1 Lecture 13 (April 29th)

Proposition. We shall often use the following power series representations.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

where $z \in \boldsymbol{C}$. An effective way to create a power series representation is to substitute

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

for $z \in D(0,1)$.

Example. Power series expansions allow us to calculate integrals on the complex plane. For example, consider

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

for $z \neq 0$. Then,

$$\int_{|z|=1} e^{1/z}\, dz = \sum_{n=0}^{\infty} \int_{|z|=1} \frac{1}{n! z^n}\, dz = \int_{|z|=1} \frac{1}{z}\, dz = 2\pi i$$

as each term is equal to 0 if $n \neq 1$.

Theorem. (Laurent theorem) Let f be analytic in a multiply connected domain of the form $D = \{z \in \boldsymbol{C} \mid r < |z - z_0| < R\}$. Also, let C be a POSCC in D such that z_0 is inside C. Then,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

In particular, when $n = -1$,

$$a_{-1} = \frac{1}{2\pi i} \int_C f(z)\, dz$$

We will use a_{-1} to find the integral of $f(z)$ on C in the future.

Proof. There is $R_1 < R_2$ such that $R_1 \leq |z - z_0| \leq R_2$ contains C and $R_1 \leq |z - z_0| \leq R_2$ is in D. Then f is analytic in $R_1 \leq |z - z_0| \leq R_2$. By the Cauchy integral formula, for

$R_1 < |z - z_0| < R_2$,

$$
\begin{aligned}
f(z) =& \frac{1}{2\pi i} \int_{|\xi - z_0| = R_2} \frac{f(\xi)}{\xi - z}\, d\xi - \frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{\xi - z}\, d\xi \\
=& \sum_{n=0}^{\infty} a_n (z - z_0)^n - \frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{\xi - z}\, d\xi
\end{aligned}
$$

where both are positively oriented simply curves and

$$
a_n = \frac{1}{2\pi i} \int_{|\xi - z_0| = R_2} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi
$$

For the second part,

$$
\frac{1}{z - \xi} = \frac{1}{z - z_0 - (\xi - z_0)} = \frac{1}{z - z_0} \frac{1}{1 - \left(\dfrac{\xi - z_0}{z - z_0} \right)} = \sum_{n=0}^{\infty} \frac{(\xi - z_0)^n}{(z - z_0)^{n+1}}
$$

since $|(\xi - z_0)/(z - z_0)| < 1$. As the series converges uniformly on the compact set, we have, for the second part,

$$
\begin{aligned}
\frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{\xi - z}\, d\xi =& \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{|\xi - z_0| = R_1} \frac{(\xi - z_0)^n}{(z - z_0)^{n+1}} f(\xi)\, d\xi \\
=& \frac{1}{2\pi i} \sum_{n=0}^{\infty} \Big[\int_{|\xi - z_0| = R_1} f(\xi)(\xi - z_0)^n\, d\xi \Big] (z - z_0)^{-(n+1)} \\
=& \sum_{m=-\infty}^{-1} \Big[\frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{(\xi - z_0)^m}\, d\xi \Big] (z - z_0)^m
\end{aligned}
$$

Therefore,

$$
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m - \infty}^{-1} a_m (z - z_0)^m
$$

where

$$
\begin{aligned}
a_n =& \frac{1}{2\pi i} \int_{|\xi - z_0| = R_2} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi & n \geq 0 \\
a_m =& \frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{(\xi - z_0)^{m+1}}\, d\xi & m < 0
\end{aligned}
$$

as the integrand is analytic between the concentric circles with radius R_1 and R_2, we can generalise to a curve in this domain and say

$$
a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi
$$

for $n \in \boldsymbol{Z}$. $\square$

Example. By cleverly manipulating functions to be expressed in terms of an infinite summation of a geometric sequence, we can obtain various Laurent series expansions at different regions on the complex plane. Let

$$f(z) = \frac{1}{(z-1)(z-2)}$$

be an analytic function in

(i) Consider $D_1 = \{0 < |z-1| < 1\}$

$$f(z) = -\frac{1}{(z-1)(1-(z-1))} = \sum_{n=-1}^{\infty} (z-1)^n$$

(ii) Consider $D_2 = \{0 < |z-2| < 1\}$

$$f(z) = \frac{1}{((z-2)+1)(z-1)} = \frac{1}{z-2}\sum_{n=1}^{\infty}(-1)^n(z-2)^n = \sum_{n=-1}^{\infty}(-1)^{n+1}(z-2)^n$$

(iii) Consider $D_3 = \{|z| < 1\}$

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

(iv) Consider $D_4 = \{1 < |z| < 2\}$

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

(v) Consider $D_5 = \{|z| > 2\}$

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

2 Lecture 14 (May 8)

Definition. We say that f has a singularity at z_0 if f is not differentiable at z_0. We say that f has an isolated singularity at z_0 provided that in addition to not being differentiable, there is $r > 0$ such that f is differentiable on $D^*(a, r)$

Example. All of these functions have a singularity at z_0

(i) $f(z) = \dfrac{\sin z}{z}$ at $z = 0$. Notice that when we define $f(0) = 1$, then f is entire.

(ii) $f(z) = \dfrac{e^z}{(z-1)^2}$ at $z_0 = 1$. Notice that

$$\lim_{z\to 1}(z-1)^2 f(z) = e \quad \text{and} \quad \lim_{z\to 1}|f(z)| = \infty$$

That is, the function approaches infinity as you approach the singularity.

(iii) $f(z) = e^{1/z}$ at $z_0 = 0$. Notice that

$$\lim_{x \to 0^+} e^{1/x} = +\infty \quad \text{but} \quad \lim_{x \to 0^-} e^{1/x} = 0$$

(iv) $f(z) = \dfrac{1}{\sin(\pi/z)}$ at $z_0 = 0$. For $z_n = 1/n$, f has a singularity at each $z_n = 1/n$.

In each case, we see a removable, pole, essential, and non-isolated singularity!

Definition. Let f has an isolated singularity at z_0.

(i) (Removable) If we can define $f(z_0)$ such that f is analytic at z_0, then we say that f has a removable singularity at z_0.

(ii) (Pole) If there is a $k \in \boldsymbol{N}$ such that

$$\lim_{z \to z_0} (z - z_0)^k f(z) = \alpha \neq 0$$

we say that f has a pole of order k at z_0. If $k = 1$, f is said to have a simple pole.

(iii) (Essential) If f satisfies neither of the two above, we then say that f has an essential singularity.

Corollary. If f has an isolated singularity at z_0, then the Laurent series of f,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is available in some $D^*(z_0, r)$.

Theorem. (Riemann) If f has an isolated singularity at z_0 and $|f(z)|$ is bounded and analytic on some $D^*(z_0, r)$, then the singularity at z_0 is removable.

Proof. Define h on $D(z_0, r)$ such that

$$h(z_0) = 0 \quad \text{and} \quad h(z) = (z - z_0)^2 f(z)$$

on $D(z_0, r)$. Then,

$$h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)^2 f(z)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0$$

since $f(z)$ is bounded on $D^*(z_0, r)$. Thus h is analytic on $D(z_0, r)$ so that

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

As such, we see that on $D(z_0, r)$,

$$f(z) = \frac{a_0}{(z - z_0)^2} + \frac{a_1}{(z - z_0)} + a_2 + a_3(z - z_0) + a_4(z - z_0)^2 + \dots$$

However, we know that there should be no divergent terms as f is bounded, and $a_0 = a_1 = 0$. Accordingly, we can then define

$$f(z_0) = a_2$$

to create a power series expansion that is convergent on z_0 such that f on the disk $D(z_0, r)$ is analytical. □

Theorem. (Casorati-Weierstrass) If f has an essential singularity at z_0, then for every $r > 0$, $f(D^*(z_0, r))$ is dense in $\boldsymbol{C}$.

