1 Lecture 1 (March 4th)

Definition. A complex number is expressed as $z = x + iy$ with $x, y \in \boldsymbol{R}$ with $i^2 = -1$. Here,

$$\begin{cases} x = \text{Re } z \text{ (real part)} \\ y = \text{Im } z \text{ (imaginary part)} \end{cases}$$

The cornerstone of imaginary numbers is that with the introduction of i, all operations become closed. We use the letter $\boldsymbol{C}$ to denote the set of complex numbers which is a complete field. Topologically, $\boldsymbol{C}$ and $\boldsymbol{R}^2$ have identical metrics and are topologically equivalent. In this way, continuity in both sets are defined identically. However, $\boldsymbol{C}$ has well defined multiplication and division which are very useful. Accordingly, the definition of differentiability is radically different.

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

The concept of differentiability for functions on complex numbers are defined identically as the functions on real numbers. However, there is a critical complication regarding "path".

All of differentiation of complex functions stems from the "Cauchy-Riemann equations". Once this strong condition is satisfied by functions, all miracles of complex numbers come from the theorem called the "Cauchy integral formula".

Remark. The most important function in all of mathematics is the complex function

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

for $z \in \boldsymbol{C}$. Given $z = x + iy$, we define this function using the following property

$$e^z = e^x e^{iy}$$

The equation used for the definition of e^z is a very special definition indeed which we will investigate further later on.

Definition. For any complex number z, the conjugate is defined as $\bar{z} = x - iy$. For any number of the complex plane, the conjugate is simply its reflection along the real axis. The real and imaginary parts of a number can be, through this, defined as

$$\begin{cases} \text{Re } z = \quad x = \dfrac{z + \bar{z}}{2} \\ \\ \text{Im z} = \quad y = \dfrac{z - \bar{z}}{2i} \end{cases}$$

We define addition, subtraction, and division of complex numbers like the following.

$$\begin{aligned} z_1 + z_2 =& (x_1 + x_2) + i(y_1 + y_2) \\ z_1 z_2 =& (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \\ \frac{z_1}{z_2} =& \frac{(x_1 x_2 + y_1 y_2) + i(x_1 y_2 - x_2 y_1)}{x_2^2 + y_2^2} \end{aligned}$$

Consequently,

$$\begin{aligned} \overline{z_1 + z_2} =& \bar{z_1} + \bar{z_2} \\ \overline{z_1 z_2} =& \bar{z_1} + \bar{z_2} \\ \overline{\left(\frac{z_1}{z_2}\right)} =& \frac{\bar{z_1}}{\bar{z_2}} \end{aligned}$$

Definition. We define the distance between two complex numbers (x_1, y_1) and (x_2, y_2) in $\boldsymbol{R}^2$ as

$$|z_1 - z_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For $z_0 \in \boldsymbol{C}$ and $r > 0$, we can define open and closed disks centered at z_0 with radius r as like the following

$$D_c(z_0, r) = \{z \mid |z - z_0| \leq r\} \qquad D_o(z_0, r) = \{z \mid |z - z_0| < r\}$$

Theorem. Note the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

From this we have $|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$ and $|z_1| - |z_2| \leq |z_1 - z_2|$ therefore we get

$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$$

Definition. The polar expression of a complex number $z = x + iy$ not equal to zero is given as

$$z = x + iy = re^{i\theta}$$

For $\theta \in \boldsymbol{R}$, we define

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $r^2 = |z|^2 = x^2 + y^2$ and $\tan\theta = y/x$. $r = |z|$ is called the modulus of z while $\theta = \arg z$ is called the argument of z. Moreoever, $z^n = (re^{i\theta})^n = r^n e^{in\theta}$ for $n \in \boldsymbol{Z}$ and for

$n, m \in \boldsymbol{Z}$, we have $z^n z^m = z^{n+m}$. Then,

$$\begin{aligned}
e^{i(\theta_1+\theta_2)} =& \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\\
=& \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\\
=& (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\
=& e^{i\theta_1}e^{i\theta_2}
\end{aligned}$$

Note that when $z \neq 0$, $\theta = \arg z$ is not uniquely determined. If θ is an argument of z, then $\theta + 2\pi$ is also an argument.

Theorem. (De-Moivre Theorem) When $n \in \boldsymbol{N}$, we have

$$e^{in\theta} = \cos n\theta + i\sin n\theta = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$

which can be proved by mathematical induction. A smiple corollary is that

$$\begin{aligned}
(e^{i\theta})^{-1} = \frac{1}{e^{i\theta}} =& \frac{1}{\cos\theta + i\sin\theta}\frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}\\
=& \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta = e^{-i\theta}
\end{aligned}$$

Additionally, from Euler's formula, we find

$$\begin{aligned}
e^{i(\theta+2\pi)} =& \cos(\theta+2\pi) + i\sin(\theta+2\pi)\\
=& \cos\theta + i\sin\theta = e^{i\theta}\\
e^{i2\pi} =& 1\\
e^{i\pi} + 1 =& 0
\end{aligned}$$

Every $z \neq 0$ can be uniquely expressed as

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \text{Arg } z$ $(-\pi < \text{Arg } z \leq \pi)$ is called the principle argument. Note that $\arg z = \text{Arg } z + 2n\pi$ for $n \in \boldsymbol{Z}$.

Example. Find $\text{Arg}(-2/1 + i\sqrt{3})$ (answer: $2\pi/3$).

2 Lecture 2 (March 6th)

We have learnt last class that when $z_1, z_2 \neq 0$, we express these complex numbers as $z_1 = r_1 e^{\theta_2}, z_2 = r_2 e^{\theta_2}$ and that $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$. We know that if

$z = ere^{i\theta}$ then $|z| = r$ and $\arg z = \theta + 2n\pi$ for $n \in \boldsymbol{Z}$. Therefore we can note that $\arg z_1 z_2 = \theta_1 + \theta_2 + 2n\pi = \arg z_1 + \arg z_2$. In other words, we know that the argument turns multiplication into addition.

Example. Lets say that $n \in \boldsymbol{N}$, $z_0 \neq 0$ and that e. How do you find all $z \in \boldsymbol{C}$ satisfying $z^n = z_0$ (where such z would be denoted as $z_0^{1/n}$)? The conclusion is that for every $z_0 \neq 0$, z_0^n has exactly one value for every integer. What if the exponent is not necessarily a integer?

$$\begin{cases} z_0^n & \text{has exactly one value} \\ z_0^r & \text{has } m \text{ values} \\ z_0^q & \text{has infinitely many values} \end{cases}$$

If the exponent is a quotient $r = n/m$ which is reducible, it would have exactly m roots. On the other hand, if the exponent is in $q \in \boldsymbol{R} \setminus \boldsymbol{Q}$, there would be an infinite number of roots. In all cases, there would be a single root if we impose that $-\pi < \arg z < \pi$. Note that all cases do not work for the case that the exponent is e.

Proof. First denote $z = re^{i\theta}$ and $z_0 = r_0 e^{i\theta_0}$. Then $z^n = z_0$ means that $r^n e^{in\theta} = r_0 e^{i\theta_0}$ and therefore $r^n = r_0$ and $n\theta = \theta_0 + 2k\pi$ (for $k \in \boldsymbol{Z}$) such that $r = r_0^{1/n}$ and $\theta = \theta_0/n + 2k\pi/n$. In sum, we would have

$$z = re^{i\theta} = \sqrt[n]{r_0} \exp\left[i\Big(\frac{\theta_0}{n} + \frac{2k\pi}{n}\Big)\right] = \sqrt[n]{r_0} \exp\Big(i\frac{\theta}{n}\Big) \exp\Big(i\frac{2k\pi}{n}\Big)$$

this number is indeed finite from the fact that the later term has n possible values for $k \in \boldsymbol{Z}$ (for $k = 0, \ldots, n-1$ and the other values would be redundant). $\square$

Example. Find all z satisfying $z^3 = -8i$. Then,

$$-8i = 8e^{-i\pi/2}$$

Using the formula above for $k = 0, 1, 2$,

$$z = \sqrt[3]{8} \exp\left[i\Big(\frac{-\pi/2}{3} + \frac{2k\pi}{3}\Big)\right]$$

when $k = 0$,

$$z = 2e^{-\pi i/6} = 2\Big(\cos\Big(-\frac{\pi}{6}\Big) + i\sin\Big(-\frac{\pi}{6}\Big)\Big) = \sqrt{3} - i$$

for $k = 1$,

$$z = 2e^{i\pi/2} = 2\Big(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\Big) = 2i$$

and for $k = 2$,

$$r = 2e^{i7\pi/6} = 2\Big(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\Big) = -\sqrt{3} - i$$

We now observe many theorems that follow from properties of $\boldsymbol{R}^2$ onto $\boldsymbol{C}$.

Definition. In the complex plane, $G \subset \boldsymbol{C}$ is called open provided that for every $z_0 \in G$, there is a $\delta > 0$ such that $D(z_0, \delta) \subset G$. On the other hand, $F \subset \boldsymbol{C}$ is closed if $F^C = \boldsymbol{C} \setminus F$ is open. Lastly, $K \subset \boldsymbol{C}$ is called compact provided that every open cover of K has a finite subcover.

Theorem. (Heine-Borel) Satisfied in $\boldsymbol{R}^2$, the Heine-Borel theorem works for the complex plane which states $K \subset \boldsymbol{C}$ is compact if and only if K is closed and bounded.

Definition. A sequence $\{z_n\}$ in $\boldsymbol{C}$ is said to converge to α provided that for every $\varepsilon > 0$, there exists an $N \in \boldsymbol{N}$ such that if $n > N$, $|z_n - \alpha| < \varepsilon$. This is equivalent to saying that any given disk around α contains all but a finite number of z_n. If this is true, we denote this as

$$\lim_{n \to \infty} z_n = \alpha$$

Definition. Let $z_n = x_n + iy_n$ and $\alpha = a + ib$. $\{z_n\}$ converges to α if and only if $\{x_n\}$ converges to a and $\{y_n\}$ converges to b.

Definition. $\{z_n\}$ is called a Cauchy sequence provided that for every $\varepsilon > 0$ there is $N \in \boldsymbol{N}$ such that if $n, m > N$ then $|z_n - z_m| < \varepsilon$.

Theorem. We say that a sequence $\{z_n\}$ converges if and only if it is a Cauchy sequence.

Remark. We know that

$$C(I) \supset D(I) \supset C^1(D) \supset \ldots \supset C^\infty \supset \text{Analytic}$$

however, for complex functions, we have

$$C(G) \supset D(G) = \text{Analytic} = \text{Polynomial}$$

We at this point make three quick definitions.

Definition. (i) A set $G \subset \boldsymbol{C}$ is called *polygonally connected* provided that if $z_1, z_2 \in G$ then there is a polygonal line in G connecting z_1 and z_2.

(ii) Also, $G \subset \boldsymbol{C}$ is called *simply connected* if every closed curve in G contains points of G only (that is, G has no holes)

(iii) If G is open and connected, then G is called a *domain.*

3 Lecture 3 (March 11th)

Definition. Let D be a domain (open and connected set). A function is normally defined as $f : D \to \boldsymbol{C}$. Then for $u = \text{Re } f$ and $v = \text{Im } f$,

$$f(x+iy) = u(x,y) + iv(x,y)$$

and $u, v : D \to \boldsymbol{R}$. We say that $\lim_{z \to z_0} f(z) = w_0$ if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $z \in D^*(z_0, \delta)$ (a deleted neighborhood of $0 < |z - z_0| < \delta$) then $f(z) \in D(w_0, \varepsilon)$ ($|f(z) - w_0| < \varepsilon$).

If $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ and $w_0 = u_0 + iv_0$ then

$$f(z) - w_0 = [u(x,y) - u_0] + i[v(x,y) - v_0]$$

and

$$|f(z) - w_0| = \sqrt{(u(x,y) - u_0)^2 + (v(x,y) - v_0)^2}$$

so that $\lim_{z \to z_0} f(z) = w_0$ if and only if $\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$.

Theorem. $f(z)$ be continuous at z_0 if and only if $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0).

Definition. If u and v belongs to $C^1(D)$ (u, v and their partial derivatives are all continuous). Then we say that $f \in C^1(D)$. We note that this assumption is very weak. In advanced calculus, we defined differentiability as the following. If $u \in C^1(D)$ and $(x_0, y_0) \in D$, then

$$u(x,y) - u(x_0,y_0) = u_x(x_0,y_0)(x - x_0) + u_y(x_0,y_0)(y - y_0) + \varepsilon_1(x,y)(x - x_0) + \varepsilon_2(x,y)(y - y_0)$$

for (x,y) in a neighborhood of (x_0,y_0) where $\lim_{(x,y) \to (x_0,y_0)} \varepsilon_1 = \lim_{(x,y) \to (x_0,y_0)} \varepsilon_2 = 0$. This implies that the following expression is possible

$$\Delta u = u_x + u_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Example. If $f(z) = z^2$ then $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.

When $f(z) = \bar{z} = x - iy$ then $u(x,y) = x$ and $v(x,y) = -y$

When $f(z) = |z|^2 = z\bar{z} = x^2 + y^2$ then $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$.

For these functions, $f \in C^\infty(D)$.

Definition. We define

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

if it exists. In this case, we call $f'(z_0)$ the complex derivative of f at z_0. If D is open and

$f'(z_0)$ exists for all $z_0 \in D$, then we say f is differentiable on D. From this definition, the product rule, quotient rule, and chain rule all exist and are defined in the same way as in calculus 1.

Example. Where is $f(z) = \bar{z}$ differentiable?

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h\to 0} \frac{\bar{h}}{h}$$

The above limit corresponds to

$$\begin{cases} 1 & \text{if } h \in \boldsymbol{R} \\ -1 & \text{if } h \in i\boldsymbol{R} \end{cases}$$

where $i\boldsymbol{R}$ denotes pure imaginary. The function is differentiable nowhere!

Example. Where is $f(z) = |z|^2 = z\bar{z}$ differentiable?

$$\begin{aligned} \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{|z+h|^2 - |z|^2}{h} \\ &= \lim_{h\to 0} \frac{(z+h)(\bar{z}+\bar{h}) - z\bar{z}}{h} \\ &= \lim_{h\to 0} \left(\bar{z} + z\frac{\bar{h}}{h} + \bar{h}\right) \end{aligned}$$

Notice how this limit exists only when $z = 0$ and $f'(0) = 0$!

In this way, our prior notion of differentiability doesn't really work alone. We again emphasize that this is because the limit differs by direction. Let's try this differently.

Definition. f is analytic at z_0 if there is $\delta > 0$ such that f is differentiable on $D(z_0, \delta)$. If f is analytic at every $z_0 \in D$ then we say that f is entire on D. We make this definition because if $D \subset \boldsymbol{C}$ is open then f is differentiable on D if and only if f is analytic on D. Therefore, $f(z) = \bar{z}$ is nowhere differentiable and $f(z) = |z|^2$ is differentiable at 0 but analytic nowhere.

Theorem. (i) If f is continuous on a connected set D then $f(D)$ is connected.

(ii) If f is continuous on a compact set K then $f(D)$ is compact

(iii) If f is continuous on a compact set K then $|f|$ takes a maximum and minimum on K.

Theorem. (Cauchy-Riemann equation) The conclusion is that if f is differentiable in an open set D such that $f(x+iy) = u(x,y) + iv(x,y)$ then $u_x = v_y$ and $v_x = -u_y$ on D. The converse is also true if $f \in C^1(D)$. f is analytic $\iff$ $u_x = v_y$ and $v_x = -u_y$. Notice how from the left to the right, we need nothing but coming back we need $C^1(D)$.

