

## 1 Lecture 14 (April 29th)

**Definition.** (Convergence in measure) For measurable functions on  $E$ , we say  $f_n \rightarrow f$  in measure on  $E$  provided that for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} m(\{x \in E \mid |f_n(x) - f(x)| > \varepsilon\}) = 0$$

**Definition.** ( $L^1$  convergence) For  $f_n \in L^1(E)$  and a measurable  $f$  on  $E$ , we say that  $f_n \rightarrow f$  in  $L^1(E)$  provided that

$$\lim_{n \rightarrow \infty} \int_E |f_n - f| dm = 0$$

**Theorem.** If  $f_n \rightarrow f$  and  $f_n \rightarrow g$  in measure (or in  $L^1(E)$ ) in  $E$ , then  $f = g$  almost everywhere on  $E$ .

*Proof.* Suppose that  $f_n \rightarrow f$  and  $f_n \rightarrow g$  in measure. First note that

$$\{x \in E \mid |f(x) - g(x)| > 0\} = \bigcup_{n=1}^{\infty} \left\{x \in E \mid |f(x) - g(x)| \geq \frac{1}{n}\right\}$$

Meanwhile, for each  $n$ , due the triangle inequality,

$$|f(x) - f_k(x)| + |f_k(x) - g(x)| \geq |f(x) - g(x)| \geq \frac{1}{n}$$

and one of the two terms in the left must be greater than  $1/2n$ . We now see the following inclusion for  $x \in E$ :

$$\left\{x \mid |f(x) - g(x)| \geq \frac{1}{n}\right\} \subset \left\{x \mid |f(x) - f_k(x)| \geq \frac{1}{2n}\right\} \cup \left\{x \mid |g(x) - f_k(x)| \geq \frac{1}{2n}\right\}$$

See how for each term, we can choose  $k_1$  and  $k_2$  such that they are measure zero, and we see that by taking  $\max\{k_1, k_2\}$ , their infinite union is measure zero too.  $\square$

**Example.** To this point we have learned the following convergences:

- (i)  $f_n \rightarrow f$  uniformly
- (ii)  $f_n \rightarrow f$  pointwise almost everywhere on  $E$
- (iii)  $f_n \rightarrow f$  almost uniformly on  $E$
- (iv)  $f_n \rightarrow f$  in measure
- (v)  $f_n \rightarrow f$  in  $L^1(E)$

Consider the following sequences.

- (i)  $f_n = \mathbf{1}_{(n,\infty)}$  (uniform, pointwise almost everywhere)
- (ii)  $f_n = n\mathbf{1}_{(0,1/n)}$  or  $f_n = \mathbf{1}_{(0,n)}/n$  (pointwise almost everywhere, almost uniformly, in measure) (uniformly, almost everywhere, almost uniformly, and in measure)
- (iii)  $f_n = \mathbf{1}_{I_n}$  where

$$I_1 = [0, 1/2] \quad I_2 = [1/2, 1] \quad I_3 = [0, 1/3] \quad I_4 = [2/3, 1] \quad I_5 = [0, 1/4]$$

and etcetera. (uniformly, almost everywhere, almost uniformly, and in measure)

**Theorem.** If  $f_n \rightarrow f$  in measure on  $E$ , then there is a subsequence  $\{f_{n_k}\}$  which converges pointwise almost everywhere to  $f$ .

*Proof.* If  $f_n \rightarrow f$  in measure the following are both true.

- (i) For every  $\varepsilon > 0$  there is  $N$  such that if  $n > N$  then

$$m(\{x \in E \mid |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$$

- (ii) For every  $\varepsilon > 0$  there is  $N$  such that if  $n > N$  then

$$m(\{x \in E \mid |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon^2$$

For all  $k \in \mathbf{N}$  there exists  $n_k$ 's such that  $n_{k+1} > n_k$  and

$$m(\{x \in E \mid |f_{n_k}(x) - f(x)| > 1/k\}) < 1/k^2$$

Let  $E_k = \{x \in E \mid |f_{n_k}(x) - f(x)| > 1/k\}$ . We will now show that  $f_{n_k} \rightarrow f$  almost everywhere on  $E$ . Since  $\sum_{n=1}^{\infty} m(E_k) \leq \sum_{k=1}^{\infty} 1/k^2 < \infty$ , by the Borel-Cantelli lemma, almost all  $x \in E$  belongs to at most finitely many  $E_k$ 's. So,  $f_{n_k}$  almost everywhere on  $E$ .  $\square$

**Theorem.** (i) If  $f_n \rightarrow f$  almost uniformly on  $E$ , then  $f_n \rightarrow f$  in measure.

- (ii) If  $f_n \rightarrow f$  in  $L^1(E)$  then  $f_n \rightarrow f$  in measure.

*Proof.* We prove (ii) first. By Chebychev, for each  $\varepsilon$ ,

$$m(\{x \in E \mid |f_n(x) - f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_E |f_n - f| dm$$

where the right hand side approaches 0 as  $n \rightarrow \infty$ .

Now for (i), suppose  $f_n$  doesn't converge to  $f$  in measure. Then there exists  $\varepsilon, \delta > 0$  such that

$$m(\{x \in E \mid |f_n(x) - f(x)| > \varepsilon\}) \geq \delta$$

for infinitely many  $n$ 's so that there is a subsequence  $\{f_{n_k}\}$  such that

$$m(\{x \in E \mid |f_{n_k}(x) - f(x)| > \varepsilon\}) \geq \delta$$

for all  $k \in \mathbf{N}$ . There is no  $A \subset E$  such that  $m(A) < \delta/2$  and  $f_{n_k} \rightarrow f$  uniformly on  $A^c = E \setminus A$ . Hence  $f_n$  does not converge to  $f$  almost uniformly on  $E$ .  $\square$

## 2 Lecture 15 (May 8th)

**Definition.** (Norm) A norm is a function  $\|\cdot\| : X \rightarrow [0, \infty)$  from a vector space of  $\mathbf{R}$  that satisfies the following properties:

- (i) (Nonnegativity)  $\|x\| \geq 0$  for all  $x \in X$  and  $\|x\| = 0$  if and only if  $x = \mathbf{0}$  (that is,  $x$  is the zero vector)
- (ii) (Positive homogeneity)  $\|\alpha x\| = |\alpha| \|x\|$  for all  $\alpha \in \mathbf{R}$  and  $x \in X$
- (iii) (The triangle inequality)  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x$  and  $y$  in  $X$

Note that  $d(x, y) = \|x - y\|$  is a metric on  $X$ . Therefore, all normed vector spaces are a metric space.

**Definition.** (Banach space) A metric space is called complete if every Cauchy sequence in  $X$  converges (in  $X$ ) with respect to the metric. A normed vector space that is complete with respect to its norm that acts as a metric is called a Banach space.

**Example.** (Examples of normed vector spaces)

- (i) ( $C([0, 1])$  with uniform norm)  $C([0, 1]) = \{f \mid f \text{ is continuous on } [0, 1]\}$  maybe given the norm  $\|f\| = \max_{[0, 1]} |f(x)|$ . With respect to this norm,  $C([0, 1])$  is also a Banach space. This is because

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|$$

and all Cauchy sequences converge uniformly to some  $f$  that is continuous.

- (ii) ( $C([0, 1])$  with  $L^1$  norm)  $C([0, 1])$  may also given the norm  $\|f\| = \int_0^1 |f(x)| dx$ . With respect to this norm,  $C([0, 1])$  is not Banach space. This is because Cauchy sequences such as  $\{f_n\} = \{x_n\}$  do not converge to a continuous function.
- (iii) ( $C^1([0, 1])$  with uniform norm)  $C^1([0, 1])$  is a normed vector space with respect to the uniform norm but it is not complete. For example, take  $f_n(x) = \sqrt{x + 1/n}$ .

$f_n \rightarrow f(x) = \sqrt{x}$  uniformly on  $[0, 1]$  which implies that it is a Cauchy sequence, but it is not in  $C^1([0, 1])$ .

**Definition.** ( $L^1(E)$ ) For a measure space  $(E, X, m)$ , the space  $L^1(E)$  is the vector space of measurable functions endowed with the function  $\|f\|_1 = \int_E |f| dm$ . Notice that

$$\begin{aligned} \int_E |f| dm = 0 & \quad \text{then} \quad f = 0 \quad \text{almost everywhere on } E \\ \int_E |\alpha f| dm &= |\alpha| \int_E |f| dm \quad \text{for} \quad \alpha \in \mathbf{R}, f \in L^1(E) \\ \int_E |f + g| dm &\leq \int_E |f| dm + \int_E |g| dm \quad \text{for} \quad f, g \in L^1(E) \end{aligned}$$

Given this knowledge, if  $f, g : E \rightarrow \mathbf{R}$  satisfies  $f, g \in L^1(E)$  and  $f = g$  almost everywhere on  $E$ , define  $f = g$  as an element of  $L^1(E)$ . Then,  $L^1(E)$  is a normed vector space with respect to the norm  $\|f\|_1 = \int_E |f| dm$ .

**Definition.** ( $L^p(E)$ ) Let  $1 < p < \infty$ . For a measure space  $(E, X, m)$ , we define  $L^p(E)$  as the set of measurable functions of a measure space satisfying

$$\int_E |f|^p dm < \infty$$

If  $f, g \in L^p(E)$  and  $f = g$  almost everywhere on  $E$ , then define  $f$  and  $g$  as the same element of  $L^p(E)$ . To satisfy  $\|\alpha f\|_p = |\alpha| \|f\|_p$ , we define

$$\|f\|_p = \left( \int_E |f|^p dm \right)^{1/p}$$

We will prove later on that  $L^p(E)$  is a Banach space for  $1 \leq p \leq \infty$ .

**Definition.** If  $X$  and  $Y$  are normed vector spaces,  $T : X \rightarrow Y$  is called a linear operator provided that  $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$  for  $\alpha, \beta \in \mathbf{R}$  and  $x, y \in X$ .

**Theorem.** If  $T : X \rightarrow Y$  is a linear operator, then the following are equivalent.

- (i)  $T$  is continuous on  $X$
- (ii)  $T$  is uniformly continuous on  $X$
- (iii)  $T$  is continuous at  $\mathbf{0}$

**Definition.**  $T : X \rightarrow Y$  is called a bounded linear operator provided that  $T$  is linear and there is  $M > 0$  such that

$$\|T(x)\|_Y \leq M \|x\|_X$$

for all  $x \in X$ . Intuitively, bounded linear operators are operators that do not “blow up” small inputs.

