Chapter 2 Problem Set

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2.1.1 Verify that ℓ^2 is a normed vector space. Show that for all sequences $x, y \in \ell^2$, the inner product $\langle x,y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$ is defined and satisfies the requisite properties.

$\ell^2 = \{ \{x_n\} \mid \sum_{n=1}^{\infty} |x_n|^2 < \infty \}$ is a normed vector space with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}.$$

Verify that the definition on pg. 60 of a normed space is satisfied.

1. Since $|x_n|^2 \geq 0 \ \forall n$, $\|x\|^2 \geq 0$ and so $\|x\| \geq 0$. Also $x = 0$ iff $x_n = 0$ $\forall n$ iff $|x_n|^2 = 0 \ \forall n$ iff $\|x\| = 0$.

2. $\|\alpha x\| = \left(\sum_{n=1}^{\infty} |\alpha x_n|^2 \right)^{1/2} = |\alpha| \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} = |\alpha| \|x\|$.

3. $\|x+y\|^2 = \sum_{n=1}^{\infty} |x_n + y_n|^2 \leq \sum_{n=1}^{\infty} (|x_n| + |y_n|)^2 \leq \sum_{n=1}^{\infty} |x_n|^2 + \sum_{n=1}^{\infty} |y_n|^2$

   $= \|x\|^2 + \|y\|^2 \leq (\|x\| + \|y\|)^2$

   $0 \leq \|x+y\|^2 \leq (\|x\| + \|y\|)^2$ iff $\|x+y\| \leq \|x\| + \|y\|$.

Verify that the definition on pg. 64 of an inner product is satisfied.

1. $\langle x,y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} = \sum_{i=1}^{\infty} \overline{\overline{x_i} y_i} = \overline{\sum_{i=1}^{\infty} y_i \overline{x_i}} = \overline{\langle y,x\rangle}$.

2. $\langle \alpha x, y\rangle = \sum_{i=1}^{\infty} \alpha x_i \overline{y_i} = \alpha \sum_{i=1}^{\infty} x_i \overline{y_i} = \alpha \langle x,y\rangle$

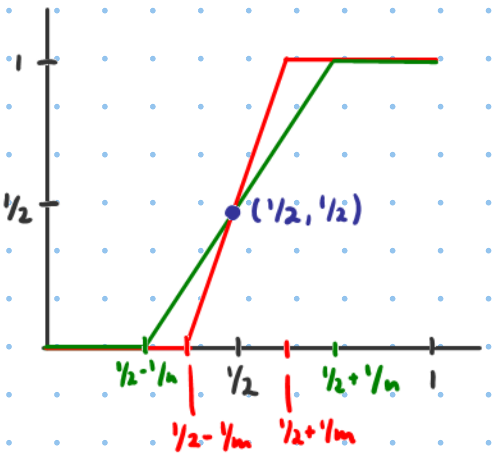
3. $\langle x+y, z\rangle = \sum_{i=1}^{\infty} (x_i + y_i)\overline{z_i} = \sum_{i=1}^{\infty} x_i \overline{z_i} + \sum_{i=1}^{\infty} y_i \overline{z_i} = \langle x,y\rangle + \langle y,z\rangle$.

4. $\langle x,x\rangle = \sum_{i=1}^{\infty} x_i \overline{x_i} = \sum_{i=1}^{\infty} |x_i|^2 = \|x\|^2 \geq 0$. Since $\|x\|^2 \geq 0$ iff $\|x\| \geq 0$ (with equality iff $x = 0$), $\langle x,x\rangle \geq 0$ with $\langle x,x\rangle = 0$ iff $x = 0$.

2.1.6 Show that the sequence of functions $\{f_n(t)\}$ is Cauchy in $L^2[0,1]$ but not in $C[0,1]$ (uniform norm).

$$f_n(t) = \begin{cases} 0 & , \quad 0 \le t < 1/2 - 1/n \\ 1/2 + n(t - 1/2)/2 & , \quad 1/2 - 1/n \le t \le 1/2 + 1/n \\ 1 & , \quad 1/2 + 1/n \le t \le 1 \end{cases}$$

Consider $\{f_n(t)\}$ for $n \geq 2$. The behavior of $f_n(t)$ matches Figure 2.1 for $n \geq 3$.



Let $m = n + k$, $m, n, k \in \mathbb{Z}$, $n \geq 3$, $k \geq 1$.

For $0 \le t \le 1/2 - 1/n$, $1/2 + 1/n \le t \le 1$, $t = 1/2$, $f_m(t) = f_n(t)$
For $1/2 - 1/n < t < 1/2$, $f_n(t) > f_m(t)$
For $1/2 < t < 1/2 + 1/n$, $f_m(t) > f_n(t)$

Since $f_m(t) = 0$ on $t < 1/2 - 1/m$ and $f_m(t) = 1$ on $1/2 + 1/m < t$ and the slope of $f_m(t)$ is greater than the slope of $f_n(t)$ on $1/2 - 1/m < t < 1/2 + 1/m$, $|f_m(t) - f_n(t)|$ is maximized by $t = 1/2 \pm 1/m$ (only).

$$|f_m(t) - f_n(t)| \le f_n(1/2 - 1/m) - f_m(1/2 - 1/m) = f_n(1/2 - 1/m)$$

$$= 1/2 + n/2\,(1/2 - 1/m - 1/2) = \frac{1}{2} - \frac{n}{2}\frac{1}{n+k} \quad \text{for all } t \in [0,1].$$

To show that $\{f_n(t)\}$ is Cauchy in $L^2[0,1]$ let $\varepsilon > 0$ and pick $N > 0$ such that $1/N < \varepsilon$. Then for any $m, n \geq N$, $m = n + k$

$$\|f_m - f_n\|_2^2 = \int_0^1 |f_m(t) - f_n(t)|\, dt \le \int_{1/2 - 1/n}^{1/2 + 1/n} \left(\frac{1}{2} - \frac{n}{2}\frac{1}{n+k}\right)^2 dt = \left(\frac{1}{2}\frac{k}{n+k}\right)^2 \frac{2}{n}$$

$$= \frac{1}{2}\frac{k^2}{n(n+k)^2} \le \frac{1}{2}\frac{k^2}{nk^2} \le \frac{1}{n} < \varepsilon.$$

Conclude that since $\|f_m - f_n\|^2 \to 0$, $\|f_m - f_n\| \to 0$. This means $\{f_n(t)\}$ is Cauchy in $L^2[0,1]$.