Proof. Suppose that f is differentiable at $z_0 = x_0 + iy_0$. Then for $h, k \in \boldsymbol{R}$,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0) - f(z_0)}{h} = \lim_{k \to 0} \frac{f(z_0 + ik) - f(z_0)}{ik}$$

where

$$\begin{aligned} f(z_0 + h) - f(z_0) =& u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0) \\ =& u(x_0 + h, y_0) - u(x_0, y_0) + i[v(x_0 + h, y_0) - v(x_0, y_0)] \\ f'(z_0) =& \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i \frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} \\ =& u_x(x_0, y_0) + iv_x(x_0, y_0) \end{aligned}$$

that is, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. □

4 Lecture 4 (March 18th)

Recall.

$$f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

if $f'(z)$ exists and $\Delta z = h \in \boldsymbol{R}$

$$f'(z) = u_x + iv_x$$

where $f = u + iv$. If $\Delta = ih$ ($h \in \boldsymbol{R}$), then

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} =& \lim_{h \to 0} \frac{u(x, y + h) + iv(x, y + h) - u(x, y) - iv(x, y)}{ih} \\ =& -i \lim_{h \to 0} \frac{u(x, y + h) - u(x, y)}{h} + \lim_{h \to 0} \frac{v(x, y + h) - v(x, y)}{h} \\ =& -iu_y(x, y) + v_y(x, y) \end{aligned}$$

We desire that these two expressions are equal, and we get $u_x = v_y$ and $v_x = -u_y$. In sum, if f is differentiable at $z_0 = x_0 + iy_0$, then $u_x = v_y$ and $v_x = -u_y$ at (x_0, y_0). This equality is called the Cauchy-Riemann equation.

Example. Note that the function $f(z) = \bar{z} = x - iy$ is a C^∞ function everywhere, but nowhere differentiable.

Theorem. If $f \in C^1(D)$ where D is open, and $u_x = v_y$ and $v_x = -u_y$, then f is differentiable (or analytic) in D.

Recall. If $f = u + iv \in C^1(D)$, then u, v satisfies

$$\begin{cases} \Delta u = u_x \Delta x + u_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\ \Delta v = v_x \Delta x + v_y \Delta y + \eta_1 \Delta x + \eta_2 \Delta y \end{cases}$$

where $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Proof. If $\Delta z = h + ik$ where $h, k \in \boldsymbol{R}$, then

$$
\begin{aligned}
f(z_0 + \Delta z) - f(z) =& u(x_0 + h, y_0 + k) - u(x_0, y_0) + i[v(x_0 + h, y_0 + k) - v(x_0, y_0)] \\
=& u_x(x_0, y_0)h + \underbrace{u_y(x_0, y_0)}_{-v_x(x_0,y_0)} k + \varepsilon_1 h + \varepsilon_2 k \\
&+ i[v_x(x_0, y_0)h + \underbrace{v_y(x_0, y_0)}_{u_x(x_0,y_0)} k + \eta_1 \Delta x + \eta_2 \Delta y] \\
=& u_x(x_0, y_0)(h + ik) + i[v_x(x_0, y_0)(h + ik)] + \varepsilon_1 h + \varepsilon_2 k + i[\eta_1 h + \eta_2 k] \\
=& u_x(x_0, y_0)\Delta z + iv_x(x_0, y_0)\Delta z
\end{aligned}
$$

where $z_0 = x_0 + iy_0 \in D$. Then,

$$
\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z)}{\Delta z} =& \lim_{\Delta z \to 0} [u_x(x_0, y_0) + iv_x(x_0, y_0)] + \lim_{\Delta z \to 0} \frac{\varepsilon_1 h + \varepsilon_2 k + i(\eta_1 h + \eta_2 k)}{h + ik} \\
=& u_x(x_0, y_0) + iv_x(x_0, y_0) + \lim_{\Delta z \to 0} \frac{h}{h + ik}\varepsilon_1 + \lim_{\Delta z \to 0} \frac{k}{h + ik}\varepsilon_2 + \dots \\
=& u_x + iv_x
\end{aligned}
$$

The limit thus exists at z_0! The latter parts all go to zero as $h \to 0$ and $k \to 0$ because

$$
\left| \frac{h}{h + ik} \right| = \frac{|h|}{\sqrt{h^2 + k^2}} \leq 1
$$

$\square$

Example. For $z = x + iy$, define $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$. Then, if $f(z) = e^z$ then $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. We will show that these satisfy the C-R conditions.

$$
\begin{cases}
u_x = e^x \cos y = v_y \\
v_x = e^x \sin y = -u_y
\end{cases}
$$

The function f is also thus analytic on $\boldsymbol{C}$ and $f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y = e^x(\cos x + i \sin y) = e^x e^{iy} = e^z$. Simply put, if $f(z) = e^z$ then $f'(z) = e^z$.

Definition. We now see the C-R equation in polar coordinates. Let $f = u + iv$ where $z = x + iy = re^{i\theta}$. If $u, v \in C^1(D)$ satisfies the C-R equation,

$$\begin{cases} u_r = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} = u_x\dfrac{\partial x}{\partial r} + u_y\dfrac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta \\ u_\theta = u_x\dfrac{\partial x}{\partial \theta} + u_y\dfrac{\partial y}{\partial \theta} = -u_x r\sin\theta + u_y r\cos\theta \\ v_r = v_x\dfrac{\partial x}{\partial r} + v_y\dfrac{\partial y}{\partial r} = v_x\cos\theta + v_y\sin\theta = -u_y\cos\theta + u_x\sin\theta \\ v_\theta = u_x\dfrac{\partial x}{\partial \theta} + u_y\dfrac{\partial y}{\partial \theta} = u_y r\sin\theta + u_x r\cos\theta \end{cases}$$

We conclude that

$$\begin{cases} v_\theta = r u_r \\ u_\theta = -r v_r \end{cases}$$

Example. For a multivariable function $u(x_1, \dots, x_n) \in C^2$, we can define the Laplacian as

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$$

If $\Delta u = 0$ then we call u harmonic.

If $f \in C^2(D)$ with $f = u + iv$ is analytic on D, then from the Cauchy-Riemann equation we have $u_x = v_y$ and $v_x = -u_y$. Thus $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. As we know that $v_{xy} = v_{yx}$, we conclude that on D,

$$u_{xx} + u_{yy} = 0$$

Similarly,

$$v_{xx} + v_{yy} = 0$$

on D.

Theorem. We have proved the following theorem. If $f = u + iv \in C^2(D)$ is analytic, then u, v are harmonic.

Theorem. If $u \in C^2(\boldsymbol{R}^2)$ is harmonic, then $u = \operatorname{Re} f$ for some entire function f.

Example. If $u(x, y) = y^3 - 3x^2 y$,

$$\begin{cases} u_x = -6xy \qquad u_{xx} = -6x \\ u_y = 3y^2 - 3x^2 \qquad u_{yy} = 6y \end{cases}$$

and we can conclude that on $\boldsymbol{R}^2$, $u_{xx} + u_{yy} = 0$.

For a given u, how do we find $v \in C^2(\boldsymbol{R}^2)$, called harmonic conjugate, that allows $x+iv = f$ to be entire (analytic)? We require that $v_x = -u_y$ and $v_y = u_x$.

5 Lecture 5 (March 20th)

Example. Determine c so that $u(x, y) = cx^2 + x + e^{-x} \cos y$ is harmonic and for such case find an entire function f satisfying $u = \text{Re } f$.

Proof. $u_{xx} + u_{yy} = 2c + e^{-x} \cos y - e^{-x} \cos y = 2c = 0$. To obtain the harmonic conjugate, let $f = u - iv$ and $u_x = v_y$ with $u_y = -v_x$.

$$u_x = 1 + e^{-x} \cos y = v_y \qquad \Longrightarrow \qquad v(x, y) = y - e^{-x} \sin y + \phi(x)$$

this implies

$$v_x(x, y) = e^{-x} \sin y + \phi'(x) = -u_y = e^{-x} \sin y$$

so that $\phi'(x)$ and $\phi(x) = k$. The entire function f is $u+iv = x+e^x \cos y+i(y-e^{-x} \sin y+k)$. To obtain the function in terms of z, a good tip is to put $y = 0$ and obtain $f(x)$. Replacing x with z grants us the function we wanted.

$$f(z) = z + e^{-z}$$

$\square$

Recall. If f is differentiable on an interval I such that $f' \neq 0$ at any $z \in I$, then f is 1-1 on I. However, note that the function $f(z) = e^z$ has a non-zero derivative but still is not 1-1.

Theorem. If f is analytic in a domain D (open & connected) with $f'(z) = 0$ for all $z \in D$, then f is constant in D.

Proof. If $f = u+iv$ then by C-R equations, $u_x = v_y$ and $v_x = -u_y$. Then, $f' = u_x+iv_x = 0$ and $u_x, v_x = 0$. By a similar reasoning, $u_y, v_y = 0$ on D. Therefore, f is constant on D. $\square$

Theorem. If f, g are differentiable at z_0 with $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

Proof.

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}}{\lim_{z\to z_0} \dfrac{g(z)-g(z_0)}{z-z_0}} = \lim_{z\to z_0} \frac{\dfrac{f(z)-f(z_0)}{z-z_0}}{\dfrac{g(z)-g(z_0)}{z-z_0}} = \lim_{z\to z_0} \frac{f(z)}{g(z)}$$

□

Chapter 3 Elementary Functions

$$f(z) = e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$$

Note that $f(z + 2\pi i) = f(z)$.

Remark.

$$\begin{cases} w = re^{i\theta} \implies |e^{iw^2}| = |e^{ir^2 e^{i2\theta}}| = |e^{ir^2(\cos 2\theta + i\sin 2\theta)}| = e^{-r^2 \sin 2\theta} \\ z = x + iy \implies |e^{z^2}| = |e^{(x^2-y^2)+2ixy}| = e^{x^2-y^2} \end{cases}$$

Also, if $|e^z| < 0$ means that $x < 0$.

Example. Find all $z = x + iy$ satisfying

$$e^z = 1 + i$$

The answer is $e^z = e^x e^{iy} = 1 + i = \sqrt{2}e^{i\pi/4}$. $e^x = \sqrt{2}$ and $e^{iy} = e^{i\pi/4}$ making $x = \ln\sqrt{2} = \ln 2/2$ and $y = \pi/4 + 2n\pi$.

6 Lecture 6 (March 25th)

Definition. If $z \neq 0$ and $e^w = z$, we then define $w = \log z$ $(= \ln|z| + i\text{arg } z)$. We note that $\log z$ is a multi-valued function. For this reason, we define the principle value of the logarithm as

$$\text{Log } z = \ln|z| + \ i\text{Arg } z$$

which is uniquely defined for $z \neq 0$ and $-\pi \leq \text{Arg } z \leq \pi$. We very often use the principle branch of the logarithm defined as

$$\log\ z = \ln|z| + i \text{ arg } z$$

for $-\pi < \text{arg } z < \pi$. The motivation for this inequality is continuity. The function Log z is not continuous at Arg $z = \pi$, which corresponds to the negative x-axis. We can define different branches of the logarithm where $\log z = \ln|z| + i\theta$ given the condition $\alpha < \theta < \alpha + 2\pi$. In the case of the principle branch, $\alpha = -\pi$. Notice how in this domain, the logarithm function becomes continuous.

Let's now check the differentiability of a branch of the logarithm. The Cauchy-Riemann equations in polar coordinates were $u_\theta = -rv_r$ and $v_\theta = ru_r$. With $u = \ln r$ and $v = \theta$, we find that the conditions are indeed satisfied.

Remark. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, we know that

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

and thus

$$\begin{aligned}\log z_1 z_2 &= \ln r_1 r_2 + i(\theta_1 + \theta_2 + 2n\pi) \\ &= \ln r_1 + i(\theta_1 + 2k\pi) + \ln r_2 + i(\theta_2 + 2n\pi) = \log z_1 + \log z_2\end{aligned}$$

by setting $z_1 = z_1 z_2 / z_2$, we can also get

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$$

Example.

$$\log(-1) = \ln|-1| + i \arg(-1) = i(2n+1)\pi$$

Recall. z^c for $c \notin \boldsymbol{Q}$ is defined as

$$z^c = e^{c \log z}$$

We have previously mentioned that z^m ($m \in \boldsymbol{Z}$) is single valued and that $z^{n/m}$ (n/m is irreducible) has m distinct values. These rules are satisfied by the definition above. We additionally note that we cannot apply power rules when the base isn't e.

Example.

$$i^{2i} = e^{2i \log i} = e^{2i \frac{(1+4n)\pi i}{2}} = e^{-(1+4n)\pi}$$

as

$$\log i = \ln|i| + i \arg(i) = \frac{(1+4n)\pi i}{2}$$

Remark. For $c \notin \boldsymbol{Z}$, z^c is differentiable on any branch of the logarithm and $z_1^{c_1} z^{c_2} = z^{c_1+c_2}$ (since $e^{\alpha_1+\alpha_2} = e^{\alpha_1} e^{\alpha_2}$). Therefore,

$$\frac{d}{dz} z^c = \frac{d}{dz} e^{c \log z} = e^{c \log z} \frac{c}{z} = z^c \frac{c}{z} = c z^{c-1}$$

Definition. We define the trigonometric functions as

$$\begin{cases} \sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \\ \cos z = \dfrac{e^{iz} + e^{-iz}}{2} \\ \tan z = \dfrac{\sin z}{\cos z} \end{cases}$$

These functions are entire and satisfy

$$\begin{cases} \dfrac{d}{dz} \sin z = \cos z \\ \dfrac{d}{dz} \cos z = -\sin z \end{cases}$$

Later, we will learn Liouville's theorem that states that bounded entire functions are constant. We will also later learn that all trigonometric identities work in the complex plane too through the identity theorem.

Definition. In a similar fashion, we define the hyperbolic sine and cosine functions as

$$\begin{cases} \sinh z = \dfrac{e^{z} - e^{-z}}{2} \\ \cosh z = \dfrac{e^{z} + e^{-z}}{2} \end{cases}$$

Like the above, all identities for real hyperbolic function hold for complex hyperbolic functions (again, due to the identity theorem).

Chapter 4 Integrals

Definition. All complex integrals are of the form

$$\int_C f(z)\, dz$$

where C is a contour (range or image of continuous, piecewise C^1 curves on $[a, b]$) or disjoint union of contours. We require that $f(z)$ is continuous on the contour C (or bounded discontinuous at finitely many points in C). Lastly, we define as $dz = z'(t)dt$. There are two ways to express $\int_C f(z)\, dz$.

7 Lecture 7 (March 27th)

Recall. We have previously seen

$$\int_C f(z)\, dz$$

where C is a contour given by $z = z(t)$ with $z \le t \le b$ and f is continuous on C. Note that $dz = z'(t)dt = dx + idy$. We could express this integral in two forms,

(i)

$$\int_C f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt$$

(ii)

$$\int_C (u + iv)(dx + idy) = \int_C udx - vdy + i \int_C vdx + udy$$

The first is of the form $\int_a^b w(t)\, dt$ where $w(t) = w_1(t) + iw_2(t)$ where $w_1(t)$ and $w_2(t)$ have real values, and the second is of the line integral form. From the second form, we learn that

$$\int_{-C} f(z)\, dz = -\int_C f(z)\, dz$$

where $-C$ denotes the same contour as C but opposite direction. The central theme of complex functions is evaluating the integral in two ways, one through the residue theorem from Cauchy's formula and one through reparametrization. There is great power in comparing these two results.

