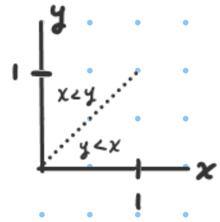
Chapter 3 Problem Set

3.1.1 Verify that the solution of $\frac{d^2u}{dt^2} = f(x)$, $u(0)=0$, $u(1)=0$ is given by $u(x) = \int_0^1 K(x,y) f(y)\,dy$ where

$$K(x,y) = \begin{cases} y(x-1) & 0 \le y < x \le 1 \\ x(y-1) & 0 \le x < y < 1 \end{cases}$$

$$\frac{du}{dx} = \frac{d}{dx}\int_0^1 K(x,y) f(y)\,dy = \int_0^1 f(y) \frac{\partial}{\partial x} K(x,y)\,dy \quad \text{(Leibniz integral rule)}$$

$$= \int_0^x f(y) \frac{\partial}{\partial x}[y(x-1)]\,dy + \int_x^1 f(y) \frac{\partial}{\partial x}[x(y-1)]\,dy$$



$$= \int_0^x y f(y)\,dy + \int_x^1 (y-1) f(y)\,dy$$

$$= \int_0^1 y f(y)\,dy - \int_x^1 f(y)\,dy$$

$$= \int_0^1 y f(y)\,dy + \int_1^x f(y)\,dy$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\int_0^1 y f(y)\,dy + \frac{d}{dx}\int_1^x f(y)\,dy = f(x) \quad \text{(FTC)}$$

$$u(0) = \int_0^1 K(0,y) f(y)\,dy = \int_0^1 0 \cdot (y-1) f(y)\,dy = \int_0^1 0\,dy = 0$$

$$u(1) = \int_0^1 K(1,y) f(y)\,dy = \int_0^1 y(1-1) f(y)\,dy = \int_0^1 0\,dy = 0$$

3.3.1 Find the solutions of

$$u(x) - \lambda \int_0^{2\pi} \sum_{j=1}^{n} \frac{\cos jt \cos jx}{j} u(t)\,dt = \sin^2 x \,, \quad n \geq 2$$

for all values of λ. Find the resolvent kernel for this equation. Find the least squares solution if necessary.

This is a Fredholm integral equation of the second kind with a separable integral kernel: $K(x,t) = \lambda/j \cos jt \cos jx = \lambda M_j(x) N_j(t)$ with $M_j(x) = \cos jx$ and $N_j(t) = 1/j \cos jt$.

$$u(x) = \sin^2 x + \lambda \sum_{j=1}^{n} \cos jx \int_0^{2\pi} \frac{1}{j} \cos jt\, u(t)\,dt$$

Find the c_j to determine $u(x)$

$$u(x) = \sin^2 x + \lambda \sum_{j=1}^{n} c_j \cos jx \,, \quad c_j \equiv \int_0^{2\pi} \frac{1}{j} \cos jt\, u(t)\,dt$$

Multiply through by $\frac{1}{k}\cos kx$ for $k \in \{1, \dots, n\}$ and integrate.

$$\underbrace{\int_0^{2\pi} u(x) \frac{1}{k}\cos kx\,dx}_{c_k} = \underbrace{\int_0^{2\pi} \sin^2 x \frac{1}{k}\cos kx\,dx}_{b_k} + \lambda \sum_{j=1}^{n} c_j \underbrace{\int_0^{2\pi} \frac{1}{k}\cos jx \cos kx\,dx}_{a_{kj}} \quad (k \in \{1,\dots,n\})$$

$$c_k = b_k + \lambda \sum_{j=1}^{n} a_{kj} c_j \,, \quad k = 1, 2, \dots, n$$

$$\vec{C} = \vec{B} + \lambda A \vec{C} \rightarrow \vec{C} = (I - \lambda A)^{-1} B$$

Since $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, b_2 may not be zero. Checking this:

$$b_2 = \int_0^{2\pi} \sin^2 x \frac{1}{2} \cos 2x\,dx = \frac{1}{4}\int_0^{2\pi} (1 - \cos 2x)\cos 2x\,dx = -\pi/4.$$

