1. Which of the following Diophantine equations cannot be solved?
(a) $6x + 51y = 22$.
(b) $33x + 14y = 115$.
(c) $14x + 35y = 93$.

$\Big\}$ for $x, y \in \mathbb{Z}$

Theorem 2.9. The linear Diophantine equation $ax + by = c$ has a solution if and only if $d \mid c$, where $d = \gcd(a, b)$. If x_0, y_0 is any particular solution of this equation, then all other solutions are given by

$$x = x_0 + \left(\frac{b}{d}\right)t \qquad y = y_0 - \left(\frac{a}{d}\right)t$$

where t is an arbitrary integer.

(a) $\gcd(51, 6) = 3$, $3 \nmid 22$. Cannot be solved.

(b) $\gcd(33, 14) = 1$, $1 \mid 115$. Can be solved.

(c) $\gcd(35, 14) = 7$, $7 \nmid 93$. Cannot be solved.

2. Determine all solutions in the integers of the following Diophantine equations:
(a) $56x + 72y = 40$.
(b) $24x + 138y = 18$.
(c) $221x + 35y = 11$.

(a)
$$72 = 56 \cdot 1 + 16$$
$$56 = 16 \cdot 3 + 8$$
$$16 = 8 \cdot 2 + 0$$

$\gcd(72, 56) = 8$ and $8 \mid 40$, a solution exists.

$$8 \cdot 5 = 40$$
$$(56 - 16 \cdot 3) \cdot 5 = 40$$
$$56 \cdot 5 - 16 \cdot 15 = 40$$
$$56 \cdot 5 - (72 - 56) \cdot 15 = 40$$
$$56 \cdot 20 + 72 \cdot (-15) = 40$$

$(x_0, y_0) = (20, -15)$

$$\left\{ \begin{cases} x = 20 + 9t \\ y = -15 - 7t \end{cases}, \ t \in \mathbb{Z} \right\}$$

(b)
$$138 = 24 \cdot 5 + 18$$
$$24 = 18 \cdot 1 + 6$$
$$18 = 6 \cdot 3 + 0$$

$\gcd(138, 24) = 6$ and $6 \mid 18$, a solution exists.

$$6 \cdot 3 = 18$$
$$(24 - 18) \cdot 3 = 18$$
$$24 \cdot 3 + (138 - 24 \cdot 5) \cdot (-3) = 18$$
$$24 \cdot 18 + 138 \cdot (-3) = 18$$

$(x_0, y_0) = (18, -3)$

$$\{ x = 18 + 23t, \ y = -3 - 4t, \ t \in \mathbb{Z} \}$$

(c) $221 = 35 \cdot 6 + 11$
$35 = 11 \cdot 3 + 2$
$11 = 2 \cdot 5 + 1$
$2 = 1 \cdot 2 + 0$

$\gcd(221, 35) = 1$ and $1 \mid 11$, a solution exists.

$1 \cdot 11 = 11$
$(11 - 2 \cdot 5) \cdot 11 = 11$
$11 \cdot 11 + 2 \cdot (-55) = 11$
$11 \cdot 11 + (35 - 11 \cdot 3)(-55) = 11$
$11 \cdot 176 + 35 \cdot (-55) = 11$
$(221 - 35 \cdot 6)(176) + 35(-55) = 11$
$221 \cdot 176 + 35 \cdot (-1{,}111) = 11$ $\quad (x_0, y_0) = (176, -1111)$

$\{x = 176 + 35t,\ y = -1{,}111 - 221t,\ t \in \mathbb{Z}\}$

3. Determine all solutions in the positive integers of the following Diophantine equations:
(a) $18x + 5y = 48$.
(b) $54x + 21y = 906$.
(c) $123x + 360y = 99$.
(d) $158x - 57y = 7$.