Proof. Suppose not, then there is $w_0 \in \boldsymbol{C}$ and $\delta > 0$ such that $f(D^*(z_0, r)) \cap D(w_0, \delta) = \varnothing$. Then $|f(z) - w_0| \geq \delta$ for all $z \in D^*(z_0, r)$. Define

$$g(z) = \frac{1}{f(z) - w_0}$$

on $D^*(z_0, r)$. Then $|g(z)| \leq 1/\delta$ for all $z \in D^*(z_0, r)$. By the previous theorem, we can define $g(z_0)$ so that $g(z)$ is analytic on $D^*(z_0, r)$. Let

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

on $D^*(z_0, r)$. That is,

$$\frac{1}{f(z) - w_0} = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

on $D^*(z_0, r)$. In such a case,

$$\lim_{z \to z_0} \frac{1}{f(z) - w_0} = a_0$$

Suppose, firstly, that $a_0 \neq 0$. Then we have

$$f(z) = \frac{1}{g(z)} + w_0$$

and f has a removable singularity at z_0.

In the second case, let $a_0 = 0$. Take $k \in \boldsymbol{N}$ to be the smallest k integer such that $a_k \neq 0$. Then,

$$\frac{1}{f(z) - w_0} = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \dots$$

at $D^*(z_0, r)$. Thus,

$$\frac{1}{(f(z) - w_0)(z - z_0)^k} = a_k + a_{k+1}(z - z_0) + \dots$$

on $D^*(z_0, r)$. We see that

$$\lim_{z \to z_0} \frac{1}{(f(z) - w_0)(z - z_0)^k} = a_k \neq 0$$

with

$$\lim_{z \to z_0} (f(z) - w_0)(z - z_0)^k = \lim_{z \to z_0} (z - z_0)^k f(z) + \lim_{z \to z_0} w_0(z - z_0)^k = \frac{1}{a_k} \neq 0$$

Therefore, in the second case, f has a pole of order k. $\square$

Theorem. (Picard's Great Theorem) If f has an essential singularity at z_0 then for every $r > 0$, $f(D^*(z_0, r))$ takes every complex number (except possibly one) infinitely many times.

Theorem. If f has a pole of order $k \in \boldsymbol{N}$ at z_0, then $f(z) = \sum_{n=-k}^{\infty} a_n(z - z_0)^n$ on some $D^*(z_0, r)$.

Proof. Since $\lim_{z \to z_0} (z - z_0)^k f(z) = \alpha \neq 0$, $(z - z_0)^k f(z)$ has a removable singularity at z_0 so that

$$(z - z_0)^k f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

on some $D^*(z_0, r)$. Thus

$$f(z) = \sum_{n=-k}^{\infty} c_{n+k}(z - z_0)^n$$

on some $D^*(z_0, r)$. $\square$

Corollary. If f has a simple pole at z_0, then

$$\mathrm{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$$

as

$$\begin{aligned}f(z) =& a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \dots \\ (z-z_0)f(z) =& a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \dots\end{aligned}$$

3 Lecture 15 (May 13th)

Remark. If f has a simple pole at z_0, then

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z)$$

We have previously remarked that L'Hospital's theorem works in the complex plane.

Example. Let $f(z) = \pi\cot\pi z = \pi\frac{\cos\pi z}{\sin\pi z}$. This function has a simple pole at each $n \in \boldsymbol{Z}$.

$$\operatorname*{Res}_{z=n} f(z) = \lim_{z\to n}(z-n)\pi\cot\pi z = \pi\cos n\pi \lim_{z\to n}\frac{z-n}{\sin\pi z} = \pi\cos n\pi \lim_{z\to n}\frac{1}{\pi\cos\pi z} = 1$$

where in the second last line, we have used L'Hospital's theorem.

Example. Observe that

$$\operatorname*{Res}_{z=\pi i}\left(\frac{1}{e^z+1}\right) = \lim_{z\to\pi i}(z-\pi i)\frac{1}{e^z+1} = \lim_{z\to\pi i}\frac{1}{e^z} = -1$$

Theorem. (Residue theorem 1) Let f be analytic inside and on a POSCC except for an isolated singularity at z_0 inside C. Then

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \operatorname*{Res}_{z=z_0} f(z)$$

Proof. Note that due to the Cauchy theorem,

$$\int_C f(z)\,dz = \int_{|z-z_0|=\delta} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z)$$

□

Example. Evaluate

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$$

by using the residue theorem.

Proof. Let the function $f(z) = 1/(1+z^2)$ be on D_R, a positively oriented upper half circle with radius $R > 1$. Then by the residue theorem,

$$\int_{C_R} f(z)\, dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = 2\pi i \lim_{z\to i}(z-i)\frac{1}{1+z^2} = 2\pi i \times \frac{1}{2i} = \pi$$

for all $R > 1$. On the other hand, by parametrization,

$$\int_{C_R} f(z)\, dz = \int_{-R}^{R} \frac{1}{1+x^2}\, dx + \int_0^{\pi} \frac{1}{1+R^2e^{2it}} iRe^{it}\, dt$$

where we substituted $z = Re^{it}$. We then have

$$\Big|\int_0^{\pi} \frac{1}{1+R^2e^{2it}} iRe^{it}\, dt\Big| \leq \int_0^{\pi} \frac{R}{R^2-1}\, dt = \frac{\pi R}{R^2-1}$$

which $\to 0$ as $R \to \infty$. We thus found that the integral is equal to π. □

Example. Consider

$$f(z) = \frac{e^{\alpha z}}{1+e^z}$$

for $0 < \alpha < 1$ on the rectangular contour with base $2R$ and height 2π with its base along the x-axis. Our aim is to evaluate

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x}\, dx$$

Proof.

$$\operatorname*{Res}_{z=\pi i} f(z) = \lim_{z\to\pi i}(z-\pi i)\frac{e^{\alpha z}}{1+e^z} = e^{\alpha\pi i}(-1)$$

Thus by the residue theorem,

$$\int_{C_R} \frac{e^{\alpha z}}{1+e^z}\, dz = 2\pi i(-e^{\alpha\pi i})$$

for all $R > 1$. On the other hand,

$$\int_{C_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_0^{2\pi} f(R+iy)i\, dy + \int_R^{-R} f(x+2\pi i)\, dx + \int_{2\pi}^{0} f(-R+iy)i\, dy$$

or, equivalently,

$$\int_{C_R} f(z)\, dz = \int_{-R}^{R} \frac{e^{\alpha x}}{1+e^x}\, dx + \underbrace{\int_0^{2\pi} \frac{e^{\alpha(R+iy)}}{1+e^{R+iy}} i\, dy}_{\text{II}} - \int_R^{-R} \frac{e^{\alpha(x+2\pi i)}}{1+e^{x+2\pi i}}\, dx - \underbrace{\int_0^{2\pi} \frac{e^{\alpha(-R+iy)}}{1+e^{-R+iy}} i\, dy}_{\text{IV}}$$

