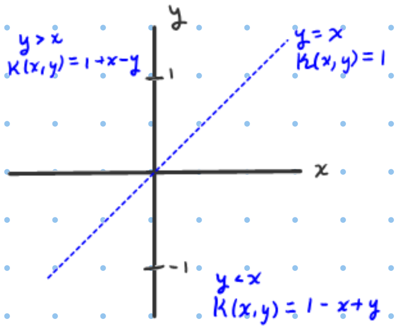
3.4.3 Find the eigenvalues and eigenfunctions of the integral operator

$$\mathbb{K}u = \int_{-1}^{1} K(x,y)\,u(y)\,dy, \quad K(x,y) = 1 - |x-y|$$

Find $(\lambda, \phi(x))$ s.t. $\lambda\phi(x) = \mathbb{K}\phi$ $(\phi \not\equiv 0)$.



$$\lambda\phi(x) = \int_{-1}^{1} K(x,y)\,\phi(y)\,dy$$

$$= \int_{-1}^{0} (1-x+y)\,\phi(y)\,dy + \int_{0}^{1} (1+x-y)\,\phi(y)\,dy$$

$$\lambda\phi'(x) = -\int_{-1}^{0} \phi(y)\,dy + \int_{0}^{1} \phi(y)\,dy = \text{constant}$$

$\therefore\ \lambda\phi(x) = Cx + D$ for constants C, D if $\underline{\lambda \neq 0}$.

Put $\lambda\phi(x) = Cx + D$ into the integral equation:

$$Cx + D = \lambda\phi(x) = \int_{-1}^{1} K(x,y)\phi(y)\,dy = \frac{1}{\lambda}\int_{-1}^{0}(1-x+y)(Cy+D)\,dy + \frac{1}{\lambda}\int_{0}^{1}(1+x-y)(Cy+D)\,dy$$

$$= \frac{1}{\lambda}\frac{1}{6}\left\{(3Cx - 3C - 6Dx + 3D) + (3Cx + 3C + 6Dx + 3D)\right\}$$

$$= \frac{1}{\lambda}(Cx + D) \implies \text{Either } C = D = 0 \text{ or } \lambda = 1$$

If $C = D = 0$ we have $\phi(x) \equiv 0$. But $\phi(x)$ is nonzero by assumption. $\therefore\ \lambda = 1$. Let $(\lambda_1, \phi_1(x)) = (1, x)$ and $(\lambda_2, \phi_2(x)) = (1, 1)$. These are the eigenvalues and corresponding eigenvectors of the integral operator.

Check: Using the previous calculation with $(C, D) = (1, 0)$ (resp. $(C, D) = (0, 1)$).

$$\mathbb{K}\phi_1 = \int_{-1}^{1} K(x,y)\,y\,dy = \frac{1}{\lambda}x = \frac{1}{1}x = 1\cdot x = \lambda\phi_1(x)$$

$$\mathbb{K}\phi_2 = \int_{-1}^{1} K(x,y)\,dy = \frac{1}{\lambda}\cdot 1 = 1\cdot 1 = \lambda\phi_2(x)$$

Note that $C\phi_1$, $D\phi_2$ are also eigenfunctions for any $C \neq 0$, $D \neq 0$.

If $\lambda = 0$, $0 = \lambda\phi(x) = \int_{-1}^{0}(1-x+y)\phi(y)\,dy + \int_{0}^{1}(1+x-y)\phi(y)\,dy$. In particular, at $x = 0$,

$$0 = \lambda\phi(0) = \int_{-1}^{0}(1+y)\phi(y)\,dy + \int_{0}^{1}(1-y)\phi(y)\,dy = \int_{0}^{1}(1-y)\phi(-y)\,dy + \int_{0}^{1}(1-y)\phi(y)\,dy$$

Since $1 - y > 0$ for $0 < y < 1$, this implies $\phi(-y) = -\phi(y)$, $0 < y < 1$. This might be used to either rule out $\lambda = 0$ or find the eigenfunctions corresponding to $\lambda = 0$. Uncertain, but I suspect 0 is not an eigenvalue.

3.5.3 Show that the integral equation is equivalent to a differential equation, find the resolvent (or pseudo-resolvent) operator, and solve the integral equation.

$$u(x) = 1 + x + \int_0^x (x-t)\,u(t)\,dt$$

- $u(x) = 1 + x + x\int_0^x u(t)\,dt - \int_0^x t\,u(t)\,dt \quad , \quad u(0) = 1$

  $u'(x) = 1 + \int_0^x u(t)\,dt + x\,u(x) - x\,u(x) = 1 + \int_0^x u(t)\,dt \,,\; u'(0) = 1$

  $u''(x) = u(x) \quad , \quad u_0 = 1 \,,\; u_0' = 1$

  $u(x) = c_1 e^{x} + c_2 e^{-x}$ $\qquad \begin{array}{l} 1 = u(0) = c_1 + c_2 \\ 1 = u'(0) = c_1 - c_2 \end{array} \longrightarrow \begin{array}{l} c_1 = 1 \\ c_2 = 0 \end{array}$

  $u(x) = e^{x}$

- $K_1(x,t) = K(x,t) = x - t$

  $K_2(x,t) = \int_t^x K(x,s)\,K_1(s,t)\,ds = \int_t^x (x-s)(s-t)\,ds$

  $= \int_t^x \{xs - xt - s^2 + st\}\,ds = \frac{1}{2}xs^2 - xts - s^3/3 + s^2t/2 \Big|_t^x$

  $= \frac{1}{2}x^3 - x^2t - x^3/3 + x^2t/2$
  $- \frac{1}{2}xt^2 + xt^2 + t^3/3 - t^3/2$