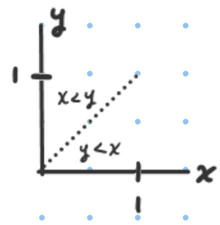
Chapter 3 Problem Set

3.1.1 Verify that the solution of $\frac{d^2u}{dt^2} = f(x)$, $u(0)=0$, $u(1)=0$ is given by $u(x) = \int_0^1 K(x,y) f(y)\, dy$ where

$$K(x,y) = \begin{cases} y(x-1) & 0 \le y < x \le 1 \\ x(y-1) & 0 \le x < y < 1 \end{cases}$$

$$\frac{du}{dx} = \frac{d}{dx}\int_0^1 K(x,y) f(y)\, dy = \int_0^1 f(y) \frac{\partial}{\partial x} K(x,y)\, dy \quad \text{(Leibniz integral rule)}$$

$$= \int_0^x f(y) \frac{\partial}{\partial x}[y(x-1)]\, dy + \int_x^1 f(y) \frac{\partial}{\partial x}[x(y-1)]\, dy$$



$$= \int_0^x y f(y)\, dy + \int_x^1 (y-1) f(y)\, dy$$

$$= \int_0^1 y f(y)\, dy - \int_x^1 f(y)\, dy$$

$$= \int_0^1 y f(y)\, dy + \int_1^x f(y)\, dy$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\int_0^1 y f(y)\, dy + \frac{d}{dx}\int_1^x f(y)\, dy = f(x) \quad \text{(FTC)}$$

$$u(0) = \int_0^1 K(0,y) f(y)\, dy = \int_0^1 0 \cdot (y-1) f(y)\, dy = \int_0^1 0\, dy = 0$$

$$u(1) = \int_0^1 K(1,y) f(y)\, dy = \int_0^1 y(1-1) f(y)\, dy = \int_0^1 0\, dy = 0$$

3.3.1 Find the solutions of

$$u(x) - \lambda \int_0^{2\pi} \sum_{j=1}^{n} \frac{\cos jt \cos jx}{j} u(t)\,dt = \sin^2 x \,, \quad n \geq 2$$

for all values of λ. Find the resolvent kernel for this equation. Find the least squares solution if necessary.

This is a Fredholm integral equation of the second kind with a separable integral kernel: $K(x,t) = \lambda/j \cos jt \cos jx = \lambda M_j(x) N_j(t)$ with $M_j(x) = \cos jx$ and $N_j(t) = 1/j \cos jt$.

$$u(x) = \sin^2 x + \lambda \sum_{j=1}^{n} \cos jx \int_0^{2\pi} \tfrac{1}{j} \cos jt \, u(t)\,dt$$

Find the c_j to determine $u(x)$

$$u(x) = \sin^2 x + \lambda \sum_{j=1}^{n} c_j \cos jx \,, \quad c_j \equiv \int_0^{2\pi} \tfrac{1}{j} \cos jt \, u(t)\,dt$$

Multiply through by $\frac{1}{k}\cos kx$ for $k \in \{1, \dots, n\}$ and integrate.

$$\underbrace{\int_0^{2\pi} u(x) \tfrac{1}{k} \cos kx \, dx}_{c_k} = \underbrace{\int_0^{2\pi} \sin^2 x \tfrac{1}{k} \cos kx \, dx}_{b_k} + \lambda \sum_{j=1}^{n} c_j \underbrace{\int_0^{2\pi} \tfrac{1}{k} \cos jx \cos kx \, dx}_{a_{kj}} \quad (k \in \{1, \dots, n\})$$

$$c_k = b_k + \lambda \sum_{j=1}^{n} a_{kj} c_j \,, \quad k = 1, 2, \dots, n$$

$$\vec{C} = \vec{B} + \lambda A \vec{C} \rightarrow \vec{C} = (I - \lambda A)^{-1} B$$

Since $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, b_2 may not be zero. Checking this:

$$b_2 = \int_0^{2\pi} \sin^2 x \tfrac{1}{2} \cos 2x \, dx = \tfrac{1}{4} \int_0^{2\pi} (1 - \cos 2x) \cos 2x \, dx = -\pi/4.$$