To show that $\{f_n(t)\}$ is not Cauchy in $C[0,1]$, show $\exists\, \varepsilon > 0$ s.t. $\forall N > 0$, $\exists\, m, n \geq N$ s.t. $\|f_m - f_n\| \geq \varepsilon$. In fact we can show that any $0 < \varepsilon < 1/2$ works.

Let $0 < \varepsilon < 1/2$ and let N be arbitrary. Let $n = N$ and take $m = n + k$ with $k \geq 2\varepsilon n/(1 - 2\varepsilon)$ (solve $1/2 - n/2 \cdot 1/(n+k) \geq \varepsilon$ for k)†. This produces $m, n \geq N$ such that

$$\|f_m - f_n\|_\infty = \max_{t \in [0,1]} |f_m(t) - f_n(t)| = \frac{1}{2} - \frac{n}{2}\frac{1}{n+k} \geq \varepsilon.$$

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† $1 - n/(n+k) \geq 2\varepsilon \iff (1 - 2\varepsilon)(n+k) \geq n \iff k(1 - 2\varepsilon) + n - 2\varepsilon n \geq n \iff k \geq 2\varepsilon n/(1 - 2\varepsilon)$

2.1.6

a) Suppose that for any function $f(t)$, $\| |f(t)| \| = \|f(t)\|$ and that if $0 < f \leq g$, then $\|f\| \leq \|g\|$. If $\{g_n(t)\}$ is a sequence of functions for which $\lim g_n(t) = g(t)$, show that $\lim |g_n(t)| = |g(t)|$.

b) Give some examples of norms for which the two assumptions hold.

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a) $|u| \leq |u-v| + |v|$ & $|v| \leq |v-u| + |u|$ $\Rightarrow$ $||u|-|v|| \leq |u-v|$.
$|u|-|v| \leq |u-v|$ $\quad$ $|u|-|v| \geq -|u-v|$

Since $0 \leq \| |g_n(t)| - |g(t)| \| \leq \| |g_n(t) - g(t)| \| = \|g_n(t) - g(t)\| \to 0$ as $n \to \infty$, $\| |g_n(t)| - |g(t)| \| \to 0$ as $n \to \infty$. Equivalently, $\lim_{n\to\infty} |g_n(t)| = |g(t)|$.

b) For the uniform norm on $C[a,b]$ and $t \in [a,b]$

- $\| |f(t)| \| = \sup_t | |f(t)| | = \sup_t |f(t)| = \|f(t)\|$
- If $0 < f \leq g$, $\|f\| = \sup_t |f(t)| \leq \sup_t |g(t)| = \|g\|$.

For $L^p[a,b]$, $p \geq 1$

- $\| |f(t)| \| = \left(\int_a^b | |f(t)| |^p \, dt \right)^{1/p} = \left(\int_a^b |f(t)|^p dt \right)^{1/p} = \|f\|$.
- If $0 < f \leq g$, $\|f\| = \left(\int_a^b |f(t)|^p dt \right)^{1/p} \leq \left(\int_a^b |g(t)|^p \, dt \right)^{1/p} = \|g\|$.

2.1.11 For $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$, show that $\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy \neq \int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx$. What went wrong?

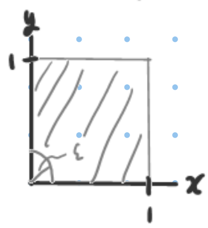
Note that $\partial/\partial x \left[-\frac{x}{x^2+y^2} \right] = f(x,y)$ and $\partial/\partial y \left[\frac{y}{x^2+y^2} \right] = f(x,y)$.

$$\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy = \int_0^1 \left(-\frac{x}{x^2+y^2} \Big|_0^1 \right) dy = \int_0^1 -\frac{1}{1+y^2}\, dy = -\arctan 1 + \arctan 0 = -\pi/4.$$

$$\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx = \int_0^1 \left(\frac{y}{x^2+y^2} \Big|_0^1 \right) dx = \int_0^1 \frac{1}{1+x^2}\, dx = \arctan 1 - \arctan 0 = \pi/4.$$

Property 8b of Lebesgue integration (pg 65) states that if f is integrable, then $\int_{m \times m'} f = \int_m \left(\int_{m'} f \right) = \int_{m'} \left(\int_m f \right)$ so that it might at first appear that there is a contradiction. In this case, however, f is not integrable:

$$\|f\|^2 = \iint_{[0,1]\times[0,1]} |f(x,y)|\, dA = \iint_{[0,1]\times[0,1]} \left| \frac{x^2-y^2}{(x^2+y^2)^2} \right| dA$$
$$\geq \int_0^{\pi/2} \int_0^{\varepsilon} \left| \frac{r^2\cos^2\theta - r^2\sin^2\theta}{(r^2\cos^2\theta + r^2\sin^2\theta)^2} \right| r\, dr\, d\theta$$
$$= \int_0^{\pi/2} \int_0^{\varepsilon} \frac{|\cos 2\theta|}{r}\, dr\, d\theta = \int_0^{\pi/2} |\cos 2\theta|\, d\theta \int_0^{\varepsilon} 1/r\, dr$$
$$= 1 \cdot \left(\ln \varepsilon - \lim_{a\to 0^+} \ln a \right) = \infty.$$



2.2.4 Show that the trigonometric functions† form a complete orthogonal basis for any Sobolev space $H^n[0,2\pi]$ whose functions satisfy $f(0)=f(2\pi)$. What is the difference between n terms of an L^2 and an H^1 Fourier approximation of some piecewise C^1 function? Illustrate by an example.