For $k \neq 2$, $b_k = \frac{1}{k}\int_0^{2\pi} \sin^2 x \cos kx\,dx = \frac{1}{2k}\int_0^{2\pi} \cos kx\,dx - \frac{1}{2k}\int_0^{2\pi} \cos 2x \cos kx\,dx = 0$

$$a_{kj} = \int_0^{2\pi} \frac{1}{k} \cos kx \cos jx\,dx = \begin{cases} 0 \,, & k \neq j \\ \int_0^{2\pi} \frac{1}{k} \cos^2 kx\,dx \,, & k = j \end{cases} = \begin{cases} 0 \,, & k \neq j \\ \pi/k \,, & k = j \end{cases}$$

$$\frac{1}{k}\int_0^{2\pi} \cos^2 kx\,dx = \frac{1}{2k}\int_0^{2\pi} (1 - \cos 2kx)\,dx = \frac{2\pi}{2k} - \left(\frac{1}{2k}\sin 2kx\right)\Big|_0^{2\pi} = \frac{\pi}{k}$$

$$I - \lambda A = \begin{bmatrix} 1-\lambda\pi & & & \\ & 1-\lambda\pi/2 & & \\ & & \ddots & \\ & & & 1-\lambda\pi/n \end{bmatrix} \quad (I - \lambda A)^{-1} = \begin{bmatrix} (1-\lambda\pi)^{-1} & & & \\ & (1-\lambda\pi/2)^{-1} & & \\ & & \ddots & \\ & & & (1-\lambda\pi/n)^{-1} \end{bmatrix}$$

for $\lambda \neq 1/\pi, 2/\pi, \dots, n/\pi$. Since $(I - \lambda A)^{-1}$ is diagonal and $b_k = 0$ for $k \neq 2$, we have $c_k = 0$ for $k \neq 2$ and $c_2 = (1 - \lambda\pi/2)^{-1}(-\pi/4)$

$$\therefore \boxed{u(x) = \sin^2 x - \frac{\lambda\pi}{4} \frac{1}{1 - \lambda\pi/2} \cos 2x = \sin^2 x - \frac{\lambda\pi}{4 - 2\lambda\pi} \,, \quad \lambda \neq 2/\pi}$$

3.4.1

a. Find the eigenfunctions for the integral operator

$$Ku = \int_0^1 K(x,\xi)\,u(\xi)\,d\xi$$

$$K(x,\xi) = \begin{cases} x(1-\xi) & 0 \le x < \xi \le 1 \\ \xi(1-x) & 0 \le \xi < x \le 1 \end{cases}$$

b. Find the expansion of $f(x)$ in terms of the eigenfunctions of K. Is there a solution of $Ku = f$?

$$f(x) = \begin{cases} x/2 & 0 \le x \le 1/2 \\ \frac{1-x}{2} & 1/2 \le x \le 1 \end{cases}$$

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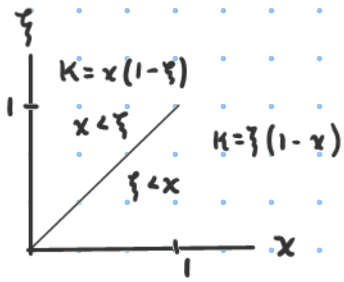
a. Determine eigenvalues λ, eigenfunctions $\phi(x) \not\equiv 0$ satisfying $K\phi = \lambda\phi$

$$\int_0^1 K(x,\xi)\,\phi(\xi)\,d\xi = \lambda\phi(x)$$



$$\int_0^x \xi(1-x)\,\phi(\xi)\,d\xi + \int_x^1 x(1-\xi)\,\phi(\xi)\,d\xi = \lambda\phi(x)$$

$$(1-x)\int_0^x \xi\,\phi(\xi)\,d\xi + x\int_x^1 (1-\xi)\,\phi(\xi)\,d\xi = \lambda\phi(x)$$

$$-\int_0^x \xi\,\phi(\xi)\,d\xi + (1-x)x\phi(x) + \int_x^1 (1-\xi)\phi(\xi)\,d\xi - x(1-x)\phi(x) = \lambda\phi'(x)$$

$$\int_x^1 (1-\xi)\phi(\xi)\,d\xi - \int_0^x \xi\,\phi(\xi)\,d\xi = \lambda\phi'(x)$$

$$-(1-x)\phi(x) - x\phi(x) = \lambda\phi''(x) \longrightarrow \lambda\phi''(x) + \phi(x) = 0$$

$\lambda \neq 0$ since if $\lambda = 0$, $0\phi''(x) + \phi(x) = 0$ implies $\phi(x) \equiv 0$, contradicting the assumption that $\phi \not\equiv 0$ as an eigenfunction.

