PROBLEMS 12.2

1. Show that the equation $x^2 + y^2 = z^3$ has infinitely many solutions for x, y, z positive integers.
[*Hint:* For any $n \geq 2$, let $x = n(n^2 - 3)$ and $y = 3n^2 - 1$.]
2. Prove the theorem: The only solutions in nonnegative integers of the equation $x^2 + 2y^2 = z^2$, with $\gcd(x, y, z) = 1$, are given by
$$x = \pm(2s^2 - t^2) \qquad y = 2st \qquad z = 2s^2 + t^2$$
where s, t are arbitrary nonnegative integers.
[*Hint:* If u, v, w are such that $y = 2w$, $z + x = 2u$, $z - x = 2v$, then the equation becomes $2w^2 = uv$.]
3. In a Pythagorean triple x, y, z, prove that not more than one of x, y, or z can be a perfect square.
4. Prove each of the following assertions:
(a) The system of simultaneous equations
$$x^2 + y^2 = z^2 - 1 \qquad \text{and} \qquad x^2 - y^2 = w^2 - 1$$
has infinitely many solutions in positive integers x, y, z, w.
[*Hint:* For any integer $n \geq 1$, take $x = 2n^2$ and $y = 2n$.]
(b) The system of simultaneous equations
$$x^2 + y^2 = z^2 \qquad \text{and} \qquad x^2 - y^2 = w^2$$
admits no solution in positive integers x, y, z, w.
(c) The system of simultaneous equations
$$x^2 + y^2 = z^2 + 1 \qquad \text{and} \qquad x^2 - y^2 = w^2 + 1$$
has infinitely many solutions in positive integers x, y, z, w.
[*Hint:* For any integer $n \geq 1$, take $x = 8n^4 + 1$ and $y = 8n^3$.]
5. Use Problem 4 to establish that there is no solution in positive integers of the simultaneous equations
$$x^2 + y^2 = z^2 \qquad \text{and} \qquad x^2 + 2y^2 = w^2$$
[*Hint:* Any solution of the given system also satisfies $z^2 + y^2 = w^2$ and $z^2 - y^2 = x^2$.]
6. Show that there is no solution in positive integers of the simultaneous equations
$$x^2 + y^2 = z^2 \qquad \text{and} \qquad x^2 + z^2 = w^2$$
hence, there exists no Pythagorean triangle whose hypotenuse and one of whose sides form the sides of another Pythagorean triangle.
[*Hint:* Any solution of the given system also satisfies $x^4 + (wy)^2 = z^4$.]
7. Prove that the equation $x^4 - y^4 = 2z^2$ has no solutions in positive integers x, y, z.
[*Hint:* Because x, y must be both odd or both even, $x^2 + y^2 = 2a^2$, $x + y = 2b^2$, $x - y = 2c^2$ for some a, b, c; hence, $a^2 = b^4 + c^4$.]
8. Verify that the only solution in relatively prime positive integers of the equation $x^4 + y^4 = 2z^2$ is $x = y = z = 1$.
[*Hint:* Any solution of the given equation also satisfies the equation
$$z^4 - (xy)^4 = \left(\frac{x^4 - y^4}{2}\right)^2.]$$

9. Prove that the Diophantine equation $x^4 - 4y^4 = z^2$ has no solution in positive integers x, y, z.
[*Hint:* Rewrite the given equation as $(2y^2)^2 + z^2 = (x^2)^2$ and appeal to Theorem 12.1.]

10. Use Problem 9 to prove that there exists no Pythagorean triangle whose area is twice a perfect square.
[*Hint:* Assume to the contrary that $x^2 + y^2 = z^2$ and $\frac{1}{2}xy = 2w^2$. Then $(x + y)^2 = z^2 + 8w^2$, and $(x - y)^2 = z^2 - 8w^2$. This leads to $z^4 - 4(2w)^4 = (x^2 - y^2)^2$.]

11. Prove the theorem: The only solutions in positive integers of the equation

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2} \qquad \gcd(x, y, z) = 1$$

are given by

$$x = 2st(s^2 + t^2) \qquad y = s^4 - t^4 \qquad z = 2st(s^2 - t^2)$$

where s, t are relatively prime positive integers, one of which is even, with $s > t$.

12. Show that the equation $1/x^4 + 1/y^4 = 1/z^2$ has no solution in positive integers.