1 Lecture 14 (April 29th)

Definition. (Convergence in measure) For measurable functions on E, we say $f_n \to f$ in measure on E provided that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} m(\{x \in E \mid |f_n(x) - f(x)| > \varepsilon\}) = 0$$

Definition. (L^1 convergence) For $f_n \in L^1(E)$ and a measurable f on E, we say that $f_n \to f$ in $L^1(E)$ provided that

$$\lim_{n\to\infty} \int_E |f_n - f|\, dm = 0$$

Theorem. If $f_n \to f$ and $f_n \to g$ in measure (or in $L^1(E)$) in E, then $f = g$ almost everywhere on E.

Proof. Suppose that $f_n \to f$ and $f_n \to g$ in measure. First note that

$$\{x \in E \,|\, |f(x) - g(x)| > 0\} = \bigcup_{n=1}^{\infty} \left\{x \in E \,\middle|\, |f(x) - g(x)| \geq \frac{1}{n}\right\}$$

Meanwhile, for each n, due the triangle inequality,

$$|f(x) - f_k(x)| + |f_k(x) - g(x)| \geq |f(x) - g(x)| \geq \frac{1}{n}$$

and one of the two terms in the left must be greater than $1/2n$. We now see the following inclusion for $x \in E$:

$$\left\{x \,\middle|\, |f(x) - g(x)| \geq \frac{1}{n}\right\} \subset \left\{x \,\middle|\, |f(x) - f_k(x)| \geq \frac{1}{2n}\right\} \cup \left\{x \,\middle|\, |g(x) - f_k(x)| \geq \frac{1}{2n}\right\}$$

See how for each term, we can choose k_1 and k_2 such that they are measure zero, and we see that by taking $\max\{k_1, k_2\}$, their infinite union is measure zero too. $\square$

Example. To this point we have learned the following convergences:

(i) $f_n \to f$ uniformly

(ii) $f_n \to f$ pointwise almost everywhere on E

(iii) $f_n \to f$ almost uniformly on E

(iv) $f_n \to f$ in measure

(v) $f_n \to f$ in $L^1(E)$

Consider the following sequences.

(i) $f_n = \mathbf{1}_{(n,\infty)}$ (uniform, pointwise almost everywhere)

(ii) $f_n = n\mathbf{1}_{(0,1/n)}$ or $f_n = \mathbf{1}_{(0,n)}/n$ (pointwise almost everywhere, almost uniformly, in measure) (uniformly, almost everywhere, almost uniformly, and in measure)

(iii) $f_n = \mathbf{1}_{I_n}$ where

$$I_1 = [0, 1/2] \quad I_2 = [1/2, 1] \quad I_3 = [0, 1/3] \quad I_4 = [2/3, 1] \quad I_5 = [0, 1/4]$$

and etcetera. (uniformly, almost everywhere, almost uniformly, and in measure)

Theorem. If $f_n \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ which converges pointwise almost everywhere to f.

Proof. If $f_n \to f$ in measure the following are both true.

(i) For every $\varepsilon > 0$ there is N such that if $n > N$ then

$$m(\{x \in E \,|\, |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$$

(ii) For every $\varepsilon > 0$ there is N such that if $n > N$ then

$$m(\{x \in E \,|\, |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon^2$$

For all $k \in \boldsymbol{N}$ there exists n_k's such that $n_{k+1} > n_k$ and

$$m(\{x \in E \,|\, |f_{n_k}(x) - f(x)| > 1/k\}) < 1/k^2$$

Let $E_k = \{x \in E \,|\, |f_{n_k}(x) - f(x)| > 1/k\}$. We will now show that $f_{n_k} \to f$ almost everywhere on E. Since $\sum_{n=1}^{\infty} m(E_k) \leq \sum_{k=1}^{\infty} 1/k^2 < \infty$, by the Borel-Cantelli lemma, almost all $x \in E$ belongs to at most finitely many E_k's. So, f_{n_k} almost everywhere on E. $\square$

Theorem. (i) If $f_n \to f$ almost uniformly on E, then $f_n \to f$ in measure.

(ii) If $f_n \to f$ in $L^1(E)$ then $f_n \to f$ in measure.

Proof. We prove (ii) first. By Chebychev, for each ε,

$$m(\{x \in E \,|\, |f_n(x) - f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_E |f_n - f| \, dm$$

where the right hand side approaches 0 as $n \to \infty$.

