From "Principles of Applied Mathematics" by James P. Keener.

Chapter 1 Finite Dimensional Vector Spaces

Section 1.1 Linear Vector Spaces

Problem 1

Show that in any inner product space $||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$.

Answer:

$$\begin{aligned}
||x+y||^2 + ||x-y||^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x, x+y\rangle + \langle y, x+y\rangle + \langle x, x-y\rangle - \langle y, x-y\rangle \\
&= \overline{\langle x+y, x\rangle} + \overline{\langle x+y, y\rangle} + \overline{\langle x-y, x\rangle} - \overline{\langle x-y, y\rangle} \\
&= \overline{\langle x, x\rangle + \langle y, x\rangle} + \overline{\langle x, y\rangle + \langle y, y\rangle} + \overline{\langle x, x\rangle - \langle y, x\rangle} - \overline{\langle x, y\rangle - \langle y, y\rangle} \\
&= \overline{\langle x, x\rangle} + \overline{\langle y, x\rangle} + \overline{\langle x, y\rangle} + \overline{\langle y, y\rangle} + \overline{\langle x, x\rangle} - \overline{\langle y, x\rangle} - \overline{\langle x, y\rangle} + \overline{\langle y, y\rangle} \\
&= 2\overline{\langle x, x\rangle} + 2\overline{\langle y, y\rangle} \\
&= 2\langle x, x\rangle + 2\langle y, y\rangle \\
&= 2||x||^2 + 2||y||^2.
\end{aligned}$$

Problem 5

Verify that the choice $\alpha = \frac{\langle x,y\rangle}{||y||^2}$ makes $||x-\alpha y||^2$ as small as possible. Show that $|\langle x, y\rangle|^2 = ||x||^2||y||^2$ if and only if x and y are linearly dependent.

$$\begin{aligned}
0 &= \frac{\partial}{\partial\alpha}||x-\alpha y||^2 \\
&= \frac{\partial}{\partial\alpha}\left[||x||^2 - 2\mathrm{Re}\left(\langle x, \alpha y\rangle\right) + |\alpha|^2||y||^2\right] \\
&= 0 - 2\frac{\partial}{\partial\alpha}\mathrm{Re}\left(\overline{\alpha}\langle x, y\rangle\right) + 2\alpha||y||^2 \\
&= -2\mathrm{Re}\left(\frac{\partial}{\partial\alpha}\overline{\alpha}\langle x, y\rangle\right) + 2\alpha||y||^2 \\
&= -2\mathrm{Re}\left(\overline{\frac{\partial}{\partial\alpha}\alpha\overline{\langle x, y\rangle}}\right) + 2\alpha||y||^2 \\
&= -2\mathrm{Re}\left(\overline{\overline{\langle x, y\rangle}}\right) + 2\alpha||y||^2 \\
&= -2\mathrm{Re}\left(\langle x, y\rangle\right) + 2\alpha||y||^2 \\
\implies \alpha &= \frac{\mathrm{Re}\langle x, y\rangle}{||y||^2}.
\end{aligned}$$

Since $||x-y||^2$ is quadratic in α with $||y||^2 > 0$, this value of α minimizes $||x-\alpha y||^2$. If x and y are real valued vectors or if at least $\langle x, y\rangle$ is a real number, $\alpha = \langle x, y\rangle/||y||^2$ makes $||x-\alpha y||^2$ as small as possible.

Section 1.2 Spectral Theory for Matrices

Problem 2

Prove that two symmetric matrices are equivalent if and only if they have the same eigenvalues.

Answer: Suppose that A and B are symmetric matrices and that $A = M^{-1}BM$. Suppose λ is an eigenvalue of A with corresponding eigenvector v.

$$\begin{aligned} Av &= \lambda v \\ M^{-1}BMv &= \lambda v \\ B(Mv) &= \lambda(Mv) \end{aligned}$$

This shows that λ is also an eigenvalue of B (and that Mv is an eigenvector corresponding to λ). A similar calculation shows that any eigenvalue of B is also an eigenvalue of A. Conclude that the similar matrices A and B have the same eigenvalues.