**Proposition.** If  $T$  is continuous at  $\mathbf{0}$ , then  $T$  is bounded linear.

*Proof.* By continuity, we see that for every  $\varepsilon > 0$ , there is  $\delta > 0$  such that if  $\|x\| < \delta$  then  $\|T(x)\| < \varepsilon$ . Take any  $w \in X \setminus \{\mathbf{0}\}$  and we see that the vector  $\delta w/2\|w\| \in X$  has a norm  $\delta/2$ . This implies that

$$\left\| T\left(\frac{\delta w}{2\|w\|}\right) \right\| < \varepsilon \quad \text{and} \quad T(w) \leq \frac{2\varepsilon}{\delta} \|w\|$$

for all  $w \neq \mathbf{0}$ . □

### 3 Lecture 16 (May 13th)

**Definition.** If  $X$  and  $Y$  are normed vector spaces, we define  $\mathcal{L}(X, Y)$  as the set of all bounded linear operators from  $X$  to  $Y$ . We call  $X^* = \mathcal{L}(X, \mathbf{R})$  the dual of  $X$ . The elements of  $X^* = \mathcal{L}(X, \mathbf{R})$  are called bounded linear functionals.

**Theorem.** For  $T \in \mathcal{L}(X, Y)$ , let norm of  $T$  be defined as  $\|T\| = \sup_{\|x\|=1} \|T(x)\|$ . If  $Y$  is a Banach space, then  $\mathcal{L}(X, Y)$  is a Banach space. In particular,  $X^* = \mathcal{L}(X, \mathbf{R})$  is a Banach space.

**Remark.** For  $x \neq 0$ , observe that

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \|T\| \implies \|T(x)\| \leq \|T\| \|x\|$$

As  $\|T(x)\| \leq M \|x\|$  we also have that

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq M \implies \|T\| \leq M$$

This tells us the following equivalence of definitions,

$$\|T\| = \inf(\{M \geq 0 \mid \|T(x)\| \leq M \|x\|\})$$

for every  $x \in X$ . This tells us that the norm measures how much a vector can be scaled and stretched.

**Theorem.** The proof that  $\mathcal{L}(X, Y)$  is a normed vector space with the aforementioned norm and will be for now will be skipped. If  $Y$  is a Banach space, we'll show that  $\mathcal{L}(X, Y)$  is a Banach space. Given a Cauchy sequence  $\{T_n\}$ , we have to show that there is  $T \in \mathcal{L}(X, Y)$  such that  $\lim_{n \rightarrow \infty} \|T_n - T\| = 0$ .

*Proof.* We define the target linear operator pointwise. For each  $x \in X$ ,

$$\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\| \leq \|T_n - T_m\| \|x\|$$

and we see that  $\{T_n(x)\}$  is a Cauchy sequence in  $Y$ . Since  $Y$  is complete, there is  $y \in Y$  such that  $\lim_{n \rightarrow \infty} T_n(x) = y$ . Define  $y = T(x)$ , that is,  $T(x) = \lim_{n \rightarrow \infty} T_n(x)$  in  $Y$ . Then

$$\begin{aligned} T(\alpha x) &= \lim_{n \rightarrow \infty} T_n(\alpha x) = \alpha \lim_{n \rightarrow \infty} T_n(x) = \alpha T(x) \\ T(x + y) &= \lim_{n \rightarrow \infty} T_n(x + y) = T(x) + T(y) \end{aligned}$$

and therefore  $T$  is linear. To show  $T \in \mathcal{L}(X, Y)$  we need to still prove that  $T$  is bounded. Take any  $\varepsilon > 0$ , there is  $N \in \mathbf{N}$  such that if  $n, m \geq N$  then  $\|T_n - T_m\| < \varepsilon$ . This implies that, for such  $N$ , if  $n \geq N$  then  $\|T_n\| \leq \|T_N\| + \varepsilon$ . Thus for each  $x \in X$ ,

$$\|T(x)\| = \lim_{n \rightarrow \infty} \|T_n(x)\| \leq \lim_{n \rightarrow \infty} [\|T_n\| \|x\|] \leq [\|T_N\| + \varepsilon] \|x\|$$

Therefore, the norm of the target is bounded with

$$\|T\| \leq \|T_N\| + \varepsilon$$

Lastly, we prove that  $T$  is the limit of the Cauchy sequence. Take any  $\varepsilon > 0$ . Then, there exists an  $N$  such that if  $n > N$ ,

$$\|T_n(x) - T(x)\| = \lim_{m \rightarrow \infty} \|T_n(x) - T_m(x)\| \leq \lim_{m \rightarrow \infty} \|T_n - T_m\| \|x\| \leq \varepsilon \|x\|$$

and  $\|T_n - T\| \leq \varepsilon$ . In sum, we have followed process of, for an arbitrary Cauchy sequence, finding a target, proving that the target is linear and bounded, and lastly proving that the target is indeed the limit of the Cauchy sequence.  $\square$

**Proposition.**  $\text{Lip}_\alpha([0, 1])$  is a Banach space for  $0 < \alpha < 1$  where  $\text{Lip}_\alpha([0, 1])$  is the set of all  $f \in C([0, 1])$  such that

$$M_f = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^\alpha} < \infty$$

with the norm  $\|f\| = |f(0)| + M_f$ .

**Corollary.** If  $X$  is a normed vector space, then

$$X^* = \mathcal{L}(X, \mathbf{R})$$

is a Banach space with the operator norm.

**Corollary.** Let  $L^p(E)$  denote the set of all measurable functions on  $E$  with

$$\|f\|_p = \left( \int_E |f|^p dm \right)^{1/p} < \infty$$

where we define  $f, g \in L^p(E)$  to be equivalent if  $f = g$  almost everywhere on  $E$ .

*Proof.* To show that  $L^p(E)$  is a normed vector space for  $1 < p < \infty$  we simply need to prove that the function  $\|\cdot\|_p$  is a norm. Note how

(i) If  $\|f\|_p = 0$  then  $f = 0$  almost everywhere on  $E$  so that  $f$  is a zero vector in  $L^p(E)$ .

(ii) If  $f \in L^p(E)$  and  $\alpha \in \mathbf{R}$ , then

$$\|\alpha f\|_p = \left( \int_E |\alpha f|^p dm \right)^{1/p} = |\alpha| \left( \int_E |f|^p dm \right)^{1/p} = |\alpha| \|f\|_p$$

(iii) If  $f, g \in L^p(E)$  then  $\|f+g\|_p \leq \|f\|_p + \|g\|_p$  which is called the Minkowski inequality.

Due to its difficulty, we omit the proof for now.

□

**Lemma.** For  $a, b \geq 0$  and  $0 < \lambda < 1$  the equality  $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$  holds.

*Proof.* For  $t \geq 0$ , define  $\phi(t) = \lambda t - t^\lambda$ . The derivative is given as  $\phi'(t) = \lambda - \lambda t^{\lambda-1} = \lambda(1 - t^{\lambda-1})$ . With a minimum at  $t = 1$  we know that  $\phi(t) \geq \phi(1)$  for all  $t > 0$ . Therefore,

$$\lambda t - t^\lambda \geq \lambda - 1 \quad \text{so that} \quad t^\lambda \leq \lambda t + (1 - \lambda)$$

for all  $t \geq 0$ . Notice that the equality holds when  $t = 1$ . Now put  $t = a/b$  to get

$$\left(\frac{a}{b}\right)^\lambda \leq \lambda \left(\frac{a}{b}\right) + (1 - \lambda)$$

so that multiplying  $b$  on both sides yields the sought inequality.

□

**Corollary.** Let  $1 < p < \infty$  satisfy  $1/p + 1/q = 1$ . Put  $\lambda = 1/p$ ,  $a = A^p$ , and  $b = B^p$  for  $A, B \geq 0$ . Then,

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b$$

becomes

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}$$

where the equality holds when  $A^p = B^q$ .

**Theorem.** (Hölder's inequality) Let  $1 \leq p < \infty$ . If  $f \in L^p(E)$  and  $g \in L^q(E)$  with  $1/p + 1/q = 1$ , then  $fg \in L^1(E)$  and  $\|fg\|_1 \leq \|f\|_p \|g\|_q$ . That is,

$$\int_E |fg| dm \leq \left( \int_E |f|^p dm \right)^{1/p} \left( \int_E |g|^q dm \right)^{1/q}$$

*Proof.* From  $AB \leq A^p/p + B^q/q$  we put  $A = |f(x)|/||f||_p$  and  $B = |g(x)|/||g||_q$  for  $x \in E$  to get

$$\frac{|f(x)g(x)|}{||f||_p ||g||_q} \leq \frac{1}{p} \frac{|f(x)|^p}{||f||_p^p} + \frac{1}{q} \frac{|g(x)|^q}{||g||_q^q}$$

then we integrate.

$$\begin{aligned} \frac{1}{||f||_p ||g||_q} \int_E |f(x)g(x)| dm(x) &\leq \frac{1}{p} \frac{1}{||f||_p^p} \int_E |f(x)|^p dm(x) + \frac{1}{q} \frac{1}{||g||_q^q} \int_E |g(x)|^q dm(x) \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

This completes the proof.  $\square$

**Theorem.** (Minkowski inequality) Let  $1 \leq p < \infty$ . If  $f, g \in L^p(E)$  then  $||f + g||_p \leq ||f||_p + ||g||_p$ .

*Proof.* If  $f, g \in L^p(E)$  then  $|f(x) + g(x)| \leq 2 \max(|f(x)|, |g(x)|)$  which implies that

$$\int_E |f + g|^p dm \leq 2^p \int_E (|f|^p + |g|^p) dm < \infty$$

Then

$$\begin{aligned} \int_E |f + g|^p dm &= \int_E |f + g| |f + g|^{p-1} dm \leq \int_E (|f| + |g|) |f + g|^{p-1} dm \\ &= \int_E |f| |f + g|^{p-1} dm + \int_E |g| |f + g|^{p-1} dm \end{aligned}$$

where, from Hölder's inequality,

$$\begin{aligned} \int_E |f| |f + g|^{p-1} dm &\leq ||f||_p \left[ \int_E |f + g|^{(p-1)q} dm \right]^{1/q} = ||f||_p \left[ \int_E |f + g|^p dm \right]^{1/q} \\ &= ||f||_p ||f + g||_p^{p/q} \end{aligned}$$

For the justification of the use of the inequality, we must show that each integrand are elements of  $L^p$  and  $L^q$ . Note that

$$\int_E \left[ |f + g|^{p-1} \right]^q dm = \int_E |f + g|^{pq-q} dm = \int_E |f + g|^p dm < \infty$$

We now see that, in parallel,

$$\int_E |g| |f + g|^{p-1} dm \leq ||g||_p ||f + g||_p^{p/q}$$

and therefore

$$||f + g||_p^p \leq \left[ ||f||_p + ||g||_p \right] ||f + g||_p^{p/q}$$



Dividing both sides by  $\|f + g\|_p^{p/q}$ ,

$$\|f + g\|_q^{p-p/q} \leq \|f\|_p + \|g\|_p \quad \text{and} \quad \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

now we seen how for  $p \geq 1$ ,  $L^p(E)$  is a normed vector space. Next class, we show that, in addition to this,  $L^p(E)$  is complete.  $\square$