Note that

$$|\,\mathrm{II}\,| \le \int_0^{2\pi} \frac{e^{\alpha R}}{e^R - 1}\,dy = \frac{2\pi e^{\alpha R}}{e^R - 1} \to 0$$
$$|\,\mathrm{IV}\,| \le \int_0^{2\pi} \frac{e^{-\alpha R}}{1 - e^{-R}}\,dy = \frac{2\pi e^{-\alpha R}}{1 - e^{-R}} \to 0$$

as $R \to \infty$ because $0 < \alpha < 1$. Accordingly,

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = (1 - e^{2\pi\alpha i}) \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1 + e^x}\,dx = -2\pi i e^{\alpha\pi i}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1 + e^x} = -\frac{2\pi i e^{\alpha\pi i}}{1 - e^{2\pi\alpha i}} = \frac{2\pi i}{e^{\alpha\pi i} - e^{-\alpha\pi i}} = \frac{\pi}{\sin\alpha\pi}$$

□

Remark. Here, substitute $e^x = t$. Then, $dt = e^x\,dx = t\,dx$.

$$\int_0^{\infty} \frac{t^{\alpha-1}}{1+t}\,dt = \frac{\pi}{\sin\alpha\pi}$$

Now, again let $t = x^\beta$ for $0 < \beta < \infty$ to obtain $dt = \beta x^{\beta-1}\,dx$.

$$\int_0^{\infty} \frac{x^{\alpha\beta-1}}{1+x^\beta}\,dx = \frac{1}{\beta}\frac{\pi}{\sin\alpha\pi}$$

Now as $0 < \beta < \infty$, let $\alpha = 1/\beta$, getting

$$\int_0^{\infty} \frac{1}{1+x^\beta}\,dx = \frac{1}{\beta}\frac{\pi}{\sin\pi/\beta}$$

Telling us that

$$\int_0^{\infty} \frac{1}{1+x^\beta}\,dx$$

converges when $\beta > 1$. This example readily shows the beauty of complex integration, with parametrized integrals leaving us with powerful results.

Theorem. (Residue theorem 2) Let f be analytic inside and on a POSCC C except for finite isolated singularities at $z_1, \ldots, z_n$ inside C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Proof. As the singularities are isolated, there is $r > 0$ such that

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{|z-z_k|=r} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

by the Cauchy theorem where $|z - z_r| = k$ is POS (positively oriented and simple). $\square$

Definition. (Zeta function) The zeta function is given as

$$\xi(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

4 Lecture 16 (May 13th)

Example. Consider the Laurent series of $f(z) = \pi \cot \pi z$ at $z = 0$. From

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots \qquad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

we have

$$\begin{aligned} \pi \cot \pi z =& \pi \frac{\cos \pi z}{\sin \pi z} \\ =& \pi \frac{1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \dots}{\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \dots} \\ =& \pi \frac{1}{\pi z} \frac{1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24}}{1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{120}} \end{aligned}$$

After preforming long division, we get

$$\pi \cot \pi z = \frac{1}{z}\Big[1 - \frac{\pi^2 z^2}{3} - \frac{\pi^4 z^4}{45} - \frac{2\pi^6 z^6}{945} + \dots\Big]$$

From this, we see how

$$g(z) = \frac{\pi \cot \pi z}{z^2} \implies \operatorname*{Res}_{z=0} g(z) = -\frac{\pi^2}{3}$$

and

$$h(z) = \frac{\pi \cot \pi z}{z^4} \implies \operatorname*{Res}_{z=0} h(z) = -\frac{\pi^4}{45}$$

We can continue this indefinitely,

$$\operatorname*{Res}_{z=0} \frac{\pi \cot \pi z}{z^6} = -\frac{2\pi^6}{945}$$

The reason why this is important is as follows.

Definition. (Squares lemma) In advanced mathematics, we often see the square contour C_N, with edges at $\pm(N+1/2)i$ and $\pm(N+1/2)$ (taking $N \in \boldsymbol{N}$). There is an upper bound M $(= 2)$ such that

$$|\cot \pi z| \leq M$$

for all $z \in C_N$ and for all $N \in \boldsymbol{N}$.

Proof. Take $z = x + iy$. Consider cutting the square in three parts with $y = 1/2$ and $y = -1/2$. We show that ff $y > 1/2$, then $|\cot \pi z| \leq 2$ and if $y < -1/2$ then $|\cot \pi z| \leq 2$. We also show that on the left and right edges of the middle cut, $|\cot \pi z| \leq 1$. Note that

$$\cot \pi z = \frac{\cos \pi z}{\sin \pi z} = \frac{\dfrac{e^{i\pi z} + e^{-i\pi z}}{2}}{\dfrac{e^{i\pi z} - e^{-i\pi z}}{2i}}$$

and that

$$|\cot \pi z| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right| = \left|\frac{e^{i\pi x}e^{-\pi y} + e^{-i\pi x}e^{\pi y}}{e^{i\pi x}e^{-\pi y} - e^{i\pi x}e^{\pi y}}\right|$$

We see that

$$|e^{i\pi z}| = |e^{i\pi(x+iy)}| = e^{-\pi y} \quad \text{and} \quad |e^{-i\pi z}| = e^{\pi y}$$

assuming $y > 1/2$, we automatically have $e^{\pi y} > e^{-\pi y}$ and that

$$|\cot \pi z| \leq \frac{e^{-\pi y} + e^{\pi y}}{e^{\pi y} - e^{-\pi y}} = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}} < 2$$

Meanwhile, if $y < -1/2$, then

$$|\cot \pi z| \leq \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} - e^{\pi y}} = \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}} < 2$$

Now if $-1/2 \leq y \leq 1/2$ and $z = (N + 1/2) + iy$,

$$|\cot \pi z| = \left|\cot \pi\Big(N + \frac{1}{2} + iy\Big)\right| = \left|\cot\left(\frac{\pi}{2} + i\pi y\right)\right| = |\tan(i\pi y)| = \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right| \leq 1$$

Also, if $-1/2 \leq y \leq 1/2$ and $z = -(N + 1/2) + iy$, the same thing happens and

$$|\cot \pi z| = |\tan(i\pi y)| \leq 1$$

□

Example. Consider $g(z) = \pi \cot \pi z / z^2$ on the square contour C_N. We can use both (1) the residue theorem and (2) parameterisation. By the residue theorem,

$$\begin{aligned}\int_{C_N} \frac{\pi \cot \pi z}{z^2}\, dz =& 2\pi i \sum_{k=-n}^{n} \operatorname*{Res}_{z=k} g(z) \\ =& 2\pi i \Big[\operatorname*{Res}_{z=0} g(z) + 2 \sum_{k=1}^{N} \operatorname*{Res}_{z=k} g(z) \Big] \\ =& 2\pi i \Big[-\frac{\pi^2}{3} + 2 \sum_{k=1}^{N} \frac{1}{k^2} \Big]\end{aligned}$$

where

$$\operatorname*{Res}_{z=k, k\neq 0} g(z) = \lim_{z\to k} (z-k) \frac{\pi \cot \pi z}{z^2} = \frac{1}{k^2}$$

and

$$\operatorname*{Res}_{z=0} g(z) = -\frac{\pi^2}{3}$$

On the other hand,

$$\Big| \int_{C_N} g(z)\, dz \Big| = \Big| \int_{C_N} \frac{\pi \cot \pi z}{z^2} \Big| \leq \frac{M}{N^2}(8N+2) \to 0$$

with M being less than 2π. We thus find

$$2\pi i \Big[2 \sum_{k=1}^{\infty} \frac{1}{k^2} - \frac{\pi^2}{3} \Big] = 0$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Also, we can find

$$\int_{C_N} \frac{\pi \cot \pi z}{z^4}\, dz = 2\pi i \Big[2 \sum_{k=1}^{N} \frac{1}{k^4} - \frac{\pi^4}{45} \Big]$$

which leads to

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

In this manner,

$$\xi(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}}$$

can be found.