Section 1: Orthogonality

$H^n[0,2\pi]$ is the space of functions in $L^2[0,2\pi]$ whose derivatives through order n are also in $L^2[0,2\pi]$. The inner product is

$$\langle f,g\rangle_{H^n}=\int_0^{2\pi}\Big(\sum_{j=0}^{n} f^{(j)}(x)\overline{g^{(j)}(x)}\Big)\,dx \quad ^{\dagger\dagger}$$

The trigonometric functions are orthogonal with respect to the $L^2[0,2\pi]$ inner product $\langle f,g\rangle_{L^2}=\int_0^{2\pi} f(x)\overline{g(x)}\,dx$ (pg 74). Use this to show that the trigonometric functions are also orthogonal with respect to the $H^n[0,2\pi]$ inner product. Let $f,g\in T$.

Case 1 $f(x)=\cos kx$, $g(x)=\cos jx$, $k,j\in\{0,1,2,\dots\}$, $k\neq j$

$$\langle f,g\rangle_{H^n}=\begin{cases}\int_0^{2\pi}\left[\cos kx\cos jx+kj\sin kx\sin jx+\dots+(kj)^n\cos kx\cos jx\right]dx, & n \text{ even}\\ \int_0^{2\pi}\left[\cos kx\cos jx+kj\sin kx\sin jx+\dots+(kj)^n\sin kx\sin jx\right]dx, & n \text{ odd}\end{cases}$$

$$=\begin{cases}\int_0^{2\pi}\cos kx\cos jx\,dx+\dots+(kj)^n\int_0^{2\pi}\cos kx\cos jx\,dx, & n \text{ even}\\ \int_0^{2\pi}\cos kx\cos jx\,dx+\dots+(kj)^n\int_0^{2\pi}\sin kx\sin jx\,dx, & n \text{ odd}\end{cases}$$

$= 0$ (either n odd or n even).

Case 2 $f(x)=\sin kx$, $g(x)=\sin jx$, $k,j\in\{1,2,\dots\}$, $k\neq j$

$$\langle f,g\rangle_{H^n}=\begin{cases}\int_0^{2\pi}\left[\sin kx\sin jx+kj\cos kx\cos jx+\dots+(kj)^n\sin kx\sin jx\right]dx, & n \text{ even}\\ \int_0^{2\pi}\left[\sin kx\sin jx+kj\cos kx\cos jx+\dots+(kj)^n\cos kx\cos jx\right]dx, & n \text{ odd}\end{cases}$$

$$=\begin{cases}\int_0^{2\pi}\sin kx\sin jx\,dx+\dots+(kj)^n\int_0^{2\pi}\sin kx\sin jx\,dx, & n \text{ even}\\ \int_0^{2\pi}\sin kx\sin jx\,dx+\dots+(kj)^n\int_0^{2\pi}\cos kx\cos jx\,dx, & n \text{ odd}\end{cases}$$

$= 0$ (either n odd or n even).

† Here the trigonometric functions means the set $T=\{\sin(nx),\cos(nx)\}_{n=0}^{\infty}$ for $x\in[0,2\pi]$. Exclude $\sin(0x)$ for independence.

†† Let $\langle\cdot,\cdot\rangle_{L^2}$, $\langle\cdot,\cdot\rangle_{H^n}$ denote the $L^2[0,2\pi]$ and $H^n[0,2\pi]$ inner products respectively.

Case 3 $f(x) = \cos kx$, $g(x) = \sin jx$, $k, j \in \{0, 1, 2, \ldots\}$, $j \neq 0$

$$\langle f, g \rangle_{H^n} = \begin{cases} \int_0^{2\pi} [\cos kx \sin jx - kj \sin kx \cos jx + \ldots + (kj)^n \cos kx \sin jx]\, dx, & n \text{ even} \\ \int_0^{2\pi} [\cos kx \sin jx - kj \sin kx \cos jx + \ldots - (kj)^n \sin kx \cos jx]\, dx, & n \text{ odd} \end{cases}$$

$$= \begin{cases} \int_0^{2\pi} \cos kx \sin jx\, dx - kj \int_0^{2\pi} \sin kx \cos jx\, dx + \ldots + (kj)^n \int_0^{2\pi} \cos kx \sin jx\, dx, & n \text{ even} \\ \int_0^{2\pi} \cos kx \sin jx\, dx - kj \int_0^{2\pi} \sin kx \cos jx\, dx + \ldots - (kj)^n \int_0^{2\pi} \sin kx \cos jx\, dx, & n \text{ odd} \end{cases}$$

$$= 0$$

Case 4 $f(x) = g(x) = \cos 0x = 1$ $\quad \langle f, g \rangle_{H^n} = \int_0^{2\pi} 1\, dx = 2\pi$

Case 5 $f(x) = g(x) = \cos mx$, $m \in \{1, 2, 3, \ldots\}$

$$\langle f, g \rangle_{H^n} = \begin{cases} \int_0^{2\pi} [\tfrac{1}{2} + \tfrac{1}{2}\cos 2mx - m \sin 2mx - 2m^2 \cos 2mx + 4m^3 \sin 2mx \\ \quad + 8m^4 \cos 2mx - 16m^5 \sin mx + \ldots + 2^{n-1} m^n \cos mx]\, dx, & n \in \{4, 8, 12, \ldots\} \\ \int_0^{2\pi} [\tfrac{1}{2} + \tfrac{1}{2}\cos 2mx - m \sin 2mx - 2m^2 \cos 2mx + 4m^3 \sin 2mx \\ \quad + 8m^4 \cos 2mx - 16m^5 \sin mx + \ldots - 2^{n-1} m^n \cos mx]\, dx, & n \in \{2, 6, 10, \ldots\} \\ \int_0^{2\pi} [\tfrac{1}{2} + \tfrac{1}{2}\cos 2mx - m \sin 2mx - 2m^2 \cos 2mx + 4m^3 \sin 2mx \\ \quad + 8m^4 \cos 2mx - 16m^5 \sin mx + \ldots + 2^{n-1} m^n \sin mx]\, dx, & n \in \{3, 7, 11, \ldots\} \\ \int_0^{2\pi} [\tfrac{1}{2} + \tfrac{1}{2}\cos 2mx - m \sin 2mx - 2m^2 \cos 2mx + 4m^3 \sin 2mx \\ \quad + 8m^4 \cos 2mx - 16m^5 \sin mx + \ldots + 2^{n-1} m^n \sin mx]\, dx, & n \in \{1, 5, 9, \ldots\} \end{cases}$$

$$= \int_0^{2\pi} \tfrac{1}{2}\, dx = \pi \quad (\text{any } n \in \{1, 2, 3, \ldots\})$$