For $k \neq 2$, $b_k = \frac{1}{k} \int_0^{2\pi} \sin^2 x \cos kx \, dx = \frac{1}{2k} \int_0^{2\pi} \cos kx \, dx - \frac{1}{2k} \int_0^{2\pi} \cos 2x \cos kx \, dx = 0$

$$a_{kj} = \int_0^{2\pi} \tfrac{1}{k} \cos kx \cos jx \, dx = \begin{cases} 0 \,, & k \neq j \\ \int_0^{2\pi} \frac{1}{k} \cos^2 kx \, dx \,, & k = j \end{cases} = \begin{cases} 0 \,, & k \neq j \\ \pi/k \,, & k = j \end{cases}$$

$$\tfrac{1}{k} \int_0^{2\pi} \cos^2 kx \, dx = \tfrac{1}{2k} \int_0^{2\pi} (1 - \cos 2kx)\,dx = \frac{2\pi}{2k} - \left(\frac{1}{2k} \sin 2kx\right)\Big|_0^{2\pi} = \frac{\pi}{k}$$

$$I - \lambda A = \begin{bmatrix} 1-\lambda\pi & & & \\ & 1-\lambda\pi/2 & & \\ & & \ddots & \\ & & & 1-\lambda\pi/n \end{bmatrix} \quad (I - \lambda A)^{-1} = \begin{bmatrix} (1-\lambda\pi)^{-1} & & & \\ & (1-\lambda\pi/2)^{-1} & & \\ & & \ddots & \\ & & & (1-\lambda\pi/n)^{-1} \end{bmatrix}$$

for $\lambda \neq 1/\pi, 2/\pi, \dots, n/\pi$. Since $(I - \lambda A)^{-1}$ is diagonal and $b_k = 0$ for $k \neq 2$, we have $c_k = 0$ for $k \neq 2$ and $c_2 = (1 - \lambda\pi/2)^{-1}(-\pi/4)$

$$\therefore \boxed{u(x) = \sin^2 x - \frac{\lambda\pi}{4} \frac{1}{1 - \lambda\pi/2} \cos 2x = \sin^2 x - \frac{\lambda\pi}{4 - 2\lambda\pi} \,, \quad \lambda \neq 2/\pi}$$

3.4.1

a. Find the eigenfunctions for the integral operator

$$Ku = \int_0^1 K(x,\xi)\,u(\xi)\,d\xi$$

$$K(x,\xi) = \begin{cases} x(1-\xi) & 0 \le x < \xi \le 1 \\ \xi(1-x) & 0 \le \xi < x \le 1 \end{cases}$$

b. Find the expansion of $f(x)$ in terms of the eigenfunctions of K. Is there a solution of $Ku = f$?

$$f(x) = \begin{cases} x/2 & 0 \le x \le 1/2 \\ \frac{1-x}{2} & 1/2 \le x \le 1 \end{cases}$$

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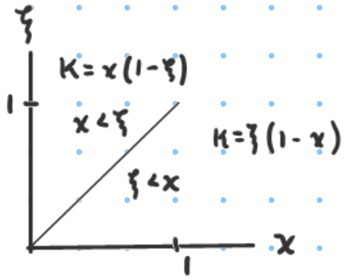
a. Determine eigenvalues λ, eigenfunctions $\phi(x) \not\equiv 0$ satisfying $K\phi = \lambda\phi$

$$\int_0^1 K(x,\xi)\,\phi(\xi)\,d\xi = \lambda\phi(x)$$



$$\int_0^x \xi(1-x)\,\phi(\xi)\,d\xi + \int_x^1 x(1-\xi)\,\phi(\xi)\,d\xi = \lambda\phi(x)$$

$$(1-x)\int_0^x \xi\,\phi(\xi)\,d\xi + x\int_x^1 (1-\xi)\,\phi(\xi)\,d\xi = \lambda\phi(x)$$

$$-\int_0^x \xi\,\phi(\xi)\,d\xi + (1-x)x\phi(x) + \int_x^1 (1-\xi)\phi(\xi)\,d\xi - x(1-x)\phi(x) = \lambda\phi'(x)$$

$$\int_x^1 (1-\xi)\phi(\xi)\,d\xi - \int_0^x \xi\,\phi(\xi)\,d\xi = \lambda\phi'(x)$$

$$-(1-x)\phi(x) - x\phi(x) = \lambda\phi''(x) \longrightarrow \lambda\phi''(x) + \phi(x) = 0$$