Example. Let C be given by $z = 3e^{it}$ where $0 \le t \le \pi$. Find $\int_C z^{1/2}\, dz$.

$$\int_0^\pi \left(3e^{it}\right)^{1/2} i3e^{it}\, dt = i3\sqrt{3}\int_0^\pi e^{3ti/2}\, dt = i3\sqrt{3}\Big[\frac{2}{3i}e^{3ti/2}\Big]_0^\pi = -2\sqrt{3}(1+i)$$

Theorem.

$$\Big|\int_a^b w(t)\, dt\Big| \le \int_a^b |w(t)|\, dt$$

Proof. Let $\int_a^b w(t)\, dt = re^{i\theta}$. Then,

$$\Big|\int_a^b w(t)\, dt\Big| = r = e^{-i\theta}\int_a^b w(t)\, dt = \mathrm{Re}\int_a^b e^{-i\theta}w(t)\, dt = \int_a^b \mathrm{Re}\left(e^{-i\theta}w(t)\right)dt$$

As we know that $x = \text{Re } z \leq |z| = \sqrt{x^2+y^2}$, we have proven the statement for the real part. $\square$

Theorem.

$$\Big|\int_C f(z)\ dz\Big| \leq ML$$

where $M = \max_{z\in C}|f(z)|$ and L is the length of C.

Proof. The proof follows from the previous theorem.

$$\begin{aligned}\Big|\int_C f(z)\ dz\Big| =& \Big|\int_a^b f(z(t))z'(t)dt\Big|\\ \leq& \int_a^b |f(z(t))||z'(t)|\ dt \leq M\int_a^b |z'(t)|\ dt\\ =& M\int_a^b \sqrt{\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2}dt = ML\end{aligned}$$

$\square$

Example. Let C denote $z = e^{it}$ with $0 \leq t \leq \pi$. Then,

$$\Big|\int_C \frac{2z+1}{3+z^2}\ dz\Big| \leq \frac{3}{2} \times l(C) = \frac{3\pi}{2}$$

on C with

$$\Big|\frac{2z+1}{3+z^2}\Big| \leq \frac{2|z|+1}{3-|z|^2} = \frac{3}{2}$$

noting that $|z| = 1$.

Example. Given $C_R = Re^{it}$ with $0 \leq t \leq \pi/4$ and $f(z) = e^{iz^2}$, show that

$$\lim_{R\to\infty}\int_{C_R} f(z)\ dz = 0$$

Proof. On C_R, note that $|f(z)| = |e^{iz^2}| = e^{-R^2\sin 2t}$. Also, $dz = iRe^{it}\ dt$. We then have

$$\Big|\int_{C_R} e^{iz^2}\ dz\Big| \leq \int_0^{\pi/4} e^{-R^2\sin 2t}R\ dt = R\int_0^{\pi/4} e^{-R^2\sin 2t}\ dt = \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\theta}\ d\theta$$

Recall that for $0 \leq x \leq \pi/2$,

$$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1$$

We thus conclude that

$$I \leq \frac{R}{2}\int_0^{\pi/2} e^{-R^2 2t/\pi}\, dt = \frac{R}{2}\Big[-\frac{\pi}{2R^2}e^{-2R^2 t/\pi}\Big]_0^{\pi/2} = \frac{\pi}{4R}(1-e^{-R^2})$$

and that the integral converges to zero as $R \to \infty$. □

Theorem. If C is a closed contour and there is an analytic function f on C such that $F' = f$ then $\int_C f(z)\, dz = 0$.

Example. Let C be any closed contour (not passing through the origin) and $n \in \boldsymbol{Z}$. We have $\int_C z^n\, dz = 0$ for all $n \neq -1$. We cannot allow $n = -1$ as

$$\frac{d}{dz}\frac{1}{n+1}z^{n+1} = z^n$$

on C.

Proof.

$$\begin{aligned}\int_C f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt = \int_a^b & F'(z(t))z'(t)\, dt \\ =& F(z(b)) - F(z(a)) = 0\end{aligned}$$

as $z(a) = z(b)$. □

Example. Consider these three contours where the function $\int_C 1/z\, dz$ is to be integrated.

$$\begin{cases} C_1: & z = z(t) = e^{it} \quad 0 \leq t \leq 2\pi \\ C_2: & z = z(t) = e^{it} \quad 0 \leq t \leq 4\pi \\ C_3: & z = z(t) = e^{-it} \quad 0 \leq t \leq 2\pi \end{cases}$$

We obtain

$$\begin{cases} \int_{C_1} \frac{1}{z}\, dz = \int_{C_1} \frac{1}{e^{it}} ie^{it} dt = 2\pi i \\ \int_{C_2} \frac{1}{z}\, dz = 4\pi i \\ \int_{C_3} \frac{1}{z}\, dz = -2\pi i \end{cases}$$

Definition. A contour $C = z : [a, b] \to \boldsymbol{C}$ is called positive oriented if it is counterclockwise. A contour is simple if $z(x) \neq z(y)$ for $x, y \in [a, b)$ with $x \neq y$. A contour is denoted POSCC if it is a positvely oriented simple closed contour.

Definition. Let C be a POSCC (passing the origin) with $n \in \boldsymbol{Z}$. Then,

$$\int_C z^n \, dz = \begin{cases} 0 & \text{if} \quad n \neq -1 \\ 0 & \text{if} \quad n = -1 \quad \text{and } 0 \text{ is not in } C \\ 2\pi i & \text{if} \quad n = -1 \quad \text{and } 0 \text{ is in } C \end{cases}$$

8 Lecture 8 (April 1st)

Definition. For a closed contour C, we define $\Delta_C \arg z$ as the net change of the argument as z traverses C. Then,

$$\frac{1}{2\pi} \Delta_C \arg z$$

is the number of times C winds up the origin in the positive sense. On another note, we know that for a closed contour $0 \in C$,

$$\int_C z^n \, dz = 0 \qquad \text{for} \qquad z \in \boldsymbol{Z} \setminus \{-1\}$$

since

$$\left[\frac{1}{z+1} z^{n+1}\right]' = z^n$$

and

$$\int_C f(z) \, dz = \int_a^b f(z(t)) z'(t) \, d = F(z(b)) - F(z(a))$$

for $F' = f$ on C. Continuing this argument, we have

$$\begin{aligned} \int_C \frac{1}{z} \, dz =& \log z(b) - \log z(a) = \ln |z(b)| + i\text{arg } z(b) - [\ln |z(a)| + i\text{arg } z(a)] \\ =& i[\arg z(b) - \arg z(a)] = i \Delta_C \arg z \end{aligned}$$

for a closed contour $C : z = z(t)$, $a \leq t \leq b$ and $z(a) = z(b)$. In conclusion, for a closed contour where $0 \notin C$,

$$\frac{1}{2\pi i} \int_C \frac{1}{z} \, dz$$

is the number of times C winds up 0 in the positive sense.

Theorem. If C is a POSCC (positively oriented simple closed curve) and $0 \notin C$), for

$n \in \boldsymbol{Z}$,

$$\int_C z^n \, dz = \begin{cases} 0 & n \neq -1 \\ 0 & n = -1, \text{ origin is outside } C \\ 2\pi i & n = -1, \text{ origin is inside } C \end{cases}$$

If C is a POSCC and $a \notin C$, for $z \in \boldsymbol{Z}$,

$$\int_C (z-a)^n \, dz = \begin{cases} 0 & n \neq -1 \\ 0 & n = -1, \ a \text{ is outside } C \\ 2\pi i & n = -1, \ a \text{ is inside } C \end{cases}$$

We can obtain this result by letting $z - a = w$ and $dz = dw$. Many (or most of) contour integrals are independent of path (dependent only on endpoints).

Example. Let C be a contour in Re $z \geq 0$ $(-\pi < \arg z < \pi)$ and $0 \notin C$. For C that extends from $-2i$ to $2i$, evaluate

$$\int_C \frac{1}{z} \, dz$$

Proof. This is equal to $\log 2i - \log -2i = \pi i$. If we contrarily limit the argument to $0 < \arg z < 2\pi$, we get $-\pi i$. $\square$

Example. Let C be a contour in Im $z \geq 0$ with $0 \notin C$. Let C be from 3 to -3. Evaluate

$$\int_C z^{1/2} \, dz$$

Proof. Limiting the argument by creating a branch at $-\pi/2$, that is, limiting the argument to $-\pi/2 < \arg z < 3\pi/2$, $z^{1/2}$ is analytic with $(2z^{3/2}/3)' = z^{1/2}$, and we have

$$I = F(-3) - F(3) = \frac{2}{3}\left(3e^{\pi i}\right)^{3/2} - \frac{2}{3}\left(3^{3/2}\right) = \frac{2}{3}3\sqrt{3}\left[e^{3\pi i/2} - 1\right] = 2\sqrt{3}(-i-1)$$

$\square$

Theorem. (Cauchy's theorem) Let D be a simply connected domain. If f is a C^1 analytic function in D, then

$$\int_C f(z) \, dz = 0$$

for all simply closed contours C in D.

Proof. Assume C is a positive-oriented. Then

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

By Green's theorem, the interior of the contour becomes a simply connected set and

$$\int u\,dx - v\,dy = \iint_\Omega \Big(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)dxdy = 0$$

where Ω is the interior of C. Note well that Cauchy's theorem is strictly the property of the domain of the function whereas the theorem we dealt with beforehand is a result of line integrals and furthermore the fundamental theorem of calculus. Again, Cauchy's theorem is about the boundary while line integrals are about the existence of the anti-derivative! $\square$

Remark. For P and $Q \in C^1(\bar{D})$,

$$\iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy = \int_{\partial D} P\,dx + Q\,dy$$

9 Lecture 9 (April 3rd)

Theorem. (Cauchy theorem) Let D be a simply connected domain and let f be analytic on D. Then,

$$\int_C f(z)\,dz = 0$$

for every simple closed contour C in D. Note that the condition implies that f is analytic on $\bar{D}$ for a bounded domain D.

Remark. There are two cases for which Cauchy theorem is true.

(i) D is simply connected and ∂D is a simple closed contour

(ii) D is multiply connected, ∂D is a union of disjoint simple closed contour then

$$\int_{\partial D} f(z)\,dz = 0$$

where ∂D has a positive orientation.

The second one can be seen as a corollary of the first and can be proven as

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

and multiply connected domain can be seen as an amalgamation of a single one.

Example. (i)

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

(ii)

$$\int_0^\infty \sin x^2\, dx$$

both of the above converge by the alterating series theorem. We can find these values using expressing them as analytic complex integrals.

Example. Let $f(z) = e^{iz^2}$ (entire function). Let C_R be a positively oriented contour given by one eighth on a circle with radius R.

(i) By the Cauchy theorem,

$$\int_{C_R} e^{iz^2}\, dz = 0$$

for all $R > 0$

(ii) By parametrisation,

$$\begin{aligned}
\int_{C_R} e^{iz^2}\, dz &= \int_{\mathrm{I}} e^{iz^2}\, dz + \int_{\mathrm{II}} e^{iz^2}\, dz + \int_{\mathrm{III}} e^{iz^2}\, dz \\
&= \int_0^R e^{ix^2}\, dx + \int_{\mathrm{II}} e^{iz^2} dz - \int_0^R \exp\left[i\left(xe^{\pi i/4}\right)^2\right] e^{\pi i/4}\, dx \\
&= \int_0^R e^{ix^2}\, dx + \int_{\mathrm{II}} e^{iz^2}\, dz - e^{\pi i/4}\int_0^R e^{-x^2}\, dx
\end{aligned}$$

At this point, take $R \to \infty$ and we have:

$$0 = \int_0^\infty e^{ix^2}\, dx - e^{\pi i/4}\int_0^\infty e^{-x^2}\, dx$$

This implies that

$$\int_0^\infty e^{ix^2}\, dx = \int_0^\infty = e^{\pi i/4}\frac{\sqrt{\pi}}{2}$$

and finally

$$\int_0^\infty \cos x^2\, dx = \frac{\sqrt{2}}{2}\frac{\sqrt{\pi}}{2} = \int_0^\infty \sin x^2\, dx$$

Example. Consider an upper half hemisphere with outer radius R and inner radius ε, $C_{\varepsilon,R}$, which is positively oriented. Also, take $f(z) = e^{iz}/z$. Again, we have two methods:

(i) By the Cauchy theorem,

$$\int_{C_{\varepsilon,R}} f(z)\, dz = 0$$

for every $0 < \varepsilon < R < \infty$

(ii) By parametrisation,

$$\int_{C_{\varepsilon,R}} f(z)\,dz = \int_{\mathrm{I}} f(z)\,dz + \ldots + \int_{\mathrm{IV}} f(z)\,dz$$

By taking $\varepsilon \to 0^+$ and $R \to \infty$, we achieve

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Theorem. (Cauchy Integral Formula) Let f be analytic inside and on a POSCC C then for every z_0 inside C, we have

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz$$

Proof. Take any $\varepsilon > 0$. We'll show that

$$\left| f(z_0) - \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz \right| < \varepsilon$$

Since f is continuous at z_0 there is $\delta > 0$ such that if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$. Take $r < \delta$ and apply the Cauchy theorem to $f(z)/z - z_0$ on a positively oriented shell around z_0 with inner radius r. Then, the function is analytic with that shell and

$$0 = \int_C \frac{f(z)}{z - z_0}\,dz - \int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\,dz$$

where the right integral is evaluated contour clockwise. Then,

$$\begin{aligned}\int_C \frac{f(z)}{z - z_0}\,dz &= \int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\,dz \\ &= \int_{|z-z_0|=r} \frac{f(z) - f(z_0)}{z - z_0}\,dz + \int_{|z-z_0|=r} \frac{f(z_0)}{z - z_0}\,dz\end{aligned}$$

We therefore arrive at the equality

$$\int_C \frac{f(z)}{z - z_0}\,dz - 2\pi i f(z_0) = \int_{|z-z_0|=r} \frac{f(z) - f(z_0)}{z - z_0}\,dz$$

and by applying the modulus function we have

$$\left| \int_C \frac{f(z)}{z - z_0}\,dz - 2\pi i f(z_0) \right| \leq \int_{|z-z_0|=r} \frac{\varepsilon}{r}|dz| = 2\pi\varepsilon$$

dividing each side by $2\pi = |2\pi i|$ completes the proof. There are two ideas inside this proof:

(1) the integral of $1/z - z_0$ and (2) the continuity of $f(z)$ at z_0. $\square$

Remark. Let D be a bounded simply connected domain whose boundary ∂D is a closed contour. If f is analytic on $\bar{D} = D \cup \partial D$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z}\, d\xi$$

for all $z \in D$. Notice how the value of f on D is determined by its boundary values. From this, we have a power series representation of f.

10 Lecture 10 (April 8th)

Theorem. (Cauchy Formula) Let f be analytic in $\bar{D} = D \cup \partial D$ for a simply connected domain D whose boundary is POSCC ∂D. Then for every $z \in D$ we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} d\xi\, \frac{f(\xi)}{\xi - z}$$

Example. Consider the following integral

$$\int_{|z-i|=1/2} dz\, \frac{3z^2}{(z-2)(z^2+1)} = 2\pi i f(i) = 2\pi i \frac{3(i)^2}{(i-2)(2i)}$$

where

$$f(z) = \frac{3z^2}{(z-2)(z+i)}$$

is analytic in $|z - i| = 1/2$.

Theorem. (Cauchy's differentiation formula) Consider how

$$\begin{aligned}
\frac{d}{dz}(a-z)^{-1} =& (a-z)^{-2} \\
\frac{d}{dz}(a-z)^{-2} =& 2(a-z)^{-3} \\
\frac{d}{dz}2(a-z)^{-3} =& 6(a-z)^{-4} \\
&\vdots \\
\frac{d^n}{dz^n}(a-z)^{-n} =& n!(a-z)^{-(n+1)}
\end{aligned}$$

Employing this to Cauchy's formula,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} d\xi\, \frac{f(\xi)}{(\xi - z)^{n+1}}$$

where f is analytic in D and $z \in D$.