#### 4 Lecture 17 (May 15th)

**Theorem.** Take  $1 \leq p < \infty$ . Then  $L^p(E)$  is complete.

*Proof.* Let  $\{f_n\}$  be a Cauchy sequence in  $L^p(E)$ . Then there is a subsequence  $\{f_{n_k}\}$  such that

$$\|f_{n_{k+1}} - f_{n_k}\|_p < \frac{1}{k^2}$$

If so, we'll show that

(i)  $f_{n_k}$  converges pointwise almost everywhere on  $E$

(ii) If  $f(x) = \lim_{n \rightarrow \infty} f_{n_k}(x)$  almost everywhere then  $f_n \rightarrow f$  in  $L^p(E)$

(STEP 1) We know that  $f \in L^+(E)$  and  $\int_E f \, dm < \infty$  then  $m(\{x \in E \mid f(x) = \infty\}) = 0$ . Notice how

$$\sum_{i=1}^{k-1} (f_{n_{i+1}}(x) - f_{n_i}(x)) = f_{n_k}(x) - f_{n_1}(x)$$

Now define  $g_n(x) = \sum_{k=1}^n |f_{n_{k+1}}(x) - f_{n_k}(x)|$  and  $g(x) = \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$ . Then by the Minkowski inequality,

$$\|g_n\|_p \leq \sum_{k=1}^n \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < 2$$

As  $g_n \nearrow g$ , we see that  $g_n^p \nearrow g^p$ , and due to the monotone convergence theorem,

$$\int_E |g|^p \, dm = \lim_{n \rightarrow \infty} \int_E |g_n|^p \, dm < 2^p$$

In otherwords,  $m(\{x \in E \mid g(x) = \infty\}) = 0$  and it converges almost everywhere on  $E$  which implies that, by defintiion,

$$\sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

converges almost everywhere on  $E$ . Since

$$f_{n_k}(x) = f_{n_1}(x) + \sum_{i=1}^{k-1} (f_{n_{i+1}}(x) - f_{n_i}(x))$$

We see that  $\lim_{k \rightarrow \infty} f_{n_k}(x)$  converges pointwise almost everywhere on  $E$ .

(STEP 2) Now define  $f(x) = \lim_{k \rightarrow \infty} f_{n_k}(x)$  where it converges and 0 elsewhere. We'll show that  $f \in L^p(E)$  and  $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$ . Notice how for some  $N$ , if  $n \geq N$  then

$$\int_E |f - f_n|^p dm \leq \liminf_{k \rightarrow \infty} \int_E |f_{n_k} - f_n|^p dm < \varepsilon^p$$

or equivalently, if  $n \geq N$  then  $\|f - f_n\|_p < \varepsilon$ . Additionally, for  $n \geq N$ ,  $f$  can be expressed as

$$f = \underbrace{f - f_n}_{\in L^p} + \underbrace{f_n}_{\in L^p}$$

proving that  $f \in L^p$ . □

**Definition.**  $f \in L^\infty(E)$  provided that there is  $M \geq 0$  such that  $|f(x)| \leq M$  almost everywhere on  $E$ . The norm  $\|f\|_\infty$  is defined as the infimum of such upper bounds  $M$ . If  $f \in L^\infty(E)$ ,  $f$  is called essentially bounded and measurable on  $E$  and  $\|f\|_\infty$  is called the essential supremum of  $f$  on  $E$ .

**Remark.** (i)  $|f(x)| \leq \|f\|_\infty$  almost everywhere on  $E$

(ii) If the zero vector of  $L^\infty(E)$  is defined by zero function almost everywhere then  $L^\infty(E)$  is a Banach space

**Definition.**  $l^p(\mathbf{N})$  is the space  $L^p(\mathbf{N})$  with respect to the counting measure  $\lambda$ . We show that elements in  $l^p(\mathbf{N})$  are bounded sequences. For  $f \in l^1(\mathbf{N}, \lambda)$  define

$$f_n(k) = \begin{cases} |f(k)| & 1 \leq k \leq n \\ 0 & n < k \end{cases}$$

such that  $f_n(k)$  is a simple function in  $L^+(\mathbf{N})$  with  $f_n \nearrow |f|$ . By MCT,

$$\int_{\mathbf{N}} |f| d\lambda = \lim_{n \rightarrow \infty} \int f_n(k) d\lambda = \lim_{n \rightarrow \infty} \sum_{k=1}^n |f(k)| = \sum_{k=1}^{\infty} |f(k)|$$

A sequence  $\{a_n\}$  ( $a_n = f(n)$ ) therefore belongs to  $l^p(\mathbf{N})$  if

$$\|\{a_n\}\|_p = \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} < \infty$$

and  $|a_n|^p$  converges absolutely. As an extension,  $\{a_n\} \in l^\infty(\mathbf{N})$  if

$$\|\{a_n\}\|_\infty = \sup_{n \in \mathbf{N}} |a_n| < \infty$$

and is a bounded sequence.

**Definition.** Let  $\mathbf{x} = \{a_n\}$  be a sequence.  $\mathbf{x} \in l^2(\mathbf{N})$  if  $\|\mathbf{x}\|^2 = \sum_{k=1}^{\infty} |a_n|^2 < \infty$ . Consider the sequence  $\{e_n\} \subset l^2(\mathbf{N})$  (an orthonormal basis for  $l^2(\mathbf{N})$ ) then

$$\begin{cases} \|e_n\| = 1 \\ \|e_j - e_k\| = \sqrt{2} \quad j \neq k \end{cases}$$

this implies that  $\{e_n\}$  is a bounded sequence in  $l^2(\mathbf{N})$  but that there is no subsequence that is a Cauchy sequence. To summarise,  $\{e_n\}$  is a

- (i) Bounded sequence with no convergence subsequence in  $l^2(\mathbf{N})$
- (ii) Bounded infinite set with no limit point
- (iii) Closed and bounded but not compact

**Example.** (Unbounded linear operators) Consider  $X : C^1([0, 1])$  and  $Y : C([0, \pi])$  both with the uniform norm

$$\|f\| = \max_{[0,1]} |f(x)|$$

Let  $T : X \rightarrow Y$  be defined by  $Tf = f'$  which is linear. Take

$$f_n(x) = x^n \quad \text{and} \quad g_n(x) = \sin(nx)$$

Then,  $f_n, g_n \in X$  with  $\|f_n\| = 1$  and  $\|g_n\| = 1$  for all  $n$ . Note that  $T(f_n)(x) = nx^{n-1}$  so that  $\|Tf_n\| = n$  and  $T(g_n)(x) = n \cos nx$  so that  $T(g_n)(0) = n$ . Therefore, there is no  $M > 0$  so that

$$n = \|Tf_n\| \leq M\|f_n\| = M$$

nor

$$n = \|Tg_n\| \leq M\|g_n\| = M$$

for all  $n \in \mathbf{N}$ .

**Definition.** (Separable) A normed vector space  $X$  is called separable if  $X$  has a countable dense subset.

**Remark.** For  $1 \leq p < \infty$ ,  $L^p(E)$  is separable but  $L^\infty(E)$  is not separable.

*Proof.* Suppose that  $\{f_n\}$  is a countable dense subset of  $L^\infty([a, b])$ . Then, for every  $a < x < b$  there is  $f_{n(x)}$  such that

$$\|\mathbf{1}_{[a,x]} - f_{n(x)}\|_\infty < \frac{1}{2}$$

Suppose  $a < x < y < b$  satisfies  $f_{n(x)} = f_{n(y)}$ . Then,

$$\begin{aligned} 1 &= \|\mathbf{1}_{[a,x]} - \mathbf{1}_{[a,y]}\|_\infty \\ &= \|\mathbf{1}_{[a,x]} - f_{n(x)} + f_{n(y)} - \mathbf{1}_{[a,y]}\|_\infty \\ &\leq \|\mathbf{1}_{[a,x]} - f_{n(x)}\|_\infty + \|f_{n(y)} - \mathbf{1}_{[a,y]}\|_\infty \\ &< \frac{1}{2} + \frac{1}{2} < 1 \end{aligned}$$

which is a contradiction. We thus found that all dense subsets of  $L^\infty(E)$  are uncountable.  $\square$

**Definition.** We define  $C_c^\infty(E)$  as the space of smooth functions with a compact support inside  $E$ . By support we mean the closure of the set where  $f(x) \neq 0$ .

$$\text{supp}(f) = \overline{\{x \in E \mid f(x) \neq 0\}}$$

We remark that  $C_c^\infty(E)$  is dense in  $L^p(E)$  for  $1 \leq p < \infty$ .

## 5 Lecture 18 (May 20th)

**Theorem.** If  $m(E) < \infty$  and  $1 < p < q < \infty$  then  $L^\infty(E) \subset L^q(E) \subset L^p(E) \subset L^1(E)$ .

*Proof.* If  $\|f\|_\infty < \infty$  then  $\int_E |f|^p dm \leq \int_E \|f\|_\infty^p dm = \|f\|_\infty^p \cdot m(E) < \infty$  such that  $f \in L^p(E)$ . Meanwhile, if  $p < q$  and  $f \in L^q(E)$ , we notice that

$$\frac{q}{p} \quad \text{and} \quad c = \frac{1}{1 - \frac{q}{p}}$$

are conjugate exponents. Subsequently,

$$\begin{aligned} \int_E |f|^p dm &= \int_E |f|^p \cdot 1 dm \leq \left[ \int_E (|f|^p)^{q/p} \right]^{p/q} \left[ \int_E 1 dm \right]^c \\ &= \|f\|_q^p \cdot m(E)^c < \infty \end{aligned}$$

by taking  $p = q/p$  and using Hölder's inequality.  $\square$

**Theorem.** If  $m(E) < \infty$  and  $f \in L^\infty(E)$  then  $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$

*Proof.*

$$\int_E |f|^p dm \leq \int_E \|f\|_\infty^p dm = \|f\|_\infty^p \cdot m(E)$$

so that

$$\left( \int_E |f|^p dm \right)^{1/p} \leq \|f\|_\infty \cdot m(E)^{1/p} \quad \text{and} \quad \limsup_{p \rightarrow \infty} \|f\|_p \leq \|f\|_\infty$$

where the measure goes to 1 as  $p \rightarrow \infty$ . Take any  $\varepsilon > 0$ , we'll show that

$$\liminf_{p \rightarrow \infty} \|f\|_p \geq \|f\|_\infty - \varepsilon$$

Define

$$E_\varepsilon = \{x \in E \mid |f(x)| > \|f\|_\infty - \varepsilon\}$$

Then,  $0 < m(E_\varepsilon) \leq m(E) < \infty$  and

$$\begin{aligned} \|f\|_p &= \left( \int_E |f|^p dm \right)^{1/p} = \left( \int_{E_\varepsilon} |f|^p dm \right)^{1/p} \geq \left( \int_{E_\varepsilon} (\|f\|_\infty - \varepsilon)^p dm \right)^{1/p} \\ &= \left[ (\|f\|_\infty - \varepsilon)^p m(E_\varepsilon) \right]^{1/p} = (\|f\|_\infty - \varepsilon) m(E_\varepsilon)^{1/p} \end{aligned}$$

where  $m(E_\varepsilon)^{1/p}$  again goes to 1 as  $p \rightarrow \infty$ . Thus,

$$\liminf_{p \rightarrow \infty} \|f\|_p \geq \|f\|_\infty - \varepsilon$$

□

**Theorem.** Let  $1 < p < q < r < \infty$ . If  $f \in L^p(E)$  and  $f \in L^r(E)$ , then  $f \in L^q(E)$ . Furthermore,  $\|f\|_q < \max(\|f\|_p, \|f\|_r)$ .