$\lambda \neq 0$ since if $\lambda = 0$, $0\phi''(x) + \phi(x) = 0$ implies $\phi(x) \equiv 0$, contradicting the assumption that $\phi \not\equiv 0$ as an eigenfunction.

$$\lambda\phi(0) = \int_0^0 \xi(1-0)\phi(\xi)\,d\xi + \int_0^1 0\cdot(1-\xi)\,\phi(\xi)\,d\xi = 0 \longrightarrow \phi(0) = 0$$

$$\lambda\phi(1) = \int_0^1 \xi(1-1)\phi(\xi)\,d\xi + \int_1^1 1\cdot(1-\xi)\phi(\xi)\,d\xi = 0 \longrightarrow \phi(1) = 0$$

Eigenpairs (λ, ϕ) satisfy the BVP $\lambda\phi''(x) + \phi(x) = 0$, $\lambda \neq 0$, $\phi(0) = \phi(1) = 0$. Using a characteristic equation $\lambda r^2 + 1 = 0 \rightarrow r = \pm 1/\sqrt{-\lambda}$.

$\lambda < 0$: $\phi(x) = c_1 e^{x/\sqrt{-\lambda}} + c_2 e^{-x/\sqrt{-\lambda}}$, $0 = \phi(0) = c_1 + c_2 \rightarrow c_2 = -c_1$

$0 = \phi(1) = c_1 e^{1/\sqrt{-\lambda}} - c_1 e^{-1/\sqrt{-\lambda}} = 2c_1 \sinh(1/\sqrt{-\lambda})$

Since $1/\sqrt{-\lambda} \neq 0$, $\sinh(1/\sqrt{-\lambda}) \neq 0$ so $c_1 = 0$, $c_2 = 0$, $\phi(x) \equiv 0$. This contradicts the assumption $\phi(x)$ is an eigenfunction.

$\therefore$ Only $\lambda > 0$ is possible.

$\lambda > 0$: $r = \pm \frac{1}{\sqrt{\lambda}} i$, $\phi(x) = c_1 \sin x/\sqrt{\lambda} + c_2 \cos x/\sqrt{\lambda}$

$0 = \phi(0) = c_2$

$0 = \phi(1) = c_1 \sin 1/\sqrt{\lambda} \Rightarrow 1/\sqrt{\lambda} = n\pi, \ n \in \mathbb{Z}$ or $c_1 = 0$

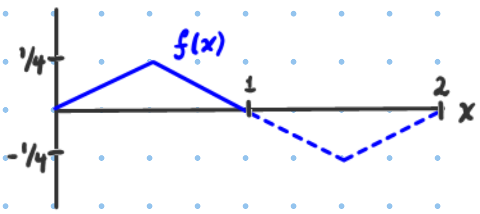
Since $c_1 = c_2 = 0$ would give $\phi \equiv 0$ again, $c_1 \neq 0$ so $1/\sqrt{\lambda} = n\pi$

The eigenvalues of K are $\lambda_n = \frac{1}{n^2\pi^2}$, $n = 1, 2, 3, \ldots$ (since $n^2 = (-n)^2$ and $\sin(0) = 0$ only $n = 1, 2, 3, \ldots$ are needed for eigenfunctions) with corresponding eigenfunctions $\phi_n(x) = \sin(n\pi x)$ (omit c_1 since any nonzero multiple of an eigenfunction is also an eigenfunction for the same eigenvalue).

$$\boxed{(\lambda_n, \phi_n) = \left(1/n^2\pi^2, \ \sin(n\pi x)\right), \ n = 1, 2, 3, \ldots}$$

b.

$$f(x) = \begin{cases} x/2, & 0 \le x < 1/2 \\ (1-x)/2, & 1/2 \le x \le 1 \\ (1-x)/2, & 1 \le x \le 3/2 \\ (x-2)/2, & 3/2 \le x \le 2 \end{cases}$$



In order to expand $f(x)$ in terms of $\{\sin n\pi x\}_{n=1}^{\infty}$, replace $f(x)$ with an odd extension and suppose $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$ on $0 < x < 2$.