(a) $18 = 5 \cdot 3 + 3$
$5 = 3 \cdot 1 + 2$
$3 = 2 \cdot 1 + 1$
$2 = 1 \cdot 2$
$\gcd(18, 5) = 1$

$1 \cdot 48 = 48$
$3 \cdot 48 + 2 \cdot (-48) = 48$
$3 \cdot 48 + (5 - 3)(-48) = 48$
$3 \cdot 96 + 5(-48) = 48$
$(18 - 5 \cdot 3)96 + 5(-48) = 48$
$18 \cdot 96 + 5 \cdot (-336) = 48$

$x = 96 + 5t,\ y = -336 - 18t,\ t \in \mathbb{Z}$
$x, y > 0$ for $t = -19 \longrightarrow (x, y) = (1, 6)$

(b) $54 = 21 \cdot 2 + 12$
$21 = 12 \cdot 1 + 9$
$12 = 9 \cdot 1 + 3$
$9 = 3 \cdot 3$
$\gcd(54, 21) = 3$

$3 \cdot 302 = 906$
$(12 - 9)302 = 906$
$12 \cdot 302 + 9(-302) = 906$
$12 \cdot 302 + (21 - 12)(-302) = 906$
$12(604) + 21(-302) = 906$
$(54 - 2 \cdot 21)(604) + 21(-302) = 906$
$54 \cdot 604 + 21 \cdot (-1510) = 906$

$x = 604 + 7t,\ y = -1510 - 18t,\ t \in \mathbb{Z}$
$x, y > 0$ for $t = -84, -85, -86$

$(x, y) = (16, 2), (9, 20), (2, 38)$

(c) $123x + 360y = 99$

$360 = 123 \cdot 2 + 114$
$123 = 114 \cdot 1 + 9$
$114 = 9 \cdot 12 + 6$
$9 = 6 \cdot 1 + 3$
$6 = 3 \cdot 2$
$\gcd(360, 123) = 3$

$3 \cdot 33 = 99$
$9 \cdot 33 + 6 \cdot (-33) = 99$
$9 \cdot 33 + (114 - 9 \cdot 12)(-33) = 99$
$9 \cdot 429 + 114(-33) = 99$
$(123 - 114) 429 + 114(-33) = 99$
$123(429) + 114(-462) = 99$
$123(429) + (360 - 2 \cdot 123)(-462) = 99$
$123(1353) + 360(-462) = 99$
$(x_0, y_0) = (1353, -462)$

$x = 1353 + 120t$, $y = -462 - 41t$, $t \in \mathbb{Z}$

$1353 + 120t > 0$
$-11.3 \approx -\frac{1353}{120} < t$

$-462 - 41t > 0$
$-11.3 \approx \frac{-462}{41} > t$

$t \leq -12$
$t \geq -11$

No positive integer solutions exist.

(d) $158x - 57y = 7$

$158 = 57 \cdot 2 + 44$
$57 = 44 \cdot 1 + 13$
$44 = 13 \cdot 3 + 5$
$13 = 5 \cdot 2 + 3$
$5 = 3 \cdot 1 + 2$
$3 = 2 \cdot 1 + 1$
$2 = 1 \cdot 2$
$1 = \gcd(158, -57) = \gcd(158, 57)$

$1 \cdot 7 = 7$
$3 \cdot 7 + 2 \cdot (-7) = 7$
$3 \cdot 7 + (5 - 3)(-7) = 7$
$3 \cdot 14 + 5(-7) = 7$
$(13 - 5 \cdot 2) 14 + 5(-7) = 7$
$13(14) + 5(-35) = 7$
$13(14) + (44 - 13 \cdot 3)(-35) = 7$
$13(119) + 44(-35) = 7$
$(57 - 44)(119) + 44(-35) = 7$
$57(119) + 44(-154) = 7$
$57(119) + (158 - 57 \cdot 2)(-154) = 7$
$158(-154) + 57(427) = 7$
$158(-154) - 57(-427) = 7$
$(x_0, y_0) = (-154, -427)$

$x = -154 - 57t$, $y = -427 - 158t$, $t \in \mathbb{Z}$
$x, y > 0$ for $t \geq -3$.