$$\lambda\phi(0) = \int_0^0 \xi(1-0)\phi(\xi)\,d\xi + \int_0^1 0\cdot(1-\xi)\,\phi(\xi)\,d\xi = 0 \longrightarrow \phi(0) = 0$$
$$\lambda\phi(1) = \int_0^1 \xi(1-1)\phi(\xi)\,d\xi + \int_1^1 1\cdot(1-\xi)\phi(\xi)\,d\xi = 0 \longrightarrow \phi(1) = 0$$

Eigenpairs (λ, ϕ) satisfy the BVP $\lambda\phi''(x) + \phi(x) = 0$, $\lambda \neq 0$, $\phi(0) = \phi(1) = 0$. Using a characteristic equation $\lambda r^2 + 1 = 0 \rightarrow r = \pm 1/\sqrt{-\lambda}$.

$\lambda < 0$: $\phi(x) = c_1 e^{x/\sqrt{-\lambda}} + c_2 e^{-x/\sqrt{-\lambda}}$, $\quad 0 = \phi(0) = c_1 + c_2 \rightarrow c_2 = -c_1$

$\quad 0 = \phi(1) = c_1 e^{1/\sqrt{-\lambda}} - c_1 e^{-1/\sqrt{-\lambda}} = 2c_1 \sinh(1/\sqrt{-\lambda})$

Since $1/\sqrt{-\lambda} \neq 0$, $\sinh(1/\sqrt{-\lambda}) \neq 0$ so $c_1 = 0$, $c_2 = 0$, $\phi(x) \equiv 0$. This contradicts the assumption $\phi(x)$ is an eigenfunction.
$\therefore$ Only $\lambda > 0$ is possible.

$\lambda>0$: $r = \pm \frac{1}{\sqrt{\lambda}} i$, $\quad \phi(x) = c_1 \sin x/\sqrt{\lambda} + c_2 \cos x/\sqrt{\lambda}$

$$0 = \phi(0) = c_2$$
$$0 = \phi(1) = c_1 \sin 1/\sqrt{\lambda} \Rightarrow 1/\sqrt{\lambda} = n\pi,\ n \in \mathbb{Z} \text{ or } c_1 = 0$$

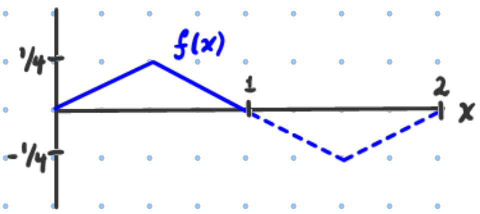
Since $c_1 = c_2 = 0$ would give $\phi \equiv 0$ again, $c_1 \neq 0$ so $1/\sqrt{\lambda} = n\pi$

The eigenvalues of K are $\lambda_n = \frac{1}{n^2\pi^2}$, $n = 1, 2, 3, \ldots$ (since $n^2 = (-n)^2$ and $\sin(0) = 0$ only $n = 1, 2, 3, \ldots$ are needed for eigenfunctions) with corresponding eigenfunctions $\phi_n(x) = \sin(n\pi x)$ (omit c_1 since any nonzero multiple of an eigenfunction is also an eigenfunction for the same eigenvalue).

$$\boxed{(\lambda_n, \phi_n) = \left(1/n^2\pi^2,\ \sin(n\pi x)\right),\ n = 1, 2, 3, \ldots}$$

b.

$$f(x) = \begin{cases} x/2, & 0 \le x < 1/2 \\ (1-x)/2, & 1/2 \le x \le 1 \\ (1-x)/2, & 1 \le x \le 3/2 \\ (x-2)/2, & 3/2 \le x \le 2 \end{cases}$$



In order to expand $f(x)$ in terms of $\{\sin n\pi x\}_{n=1}^{\infty}$, replace $f(x)$ with an odd extension and suppose $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$ on $0 < x < 2$.

$$\int_0^2 \sin n\pi x \sin m\pi x \, dx = \frac{1}{2}\int_0^2 \left(\cos(n-m)\pi x - \cos(n+m)\pi x\right) dx$$

$$= \left(\frac{1}{2}\frac{1}{n-m}\sin(n-m)\pi x - \frac{1}{2}\frac{1}{n+m}\sin(n+m)\pi x\right)\Big|_0^2 = 0, \quad m \neq n.$$

$$\int_0^2 \sin^2 n\pi x \, dx = \frac{1}{2}\int_0^2 (1 - \cos 2n\pi x)\, dx = 1 - \frac{1}{2n\pi}(\sin 4n\pi - \sin 0) = 1.$$