Case 6 $f(x) = g(x) = \sin mx$, $m \in \{1, 2, 3, \ldots\}$

$$\langle f, g \rangle_{H^n} = \begin{cases} \int_0^{2\pi} [\tfrac{1}{2} - \tfrac{1}{2}\cos 2mx + m \sin 2mx + 2m^2 \cos 2mx - 4m^3 \sin 2mx \\ \quad - 8m^4 \cos 2mx + 16m^5 \sin mx + \ldots + 2^{n-1} m^n \cos mx]\, dx, & n \in \{2, 6, 10, \ldots\} \\ \int_0^{2\pi} [\tfrac{1}{2} - \tfrac{1}{2}\cos 2mx + m \sin 2mx + 2m^2 \cos 2mx - 4m^3 \sin 2mx \\ \quad - 8m^4 \cos 2mx + 16m^5 \sin mx + \ldots - 2^{n-1} m^n \cos mx]\, dx, & n \in \{4, 8, 12, \ldots\} \\ \int_0^{2\pi} [\tfrac{1}{2} - \tfrac{1}{2}\cos 2mx + m \sin 2mx + 2m^2 \cos 2mx - 4m^3 \sin 2mx \\ \quad - 8m^4 \cos 2mx + 16m^5 \sin mx + \ldots + 2^{n-1} m^n \sin mx]\, dx, & n \in \{1, 5, 9, \ldots\} \\ \int_0^{2\pi} [\tfrac{1}{2} - \tfrac{1}{2}\cos 2mx + m \sin 2mx + 2m^2 \cos 2mx - 4m^3 \sin 2mx \\ \quad - 8m^4 \cos 2mx + 16m^5 \sin mx + \ldots - 2^{n-1} m^n \sin mx]\, dx, & n \in \{3, 7, 11, \ldots\} \end{cases}$$

$$= \int_0^{2\pi} \tfrac{1}{2}\, dx = \pi \quad (\text{any } n \in \{1, 2, 3, \ldots\})$$

Summarizing the 6 cases:

$$\langle f,g\rangle_{H^n} = \int_0^{2\pi}\left(\sum_{j=0}^{n} f^{(j)}(x)\overline{g^{(j)}(x)}\right)dx$$
$$= \sum_{j=0}^{n}\int_0^{2\pi} f^{(j)}(x)g^{(j)}(x)\,dx$$
$$= \begin{cases} 0 \quad , & f \neq g \\ \int_0^{2\pi}(f(x))^2dx \;, & f = g \;* \end{cases}$$

$*\int_0^{2\pi}(f(x))^2dx = \pi$ if
$f(x) = \sin(mx)$, $m = 1, 2, 3, \ldots$
$f(x) = \cos(mx)$, $m = 1, 2, 3, \ldots$

$\int_0^{2\pi}(f(x))^2dx = 2\pi$ if
$f(x) = \cos(0\cdot x) = 1$.

This shows the elements of T are orthogonal wrt the $H^n[0,2\pi]$ inner product.

Section 2: Completeness

Definition:
An orthonormal set $\{\phi_i\}_{i=1}^{\infty}$ is complete if $\Sigma_{i=1}^{\infty}\langle f,\phi_i\rangle\phi_i = f$ for every f in the Hilbert space H (pg 70).

Note that the Sobolev space $H^n[0,2\pi]$ is a Hilbert Space so this definition may be applied to orthonormal subsets of $H^n[0,2\pi]$.

However, the set T is not orthonormal so the definition must be extended to orthogonal sets otherwise it is by this definition above impossible for the trigonometric functions to be complete. One could alternatively redefine the elements of T to normalize (divide $\cos(0x) = 1$ by 2π and divide all other elements by 2π). Let us assume the author intended that the definition be extended to orthogonal (not necessarily orthonormal) sets.

Part 4 of Theorem 2.2 provides an equivalent condition:

$$\boxed{\text{A set } \{\phi_i\}_{i=1}^{\infty} \text{ is complete} \iff \text{If } \langle f,\phi_i\rangle = 0 \text{ for all } i, \text{ then } f = 0.}$$

In this case the ϕ_i are the functions $\cos mx$, $m = 0, 1, \ldots$ or $\sin mx$, $m = 1, 2, \ldots$
We will use this result to show that T is complete. Let $f \in H^n[0,2\pi]$ with f 2π-periodic and suppose $\langle f, \cos mx\rangle_{H^n} = 0$ $(m = 0, 1, \ldots)$ and $\langle f, \sin mx\rangle_{H^n} = 0$ $(m = 1, 2, \ldots)$. Using a lemma (see the next page),

$$0 = \langle f, \cos mx\rangle_{H^n} = (1 + m^2 + \ldots + m^{2n})\int_0^{2\pi} f(x)\cos mx\,dx = \left(\sum_{j=0}^{n} m^{2j}\right)\langle f, \cos mx\rangle_{L^2}$$
$$0 = \langle f, \sin mx\rangle_{H^n} = (1 + m^2 + \ldots + m^{2n})\int_0^{2\pi} f(x)\sin mx\,dx = \left(\sum_{j=0}^{n} m^{2j}\right)\langle f, \sin mx\rangle_{L^2}$$

Since $\Sigma_{j=0}^{n} m^{2j} > 0$ for $m = 1, 2, 3, \ldots$ (note $\langle f, \cos(0x)\rangle_{H^n} = \langle f, \cos(0x)\rangle_{L^2}$), this shows that $0 = \langle f, \cos mx\rangle_{L^2} = \langle f, \sin mx\rangle_{L^2}$ $(m = 0, 1, \ldots)$. Since the elements of T are a basis for $L^2[0,2\pi]$, $\exists$ coefficients a_m, b_m s.t.