Example. Find

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\ldots=\sum_{k=1}^{\infty}\frac{1}{k^2}-2\cdot\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi}{6}-\frac{\pi}{12}=\frac{\pi}{12}$$

and

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots=\sum_{k=1}^{\infty}\frac{1}{(2k)^2}=\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi}{24}$$

Example. Evaluate

$$\sum_{k=-\infty}^{\infty}\frac{1}{(n+a)^2}$$

for $a \notin \boldsymbol{Z}$

Proof. Define, on C_N

$$f(z)=\frac{\pi\cot\pi z}{(z+a)^2}$$

on C_N. Then

$$0=2\pi i\Big[\sum_{n=-\infty}^{\infty}\operatorname*{Res}_{z=n}f(z)+\operatorname*{Res}_{z=-a}f(z)\Big]$$

where

$$\operatorname*{Res}_{z=-a}f(z)=\lim_{z\to -a}[\pi\cot\pi z]'=-\pi^2\csc^2\pi a$$

Noting that $f(z)$ has a pole of order 2, we then have

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n+a)^2}=\Big(\frac{\pi}{\sin\pi a}\Big)^2$$

□

5 Lecture 17 (May 20th)

Remark. If f has a simple pole at z_0, then

$$\operatorname*{Res}_{z=z_0}f(z)=\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to z_0}\frac{p(z)}{(q(z)-q(z_0))/(z-z_0)}=\frac{p(z_0)}{q'(z_0)}$$

if we have $f(z)=p(z)/q(z)$ with $q(z_0)=0, p(z_0)\neq 0$.

Example. For example, take

$$f(z)=\frac{e^{iz}z^3}{z^4+16}$$

in $\operatorname{Im} z \geq 0$. Observe that

$$
\begin{aligned}
z^4 = -16 =& 2^4 e^{\pi i} \\
z =& 2 \exp\left(\frac{\pi i}{4} + \frac{2k\pi i}{4}\right)
\end{aligned}
$$

for $k = 0, 1, 2, 3, \ldots$. Then

$$
z_1 = 2\exp\left(\frac{\pi i}{4}\right) = \sqrt{2} + i\sqrt{2} \quad z_2 = 2\exp\left(\frac{3\pi i}{4}\right) = -\sqrt{2} + i\sqrt{2}
$$

in $\operatorname{Im} z \geq 0$. f therefore has simple poles at z_1, z_2 in the domain, and

$$
\operatorname*{Res}_{z=z_1} f(z) = \frac{e^{iz_1} z_1^3}{4z_1^3} = \frac{1}{4}e^{iz_1} = \frac{1}{4}e^{-\sqrt{2}+i\sqrt{2}}
$$

or

$$
\operatorname*{Res}_{z=z_2} f(z) = \frac{e^{iz_2} z_2^3}{4z_2^3} = \frac{1}{4}e^{iz_2} = \frac{1}{4}e^{-\sqrt{2}-i\sqrt{2}}
$$

Theorem. In complex analysis, we often use upper hemispheres and shells. There are two important theorems regarding the contour integrals at the rim C_R.

(i) Take $f(z) = \dfrac{p(z)}{q(z)}$ ($q \neq 0$ on C_R) with $\deg q(z) \geq$ p(z) +2. Then

$$
\lim_{R\to\infty} \int_{C_R} \frac{p(z)}{q(z)}\, dz = 0
$$

(ii) The Jordan lemma, which stems from the fact that

$$
\Big| \int_{C_R} e^{iz}\, dz \Big| < \pi
$$

for all $R > 0$. Or, more generally,

$$
\Big| \int_{C_R} e^{iaz}\, dz \Big| < \frac{\pi}{a}
$$

for $a > 0$.

Proof.

$$\int_{C_R} e^{iz}\,dz = \int_0^\pi e^{iRe^{it}} iRe^{it}\,dt$$

$$\Big|\int_{C_R} e^{iz}\,dz\Big| \leq \int_0^\pi \left|e^{iRe^{it}} iRe^{it}\right|\,dt$$

$$=R\int_0^\pi e^{-R\sin t}\,dt = 2R\int_0^{\pi/2} e^{-R\sin t}\,dt \leq \pi(1-e^{-R})$$

Noting that

$$\sin t \geq \frac{2}{\pi}t$$

for $0 \leq t \leq \pi/2$. □

Theorem. (Jordan lemma) If $|f(z)| \leq M_R$ for $z \in C_R$ and $\lim_{R\to\infty} M_R = 0$, then

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}\,dz = 0$$

for $a > 0$ since

$$\Big|\int_{C_R} f(z)e^{iaz}\,dz\Big| \leq M_R R\int_0^\pi e^{-aR\sin t}\,dt \leq \frac{M_R\pi}{a} \to 0$$

Example. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

Let

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

defined on the upper hemisphere with radius R (whole contour C_R and $R > 2$). By the residue theorem,

$$\int_{C_R} f(z)\,dz = 2\pi i\Big(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=2i} f(z)\Big)$$

We find

$$\operatorname*{Res}_{z=i} f(z) = \operatorname*{Res}_{z=i} \frac{\frac{z^2}{z^2+4}}{z^2+1} = -\frac{1}{6i}$$

and

$$\operatorname*{Res}_{z=2i} f(z) = \operatorname*{Res}_{z=2i} \frac{\frac{z^2}{z^2+1}}{z^2+4} = \frac{1}{3i}$$

Meanwhile,

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = 0$$

since $\deg(z^2+1)(z^2+4) = \deg z^2 + 2$, and the above becomes

$$\int_{C_R} f(z)\,dz = 2\pi i\Big[-\frac{1}{6i} + \frac{1}{3i}\Big] = \frac{\pi}{3}$$

Example. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{x^4+16}\,dx$$

Define

$$f(z) = \frac{e^{iz}z^3}{z^4+16} \quad \text{while} \quad \operatorname{Im} f(z) = \frac{x^3 \sin x}{x^4+16}$$

on the upper hemisphere C_R with $R > 2$. Then,

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{1}{4}e^{-\sqrt{2}+i\sqrt{2}} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{1}{4}e^{-\sqrt{2}-i\sqrt{2}}$$

Finishing off,

$$\begin{aligned}\int_{C_R} f(z)\,dz = 2\pi i\Big[\frac{1}{4}e^{-\sqrt{2}+i\sqrt{2}} + \frac{1}{4}e^{-\sqrt{2}+i\sqrt{2}}\Big] \\ &= \frac{\pi i}{2}e^{-\sqrt{2}}\Big(e^{i\sqrt{2}} + e^{-i\sqrt{2}}\Big) \\ &= \pi i e^{-\sqrt{2}} \cos\sqrt{2}\end{aligned}$$

Meanwhile,

$$\begin{aligned}\lim_{R\to\infty}\int_{C_R} f(z)\,dz &= \int_{-\infty}^{\infty} \frac{e^{ix}x^3}{x^4+16}\,dx + \lim_{R\to\infty}\int_0^{\pi} \cdot\,dz \\ &= \int_{-\infty}^{\infty} \frac{x^3\cos x}{x^4+16}\,dx + i\int_{-\infty}^{\infty}\frac{x^3 \sin x}{x^4+16}\,dx\end{aligned}$$

where the term on the first line vanishes due to Jordan's lemma.

