Definition. Consider the following Lagrangian density:

$$\mathcal{L} = \Big(R + 4g^2 \sum_i (e^{\phi_i} + e^{-\phi_i} + \chi_i^2 e^{\phi_i})\Big) \star 1 - \frac{1}{2} \sum_i \Big(d\phi_i \wedge \star d\phi_i + e^{2\phi_i} d\chi_i \wedge \star d\chi_i\Big) + \mathcal{L}_{KinA} + \mathcal{L}_{CS}$$

Definition. We define the truncation limit as the following set of limits

$$\begin{cases} \phi_1 \to \xi \\ \chi_1 \to \chi \\ A_1 \text{ and } A_2 \to A_3 \\ A_3 \text{ and } A_4 \to A_1 \end{cases}$$

While the remaining fields are put to 0. We can additionally impose the following limits such that the equations of motion would reduce to the one in Cassani's paper.

$$\begin{cases} A_i \to \dfrac{1}{\sqrt{2}} A_i \\ g \to \dfrac{1}{2} g \end{cases}$$

Proposition. (Variation with respect to scalar fields ϕ_i) The variation of the Lagrangian density with respect to the fields ϕ_i are

$$\delta\mathcal{L} = \delta\phi_i \Big[4g^2(e^{\phi_i} - e^{-\phi_i} + \chi_i^2 e^{\phi_i}) \star 1 + d(\star d\phi_i) - e^{2\phi_i} d\chi_i \wedge \star d\chi_i + \frac{\delta\mathcal{L}_{KinA}}{\delta\phi_i} + \frac{\delta\mathcal{L}_{CS}}{\delta\phi_i}\Big]$$

where we omitted the expressions for $\delta\mathcal{L}_{KinA}/\delta\phi_i$ and $\delta\mathcal{L}_{CS}/\delta\phi_i$ for time being. We can easily see how, under the truncation limit,

$$\delta_{\phi_2}\mathcal{L} \to 0 \quad \text{and} \quad \delta_{\phi_3}\mathcal{L} \to 0$$

For the variation with respect to the field ϕ_1, however, we have, along with the Cassani scaling conventions,

$$\frac{\delta\mathcal{L}_{KinA}}{\delta\phi_1} \to \frac{e^{-2\xi}(-e^{4\xi} \star F_1 \wedge F_1 + (1 + e^{2\xi}\chi^2)^2 \star F_3 \wedge F_3)}{(1 + e^{2\xi}\chi^2)^2} \tag{†}$$

and

$$\frac{\delta\mathcal{L}_{CS}}{\delta\phi_1} \to \frac{e^{3\xi}\chi(3 + e^{2\xi}\chi^2) \star F_1 \wedge F_1}{2(1 + e^{2\xi}\chi^2)^2} - \frac{1}{2} e^{-\xi}\chi(-1 + e^{2\xi}\chi^2) \star F_3 \wedge F_3 \tag{††}$$

We therefore have, in the Cassani paper's conventions,

$$\delta\mathcal{L} = \delta\xi\Big[g^2(e^{\xi} - e^{\xi} + \chi^2 e^{\xi}) \star 1 + d(\star d\xi) - e^{2\xi} d\chi \wedge \star d\chi + (\dagger) + (\dagger\dagger)\Big]$$