$$\int_0^2 \sin n\pi x \sin m\pi x \, dx = \frac{1}{2}\int_0^2 \left(\cos(n-m)\pi x - \cos(n+m)\pi x\right) dx$$

$$= \left(\frac{1}{2}\frac{1}{n-m}\sin(n-m)\pi x - \frac{1}{2}\frac{1}{n+m}\sin(n+m)\pi x\right)\Big|_0^2 = 0, \quad m \neq n.$$

$$\int_0^2 \sin^2 n\pi x \, dx = \frac{1}{2}\int_0^2 (1 - \cos 2n\pi x)\, dx = 1 - \frac{1}{2n\pi}(\sin 4n\pi - \sin 0) = 1.$$

$$b_n = \int_0^2 b_n \sin^2 n\pi x \, dx = \int_0^2 (b_1 \sin \pi x + b_2 \sin 2\pi x + \ldots) \sin n\pi x \, dx = \int_0^2 f(x) \sin n\pi x$$

$$= 2\int_0^1 f(x) \sin n\pi x = \int_0^{1/2} x \sin n\pi x \, dx + \int_{1/2}^1 (1-x) \sin n\pi x$$

$$= \left\{-\frac{1}{n\pi} x \cos n\pi x\Big|_0^{1/2} + \left(\frac{1}{n\pi}\right)^2 (\sin n\pi x)\Big|_0^{1/2}\right\} + \left\{-\frac{1}{n\pi}(1-x)\cos n\pi x\Big|_{1/2}^1 - \left(\frac{1}{n\pi}\right)^2 (\sin n\pi x)\Big|_{1/2}^1\right\}$$

$$= -\frac{1}{n\pi}\cos n\pi/2 + \frac{1}{(n\pi)^2}\sin n\pi/2 + \frac{1}{n\pi}\cos n\pi/2 + \frac{1}{(n\pi)^2}\sin n\pi/2 = \frac{2}{n^2\pi^2}\sin n\pi/2$$

$$b_n = \begin{cases} 2/n^2\pi^2, & n = 1, 5, 9, \ldots \\ -2/n^2\pi^2, & n = 3, 7, 11, \ldots \\ 0, & n \text{ even} \end{cases}$$

$$\boxed{\begin{aligned} f(x) &= \frac{2}{\pi^2}\left\{\sin \pi x - \frac{1}{3^2}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x - \ldots\right\} \\ &= \frac{2}{\pi^2}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j-1)^2}\sin(2j-1)\pi x \end{aligned}}$$

Let $u(\xi) = \delta(\xi - 1/2)$. Then $Ku = f$:

$$Ku = \int_0^1 K(x,\xi)\delta(\xi - 1/2)\, d\xi = K(x, 1/2) = \begin{cases} x/2, & 0 \le x \le 1/2 \\ (1-x)/2, & 1/2 \le x \le 1 \end{cases} = f$$

3.4.2 Find the eigenvalues and eigenfunctions for the integral operator $Ku = \int_0^\pi k(x,\xi)u(\xi)\,d\xi$ where

a. $k(x,\xi) = x\xi$ b. $k(x,\xi) = \sin x \sin\xi + \alpha\cos x\cos\xi$

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a. $\lambda\phi(x) = K\phi = \int_0^\pi x\xi\,\phi(\xi)\,d\xi = x\int_0^\pi \xi\,\phi(\xi)\,d\xi$

$\phi(x) = x$, $\lambda = \int_0^\pi \xi\,\phi(\xi)\,d\xi = \int_0^\pi \xi^2\,d\xi = \pi^3/3$

Check: $K\phi = \int_0^\pi x\xi^2\,d\xi = \frac{\pi^3}{3}x = \lambda x = \lambda\phi(x)$

b. $\lambda\phi(x) = \int_0^\pi (\sin x\sin\xi + \alpha\cos x\cos\xi)\,\phi(\xi)\,d\xi$

$\lambda\phi(x) = \sin x\int_0^\pi \phi(\xi)\sin\xi\,d\xi + \cos x\int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi$

$\lambda\phi'(x) = \cos x\int_0^\pi \phi(\xi)\sin\xi\,d\xi - \sin x\int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi$

$\lambda\phi''(x) = -\sin x\int_0^\pi \phi(\xi)\sin\xi\,d\xi - \cos x\int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi = -\lambda\phi(x)$

$\lambda\phi'' + \lambda\phi = 0$

For $\lambda \neq 0$, $\phi(x) = A\cos x + B\sin x$ for some constants A & B. Suppose $\lambda = 0$.