Now suppose that A and B have the same eigenvalues. Since A is symmetric, $A = C^{-1}\Lambda C$ where Λ is the diagonal matrix containing the eigenvalues of A (we can assume that the eigenvalues in Λ can be put into descending order using permutation matrices if necessary) and C contains eigenvectors corresponding to the eigenvalues of A. Since B is symmetric and has the same eigenvalues as A, $B = D^{-1}\Lambda D$ where D contains the eigenvectors corresponding to the eigenvalues of B.

$$CAC^{-1} = \Lambda = DBD^{-1} \implies A = C^{-1}DBD^{-1}C = (D^{-1}C)^{-1}B(D^{-1}C) = M^{-1}BM, \quad M = D^{-1}C.$$

Problem 7

Find the spectral representation of the matrix

$$A = \begin{pmatrix} 7 & 2 \\ -2 & 2 \end{pmatrix} .$$

Illustrate how $Ax = b$ can be solved geometrically using the appropriately chosen coordinate system on a piece of graph paper.

Answer: Find the eigenvalues of A: $0 = (7-\lambda)(2-\lambda)+4 = (\lambda-6)(\lambda-3) \implies \lambda = 6, 3$. Take $x = (2, -1)^T/\sqrt{3}$ as an eigenvector corresponding to $\lambda = 6$ and $y = (1, -2)^T/\sqrt{3}$ as an eigenvector corresponding to $\lambda = 3$. Let $M = (x, y)$. A spectral representation of A is given by

$$A = M^{-1}\Lambda M = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}.$$

Solving $Az = b$ is the same as solving $\Lambda(Mz) = Mb$ for $Mz = z'$ and then calculating $z = M^{-1}z'$. Since Λ is diagonal, the ith element of Mz is found by dividing the ith element of Mb by λ_i. Choose the coordinate system that has the eigenvectors x and y as axes so that solving any system amounts to solving $\Lambda z' = b'$. That is, find the coordinates of b' with respect to x and y.

Problem 9

The sets of vectors $\{\phi_i\}_{i=1}^n, \{\psi_i\}_{i=1}^n$ are biorthogonal if $\langle \phi_i, \psi_j \rangle = \delta_{ij}$. Suppose $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ are biorthogonal.

a. Show that $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ each form a linearly independent set.

b. Show that any vector in $\mathbb{R}^n$ can be written as a linear combination of $\{\phi_i\}$ as

$$x = \sum_{i=1}^{n} \alpha_i \phi_i, \quad \alpha_i = \langle x, \psi_i \rangle.$$

c. Re-express the result from part b in matrix form; that is, show that

$$x = \sum_{i=1}^{n} P_i x$$

where P_i are projection matrices with the properties that $P_i^2 = P_i$ and $P_i P_j = 0$ for $i \neq j$. Express the matrix P_i in terms of the vectors ϕ_i and ψ_i.

Answer:

a. Suppose $\sum_{i=1}^{n} \alpha_i \phi_i = 0$. For $j = 1, \dots, n$, taking the inner product with ψ_j on each side of this equation gives:

$$\alpha_j = \alpha_j \langle \phi_j, \psi_j \rangle = \left\langle \sum_{i=1}^{n} \alpha_i \phi_i, \psi_j \right\rangle = \langle 0, \psi_j \rangle = 0.$$

The only way to write the zero vector as a linear combination of the ϕ_i is when all coefficients $\alpha_i = 0$. Therefore $\{\phi_i\}_{i=1}^{n}$ is a linearly independent set. A similar calculation shows that if $\sum_{i=1}^{n} \beta_i \psi_i = 0$, then $\beta_i = 0$ for each i. Therefore $\{\psi_i\}_{i=1}^{n}$ is also a linearly independent set.

b. This was not explicitly stated but assuming that the ϕ_i are vectors in $\mathbb{R}^n$, $\{\phi_i\}_{i=1}^{n}$ is a set of n linearly independent vectors in $\mathbb{R}^n$ so the vectors in $\{\phi_i\}_{i=1}^{n}$ form a basis for $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, there exist α_i, $i = 1, \dots, n$ such that $x = \sum_{i=1}^{n} \alpha_i \phi_i$. Use the biorthogonality of $\{\phi_i\}_{i=1}^{n}$ and $\{\psi_i\}_{i=1}^{n}$ to determine the value of each α_i.

$$\langle x, \psi_i \rangle = \left\langle \sum_{k=1}^{n} \alpha_k \phi_k, \psi_i \right\rangle = \langle \alpha_i \phi_i, \psi_i \rangle = \alpha_i, \quad \text{for each } i = 1, \dots, n.$$

c.

