big|_{-N}^{N}$$
$$-\int_{-N}^{N}\phi'\left(x+2\sum{}''\quad{}''\right)dx$$
$$= 2N\phi - \left(x^2\right.$$

Solve the following differential equations in the sense of distribution.

4.1.5 $x^2 \frac{du}{dx} = 0$

Find a distribution u whose action satisfies $\langle x^2 u', \phi \rangle = 0$ for any test function ϕ. By definition, $\phi \in C^\infty(\mathbb{R})$ with compact support.

$$0 = \langle x^2 u', \phi \rangle = \int_{-\infty}^{\infty} x^2 u' \phi \, dx = \int_{-\infty}^{\infty} (x^2\phi) u' \, dx = x^2 \phi u \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (x^2\phi)' u \, dx$$

$$= -\int_{-\infty}^{\infty} (x^2\phi)' u \, dx = -\langle u, (x^2\phi)' \rangle.$$

$0 = \langle u, \psi \rangle$ for any test function ψ of the form $\psi = (x^2\phi)'$.

$$\int_{-\infty}^{\infty} \psi \, dx = x^2 \phi \Big|_{-\infty}^{\infty} = 0$$

$$\int_{0}^{\infty} \psi \, dx = x^2 \phi \Big|_{0}^{\infty} = 0$$

$$\psi(x) = (x^2\phi)' = 2x\phi(x) + x^2\phi'(x) \quad \rightarrow \quad \psi(0) = 0.$$

Pick out test functions ϕ_0, ϕ_1, ϕ_2 s.t.

$$\int_0^\infty \phi_0 \, dx = 1 \qquad \int_0^\infty \phi_1 \, dx = 0 \qquad \int_0^\infty \phi_2 \, dx = 0$$

$$\int_{-\infty}^\infty \phi_0 \, dx = 0 \qquad \int_{-\infty}^\infty \phi_1 \, dx = 1 \qquad \int_{-\infty}^\infty \phi_2 \, dx = 0$$

$$\phi_0(0) = 0 \qquad \phi_1(0) = 0 \qquad \phi_2(0) = 1$$

We can write any test function as a linear combination of ϕ_0, ϕ_1, ϕ_2 and a remainder which has the form of ψ:

$$\phi(x) = \phi_0(x)\int_0^\infty \phi \, dx + \phi_1(x)\int_{-\infty}^\infty \phi \, dx + \phi_2(x)\phi(0) + \psi(x) \quad \text{where}$$

$$\psi(x) = \phi(x) - \left(\phi_0(x)\int_0^\infty \phi \, dx + \phi_1(x)\int_{-\infty}^\infty \phi \, dx + \phi_2(x)\phi(0)\right)$$

$$\langle u, \phi \rangle = \langle u, \phi_0 \rangle \int_0^\infty \phi \, dx + \langle u, \phi_1 \rangle \int_{-\infty}^\infty \phi \, dx + \langle u, \phi_2 \rangle \phi(0) + \langle u, \psi \rangle$$

$$= \langle u, \phi_0 \rangle \int_0^\infty \phi \, dx + \langle u, \phi_1 \rangle \int_{-\infty}^\infty \phi \, dx + \langle u, \phi_2 \rangle \phi(0)$$

$$= \langle u, \phi_0 \rangle \langle H, \phi \rangle + \langle u, \phi_1 \rangle \langle 1, \phi \rangle + \langle u, \phi_2 \rangle \langle \delta, \phi \rangle$$

$$= c_1 \langle H, \phi \rangle + c_2 \langle 1, \phi \rangle + c_3 \langle \delta, \phi \rangle$$

Identify the action of u as that of $\boxed{u(x) = c_1 H(x) + c_2 + c_3 \delta(x)}$.

$\underline{4.1.6}$ $\frac{d^2u}{dx^2} = \delta''(x)$

Find a distribution u whose action satisfies $\langle u'', \phi \rangle = \langle \delta'', \phi \rangle = \phi''(0)$ for any test function ϕ. By definition, $\phi \in C^\infty(\mathbb{R})$ with compact support.

$$\int_{-\infty}^{\infty} u''\phi = \cancel{u'\phi\Big|_{-\infty}^{\infty}}^{0} - \int_{-\infty}^{\infty} u'\phi' = -\int_{-\infty}^{\infty} u'\phi' = -\left(u\phi'\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u\phi''\right)$$

$$= \int_{-\infty}^{\infty} u\phi'' = \phi''(0)$$

$$\langle \delta'', \phi \rangle = -\langle \delta', \phi' \rangle$$
$$= \langle \delta, \phi'' \rangle = \phi''(0)$$

$$\phi''(0) = \int u\phi'' =$$

§ 4.2 Green's Functions

4.2.1 Construct the Green's function for the following equation.

$u'' = f(x)$, $u(0) = u'(1) = 0$

$Lu = f$ $\quad L = \frac{d^2}{dx^2}$ (self adjoint operator so $L = L^\dagger$)

$L^\dagger G = \delta(x-\xi)$ $\quad L^\dagger G = LG = G_{xx} = \delta(x-\xi)$ $\quad G(0) = G_x(1) = 0$

$\int_{-\xi}^{\xi} G_{xx}\, dx = \int_{-\xi}^{\xi} \delta(x-\xi)\, dx$ (for a fixed $\xi \in \mathbb{R}$)

$[G_x]_{-\xi}^{\xi} = 1 \rightarrow$ G_x has a jump of 1 at ξ (discontinuous derivative)

Assume G is continuous at ξ

For $x < \xi$ we have $G_{xx} = 0$ $(\delta(x-\xi) = 0,\ x < \xi) \rightarrow G = Ax + B$
For $x > \xi$ we have $G_{xx} = 0$ $(\delta(x-\xi) = 0,\ x > \xi) \rightarrow G = Cx + D$

For $\xi \neq 0$, $G(0) = 0$ implies $B = D = 0$ so $G = Ax,\ x < \xi$
$G = Cx,\ x > \xi$