$0 = \lambda\phi(x) = \cos x\int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi + \sin x\int_0^\pi \phi(\xi)\sin\xi\,d\xi$

$\tan x = -\dfrac{\int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi}{\int_0^\pi \phi(\xi)\sin\xi\,d\xi} = \text{constant}$

This equality holds for any x. Since tangent is not constant, $\lambda = 0$ produces a contradiction.

Since $\lambda \neq 0$, $\phi(x) = A\cos x + B\sin x$. Using the original integral equation,

$A\cos x + B\sin x = \phi(x) = \frac{1}{\lambda}\cos x\int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi + \frac{1}{\lambda}\sin x\int_0^\pi \phi(\xi)\sin\xi\,d\xi$

This implies $A = \alpha/\lambda\int_0^\pi \phi(\xi)\cos\xi\,d\xi$ and $B = 1/\lambda\int_0^\pi \phi(\xi)\sin\xi\,d\xi$.

$A = \alpha/\lambda\int_0^\pi \phi(\xi)\cos\xi\,d\xi$

$= \alpha/\lambda\int_0^\pi (A\cos\xi + B\sin\xi)\cos\xi\,d\xi$

$= \alpha/\lambda\int_0^\pi A\cos^2\xi\,d\xi$ †

$\lambda = \alpha\int_0^\pi \cos^2\xi\,d\xi = \alpha\pi/2$

or

$B = 1/\lambda\int_0^\pi \phi(\xi)\sin\xi\,d\xi$

$= 1/\lambda\int_0^\pi (A\cos\xi + B\sin\xi)\sin\xi\,d\xi$

$= 1/\lambda\int_0^\pi B\sin^2\xi\,d\xi$ †

$\lambda = \int_0^\pi \sin^2\xi\,d\xi = \pi/2$

† Using the orthogonality of sine and cosine on $[0,\pi]$ (Problem Set 2).

We could have noted from $\phi'' + \phi = 0$ that $\phi(x) = A\cos x$ and $\phi(x) = B\sin x$ each satisfy the differential equation. Following the process above for each case implies $\lambda = \alpha\pi/2$ is the eigenvalue corresponding to $\cos x$ and $\lambda = \pi/2$ the eigenvalue corresponding to $\sin x$.

$$(\lambda_1, \phi_1) = \left(\frac{\alpha\pi}{2}, \frac{2}{\alpha\pi}\cos x\right) \qquad (\lambda_2, \phi_2) = \left(\frac{\pi}{2}, \frac{2}{\pi}\sin x\right)$$

Check:

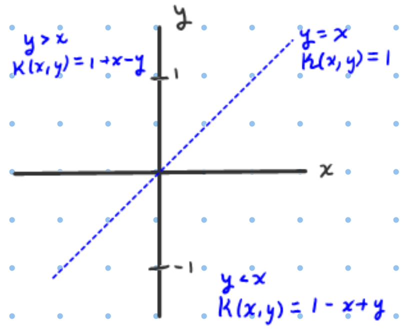
$$K\phi_1 = \int_0^\pi (\sin x \sin \xi + \alpha \cos x \cos \xi)\frac{2}{\alpha\pi}\cos\xi \, d\xi = \frac{2}{\alpha\pi}\cos x \int_0^\pi \alpha \cos^2\xi \, d\xi = \frac{\alpha\pi}{2}\cdot\frac{2}{\alpha\pi}\cos x = \lambda_1\phi_1$$

$$K\phi = \int_0^\pi (\sin x \sin \xi + \alpha \cos x \cos \xi)\frac{2}{\pi}\sin\xi \, d\xi = \frac{2}{\pi}\sin x \int_0^\pi \sin^2\xi \, d\xi = \frac{\pi}{2}\cdot\frac{2}{\pi}\sin x = \lambda_2\phi_2$$

3.4.3 Find the eigenvalues and eigenfunctions of the integral operator

$$\mathbb{K}u = \int_{-1}^{1} K(x,y)\,u(y)\,dy, \quad K(x,y) = 1-|x-y|$$

Find $(\lambda, \phi(x))$ s.t. $\lambda\phi(x) = \mathbb{K}\phi \quad (\phi \not\equiv 0)$.



$$\lambda\phi(x) = \int_{-1}^{1} K(x,y)\,\phi(y)\,dy$$
$$= \int_{-1}^{0} (1-x+y)\,\phi(y)\,dy + \int_{0}^{1} (1+x-y)\,\phi(y)\,dy$$

$$\lambda\phi'(x) = -\int_{-1}^{0} \phi(y)\,dy + \int_{0}^{1} \phi(y)\,dy = \text{constant}$$

$\therefore \lambda\phi(x) = Cx + D$ for constants C, D if $\underline{\lambda \neq 0}$.

