Lemma: $\int_0^{2\pi} \sum_{j=0}^{n} f^{(j)}(x)(\cos(mx))^{(j)}\,dx = \left(\sum_{j=0}^{n} m^{2j}\right)\int_0^{2\pi} f(x)\cos(mx)$, $n=0,1,2,\ldots$

$\int_0^{2\pi} \sum_{j=0}^{n} f^{(j)}(x)(\sin(mx))^{(j)}\,dx = \left(\sum_{j=0}^{n} m^{2j}\right)\int_0^{2\pi} f(x)\sin(mx)$, $n=0,1,2,\ldots$

For $f \in H^n[0,2\pi]$ with f 2π-periodic.

Proof.

Part 1

Since f is 2π periodic, $f^{(j)}$ is 2π-periodic for $j=1,2,\ldots,n$. To see this, begin with the first derivative:

$$f'(x+2\pi) = \lim_{h\to 0} \frac{f(x+2\pi+h)-f(x+2\pi)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

Apply this reasoning iteratively to show that $f^{(2)}, f^{(3)}, \ldots, f^{(n)}$ are each 2π-periodic. In particular, $f^{(j)}(0) = f^{(j)}(2\pi)$ is used repeatedly in part 2.

Part 2

$$\boxed{\int_0^{2\pi} f^{(j)}(x)\,(\cos mx)^{(j)}\,dx = m^{2j}\int_0^{2\pi} f(x)\cos mx\,dx}$$

Case 1: $j = 4, 8, 12, \ldots$

$$\begin{aligned}
\int_0^{2\pi} f^{(j)}(x)(\cos mx)^{(j)}\,dx &= \int_0^{2\pi} f^{(j)}(x) m^j \cos mx\,dx\\
&= m^j\left[f^{(j-1)}(x)\cos mx\Big|_0^{2\pi} + m\int_0^{2\pi} f^{(j-1)}(x)\sin mx\,dx\right]\\
&= m^j m\int_0^{2\pi} f^{(j-1)}(x)\sin mx\,dx \qquad (f^{(j-1)}(0)=f^{(j-1)}(2\pi))\\
&= m^j m\left[f^{(j-2)}(x)\sin mx\Big|_0^{2\pi} - m\int_0^{2\pi} f^{(j-2)}(x)\cos mx\,dx\right]\\
&= -m^j m^2\int_0^{2\pi} f^{(j-2)}(x)\cos mx\,dx\\
&= -m^j m^2\left[f^{(j-3)}(x)\cos mx\Big|_0^{2\pi} + m\int_0^{2\pi} f^{(j-3)}(x)\sin mx\,dx\right]\\
&= -m^j m^3\int_0^{2\pi} f^{(j-3)}(x)\sin mx\,dx\\
&= -m^j m^3\left[f^{(j-4)}(x)\sin mx\Big|_0^{2\pi} - m\int_0^{2\pi} f^{(j-4)}\cos mx\,dx\right]\\
&= m^j m^4\int_0^{2\pi} f^{(j-4)}(x)\cos mx\,dx\\
&\;\vdots\\
&= m^j m^j\int_0^{2\pi} f^{(j-j)}(x)\cos mx\,dx\\
&= m^{2j}\int_0^{2\pi} f(x)\cos mx\,dx
\end{aligned}$$

Case 2: $j = 3, 7, 11, \ldots$

$$\begin{aligned}
\int_0^{2\pi} f^{(j)}(x)(\cos mx)^{(j)}\,dx &= \int_0^{2\pi} f^{(j)}(x) m^j \sin mx\,dx\\
&= -m^j m\int_0^{2\pi} f^{(j-1)}(x)\cos mx\,dx\\
&= -m^j m^2\int_0^{2\pi} f^{(j-2)}(x)\sin mx\,dx\\
&= m^j m^3\int_0^{2\pi} f^{(j-3)}(x)\cos mx\,dx\\
&= m^j m^4\int_0^{2\pi} f^{(j-4)}(x)\sin mx\,dx\\
&\;\vdots\\
&= m^j m^j\int_0^{2\pi} f^{(j-j)}(x)\cos mx\,dx
\end{aligned}$$

$\cos mx\,dx$

Case 3 : $j = 2, 6, 10, \ldots$

$$\begin{aligned}\int_0^{2\pi} f^{(j)}(x)(\cos mx)^{(j)}\,dx &= \int_0^{2\pi} f^{(j)}(x)(-m^j)\cos mx\,dx \\ &= -m^j m \int_0^{2\pi} f^{(j-1)}(x)\sin mx\,dx \\ &= m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\cos mx\,dx \\ &= \\ &\vdots \\ &= m^{2j}\int_0^{2\pi} f(x)\cos mx\,dx\end{aligned}$$