Now for (i), suppose f_n doesn't converge to f in measure. Then there exists $\varepsilon, \delta > 0$ such that

$$m(\{x \in E \mid |f_n(x) - f(x)| > \varepsilon\}) \geq \delta$$

for infinitely many n's so that there is a subsequence $\{f_{n_k}\}$ such that

$$m(\{x \in E \mid |f_{n_k}(x) - f(x)| > \varepsilon\}) \geq \delta$$

for all $k \in \boldsymbol{N}$. There is no $A \subset E$ such that $m(A) < \delta/2$ and $f_{n_k} \to f$ uniformly on $A^c = E \setminus A$. Hence f_n does not converge to f almost uniformly on E. □

2 Lecture 15 (May 8th)

Definition. A normed vector space is called complete if every Cauchy sequence in X converges (in X) with respect to the norm.

Definition. A Banach space is defined as a complete normed vector space. A norm is a function $||\cdot|| : X \to [0, \infty)$ from a vector space of $\boldsymbol{R}$ that satisfies the following properties:

(i) (Nonnegativity) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = \mathbf{0}$ (that is, x is the zero vector)

(ii) (Positive homogenity) $||\alpha x|| = |\alpha|\,||x||$ for all $\alpha \in \boldsymbol{R}$ and $x \in X$

(iii) (The triangle inequality) $||x + y|| \leq ||x|| + ||y||$ for all x and y in X

Note that $d(x, y) = ||x - y||$ is a metric on X. Therefore, all normed vector spaces are a metric space.

Example. (Examples of normed vector spaces)

(i) $C([0,1]) = \{f \mid f \text{ is continuous on } [0,1]\}$ maybe given the norm $||f|| = \max_{[0,1]} |f(x)|$ (this is called the uniform norm).

(ii) $C([0,1])$ may also given the norm $||f|| = \int_0^1 |f(x)|\,dx$.

Proposition. $C([0,1])$ is a Banach space with respect to the uniform norm since a Cauchy sequence $\{f_n\}$ in $C([0,1])$ converges to some $f \in C([0,1])$.

Proof. Notice that

$$|f_n(x) - f_m(x)| \leq ||f_n(x) - f_m(x)||$$

for all $x \in [0,1]$, and that $\{f_n\}$ is uniformly Cauchy on $[0,1]$. This means that the sequence converges uniformly to some f that is continuous. □

Remark. $C([0,1])$ is not a Banach space with respect to the norm $||f|| = \int_0^1 |f(x)|\,dx$. For example, take $\{f_n\} = \{x_n\}$.

Remark. $C^1([0,1])$ is a normed vector space with the uniform norm but it is not complete. For example, take $f_n(x) = \sqrt{x + 1/n}$. $f_n \to f(x) = \sqrt{x}$ uniformly on $[0,1]$ which is not in $C^1([0,1])$.

Definition. For a measure space (E, X, m), the space $L^1(E)$ is the vector space of measurable functions endowed with the function $||f||_1 = \int_E |f|\,dm$. Notice that

$$\int_E |f|\,dm = 0 \quad \text{then} \quad f = 0 \;\; \text{almost everywhere on } E$$
$$\int_E |\alpha f|\,dm = |\alpha| \int_E |f|\,dm \quad \text{for} \quad \alpha \in R, f \in L^1(E)$$
$$\int_E |f+g|\,dm \leq \int_E |f|\,dm + \int_E |g|\,dm \quad \text{for} \quad f, g \in L^1(E)$$

If $f, g : E \to \boldsymbol{R}$ satisfies $f, g \in L^1(E)$ and $f = g$ almost everywhere on E, we define $f = g$ as an element of $L^1(E)$. Then, $L^1(E)$ is a normed vector space with respect to the norm $||f||_1 = \int_E |f|\,dm$.

Definition. For $1 < p < \infty$, we define $L^p(E)$ as the set of measurable functions of a measure space satifying

$$\int_E |f|^p\,dm < \infty$$

If $f, g \in L^p(E)$ and $f = g$ almost everywhere on E, then f and g are defined to be the same element of $L^p(E)$. To satisfy $||\alpha f||_p = |\alpha|\,||f||_p$, we define

$$||f||_p = \Big(\int_E |f|^p\,dm\Big)^{1/p}$$

We will prove later on that $L^p(E)$ is a Banach space for $1 \leq p \leq \infty$.

Definition. If X and Y are normed vector spaces, $T : X \to Y$ is called a linear operator provided that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for $\alpha, \beta \in \boldsymbol{R}$ and $x, y \in X$.

Theorem. If $T : X \to Y$ is a linear operator, then the following are equivalent.

(i) T is continuous on X

(ii) T is uniformly continuous on X

(iii) T is continuous at $\mathbf{0}$

Definition. $T : X \to Y$ is called a bounded linear operator provided that T is linear and there is $M > 0$ such that

$$||T(x)||_Y \leq M||x||_X$$

for all $x \in X$. Intuitively, bounded linear operators are operators that do not "blow up" small imputs.

Proposition. If T is continuous at $\mathbf{0}$, then T is bounded linear.