Put $\lambda\phi(x) = Cx+D$ into the integral equation:

$$Cx+D = \lambda\phi(x) = \int_{-1}^{1} K(x,y)\phi(y)\,dy = \frac{1}{\lambda}\int_{-1}^{0}(1-x+y)(Cy+D)\,dy + \frac{1}{\lambda}\int_{0}^{1}(1+x-y)(Cy+D)\,dy$$
$$= \frac{1}{\lambda}\frac{1}{6}\{(3Cx - 3C - 6Dx + 3D) + (3Cx + 3C + 6Dx + 3D)\}$$
$$= \frac{1}{\lambda}(Cx+D) \implies \text{Either } C=D=0 \text{ or } \lambda = 1$$

If $C=D=0$ we have $\phi(x) \equiv 0$. But $\phi(x)$ is nonzero by assumption. $\therefore \lambda = 1$. Let $(\lambda_1, \phi_1(x)) = (1, x)$ and $(\lambda_2, \phi_2(x)) = (1, 1)$. These are the eigenvalues and corresponding eigenvectors of the integral operator.

Check: Using the previous calculation with $(C,D) = (1,0)$ (resp. $(C,D) = (0,1)$).

$$\mathbb{K}\phi_1 = \int_{-1}^{1} K(x,y)\,y\,dy = \frac{1}{\lambda}x = \frac{1}{1}x = 1\cdot x = \lambda\phi_1(x)$$
$$\mathbb{K}\phi_2 = \int_{-1}^{1} K(x,y)\,dy = \frac{1}{\lambda}\cdot 1 = 1\cdot 1 = \lambda\phi_2(x)$$

Note that $C\phi_1$, $D\phi_2$ are also eigenfunctions for any $C \neq 0$, $D \neq 0$.

If $\lambda = 0$, $0 = \lambda\phi(x) = \int_{-1}^{0}(1-x+y)\phi(y)\,dy + \int_{0}^{1}(1+x-y)\phi(y)\,dy$. In particular, at $x=0$,

$$0 = \lambda\phi(0) = \int_{-1}^{0}(1+y)\phi(y)\,dy + \int_{0}^{1}(1-y)\phi(y)\,dy = \int_{0}^{1}(1-y)\phi(-y)\,dy + \int_{0}^{1}(1-y)\phi(y)\,dy$$

Since $1-y > 0$ for $0<y<1$, this implies $\phi(-y) = -\phi(y)$, $0<y<1$. This might be used to either rule out $\lambda = 0$ or find the eigenfunctions corresponding to $\lambda = 0$. Uncertain, but I suspect 0 is not an eigenvalue.

<u>3.5.1</u> Show that the integral equation is equivalent to a differential equation, find the resolvent (or pseudo-resolvent) operator, and solve the integral equation.

$$u(x) = 1 + \int_0^x u(t)\,dt$$

- $\frac{d}{dx} u(x) = \frac{d}{dx}\left\{1 + \int_0^x u(t)\,dt\right\}, \quad u(0) = 1 + \int_0^0 u(t)\,dt$

  $u'(x) = u(x), \quad u(0) = 1$

  From this we know already that $u(t) = e^t$.

- $u(x) = 1 + \int_0^x u(t)\,dt$ is a Volterra integral equation of the second kind:

  $$u(x) = F(x) + \lambda \int_0^x K(x,t)\,u(t)\,dt$$

  In this case $F(x) = 1$, $\lambda = 1$, $K(x,t) = 1$. One way to find the resolvent kernel $r(x,t,\lambda)$ is to first find the iterated kernels $K_n, n = 1, 2, \ldots$ and then calculate $r(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,t)$. Here $\lambda = 1$.