$$f(x) = \sum_{m=0}^{\infty} a_m\langle f, \cos mx\rangle_{L^2}\cos mx + \sum_{m=1}^{\infty} b_m\langle f, \sin mx\rangle_{L^2}\sin mx = \sum_{m=0}^{\infty} 0 + \sum_{m=1}^{\infty} 0 = 0.$$

Lemma: $\int_0^{2\pi} \sum_{j=0}^{n} f^{(j)}(x)(\cos(mx))^{(j)}\,dx = \left(\sum_{j=0}^{n} m^{2j}\right)\int_0^{2\pi} f(x)\cos(mx)$, $n=0,1,2,\dots$

$\int_0^{2\pi} \sum_{j=0}^{n} f^{(j)}(x)(\sin(mx))^{(j)}\,dx = \left(\sum_{j=0}^{n} m^{2j}\right)\int_0^{2\pi} f(x)\sin(mx)$, $n=0,1,2,\dots$

For $f \in H^n[0,2\pi]$ with f 2π-periodic.

Proof.

Part 1

Since f is 2π periodic, $f^{(j)}$ is 2π-periodic for $j=1,2,\dots,n$. To see this, begin with the first derivative:

$$f'(x+2\pi) = \lim_{h\to 0} \frac{f(x+2\pi+h)-f(x+2\pi)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

Apply this reasoning iteratively to show that $f^{(2)}, f^{(3)}, \dots, f^{(n)}$ are each 2π-periodic. In particular, $f^{(j)}(0) = f^{(j)}(2\pi)$ is used repeatedly in part 2.

Part 2

$$\boxed{\int_0^{2\pi} f^{(j)}(x)\,(\cos mx)^{(j)}\,dx = m^{2j}\int_0^{2\pi} f(x)\cos mx\,dx}$$

Case 1: $j=4,8,12,\dots$

$$\begin{aligned}
\int_0^{2\pi} f^{(j)}(x)(\cos mx)^{(j)}\,dx &= \int_0^{2\pi} f^{(j)}(x)m^j\cos mx\,dx\\
&= m^j\left[f^{(j-1)}(x)\cos mx\Big|_0^{2\pi} + m\int_0^{2\pi} f^{(j-1)}(x)\sin mx\,dx\right]\\
&= m^j m\int_0^{2\pi} f^{(j-1)}(x)\sin mx\,dx \qquad (f^{(j-1)}(0)=f^{(j-1)}(2\pi))\\
&= m^j m\left[f^{(j-2)}(x)\sin mx\Big|_0^{2\pi} - m\int_0^{2\pi} f^{(j-2)}(x)\cos mx\,dx\right]\\
&= -m^j m^2\int_0^{2\pi} f^{(j-2)}(x)\cos mx\,dx\\
&= -m^j m^2\left[f^{(j-3)}(x)\cos mx\Big|_0^{2\pi} + m\int_0^{2\pi} f^{(j-3)}(x)\sin mx\,dx\right]\\
&= -m^j m^3\int_0^{2\pi} f^{(j-3)}(x)\sin mx\,dx\\
&= -m^j m^3\left[f^{(j-4)}(x)\sin mx\Big]_0^{2\pi} - m\int_0^{2\pi} f^{(j-4)}\cos mx\,dx\right]\\
&= m^j m^4\int_0^{2\pi} f^{(j-4)}(x)\cos mx\,dx\\
&\ \vdots\\
&= m^j m^j\int_0^{2\pi} f^{(j-j)}(x)\cos mx\,dx\\
&= m^{2j}\int_0^{2\pi} f(x)\cos mx\,dx
\end{aligned}$$

Case 2: $j=3,7,11,\dots$

$$\begin{aligned}
\int_0^{2\pi} f^{(j)}(x)(\cos mx)^{(j)}\,dx &= \int_0^{2\pi} f^{(j)}(x)m^j\sin mx\,dx\\
&= -m^j m\int_0^{2\pi} f^{(j-1)}(x)\cos mx\,dx\\
&= -m^j m^2\int_0^{2\pi} f^{(j-2)}(x)\sin mx\,dx\\
&= m^j m^3\int_0^{2\pi} f^{(j-3)}(x)\cos mx\,dx\\
&= m^j m^4\int_0^{2\pi} f^{(j-4)}(x)\sin mx\,dx\\
&\ \vdots\\
&= m^j m^j\int_0^{2\pi} f^{(j-j)}(x)\cos mx\,dx
\end{aligned}$$

$\cos mx\,dx$

Case 3 : $j = 2, 6, 10, \ldots$

$$\int_0^{2\pi} f^{(j)}(x)(\cos mx)^{(j)}dx = \int_0^{2\pi} f^{(j)}(x)(-m^j)\cos mx\,dx$$
$$= -m^j m \int_0^{2\pi} f^{(j-1)}(x)\sin mx\,dx$$
$$= m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\cos mx\,dx$$
$$= \vdots$$
$$= m^{2j}\int_0^{2\pi} f(x)\cos mx\,dx$$

Case 4: $j = 1, 5, 9, \ldots$

$$\int_0^{2\pi} f^{(j)}(x)(\cos mx)^{(j)}dx = \int_0^{2\pi} f^{(j)}(x)(-m^j)\sin mx\,dx$$
$$= m^j m \int_0^{2\pi} f^{(j-1)}(x)\cos mx\,dx$$
$$= m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\sin mx\,dx$$
$$= -m^j m^3 \int_0^{2\pi} f^{(j-3)}(x)\cos mx\,dx$$
$$\vdots$$
$$= m^{2j}\int_0^{2\pi} f(x)\cos mx\,dx$$

$$\boxed{\int_0^{2\pi} f^{(j)}(x)(\sin mx)^{(j)}dx = m^{2j}\int_0^{2\pi} f(x)\sin mx\,dx}$$