*Proof.* Let  $q = \lambda p + (1 - \lambda)r$  for some  $\lambda \in (0, 1)$ . Then,

$$\begin{aligned} \int_E |f|^q dm &= \int_E |f|^{\lambda p} \cdot |f|^{(1-\lambda)r} dm \leq \left( \int_E [|f|^{\lambda p}]^{1/\lambda} dm \right)^\lambda \left( \int_E [|f|^{(1-\lambda)r}]^{1/(1-\lambda)} dm \right)^{1-\lambda} \\ &= \|f\|_p^{\lambda p} \|f\|_r^{(1-\lambda)r} \leq [\max(\|f\|_p, \|f\|_r)]^{\lambda p + (1-\lambda)r} = [\max(\|f\|_p, \|f\|_r)]^q \end{aligned}$$

□

**Remark.** For  $p > 0$ ,

$$\int_0^1 \frac{1}{x^p} dx < \infty \quad \text{iff} \quad 0 < p < 1 \quad \text{and} \quad \int_1^\infty \frac{1}{x^p} < \infty \quad \text{iff} \quad 1 < p$$

**Example.** Given  $a > 0$ , find  $f$  on  $(0, \infty) = E$  such that  $f \in L^p((0, \infty))$  if and only if  $p \in (a, b)$ .

*Proof.* Simply take

$$f(x) = \begin{cases} x^{-1/b} & x \in (0, 1) \\ x^{-1/a} & x \in (1, \infty) \end{cases} = x^{-1/b} \mathbf{1}_{(0,1)} + x^{-1/a} \mathbf{1}_{(1,\infty)}$$

□

**Example.** Note how

$$\int_1^\infty \frac{1}{[x(1 + \ln x)^2]^p} dx = \int_0^\infty \frac{e^{t(1-p)}}{(1+t)^{2p}} dt < \infty$$

if and only if  $p \geq 1$ , where we substituted  $t = \ln x$ . Also,

$$\int_0^1 \frac{1}{[x(1 - \ln x)^2]^p} dx = \int_\infty^0 \frac{e^{t(p-1)}}{[1+t]^{2p}} (-dt) = \int_0^\infty \frac{e^{t(p-1)}}{(1+t)^{2p}} dt < \infty$$

if and only if  $0 < p \leq 1$ , with the substitution being  $t = -\ln x$ .

**Example.** Given  $a > 0$ , find  $f$  on  $(0, \infty) = E$  such that  $f \in L^p((0, \infty))$  if and only if  $p \in [a, b]$  ( $0 < a < b$ ).

*Proof.* Simply take this time

$$f(x) = [x(1 - \ln x)^2]^{-1/b} \mathbf{1}_{(0,1)} + [x(1 + \ln x)^2]^{-1/a} \mathbf{1}_{[0,\infty)}$$

□

**Example.** For  $\alpha \in (0, \infty)$ , find  $f$  so that  $f \in L^p((0, \infty))$  if and only if  $p = \alpha$ . The  $f$  is given by the expression above with  $\alpha = a = b$ . We now find that there is a function that is  $L^p((0, \infty))$  only when  $p$  is a given number. There is no subset relation when the measure of the space is infinite!

**Theorem.** For  $1 < p < \infty$  and  $1/p + 1/q = 1$ , let  $T : L^p(E) \rightarrow \mathbf{R}$  be defined by

$$T(f) = \int_E fg \, dm$$

for some  $g \in L^q(E)$ . Then,  $T$  is a bounded linear functional on  $L^p(E)$  with  $\|T\| = \|g\|_q$ .

**Remark.** First of all,

$$T(\alpha_1 f_1 + \alpha_2 f_2) = \int_E (\alpha_1 f_1 + \alpha_2 f_2) g \, dm = \alpha_1 \int_E f_1 g \, dm + \alpha_2 \int_E f_2 g \, dm = \alpha_1 T(f_1) + \alpha_2 T(f_2)$$

and by Holder's inequality,

$$|T(f)| = \left| \int_E fg \, dm \right| \leq \|f\|_p \|g\|_q$$

so that  $\|T\| \leq \|g\|_q$ . We now want to prove that  $\|T\| \geq \|g\|_q$ . Let

$$f = \frac{|g|^q}{g}$$

we can easily see that  $f \in L^p(E)$  (given  $g \neq 0$  and if  $g = 0$  we define  $f = 0$ ). Now see that  $fg = |g|^q$  and

$$T(f) = \int_E fg \, dm = \int_E |g|^q \, dm = \|g\|_q^q = \|g\|_q \|g\|_q^{q-1} = \|f\|_p \|g\|_q$$

given the fact that

$$\|g\|_q^{q-1} = \left( \int_E |g|^q \, dm \right)^{(q-1)/q} = \left( \int_E |f|^p \, dm \right)^{1/p} = \|f\|_p$$

We now see how

$$\left| T\left(\frac{f}{\|f\|_p}\right) \right| \geq \|g\|_q$$

As the norm of an operator is defined by the supremum of which the left hand side is an element of,

$$\|T\| \geq \left| T\left(\frac{f}{\|f\|_p}\right) \right| \geq \|g\|_q$$

Finally, due to the previous remark,  $\|T\| = \|g\|_q$ .

**Corollary.** Let  $T : L^1(E) \rightarrow \mathbf{R}$  be defined by

$$T(f) = \int_E fg \, dm$$

for some  $g \in L^\infty(E)$ .  $T$  is a bounded linear functional on  $L^1(E)$  with  $\|T\| = \|g\|_\infty$ .

## 6 Lecture 19 (May 22nd)

**Recall.** We have seen how integral transforms of the form

$$T(f) = \int_E fg \, dm$$

are bounded linear operators with a norm of  $\|T\| = \|g\|_q$ . We see that, suprisingly, all bounded linear operators can be realised to be of this form.

**Theorem.** (Riesz representation theorem) If  $1 \leq p < \infty$  and  $T$  is a bounded linear functional on  $L^p(\mu)$ , there is a unique  $g \in L^q(\mu)$  (where  $q$  is the conjugate exponent of  $p$ ) such that

$$T(f) = \int fg \, d\mu$$

for all  $f \in L^p(\mu)$  and  $\|T\| = \|g\|_q$ . In particular, if  $\mu$  is a  $\sigma$ -finite measure on  $X$  and  $T$  is

a bounded linear functional on  $L^1(\mu)$  then there is a unique  $g \in L^\infty(\mu)$  such that

$$T(f) = \int_X fg \, d\mu$$

for all  $f \in L^1(\mu)$  and  $\|T\| = \|g\|_\infty$ .

**Definition.** If  $\mu$  is a counting measure on  $\mathbf{N}$ , we define three vector spaces.

(i)  $l^p(\mathbf{N})$  the set of sequence  $\mathbf{x} = (x_1, x_2, \dots)$  with the norm

$$\|\mathbf{x}\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

for  $p \geq 1$ .

(ii)  $l^\infty(\mathbf{N})$  is the set of all bounded sequences with the supremum norm.

(iii)  $C_0(\mathbf{N})$  the sequence  $\mathbf{x}$  which converges to zero with the supremum norm.

**Definition.** A linear map  $T : l^1(\mathbf{N}) \rightarrow \mathbf{R}$  is defined by

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n$$

where  $\mathbf{y} = (y_1, y_2, \dots) \in l^\infty(\mathbf{N})$ . We now try to show that the norm of  $\mathbf{y}$  is the norm of the linear operator. Observe that

$$|T(\mathbf{x})| \leq \|\mathbf{y}\|_\infty \sum_{n=1}^{\infty} |x_n| = \|\mathbf{x}\|_1 \|\mathbf{y}\|_\infty$$

so that  $\|T\| \leq \|\mathbf{y}\|_\infty$ . To show that  $\|T\| = \|\mathbf{y}\|_\infty$ , suppose  $y = \{y_n\}$ . Then we put

$$\mathbf{x} = \frac{|y_k|}{y_k} \mathbf{e}_k \quad \text{for } y_k \neq 0 \quad \text{implying} \quad \|\mathbf{x}\|_1 = 1$$

and

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n = |y_k|$$

This tells us that

$$|y_k| \leq \|T\|$$

for every  $k \in \mathbf{N}$  and  $\|T\| \geq \|\mathbf{y}\|_\infty$ .

**Theorem.** (Riesz representation theorem for  $l^1(\mathbf{N})$ ) If  $T$  is a bounded linear functional



on  $l^1(\mathbf{N})$  then we'll show  $\mathbf{y} = (y_1, y_2, \dots) \in l^\infty(\mathbf{N})$  such that

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n$$

with  $\|T\| = \|\mathbf{y}\|_\infty$ .

*Proof.* To do that just define  $T(\mathbf{e}_k) = y_k$ . Then by continuity and linearity of  $T$ ,

$$T(\mathbf{x}) = T\left(\sum_{k=1}^{\infty} x_k \mathbf{e}_k\right) = \sum_{k=1}^{\infty} T(x_k \mathbf{e}_k) = \sum_{k=1}^{\infty} x_k T(\mathbf{e}_k) = \sum_{k=1}^{\infty} x_k y_k$$

To prove that uniqueness, do this again, and we'll find

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} x_n z_n$$

and by putting  $\mathbf{x} = \mathbf{e}_k$  we find  $y_k = z_k$ . □

**Remark.** We have previously shown that, in the limited case of  $p = 1$ ,

$$l^p(\mathbf{N})^* = l^q(\mathbf{N})$$

for  $1 \leq p < \infty$  and  $1/p + 1/q = 1$ . Also, for the Lebesgue measure,

$$L^p(E)^* = L^q(E)$$

for  $1/p + 1/q = 1$  with  $1 \leq p < \infty$ . We now show that

$$C_0(\mathbf{N})^* = l^1(\mathbf{N})$$

and also that

$$C_0(E)^* = M(E)$$

Notice that, importantly, the dual of  $C_0(\mathbf{N})$  is  $l^1(\mathbf{N})$  while the dual of  $l^1(\mathbf{N})$  is  $l^\infty(\mathbf{N})$ . The dual of a dual is not itself!