$$\begin{aligned}
x &= \sum_{i=1}^{n} \alpha_i \phi_i \\
&= \sum_{i=1}^{n} \langle x, \psi_i \rangle \phi_i \\
&= \begin{pmatrix} \phi_1 & \dots & \phi_n \end{pmatrix} \begin{pmatrix} \langle x, \psi_1 \rangle \\ \vdots \\ \langle x, \psi_n \rangle \end{pmatrix} \\
&= \begin{pmatrix} \phi_1 & \dots & \phi_n \end{pmatrix} \begin{pmatrix} \psi_1^T \\ \vdots \\ \psi_n^T \end{pmatrix} x \\
&= \sum_{i=1}^{n} \phi_i \psi_i^T x \\
&= \sum_{i=1}^{n} P_i x, \quad P_i = \phi_i \psi_i^T
\end{aligned}$$

$$P_i^2 = (\phi_i\psi_i^T)(\phi_i\psi_i^T) = \phi_i(\psi_i^T\phi_i)\psi_i^T) = \phi_i\psi_i^T = P_i \text{ and } P_iP_j = \phi_i\psi_i^T\phi_j\psi_j^T = \phi_i \cdot 0 \cdot \psi_j^T = 0 \text{ for } i \neq j.$$

Problem 10

a. Suppose the eigenvalues of A are distinct. Show that the eigenvectors of A and the eigenvectors of A^* form a biorthogonal set.

b. Suppose $A\phi = \lambda_i\phi_i$ and $A^*\psi_i = \overline{\lambda_i}\psi_i$, $i = 1, \ldots, n$ and that $\lambda_i \neq \lambda_j$ for $i \neq j$. Prove that $A = \sum_{i=1}^n \lambda_i P_i$ where $P_i = \phi_i\psi_i^*$ is a projection matrix.

c. Express the matrices C and C^{-1} where $A = C\Lambda C^{-1}$, in terms of ϕ_i and ψ_i.

Answer:

a. If λ is an eigenvalue of A then $\overline{\lambda}$ is an eigenvalue of A^*.

$$\lambda_j\langle\phi_i,\psi_j\rangle = \langle\phi_i,\overline{\lambda_j}\psi_j\rangle = \langle\phi_i,A^*\psi_j\rangle = \langle A\phi_i,\psi_j\rangle = \langle\lambda_i\phi_i,\psi_j\rangle = \lambda_i\langle\phi_i,\psi_j\rangle \implies (\lambda_j - \lambda_i)\langle\phi_i,\psi_j\rangle = 0$$

From this conclude that $\langle\phi_i,\psi_j\rangle = 0$ if $\lambda_i \neq \lambda_j$.

Since A has n distince eigenvalues, A^* has n distinct eigenvalues. By Theorems 1.2 and 1.3 both are diagonalizable.

$$\begin{aligned} \Lambda &= M^{-1}AM \\ \Lambda^* &= M^*A^*(M^{-1})^* = N^{-1}A^*N, \quad N = (M^{-1})^* \end{aligned}$$

M contains the eigenvectors $\{\phi_i\}_{i=1}^n$ of A and N contains the eigenvectors $\{\psi_i\}_{i=1}^n$ of A^*.

$$I = M^{-1}M = N^*M = \begin{pmatrix} \psi_1^* \\ \vdots \\ \psi_n^* \end{pmatrix} \begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} = \begin{pmatrix} \langle\psi_1,\phi_1\rangle & & \\ & \ddots & \\ & & \langle\psi_n,\phi_n\rangle \end{pmatrix}$$

Conclude that $\langle\psi_i,\phi_i\rangle = 1$ for $i = 1, \ldots, n$.

b.
$$A = M\Lambda M^{-1} = M\Lambda N^* = \begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \psi_1^* \\ \vdots \\ \psi_n^* \end{pmatrix} = \sum_{i=1}^n \lambda_i\phi_i\psi_i^* = \sum_{i=1}^n \lambda_i P_i.$$

The P_i are projection matrices since $P_i^2 = P_i$.

c. Replacing M with C in the work above,

$$A = C\Lambda C^{-1}, \quad C = \begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} \psi_1^* \\ \vdots \\ \psi_n^* \end{pmatrix}$$

Section 1.3 Geometrical Significance of Eigenvalues

Problem 3

Show that the intermediate eigenvalue of λ_2 of A is not positive when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{pmatrix}.$$

Answer: No complete answer was found. First I calculated $Q(x) = \langle Ax, x \rangle$, which gives a quadratic form in x_1, x_2, x_3. Then converted to polar coordinates but with $\rho = 1$ and calculated $\frac{\partial Q}{\partial \phi}, \frac{\partial Q}{\partial \theta}$. This got too complicated to continue without more time.

Section 1.4 Fredholm Alternative Theorem

Problem 1

Under what conditions do these matrices have solutions $Ax = b$? Are they unique?

a.