Case 1: $j = 4, 8, 12, \ldots$

$$\int_0^{2\pi} f^{(j)}(x)(\sin mx)^{(j)}dx = m^j \int_0^{2\pi} f^{(j)}(x)\sin mx\,dx$$
$$= -m^j m \int_0^{2\pi} f^{(j-1)}(x)\cos mx\,dx$$
$$= -m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\sin mx\,dx$$
$$= m^j m^3 \int_0^{2\pi} f^{(j-3)}(x)\cos mx\,dx$$
$$\vdots$$
$$= m^{2j}\int_0^{2\pi} f(x)\sin mx\,dx$$

Case 2 : $j = 3, 7, 11, \ldots$

$$\int_0^{2\pi} f^{(j)}(x)(\sin mx)^{(j)}dx = -m^j \int_0^{2\pi} f^{(j)}(x)\cos mx\,dx$$
$$= -m^j m \int_0^{2\pi} f^{(j-1)}(x)\sin mx\,dx$$
$$= m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\cos mx\,dx$$
$$= m^j m^3 \int_0^{2\pi} f^{(j-3)}(x)\sin mx\,dx$$
$$\vdots$$
$$= m^{2j}\int_0^{2\pi} f(x)\sin mx\,dx$$

Case 3 : $j = 2, 6, 10, \ldots$

$$\int_0^{2\pi} f^{(j)}(x)(\sin mx)^{(j)}dx = -m^j \int_0^{2\pi} f^{(j)}(x)\sin mx\,dx$$
$$= m^j m \int_0^{2\pi} f^{(j-1)}(x)\cos mx\,dx$$
$$= m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\sin mx\,dx$$
$$\vdots$$
$$= m^{2j}\int_0^{2\pi} f(x)\sin mx\,dx$$

Case 4: $j = 1, 5, 9, \ldots$

$$\begin{aligned}\int_0^{2\pi} f^{(j)}(x)\,(\sin mx)^{(j)}\,dx &= m^j \int_0^{2\pi} f^{(j)}(x)\cos mx\,dx\\ &= m^j m \int_0^{2\pi} f^{(j-1)}(x)\sin mx\,dx\\ &= -m^j m^2 \int_0^{2\pi} f^{(j-2)}(x)\cos mx\,dx\\ &\;\vdots\\ &= m^{2j}\int_0^{2\pi} f(x)\sin mx\,dx\end{aligned}$$

Part 3 (Induction)

The identities in the lemma clearly hold for $n=0$ $\left(H^0[0,2\pi]\right)$.
Assume the identities hold for $n-1$ with $n \geq 1$

$$\begin{aligned}\int_0^{2\pi} \sum_{j=0}^{n} f^{(j)}(x)(\cos(mx))^{(j)}dx &= \int_0^{2\pi} \sum_{j=0}^{n-1} f^{(j)}(x)(\cos(mx))^{(j)}dx + \int_0^{2\pi} f^{(n)}(x)(\cos mx)^{(n)}dx\\ &= \left(\sum_{j=0}^{n-1} m^{2j}\right)\int_0^{2\pi} f(x)\cos mx\,dx + m^{2n}\int_0^{2\pi} f(x)\cos mx\,dx\\ &= \left(\sum_{j=0}^{n} m^{2j}\right)\int_0^{2\pi} f(x)\cos mx\,dx\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} \sum_{j=0}^{n} f^{(j)}(x)(\sin(mx))^{(j)}dx &= \int_0^{2\pi} \sum_{j=0}^{n-1} f^{(j)}(x)(\sin(mx))^{(j)}dx + \int_0^{2\pi} f^{(n)}(x)(\sin mx)^{(n)}dx\\ &= \left(\sum_{j=0}^{n-1} m^{2j}\right)\int_0^{2\pi} f(x)\sin mx\,dx + m^{2n}\int_0^{2\pi} f(x)\sin mx\,dx\\ &= \left(\sum_{j=0}^{n} m^{2j}\right)\int_0^{2\pi} f(x)\sin mx\,dx\end{aligned}$$

Section 3: Basis

We want to show that any $f \in H^n[0,2\pi]$ with $f(0) = f(2\pi)$ can be written as a linear combination of the trigonometric functions. But $f \in H^n[0,2\pi]$ means $f \in L^2[0,2\pi]$ by the definition of $H^n[0,2\pi]$. We already know that the trigonometric functions are a basis for $L^2[0,2\pi]$ and therefore f can be written as a linear combination of the trigonometric functions (the Fourier series of f converges to f in $L^2[0,2\pi]$).

2.2.6* Suppose $\{\phi_n(x)\}_{n=0}^{\infty}$ is a set of orthonormal polynomials relative to the L^2 inner product with positive weight function $\omega(x)$ on the domain $[a, b]$, and the suppose $\phi_n(x)$ is a polynomial of degree n with leading coefficient $K_n x^n$.

a. Show that $\phi_n(x)$ is orthogonal to any polynomial of degree $m < n$.

b. Show that the polynomials satisfy the recurrence relation

$$\phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) + C_n \phi_{n-1}(x)$$

where $A_n = K_{n+1}/K_n$. Express B_n & C_n in terms of A_n, A_{n-1}, and ϕ_n.

c. Evaluate A_n, B_n, and C_n for the Legendre and Chebyshev polynomials

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a. Let $p(x)$ be a polynomial of degree $m < n$ defined on $[a, b]$.
The vectors $\{\phi_i\}_{i=0}^{m}$ are linearly independent by mutual orthogonality:

$$0 = \sum_{i=0}^{m} \alpha_i \phi_i \Rightarrow 0 = \langle 0, \phi_K \rangle = \langle \sum_{i=0}^{m} \alpha_i \phi_i, \phi_K \rangle = \alpha_K \|\phi_K\|^2 = \alpha_K, \quad K \in \{0, \dots, m\}$$