**Theorem.** Recall that  $C_0(\mathbf{N})$  is defined as the set of infinite sequences that converge to 0 provided the supremum norm. If  $T : C_0(\mathbf{N}) \rightarrow \mathbf{R}$  is defined by

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n$$

for some  $\mathbf{y} = (y_1, \dots, y_n, \dots) \in l^1(\mathbf{N})$ , then  $T$  is continuous and linear with  $\|T\| = \|\mathbf{y}\|_1$ .

Conversely, if  $T$  is a bounded linear functional, then there is a unique  $\mathbf{y} \in l^1(\mathbf{N})$  such that

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n$$

with  $\|T\| = \|\mathbf{y}\|_1$ . The latter converse follows from just defining  $T(e_k) = y_k$ .

*Proof.*  $T$  is obviously linear.

$$|T(x)| = \left| \sum_{n=1}^{\infty} x_n y_n \right| \leq \sum_{n=1}^{\infty} |x_n| |y_n| \leq \|\mathbf{x}\|_{\infty} \|\mathbf{y}\|_1$$

This implies that  $\|T\| \leq \|\mathbf{y}\|_1$ . Conversely, for each  $n$ , we define  $\mathbf{x}_n \in C_0(\mathbf{N})$  as  $\mathbf{x}_n = \{x_{n,k}\}_{k=1}^{\infty}$  where

$$x_{n,k} = \begin{cases} \frac{|y_k|}{y_k} & \text{if } 1 \leq k \leq n \text{ and } y_k \neq 0 \\ 0 & \text{if } k > n \text{ and } y_k = 0 \end{cases}$$

Notice how  $\lim_{k \rightarrow \infty} x_{n,k} = 0$ . Then  $\|\mathbf{x}_n\|_{\infty} = 1$  for each  $n$ , and

$$T(\mathbf{x}_n) = \sum_{k=1}^{\infty} x_{n,k} y_k = \sum_{k=1}^n |y_k|$$

Thus,  $\sum_{k=1}^n |y_k| \leq \|T\|$  for every  $n \in \mathbf{N}$ . Therefore,  $\|T\| \geq \|\mathbf{y}\|_1$ . For the converse, we simply define  $T(\mathbf{e}_k) = y_k$ . Then by the continuity and linearity,  $T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n$ .  $\square$

**Example.** We see another example where which a sequence that is bounded fails to have a subsequence that converges. Define  $f_n(x) = \sin nx$  for  $x \in [0, 2\pi]$ . This implies that  $|f_n(x)| \leq 1$  for all  $x \in [0, 2\pi]$  and for all  $n \in \mathbf{N}$ . Suppose  $\{f_n\}$  has a subsequence  $\{f_{n_k}\}$  which converges pointwise on  $[0, 2\pi]$ . Then,  $\lim_{k \rightarrow \infty} \sin n_k x = \lim_{k \rightarrow \infty} \sin n_{k+1} x$ , that is,

$$\lim_{k \rightarrow \infty} (\sin n_{k+1} x - \sin n_k x) = 0$$

for all  $x \in [0, 1]$ . Or, equivalently,  $\lim_{k \rightarrow \infty} (\sin n_{k+1} x - \sin n_k x)^2 = 0$ . By LDCT,

$$\lim_{k \leftarrow \infty} \int_0^{2\pi} (\sin n_{k+1} x - \sin n_k x)^2 dx = 0$$

When this is actually computed, we have  $2\pi$  for all  $k$ . To elaborate further, observe that the above is equal to

$$\int_0^{2\pi} (\sin^2 n_{k+1} x + \sin^2 n_k x - 2 \sin n_{k+1} x \sin n_k x) dx$$

and that the first two terms become  $\pi$  each and the last term vanishes.

**Definition.** Simply put, weak\* convergence is pointwise convergence (for a sequence of bounded linear operators). For  $T_n \in X^*$  and  $T \in X^*$ ,  $T_n \rightarrow T$  weak\* in  $X^*$  provided that  $\lim_{n \rightarrow \infty} T_n(x) = T(x)$  for every  $x \in X$ .

**Theorem.** (Arzela-Ascoli theorem) (Important!) Let  $X$  be a separable normed vector space. Then every bounded sequence in  $X^*$  has a weak\* convergent subsequence.

**Corollary.** The following is an application for the above theorem in  $L^p(E) = L^q(E)^*$ . For  $1 \leq p < \infty$ , let  $f_n \in L^p(E)$  with  $\|f_n\|_p \leq M$  for all  $n \in \mathbf{N}$ . Then  $\{f_n\}$  is bounded in  $L^q(E)^*$ . Then there is  $f \in L^p(E)$  and a subsequence  $\{f_{n_k}\}$  such that  $f_{n_k} \rightarrow f$  weak\*. Notice that by the Riesz-representation theorem, there is a bounded linear functional

$$T(g) = \int_E gf \, dm$$

for  $g \in L^q(E)$  so that

$$\lim_{k \rightarrow \infty} \int_E f_{n_k} g \, dm = \lim_{k \rightarrow \infty} T_{n_k}(g) = T(g) = \int_E fg \, dm$$

for all  $g \in L^q(E)$ .

## 7 Lecture 20 (May 27th)

**Definition.** A separable metric space  $X$  is a space that has a countable dense subset.

**Definition.** For  $f_n : X \rightarrow \mathbf{R}$ ,  $\{f_n\}$  is pointwise bounded on  $X$  provided that for every  $x \in X$  there is  $M_x > 0$  such that  $|f_n(x)| \leq M_x$  for all  $n \in \mathbf{N}$ .

**Example.** Let  $X, Y$  be normed vector spaces. Let  $T_n : X \rightarrow Y$  be a bounded linear map such that  $\|T_n\| \leq M$  for all  $n \in \mathbf{N}$ . Then for every  $x \in X$ , we have

$$\|T_n(x)\| \leq \|T_n\| \|x\| \leq M \|x\|$$

Thus  $\{T_n\}$  is pointwise bounded on  $X$ .

**Definition.**  $\{f_n\}$  is equicontinuous on  $X$  provided that for every  $\varepsilon > 0$ , there is  $\delta > 0$  such that if  $x, y \in X$  satisfies  $d(x, y) < \delta$ , then  $|f_n(x) - f_n(y)| < \varepsilon$  for all  $n \in \mathbf{N}$ .

**Example.** Consider  $f_n : \mathbf{R} \rightarrow \mathbf{R}$  where  $f_n = nx$ . As

$$|f_n(x) - f_n(y)| = n|x - y|$$

and each  $f_n$  is uniformly continuous on  $\mathbf{R}$ . However,  $\{f_n\}$  is not equicontinuous on  $\mathbf{R}$ .

**Example.** The function  $f_n(x) = \sin nx$  is not equicontinuous on  $[0, 2\pi]$ .

**Example.** Let  $T_n : X \rightarrow Y$  be norm bounded, that is,  $\|T_n\| \leq M$  for all  $n \in \mathbf{N}$ . This implies that

$$\|T_n(x) - T_n(y)\| = \|T_n(x - y)\| \leq M\|x - y\|$$

so that  $\{T_n\}$  is equicontinuous on  $X$ .

**Theorem.** (Arzela-Ascoli theorem) Consider a sequence in  $X^*$ , that is,  $f_n : X \rightarrow \mathbf{R}$  for  $n \in \mathbf{N}$  where  $X$  is a separable metric space. Take  $\{f_n\}$  to be pointwise bounded and equicontinuous on  $X$ . Then,  $\{f_n\}$  has a subsequence  $\{f_{n_k}\}$  which converges uniformly on all compact subsets of  $X$ .

*Proof.* Let  $E = \{x_1, x_2, \dots\}$  be a countable dense subset of  $X$ .

$$\begin{array}{cccc} f_{1,1}, & f_{2,1} & f_{3,1}, & \dots \\ f_{1,2}, & f_{2,2} & f_{3,2}, & \dots \\ & \vdots & & \\ f_{1,k}, & f_{2,k} & f_{3,k}, & \dots \end{array}$$

(STEP 1) We construct a subsequence of  $\{f_n\}$  which converges at every  $x_k \in E$ . This requires only pointwise boundedness. Note that  $\{f_n(x_1)\}$  is a bounded sequence on  $\mathbf{R}$ , so that there is a subsequence  $\{f_{n,1}\}$  of  $\{f_n\}$  which converges at  $x_1$ . Likewise,  $\{f_{n,1}(x_2)\}$  is a bounded sequence in  $\mathbf{R}$  so that there is a subsequence  $\{f_{n,2}\}$  of  $\{f_{n,1}\}$  converges at  $x_2$  (it also converges at  $x_1$ ). Continuing, we find  $\{f_{n,k}\}$  which converges at  $\{x_{k+1}\}$  and also  $x_1, x_2, \dots, x_k$ . Now, take the diagonal sequence  $\{f_{n,n}\}_{n=1}^\infty$ . Then we'll show that  $\lim_{n \rightarrow \infty} f_{n,n}(x_k)$  converges at every  $x_k \in E$ . Take any  $x_k \in E$  then  $\{f_{n,n}\}_{n=k}^\infty$  is a subsequence of  $\{f_{n,k}\}_{n=k}^\infty$ .

Define  $g_n = f_{n,n}$ . Let  $K$  be any compact subset of  $X$ . We'll show that  $\{g_n\}$  converges uniformly on  $K$ . Equivalently, we'll show that the sequence  $\{g_n\}$  is uniformly Cauchy on  $K$ . Take any  $\varepsilon > 0$ , we will find  $N \in \mathbf{N}$  such that if  $n, m > N$  then  $|g_m(x) - g_n(x)| < \varepsilon$  for all  $x \in K$ .

(STEP 2) We show that the above subsequence converges uniformly on all compact subsets of  $X$ . This requires only equicontinuity. By equicontinuity of  $\{g_n\}$ , there is  $\delta > 0$  such that if  $d(p, q) < \delta$ , then  $|g_n(p) - g_n(q)| < \varepsilon/3$  for all  $n \in \mathbf{N}$ . Notice that

$$\mathcal{F} = \left\{ B\left(x, \frac{\delta}{2}\right) \mid x \in K \right\}$$

is an open cover of  $K$  so that is a finite subcover  $\{B_1, B_2, \dots, B_M\}$  all with radius  $\delta/2$ . Recall that  $E$  was a countable dense subset in  $X$  such that there is a subsequence of  $\{f_n\}$  that converges. Since  $E = \{x_n\}$  is dense in  $X$ , for every  $1 \leq k \leq M$ , there is  $x_k \in E \cap B_k$ .