Proof. By continuity, we see that for every $\varepsilon > 0$, there is $\delta > 0$ such that if $||x|| < \delta$ then $||T(x)|| < \varepsilon$. Take any $w \in X \setminus \{\mathbf{0}\}$ and we see that the vector $\delta w/2||w|| \in X$ has a norm $\delta/2$. This implies that

$$||T\Big(\frac{\delta w}{2||w||}\Big)|| < \varepsilon \quad \text{and} \quad T(w) \leq \frac{2\varepsilon}{\delta}||w||$$

for all $w \neq \mathbf{0}$. □

3 Lecture 16 (May 13th)

Definition. If X and Y are normed vector spaces, we define $\mathcal{L}(X, Y)$ as the set of all bounded linear operators from X to Y. We call $X^* = \mathcal{L}(X, \boldsymbol{R})$ the dual of X. The elements of $X^* = \mathcal{L}(X, \boldsymbol{R})$ are called bounded linear functionals.

Theorem. For $T \in \mathcal{L}(X, Y)$, let norm of T be defined as $||T|| = \sup_{||x||=1} ||T(x)||$. If Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space. In particular, $X^* = \mathcal{L}(X, \boldsymbol{R})$ is a Banach space.

Remark. For $x \neq 0$, observe that

$$\Big|\Big|T\Big(\frac{x}{||x||}\Big)\Big|\Big| \leq ||T|| \implies ||T(x)|| \leq ||T||\,||x||$$

As $||T(x)|| \leq M\,||x||$ we also have that

$$\Big|\Big|T\Big(\frac{x}{||x||}\Big)\Big|\Big| \leq M \implies ||T|| \leq M$$

This tells us the following equivalence of defintions,

$$||T|| = \inf(\{M \geq 0 \,|\, ||T(x)|| \leq M\,||x||\})$$

for every $x \in X$. This tells us that the norm measures how much a vector can be scaled and stretched.

Theorem. The proof that $\mathcal{L}(X, Y)$ is a normed vector space with the aforementioned norm and will be for now will be skipped. If Y is a Banach space, we'll show that $\mathcal{L}(X, Y)$ is a Banach space. Given a Cauchy sequence $\{T_n\}$, we have to show that there is $T \in \mathcal{L}(X, Y)$ such that $\lim_{n\to\infty} ||T_n - T|| = 0$.

Proof. We define the target linear operator pointwise. For each $x \in X$,

$$||T_n(x) - T_m(x)|| = ||(T_m - T_n)(x)|| \leq ||T_n - T_m||\,||x||$$

and we see that $\{T_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, there is $y \in Y$ such that $\lim_{n\to\infty} T_n(x) = y$. Define $y = T(x)$, that is, $T(x) = \lim_{n\to\infty} T_n(x)$ in Y. Then

$$T(\alpha x) = \lim_{n\to\infty} T_n(\alpha x) = \alpha \lim_{n\to\infty} T_n(x) = \alpha T(x)$$
$$T(x+y) = \lim_{n\to\infty} T_n(x+y) = T(x) + T(y)$$

and therefore T is linear. To show $T \in \mathcal{L}(X,Y)$ we need to still prove that T is bounded. Take any $\varepsilon > 0$, there is $N \in \boldsymbol{N}$ such that if $n, m \geq N$ then $||T_n - T_m|| < \varepsilon$. This implies that, for such N, if $n \geq N$ then $||T_n|| \leq ||T_N|| + \varepsilon$. Thus for each $x \in X$,

$$||T(x)|| = \lim_{n\to\infty} ||T_n(x)|| \leq \lim_{n\to\infty} \Big[||T_n||\,||x||\Big] \leq \Big[||T_N|| + \varepsilon\Big]||x||$$

Therefore, the norm of the target is bounded with

$$||T|| \leq ||T_N|| + \varepsilon$$

Lastly, we prove that T is the limit of the Cauchy sequence. Take any $\varepsilon > 0$. Then, there exists an N such that if $n > N$,

$$||T_n(x) - T(x)|| = \lim_{m\to\infty} ||T_n(x) - T_m(x)|| \leq \lim_{m\to\infty} ||T_n - T_m||\,||x|| \leq \varepsilon ||x||$$

and $||T_n - T|| \leq \varepsilon$. In sum, we have followed process of, for an arbitrary Cauchy sequence, finding a target, proving that the target is linear and bounded, and lastly proving that the target is indeed the limit of the Cauchy sequence. □

Proposition. $\mathrm{Lip}_\alpha([0,1])$ is a Banach space for $0 < \alpha < 1$ where $\mathrm{Lip}_\alpha([0,1])$ is the set of all $f \in C([0,1])$ such that

$$M_f = \sup_{s\neq t} \frac{|f(s) - f(t)|}{|s-t|^\alpha} < \infty$$

with the norm $||f|| = |f(0)| + M_f$.