  $K_1(x,t) = K(x,t) = 1$

  $K_2(x,t) = \int_t^x K(x,s)\,K_1(s,t)\,ds = \int_t^x ds = x - t$

  $K_3(x,t) = \int_t^x K(x,s)\,K_2(s,t)\,ds = \int_t^x (s-t)\,ds = \frac{1}{2}(x-t)^2$

  $K_4(x,t) = \int_t^x K(x,s)\,K_3(s,t)\,ds = \int_t^x \frac{1}{2}(s-t)^2\,ds = \frac{1}{3\cdot 2}(x-t)^3$

  Claim: $K_n(x,t) = \frac{1}{(n-1)!}(x-t)^{n-1}, \quad n \in \{1, 2, \ldots$

  Proof. The base case $(n=1)$ has already been established. Assume the equality holds for some $n \geq 1$.

  $K_{n+1}(x,t) = \int_t^x K(x,s)\,K_n(s,t)\,ds = \int_t^x 1 \cdot \frac{(s-t)^{n-1}}{(n-1)!}\,ds = \frac{(x-t)^n}{n!}$ ■

  Now $r(x,t,1) = \sum_{n=1}^{\infty} K_n(x,t) = \sum_{n=1}^{\infty} \frac{(x-t)^{n-1}}{(n-1)!} = \sum_{j=0}^{\infty} \frac{(x-t)^j}{j!} = e^{x-t}$.

<u>Definition:</u> $r(x,t,\lambda)$ is the resolvent kernel of $\phi(x) = f(x) + \lambda \int_a^x K(x,t)\phi(s)\,ds$ if the solution $\phi(x)$ can be written as $\phi(x) = f(x) + \lambda \int_a^x r(x,t,\lambda) f(t)\,dt$.

$$\therefore \quad u(x) = 1 + \int_0^x e^{x-t}\,dt = 1 - \left(e^{x-t}\Big|_{t=0}^{x}\right) = e^x$$

3.5.5 Find the resolvent kernel and solve the integral equation:

$$u(x) = \tfrac{5x}{6} + \tfrac{1}{2}\int_0^1 xt\, u(t)\, dt$$

Define the iterated kernels.

$$K_1(x,t) = K(x,t) = xt$$

$$K_2(x,t) = \int_0^1 K(x,s) K_1(s,t)\, ds = \int_0^1 (xs)(st)\, ds = \tfrac{1}{3} xt$$

$$K_3(x,t) = \int_0^1 K(x,s) K_2(s,t)\, ds = \tfrac{1}{3}\int_0^1 (xs)(st)\, ds = \left(\tfrac{1}{3}\right)^2 xt$$

$$\vdots$$

* $K_n(x,t) = \int_0^1 K(x,s) K_{n-1}(s,t)\, ds = \left(\tfrac{1}{3}\right)^{n-1} xt \,, \; n \in \{1,2,3,\ldots\}$

*Proof. The base case $(n=1)$ has been established already. Assuming the result holds for some $n \geq 1$,

$$K_{n+1}(x,t) = \int_0^1 K(x,s) K_n(s,t)\, ds = \int_0^1 (xs)\left(\tfrac{1}{3}\right)^{n-1}(st)\, ds = \left(\tfrac{1}{3}\right)^n xt.$$

Define the resolvent kernel $r(x,t,\lambda) = r(x,t,\tfrac{1}{2})$ by

$$r(x,t,\tfrac{1}{2}) = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^{n-1} K_n(x,t) = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^{n-1}\left(\tfrac{1}{3}\right)^{n-1} xt = \frac{6xt}{5}$$

Calculate the solution using the resolvent kernel.

$$u(x) = f(x) + \lambda \int_0^h r(x,t,\lambda) f(t)\, dt = \tfrac{5}{6}x + \tfrac{1}{2}\int_0^1 \left(\tfrac{6}{5}xt\right)\left(\tfrac{5}{6}t\right) dt = \tfrac{5}{6}x + \tfrac{1}{6}x = x.$$

Check:

$$\tfrac{5}{6}x + \tfrac{1}{2}\int_0^1 (xt)(t)\, dt = \tfrac{5}{6}x + \tfrac{1}{2}\int_0^1 xt^2\, dt = \tfrac{5}{6}x + \tfrac{1}{6}x = x = u(x)$$