MME 529 Homework 3

Mersenne Primes

1. Why $2^n - 1$ and not $3^n - 1$ or $4^n - 1$ or $5^n - 1$ or $6^n - 1$?

Since $3^n - 1$ and $5^n - 1$ are even $\forall n$, these will not produce primes at all (except $3^1 - 1 = 2$).

Note that $4^n - 1 = 2^{2n} - 1$. In HW 1 #11 we showed that numbers of the form $2^{pq} - 1$ have a nontrivial factorization using the geometric series. That means $4^n - 1$ must always be composite (except when $n = 1$ and $2^{2n} - 1 = 2^2 - 1 = 3$).

Since 6^n always has 6 as its last digit, $6^n - 1$ is always divisible by 5 and so $6^n - 1$ is composite for all $n \geq 2$.

2. Omitted from these solutions.

3. Can you find a Mersenne Prime where the exponent is itself a Mersenne prime? (we know the exponent must be prime from class)

Yes. Note that $2^2 - 1 = 3$ is a Mersenne prime and

$$2^{2^2 - 1} - 1 = 2^3 - 1 = 7$$

is a Mersenne Prime with the Mersenne prime exponent $2^2 - 1 = 3$.

Fibonacci

4. Get the Binet formula out. For the cases n=3 and 4, verify that you really do get F_3 and F_4

If F_n is the nth Fibonacci number, then

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$F_3 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3\right]$$

$$= \left[\left(1 + 3\sqrt{5} + 3\sqrt{5}^2 + \sqrt{5}^3\right) - \left(1 - 3\sqrt{5} + 3\sqrt{5}^2 - \sqrt{5}^3\right)\right] / 2^3\sqrt{5}$$

$$= \left[2 \cdot 3\sqrt{5} + 2\sqrt{5}^3\right] / 2^3\sqrt{5}$$

$$= 16\sqrt{5} \,/\, 8\sqrt{5} = 2.$$

$$F_4 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^4 - \left(\frac{1-\sqrt{5}}{2}\right)^4\right]$$

$$= \left[\left(1 + 4\sqrt{5} + 6\sqrt{5}^2 + 4\sqrt{5}^3 + \sqrt{5}^4\right) - \left(1 - 4\sqrt{5} + 6\sqrt{5}^2 - 4\sqrt{5}^3 + \sqrt{5}^4\right)\right] / 2^4\sqrt{5}$$

$$= \left[2\cdot 4\sqrt{5} + 2\cdot 4\sqrt{5}^3\right] / 2^4\sqrt{5}$$

$$= \left[8\sqrt{5} + 40\sqrt{5}\right] / 16\sqrt{5} = 48/16 = 3$$

Compare these with the 3^{rd} and 4^{th} Fibonacci numbers: 1, 1, <u>2</u>, <u>3</u> ✓

5. a) develop polynomial expressions for

   $1/\phi$, $1/\phi^2$, $1/\phi^3$, $1/\phi^4$, and $1/\phi^5$

   with simple algebra. ($1/\phi = \phi - 1$, just to get your going)

   b) can you see a formula emerging for $1/\phi^n$?

a) $\phi = (1+\sqrt{5})/2$

$$1/\phi = 2/(1+\sqrt{5}) = 2(1-\sqrt{5})/(1-5)$$
$$= (2\sqrt{5}-2)/4$$
$$= (\sqrt{5}-1)/2$$
$$= (1+\sqrt{5}-2)/2$$
$$= (1+\sqrt{5})/2 - 1 = \phi - 1$$

$$1/\phi^2 = (1/\phi)^2 = (\phi-1)^2 = \phi^2 - 2\phi - 1$$

$$1/\phi^3 = (\phi-1)^3 = \phi^3 - 3\phi^2 + 3\phi - 1$$

$$1/\phi^4 = (\phi-1)^4 = \phi^4 - 4\phi^3 + 6\phi^2 - 4\phi + 1$$

$$1/\phi^5 = (\phi-1)^5 = \phi^5 - 5\phi^4 + 10\phi^3 - 10\phi^2 + 5\phi - 1$$

b) $$1/\phi^n = (\phi-1)^n = \sum_{k=0}^{n} \binom{n}{k} \phi^{n-k}(-1)^k$$

by the Binomial Theorem.