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 5 \end{pmatrix}$$

b.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Answer: By Theorem 1.10 (Fredholm Alternative), $Ax = b$ has a solution if and only if $\langle b, v \rangle = 0$ for every vector v satisfying $A^* v = 0$. By Theorem 1.9, the solution to $Ax = b$ (if it exists) is unique if and only if the only solution to $Ax = 0$ is $x = 0$.

a. Find the nullspace of A^*. That is, find the set of vectors v such that $A^* v = 0$.

$$(A^*|0) = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 3 & 5 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \implies v = r \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad r \in \mathbb{C}.$$

$$0 = \langle b, v \rangle = r(b_1 + b_2 - b_3)$$

The equation $Ax = b$ has a solution if and only if the components of b satisfy $b_1 + b_2 - b_3 = 0$. The solution x of $Ax = b$ is unique if and only if the equation $Ay = 0$ has $y = 0$ as its only solution.

$$(A|0) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 2 & 5 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies y = 0.$$

There exists a unique solution x of the equation $Ax = b$ if $b_1 + b_2 - b_3 = 0$.

b.

$$(A^*|0) = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \sim (I|0) \implies v = 0.$$

The nullspace of A^* contains only the zero vector. This means that for any b, $0 = \langle b, v\rangle$ for every v such that $A^*v = 0$. The equation $Ax = b$ has a solution for any b.

$$(A|0) = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim (I|0) \implies y = 0.$$

The only solution to $Ay = 0$ is $y = 0$. Conclude that for any b there exists a unique solution x of the equation $Ax = b$.

Problem 4

A square matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if $\langle Ax, x\rangle > 0$ for all $x \neq 0$. Use the Fredholm alternative to prove that a positive definite matrix is uniquely invertible.

Answer: Suppose $0 = A^*v = A^Tv$. Then $0 = v^TA^Tv = \langle Av, v\rangle$. Since A is positive definite, $v = 0$. For any $b \in \mathbb{R}^n$, $0 = \langle b, v\rangle = \langle b, 0\rangle$. By the Fredholm alternative there exists a solution x of the equation $Ax = b$ for any $b \in \mathbb{R}^n$. In particular there exist $x_1, \ldots, x_n$ such that $Ax_i = e_i$ for $i = 1, \ldots, n$ where e_i is the ith standard basis vector for $\mathbb{R}^n$. Suppose $Ay = 0$. Then $0 = y^TAy = \langle y, Ay\rangle = \overline{\langle Ay, y\rangle} = \langle Ay, y\rangle$. Since A is positive definite, $y = 0$. For $i = 1, \ldots, n$ there exists by the above and Theorem 1.9 a unique solution x_i of the equation $Ax_i = e_i$. Therefore A is uniquely invertible with $A^{-1} = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix}$.

Section 1.5 Least Squares Solutions - Pseudo Inverses

Problem 2

Verify that the pseudo-inverse of an $m \times n$ diagonal matrix D with $d_{ij} = \sigma_i \delta_{ij}$ is the $n \times m$ diagonal matrix D' with $d'_{ij} = \frac{1}{\sigma_i}\delta_{ij}$ whenever $\sigma_i \neq 0$ and $d'_{ij} = 0$ otherwise.

Answer: The definition of a least square pseudo inverse A' of A requires that $x = A'b$ satisfies:

1. $A^*Ax = A^*b$
2. $\langle x, w\rangle = 0$ for every w satisfying $Aw = 0$.

If $\sigma_i \neq 0$ for each i,

$$\begin{aligned} D^*Dx &= D^*b \\ D^*DD'b &= D^*b \\ \begin{pmatrix} |\sigma_1|^2 & & \\ & \ddots & \\ & & |\sigma_n|^2 \end{pmatrix} D'b &= D^*b \\ \begin{pmatrix} |\sigma_1|^2 & & \\ & \ddots & \\ & & |\sigma_n|^2 \end{pmatrix} \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{pmatrix} b &= \begin{pmatrix} \sigma_1^* & & \\ & \ddots & \\ & & \sigma_n^* \end{pmatrix} b \end{aligned}$$

This shows that $|\sigma_i|^2 b_i/\sigma_i = \sigma_i^* b_i$, which is equivalent to $|\sigma_i|^2 b_i = |\sigma_i|^2 b_i$, which is of course true. There may be an additional $r = m - n$ columns in D' and D^* if $m > n$. Since these are r columns of zeros, multiplication by b just causes the last r components to vanish so that for these components we only need to verify that $0 = 0$, which is of course true. Finally, if there is a $\sigma_i = 0$, we have instead of $1/\sigma_i$ a 0 at the ith diagonal entry of D'. This causes the ith component of the left hand side of the equation to vanish. But also $\sigma_i^* = 0$ in this case so that the same occurs on the right hand side of this equation, so that we again only confirm that $0 = 0$. Therefore $x = D'b$ satisfies (1).