Example.

$$\int_{|z-1|=3} dz\, \frac{4-2\cos z}{\left(z-\frac{\pi}{2}\right)^2} = 2\pi i f'\left(\frac{\pi}{2}\right) = 2\pi i \cdot 2\sin\frac{\pi}{2} = 4\pi i$$

Here, $f(z) = 4 - 2\cos z$ implies that $f'(z) = 2\sin z$ and $n = 1$, $\partial D \;:\; |z-1| = 3$ and $z_0 = \pi/2$.

Example.

$$\int_{|z|=3} dz\, \frac{4z^5 - 7z^2 + 2}{(z-1)^3} = 2\pi i \frac{1}{2} f''(1)$$

where $f(z) = 4z^5 - 7z^2 + 2$.

Remark. There are functions that are infinitely differentiable but not analytic (Taylor series representable).

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Notice how $f^{(n)}(0) = 0$ and $f \in C^\infty(\boldsymbol{R})$ but not power series representable at 0.

Remark. Notationwise, we write

$$D(z_0, r) = \{z \in D \mid |z - z_0| < r\}$$

Proof. We prove Cauchy's differentiation formula by first showing a power series representation of the original Cauchy's formula. In this process, we will need to show uniform convergence of the series. Then, we will differentiate the power series to obtain the conclude the proof. We first state that $z \in D(z_0, r)$ and $\bar{D}(z_0, r) \subset D$. For $|z| < 1$ we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

and

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 - (z - z_0)} = \frac{1}{\xi - z_0} \frac{1}{1 - \dfrac{z - z_0}{\xi - z_0}}$$

where

$$\left|\frac{z - z_0}{\xi - z_0}\right| < 1$$

becoming

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}}$$

Then, for f analytic in D and $\bar{D}(z_0, r) \subset D$ and $z \in D(z_0, r)$ we have

$$\begin{aligned} f(z) =& \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z} \, d\xi \\ =& \frac{1}{2\pi i} \int_{\partial D} f(\xi) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} \, d\xi \\ =& \frac{1}{2\pi i} \sum_{n=0}^{\infty} \Big[\int_{\partial D} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \Big] (z - z_0)^n \end{aligned}$$

We can take out the summation due to uniform convergence. We also note that

$$f(z) = \sum_{n=0}^{\infty} (a - a_0)^n \quad \text{implies} \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Two points will be later investigated: (1) that the series converges uniformly and that the summation can be taken out and (2) the implication above. □

Definition. For $E \subset \boldsymbol{C}$ and $f_n : E \to \boldsymbol{C}$ with a limit of $f : E \to \boldsymbol{C}$ we say that $f_n \to f$ uniformly on E provided that

$$T_n = \sup_{z \in E} |f_n(z) - f(z)| \to 0$$

Theorem. If f_n is continuous on E for each n and $f_n \to f$ uniformly on E, then f is continuous on E.

Theorem. (Weierstrass M-test) For $f_n : E \to \boldsymbol{C}$, for each n, there is $M_n \geq 0$ such that $|f_n(z)| \leq M_n$ for all $z \in E$ and $\sum_{n=1}^{\infty} M_n < \infty$ then $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

Theorem. If $\{f_n\}$ is continuous on a contour C such that $f_n \to f$ uniformly on C then

$$\int_C f(z) \, dz = \lim_{n \to \infty} \int_C f_n(z) \, dz$$

Proof. Let L be the length of C. Take any $\varepsilon > 0$ there is $N \in \boldsymbol{N}$ such that if $n > N$ then $|f_n(z) - f(z)| < \varepsilon / L$ for all $z \in \boldsymbol{C}$. Then if $n > N$, we have

$$\Big| \int_C f_n \, dz - \int_C f \, dz \Big| = \Big| \int_C (f_n - f) \, dz \Big| \leq \int_C |f_n - f| \, |dz| < \frac{\varepsilon}{L} \cdot L = \varepsilon$$

□

Remark. In our setting, $z \in D(z_0, r)$ and $\bar{D}(z_0, r) \subset D$ with

$$f(z) = \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\xi - z_0)^{n+1}} f(\xi)\, d\xi$$

For fixed $z \in D(z_0, r)$,

$$\begin{gathered} \left| \frac{z-z_0}{\xi - z_0} \right| < \frac{|z-z_0|}{r} \\ \sum_{n=1}^{\infty} \left| \frac{z-z_0}{\xi - z_0} \right|^n \leq \sum_{n=1}^{\infty} \left(\frac{|z-z_0|}{r} \right)^n < \infty \end{gathered}$$

for all $\xi \in \partial D$ and the series converges uniformly on C with respect to ξ.

Example. If g is continuous on D and

$$f(z) = \int_{\partial D} \frac{g(\xi)}{\xi - z}\, d\xi$$

then f is power series representable for the same reason.

11 Lecture 11 (April 11)

Recall. (Cauchy integral formula)

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z}\, d\xi = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for $z \in D(z_0, r) \subset D$.

Theorem. (Liouville's theorem) Bounded entire functions are constant.

Proof. Let f be an entire function such that $|f(z)| < M$ for all $z \in \boldsymbol{C}$. Take any $\varepsilon > 0$ and any $z_0 \in \boldsymbol{C}$. It is enough to show $|f'(z_0)| < \varepsilon$. Take $R > 0$ so that $M/R < \varepsilon$. By the Cauchy integral formula,

$$f'(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}\, dz$$

This implies that

$$|f'(z_0)| \leq \frac{1}{2\pi} \int_{|z-z_0|=R} \frac{M}{|z-z_0|^2}\, |dz| = \frac{1}{2\pi} \frac{M}{R^2} 2\pi R = \frac{M}{R} < \varepsilon$$

$\square$

Theorem. (Fundamental theorem of algebra) A complex polynomial

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

with $a_n \neq 0$ has a zero in $\boldsymbol{C}$.

Proof. Suppose there is no $z \in \boldsymbol{C}$ with $p_n(z) = 0$. Then $g(z) = 1/p_n(z)$ is a bounded entire function which is a contradiction. $\square$

Corollary. If f is a non-constant entire function, then $f(\boldsymbol{C})$ is dense in $\boldsymbol{C}$.

Proof. Suppose there exists $\delta > 0$, $w_0 \in \boldsymbol{C}$ such that $f(\boldsymbol{C}) \cap D(w_0, \delta) = \emptyset$. Then

$$g(z) = \frac{1}{f(z) - w_0}$$

satisfies $|g(z)| \leq 1/\delta$ for all $z \in \boldsymbol{C}$. By Liouville's theorem, $g(z) = \boldsymbol{C}$ which imlpies that $f(z) = w_0 + 1/C$ is constant. $\square$

Theorem. (Picard's theorem 1) If f is non-constant entire function there is $w_0 \in \boldsymbol{C}$ such that

$$\boldsymbol{C} \setminus \{w_0\}$$

For example, for $f(z) = e^z$, $w_0 = 0$.

Remark. We have a brief summary of power series. For a power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$,

(i) there is $R \in [0, \infty]$ such that the series converges absolutely on $D(z_0, R)$. If $R = 0$ the series converges only at z_0 and if $R = \infty$ the series converges on all of $\boldsymbol{C}$

(ii) if $0 < r < R$ the series converges uniformly on $\bar{D}(z_0, r)$

(iii) on $D(z_0, R)$,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

as for a differentiable f_n, if $\sum f_n$ converges pointwise and $\sum f_n'$ converges uniformly, $(\sum f_n)' = \sum f_n'$

(iv) for $n = 0, 1, \ldots$,

$$a_n = \frac{f^n(z_0)}{n!}$$

as $f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1) a_n (z - z_0)^{n-k}$

Take

$$R = \frac{1}{\limsup |a_n|^{1/n}} \in [0, \infty]$$

By applying the root test to $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$, we have that

$$\limsup |a_n(z-z_0)^n| < 1$$

implies that the absolute of the sequence converges. We know that

$$\limsup |a_n|^{1/n}|z-z_0| = |z-z_0|/R < 1$$

and that it converges.

Recall. Let D be a bounded simply connected domain with ∂D being a POSCC and let $\bar{D}(z_0, r) \subset D$.

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z}\, d\xi = \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi\right](z-z_0)^n$$

for $z \in D(z_0, r)$. The coefficients can be obtained like the following

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi = \frac{f^{(n)}(z_0)}{n!}$$

and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi$$

Example. On $\boldsymbol{R}^2$, consider $f(x, y) = xy$. Notice how it is zero in both the x and y-axes. However, f is not constant zero.

Theorem. Let $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges on $D(z_0, R)$. If $\{a_n\}$ is a sequence of distinct points in $D(z_0, R)$ such that

(i) $a_n \to z_0$

(ii) $f(a_n) = 0$ for every $n \in \boldsymbol{N}$

then $f(z) = 0$ for all $z \in D(z_0, R)$.

Proof. For $0 = f(a_k) = \sum_{n=0}^{\infty} c_n(a_k - z_0)^n$, taking $k \to \infty$ we get have $c_0 = 0$. □

12 Lecture 12 (April 15th)

Theorem. In a disk $D(a, R)$, consider the power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

If there exists a sequence $\{a_n\} \to a$ with $f(a_n) = 0$ for all $n \in \boldsymbol{N}$, $f(z) = 0$ for all $z \in D(a, R)$

Proof. Let $f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots$. By putting $z = a_n$ we have

$$0 = f(a_n) = c_0 + c_1(a_n - a) + \ldots$$

Taking $n \to \infty$, we get $c_0 = 0$. Then,

$$f(z) = (z-a)(c_1 + c_2(z-a) + c_3(z-a)^2 + \ldots)$$

and

$$\frac{f(z)}{z-a} = c_1 + c_2(z-a) + c_3(z-a)^2 \ldots$$

now, taking $z = a_n$, then

$$0 = \frac{f(a_n)}{a_n - a} = c_1 + (a_n - a)c_2 + \ldots$$

taking $n \to \infty$, we then get $c_1 = 0$. As $c_0 = c_1 = \ldots = c_k = 0$,

$$f(z) = (z-a)^{k+1}(c_{k+1} + c_{k+2}(z-a) + c_{k+3}(z-a)^2 + \ldots)$$

so that

$$\frac{f(z)}{(z-a)^{k+1}} = c_{k+1} + (z-a)\Big[c_{k+2} + \ldots\Big]$$

put $z = a_n$ then take $n \to \infty$ we get $c_{k+1} = 0$. Therefore, $c_n = 0$ for all $n = 0, 1, 2, \ldots$. $\square$

Theorem. If f is analytic in a domain D such that there is a sequence of distinct points $\{a_n\}$ in D with $a_n \to a \in D$ and $f(a_n) = 0$ for every $n \in \boldsymbol{N}$, then $f(z) = 0$ for all $z \in D$. Remark that this time, we converge to an arbitrary point in the disk and the sequence is in a domain, not a disk.

Proof. We can use the overlapping disk technique. In the domain, we know that the function is zero on a disk with center a. Take any $b \in D$ with $a \neq b$. Then there is a polygonal line segment L from a to b. Notice that L is compact, and $d(L, \partial D) = r > 0$. Now create an overlapping disk with radius δ from a to b such that every disk contains the center of the previous disk. As there exists distinct points that converge to a point whose function values are all zero, all disks have $f(z) = 0$. $\square$

Definition. (Identity theorem) Let f and g be analytic in a domain D. If there is a sequence $\{z_n\}$ which has a limit point in D such that $f(z_n) = g(z_n)$ for every $n \in \boldsymbol{N}$, then $f(z) = g(z)$ for all $z \in D$.

Example. Notice how $f(z) = e^{i/1-z}$ is analytic in $|z| < 1$ ($D(0,1)$).

$$z_n = 1 - \frac{1}{2n\pi}$$

satisfies $|z_n| < 1$, and is a sequence in D with $f(z_n) = 1$ for all $n \in \boldsymbol{N}$. The limit point of $\{z_n\}$ is $1 \notin D(0,1)$, telling us that the identity theorem cannot be used.

Example. Let f and g be analytic in a domain D such that $f(z)g(z) = 0$ for all $z \in D$. Show that $f(z) = 0$ for all $z \in D$ or $g(z) = 0$ for all $z \in D$.

Proof. If $f(z_0) \neq 0$ then there is $\delta > 0$ such that $f(z) \neq 0$ for any $z \in D(z_0, \delta)$. Thus $g(z) = 0$ for all $z \in D(z_0, \delta)$. We can now apply the identity theorem. $\square$

Theorem. If f is analytic in D such that $\bar{D}(z_0, r) \subset D$, then by the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz$$

Then, $|z - z_0| = r$ is a positively oriented simple curve, which we can parametrise like the following.

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

The latter is the expression for the average value of f on $|z - z_0| = r$.

Theorem. If $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ in $D(a, R)$ converges, then for every $0 < r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})|^2\, dt = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}$$

Proof.

$$f(a + re^{it}) = \sum_{n=0}^{\infty} c_n (re^{it})^n = \sum_{n=0}^{\infty} c_n r^n e^{int}$$

This series converges uniformly on $[0, 2\pi]$ (since $r < R$). Therefore,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{it})|^2\, dt =& \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty} c_n r^n e^{int}\sum_{m=0}^{\infty}\bar{c}_m r^m e^{-imt}\, dt \\ =& \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{2\pi} c_n\bar{c}_m r^{n+m} e^{i(n-m)t}\, dt \\ =& \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n\bar{c}_m r^{n+m}\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)t}\, dt\end{aligned}$$

where the far right integral is 1 only when $n = m$ and otherwise 0. □

Theorem. (Maximum modulus theorem I) Let f be analytic in a domain D. Suppose that $|f(z_0)|$ is a local maximum for some $z_0 \in D$, then f is constant in D.

Proof. Suppose $\sup_{t\in[0,2\pi]} |f(z_0+re^{it})| \leq |f(z_0)|$ for some $r > 0$. If $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ in $\bar{D}(z_0, r)$ then

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|^2\, dt \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|^2\, dt = |c_0|^2$$

which means that $c_n = 0$ for all $n \geq 1$. By the identity theorem, f is constant in D. □

Theorem. (Maximum modulus theorem II) If f is analytic inside and on a bounded domain D, then $|f|$ attains its maximum on ∂D.

Proof. $|f|$ is a real valued continuous on a compact set $\bar{D}$ so that $|f|$ has its maximum in $\bar{D}$. But $|f|$ cannot take a maximum in D. □

Theorem. (Fundamental theorem of algebra) A complex polynomial

$$p_n(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$$

with $a_n \neq 0$ has a zero in $\boldsymbol{C}$.

Proof. Suppose that there is no $z_0 \in \boldsymbol{C}$ such that $p(z_0) = 0$ then $f(z) = 1/p(z)$ is a entire function with $\lim_{|z|\to\infty} |f(z)| = 0$. Take $R > 0$ such that $|f(z)| < |f(0)|$ for $|z| = R$. This contradicts the maximum modulus theorem. □

Example. Suppose f is entire with $f(0) = 1$. Let $|f(z)| \leq |e^z|$ for all $z \in \boldsymbol{C}$. What is f?

Proof. Let $g(z) = f(z)/e^z$. As this function is entire, $|g| \leq 1$. □

13 Lecture 13 (April 29th)

Proposition. We shall often use the following power series representations.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

where $z \in \boldsymbol{C}$. An effective way to create a power series representation is to substitute

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

for $z \in D(0,1)$.

Example. Power series expansions allow us to calculate integrals on the complex plane. For example, consider

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

for $z \neq 0$. Then,

$$\int_{|z|=1} e^{1/z}\, dz = \sum_{n=0}^{\infty} \int_{|z|=1} \frac{1}{n! z^n}\, dz = \int_{|z|=1} \frac{1}{z}\, dz = 2\pi i$$

as each term is equal to 0 if $n \neq 1$.