Then there is  $N \in \mathbf{N}$  such that if  $n, m > N$  then

$$|g_n(x_k) - g_m(x_k)| < \frac{\varepsilon}{3}$$

for  $1 \leq k \leq M$ . In sum, if  $n, m > N$  for some  $N$  and  $x \in K$ , then  $x \in B_j$  for some  $1 \leq j \leq M$  so that  $d(x, y) < \delta$  if  $y \in B_j$ . We finally see that,

$$\begin{aligned} |g_n(x) - g_m(x)| &= |g_n(x) - g_n(p_j) + g_n(p_j) - g_m(p_j) + g_m(p_j) - g_m(x)| \\ &\leq |g_n(x) - g_n(p_j)| + |g_n(p_j) - g_m(p_j)| + |g_m(p_j) - g_m(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

□

**Remark.** Let  $X$  be a normed vector space, and let  $X^*$  be the dual space  $\mathcal{L}(X, \mathbf{R})$ . There are three types of convergences in this book.

- (i) Norm convergence, both in  $X$  and  $X^*$
- (ii) Weak convergence in  $X$
- (iii) Weak\* convergence in  $X^*$

We now learn weak convergence in  $X$ .

**Definition.** Let  $x_n \in X$  and  $x \in X$ .  $x_n \rightarrow x$  weakly in  $X$  if

$$\lim_{n \rightarrow \infty} T(x_n) = T(x)$$

for every  $T \in X^*$ . On the other hand,  $T_n \rightarrow T$  weak\* if

$$\lim_{n \rightarrow \infty} T_n(x) = T(x)$$

for every  $x \in X$ . The latter is simply pointwise convergence in  $X$ .

**Remark.** For  $1 \leq p < \infty$  and  $1/p + 1/q = 1$ , by the Riesz representation theorem,  $(L^p)^* = L^q$  and  $(L^q)^* = L^p$ . Let  $f_n \in L^p(E)$ .  $f_n \rightarrow f$  weakly in  $L^p(E)$  means that

$$\lim_{n \rightarrow \infty} \int_E f_n g \, dm = \int_E f g \, dm$$

for every  $g \in L^q(E)$ . Notice how this is the same as  $f_n \rightarrow f$  weak\* in  $(L^q(E))^*$ .

$$\text{weak convergence in } L^p(E) = \text{weak}^* \text{ convergence in } (L^q(E))^*!$$

**Corollary.** Let  $X$  be a separable normed vector space. Every bounded sequence in  $X^*$  has a weak\* convergent subsequence.

*Proof.* If  $\{T_n\}$  is a bounded sequence in  $X^*$  ( $T_n : X \rightarrow \mathbf{R}$ ,  $\|T_n\| \leq M$ ) then  $\{T_n\}$  is a pointwise bounded and equicontinuous. Then, as every singleton set is compact, for every  $x \in X$ , there exists a subsequence  $T_{n_k}(x)$  that converges. Define  $T : X \rightarrow \mathbf{R}$  as  $T(x) = \lim_{k \rightarrow \infty} T_{n_k}(x)$ . Then,  $T \in X^*$  and  $\{T_n\}$  converges weak\*.  $\square$

**Corollary.** Assume  $1 \leq p < \infty$  and  $\|f_n\|_p < \infty$ . A bounded linear operator  $T_n : L^q(E) \rightarrow \mathbf{R}$  is defined as

$$T_n(g) = \int_E f_n g \, dm$$

where  $\|T_n\| = \|f_n\|_p$ . By the previous corollary,  $T_n$  has a weak\* convergent subsequence where

$$\lim_{n \rightarrow \infty} T_{n_k}(g) = T(g)$$

for all  $g \in L^q$ . Then,

$$\lim_{k \rightarrow \infty} \int_E f_{n_k} g \, dm = \int_E f g \, dm$$

for some  $f \in L^p(E)$ . Therefore, if there is a sequence of  $f_n \in L^p(E)$ , there exists a subsequence  $f_{n_k}$  and  $f \in L^p(E)$  such that

$$\lim_{k \rightarrow \infty} \int_E f_{n_k} g \, dm = \int_E f g \, dm$$

for all  $g \in L^q(E)$ .

## 8 Lecture 21 (May 29th)

**Theorem.** Take  $X$  to be a separable Banach space. Let  $T_n \in X^*$  with  $\|T_n\| \leq M$  for all  $n$ . There is  $\{T_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} T_{n_k}(x)$  converges for every  $x \in X$ . If we define  $T : X \rightarrow \mathbf{R}$  by  $T(x) = \lim_{k \rightarrow \infty} T_{n_k}(x)$  then obviously,  $T$  is linear in  $X$  and for  $\|x\| = 1$ ,

$$|T(x)| = \lim_{k \rightarrow \infty} |T_{n_k}(x)| \leq M$$

so that  $\|T\| \leq M$ . Therefore,  $T \in X^*$ .

**Theorem.** (Fubini theorem) Let  $f(x, y)$  be measurable on  $E \times F$ . If

$$\int_F \int_E |f(x, y)| \, dm(x) \, dm(y) < \infty$$

or

$$\int_E \int_F |f(x, y)| \, dm(y) \, dm(x) < \infty$$

(meaning that  $f \in L^1(E \times F, m \times m)$ ), then

$$\int_E \int_F f(x, y) \, dm(y) \, dm(x) = \int_F \int_E f(x, y) \, dm(x) \, dm(y)$$

If  $f(x, y) \geq 0$ , the result satisfies also.

**Theorem.** Take  $f \in L^p(E)$ ,  $1 \leq p \leq \infty$ . An integral operator is defined as

$$Tf(x) = \int_E K(x, y) f(y) dm(y)$$

If there is  $c > 0$  such that

$$\sup_{x \in E} \int_E |K(x, y)| dm(y) \leq c \quad \text{and} \quad \sup_{y \in E} \int_E |K(x, y)| dm(x) < c$$

then  $\|Tf\|_p \leq c\|f\|_p$  for all  $f \in L^p(E)$ .

*Proof.* For  $p = 1$ ,

$$\begin{aligned} \|Tf\|_1 &= \int_E \left| \int_E K(x, y) f(y) dm(y) \right| dm(x) \\ &\leq \int_E \int_E |K(x, y)| |f(y)| dm(y) dm(x) \\ &\leq c \int_E |f(y)| dm(y) \end{aligned}$$

The case is identical for  $p = \infty$ . For  $1 < p < \infty$ ,

$$\begin{aligned} |Tf(x)| &\leq \int_E |K(x, y)| |f(y)| dm(y) = \int_E |K(x, y)|^{1/p+1/q} |f(y)| dm(y) \\ &\leq \left[ \int_E |K(x, y)| dm(y) \right]^{1/q} \left[ \int_E |K(x, y)| |f(y)|^p dm(y) \right]^{1/p} \\ &\leq c^{1/q} \left[ \int_E |K(x, y)| |f(y)|^p dm(y) \right]^{1/p} \end{aligned}$$

So that

$$\begin{aligned} \int_E |Tf(x)|^p dm(x) &\leq c^{p/q} \int_E \int_E |K(x, y)| |f(y)|^p dm(y) dm(x) \\ &= c^{p/q} \int_E |f(y)|^p \int_E |K(x, y)| dm(x) dm(y) \\ &\leq c^{(p+q)/q} \int_E |f(y)|^p dm(y) \end{aligned}$$

this implies that

$$\|Tf\|_p \leq c\|f\|_p$$

□

**Definition.** Let  $f, g$  be measurable functions on  $\mathbf{R}$ . We define  $f * g$  (convolution) by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y) dy = \int_{-\infty}^{\infty} f(y)g(x-y) dy = (g * f)(x)$$

if it exists (note how it is commutative).

**Theorem.** If  $f \in L^1(\mathbf{R})$  and  $g \in L^p(\mathbf{R})$  for  $1 \leq p \leq \infty$ , then  $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ .

*Proof.* Define  $K(x-y) = f(x-y)$  in the previous theorem. □

**Remark.** Consider

$$g(t) = \begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & -1 < t < 1 \\ 0 & |t| \geq 1 \end{cases}$$

which is  $C^\infty$  with a support  $[-1, 1]$ . Define

$$\phi(x) = \frac{g(x)}{\int_{-1}^1 g(t) dt}$$

then  $\phi \in C^\infty(\mathbf{R})$  with support  $[-1, 1]$  with  $\int_{-\infty}^{\infty} \phi(x) dx = 1$ . Then

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon} \phi\left(\frac{x}{\varepsilon}\right)$$

for  $\varepsilon > 0$  supported  $[-\varepsilon, \varepsilon]$  with

$$\int_{-\infty}^{\infty} \phi_\varepsilon(x) dx = 1$$

If  $f \in L^p(\mathbf{R})$  with compact support then  $f * \phi_\varepsilon \rightarrow f$  in  $L^p$  for  $1 < p < \infty$  as  $\varepsilon \rightarrow 0$ . The convolution  $f * \phi_\varepsilon \in C_c^\infty(\mathbf{R})$  and  $C_c^\infty(\mathbf{R})$  is dense in  $L^p(\mathbf{R})$  ( $1 \leq p < \infty$ ).

**Theorem.** (Schur) If there is a non-negative measurable function  $h$  on  $E$  such that

$$\int_E |K(x, y)| h(y)^q dm(y) \leq c_1 h(x)^q$$

and

$$\int_E |K(x, y)| h(y)^p dm(x) \leq c_2 h(y)^p$$

then  $\|Tf\|_p \leq c_1^{1/q} c_2^{1/p} \|f\|_p$  for  $1 < p < \infty$ .

*Proof.* We provide a hint.

$$|Tf(x)| \leq \int_E |K(x, y)| h(y) h(y)^{-1} |f(y)| dm(y)$$



□

**Definition.** Let  $H$  be a vector space over  $\mathbf{R}$ . If there is a function  $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbf{R}$  satisfying

- (i)  $\langle x, y \rangle = \langle y, x \rangle$  for all  $x, y \in H$
- (ii)  $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$  for  $x, y \in H, \alpha \in \mathbf{R}$
- (iii)  $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$  for  $x, y, z \in H$
- (iv)  $\langle x, x \rangle \geq 0$  for all  $x \in H$  and  $\langle x, x \rangle = 0$  if and only if  $x = \mathbf{0}$ .

Then,  $H$  is called an inner product space over  $\mathbf{R}$ . If we define  $\|x\| = \langle x, x \rangle^{1/2}$ , we can show that  $\|x\|$  is a norm.