5. (a) A man has \$4.55 in change composed entirely of dimes and quarters. What are the maximum and minimum number of coins that he can have? Is it possible for the number of dimes to equal the number of quarters?

(b) The neighborhood theater charges \$1.80 for adult admissions and \$.75 for children. On a particular evening the total receipts were \$90. Assuming that more adults than children were present, how many people attended?

(c) A certain number of sixes and nines is added to give a sum of 126; if the number of sixes and nines is interchanged, the new sum is 114. How many of each were there originally?

(a) $$10x + 25y = 455$$

x = # of dimes, y = # of quarters

$$25 = 10 \cdot 2 + 5$$
$$10 = 5 \cdot 2$$

$$5 \cdot 91 = 455$$
$$(25 - 10 \cdot 2) \cdot 91 = 455$$

$$10 \cdot (-182) + 25 \cdot 91 = 455$$
$$(x_0, y_0) = (-182, 91)$$

$$x = -182 + 5t, \quad y = 91 - 2t, \quad t \in \mathbb{Z}$$

We have positive x, y for $37 \le t \le 45$.
The number of coins is:

$$x + y = 3t - 91.$$

$x+y$ is minimized in the allowable range by $t=37$.
The result is:

3 dimes, 17 quarters, 20 coins total.

(b) Let x be the number of adults and let y be the number of children.

$$180x + 75y = 9000$$

$$180 = 75 \cdot 2 + 30$$
$$75 = 30 \cdot 2 + 15$$
$$30 = 15 \cdot 2$$

216,000

$$15 \cdot 600 = 9000$$
$$(75 - 30 \cdot 2) \cdot 600 = 9000$$
$$75 \cdot 600 + 30 \cdot (-1200) = 9000$$
$$75 \cdot 600 + (180 - 75 \cdot 2)(-1200) = 9000$$
$$180 \cdot (-1200) + 75 \cdot (3000) = 9000$$

$$(x_0, y_0) = (-1200, 3000)$$

$$x = -1200 + 5t$$
$$y = 3000 - 12t$$

We have x, y positive for $240 < t < 250$
By hypothesis, $x > y$. That is,

$$5t - 1200 > 3000 - 12t$$
$$17t > 4200$$
$$t > 247.058824$$

Since $t \in \mathbb{Z}$, this means $t = 248$ or $t = 249$

$x = -1200 + 5 \cdot 248 = 40$ adults
$y = 3000 - 12 \cdot 248 = 24$ children

or

$x = -1200 + 5 \cdot 249 = 45$ adults
$y = 3000 - 12 \cdot 249 = 12$ children

Both cases result in \$90 of revenue and both cases have more adults.

$\therefore$ Either 64 or 57 people attended.

(c) Let x be the number of 6's and let y be the number of 9's.

$$6x + 9y = 126$$
$$9x + 6y = 114$$

$$18x + 27y = 378$$
$$-18x - 12y = -228$$

$$15y = 150$$
$$\underline{y = 10}$$

$$2x + 3y = 42$$

$$\underline{x = (42 - 30)/2 = 6}$$

$6 \cdot 6 + 9 \cdot 10 = 126$ ✓
$6 \cdot 10 + 9 \cdot 6 = 114$

6. A farmer purchased 100 head of livestock for a total cost of \$4000. Prices were as follow: calves, \$120 each; lambs, \$50 each; piglets, \$25 each. If the farmer obtained at least one animal of each type, how many of each did he buy?