Example. Evaluate, for $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx$$

Define

$$f(z) = \frac{e^{iz}}{z^2+a^2}$$

on C_R with $R > a$. By the residue theorem,

$$\int_{C_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=ai} f(z) = 2\pi i\frac{e^{-a}}{2ai} = \pi\frac{e^{-a}}{a}$$

On the other hand,

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = \int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+a^2}\,dx + \lim_{R\to\infty}\int_0^{\pi}\cdot\,dx = \int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,dx + i\int_{-\infty}^{\infty}\frac{\sin x}{x^2+a^2}\,dx$$

where the second term goes to zero by Jordan's lemma. So,

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,dx = \pi\frac{e^{-a}}{a}$$

Now, put $x = \beta t$ for $\beta > 0$, to obtain

$$\int_{-\infty}^{\infty}\frac{\cos\beta t}{\beta^2t^2+a^2}\beta\,dt = \frac{\pi e^{-a}}{a}$$

We succeedingly put $\alpha = a/\beta$ and we get

$$\int_{-\infty}^{\infty}\frac{\cos\beta t}{t^2+\alpha^2}\,dt = \frac{\pi}{\alpha}e^{-\alpha\beta}$$

for $\alpha, \beta > 0$. Lastly, differentiate the function with respect to t.

$$\int_{-\infty}^{\infty}\frac{-\beta\sin\beta t}{t^2+\alpha^2}\,dt = -\pi e^{-\alpha\beta}$$

For $\beta = 1$,

$$\int_{-\infty}^{\infty}\frac{t\sin t}{t^2+\alpha^2}\,dt = \pi e^{-\alpha}$$

with $\alpha > 0$. Taking $\alpha \to 0^+$,

$$\int_{-\infty}^{\infty}\frac{\sin t}{t}\,dt = \pi$$

Similarly, we can differentiate the above with respect to α instead of t to get

$$\int_{-\infty}^{\infty}\frac{x\sin x}{(x^2+a^2)^2}\,dx$$

Example. Find

$$\int_0^{\infty}\frac{\sin x}{x}\,dx$$

Define

$$f(z) = \frac{e^{iz}}{z}$$

on the shell $C_{\varepsilon,R}$ ($0 < \varepsilon < 1$ and $R > 1$) for which it is analytic. We find

$$\int_{C_{\varepsilon,R}} f(z)\,dz = \int_{-R}^{R}\frac{e^{ix}}{x}\,dx + \int_0^{\pi} f(Re^{it})iRe^{it}\,dt + \int_{-R}^{-\varepsilon}\frac{e^{ix}}{x}\,dx - \int_0^{\pi}\frac{e^{i\varepsilon e^{it}}}{\varepsilon e^{it}}i\varepsilon e^{it}\,dt$$

the second term vanishes due to the Jordan lemma while the last term becomes

$$-\int_0^\pi ie^{i\varepsilon e^{it}}\,dt \to -\pi i$$

as $\varepsilon \to 0$.

6 Lecture 18 (May 22nd)

Example. Define the function

$$f(z) = \frac{e^{iz}}{z}$$

on the upper shell with inner radius ε and outer radius R.

$$\int_{C_R} f(z)\,dz = \int_\varepsilon^R \frac{e^{ix}}{x}\,dx + \int_0^{2\pi} \frac{e^{iRe^{it}}}{Re^{it}} iRe^{it}\,dt + \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx - \int_0^\pi \frac{e^{i\varepsilon e^{it}}}{\varepsilon e^{it}} i\varepsilon e^{it}\,dt$$

Taking the imaginary part, we have

$$\int_0^\infty \frac{e^{ix} - e^{-ix}}{x}\,dx = \pi i$$

as $\varepsilon \to 0$ and $R \to \infty$. We therefore have, by definition,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Example. Find

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

Notice that $2\sin^2 x = 1 - \cos 2x = \mathrm{Re}(1 - e^{2ix})$. Then, we can define

$$f(z) = \frac{1 - e^{2iz}}{z^2}$$

on the same shell as above. For every $0 < \varepsilon < 1$ and $R > 1$ we have

$$0 = \int_{C_{\varepsilon,R}} \frac{1 - e^{2iz}}{z^2}\,dz = \int_\varepsilon^R \frac{1 - e^{2ix}}{x^2}\,dx + \int_0^\pi \frac{1 - e^{2iRe^{it}}}{(Re^{it})^2} iRe^{it}\,dt + \int_{-R}^{-\varepsilon} \frac{1 - e^{2ix}}{x^2}\,dx - 2\pi$$

We see that the second term vanishes to 0 as $R \to \infty$. Also, we used that

$$f(z) = \frac{1}{z^2}[-2iz + 2z^2 + \ldots] = -\frac{2i}{z} + 2 + \ldots$$

and that

$$\lim_{\varepsilon\to 0} \int_0^\pi f(\varepsilon e^{it}) i\varepsilon e^{it}\,dt = \lim_{\varepsilon\to 0} \int_0^\pi \frac{-2i}{\varepsilon e^{it}} i\varepsilon e^{it}\,dt = 2\pi$$

Taking the real part, we have

$$\int_0^\infty \frac{1-e^{2ix}}{x^2}\,dx + \int_0^\infty \frac{1-e^{-2ix}}{x}\,dx = 2\pi$$

and

$$2\int_0^\infty \frac{\sin^2 x}{x^2}\,dx + 2\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = 2\pi \quad \text{or} \quad \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

Example. Find

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx$$

On the branch $-\pi/2 < \arg z < 3\pi/2$, $\ln z$ is analytic. Define

$$f(z) = \frac{\log z}{(1+z^2)^2}$$

which has a pole of order 2 at $z = i$ inside $C_{\varepsilon,R}$. Recall, for a pole of order k,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(k-1)!}\lim_{z\to z_0}\Big[(z-z_0)^k f(z)\Big]^{(k-1)}$$

and we have

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\leftarrow i}\left[(z-i)^2\frac{\log z}{(1+z^2)^2}\right] = \lim_{z\to i}\left[\frac{\log z}{(z+i)^2}\right]' = \lim_{z\to i}\left[\frac{\frac{1}{z}(z+i)^2 - 2(z+i)\log z}{(z+i)^4}\right] = \frac{\pi+2i}{8}$$

Which implies that

$$\int_{C_{\varepsilon,R}} \frac{\log z}{(1+z^2)^2}\,dz = 2\pi i\frac{\pi+2i}{8}$$

for all $0 < \varepsilon < 1$ and $R > 1$.

$$|\,\mathrm{II}\,| = \left|\int_0^\pi \frac{\log Re^{it}}{(1+R^2e^{2it})^2} iRe^{it}\,dt\right| \le \int_0^\pi \frac{\log R + \pi}{(R^2-1)^2} R\,dt \to 0$$

as $R \to \infty$.

$$|\,\mathrm{IV}\,| = \left|-\int_0^\pi \frac{\log \varepsilon e^{it}}{(1+\varepsilon^2 e^{2it})^2} i\varepsilon e^{it}\,dt\right| \le \int_0^\pi \frac{\varepsilon[\log\varepsilon + \pi]}{(1-\varepsilon^2)^2}\varepsilon\,dt$$

While $\lim_{\varepsilon\to 0^+} \varepsilon \ln \varepsilon = 0$ as

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

Also,

$$|\,I\,|+|\,III\,| = \int_\varepsilon^R \frac{\ln x}{(1+x^2)^2}\,dx + i\int_\varepsilon^R \frac{\pi}{(1+x^2)^2}\,dx + \int_\varepsilon^R \frac{\ln x}{(1+x^2)^2} = 2\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx + i\int_0^\infty \frac{\pi}{(1+x^2)^2}\,dx$$

which implies that

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx = -\frac{\pi}{4}$$

7 Lecture 19 (May 27th)

Example. (Reverse parametrisation) Solve

$$\int_0^{2\pi} \frac{1}{5+4\sin t}\,dt$$

using the following equality

$$\int_0^{2\pi} f(e^{it}) i e^{it}\,dt = \int_{|z|=1} f(z)\,dz$$

Proof.