$$b_n = \int_0^2 b_n \sin^2 n\pi x \, dx = \int_0^2 (b_1 \sin\pi x + b_2 \sin 2\pi x + \ldots)\sin n\pi x \, dx = \int_0^2 f(x) \sin n\pi x$$

$$= 2\int_0^1 f(x)\sin n\pi x = \int_0^{1/2} x \sin n\pi x \, dx + \int_{1/2}^1 (1-x)\sin n\pi x$$

$$= \left\{-\frac{1}{n\pi} x \cos n\pi x\Big|_0^{1/2} + \left(\frac{1}{n\pi}\right)^2 (\sin n\pi x)\Big|_0^{1/2}\right\} + \left\{-\frac{1}{n\pi}(1-x)\cos n\pi x\Big|_{1/2}^1 - \left(\frac{1}{n\pi}\right)^2 (\sin n\pi x)\Big|_{1/2}^1\right\}$$

$$= -\frac{1}{n\pi}\cos n\pi/2 + \frac{1}{(n\pi)^2}\sin n\pi/2 + \frac{1}{n\pi}\cos n\pi/2 + \frac{1}{(n\pi)^2}\sin n\pi/2 = \frac{2}{n^2\pi^2}\sin n\pi/2$$

$$b_n = \begin{cases} 2/n^2\pi^2, & n = 1, 5, 9, \ldots \\ -2/n^2\pi^2, & n = 3, 7, 11, \ldots \\ 0, & n \text{ even} \end{cases}$$

$$\boxed{\begin{aligned} f(x) &= \frac{2}{\pi^2}\left\{\sin\pi x - \frac{1}{3^2}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x - \ldots\right\} \\ &= \frac{2}{\pi^2}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j-1)^2}\sin(2j-1)\pi x \end{aligned}}$$

Let $u(\xi) = \delta(\xi - 1/2)$. Then $Ku = f$:

$$Ku = \int_0^1 K(x,\xi)\delta(\xi - 1/2)\, d\xi = K(x, 1/2) = \begin{cases} x/2, & 0 \le x \le 1/2 \\ (1-x)/2, & 1/2 \le x \le 1 \end{cases} = f$$

3.4.2 Find the eigenvalues and eigenfunctions for the integral operator $Ku = \int_0^\pi k(x,\xi)u(\xi)\,d\xi$ where

a. $k(x,\xi) = x\xi$ b. $k(x,\xi) = \sin x \sin\xi + \alpha \cos x \cos\xi$

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a. $\lambda\phi(x) = K\phi = \int_0^\pi x\xi\,\phi(\xi)\,d\xi = x\int_0^\pi \xi\,\phi(\xi)\,d\xi$

$\phi(x) = x$, $\lambda = \int_0^\pi \xi\,\phi(\xi)\,d\xi = \int_0^\pi \xi^2\,d\xi = \pi^3/3$

Check: $K\phi = \int_0^\pi x\xi^2\,d\xi = \frac{\pi^3}{3}x = \lambda x = \lambda\phi(x)$

b. $\lambda\phi(x) = \int_0^\pi (\sin x\sin\xi + \alpha\cos x\cos\xi)\,\phi(\xi)\,d\xi$

$\lambda\phi(x) = \sin x\int_0^\pi \phi(\xi)\sin\xi\,d\xi + \cos x\int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi$

$\lambda\phi'(x) = \cos x\int_0^\pi \phi(\xi)\sin\xi\,d\xi - \sin x\int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi$

$\lambda\phi''(x) = -\sin x\int_0^\pi \phi(\xi)\sin\xi\,d\xi - \cos x\int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi = -\lambda\phi(x)$

$\lambda\phi'' + \lambda\phi = 0 \rightarrow \phi(x) = c_1\sin x + c_2\cos x,\ \lambda \neq 0$

$\lambda\phi(x) = \sin x\int_0^\pi \phi(\xi)\sin\xi\,d\xi + \cos x\int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi$

Since $\sin x, \cos x$ are orthogonal on $[0,\pi]$, $0 = \lambda\phi(x)$ implies both $\int_0^\pi \phi(\xi)\sin\xi\,d\xi = 0$ and $\int_0^\pi \phi(\xi)\cos\xi\,d\xi = 0$.