**Example.** (i)  $\mathbf{R}^n$  with the inner product  $\langle x, y \rangle = \sum_{k=1}^n x_k y_k$

(ii)  $L^2(\mu)$  with the inner product

$$\langle f, g \rangle = \int_X f g d\mu$$

(iii)  $C([0, 1])$  with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x) dx$$

**Theorem.** (Cauchy-Schwarz Inequality) For  $x, y \in H$ ,  $|\langle x, y \rangle| \leq \|x\| \|y\|$

*Proof.* For every  $t \in \mathbf{R}$ ,  $\|x - ty\|^2 \geq 0$  and

$$\|x - ty\|^2 = \langle x - ty, x - ty \rangle = \|x\|^2 - 2\langle x, y \rangle t + \|y\|^2 t^2$$

If  $y \neq 0$ , then  $D/4 \leq 0$  where  $D/4 = \langle x, y \rangle^2 - \|x\|^2 \|y\|^2$ . □

**Corollary.**  $\|x + y\| \leq \|x\| + \|y\|$  for  $x, y \in \mathbf{R}$ .

*Proof.*

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

We now have shown that  $\langle x, x \rangle^{1/2}$  indeed satisfies a norm. □

**Definition.** An inner product space  $H$  is called a Hilbert space if it is complete with respect to the norm which is induced by the inner product.

**Theorem.** A Hilbert space is a Banach space.

**Example.**  $C([0, 1])$  is a Banach space and an inner product space but is not a Hilbert space with respect to the norm

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

Technically speaking, a Hilbert space is a Banach space, not with the inner product, but satisfying the parallelogram law.

## 9 Lecture 22 (June 5th)

**Theorem.** (Parallelogram law) If  $x, y \in H$ , then

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

**Proposition.** If  $M$  is a subspace of  $H$ , then  $\bar{M}$  is a closed subspace of  $H$ .

**Definition.** (Convex subset) A convex subset of  $H$  is a subset where if  $x, y \in E$  and  $\lambda \in (0, 1)$ , we have  $\lambda x + (1 - \lambda)y \in E$ .

**Proposition.** If  $E$  is a convex subset of  $H$ , then  $x + E = \{x + y \mid y \in E\}$  is also convex.

**Definition.** ( $x^\perp$ ) For  $x \in H$ , we define  $x^\perp = \{y \in H \mid \langle x, y \rangle = 0\}$ . This is a subspace of  $H$ . If we define  $T : H \rightarrow \mathbf{R}$  by  $T(y) = \langle x, y \rangle$ , by the Cauchy Schwarz inequality,  $|T(y)| \leq \|x\| \|y\|$ . That is,  $T$  is continuous and  $x^\perp = \{y \in H \mid T(y) = 0\}$  is a closed subspace of  $H$ .

**Definition.** ( $M^\perp$ ) If  $M$  is a subspace of  $H$ , we define

$$M^\perp = \{y \in H \mid \langle x, y \rangle = 0\} = \bigcap_{x \in M} x^\perp$$

which implies that  $M^\perp$  is a closed subspace of  $H$ .

**Theorem.** If  $E$  is a non-empty closed convex subset of a Hilbert space  $H$ , then  $E$  has a unique element of smallest norm.

*Proof.* Let  $\delta = \inf\{\|x\| \mid x \in E\}$ . We'll show that there is a unique  $x_0 \in E$  with  $\|x_0\| = \delta$ .

Let  $x, y \in E$ . Then,  $x/2, y/2 \in H$ . Apply the parallelogram law and we have

$$\left\| \frac{x}{2} + \frac{y}{2} \right\|^2 + \left\| \frac{x}{2} - \frac{y}{2} \right\|^2 = 2 \left( \left\| \frac{x}{2} \right\|^2 + \left\| \frac{y}{2} \right\|^2 \right)$$

so that

$$\|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4 \left\| \frac{x}{2} + \frac{y}{2} \right\|^2$$

for  $x, y \in E$ . By the convexity of  $E$ , the last term is in  $E$ . We then know that

$$\|x - y\|^2 \leq 2(\|x\|^2 + \|y\|^2) - 4\delta^2$$

If  $\|x\| = \|y\| = \delta$ , then  $\|x - y\|^2 \leq 0$  such that  $x = y$ . This tells us that  $x_0$  is unique if it exists.

By definition of  $\delta$ , there is a sequence  $y_n \in E$  such that  $\lim_{n \rightarrow \infty} \|y_n\| = \delta$ . Put  $y_n, y_m$  in the inequality above and we have

$$\|y_n - y_m\|^2 \leq 2(\|y_n\|^2 + \|y_m\|^2) - 4\delta^2$$

so that  $\{y_n\}$  is a Cauchy sequence in  $E \subset H$ . Since  $H$  is complete, there is  $x_0 \in H$  such that  $\lim_{n \rightarrow \infty} \|y_n - x_0\| = 0$ . Since  $E$  is closed,  $x_0 \in E$ . Finally,

$$\lim_{n \rightarrow \infty} \|x_0\| = \lim_{n \rightarrow \infty} \|y_n\| = \delta$$

since  $\|\cdot\|$  is continuous. □

**Example.** In  $L^1([0, 1])$ , define

$$E = \left\{ f \in L^1([0, 1]) \mid \int_0^1 f(x) dx = 1 \right\}$$

Notice that the set is convex as the integral of  $g = \lambda f + (1 - \lambda)h$  is 1. In addition to this,  $E$  is closed. To see this, we ask whether for  $f_n \in E$  and  $\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$ ,  $\int_0^1 f dx = 1$ . this is true, as

$$\left| \int_0^1 f_n dx - \int_0^1 f dx \right| \leq \int_0^1 |f_n - f| dx \rightarrow 0$$

Together, we now know that the set is a closed convex subset of  $L^1([0, 1])$  with infinitely many elements with the smallest norm.

**Example.** Consider  $f_n \in C([0, 1])$  with the uniform norm. Define

$$E = \left\{ f \in C([0, 1]) \mid \int_0^{1/2} f(x) dx - \int_{1/2}^1 f(x) dx = 1 \right\}$$

Obviously, this is a convex set. By LDCT,  $E$  is closed. The infimum of the norm is 1, while there is no single function that has this norm.

**Theorem.** (Orthogonal decomposition) (Big theorem) If  $M$  is a closed subspace of a Hilbert space  $H$  then every  $x \in H$  can be uniquely expressed as  $x = P_x + Q_x$  where  $P_x \in M$  and  $Q_x \in M^\perp$ . Indeed,  $P \in \mathcal{L}(H, M)$  and  $Q \in \mathcal{L}(H, M^\perp)$  are norm 1 linear operators with

$$\|x\|^2 = \|P_x\|^2 + \|Q_x\|^2$$

The key idea behind this theorem is that  $x + M$  is a closed convex subset of  $H$ .

*Proof.*  $x + M = \{x + y \mid y \in M\}$  is a non-empty closed convex subset of  $H$ . Define  $Q_x$  as the unique element of  $x + M$  with the smallest norm. Then define  $P_x = x - Q_x$ , and by definition,  $P_x \in M$ .

We have to show that  $Q_x \in M^\perp$  and that the decomposition is unique. The latter part is simple, as if we take  $x = p + q = p' + q'$ ,  $x - x' = y' - y \in M \cap M^\perp = \{0\}$ . To show that  $Q_x \in M^\perp$ , we need to show that  $\langle Q_x, y \rangle = 0$  for all  $y \in M$  or that  $\langle Q_x, y \rangle = 0$  for all  $y \in M$  with  $\|y\| = 1$ .

Take any  $y \in M$  with  $\|y\| = 1$ . Then for every  $\alpha \in \mathbf{R}$ ,  $Q_x - \alpha y \in x + M$  so that

$$\|Q_x\|^2 \leq \|Q_x - \alpha y\|^2 = \|Q_x\|^2 - 2\alpha \langle Q_x, y \rangle + |\alpha|^2$$

for all  $\alpha \in \mathbf{R}$ . Put  $\alpha = \langle Q_x, y \rangle$  to get

$$\|Q_x\|^2 \leq \|Q_x\|^2 - \langle Q_x, y \rangle^2$$

or that  $\langle Q_x, y \rangle^2 \leq 0$  and  $\langle Q_x, y \rangle = 0$ . □

**Corollary.** If  $M$  is a closed subspace of  $H$  with  $M \neq H$ , then there is  $x \in M^\perp$  with  $\|x\| = 1$ .

**Theorem.** (Riesz-representation theorem) If  $T$  is a bounded linear functional on a Hilbert space  $H$ , then there is a unique element  $y \in H$  such that  $T(x) = \langle x, y \rangle$  for all  $x \in H$ .

*Proof.* If  $T(x) = 0$  for all  $x \in H$  then put  $y = 0$ . If  $T(x) \neq 0$  for some  $x \in H$ ,

$$M = \{x \in H \mid T(x) = 0\}$$

is a closed subspace of  $H$  so that there is  $z \in M^\perp$  with  $\|z\| = 1$ . Put  $y = T(z)z$  and the proof is over. Take any  $x \in H$  and put  $u = T(x)z - T(z)x$  and  $T(u) = 0$  so that  $u \in M$ . Hence  $\langle u, y \rangle = 0$  as

$$\langle T(x)z - T(z)x, T(z)z \rangle = T(x) - \langle x, T(z)z \rangle$$

□

## 10 Lecture 23 (June 10th)

**Theorem.** (Riesz-representation theorem) If  $T$  is a bounded linear functional on a Hilbert space  $H$ , then there is a unique element  $y \in H$  such that  $T(x) = \langle x, y \rangle$  for all  $x \in H$ .

*Proof.* If  $T(x) = 0$  for all  $x \in H$ , then simply put  $y = 0$ . If  $T(x) \neq 0$  for some  $x \in H$ , define

$$M = \{x \in H \mid T(x) = 0\}$$

which would be a closed subspace of  $H$  so that there is  $z \in M^\perp$  with  $\|z\| = 1$ . We will show that  $y = T(z)z$  satisfies  $T(x) = \langle x, y \rangle$  for all  $x \in H$ . Take any  $x \in H$  and define  $u = T(x)z - T(z)x$ . Then,

$$T(u) = T(x)T(z) - T(z)T(x) = 0$$

implies that  $u \in M$ . Therefore,  $\langle u, z \rangle = 0$  which means that

$$0 = \langle u, z \rangle = \langle T(x)z - T(z)x, z \rangle = T(x) - \langle x, T(z)z \rangle$$

That is,  $T(x) = \langle x, y \rangle$ . In addition to this, this  $y$  is unique, as if there is  $w \in H$  such that  $T(x) = \langle x, w \rangle$  for all  $x \in H$ ,

$$\langle y, x \rangle = \langle w, x \rangle$$

for all  $x \in H$  and

$$\langle y - w, x \rangle = 0$$

By substituting  $x = y - w$ , we find that  $y = w$ . Note how  $|T(x)| \leq \|x\| \|y\|$  and how  $T(y) = \|y\|^2$  such that  $\|T\| = \|y\|$ .  $\square$

**Theorem.** Let  $T \in \mathcal{L}(H) = \mathcal{L}(H, H)$ , a linear operator between two Hilbert spaces. If we define  $L : H \rightarrow \mathbf{R}$  by

$$L(x) = \langle T(x), y \rangle$$

$L$  is linear and

$$|L(x)| \leq \|T(x)\| \|y\| \leq \|T\| \|y\| \|x\|$$

so that  $L$  is a bounded linear functional on  $H$ . By the Riez representation theorem, there is a unique  $u \in H$  such that

$$L(x) = \langle x, u \rangle$$

That is, for every  $x, y \in H$ , there is  $u \in H$  such that

$$\langle T(x), y \rangle = \langle x, u \rangle$$

We define  $T^* : H \rightarrow H$  by  $T^*(y) = u$  (implying  $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ ), then  $T^*$  is linear and

$$\|T^*(y)\|^2 = \langle T^*(y), T^*(y) \rangle = \langle T(T^*(y)), y \rangle \leq \|T\| \|T^*(y)\| \|y\|$$

which implies that

$$\|T^*(y)\| \leq \|T\| \|y\|$$

and

$$\|T^*\| \leq \|T\|$$

as  $T^{**} = T$ , the equality works vice-versa, and  $\|T^*\| = \|T\|$ .