$$\begin{aligned}
\int_0^{2\pi} \frac{dt}{5+4\sin t} &= \int_{|z|=1} \frac{1}{5+4\dfrac{z-z^{-1}}{2i}} \frac{1}{iz}\,dz \\
&= \int_{|z|=1} \frac{1}{5iz+2z^2-1}\,dz \\
&= 2\pi i \operatorname*{Res}_{z=-i/2} \frac{1}{2z^2+5iz-2} = 2\pi i \Big[\frac{1}{4z+5i}\Big]_{z=-i/2} \\
&= 2\pi i \frac{1}{3i} = \frac{2\pi}{3}
\end{aligned}$$

Where utilizing the quadratic equation, the roots of the denominator are $(-5i \pm 3i)/4$. □

Example. If $-1 < a < 1$, then

$$\int_0^{2\pi} \frac{1}{1+a\sin t}\,dt = \frac{2\pi}{\sqrt{1-a^2}} = \int_0^{2\pi} \frac{1}{1+a\cos t}\,dt$$

Example. For $n \in \boldsymbol{N}$, evaluate

$$\int_0^{2\pi} \sin^{2n} t\,dt$$

Let $z = e^{it}$ for $0 \le t \le 2\pi$. Then, $\sin t = z - z^{-1}/2i$ and $dt = dz/iz$ so that

$$\int_0^{2\pi} \sin^{2n} t\,dt = \int_{|z|=1} \Big(\frac{z-1/z}{2i}\Big)^{2n} \frac{dz}{iz} = 2\pi i \operatorname*{Res}_{z=0} f(z)$$

where

$$f(z) = \frac{1}{(2i)^{2n}} \frac{1}{iz} (z - \frac{1}{z})^{2n}$$

Here,

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(2i)^{2n}} \frac{1}{i} {}^{2n}C_n (-1)^n = \frac{(2n)!}{(n!)^2 2^{2n}} \frac{1}{i}$$

The result of the integral is therefore

$$2\pi \frac{(2n)!}{(n!)^2 2^{2n}}$$

Definition. (Fourier transform) With $(\hat{f})^{\vee} = f$, the Fourier transform is defined as

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i x t}\, dt \quad \text{and} \quad f^{\vee}(t) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x t}\, dx$$

if they exist.

Example. For $f(x) = e^{-\pi x^2}$, find the Fourier transform.

$$\widehat{f}(x) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-2\pi i x t}\, dt$$

Notice that

$$e^{-\pi t^2} e^{-2\pi x t} = e^{-\pi(t^2 + 2ixt)} = e^{-\pi(t+ixt)^2 - \pi x^2}$$

Therefore,

$$\widehat{f}(x) = e^{-\pi x^2} \int_{-\infty}^{\infty} e^{-\pi(t+ix)^2}\, dt$$

For $x \neq 0$, in finding the integral, we can take the square contour with height x and width $2R$ for the following function:

$$0 = \int_{C_R} e^{-\pi z^2}\, dz = \int_{-R}^{R} e^{-\pi t^2}\, dt + \int_{0}^{x} e^{-\pi(R+iy)^2} i\, dy - \int_{-R}^{R} e^{-\pi(t+ix)^2}\, dx + \int_{x}^{0} e^{-\pi(-R+iy)^2} i\, dy$$

We can see that the second and last term approaches 0 as $R \to \infty$.

$$\int_{-\infty}^{\infty} e^{-\pi(t+ix)^2}\, dt = \int_{-\infty}^{\infty} e^{-\pi t^2}\, dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2}\, ds = 1$$

taking $s = \sqrt{\pi} t$. We see that the Fourier transform of the function is itself.

Remark. For the remaining time being, we study (1) the argument principle, which uses Rouches theorem, and (2) the Poisson integral.

Theorem. Let f be analytic inside and on a POSCC C with $f \neq 0$ on C. Then f has finitely many zeros inside C. Then there exists $z_1, \ldots, z_n$ inside C and $\alpha_k \in \boldsymbol{N}$ $(1 \leq k \leq n)$

such that

$$f(z) = (z - z_1)^{\alpha} \dots (z - z_n)^{\alpha_n} F(z)$$

where F is analytic with no zeros inside and on C. Then on C,

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{\alpha_k}{z - z_k} + \frac{F'(z)}{F(z)}$$

Thus

$$\int_C \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{n} \int_C \frac{\alpha_k}{z - z_k}\, dz + \int_C \frac{F'(z)}{F(z)}\, dz$$

Proof. If $f(z_0) = 0$ for some z_0 inside C, then $f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \dots$ in a neighborhood of z_0. Therefore,

$$f(z) = (z - z_0)\Big(a_1 + a_2(z - z_0) + \dots\Big)$$

such that $g(z) = f(z)/(z - z_0)$ has a removable singularity at z_0. As g is analytic inside and on C, $f(z) = (z - z_0)g(z)$ for some analytic function g inside and on C. □

8 Lecture 20 (May 29th)

Theorem. (Argument principle) The number of times $f(C)$ winds up the origin in the positive sense is

$$\frac{1}{2\pi} \Delta_C f(z)$$

which in turn equal to the number of zeros inside C.

Proof. Let f be analytic inside and on a POSCC C with no zeros on C. If f has zeros at $z_1, z_2, \dots, z_n$ inside C with multiplicity $\alpha_1, \alpha_2, \dots, \alpha_n$ respectively, then

$$f(z) = (z - z_1)^{\alpha_1} \dots (z - z_n)^{\alpha_n} F(z)$$

with F being analytic with no zeros inside and on C. Thus on C,

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{n} \frac{\alpha_k}{z - z_k} + \frac{F'(z)}{F(z)}$$

Where $F'(z)/F(z)$ is analytic inside and on C. Thus

$$\int_C \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{n} \int_C \frac{\alpha_k}{z - z_k}\, dz + \int_C \frac{F'(z)}{F(z)}\, dz = 2\pi i \sum_{k=1}^{n} \alpha_k$$

Thus

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$$

is the number of zeros of f inside C. On the other hand, just on C,

$$\int_C \frac{f'(z)}{f(z)}\,dz = \Big[\log f(z)\Big]_C = \Big[\ln|f(z)| + i\arg f(z)\Big]_C = i\varDelta_C \arg f(z)$$

where $\varDelta_C \arg f(z)$ is the change of $\arg f(z)$ as z tranverses C. This implies that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\varDelta_C \arg f(z)$$

The RHS is the number of times $f(C)$ winds up the origin in the positive sense. □

Theorem. (Rouche's theorem) Let f and g be analytic inside and on a POSCC C. If $|f(z)| > |g(z)|$ for every $z \in C$, then $f+g$ and f have the same number of zeros (counting multiplicities) inside C.

Proof. On C,

$$f(z) + g(z) = f(z)\Big[1 + \frac{g(z)}{f(z)}\Big]$$

so that

$$\varDelta_C \arg(f(z)+g(z)) = \varDelta_C \arg f(z) + \varDelta_C \arg\Big(1 + \frac{g(z)}{f(z)}\Big)$$

As the modulus of $g(z)/f(z) < 1$, the second term vanishes, and the arguments are equal and the number of zeros are equal also. □

Recall. (Fundamental theorem of algebra) Let $P_n(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0 = a_n z^n + P_{n-1}(z)$ where $a_n \neq 0$. Assume $r \geq 1$. If $|z| = r$, then

$$\Big|\frac{P_{n-1}(z)}{a_n z^n}\Big| = \frac{|P_{n-1}(z)|}{|a_n|r^n} \leq \frac{(|a_0| + |a_1| + \ldots + |a_{n-1}|)r^{n-1}}{|a_n|r^n} = \frac{|a_0| + |a_1| + \ldots + |a_{n-1}|}{|a_n|r} < 1$$

if

$$r > \frac{|a_0| + \ldots + |a_{n-1}|}{|a_n|}$$

We can now apply Rouche's theorem for $f(z) = a_n z^n$, $g(z) = P_{n-1}(z)$ on $C : |z| = r$, and $r > (|a_0| + \ldots)/|a_n|$. We see that $P_n(z)$ has n zeros inside $|z| = r$.