Since $\{\phi_i\}_{i=0}^{m}$ are $m+1$ vectors on the $m+1$ dimensional vector space of polynomials of degree at most m, they form a basis and $\exists \alpha_i$ s.t. $p(x) = \alpha_0 \phi_0 + \dots + \alpha_m \phi_m$. But then since ϕ_n is orthogonal to each of $\phi_0, \dots, \phi_m$,

$$\langle p, \phi_n \rangle = \langle \alpha_0 \phi_0 + \dots + \alpha_m \phi_m, \phi_n \rangle = \alpha_0 \langle \phi_0, \phi_n \rangle + \dots + \alpha_m \langle \phi_m, \phi_n \rangle = 0.$$

b. When considering three consecutive ϕ_K, I propose the following general notation to help avoid confusion about the polynomial coefficients:

$$\phi_{n+1}(x) = K_{n+1} x^{n+1} + a_n x^n + \dots + a_0$$
$$\phi_n(x) = K_n x^n + b_{n-1} x^{n-1} + \dots + b_0$$
$$\phi_{n-1}(x) = K_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \dots + c_0$$

Consider the case $n = 1$. Unfortunately, starting with $n=0$ won't allow us to find a general pattern. Since $x\phi_1$ is a degree 2 polynomial and $\{\phi_0, \phi_1, \phi_2\}$ are a basis for the set of degree 2 polynomials, there exist constants $\alpha_0, \alpha_1, \alpha_2$ such that

$$x\phi_1(x) = \alpha_0 \phi_0(x) + \alpha_1 \phi_1(x) + \alpha_2 \phi_2(x)$$
$$\phi_2(x) = (1/\alpha_2\, x - \alpha_1/\alpha_2)\phi_1(x) - \alpha_0/\alpha_2\, \phi_0(x)$$

If $\alpha_2 = 0$, $x\phi_1(x)$ would be of degree less than 2 which would imply $\phi_1(x)$ is of degree less than 1, a contradiction. So $\alpha_2 \neq 0$.

$\therefore \phi_2(x)$ can be written in the form $\phi_2(x) = (A_1 x + B_1)\phi_1(x) + C_1\phi_0(x)$ for some constants A_1, B_1, C_1.

$$1 = \langle \phi_0, \phi_0 \rangle = \int_a^b \phi_0^2(x)\,\omega(x)\,dx = K_0^2 \int_a^b \omega(x)\,dx \Rightarrow \boxed{1/K_0^2 = \int_a^b \omega(x)\,dx}$$

$$0 = \langle \phi_1, \phi_0 \rangle = \int_a^b \phi_1(x)\phi_0(x)\,\omega(x)\,dx = \int_a^b (K_1 x + b_0) K_0\,\omega(x)\,dx$$

$$= K_0 K_1 \int_a^b x\,\omega(x)\,dx + b_0 K_0 \int_a^b \omega(x) = K_0 K_1 \int_a^b x\,\omega(x)\,dx + b_0/K_0$$

$$\Rightarrow \boxed{\int_a^b x\,\omega(x)\,dx = -b_0/K_0^2 K_1}$$

$$1 = \langle \phi_1, \phi_1 \rangle = \int_a^b \phi_1^2(x)\,\omega(x)\,dx = \int_a^b (K_1 x + b_0)^2\,\omega(x)\,dx$$

$$= K_1^2 \int_a^b x^2\,\omega(x)\,dx + 2K_1 b_0 \int_a^b x\,\omega(x)\,dx + b_0^2 \int_a^b \omega(x)\,dx$$

$$= K_1^2 \int_a^b x^2\,\omega(x)\,dx + 2K_1 b_0\left(-b_0/K_0^2 K_1\right) + b_0^2/K_0^2$$

$$= K_1^2 \int_a^b x^2\,\omega(x)\,dx - b_0^2/K_0^2 \Rightarrow \boxed{\int_a^b x^2\,\omega(x)\,dx = \frac{K_0^2 + b_0^2}{K_0^2 K_1^2}}$$

$$0 = \langle \phi_2, \phi_1 \rangle = \int_a^b (K_2 x^2 + a_1 x + a_0)(K_1 x + b_0)\,\omega(x)\,dx$$

$$= K_1 K_2 \int_a^b x^3\,\omega(x)\,dx + (K_2 b_0 + a_1 K_1)\int_a^b x^2\,\omega(x)\,dx$$

$$+ (a_1 b_0 + a_0 K_1)\int_a^b x\,\omega(x)\,dx + a_0 b_0 \int_a^b \omega(x)\,dx$$

$$= K_1 K_2 \int_a^b x^3\,\omega(x)\,dx + (K_2 b_0 + a_1 K_1)\frac{K_0^2 + b_0^2}{K_0^2 K_1^2}$$

$$+ (a_1 b_0 + a_0 K_1)\left(-b_0/K_0^2 K_1\right) + a_0 b_0/K_0^2$$

$$\int_a^b x^3\,\omega(x)\,dx = \frac{(-K_2 b_0 - a_1 K_1)(K_0^2 + b_0^2)}{K_0^2 K_1^3 K_2} + \frac{(a_1 b_0 + a_0 K_1)\,b_0}{K_0^2 K_1^2 K_2} - \frac{a_0 b_0}{K_0^2 K_1 K_2}$$

$$= \frac{(-K_2 b_0 - a_1 K_1)(K_0^2 + b_0^2) + a_1 b_0^2 K_1 + a_0 b_0 K_1^2 - a_0 b_0 K_1^2}{K_0^2 K_1^3 K_2}$$

$$= \frac{-K_0^2 K_2 b_0 - K_2 b_0^3 - a_1 K_1 K_0^2 - a_1 K_1 b_0^2 + a_1 b_0^2 K_1}{K_0^2 K_1^3 K_2}$$

$$\boxed{\int_a^b x^3\,\omega(x)\,dx = -\frac{K_0^2 K_2 b_0 + K_2 b_0^3 + a_1 K_1 K_0^2}{K_0^2 K_1^3 K_2} = -\frac{a_1}{K_1^2 K_2} - \frac{b_0^3}{K_0^2 K_1^3} - \frac{b_0}{K_1^3}}$$

$$1 = \langle \phi_2, \phi_2 \rangle = \int_a^b \phi^2(x)\,\omega(x)\,dx = \int_a^b (K_2 x^2 + a_1 x + a_0)^2\,\omega(x)\,dx$$

$$= K_2^2 \int_a^b x^4\,\omega(x)\,dx + 2K_2 a_1 \int_a^b x^3\,\omega(x)\,dx + (2K_2 a_0 + a_1^2)\int_a^b x^2\,\omega(x)\,dx$$
$$+ 2a_0 a_1 \int_a^b x\,\omega(x)\,dx + a_0^2 \int_a^b \omega(x)\,dx$$

$$\int_a^b x^4 \omega(x)\,dx = \frac{1}{K_0^2 K_1^3 K_2^2} \left\{ K_0^2 K_1^3 + a_1^2 \quad \cdots \right.$$