Theorem. (Laurent theorem) Let f be analytic in a multiply connected domain of the form $D = \{z \in \boldsymbol{C} \mid r < |z - z_0| < R\}$. Also, let C be a POSCC in D such that z_0 is inside C. Then,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

In particular, when $n = -1$,

$$a_{-1} = \frac{1}{2\pi i} \int_C f(z)\, dz$$

We will use a_{-1} to find the integral of $f(z)$ on C in the future.

Proof. There is $R_1 < R_2$ such that $R_1 \leq |z - z_0| \leq R_2$ contains C and $R_1 \leq |z - z_0| \leq R_2$ is in D. Then f is analytic in $R_1 \leq |z - z_0| \leq R_2$. By the Cauchy integral formula, for

$R_1 < |z - z_0| < R_2$,

$$\begin{aligned} f(z) =& \frac{1}{2\pi i} \int_{|\xi - z_0| = R_2} \frac{f(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{\xi - z} \, d\xi \\ =& \sum_{n=0}^{\infty} a_n (z - z_0)^n - \frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{\xi - z} \, d\xi \end{aligned}$$

where both are positively oriented simply curves and

$$a_n = \frac{1}{2\pi i} \int_{|\xi - z_0| = R_2} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi$$

For the second part,

$$\frac{1}{z - \xi} = \frac{1}{z - z_0 - (\xi - z_0)} = \frac{1}{z - z_0} \frac{1}{1 - \left(\dfrac{\xi - z_0}{z - z_0} \right)} = \sum_{n=0}^{\infty} \frac{(\xi - z_0)^n}{(z - z_0)^{n+1}}$$

since $|(\xi - z_0)/(z - z_0)| < 1$. As the series converges uniformly on the compact set, we have, for the second part,

$$\begin{aligned} \frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{\xi - z} \, d\xi =& \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{|\xi - z_0| = R_1} \frac{(\xi - z_0)^n}{(z - z_0)^{n+1}} f(\xi) \, d\xi \\ =& \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left[\int_{|\xi - z_0| = R_1} f(\xi)(\xi - z_0)^n \, d\xi \right] (z - z_0)^{-(n+1)} \\ =& \sum_{m=-\infty}^{-1} \left[\frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{(\xi - z_0)^m} \, d\xi \right] (z - z_0)^m \end{aligned}$$

Therefore,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m-\infty}^{-1} a_m (z - z_0)^m$$

where

$$\begin{aligned} a_n =& \frac{1}{2\pi i} \int_{|\xi - z_0| = R_2} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi & n \geq 0 \\ a_m =& \frac{1}{2\pi i} \int_{|\xi - z_0| = R_1} \frac{f(\xi)}{(\xi - z_0)^{m+1}} \, d\xi & m < 0 \end{aligned}$$

as the integrand is analytic between the concentric circles with radius R_1 and R_2, we can generalise to a curve in this domain and say

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi$$

for $n \in \boldsymbol{Z}$. $\square$

Example. By cleverly manipulating functions to be expressed in terms of an infinite summation of a geometric sequence, we can obtain various Laurent series expansions at different regions on the complex plane. Let

$$f(z) = \frac{1}{(z-1)(z-2)}$$

be an analytic function in

(i) Consider $D_1 = \{0 < |z-1| < 1\}$

$$f(z) = -\frac{1}{(z-1)(1-(z-1))} = \sum_{n=-1}^{\infty} (z-1)^n$$

(ii) Consider $D_2 = \{0 < |z-2| < 1\}$

$$f(z) = \frac{1}{((z-2)+1)(z-1)} = \frac{1}{z-2}\sum_{n=1}^{\infty}(-1)^n(z-2)^n = \sum_{n=-1}^{\infty}(-1)^{n+1}(z-2)^n$$

(iii) Consider $D_3 = \{|z| < 1\}$

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

(iv) Consider $D_4 = \{1 < |z| < 2\}$

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

(v) Consider $D_5 = \{|z| > 2\}$

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

14 Lecture 14 (May 8)

Definition. We say that f has a singularity at z_0 if f is not differentiable at z_0. We say that f has an isolated singularity at z_0 provided that in addition to not being differentiable, there is $r > 0$ such that f is differentiable on $D^*(a, r)$

Example. All of these functions have a singularity at z_0

(i) $f(z) = \dfrac{\sin z}{z}$ at $z = 0$. Notice that when we define $f(0) = 1$, then f is entire.

(ii) $f(z) = \dfrac{e^z}{(z-1)^2}$ at $z_0 = 1$. Notice that

$$\lim_{z\to 1}(z-1)^2 f(z) = e \quad \text{and} \quad \lim_{z\to 1}|f(z)| = \infty$$

That is, the function approaches infinity as you approach the singularity.

(iii) $f(z) = e^{1/z}$ at $z_0 = 0$. Notice that

$$\lim_{x \to 0^+} e^{1/x} = +\infty \quad \text{but} \quad \lim_{x \to 0^-} e^{1/x} = 0$$

(iv) $f(z) = \dfrac{1}{\sin(\pi/z)}$ at $z_0 = 0$. For $z_n = 1/n$, f has a singularity at each $z_n = 1/n$.

In each case, we see a removable, pole, essential, and non-isolated singularity!

Definition. Let f has an isolated singularity at z_0.

(i) (Removable) If we can define $f(z_0)$ such that f is analytic at z_0, then we say that f has a removable singularity at z_0.

(ii) (Pole) If there is a $k \in \boldsymbol{N}$ such that

$$\lim_{z \to z_0} (z - z_0)^k f(z) = \alpha \neq 0$$

we say that f has a pole of order k at z_0. If $k = 1$, f is said to have a simple pole.

(iii) (Essential) If f satisfies neither of the two above, we then say that f has an essential singularity.

Corollary. If f has an isolated singularity at z_0, then the Laurent series of f,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is available in some $D^*(z_0, r)$.

Theorem. (Riemann) If f has an isolated singularity at z_0 and $|f(z)|$ is bounded and analytic on some $D^*(z_0, r)$, then the singularity at z_0 is removable.

Proof. Define h on $D(z_0, r)$ such that

$$h(z_0) = 0 \quad \text{and} \quad h(z) = (z - z_0)^2 f(z)$$

on $D(z_0, r)$. Then,

$$h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)^2 f(z)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0$$

since $f(z)$ is bounded on $D^*(z_0, r)$. Thus h is analytic on $D(z_0, r)$ so that

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

As such, we see that on $D(z_0, r)$,

$$f(z) = \frac{a_0}{(z - z_0)^2} + \frac{a_1}{(z - z_0)} + a_2 + a_3(z - z_0) + a_4(z - z_0)^2 + \dots$$

However, we know that there should be no divergent terms as f is bounded, and $a_0 = a_1 = 0$. Accordingly, we can then define

$$f(z_0) = a_2$$

to create a power series expansion that is convergent on z_0 such that f on the disk $D(z_0, r)$ is analytical. □

Theorem. (Casorati-Weierstrass) If f has an essential singularity at z_0, then for every $r > 0$, $f(D^*(z_0, r))$ is dense in $\boldsymbol{C}$.

Proof. Suppose not, then there is $w_0 \in \boldsymbol{C}$ and $\delta > 0$ such that $f(D^*(z_0, r)) \cap D(w_0, \delta) = \varnothing$. Then $|f(z) - w_0| \geq \delta$ for all $z \in D^*(z_0, r)$. Define

$$g(z) = \frac{1}{f(z) - w_0}$$

on $D^*(z_0, r)$. Then $|g(z)| \leq 1/\delta$ for all $z \in D^*(z_0, r)$. By the previous theorem, we can define $g(z_0)$ so that $g(z)$ is analytic on $D^*(z_0, r)$. Let

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

on $D^*(z_0, r)$. That is,

$$\frac{1}{f(z) - w_0} = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

on $D^*(z_0, r)$. In such a case,

$$\lim_{z \to z_0} \frac{1}{f(z) - w_0} = a_0$$

Suppose, firstly, that $a_0 \neq 0$. Then we have

$$f(z) = \frac{1}{g(z)} + w_0$$

and f has a removable singularity at z_0.

In the second case, let $a_0 = 0$. Take $k \in \boldsymbol{N}$ to be the smallest k integer such that $a_k \neq 0$. Then,

$$\frac{1}{f(z) - w_0} = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \dots$$

at $D^*(z_0, r)$. Thus,

$$\frac{1}{(f(z) - w_0)(z - z_0)^k} = a_k + a_{k+1}(z - z_0) + \dots$$

on $D^*(z_0, r)$. We see that

$$\lim_{z \to z_0} \frac{1}{(f(z) - w_0)(z - z_0)^k} = a_k \neq 0$$

with

$$\lim_{z \to z_0} (f(z) - w_0)(z - z_0)^k = \lim_{z \to z_0} (z - z_0)^k f(z) + \lim_{z \to z_0} w_0(z - z_0)^k = \frac{1}{a_k} \neq 0$$

Therefore, in the second case, f has a pole of order k. □

Theorem. (Picard's Great Theorem) If f has an essential singularity at z_0 then for every $r > 0$, $f(D^*(z_0, r))$ takes every complex number (except possibly one) infinitely many times.

Theorem. If f has a pole of order $k \in \boldsymbol{N}$ at z_0, then $f(z) = \sum_{n=-k}^{\infty} a_n(z - z_0)^n$ on some $D^*(z_0, r)$.

Proof. Since $\lim_{z \to z_0} (z - z_0)^k f(z) = \alpha \neq 0$, $(z - z_0)^k f(z)$ has a removable singularity at z_0 so that

$$(z - z_0)^k f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

on some $D^*(z_0, r)$. Thus

$$f(z) = \sum_{n=-k}^{\infty} c_{n+k}(z - z_0)^n$$

on some $D^*(z_0, r)$. □

Corollary. If f has a simple pole at z_0, then

$$\mathrm{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$$

as

$$
\begin{aligned}
f(z) =& a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \dots \\
(z-z_0)f(z) =& a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \dots
\end{aligned}
$$

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Remark. If f has a simple pole at z_0, then

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z)$$

We have previously remarked that L'Hospital's theorem works in the complex plane.

Example. Let $f(z) = \pi\cot\pi z = \pi\dfrac{\cos\pi z}{\sin\pi z}$. This function has a simple pole at each $n \in \boldsymbol{Z}$.

$$\operatorname*{Res}_{z=n} f(z) = \lim_{z\to n}(z-n)\pi\cot\pi z = \pi\cos n\pi \lim_{z\to n}\frac{z-n}{\sin\pi z} = \pi\cos n\pi \lim_{z\to n}\frac{1}{\pi\cos\pi z} = 1$$

where in the second last line, we have used L'Hospital's theorem.

Example. Observe that

$$\operatorname*{Res}_{z=\pi i}\left(\frac{1}{e^z+1}\right) = \lim_{z\to\pi i}(z-\pi i)\frac{1}{e^z+1} = \lim_{z\to\pi i}\frac{1}{e^z} = -1$$

Theorem. (Residue theorem 1) Let f be analytic inside and on a POSCC except for an isolated singularity at z_0 inside C. Then

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \operatorname*{Res}_{z=z_0} f(z)$$

Proof. Note that due to the Cauchy theorem,

$$\int_C f(z)\,dz = \int_{|z-z_0|=\delta} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z)$$

□

Example. Evaluate

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$$

by using the residue theorem.

Proof. Let the function $f(z) = 1/(1+z^2)$ be on D_R, a positively oriented upper half circle with radius $R > 1$. Then by the residue theorem,

$$\int_{C_R} f(z)\, dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = 2\pi i \lim_{z\to i}(z-i)\frac{1}{1+z^2} = 2\pi i \times \frac{1}{2i} = \pi$$

for all $R > 1$. On the other hand, by parametrization,

$$\int_{C_R} f(z)\, dz = \int_{-R}^{R} \frac{1}{1+x^2}\, dx + \int_0^{\pi} \frac{1}{1+R^2e^{2it}} iRe^{it}\, dt$$

where we substituted $z = Re^{it}$. We then have

$$\Big|\int_0^{\pi} \frac{1}{1+R^2e^{2it}} iRe^{it}\, dt\Big| \le \int_0^{\pi} \frac{R}{R^2-1}\, dt = \frac{\pi R}{R^2-1}$$

which $\to 0$ as $R \to \infty$. We thus found that the integral is equal to π. $\square$

Example. Consider

$$f(z) = \frac{e^{\alpha z}}{1+e^z}$$

for $0 < \alpha < 1$ on the rectangular contour with base $2R$ and height 2π with its base along the x-axis. Our aim is to evaluate

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x}\, dx$$

Proof.

$$\operatorname*{Res}_{z=\pi i} f(z) = \lim_{z\to\pi i}(z-\pi i)\frac{e^{\alpha z}}{1+e^z} = e^{\alpha\pi i}(-1)$$

Thus by the residue theorem,

$$\int_{C_R} \frac{e^{\alpha z}}{1+e^z}\, dz = 2\pi i(-e^{\alpha\pi i})$$

for all $R > 1$. On the other hand,

$$\int_{C_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_0^{2\pi} f(R+iy)i\, dy + \int_R^{-R} f(x+2\pi i)\, dx + \int_{2\pi}^{0} f(-R+iy)i\, dy$$

or, equivalently,

$$\int_{C_R} f(z)\, dz = \int_{-R}^{R} \frac{e^{\alpha x}}{1+e^x}\, dx + \underbrace{\int_0^{2\pi} \frac{e^{\alpha(R+iy)}}{1+e^{R+iy}} i\, dy}_{\text{II}} - \int_R^{-R} \frac{e^{\alpha(x+2\pi i)}}{1+e^{x+2\pi i}}\, dx - \underbrace{\int_0^{2\pi} \frac{e^{\alpha(-R+iy)}}{1+e^{-R+iy}} i\, dy}_{\text{IV}}$$

Note that

$$|\,\mathrm{II}\,| \le \int_0^{2\pi} \frac{e^{\alpha R}}{e^R - 1}\,dy = \frac{2\pi e^{\alpha R}}{e^R - 1} \to 0$$
$$|\,\mathrm{IV}\,| \le \int_0^{2\pi} \frac{e^{-\alpha R}}{1 - e^{-R}}\,dy = \frac{2\pi e^{-\alpha R}}{1 - e^{-R}} \to 0$$

as $R \to \infty$ because $0 < \alpha < 1$. Accordingly,

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = (1 - e^{2\pi\alpha i}) \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1 + e^x}\,dx = -2\pi i e^{\alpha\pi i}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1 + e^x} = -\frac{2\pi i e^{\alpha\pi i}}{1 - e^{2\pi\alpha i}} = \frac{2\pi i}{e^{\alpha\pi i} - e^{-\alpha\pi i}} = \frac{\pi}{\sin\alpha\pi}$$

$\square$

Remark. Here, substitute $e^x = t$. Then, $dt = e^x\,dx = t\,dx$.

$$\int_0^{\infty} \frac{t^{\alpha-1}}{1+t}\,dt = \frac{\pi}{\sin\alpha\pi}$$

Now, again let $t = x^\beta$ for $0 < \beta < \infty$ to obtain $dt = \beta x^{\beta-1}\,dx$.

$$\int_0^{\infty} \frac{x^{\alpha\beta-1}}{1+x^\beta}\,dx = \frac{1}{\beta}\frac{\pi}{\sin\alpha\pi}$$

Now as $0 < \beta < \infty$, let $\alpha = 1/\beta$, getting

$$\int_0^{\infty} \frac{1}{1+x^\beta}\,dx = \frac{1}{\beta}\frac{\pi}{\sin\pi/\beta}$$

Telling us that

$$\int_0^{\infty} \frac{1}{1+x^\beta}\,dx$$

converges when $\beta > 1$. This example readily shows the beauty of complex integration, with parametrized integrals leaving us with powerful results.

