Chapter 1 Finite Dimensional Vector Spaces

Section 1.1

1. Show that in any inner product space

$$\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Interpret this geometrically in $\mathbb{R}^2$.

Answer: $\|x+y\|^2+\|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

$$= \langle x, x+y\rangle + \langle y, x+y\rangle + \langle x, x-y\rangle - \langle y, x-y\rangle$$

$$= \overline{\langle x+y, x\rangle} + \overline{\langle x+y, y\rangle} + \overline{\langle x-y, x\rangle} - \overline{\langle x-y, y\rangle}$$

$$= \overline{\langle x,x\rangle} + \overline{\langle y,x\rangle} + \overline{\langle x,y\rangle} + \overline{\langle y,y\rangle}$$

$$+ \overline{\langle x,x\rangle} - \overline{\langle y,x\rangle} - \overline{\langle x,y\rangle} + \overline{\langle y,y\rangle}$$

$$= \|x\|^2 + 2\mathrm{Re}\langle x,y\rangle + \|y\|^2 + \|x\|^2 - 2\mathrm{Re}\langle x,y\rangle + \|y\|^2$$

$$= 2\|x\|^2 + 2\|y\|^2$$

5. Verify that $\alpha = \frac{\langle x,y\rangle}{\|y\|^2}$ makes $\|x-\alpha y\|^2$ as small as possible. Show that $|\langle x,y\rangle|^2 = \|x\|^2\|y\|^2$ iff x and y are linearly independent.

Answer: If $x, y \in \mathbb{R}^n$

$$0 = \frac{\partial}{\partial\alpha}\|x-\alpha y\|^2 = \frac{\partial}{\partial\alpha}\left[\|x\|^2 - 2\alpha\langle x,y\rangle + \alpha^2\|y\|^2\right]$$

$$0 = -2\langle x,y\rangle + 2\alpha\|y\|^2$$

$$\alpha = \frac{\langle x,y\rangle}{\|y\|^2}$$ is a critical point of $\|x-\alpha y\|^2$.

Since $\|x-\alpha y\|^2 = \|x\|^2 - 2\alpha\langle x,y\rangle + \alpha^2\|y\|^2$ is quadratic in α and $\|y\|^2 > 0$, this value of α is a minimum.

If $x, y \notin \mathbb{R}^n$, ?

Section 1.2

2\. a. Prove that two symmetric matrices are equivalent iff they have the same eigenvalues.

b. Show that if A and B are equivalent, $\det A = \det B$.

Answer:

a. Suppose A and B are equivalent with $A = M^{-1}BM$ and λ is an eigenvalue of A with eigenvector x.

$$Ax = \lambda x$$
$$M^{-1}BMx = \lambda x$$
$$B(Mx) = \lambda(Mx)$$

This means λ is an eigenvalue of B with eigenvector Mx. Similarly, if μ is an eigenvalue of B with eigenvector y, μ is an eigenvalue of A with eigenvector $M^{-1}y$. Therefore, A and B have the same eigenvalues.

Suppose A and B have the same eigenvalues. If C is the matrix of eigenvectors of A and D is the matrix of eigenvectors of B, we have by Theorem 1.2 (3):

$C^{-1}AC = \Lambda = D^{-1}BD$, Λ is the diagonal matrix of eigenvalues.

$$A = CD^{-1}BDC^{-1} = (DC^{-1})^{-1}B(DC^{-1})$$

b. If A and B are equivalent with $A = M^{-1}BM$,

$$\det A = \det(M^{-1}BM) = \det(M^{-1})\det B \det M = \frac{\det B \det M}{\det M} = \det B.$$

7\. Find the spectral representation of $A = \begin{pmatrix} 7 & 2 \\ -2 & 2 \end{pmatrix}$. Illustrate how $Ax = b$ can be solved geometrically using the appropriately chosen coordinate system.

Answer: $0 = (7-\lambda)(2-\lambda) + 4 = \lambda^2 - 9\lambda + 18 = (\lambda - 6)(\lambda - 3) \longrightarrow \lambda = 3, 6.$

$$\left(\begin{array}{cc|c} 7-6 & 2 & 0 \\ -2 & 2-6 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 0 \\ -2 & -4 & 0 \end{array}\right) \longrightarrow x_1 = -2x_2. \text{ Let } x = (2,-1)^T/\sqrt{3}.$$

$$\left(\begin{array}{cc|c} 7-3 & 2 & 0 \\ -2 & 2-3 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 4 & 2 & 0 \\ -2 & -1 & 0 \end{array}\right) \longrightarrow y_2 = -2y_1. \text{ Let } y = (1,-2)^T/\sqrt{3}.$$

$$M = \begin{pmatrix} x & y \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}, \quad M^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A = M^{-1}\Lambda M = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}.$$

9. Suppose the sets of vectors $\{\phi_i\}_{i=1}^n$, $\{\psi_i\}_{i=1}^n$ are biorthogonal, meaning $\langle \phi_i, \psi_j \rangle = \delta_{ij}$.

a. Show that $\{\phi\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ each form a linearly independent set.

b. Show that any vector $x \in \mathbb{R}^n$ can be written

$$x = \sum_{i=1}^n \alpha_i \phi_i, \quad \alpha_i = \langle x, \psi_i \rangle.$$

c. Show $x = \sum_{i=1}^n P_i x$, where P_i are projection matrices with $P_i^2 = P_i$ and $P_i P_j = 0$ for $i \neq j$. Express P_i in terms of ϕ_i and ψ_i.

Answer:

a. Suppose $c_1\phi_1 + \ldots + c_n\phi_n = \vec{0}$. Then, for each $i = 1, \ldots, n$

$c_i = c_i \langle \psi_i, \phi_i \rangle = \langle \psi_i, c_i\phi_i \rangle = \langle \psi_i, c_1\phi_1 + \ldots + c_n\phi_n \rangle = \langle \psi_i, \vec{0} \rangle = 0$

Since $c_1 = \ldots = c_n = 0$, conclude that the ϕ_i are independent. By similar reasoning, the ψ_i are also independent.

b. Since $\{\phi\}_{i=1}^n$ is a set of n linearly independent vectors in $\mathbb{R}^n$, $\{\phi_i\}_{i=1}^n$ is a basis for $\mathbb{R}^n$. This means there exist scalars $\alpha_1, \ldots, \alpha_n$ such that $x = \alpha_1\phi_1 + \ldots + \alpha_n\phi_n$. To determine α_i,

$\langle x, \psi_i \rangle = \langle \alpha_1\phi_1 + \ldots + \alpha_n\phi_n, \psi_i \rangle = \langle \alpha_i\phi_i, \psi_i \rangle = \alpha_i \langle \phi_i, \psi_i \rangle = \alpha_i$.

c.

10. a. Suppose the eigenvalues of A are distinct. Show that the eigenvectors of A and the eigenvectors of A^* form a biorthonormal set.

Answer: $\langle Ax, y \rangle = \langle x, Ay^* \rangle$