**Remark.** If  $T \in \mathcal{L}(H)$ , then  $N(T^*) = R(T)^\perp$  and  $N(T) = R(T^*)^\perp$  as if  $T^*(y) = 0$ ,  $\langle x, T^*(y) \rangle = 0$  for all  $x \in H$  and  $\langle T(x), y \rangle = 0$  for all  $x \in H$  and  $y \in R(T)^\perp$ .

**Definition.** (Orthonormal set in  $H$ ) If  $\{u_1, \dots, u_n\}$  is an orthonormal set in  $H$ , then it is linearly independent. The linear span of  $\{u_\alpha \mid \alpha \in I\}$  which are orthonormal is the set of all finite linear combinations of  $\{u_\alpha\}$  which is the smallest subspace of  $H$  which contains  $\{u_\alpha\}$ .

**Definition.** If  $\{u_\alpha \mid \alpha \in I\}$  is an orthonormal set in  $H$ , then for  $x \in H$  we define  $\hat{x}(\alpha) = \langle x, u_\alpha \rangle$  the Fourier coefficient of  $x$  with respect to the orthonormal set  $\{u_\alpha\}$ .

**Theorem.** Let  $\{u_\alpha \mid \alpha \in I\}$  be an orthonormal set and let  $F$  be a finite subset of  $I$ .  $M_F$  is the linear span of  $\{u_k \mid k \in F\}$ .

(i) If  $y \in M_F$  satisfies  $y = \sum_{k \in F} a_k u_k$  then  $a_k = \hat{y}(k) = \langle y, u_k \rangle$  for all  $k \in F$  and

$$\|y\|^2 = \sum_{k \in F} |\hat{y}(k)|^2$$

(ii) (Best approximation property) If  $x \in H$  and  $s_F(x) = \sum_{k \in F} \hat{x}(k) u_k$  then for every  $s \in M_F$  except  $s = s_F$ ,

$$\|x - s_F\| < \|x - s\|$$

*Proof.* (STEP 1) If  $y = \sum_{k \in F} a_k u_k$ , then for  $m \in F$ ,

$$\hat{y}(m) = \left\langle \sum_{k \in F} a_k u_k, u_m \right\rangle = a_m$$

(STEP 2) The key idea is that

$$\|x - s\|^2 = \left\| \underbrace{x - s_F}_{\in M_F^\perp} + \underbrace{s_F - s}_{\in M_F} \right\|^2 = \|x - s_F\|^2 + \|s_F - s\|^2$$

which is true as for  $m \in F$ ,

$$\hat{s}_F(m) = \left\langle \sum_{k \in F} \hat{x}(k) u_k, u_m \right\rangle = \hat{x}(m)$$

and  $\langle x - s_F, u_m \rangle = 0$  for all  $m \in F$ . This implies that

$$\|x - s\|^2 \leq \|x - s_F\|^2 + \|s_F - s\|^2$$

and by putting  $s = 0$  we have

$$\|s_F\|^2 \leq \|x\|^2$$

which implies the Bessel inequality

$$\sum_{k \in F} |\hat{x}(k)|^2 \leq \|x\|^2$$

□

## 11 Lecture 25 (June 12th)

**Definition.** An orthonormal set  $\{u_\alpha \mid \alpha \in I\}$  in a Hilbert space  $H$  is said to be complete (a complete orthonormal set, orthonormal basis, or maximal orthonormal set) provided that if  $x \in H$  satisfies

$$\langle x, u_\alpha \rangle = 0$$

for all  $\alpha \in I$ , then  $x = \mathbf{0}$ .

**Theorem.** (Axiom of choice) If  $\{u_\alpha\}$  is an orthonormal set in  $H$ , then there is a complete orthonormal set  $E$  which contains  $\{u_\alpha\}$ . As a result, all Hilbert spaces have a countable orthonormal basis.

**Theorem.** If  $E = \{u_\alpha \mid \alpha \in I\}$  is an orthonormal set in a separable Hilbert space, then  $E$  is countable.

*Proof.* Let  $D$  be a countable dense subset of  $H$ . Let  $i : E \rightarrow D$  satisfy

$$\|u - i(u)\| < \frac{1}{2}$$

If  $i(u) = i(v)$  then

$$\|u - v\| = \|u - i(u) + i(v) - v\| \leq \|u - i(u)\| + \|v - i(v)\| < 1$$

so that  $u = v$  as  $\|u - v\| = \sqrt{2}$  if they are orthonormal. Since  $i$  is one-to-one and  $D$  is countable,  $E$  is also countable. □

**Theorem.** Let  $E = \{u_n \mid n \in \mathbf{N}\}$  be an orthonormal set in a separable Hilbert space  $H$ .

(i) (Bessel inequality) For every  $x \in H$ , we have

$$\sum_{n=1}^{\infty} \langle x, u_n \rangle^2 \leq \|x\|^2$$

(ii) For  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n, \dots) \in l^2(\mathbf{N})$ ,

$$\sum_{n=1}^{\infty} \alpha_n u_n \in H$$

(iii) If  $x \in H$  and

$$y = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$$

then  $\langle x, u_n \rangle = \langle y, u_n \rangle$  for all  $n \in \mathbf{N}$

*Proof.* (STEP 1) We proved that

$$\sum_{k=1}^n \langle x, u_k \rangle^2 \leq \|x\|^2$$

for every  $n$  so that the inequality works for infinity.

(STEP 2) Let

$$x_n = \sum_{k=1}^n \alpha_k u_k$$

Then we'll show that  $x_n$  is a Cauchy sequence in  $H$ . Note that if  $n > m$  then

$$\|x_n - x_m\|^2 = \left\langle \sum_{k=m+1}^n \alpha_k u_k, \sum_{j=m+1}^n \alpha_j u_j \right\rangle = \sum_{k=m+1}^n \alpha_k^2$$

Notice that this can be made arbitrarily small by setting  $m$  to be large.

(STEP 3) By (i),  $\{\langle x, u_n \rangle\} \in l^2(\mathbf{N})$  and by (ii)  $y \in H$ . Thus for every  $m \in \mathbf{N}$

$$\langle y, u_m \rangle = \left\langle \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n, u_m \right\rangle = \lim_{k \rightarrow \infty} \sum_{n=1}^k \langle \langle x, u_n \rangle u_n, u_m \rangle = \langle x, u_m \rangle$$

□

**Theorem.** Let  $\{u_n \mid n \in \mathbf{N}\}$  be a complete orthonormal set in a separable Hilbert space  $H$ .



(i) For every  $x \in H$ , we have

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$$

(ii) (Parseval identity) If  $x, y \in H$  then

$$\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, u_n \rangle \langle y, u_n \rangle$$

especially,

$$\|x\|^2 = \sum_{n=1}^{\infty} \langle x, u_n \rangle^2$$

*Proof.* Let

$$y = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$$

then  $y \in H$  and  $\langle y, u_n \rangle = \langle x, u_n \rangle$  for all  $n \in \mathbf{N}$  so that  $\langle x - y, u_n \rangle = 0$  for all  $n \in \mathbf{N}$ . Since  $\{u_n\}$  is complete, we have  $x = y$ . Therefore,

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n \quad \left( = \sum_{n=1}^{\infty} \hat{x}(n) u_n \right)$$

We can take the above to be, for example, the Fourier series expansion of  $x$  in terms of  $\{u_n\}$ .  $\square$

**Corollary.** It is known that  $\{u_n\}_{n=-\infty}^{\infty}$  defined by

$$u_n(t) = e^{2\pi i n t}$$

is a complete orthonormal set in  $L^2([0, 1])$ . Here,  $L^2([0, 1])$  is the space of functions  $f : [0, 1] \rightarrow \mathbf{C}$  with

$$\int_0^1 |f(t)|^2 dt < \infty$$

and the norm defined as

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt$$

Then by application of the above theorem, we have for all  $f \in L^2([0, 1])$

$$f(x) = \sum_{n=-\infty}^{\infty} \left( \int_0^1 f(t) e^{-2\pi i n t} dt \right) e^{2\pi i n x}$$

in terms of  $L^2([0, 1])$ .

**Lemma.** The inner product  $\langle, \rangle : H \times H \rightarrow \mathbf{R}$  is continuous.

*Proof.* Take any  $(a, b) \in H \times H$  and  $\varepsilon > 0$ . We have to find  $\delta > 0$  so that if  $\|(x, y) - (a, b)\| < \delta$  then

$$|\langle x, y \rangle - \langle a, b \rangle| < \varepsilon$$

We see that the right hand side satisfies

$$|\langle x, y \rangle - \langle a, b \rangle| = |\langle x, y \rangle - \langle a, y \rangle + \langle a, y \rangle - \langle a, b \rangle| \leq |\langle x - a, y \rangle| + |\langle a, y - b \rangle|$$

such that

$$|\langle x, y \rangle - \langle a, b \rangle| \leq \|x - a\| \|y\| + \|a\| \|y - b\| \leq \delta(\|b\| + 1) + \|a\| \delta$$

as if  $\delta < 1$   $\|x\| \leq \|a\| + 1$  and  $\|y\| \leq \|b\| + 1$ . □

*Proof.* Observe that

$$\begin{aligned} \langle x, y \rangle &= \left\langle \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j, \sum_{k=1}^{\infty} \langle y, u_k \rangle u_k \right\rangle \\ &= \lim_{n \rightarrow \infty} \left\langle \sum_{j=1}^n \langle x, u_j \rangle u_j, \sum_{k=1}^n \langle y, u_k \rangle u_k \right\rangle = \sum_{n=1}^{\infty} \langle x, u_n \rangle \langle y, u_n \rangle \end{aligned}$$

□