Had trouble calculating. If this could be found, we could then find A_1

$$1 = \langle \phi_2, \phi_2 \rangle = \langle \phi_2, A_1 x \phi_1 + B_1 \phi_1 + C_1 \phi_0 \rangle = A_1 \langle \phi_2, x\phi_1 \rangle$$

$$= A_1 \int_a^b (K_2 x^2 + a_1 x + a_0)(K_1 x^2 + b_0 x)\,\omega(x)\,dx$$

$$= A_1 \Big[K_1 K_2 \int_a^b x^4 \omega(x)\,dx + (K_2 b_0 + a_1 K_1) \int_a^b x^3 \omega(x)\,dx$$

$$+ (a_1 b_0 + a_0 K_1) \int_a^b x^2 \omega(x)\,dx + a_0 b_0 \int_a^b x\,\omega(x)\,dx$$

$$= A_1 \Big\{$$

Should lead to $A_1 = K_2/K_1$. Then determine B_1, C_1 using a similar process. This would give the base case for a hypothesis about the recurrence formula. Proof by induction.

<u>2.2.12</u> Suppose $f(t)$ and $g(t)$ are 2π periodic functions with Fourier series representations

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ikt} \qquad g(t) = \sum_{k=-\infty}^{\infty} g_k e^{ikt}.$$

Find the Fourier series of $h(t) = \int_0^{2\pi} f(t-x) g(x)\, dx$.

$$h(t) = \sum_{k=-\infty}^{\infty} h_k e^{ikt}$$

$$h_k = \frac{1}{2\pi}\int_0^{2\pi} h(t) e^{-ikt}\, dt = \frac{1}{2\pi}\int_0^{2\pi}\left[\int_0^{2\pi} f(t-x) g(x)\, dx\right] e^{-ikt} dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(t-x) g(x) e^{-ik(t-x)} e^{-ikx}\, dx\, dt$$

$$= \frac{1}{2\pi}\int_{-x}^{2\pi - x}\int_0^{2\pi} f(y) g(x) e^{-iky} e^{-ikx}\, dx\, dy \qquad (y = t-x)$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(y) g(x) e^{-iky} e^{-ikx}\, dx\, dy \qquad \text{(periodicity)}$$

$$= \frac{1}{2\pi}\left(\int_0^{2\pi} f(y) e^{-iky}\, dy\right)\left(\int_0^{2\pi} g(x) e^{-ikx}\, dx\right)$$

$$= \frac{1}{2\pi}(2\pi f_k)(2\pi g_k)$$

$$= 2\pi f_k g_k$$

$\therefore$ The Fourier series representation of $h(t)$ is

$$h(t) = 2\pi \sum_{k=-\infty}^{\infty} f_k g_k e^{ikt}$$

2.2.14* Show that the discrete Fourier Transform $\{f_n\} \to \{g_n\}$ can be viewed as a change of coordinate system in $\mathbb{R}^n$. What is the matrix T corresponding to this change of basis?

If the n values $f = \{f_j\}_{j=0}^{n-1}$ are given we define the Discrete Fourier Transform of f to be the values $\{g_j\}_{j=0}^{n-1}$ given by

$$g_j = \sum_{k=0}^{n-1} f_k e^{2\pi i kj/n} \qquad j = 0, 1, \ldots, n-1$$

$$g_0 = f_0 + f_1 + \ldots + f_{n-1}$$

$$g_1 = f_0 + f_1 e^{2\pi i/n} + f_2 e^{(2\pi i/n)\cdot 2} + \ldots + f_{n-1} e^{(2\pi i/n)\cdot(n-1)}$$

$$g_2 = f_0 + f_1 e^{(2\pi i/n)\cdot 2} + f_2 e^{(2\pi i/n)\cdot 2\cdot 2} + \ldots + f_{n-1} e^{(2\pi i/n)\cdot 2\cdot(n-1)}$$

$$\vdots$$

$$g_{n-1} = f_0 + f_1 e^{(2\pi i/n)\cdot(n-1)} + f_2 e^{(2\pi i/n)\cdot(n-1)\cdot 2} + \ldots + f_{n-1} e^{(2\pi i/n)\cdot(n-1)^2}$$

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix} \qquad \omega = e^{2\pi i/n}$$

$\vec{g} = F_n \vec{f}$, $(F_n)_{jk} = \omega^{jk}$ $\quad j, k \in \{0, \ldots, n-1\}$ (0-indexed rows & columns)

X

Even if $\vec{f} \in \mathbb{R}^n$, $\vec{g} \notin \mathbb{R}^n$ generally. However, if $\vec{f} \in \mathbb{R}^n$ and $\vec{g} = [Re(F_n) + i\, Im(F_n)]\vec{f}$, we could consider $\vec{h} = T\vec{f}$ as a change of coordinate system in $\mathbb{R}^n$ with $T = (Re(F_n) + Im(F_n)) \in \mathbb{R}^{n\times n}$.

There exist discrete sine and cosine transforms that would map f from $\mathbb{R}^n$ to $\mathbb{R}^n$ but we have been asked about only the DFT, which is a mapping from either $\mathbb{R}^n \to \mathbb{C}^n$ or $\mathbb{C}^n \to \mathbb{C}^n$ generally.

Find a matrix $T \in \mathbb{R}^{2n\times 2n}$ such that

$$\begin{bmatrix} Re(g_1) \\ \vdots \\ Re(g_n) \\ Im(g_1) \\ \vdots \\ Im(g_n) \end{bmatrix} = T \begin{bmatrix} Re(f_1) \\ \vdots \\ Re(f_n) \\ Im(f_1) \\ \vdots \\ Im(f_n) \end{bmatrix}$$