Let x be the number of calves
y be the number of lambs
z be the number of piglets

$$x+y+z=100$$
$$120x+50y+25z=4000 \quad , \quad x,y,z>0$$

$$z = 100 - x - y$$
$$120x+50y+2500-25x-25y = 4000$$
$$95x+25y = 1500$$
$$19x+5y = 300$$

$19 = 5\cdot 3+4$
$5 = 4\cdot 1+1$
$4 = 1\cdot 4$
$\therefore \gcd(19,5)=1$

$$1\cdot 300 = 300$$
$$(5-4)\cdot 300 = 300$$
$$5\cdot 300 + 4\cdot(-300) = 300$$
$$5\cdot 300 + (19-5\cdot 3)(-300) = 300$$
$$19(-300)+5(1200) = 300$$
$$(x_0,y_0) = (-300, 1200)$$

$$x = -300+5t, \quad y = 1200-19t, \quad t\in\mathbb{Z}$$
$x,y>0$ for $61\le t\le 63$

$$\left.\begin{aligned} t=61: &\quad (x,y,z) = (5,41,54)\\ t=62: &\quad (x,y,z) = (10,22,68)\\ t=63: &\quad (x,y,z) = (15,3,82)\end{aligned}\right\} \text{ 3 possible solutions}$$

$$120\cdot 5+50\cdot 41+25\cdot 54 = 4000$$
$$120\cdot 10+50\cdot 22+25\cdot 68 = 4000 \quad \checkmark$$
$$120\cdot 15+50\cdot 3+25\cdot 82 = 4000$$

1. Suppose x is an integer and someone has applied the Fundamental Theorem of Arithmetic to it, obtaining $x = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$ What would x^3 look like? What generalization can you make out of this?

$$x^3 = p_1^{3\alpha_1} p_2^{3\alpha_2} \ldots p_n^{3\alpha_n}$$

In general,

$$x^k = p_1^{k\alpha_1} p_2^{k\alpha_2} \ldots p_n^{k\alpha_n}$$

2. If p is a prime number, does $\mathbf{px - ky = 1}$ have solutions ? **k** could be any integer here.

If $p \mid k$ (including $k = \pm p$), then $\gcd(p,k) = p$ and since $p \nmid 1$, $px - ky = 1$ has no integer solutions. If $k \neq \pm p$, then $\gcd(p,k) = 1$ and so $px - ky = 1$ does have integer solutions.

3. If **p** is a prime number, argue why **p** must divide $\mathbf{C_{p,r}}$ (binomial coefficient)

$$C_{p,r} = \binom{p}{r} = \frac{p!}{r!(p-r)!}$$

Assuming $r > 0$, if $p = r$ then $\binom{p}{r} = 1$ and this claim is false. If $p < r$, then $\binom{p}{r} = 0$ and $p \mid 0$. If $r = 0$, $\binom{p}{r} = 1$ and the claim is again false. If $r < 0$, $\binom{p}{r} = 0$ and $p \mid 0$. Finally, consider $0 < r < p$. $\binom{p}{r} = t \in \mathbb{Z}$.

$$t = \frac{p!}{r!(p-r)!} = \frac{p(p-1)\ldots(p-r)(p-r-1)\ldots 1}{r(r-1)\ldots 1\,(p-r)(p-r-1)\ldots 1}$$

$$= \frac{p(p-1)\ldots(p-r+1)}{r(r-1)\ldots 1} = \frac{a}{b}$$

Since $1, 2, \ldots, r < p$, and p is prime, $p \nmid b$. $a = tb$ with $p \mid a$ and $p \nmid b$. Thus, $p \mid t$.

4. See if you can prove the following:

if **x** divides the product **bc** and **x** is prime then **x** divides either **b** or **c**.

Make up a counterexample where **x** is *not* prime and **x** divides their product but not either one.

An equivalent statement of this claim is: If x is prime, $x \mid bc$ and $x \nmid b$, then $x \mid c$.

since $\gcd(x, b) = 1$, $\exists\ r, s \in \mathbb{Z}$ s.t.

$$xr + bs = 1$$
$$xcr + bcs = c$$
$$xcr + kxs = c \qquad (bc = kx,\ k \in \mathbb{Z})$$
$$x(cr + ks) = c$$

This shows $x \mid c$. QED.

Let $x = 6$ (not prime)
$b = 9$
$c = 2$

Then $bc = 18$ so $x \mid bc$ but $x \nmid b$, $x \nmid c$.