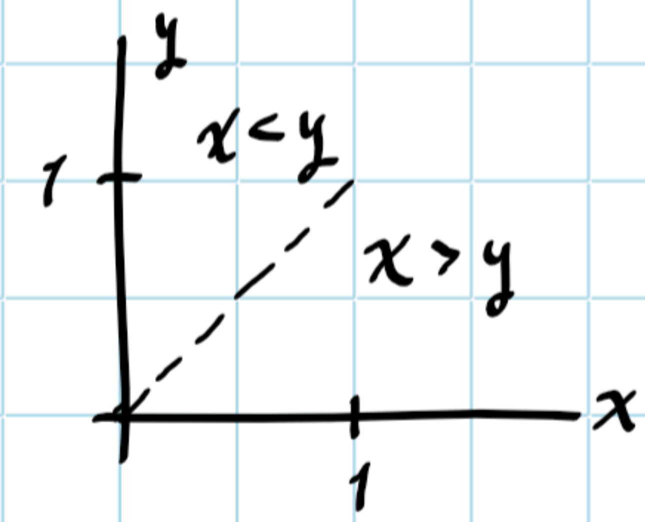
4.1 Distributions and the Delta Function

For a differential operator L, it is reasonable to expect that the inverse operator L^{-1} is an integral operator, say

$$L^{-1}(\phi) = \int_a^b g(x,t)\phi(t)\,dt$$

For example, the differential equation

$$Lu \equiv u'' = \phi(x) \qquad u(0) = u(1) = 0$$



has the solution

$$u(x) = \int_0^1 K(x,y)\phi(y)\,dy \;, \quad K(x,y) = \begin{cases} y(x-1) & 0 \le y < x \le 1 \\ x(y-1) & 0 \le x < y \le 1 \end{cases}$$

Verify:

$$u(x) = \int_0^x y(x-1)\,\phi(y)\,dy + \int_x^1 x(y-1)\,\phi(y)\,dy$$

$$\frac{du}{dx} = \int_0^x \phi(y)\frac{\partial}{\partial x}[y(x-1)]\,dy + \int_x^1 \phi(y)\frac{\partial}{\partial x}[x(y-1)]\,dy$$

$$= \int_0^x \phi(y)\,y\,dy + \int_x^1 \phi(y)\,(y-1)\,dy$$

$$= \int_0^1 \phi(y)\,y\,dy - \int_x^1 \phi(y)\,dy = \int_0^1 \phi(y)\,y\,dy + \int_1^x \phi(y)\,dy$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left\{\int_0^1 \phi(y)\,y\,dy\right\} + \frac{d}{dx}\left\{\int_1^x \phi(y)\,dy\right\} = \phi(x)$$

$$u(0) = \int_0^1 0\,\phi(y)\,dy = 0 \qquad u(1) = \int_0^1 y(1-1)\,\phi(y)\,dy = \int_0^1 0\,dy = 0 \qquad \checkmark$$

If the inverse operator is indeed an integral operator, we expect

$$\phi(x) = L(L^{-1}(\phi)) = L\left(\int_a^b g(x,t)\,\phi(t)\,dt\right) = \int_a^b L(g(x,t))\,\phi(t)\,dt$$

Written another way:

$$L(g(x,t)) = \delta(x,t) \text{ where } \delta(x,t) \text{ satisfies } \int_a^b \delta(x,t)\,\phi(t)\,dt = \phi(x)$$

for any 'reasonable' choice of $\phi(x)$. However, no such function $\delta(x,t)$ exists.

We can still make sense of $\delta(x,t)$ through the ideas of delta-sequences and the theory of distributions.

Delta Sequences

Does there exist a sequence of functions $\{S_K(x)\}$ which satisfies

$$\lim_{K\to\infty} \int_{-\infty}^{\infty} S_K(t)\phi(t)\,dt = \phi(0) \quad \forall \phi(x) \in C(\mathbb{R})$$

The existence of such a sequence implies the existence of $\{S_K(t-x)\}$ satisfying

$$\lim_{K\to\infty} \int_{-\infty}^{\infty} S_K(t-x)\phi(t) = \phi(x)$$

Example) $S_K(x) = \begin{cases} K, & -1/2K \le x \le 1/2K \\ 0, & |x| > 1/2K \end{cases}$

For any function $\phi(x)$ continuous near $x=0$,

$$\int_{-\infty}^{\infty} S_K(t)\phi(t)\,dt = \int_{-1/2K}^{1/2K} K\,\phi(t)\,dt = K\int_{-1/2K}^{1/2K} \phi(t)\,dt \longrightarrow \phi(0) \text{ as } K\to\infty$$

Example) $\{S_K(x)\}$ with $S_K(x) = \dfrac{\sin Kx}{\pi x}$ is a delta-sequence with the requirement $\phi(x)$ is Lipschitz continuous.

Delta sequences can be difficult to use. The idea would be to find a sequence

$$L(g_K(x,t)) = S_K(x-t) \rightarrow g(x,t) = \lim_{K\to\infty} L^{-1}(S_K(x-t))$$

From this obtain $L^{-1}(\phi) = \int_a^b g(x,t)\phi(t)\,dt$ for a given differential operator L. One would need to both find the $g_K(x,t)$ and prove that the limit is independent of the choice of delta sequence.

Distribution Theory

Definition A test function is a function $\phi(x) \in C^\infty(\mathbb{R})$ with compact support. The support of $\phi(x)$ is the closure of the set of x for which $\phi(x) \neq 0$. Since compact subsets of $\mathbb{R}$ are bounded, one describes $\phi(x)$ as a function which is infinitely many times continuously differential and which vanishes for sufficiently large $|x|$.

Definition A linear functional t on D is a real number $t(\phi)$ which is defined $\forall \phi \in D$ and is linear in ϕ. the notation we use for linear functionals in D is $t(\phi) \equiv \langle t, \phi\rangle$, referring to $\langle t,\phi\rangle$ as the action of the linear functional t on ϕ. Linearity means

$$\langle t, \alpha\phi_1 + \beta\phi_2\rangle = \alpha\langle t,\phi_1\rangle + \beta\langle t,\phi_2\rangle \quad \forall \phi_1,\phi_2 \in D,\ \forall \alpha,\beta \in \mathbb{R}$$