Theorem. (Residue theorem 2) Let f be analytic inside and on a POSCC C except for finite isolated singularities at $z_1, \ldots, z_n$ inside C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Proof. As the singularities are isolated, there is $r > 0$ such that

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{|z-z_k|=r} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

by the Cauchy theorem where $|z - z_r| = k$ is POS (positively oriented and simple). $\square$

Definition. (Zeta function) The zeta function is given as

$$\xi(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

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Example. Consider the Laurent series of $f(z) = \pi \cot \pi z$ at $z = 0$. From

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots \quad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

we have

$$\begin{aligned}
\pi \cot \pi z =& \pi \frac{\cos \pi z}{\sin \pi z} \\
=& \pi \frac{1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \dots}{\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \dots} \\
=& \pi \frac{1}{\pi z} \frac{1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24}}{1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{120}}
\end{aligned}$$

After preforming long division, we get

$$\pi \cot \pi z = \frac{1}{z}\Big[1 - \frac{\pi^2 z^2}{3} - \frac{\pi^4 z^4}{45} - \frac{2\pi^6 z^6}{945} + \dots\Big]$$

From this, we see how

$$g(z) = \frac{\pi \cot \pi z}{z^2} \implies \operatorname*{Res}_{z=0} g(z) = -\frac{\pi^2}{3}$$

and

$$h(z) = \frac{\pi \cot \pi z}{z^4} \implies \operatorname*{Res}_{z=0} h(z) = -\frac{\pi^4}{45}$$

We can continue this indefinitely,

$$\operatorname*{Res}_{z=0} \frac{\pi \cot \pi z}{z^6} = -\frac{2\pi^6}{945}$$

The reason why this is important is as follows.

Definition. (Squares lemma) In advanced mathematics, we often see the square contour C_N, with edges at $\pm(N+1/2)i$ and $\pm(N+1/2)$ (taking $N \in \boldsymbol{N}$). There is an upper bound M $(= 2)$ such that

$$|\cot \pi z| \leq M$$

for all $z \in C_N$ and for all $N \in \boldsymbol{N}$.

Proof. Take $z = x + iy$. Consider cutting the square in three parts with $y = 1/2$ and $y = -1/2$. We show that ff $y > 1/2$, then $|\cot \pi z| \leq 2$ and if $y < -1/2$ then $|\cot \pi z| \leq 2$. We also show that on the left and right edges of the middle cut, $|\cot \pi z| \leq 1$. Note that

$$\cot \pi z = \frac{\cos \pi z}{\sin \pi z} = \frac{\dfrac{e^{i\pi z} + e^{-i\pi z}}{2}}{\dfrac{e^{i\pi z} - e^{-i\pi z}}{2i}}$$

and that

$$|\cot \pi z| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right| = \left|\frac{e^{i\pi x}e^{-\pi y} + e^{-i\pi x}e^{\pi y}}{e^{i\pi x}e^{-\pi y} - e^{i\pi x}e^{\pi y}}\right|$$

We see that

$$|e^{i\pi z}| = |e^{i\pi(x+iy)}| = e^{-\pi y} \quad \text{and} \quad |e^{-i\pi z}| = e^{\pi y}$$

assuming $y > 1/2$, we automatically have $e^{\pi y} > e^{-\pi y}$ and that

$$|\cot \pi z| \leq \frac{e^{-\pi y} + e^{\pi y}}{e^{\pi y} - e^{-\pi y}} = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}} < 2$$

Meanwhile, if $y < -1/2$, then

$$|\cot \pi z| \leq \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} - e^{\pi y}} = \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}} < 2$$

Now if $-1/2 \leq y \leq 1/2$ and $z = (N + 1/2) + iy$,

$$|\cot \pi z| = \left|\cot \pi\left(N + \frac{1}{2} + iy\right)\right| = \left|\cot\left(\frac{\pi}{2} + i\pi y\right)\right| = |\tan(i\pi y)| = \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right| \leq 1$$

Also, if $-1/2 \leq y \leq 1/2$ and $z = -(N + 1/2) + iy$, the same thing happens and

$$|\cot \pi z| = |\tan(i\pi y)| \leq 1$$

□

Example. Consider $g(z) = \pi \cot \pi z / z^2$ on the square contour C_N. We can use both (1) the residue theorem and (2) parameterisation. By the residue theorem,

$$\begin{aligned}
\int_{C_N} \frac{\pi \cot \pi z}{z^2}\, dz =& 2\pi i \sum_{k=-n}^{n} \operatorname*{Res}_{z=k} g(z) \\
=& 2\pi i \Big[\operatorname*{Res}_{z=0} g(z) + 2 \sum_{k=1}^{N} \operatorname*{Res}_{z=k} g(z) \Big] \\
=& 2\pi i \Big[-\frac{\pi^2}{3} + 2 \sum_{k=1}^{N} \frac{1}{k^2} \Big]
\end{aligned}$$

where

$$\operatorname*{Res}_{z=k, k\neq 0} g(z) = \lim_{z\to k} (z-k) \frac{\pi \cot \pi z}{z^2} = \frac{1}{k^2}$$

and

$$\operatorname*{Res}_{z=0} g(z) = -\frac{\pi^2}{3}$$

On the other hand,

$$\Big| \int_{C_N} g(z)\, dz \Big| = \Big| \int_{C_N} \frac{\pi \cot \pi z}{z^2} \Big| \le \frac{M}{N^2}(8N+2) \to 0$$

with M being less than 2π. We thus find

$$2\pi i \Big[2 \sum_{k=1}^{\infty} \frac{1}{k^2} - \frac{\pi^2}{3} \Big] = 0$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Also, we can find

$$\int_{C_N} \frac{\pi \cot \pi z}{z^4}\, dz = 2\pi i \Big[2 \sum_{k=1}^{N} \frac{1}{k^4} - \frac{\pi^4}{45} \Big]$$

which leads to

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

In this manner,

$$\xi(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}}$$

can be found.

Example. Find

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k^2} - 2 \cdot \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12}$$

and

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi}{24}$$

Example. Evaluate

$$\sum_{k=-\infty}^{\infty} \frac{1}{(n+a)^2}$$

for $a \notin \boldsymbol{Z}$

Proof. Define, on C_N

$$f(z) = \frac{\pi \cot \pi z}{(z+a)^2}$$

on C_N. Then

$$0 = 2\pi i \Big[\sum_{n=-\infty}^{\infty} \operatorname*{Res}_{z=n} f(z) + \operatorname*{Res}_{z=-a} f(z) \Big]$$

where

$$\operatorname*{Res}_{z=-a} f(z) = \lim_{z \to -a} [\pi \cot \pi z]' = -\pi^2 \csc^2 \pi a$$

Noting that $f(z)$ has a pole of order 2, we then have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \Big(\frac{\pi}{\sin \pi a} \Big)^2$$

□

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Remark. If f has a simple pole at z_0, then

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \frac{p(z)}{(q(z) - q(z_0))/(z - z_0)} = \frac{p(z_0)}{q'(z_0)}$$

if we have $f(z) = p(z)/q(z)$ with $q(z_0) = 0, p(z_0) \neq 0$.

Example. For example, take

$$f(z) = \frac{e^{iz} z^3}{z^4 + 16}$$

in $\operatorname{Im} z \geq 0$. Observe that

$$\begin{aligned} z^4 = -16 =& 2^4 e^{\pi i} \\ z =& 2\exp\left(\frac{\pi i}{4} + \frac{2k\pi i}{4}\right) \end{aligned}$$

for $k = 0, 1, 2, 3, \ldots$. Then

$$z_1 = 2\exp\left(\frac{\pi i}{4}\right) = \sqrt{2} + i\sqrt{2} \quad z_2 = 2\exp\left(\frac{3\pi i}{4}\right) = -\sqrt{2} + i\sqrt{2}$$

in $\operatorname{Im} z \geq 0$. f therefore has simple poles at z_1, z_2 in the domain, and

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{e^{iz_1} z_1^3}{4z_1^3} = \frac{1}{4}e^{iz_1} = \frac{1}{4}e^{-\sqrt{2}+i\sqrt{2}}$$

or

$$\operatorname*{Res}_{z=z_2} f(z) = \frac{e^{iz_2} z_2^3}{4z_2^3} = \frac{1}{4}e^{iz_2} = \frac{1}{4}e^{-\sqrt{2}-i\sqrt{2}}$$

Theorem. In complex analysis, we often use upper hemispheres and shells. There are two important theorems regarding the contour integrals at the rim C_R.

(i) Take $f(z) = \dfrac{p(z)}{q(z)}$ ($q \neq 0$ on C_R) with $\deg q(z) \geq$ p(z) +2. Then

$$\lim_{R\to\infty} \int_{C_R} \frac{p(z)}{q(z)}\, dz = 0$$

(ii) The Jordan lemma, which stems from the fact that

$$\left| \int_{C_R} e^{iz}\, dz \right| < \pi$$

for all $R > 0$. Or, more generally,

$$\left| \int_{C_R} e^{iaz}\, dz \right| < \frac{\pi}{a}$$

for $a > 0$.

Proof.

$$\begin{aligned}\int_{C_R} e^{iz}\,dz = \int_0^\pi e^{iRe^{it}} iRe^{it}\,dt \\ \Big|\int_{C_R} e^{iz}\,dz\Big| \leq \int_0^\pi \left|e^{iRe^{it}} iRe^{it}\right| dt \\ =R\int_0^\pi e^{-R\sin t}\,dt = 2R\int_0^{\pi/2} e^{-R\sin t}\,dt \leq \pi(1-e^{-R})\end{aligned}$$

Noting that

$$\sin t \geq \frac{2}{\pi} t$$

for $0 \leq t \leq \pi/2$. □

Theorem. (Jordan lemma) If $|f(z)| \leq M_R$ for $z \in C_R$ and $\lim_{R\to\infty} M_R = 0$, then

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}\,dz = 0$$

for $a > 0$ since

$$\Big|\int_{C_R} f(z)e^{iaz}\,dz\Big| \leq M_R R\int_0^\pi e^{-aR\sin t}\,dt \leq \frac{M_R\pi}{a} \to 0$$

Example. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

Let

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

defined on the upper hemisphere with radius R (whole contour C_R and $R > 2$). By the residue theorem,

$$\int_{C_R} f(z)\,dz = 2\pi i\Big(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=2i} f(z)\Big)$$

We find

$$\operatorname*{Res}_{z=i} f(z) = \operatorname*{Res}_{z=i} \frac{\frac{z^2}{z^2+4}}{z^2+1} = -\frac{1}{6i}$$

and

$$\operatorname*{Res}_{z=2i} f(z) = \operatorname*{Res}_{z=2i} \frac{\frac{z^2}{z^2+1}}{z^2+4} = \frac{1}{3i}$$

Meanwhile,

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = 0$$

since $\deg(z^2+1)(z^2+4) = \deg z^2 + 2$, and the above becomes

$$\int_{C_R} f(z)\,dz = 2\pi i\Big[-\frac{1}{6i}+\frac{1}{3i}\Big] = \frac{\pi}{3}$$

Example. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3\sin x}{x^4+16}\,dx$$

Define

$$f(z) = \frac{e^{iz}z^3}{z^4+16} \quad \text{while} \quad \operatorname{Im} f(z) = \frac{x^3\sin x}{x^4+16}$$

on the upper hemisphere C_R with $R > 2$. Then,

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{1}{4}e^{-\sqrt{2}+i\sqrt{2}} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{1}{4}e^{-\sqrt{2}-i\sqrt{2}}$$

Finishing off,

$$\begin{aligned}\int_{C_R} f(z)\,dz = 2\pi i\Big[\frac{1}{4}e^{-\sqrt{2}+i\sqrt{2}}+\frac{1}{4}e^{-\sqrt{2}+i\sqrt{2}}\Big]& \\ &= \frac{\pi i}{2}e^{-\sqrt{2}}\Big(e^{i\sqrt{2}}+e^{-i\sqrt{2}}\Big) \\ &= \pi i e^{-\sqrt{2}}\cos\sqrt{2}\end{aligned}$$

Meanwhile,

$$\begin{aligned}\lim_{R\to\infty}\int_{C_R} f(z)\,dz &= \int_{-\infty}^{\infty}\frac{e^{ix}x^3}{x^4+16}\,dx + \lim_{R\to\infty}\int_0^{\pi} \cdot\, dz \\ &= \int_{-\infty}^{\infty}\frac{x^3\cos x}{x^4+16}\,dx + i\int_{-\infty}^{\infty}\frac{x^3\sin x}{x^4+16}\,dx\end{aligned}$$

where the term on the first line vanishes due to Jordan's lemma.

Example. Evaluate, for $a > 0$,

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,dx$$

Define

$$f(z) = \frac{e^{iz}}{z^2+a^2}$$

on C_R with $R > a$. By the residue theorem,

$$\int_{C_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=ai} f(z) = 2\pi i\frac{e^{-a}}{2ai} = \pi\frac{e^{-a}}{a}$$

On the other hand,

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = \int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+a^2}\,dx + \lim_{R\to\infty}\int_0^{\pi}\cdot\,dx = \int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,dx + i\int_{-\infty}^{\infty}\frac{\sin x}{x^2+a^2}\,dx$$

where the second term goes to zero by Jordan's lemma. So,

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,dx = \pi\frac{e^{-a}}{a}$$

Now, put $x = \beta t$ for $\beta > 0$, to obtain

$$\int_{-\infty}^{\infty}\frac{\cos\beta t}{\beta^2t^2+a^2}\beta\,dt = \frac{\pi e^{-a}}{a}$$

We succeedingly put $\alpha = a/\beta$ and we get

$$\int_{-\infty}^{\infty}\frac{\cos\beta t}{t^2+\alpha^2}\,dt = \frac{\pi}{\alpha}e^{-\alpha\beta}$$

for $\alpha, \beta > 0$. Lastly, differentiate the function with respect to t.

$$\int_{-\infty}^{\infty}\frac{-\beta\sin\beta t}{t^2+\alpha^2}\,dt = -\pi e^{-\alpha\beta}$$

For $\beta = 1$,

$$\int_{-\infty}^{\infty}\frac{t\sin t}{t^2+\alpha^2}\,dt = \pi e^{-\alpha}$$

with $\alpha > 0$. Taking $\alpha \to 0^+$,

$$\int_{-\infty}^{\infty}\frac{\sin t}{t}\,dt = \pi$$

Similarly, we can differentiate the above with respect to α instead of t to get

$$\int_{-\infty}^{\infty}\frac{x\sin x}{(x^2+a^2)^2}\,dx$$

Example. Find

$$\int_0^{\infty}\frac{\sin x}{x}\,dx$$

Define

$$f(z) = \frac{e^{iz}}{z}$$

on the shell $C_{\varepsilon,R}$ ($0 < \varepsilon < 1$ and $R > 1$) for which it is analytic. We find

$$\int_{C_{\varepsilon,R}} f(z)\,dz = \int_{-R}^{R}\frac{e^{ix}}{x}\,dx + \int_0^{\pi} f(Re^{it})iRe^{it}\,dt + \int_{-R}^{-\varepsilon}\frac{e^{ix}}{x}\,dx - \int_0^{\pi}\frac{e^{i\varepsilon e^{it}}}{\varepsilon e^{it}}i\varepsilon e^{it}\,dt$$

the second term vanishes due to the Jordan lemma while the last term becomes

$$-\int_0^\pi ie^{i\varepsilon e^{it}}\,dt \to -\pi i$$

as $\varepsilon \to 0$.