Remark. Consider the function $f(x) = x^2$. Notice how $f((-1,1)) = [0,1)$, and how an open set maps to a not open set.

Theorem. (Open mapping theorem) Suppose f is a non-constant analytic function in a domain D. Then, $f(D)$ is open.

Proof. Take any $w_0 \in f(D)$. We have to find $m > 0$ such that $D(w_0, m) \subset f(D)$. By defintion of w_0, there is $z_0 \in D$ such that $f(z_0) = w_0$. Since D is open, there is $\delta > 0$ such that

(i) $\bar{D}(z_0, \delta) \subset D$

(ii) $f(z) - w_0$ has no zeros in $0 < |z - z_0| \leq \delta$ by the identity theorem

Let $m = \min_{|z-z_0|=\delta} |f(z) - w_0| > 0$. Now we'll show $D(w_0, m) \subset f(D)$. Take any $w \in D(w_0, m)$. Then, $|w_0 - w| < m \leq |f(z) - w_0|$ on the contour $|z - z_0| = \delta$. By Rouche's theorem, $f(z) - w_0$ and

$$(f(z) - w_0) + (w_0 - w) = f(z) - w$$

has the same number of zeros inside $|z - z_0| = \delta$. Since $(f(z) - w_0)$ has a zero in $|z - z_0| \leq \delta$ (at z_0), $(f(z) - w)$ has a zero inside $|z - z_0| = \delta \subset D$. In otherwords, $(f(z) - w)$ has a zero in D and $w \in f(D)$. $\square$

Example. Show that all roots of $z^5 + 6z^3 + 2z + 10 = 0$ lies in $1 < |z| < 5$.

Proof. (i) On $|z| = 1$, let $f(z) = 10$, $g(z) = z^5 + 6z^3 + 2z$. $10 > |g(z)| \leq 1 + 6 + 2 \leq 9$. This implies that $f + g$ has the same numbers of zeros as 10 inside $|z| = 1$. That is, it has no zeros on $|z| = 1$.

(ii) On $|z| = 5$, let $f(z) = z^5$, $g(z) = 6z^3 + 2z + 10$. $|g(z)| \leq 6 \cdot 5^3 + 2 \cdot 5 + 10 < 5^5 = |f(z)|$. f and $f + g$ have the same number of zeros inside $|z| = 5$.

$\square$

9 Lecture 21 (June 5th)

Recall. A C^2 function $u : D \to \boldsymbol{R}$ is harmonic if $\Delta u = u_{xx} + u_{yy} = 0$ on D.

Recall. (Cauchy) If f is analytic in a simply connected domain D then

$$\int_C f(z)\, dz = 0$$

for every closed contour C in D. This is due to the fact that for every fixed $z_0 \in D$,

$$F(z) = \int_{z_0}^{z} f(\xi)\, d\xi$$

is well-defined and is independent of path.

Theorem. If f is analytic in a simply connected domain D, then $f = F'$ for some F that is analytic in D.

Proof. Take any $z_0 \in D$, and we'll show that $F(z) = \int_{z_0}^{z} f(\xi)\, d\xi$ satisfies $F' = f$ on D. Take any $w \in D$. We'll show for each $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < |h| < \delta$ then

$$\Big|\frac{F(w+h) - F(w)}{h} - f(w)\Big| < \varepsilon$$

Since D is open, there is $\delta_1 > 0$ such that $\bar{D}(w, \delta_1) \subset D$. Since f is continuous at w, there is $\delta_2 > 0$ such that if $|\xi - w| < \delta_2$ then $|f(\xi) - f(w)| < \varepsilon$. Take $\delta = \min(\delta_1, \delta_2)$. Then we'll complete the proof. If $0 < |h| < \delta$ then

$$\frac{F(w+h) - F(w)}{h} = \frac{1}{h}\Big[\int_{z_0}^{w+h} f(\xi)\, d\xi - \int_{z_0}^{w} f(\xi)\, d\xi\Big]$$

so that if $0 < |h| < \delta$, then

$$\Big|\frac{F(w+h) - F(w)}{h} - f(w)\Big| = \Big|\frac{1}{h}\int_{w}^{w+h} (f(\xi) - f(w))\, d\xi\Big| < \varepsilon$$

As the path can be made a straight line, we can reduce the right handside smaller than the length of the path and the integeral can be reduced to be smaller than ε. $\square$

Theorem. If u is harmonic in a simply connected domain D, then there is a function f that is analytic in D such that $u = \mathrm{Re}(f)$ on D.

Proof. Define $g = u_x - iu_y$ in D, and g is analytic on D since it satisfies the Cauchy-Riemann equation. Since D is simply connected, there is F such that $F' = g$ on D. Suppose that $F = U + iV$, and $F' = U_x + iV_x = U_x - iV_y = g = u_x + iu_y$ due to the Cauchy Riemann equation. Thus $U_x = u_x$ and $U_y = u_y$ in D. We see that $U = u + c$ for some $c \in \boldsymbol{R}$. Hence $u = \mathrm{Re}[F - c]$. $\square$

Example. Let $u(x, y) = \ln\sqrt{x^2 + y^2} = \ln|z|$. Then $f(x) = \log z$ is analytic in $\boldsymbol{C} \setminus (-\infty, 0]$. Additionally, $\mathrm{Re}\, f(z) = \ln|z| = u$. However, there is no g analytic in $0 < |z| < 1$ such that $u = \mathrm{Re}(g)$.

Remark. If u and v are harmonic in D, then for every $a, b \in \boldsymbol{R}$, $au + bv$ is harmonic.

Example. If u is non-constant harmonic in a domain D and $v = u^2$, $v_x = 2uu_x$ such that $v_{xx} = 2u_x^2 + 2uu_{xx}$. Therefore,

$$\Delta v = v_{xx} + v_{yy} = 2u(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2) = 2(u_x^2 + u_y^2) > 0$$

We therefore find u^2 to be never harmonic.

Remark. If f is analytic and u is harmonic, then $u \circ f$ is harmonic. The key point is that u is locally the real part of an analytic function such that $u \circ f = \mathrm{Re}(g \circ f)$.

Theorem. If u is bounded and harmonic in $\boldsymbol{C}$, then u is constant.

Proof. Since $\boldsymbol{C}$ is simply connected, there exists an entire function f such that $u = \mathrm{Re}(f)$. Then, $|\exp f(z)| = \exp u(z)$ and $\exp f(z)$ is a bounded entire function. We therefore find that $\exp f(z)$ is constant and that $f(z)$ is constant. $\square$

Theorem. If u is a positive harmonic function on $\boldsymbol{C}$, then u is constant.

Proof. There is an entire function f such that $u = \mathrm{Re}(f)$. Thus $|\exp f(z)| = \exp(u) \geq 1$. We find $g(z) = 1/\exp f(z)$ to be a bounded entire function which implies that g is constant. Then, f is constant and u is constant. $\square$

Theorem. If u is harmonic in a domain D, then u can not take a maximum in D.