Suppose w satisfies $Dw = 0$:

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}.$$

This shows that $w_i = 0$ whenever $\sigma_i \neq 0$.

$$\langle x, w\rangle = \langle D'b, w\rangle = \langle b^*, (D')^*w\rangle = \begin{pmatrix} b_1^* & \dots & b_n^* \end{pmatrix} \begin{pmatrix} (d'_{11})^* & & \\ & \ddots & \\ & & (d'_{nn})^* \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix},$$

where $(d'_{ii})^* = (1/\sigma_i)^*$ for $\sigma_i \neq 0$ and $(d'_{ii})^* = 0^* = 0$ if $\sigma_i = 0$. But this means $(D')^*w = 0$ so that $\langle x, w\rangle = b^*(D')^*w = 0$. Therefore $x = D'b$ satisfies (2) as well. Conclude that D' is a least square pseudo inverse of D.

Problem 9

Find a singular value decomposition and pseudo-inverse of

$$A = \begin{pmatrix} 2\sqrt{5} & -2\sqrt{5} \\ 3 & 3 \\ 6 & 7 \end{pmatrix}.$$

Answer:

$$A^T A = \begin{pmatrix} 65 & 31 \\ 31 & 78 \end{pmatrix} \quad \text{has eigenvalues } \sigma^2 = \frac{1}{2}(143 \pm \sqrt{4013})$$

$$v_1 = \left(\frac{1}{62}\left(\sqrt{4013} - 13\right), 1 \right)^T / \left(\frac{\sqrt{4013} - 13}{62} \right)$$

$$v_2 = \left(\frac{1}{62}\left(-\sqrt{4013} - 13\right), 1 \right)^T / \left(\frac{\sqrt{4013} + 13}{62} \right)$$

$$V = \begin{pmatrix} v_1 & v_2 \end{pmatrix}, \quad V^* = V^T = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$$

Since $V^T V = V V^T = I$, V is an orthogonal matrix.

$$A A^T = \begin{pmatrix} 40 & 0 & -2\sqrt{5} \\ 0 & 18 & 39 \\ -2\sqrt{5} & 39 & 85 \end{pmatrix} \quad \text{has eigenvalues } \sigma^2 = 0, \frac{1}{2}(143 \pm \sqrt{4013})$$

$$u_1 \approx (-0.0707906, 0.457886, 1)^T / ||-0.0707906, 0.457886, 1)^T||$$

$$u_2 \approx (25.6839, 1.78687, 1)^T / ||(25.6839, 1.78687, 1)^T||$$

$$u_3 \approx (0.111803, -2.16667, 1)^T / ||(0.111803, -2.16667, 1)^T||$$

$$U = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}, \quad U^* = U^T = \begin{pmatrix} u_1^T \\ u_2^T \\ u_3^T \end{pmatrix}$$

We have $UU^T \approx I$ and $U^T U \approx I$. If we had u_1, u_2, u_3 exactly this would become $UU^T = U^T U = I$ so that U is orthogonal as well. A full SVD of A is given by

$$A = U \Sigma V^T = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}, \quad \sigma_1 = \sqrt{\frac{1}{2}\left(143 + \sqrt{4013}\right)}, \quad \sigma_2 = \sqrt{\frac{1}{2}\left(143 - \sqrt{4013}\right)}.$$

Since A has three rows and two columns, A cannot have a left right inverse. But since the columns of A are independent, A has full column rank so that $Ax = b$ has at most one solution x for every b. Then A has a 2×3 left inverse B such that $BA = I_n$.

$$\begin{aligned} B &= (A^T A)^{-1} A^T \\ &= \begin{pmatrix} 65 & 31 \\ 31 & 78 \end{pmatrix}^{-1} \begin{pmatrix} 2\sqrt{5} & 3 & 6 \\ -2\sqrt{5} & 3 & 7 \end{pmatrix} \\ &= \frac{1}{4109} \begin{pmatrix} 218\sqrt{5} & 141 & 251 \\ -192\sqrt{5} & 102 & 269 \end{pmatrix} \end{aligned}$$

Performing the calculation BA does indeed give I_n as claimed.