18 Lecture 18 (May 22nd)

Example. Define the function

$$f(z) = \frac{e^{iz}}{z}$$

on the upper shell with inner radius ε and outer radius R.

$$\int_{C_R} f(z)\,dz = \int_\varepsilon^R \frac{e^{ix}}{x}\,dx + \int_0^{2\pi} \frac{e^{iRe^{it}}}{Re^{it}} iRe^{it}\,dt + \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx - \int_0^\pi \frac{e^{i\varepsilon e^{it}}}{\varepsilon e^{it}} i\varepsilon e^{it}\,dt$$

Taking the imaginary part, we have

$$\int_0^\infty \frac{e^{ix} - e^{-ix}}{x}\,dx = \pi i$$

as $\varepsilon \to 0$ and $R \to \infty$. We therefore have, by definition,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Example. Find

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

Notice that $2\sin^2 x = 1 - \cos 2x = \mathrm{Re}(1 - e^{2ix})$. Then, we can define

$$f(z) = \frac{1 - e^{2iz}}{z^2}$$

on the same shell as above. For every $0 < \varepsilon < 1$ and $R > 1$ we have

$$0 = \int_{C_{\varepsilon,R}} \frac{1 - e^{2iz}}{z^2}\,dz = \int_\varepsilon^R \frac{1 - e^{2ix}}{x^2}\,dx + \int_0^\pi \frac{1 - e^{2iRe^{it}}}{(Re^{it})^2} iRe^{it}\,dt + \int_{-R}^{-\varepsilon} \frac{1 - e^{2ix}}{x^2}\,dx - 2\pi$$

We see that the second term vanishes to 0 as $R \to \infty$. Also, we used that

$$f(z) = \frac{1}{z^2}[-2iz + 2z^2 + \ldots] = -\frac{2i}{z} + 2 + \ldots$$

and that

$$\lim_{\varepsilon\to 0} \int_0^\pi f(\varepsilon e^{it}) i\varepsilon e^{it}\,dt = \lim_{\varepsilon \to 0} \int_0^\pi \frac{-2i}{\varepsilon e^{it}} i\varepsilon e^{it}\,dt = 2\pi$$

Taking the real part, we have

$$\int_0^\infty \frac{1-e^{2ix}}{x^2}\,dx + \int_0^\infty \frac{1-e^{-2ix}}{x}\,dx = 2\pi$$

and

$$2\int_0^\infty \frac{\sin^2 x}{x^2}\,dx + 2\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = 2\pi \quad \text{or} \quad \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

Example. Find

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx$$

On the branch $-\pi/2 < \arg z < 3\pi/2$, $\ln z$ is analytic. Define

$$f(z) = \frac{\log z}{(1+z^2)^2}$$

which has a pole of order 2 at $z = i$ inside $C_{\varepsilon,R}$. Recall, for a pole of order k,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(k-1)!}\lim_{z\to z_0}\Big[(z-z_0)^k f(z)\Big]^{(k-1)}$$

and we have

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\leftarrow i}\Big[(z-i)^2\frac{\log z}{(1+z^2)^2}\Big] = \lim_{z\to i}\Big[\frac{\log z}{(z+i)^2}\Big]' = \lim_{z\to i}\Big[\frac{\frac{1}{2}(z+i)^2 - 2(z+i)\log z}{(z+i)^4}\Big] = \frac{\pi+2i}{8}$$

Which implies that

$$\int_{C_{\varepsilon,R}} \frac{\log z}{(1+z^2)^2}\,dz = 2\pi i\frac{\pi+2i}{8}$$

for all $0 < \varepsilon < 1$ and $R > 1$.

$$|\,\mathrm{II}\,| = \Big|\int_0^\pi \frac{\log Re^{it}}{(1+R^2e^{2it})^2} iRe^{it}\,dt\Big| \le \int_0^\pi \frac{\log R + \pi}{(R^2-1)^2} R\,dt \to 0$$

as $R \to \infty$.

$$|\,\mathrm{IV}\,| = \Big|-\int_0^\pi \frac{\log \varepsilon e^{it}}{(1+\varepsilon^2 e^{2it})^2} i\varepsilon e^{it}\,dt\Big| \le \int_0^\pi \frac{\varepsilon[\log\varepsilon + \pi]}{(1-\varepsilon^2)^2}\varepsilon\,dt$$

While $\lim_{\varepsilon\to 0^+}\varepsilon\ln\varepsilon = 0$ as

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

Also,

$$|\,I\,|+|\,III\,| = \int_\varepsilon^R \frac{\ln x}{(1+x^2)^2}\,dx + i\int_\varepsilon^R \frac{\pi}{(1+x^2)^2}\,dx + \int_\varepsilon^R \frac{\ln x}{(1+x^2)^2} = 2\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx + i\int_0^\infty \frac{\pi}{(1+x^2)^2}\,dx$$

which implies that

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,dx = -\frac{\pi}{4}$$

19 Lecture 19 (May 27th)

Example. (Reverse parametrisation) Solve

$$\int_0^{2\pi} \frac{1}{5+4\sin t}\,dt$$

using the following equality

$$\int_0^{2\pi} f(e^{it}) ie^{it}\,dt = \int_{|z|=1} f(z)\,dz$$

Proof.

$$\begin{aligned}
\int_0^{2\pi} \frac{dt}{5+4\sin t} &= \int_{|z|=1} \frac{1}{5+4\dfrac{z-z^{-1}}{2i}}\frac{1}{iz}\,dz \\
&= \int_{|z|=1} \frac{1}{5iz+2z^2-1}\,dz \\
&= 2\pi i \operatorname*{Res}_{z=-i/2} \frac{1}{2z^2+5iz-2} = 2\pi i \left[\frac{1}{4z+5i}\right]_{z=-i/2} \\
&= 2\pi i \frac{1}{3i} = \frac{2\pi}{3}
\end{aligned}$$

Where utilizing the quadratic equation, the roots of the denominator are $(-5i \pm 3i)/4$. □

Example. If $-1 < a < 1$, then

$$\int_0^{2\pi} \frac{1}{1+a\sin t}\,dt = \frac{2\pi}{\sqrt{1-a^2}} = \int_0^{2\pi} \frac{1}{1+a\cos t}\,dt$$

Example. For $n \in \boldsymbol{N}$, evaluate

$$\int_0^{2\pi} \sin^{2n} t\,dt$$

Let $z = e^{it}$ for $0 \le t \le 2\pi$. Then, $\sin t = z - z^{-1}/2i$ and $dt = dz/iz$ so that

$$\int_0^{2\pi} \sin^{2n} t\,dt = \int_{|z|=1} \left(\frac{z-1/z}{2i}\right)^{2n} \frac{dz}{iz} = 2\pi i \operatorname*{Res}_{z=0} f(z)$$

where

$$f(z) = \frac{1}{(2i)^{2n}}\frac{1}{iz}(z - \frac{1}{z})^{2n}$$

Here,

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(2i)^{2n}}\frac{1}{i}{}^{2n}C_n(-1)^n = \frac{(2n)!}{(n!)^2 2^{2n}}\frac{1}{i}$$

The result of the integral is therefore

$$2\pi\frac{(2n)!}{(n!)^2 2^{2n}}$$

Definition. (Fourier transform) With $(\hat{f})^\vee = f$, the Fourier transform is defined as

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ixt}\,dt \quad \text{and} \quad f^\vee(t) = \int_{-\infty}^{\infty} f(x)e^{2\pi ixt}\,dx$$

if they exist.

Example. For $f(x) = e^{-\pi x^2}$, find the Fourier transform.

$$\widehat{f}(x) = \int_{-\infty}^{\infty} e^{-\pi t^2}e^{-2\pi ixt}\,dt$$

Notice that

$$e^{-\pi t^2}e^{-2\pi xt} = e^{-\pi(t^2+2ixt)} = e^{-\pi(t+ixt)^2-\pi x^2}$$

Therefore,

$$\widehat{f}(x) = e^{-\pi x^2}\int_{-\infty}^{\infty} e^{-\pi(t+ix)^2}\,dt$$

For $x \neq 0$, in finding the integral, we can take the square contour with height x and width $2R$ for the following function:

$$0 = \int_{C_R} e^{-\pi z^2}\,dz = \int_{-R}^{R} e^{-\pi t^2}\,dt + \int_0^x e^{-\pi(R+iy)^2} i\,dy - \int_{-R}^{R} e^{-\pi(t+ix)^2}\,dx + \int_x^0 e^{-\pi(-R+iy)^2} i\,dy$$

We can see that the second and last term approaches 0 as $R \to \infty$.

$$\int_{-\infty}^{\infty} e^{-\pi(t+ix)^2}\,dt = \int_{-\infty}^{\infty} e^{-\pi t^2}\,dt = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-s^2}\,ds = 1$$

taking $s = \sqrt{\pi}t$. We see that the Fourier transform of the function is itself.

Remark. For the remaining time being, we study (1) the argument principle, which uses Rouches theorem, and (2) the Poisson integral.

Theorem. Let f be analytic inside and on a POSCC C with $f \neq 0$ on C. Then f has finitely many zeros inside C. Then there exists $z_1, \ldots, z_n$ inside C and $\alpha_k \in \boldsymbol{N}$ $(1 \leq k \leq n)$

such that

$$f(z) = (z - z_1)^{\alpha} \dots (z - z_n)^{\alpha_n} F(z)$$

where F is analytic with no zeros inside and on C. Then on C,

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{\alpha_k}{z - z_k} + \frac{F'(z)}{F(z)}$$

Thus

$$\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \int_C \frac{\alpha_k}{z - z_k}\,dz + \int_C \frac{F'(z)}{F(z)}\,dz$$

Proof. If $f(z_0) = 0$ for some z_0 inside C, then $f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \dots$ in a neighborhood of z_0. Therefore,

$$f(z) = (z - z_0)\Big(a_1 + a_2(z - z_0) + \dots\Big)$$

such that $g(z) = f(z)/(z - z_0)$ has a removable singularity at z_0. As g is analytic inside and on C, $f(z) = (z - z_0)g(z)$ for some analytic function g inside and on C. □

20 Lecture 20 (May 29th)

Theorem. (Argument principle) The number of times $f(C)$ winds up the origin in the positive sense is

$$\frac{1}{2\pi} \Delta_C f(z)$$

which in turn equal to the number of zeros inside C.

Proof. Let f be analytic inside and on a POSCC C with no zeros on C. If f has zeros at $z_1, z_2, \dots, z_n$ inside C with multiplicity $\alpha_1, \alpha_2, \dots, \alpha_n$ respectively, then

$$f(z) = (z - z_1)^{\alpha_1} \dots (z - z_n)^{\alpha_n} F(z)$$

with F being analytic with no zeros inside and on C. Thus on C,

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{n} \frac{\alpha_k}{z - z_k} + \frac{F'(z)}{F(z)}$$

Where $F'(z)/F(z)$ is analytic inside and on C. Thus

$$\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \int_C \frac{\alpha_k}{z - z_k}\,dz + \int_C \frac{F'(z)}{F(z)}\,dz = 2\pi i \sum_{k=1}^{n} \alpha_k$$

Thus

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$$

is the number of zeros of f inside C. On the other hand, just on C,

$$\int_C \frac{f'(z)}{f(z)}\,dz = \Big[\log f(z)\Big]_C = \Big[\ln|f(z)| + i\arg f(z)\Big]_C = i\mathit{\Delta}_C \arg f(z)$$

where $\mathit{\Delta}_C \arg f(z)$ is the change of $\arg f(z)$ as z tranverses C. This implies that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\mathit{\Delta}_C \arg f(z)$$

The RHS is the number of times $f(C)$ winds up the origin in the positive sense. □

Theorem. (Rouche's theorem) Let f and g be analytic inside and on a POSCC C. If $|f(z)| > |g(z)|$ for every $z \in C$, then $f + g$ and f have the same number of zeros (counting multiplicities) inside C.

Proof. On C,

$$f(z) + g(z) = f(z)\Big[1 + \frac{g(z)}{f(z)}\Big]$$

so that

$$\mathit{\Delta}_C \arg(f(z) + g(z)) = \mathit{\Delta}_C \arg f(z) + \mathit{\Delta}_C \arg\Big(1 + \frac{g(z)}{f(z)}\Big)$$

As the modulus of $g(z)/f(z) < 1$, the second term vanishes, and the arguments are equal and the number of zeros are equal also. □

Recall. (Fundamental theorem of algebra) Let $P_n(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0 = a_n z^n + P_{n-1}(z)$ where $a_n \neq 0$. Assume $r \geq 1$. If $|z| = r$, then

$$\Big|\frac{P_{n-1}(z)}{a_n z^n}\Big| = \frac{|P_{n-1}(z)|}{|a_n|r^n} \leq \frac{(|a_0| + |a_1| + \ldots + |a_{n-1}|)r^{n-1}}{|a_n|r^n} = \frac{|a_0| + |a_1| + \ldots + |a_{n-1}|}{|a_n|r} < 1$$

if

$$r > \frac{|a_0| + \ldots + |a_{n-1}|}{|a_n|}$$

We can now apply Rouche's theorem for $f(z) = a_n z^n$, $g(z) = P_{n-1}(z)$ on $C : |z| = r$, and $r > (|a_0| + \ldots)/|a_n|$. We see that $P_n(z)$ has n zeros inside $|z| = r$.

Remark. Consider the function $f(x) = x^2$. Notice how $f((-1,1)) = [0,1)$, and how an open set maps to a not open set.

Theorem. (Open mapping theorem) Suppose f is a non-constant analytic function in a domain D. Then, $f(D)$ is open.

Proof. Take any $w_0 \in f(D)$. We have to find $m > 0$ such that $D(w_0, m) \subset f(D)$. By defintion of w_0, there is $z_0 \in D$ such that $f(z_0) = w_0$. Since D is open, there is $\delta > 0$ such that

(i) $\bar{D}(z_0, \delta) \subset D$

(ii) $f(z) - w_0$ has no zeros in $0 < |z - z_0| \leq \delta$ by the identity theorem

Let $m = \min_{|z-z_0|=\delta} |f(z) - w_0| > 0$. Now we'll show $D(w_0, m) \subset f(D)$. Take any $w \in D(w_0, m)$. Then, $|w_0 - w| < m \leq |f(z) - w_0|$ on the contour $|z - z_0| = \delta$. By Rouche's theorem, $f(z) - w_0$ and

$$(f(z) - w_0) + (w_0 - w) = f(z) - w$$

has the same number of zeros inside $|z - z_0| = \delta$. Since $(f(z) - w_0)$ has a zero in $|z - z_0| \leq \delta$ (at z_0), $(f(z) - w)$ has a zero inside $|z - z_0| = \delta \subset D$. In otherwords, $(f(z) - w)$ has a zero in D and $w \in f(D)$. □

Example. Show that all roots of $z^5 + 6z^3 + 2z + 10 = 0$ lies in $1 < |z| < 5$.

Proof. (i) On $|z| = 1$, let $f(z) = 10$, $g(z) = z^5 + 6z^3 + 2z$. $10 > |g(z)| \leq 1 + 6 + 2 \leq 9$. This implies that $f + g$ has the same numbers of zeros as 10 inside $|z| = 1$. That is, it has no zeros on $|z| = 1$.

(ii) On $|z| = 5$, let $f(z) = z^5$, $g(z) = 6z^3 + 2z + 10$. $|g(z)| \leq 6 \cdot 5^3 + 2 \cdot 5 + 10 < 5^5 = |f(z)|$. f and $f + g$ have the same number of zeros inside $|z| = 5$.

□

21 Lecture 21 (June 5th)

Recall. A C^2 function $u : D \to \boldsymbol{R}$ is harmonic if $\Delta u = u_{xx} + u_{yy} = 0$ on D.

Recall. (Cauchy) If f is analytic in a simply connected domain D then

$$\int_C f(z)\, dz = 0$$

for every closed contour C in D. This is due to the fact that for every fixed $z_0 \in D$,

$$F(z) = \int_{z_0}^{z} f(\xi)\, d\xi$$

is well-defined and is independent of path.