Proof. Suppose that you take a maximum at $z_0 = x_0 + iy_0$ in D. Then, there is $r > 0$ such that $\bar{D}(z_0, r) \subset D$. Since $D(z_0, r)$ is simply connected, there is f that is analytic in $D(z_0, r)$ such that $u = \mathrm{Re}\, f$ on $D(z_0, r)$. Then $|\exp f(z_0)| = \exp u(z_0) =$ is a maximum in $D(z_0, r)$. Therefore, $\exp(f)$ is constant. $\square$

Theorem. (Mean value property of harmonic functions) If f is analytic in D such that $\bar{D}(z_0, R) \subset D$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

for every $0 < r \leq R$. If u is harmonic in a domain containing $\bar{D}(a, R)$, then $u = \mathrm{Re}(f)$ in $\bar{D}(z_0, R)$ resulting in

$$u(z_0) = \mathrm{Re}\, f(z_0) = \mathrm{Re}\left[\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt\right] = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt$$

for $r \leq R$. We then have, by extension,

$$\begin{aligned}
u(z_0) =& \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt \\
\int_0^R u(z_0) r\, dr =& \frac{1}{2\pi} \int_0^{2\pi} \int_0^R u(z_0 + re^{it}) r\, dr dt \\
\frac{R^2}{2} u(z_0) =& \frac{1}{2\pi} \iint_{D(z_0,R)} u(x, y)\, dA \\
u(z_0) =& \frac{1}{\pi R^2} \iint_{D(z_0,R)} u\, dA
\end{aligned}$$

and the mean value theorem works on a disk also.

Theorem. If u is bounded harmonic in $\boldsymbol{C}$, then u is constant.

Proof. Suppose that $|u(z)| \leq M$. Then, for each $z_0 \in \boldsymbol{C}$,

$$\begin{aligned} u(z_0) - u(0) =& \frac{1}{\pi R^2} \iint_{D(z_0,R)} u\, dA - \frac{1}{\pi R^2} \iint_{D(0,R)} u\, dA \\ \leq& \frac{M}{\pi R^2} (\text{symmetric difference between two discs}) \end{aligned}$$

$\square$

10 Lecture 22 (June 10th)

Remark. We use the following notation.

(i) $U = \{|z| < 1\}$ for a unit disc

(ii) $T = \{|z| = 1\}$ for a unit circle $z = re^{it}$ with $0 \leq t \leq 2\pi$

(iii) $\bar{U} = \{|z| \leq 1\}$ for a closed unit disc

Lemma. If f is analytic on $\bar{U}$, then for each $z \in U$ we have

$$f(z) = \frac{1}{2\pi i} \int_T \frac{f(\xi)}{\xi - z}\, d\xi$$

and

$$\frac{1}{2\pi i} \int_T \frac{f(\xi)\bar{z}}{1 - \bar{z}\xi}\, d\xi = 0$$

Parametrising the function $f(z)$,

$$f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{it})}{e^{it} - z} ie^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - ze^{-it}}\, dt$$

We also see that

$$0 = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{it})\bar{z}}{1 - \bar{z}e^{it}} ie^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})e^{it}\bar{z}}{1 - \bar{z}e^{it}}\, dt$$

That is,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \Big[\frac{1}{1 - ze^{-it}} + \frac{e^{it}\bar{z}}{1 - \bar{z}e^{it}} \Big]\, dt$$

Here,

$$\frac{1}{1 - ze^{-it}} + \frac{e^{it}}{1 - \bar{z}e^{it}} = \frac{1 - \bar{z}e^{it} + e^{it}\bar{z} - |z|^2}{|1 - ze^{-it}|^2} = \frac{1 - |z|^2}{|1 - ze^{-it}|^2} = \frac{1 - |z|^2}{|e^{it} - z|^2}$$

and we have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt$$

Theorem. If u is harmonic in $\bar{U}$, then for every $z \in U$ we have

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt$$

This tells us that any harmonic function can be expressed in terms of itself on a local disk.

Proof. If u is harmonic in a neighborhood of $\bar{U}$, then there is an analytic f in the neighborhood of U such that $u = \mathrm{Re}(f)$. Thus, for every $z \in U$,

$$\begin{aligned} u(z) = &\mathrm{Re}(f) = \mathrm{Re}\left[\frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt\right] \\ =&\frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left[f(e^{it}) \frac{1 - |z|^2}{|e^{it} - z|^2}\right] dt \\ =&\frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt \end{aligned}$$

□

Definition. For $h \in C(T)$, we define the Poisson integral $p[h]$ of h on U by

$$p[h](z) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{it}) \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt = \int_T h(\xi) \frac{1 - |z|^2}{|\xi - z|^2} \, d\sigma(\xi)$$

where $\sigma(T) = 1$ is a measure. Equivalently,

$$p[h](re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{it}) P_r(\theta - t) \, dt$$

where $P_r(\theta - t)$ is the Poisson kernel. Phrased differently, the above theorem states that $u = p[u]$ if it is harmonic.

Definition. Let $w = re^{it}$. Define

$$P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} = 1 + \sum_{n=1}^{\infty} r^n (e^{int} + e^{-int})$$

Note how

$$\frac{1 + w}{1 - w} = (1 + w)(1 + w + w^2 + \ldots) = 1 + 2 \sum_{n=1}^{\infty} w^n = 1 + 2 \sum_{n=1}^{\infty} r^n e^{int}$$

Thus

$$\mathrm{Re}\left(\frac{1+w}{1-w}\right) = 1 + 2\sum_{n=1}^{\infty} r^n \cos nt = P_r(t)$$

where $(1+w)/(1-w)$ is harmonic in w. We see that the ugly expression that we first used can actually be expressed as

$$P_r(t) = \mathrm{Re}\left(\frac{1+re^{it}}{1-re^{it}}\right) = \frac{1-r^2}{|1-re^{it}|^2} = \frac{1-|w|^2}{|1-w|^2}$$

We see exactly that

$$P_r(\theta - t) = \frac{1-|w|^2}{|1-re^{it(\theta-t)}|^2} = \frac{1-r^2}{|e^{it}-re^{i\theta}|^2} = \frac{1-|z|^2}{|e^{it}-z|^2}$$

where $z = re^{i\theta}$. In sum,

$$P_r(\theta - t) = \frac{1-|z|^2}{|e^{it}-z|^2}$$

Note that we can also write the Poisson kernal as

$$\frac{1-r^2}{|1-re^{it}|^2} = \frac{1-r^2}{1-2r\cos t + r^2}$$

Theorem. For $0 \le r < 1$ and $t \in \boldsymbol{R}$,

$$P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} = \mathrm{Re}\left(\frac{1+re^{it}}{1-re^{it}}\right) = \frac{1-r^2}{1-2r\cos t + r^2}$$

which implies that

$$P_r(\theta - t) = \frac{1-r^2}{1-2r\cos(\theta - t) + r^2}$$

The Poisson integral of $h \in C(T)$ is defined as

$$p[h](re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it}) P_r(\theta - t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it}) \frac{1-r^2}{1-2r\cos(\theta - t) + r^2}\, dt$$

Theorem. If $h : T \to \boldsymbol{R}$ is continuous, then $p[h]$ is harmonic in U.

Proof. Recall that $P_r(t) = \mathrm{Re}(1+re^{it})/(1-re^{it})$ so that

$$P_r(\theta - t) = \mathrm{Re}\left(\frac{1+re^{i(\theta-t)}}{1-re^{i(\theta-t)}}\right) = \mathrm{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right) = \mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)$$

which is harmonic in z. Then

$$p[h](z) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})P_r(\theta - t)\,dt = \mathrm{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} h(e^{it})\frac{e^{it}+z}{e^{it}-z}\,dt\right]$$

$\square$