Case 4: $j = 1, 5, 9, \ldots$

$$\begin{aligned}\int_0^{2\pi} f^{(j)}(x)(\cos mx)^{(j)}\,dx &= \int_0^{2\pi} f^{(j)}(x)(-m^j)\sin mx\,dx \\ &= m^j m \int_0^{2\pi} f^{(j-1)}(x)\cos mx\,dx \\ &= m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\sin mx\,dx \\ &= -m^j m^3 \int_0^{2\pi} f^{(j-3)}(x)\cos mx\,dx \\ &\vdots \\ &= m^{2j}\int_0^{2\pi} f(x)\cos mx\,dx\end{aligned}$$

$$\boxed{\int_0^{2\pi} f^{(j)}(x)(\sin mx)^{(j)}\,dx = m^{2j}\int_0^{2\pi} f(x)\sin mx\,dx}$$

Case 1: $j = 4, 8, 12, \ldots$

$$\begin{aligned}\int_0^{2\pi} f^{(j)}(x)(\sin mx)^{(j)}\,dx &= m^j \int_0^{2\pi} f^{(j)}(x)\sin mx\,dx \\ &= -m^j m \int_0^{2\pi} f^{(j-1)}(x)\cos mx\,dx \\ &= -m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\sin mx\,dx \\ &= m^j m^3 \int_0^{2\pi} f^{(j-3)}(x)\cos mx\,dx \\ &\vdots \\ &= m^{2j}\int_0^{2\pi} f(x)\sin mx\,dx\end{aligned}$$

Case 2 : $j = 3, 7, 11, \ldots$

$$\begin{aligned}\int_0^{2\pi} f^{(j)}(x)(\sin mx)^{(j)}\,dx &= -m^j \int_0^{2\pi} f^{(j)}(x)\cos mx\,dx \\ &= -m^j m \int_0^{2\pi} f^{(j-1)}(x)\sin mx\,dx \\ &= m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\cos mx\,dx \\ &= m^j m^3 \int_0^{2\pi} f^{(j-3)}(x)\sin mx\,dx \\ &\vdots \\ &= m^{2j}\int_0^{2\pi} f(x)\sin mx\,dx\end{aligned}$$

Case 3 : $j = 2, 6, 10, \ldots$

$$\begin{aligned}\int_0^{2\pi} f^{(j)}(x)(\sin mx)^{(j)}\,dx &= -m^j \int_0^{2\pi} f^{(j)}(x)\sin mx\,dx \\ &= m^j m \int_0^{2\pi} f^{(j-1)}(x)\cos mx\,dx \\ &= m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\sin mx\,dx \\ &\vdots \\ &= m^{2j}\int_0^{2\pi} f(x)\sin mx\,dx\end{aligned}$$

Case 4: $j = 1, 5, 9, \ldots$

$$\begin{aligned}\int_0^{2\pi} f^{(j)}(x)\,(\sin mx)^{(j)}\,dx &= m^j \int_0^{2\pi} f^{(j)}(x)\cos mx\,dx \\ &= m^j m \int_0^{2\pi} f^{(j-1)}(x)\sin mx\,dx \\ &= -m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\cos mx\,dx \\ &\;\vdots \\ &= m^{2j}\int_0^{2\pi} f(x)\sin mx\,dx\end{aligned}$$

Part 3 (Induction)

The identities in the lemma clearly hold for $n=0$ $\left(H^0[0,2\pi]\right)$.
Assume the identities hold for $n-1$ with $n \geq 1$

$$\begin{aligned}\int_0^{2\pi} \sum_{j=0}^{n} f^{(j)}(x)(\cos(mx))^{(j)}dx &= \int_0^{2\pi} \sum_{j=0}^{n-1} f^{(j)}(x)(\cos(mx))^{(j)}dx + \int_0^{2\pi} f^{(n)}(x)(\cos mx)^{(n)}dx \\ &= \left(\sum_{j=0}^{n-1} m^{2j}\right)\int_0^{2\pi} f(x)\cos mx\,dx + m^{2n}\int_0^{2\pi} f(x)\cos mx\,dx \\ &= \left(\sum_{j=0}^{n} m^{2j}\right)\int_0^{2\pi} f(x)\cos mx\,dx\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} \sum_{j=0}^{n} f^{(j)}(x)(\sin(mx))^{(j)}dx &= \int_0^{2\pi} \sum_{j=0}^{n-1} f^{(j)}(x)(\sin(mx))^{(j)}dx + \int_0^{2\pi} f^{(n)}(x)(\sin mx)^{(n)}dx \\ &= \left(\sum_{j=0}^{n-1} m^{2j}\right)\int_0^{2\pi} f(x)\sin mx\,dx + m^{2n}\int_0^{2\pi} f(x)\sin mx\,dx \\ &= \left(\sum_{j=0}^{n} m^{2j}\right)\int_0^{2\pi} f(x)\sin mx\,dx\end{aligned}$$