Theorem. If f is analytic in a simply connected domain D, then $f = F'$ for some F that is analytic in D.

Proof. Take any $z_0 \in D$, and we'll show that $F(z) = \int_{z_0}^{z} f(\xi)\, d\xi$ satisfies $F' = f$ on D. Take any $w \in D$. We'll show for each $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < |h| < \delta$ then

$$\Big|\frac{F(w+h) - F(w)}{h} - f(w)\Big| < \varepsilon$$

Since D is open, there is $\delta_1 > 0$ such that $\bar{D}(w, \delta_1) \subset D$. Since f is continuous at w, there is $\delta_2 > 0$ such that if $|\xi - w| < \delta_2$ then $|f(\xi) - f(w)| < \varepsilon$. Take $\delta = \min(\delta_1, \delta_2)$. Then we'll complete the proof. If $0 < |h| < \delta$ then

$$\frac{F(w+h) - F(w)}{h} = \frac{1}{h}\Big[\int_{z_0}^{w+h} f(\xi)\, d\xi - \int_{z_0}^{w} f(\xi)\, d\xi\Big]$$

so that if $0 < |h| < \delta$, then

$$\Big|\frac{F(w+h) - F(w)}{h} - f(w)\Big| = \Big|\frac{1}{h}\int_{w}^{w+h} (f(\xi) - f(w))\, d\xi\Big| < \varepsilon$$

As the path can be made a straight line, we can reduce the right handside smaller than the length of the path and the integeral can be reduced to be smaller than ε. $\square$

Theorem. If u is harmonic in a simply connected domain D, then there is a function f that is analytic in D such that $u = \mathrm{Re}(f)$ on D.

Proof. Define $g = u_x - iu_y$ in D, and g is analytic on D since it satisfies the Cauchy-Riemann equation. Since D is simply connected, there is F such that $F' = g$ on D. Suppose that $F = U + iV$, and $F' = U_x + iV_x = U_x - iV_y = g = u_x + iu_y$ due to the Cauchy Riemann equation. Thus $U_x = u_x$ and $U_y = u_y$ in D. We see that $U = u + c$ for some $c \in \boldsymbol{R}$. Hence $u = \mathrm{Re}[F - c]$. $\square$

Example. Let $u(x, y) = \ln\sqrt{x^2 + y^2} = \ln|z|$. Then $f(x) = \log z$ is analytic in $\boldsymbol{C} \setminus (-\infty, 0]$. Additionally, $\mathrm{Re}\, f(z) = \ln|z| = u$. However, there is no g analytic in $0 < |z| < 1$ such that $u = \mathrm{Re}(g)$.

Remark. If u and v are harmonic in D, then for every $a, b \in \boldsymbol{R}$, $au + bv$ is harmonic.

Example. If u is non-constant harmonic in a domain D and $v = u^2$, $v_x = 2uu_x$ such that $v_{xx} = 2u_x^2 + 2uu_{xx}$. Therefore,

$$\Delta v = v_{xx} + v_{yy} = 2u(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2) = 2(u_x^2 + u_y^2) > 0$$

We therefore find u^2 to be never harmonic.

Remark. If f is analytic and u is harmonic, then $u \circ f$ is harmonic. The key point is that u is locally the real part of an analytic function such that $u \circ f = \mathrm{Re}(g \circ f)$.

Theorem. If u is bounded and harmonic in $\boldsymbol{C}$, then u is constant.

Proof. Since $\boldsymbol{C}$ is simply connected, there exists an entire function f such that $u = \mathrm{Re}(f)$. Then, $|\exp f(z)| = \exp u(z)$ and $\exp f(z)$ is a bounded entire function. We therefore find that $\exp f(z)$ is constant and that $f(z)$ is constant. □

Theorem. If u is a positive harmonic function on $\boldsymbol{C}$, then u is constant.

Proof. There is an entire function f such that $u = \mathrm{Re}(f)$. Thus $|\exp f(z)| = \exp(u) \geq 1$. We find $g(z) = 1/\exp f(z)$ to be a bounded entire function which implies that g is constant. Then, f is constant and u is constant. □

Theorem. If u is harmonic in a domain D, then u can not take a maximum in D.

Proof. Suppose that you take a maximum at $z_0 = x_0 + iy_0$ in D. Then, there is $r > 0$ such that $\bar{D}(z_0, r) \subset D$. Since $D(z_0, r)$ is simply connected, there is f that is analytic in $D(z_0, r)$ such that $u = \mathrm{Re}\, f$ on $D(z_0, r)$. Then $|\exp f(z_0)| = \exp u(z_0) =$ is a maximum in $D(z_0, r)$. Therefore, $\exp(f)$ is constant. □

Theorem. (Mean value property of harmonic functions) If f is analytic in D such that $\bar{D}(z_0, R) \subset D$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

for every $0 < r \leq R$. If u is harmonic in a domain containing $\bar{D}(a, R)$, then $u = \mathrm{Re}(f)$ in $\bar{D}(z_0, R)$ resulting in

$$u(z_0) = \mathrm{Re}\, f(z_0) = \mathrm{Re}\left[\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt\right] = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt$$

for $r \leq R$. We then have, by extension,

$$\begin{aligned} u(z_0) =& \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt \\ \int_0^R u(z_0) r\, dr =& \frac{1}{2\pi} \int_0^{2\pi} \int_0^R u(z_0 + re^{it}) r\, dr dt \\ \frac{R^2}{2} u(z_0) =& \frac{1}{2\pi} \iint_{D(z_0,R)} u(x, y)\, dA \\ u(z_0) =& \frac{1}{\pi R^2} \iint_{D(z_0,R)} u\, dA \end{aligned}$$

and the mean value theorem works on a disk also.

Theorem. If u is bounded harmonic in $\boldsymbol{C}$, then u is constant.

Proof. Suppose that $|u(z)| \le M$. Then, for each $z_0 \in \boldsymbol{C}$,

$$\begin{aligned} u(z_0) - u(0) =& \frac{1}{\pi R^2} \iint_{D(z_0,R)} u\, dA - \frac{1}{\pi R^2} \iint_{D(0,R)} u\, dA \\ \le& \frac{M}{\pi R^2} (\text{symmetric difference between two discs}) \end{aligned}$$

$\square$

22 Lecture 22 (June 10th)

Remark. We use the following notation.

(i) $U = \{|z| < 1\}$ for a unit disc

(ii) $T = \{|z| = 1\}$ for a unit circle $z = re^{it}$ with $0 \le t \le 2\pi$

(iii) $\bar{U} = \{|z| \le 1\}$ for a closed unit disc

Lemma. If f is analytic on $\bar{U}$, then for each $z \in U$ we have

$$f(z) = \frac{1}{2\pi i} \int_T \frac{f(\xi)}{\xi - z}\, d\xi$$

and

$$\frac{1}{2\pi i} \int_T \frac{f(\xi)\bar{z}}{1 - \bar{z}\xi}\, d\xi = 0$$

Parametrising the function $f(z)$,

$$f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{it})}{e^{it} - z} ie^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - ze^{-it}}\, dt$$

We also see that

$$0 = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{it})\bar{z}}{1 - \bar{z}e^{it}} ie^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})e^{it}\bar{z}}{1 - \bar{z}e^{it}}\, dt$$

That is,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \Big[\frac{1}{1 - ze^{-it}} + \frac{e^{it}\bar{z}}{1 - \bar{z}e^{it}} \Big]\, dt$$

Here,

$$\frac{1}{1 - ze^{-it}} + \frac{e^{it}}{1 - \bar{z}e^{it}} = \frac{1 - \bar{z}e^{it} + e^{it}\bar{z} - |z|^2}{|1 - ze^{-it}|^2} = \frac{1 - |z|^2}{|1 - ze^{-it}|^2} = \frac{1 - |z|^2}{|e^{it} - z|^2}$$

and we have

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}\,dt$$

Theorem. If u is harmonic in $\bar{U}$, then for every $z \in U$ we have

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}\,dt$$

This tells us that any harmonic function can be expressed in terms of itself on a local disk.

Proof. If u is harmonic in a neighborhood of $\bar{U}$, then there is an analytic f in the neighborhood of U such that $u = \mathrm{Re}(f)$. Thus, for every $z \in U$,

$$\begin{aligned} u(z) =& \mathrm{Re}(f) = \mathrm{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}\,dt\right] \\ =& \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[f(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}\right]dt \\ =& \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}\,dt \end{aligned}$$

$\square$

Definition. For $h \in C(T)$, we define the Poisson integral $p[h]$ of h on U by

$$p[h](z) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}\,dt = \int_T h(\xi)\frac{1-|z|^2}{|\xi-z|^2}\,d\sigma(\xi)$$

where $\sigma(T) = 1$ is a measure. Equivalently,

$$p[h](re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})P_r(\theta - t)\,dt$$

where $P_r(\theta - t)$ is the Poisson kernel. Phrased differently, the above theorem states that $u = p[u]$ if it is harmonic.

Definition. Let $w = re^{it}$. Define

$$P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{int} = 1 + \sum_{n=1}^{\infty} r^n(e^{int} + e^{-int})$$

Note how

$$\frac{1+w}{1-w} = (1+w)(1+w+w^2+\ldots) = 1 + 2\sum_{n=1}^{\infty} w^n = 1 + 2\sum_{n=1}^{\infty} r^n e^{int}$$

Thus

$$\operatorname{Re}\left(\frac{1+w}{1-w}\right) = 1 + 2\sum_{n=1}^{\infty} r^n \cos nt = P_r(t)$$

where $(1+w)/(1-w)$ is harmonic in w. We see that the ugly expression that we first used can actually be expressed as

$$P_r(t) = \operatorname{Re}\left(\frac{1+re^{it}}{1-re^{it}}\right) = \frac{1-r^2}{|1-re^{it}|^2} = \frac{1-|w|^2}{|1-w|^2}$$

We see exactly that

$$P_r(\theta - t) = \frac{1-|w|^2}{|1-re^{it(\theta-t)}|^2} = \frac{1-r^2}{|e^{it}-re^{i\theta}|^2} = \frac{1-|z|^2}{|e^{it}-z|^2}$$

where $z = re^{i\theta}$. In sum,

$$P_r(\theta - t) = \frac{1-|z|^2}{|e^{it}-z|^2}$$

Note that we can also write the Poisson kernal as

$$\frac{1-r^2}{|1-re^{it}|^2} = \frac{1-r^2}{1-2r\cos t + r^2}$$

Theorem. For $0 \le r < 1$ and $t \in \boldsymbol{R}$,

$$P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} = \operatorname{Re}\left(\frac{1+re^{it}}{1-re^{it}}\right) = \frac{1-r^2}{1-2r\cos t + r^2}$$

which implies that

$$P_r(\theta - t) = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$$

The Poisson integral of $h \in C(T)$ is defined as

$$p[h](re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it}) P_r(\theta - t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it}) \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\, dt$$

Theorem. If $h : T \to \boldsymbol{R}$ is continuous, then $p[h]$ is harmonic in U.

Proof. Recall that $P_r(t) = \operatorname{Re}(1+re^{it})/(1-re^{it})$ so that

$$P_r(\theta - t) = \operatorname{Re}\left(\frac{1+re^{i(\theta-t)}}{1-re^{i(\theta-t)}}\right) = \operatorname{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right) = \operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)$$

which is harmonic in z. Then

$$p[h](z) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})P_r(\theta - t)\,dt = \text{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} h(e^{it})\frac{e^{it}+z}{e^{it}-z}\,dt\right]$$

$\square$

Definition. (Dirichlet problem) For a continuous function f, is there a g continuous inside and on C such that $f = g$ on C and g is harmonic inside C? Yes, and g is the Poisson integral of f.

Theorem. Let $f : T \to \boldsymbol{R}$ be continuous. If we define $g : \bar{U} \to \boldsymbol{R}$ by $g(z) = f(z)$ if $z \in T$ and $g(z) = p[f](z)$ if $z \in U$, then $g \in C(\bar{U})$.

23 Lecture 23 (June 12th)

Recall. We have learned multiple expressions for the Poisson integral. For a continuous function $h : T \to \boldsymbol{R}$, we define

$$\begin{aligned} p[h](z) =& \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}\,dt = \int_T h(\xi)\frac{1-|z|^2}{\xi - z}\,d\sigma \\ =& \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})\frac{1-r^2}{1-2r\cos(\theta - t)+r^2}\,dt = \text{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} h(e^{it})\frac{e^{it}+z}{e^{it}-z}\,dt\right) \\ =& \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})P_r(\theta - t)\,dt \end{aligned}$$

where the expression in the bracket is analytic and $p[h](z)$ is harmonic in U.

Theorem. If $f : T \to \boldsymbol{R}$ is continuous, define $g : \bar{U} \to \boldsymbol{R}$ by $g(z) = f(z)$ for $z \in T$ and $g(z) = p[f](z)$ for $z \in U$. Then g is continuous on $\bar{U}$.

Proof. Define for $z \in U$ and $\xi \in T$ the Poisson kernel

$$p(z, \xi) = \frac{1-|z|^2}{|\xi - z|^2}$$

then,

(i) $p(z, \xi) > 0$ for all $u \in U$ and $\xi \in T$

(ii) For all $z \in U$,

$$\int_T p(z, \xi)\,d\sigma(x) = 1$$

(iii) For every $\delta > 0$ and $\eta \in T$,

$$\lim_{z \to \eta} \int_{|\xi - \eta| > \delta} p(z, \xi)\, d\sigma(\xi) = \lim_{z \to \eta} \int_{|\xi - \eta| > \delta} \frac{1 - |z|^2}{|\xi - z|^2}\, d\sigma(\xi) = 0$$

Now, with this information, we prove that g is continuous on $\bar{U}$ as this suffices in showing that g is continuous on $\bar{U}$. Take any $\eta \in T$, and it is enough to show that g is continuous at η. Take any $\varepsilon > 0$, we have to find $\delta > 0$ so that if

$$|z - \eta| < \delta$$

, for $z \in U$ then

$$|g(z) - g(\eta)| < \varepsilon$$

For one, there is $M > 0$ such that $|f(\xi)| \leq M$ for all $\xi \in T$. For two, there is $\delta_1 > 0$ such that if $|\xi - \eta| < \delta_1$ then $|f(\xi) - f(\eta)| < \varepsilon/2$ for $\xi, \eta \in T$.

$$\begin{aligned} g(z) - g(\eta) &= \int_T f(\xi) p(z, \xi) d\sigma(\xi) - g(\eta) \\ &= \int_T [f(\xi) - f(n)] p(z, \xi)\, d\sigma(\xi) \end{aligned}$$

Thus,

$$\begin{aligned} |g(z) - g(\eta)| \leq& \int_T |f(\xi) - f(\eta)| p(z, \xi)\, d\sigma(\xi) \\ =& \int_{|\xi - \eta| \leq \delta_1} |f(\xi) - f(\eta)| p(z, \xi) d\sigma(\xi) + \int_{|\xi - \eta| < \delta_1} |f(\xi) - f(\eta)| p(z, \xi) d\sigma(\xi) \\ =& 2M \int_{|\xi - \eta| \leq \delta_1} p(\xi, z)\, d\sigma(\xi) + \frac{\varepsilon}{2} \int_{|\xi - \eta| < \delta_1} p(z, \xi) d\sigma(\xi) \end{aligned}$$

Now, take $\delta > 0$ so that if $|z - \eta| < \delta$ then

$$\int_{|\xi - \eta|} p(z, \xi)\, d\sigma(\xi) < \frac{\varepsilon}{4M}$$

Then if $|z - \eta| < \delta$ then $|g(z) - g(\eta)| < \varepsilon$. Then,

$$|g(z) - g(\eta)| < \varepsilon$$

$\square$