$\lambda\phi(x) = c_1\sin x + c_2\cos x$, $c_1 = \int_0^\pi \phi(\xi)\sin\xi\,d\xi$, $c_2 = \int_0^\pi \alpha\,\phi(\xi)\cos\xi\,d\xi$

$\phi(x) = \frac{1}{\lambda}c_1\sin x + \frac{1}{\lambda}c_2\cos x$

$\int_0^\pi \phi(x)\sin x\,dx = \frac{1}{\lambda}c_1\int \sin^2 x\,dx$, $\int_0^\pi \phi(x)\cos x\,dx = \frac{1}{\lambda}c_2\int_0 \cos^2 x$

$c_1 = c_1/\lambda\int_0^\pi \sin^2 x\,dx \quad c_1 \neq 0,$ $\qquad \frac{1}{\alpha}c_2 = \frac{c_2}{\lambda}\pi/2$

$\lambda = \int_0^\pi \frac{1}{2} - \frac{1}{2}\cos 2x\,dx = \pi/2$ $\qquad \lambda = \alpha\pi/2,\ c_2 \neq 0$

$\phi(x) = \frac{2}{\pi}\sin x,\ \frac{2}{\alpha\pi}\cos x$

$\lambda = \pi/2 \qquad \lambda = \alpha\pi/2$

$\phi = \frac{2}{\pi}\sin x$

$K\phi = \int_0^\pi (\sin x\sin\xi + \alpha\cos x\cos\xi)\frac{2}{\pi}\sin\xi\,d\xi$

$= \frac{2}{\pi}\sin x\int_0^\pi \sin^2\xi\,d\xi = \frac{\pi}{2}\frac{2}{\pi}\sin x = \frac{\pi}{2}\phi = \lambda\phi$ ✓

$\phi = \frac{2}{\alpha\pi}\cos x,\ \lambda = \alpha\pi/2$

$K\phi = \int_0^\pi (\sin x\sin\xi + \alpha\cos x\cos\xi)\phi(\xi)\,d\xi$

$= \frac{2}{\alpha\pi}\cos x\int_0^\pi \alpha\cos^2\xi\,d\xi$

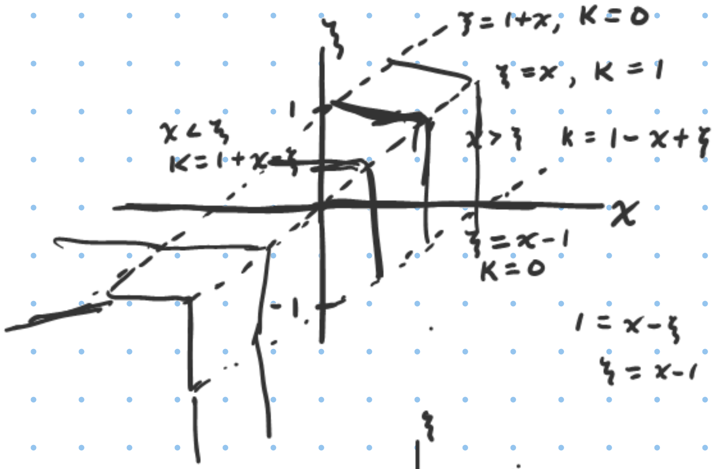
$= \left(\frac{\alpha\pi}{2}\right)\left(\frac{2}{\alpha\pi}\cos x\right)$

$= \lambda\phi$ ✓

3.4.3 Find the eigenvalues and eigenfunctions of the integral operator

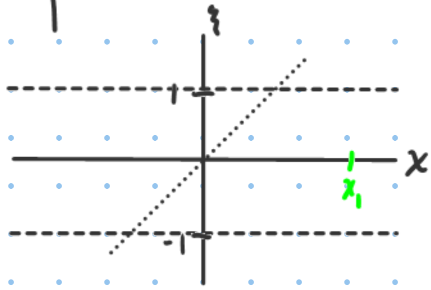
$$\underline{K}u = \int_{-1}^{1} k(x,\xi)\,u(\xi)\,d\xi, \quad k(x,\xi) = 1-|x-\xi|$$

$\int_{-1}^{1} k(x,\xi)\,u(\xi)\,d\xi$



$1 = x-\xi$

$\xi = x-1$



$\int_{-1}^{1} k(x_1,\xi)u(\xi)d\xi$

$= \int_{-1}^{1}(1-x+\xi)\,u(\xi)\,d\xi$

$\lambda\phi(x) = \int\phi\,d\xi - x\int\phi\,d\xi + \int\xi\phi\,d\xi$

$\lambda\phi'(x) = -\int\phi\,d\xi$