Corollary. If X is a normed vector space, then

$$X^* = \mathcal{L}(X, \boldsymbol{R})$$

is a Banach space with the operator norm.

Corollary. Let $L^p(E)$ denote the set of all measurable functions on E with

$$||f||_p = \Big(\int_E |f|^p \, dm\Big)^{1/p} < \infty$$

were we define $f, g \in L^p(E)$ to be equivalent if $f = g$ almost everywhere on E.

Proof. To show that $L^p(E)$ is a normed vector space for $1 < p < \infty$ we simply need to prove that the function $||\cdot||_p$ is a norm. Note how

(i) If $||f||_p = 0$ then $f = 0$ almost everywhere on E so that f is a zero vector in $L^p(E)$.

(ii) If $f \in L^p(E)$ and $\alpha \in \boldsymbol{R}$, then

$$||\alpha f||_p = \Big(\int_E |\alpha f|^p \, dm\Big)^{1/p} = |\alpha|\Big(\int_E |f|^p \, dm\Big)^{1/p} = |\alpha| \, ||f||_p$$

(iii) If $f, g \in L^p(E)$ then $||f+g||_p \leq ||f||_p + ||g||_p$ which is called the Minkowski inequality. Due to its difficulty, we omit the proof for now.

□

Lemma. For $a, b \geq 0$ and $0 < \lambda < 1$ the equality $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$ holds.

Proof. For $t \geq 0$, define $\phi(t) = \lambda t - t^\lambda$. The derivative is given as $\phi'(t) = \lambda - \lambda t^{\lambda-1} = \lambda(1 - t^{\lambda-1})$. With a minimum at $t = 1$ we know that $\phi(t) \geq \phi(1)$ for all $t > 0$. Therefore,

$$\lambda t - t^\lambda \geq \lambda - 1 \quad \text{so that} \quad t^\lambda \leq \lambda t + (1-\lambda)$$

for all $t \geq 0$. Notice that the equality holds when $t = 1$. Now put $t = a/b$ to get

$$\Big(\frac{a}{b}\Big)^\lambda \leq \lambda\Big(\frac{a}{b}\Big) + (1-\lambda)$$

so that multiplying b on both sides yields the sought inequality. □

Corollary. Let $1 < p < \infty$ satisfy $1/p + 1/q = 1$. Put $\lambda = 1/p$, $a = A^p$, and $b = B^p$ for $A, B \geq 0$. Then,

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$$

becomes

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}$$

where the equality holds when $A^p = B^q$.

Theorem. (Hölder's inequality) Let $1 \leq p < \infty$. If $f \in L^p(E)$ and $g \in L^q(E)$ with $1/p + 1/q = 1$, then $fg \in L^1(E)$ and $||fg||_1 \leq ||f||_p ||g||_q$. That is,

$$\int_E |fg| \, dm \leq \Big(\int_E |f|^p \, dm\Big)^{1/p}\Big(\int_E |g|^q \, dm\Big)^{1/q}$$

Proof. From $AB \leq A^p/p + B^q/q$ we put $A = |f(x)|/||f||_p$ and $B = |g(x)|/||g||_q$ for $x \in E$ to get

$$\frac{|f(x)g(x)|}{||f||_p||g||_q} \leq \frac{1}{p}\frac{|f(x)|^p}{||f||_p^p} + \frac{1}{q}\frac{|g(x)|^q}{||g||_q^q}$$

then we integrate.

$$\begin{aligned}\frac{1}{||f||_p||g||_q}\int_E |f(x)g(x)|\, dm(x) \leq& \frac{1}{p}\frac{1}{||f||_p^p}\int_E |f(x)|^p\, dm(x) + \frac{1}{q}\frac{1}{||g||_q^q}\int_E |g(x)|^q\, dm(x)\\ =&\frac{1}{p} + \frac{1}{q} = 1\end{aligned}$$

This completes the proof. $\square$

Theorem. (Minkowski inequality) Let $1 \leq p < \infty$. If $f, g \in L^p(E)$ then $||f + g||_p \leq ||f||_p + ||g||_p$.

Proof. If $f, g \in L^p(E)$ then $|f(x) + g(x)| \leq 2\max(|f(x)|, |g(x)|)$ which implies that

$$\int_E |f + g|^p\, dm \leq 2^p \int_E (|f|^p + |g|^p)\, dm < \infty$$

Then

$$\begin{aligned}\int_E |f + g|^p\, dm &= \int_E |f + g|\, |f + g|^{p-1}\, dm \leq \int_E (|f| + |g|)|f + g|^{p-1}\, dm\\ &= \int_E |f|\, |f + g|^{p-1}\, dm + \int_E |g|\, |f + g|^{p-1}\, dm\end{aligned}$$

where, from Hölder's inequality,

$$\begin{aligned}\int_E |f|\, |f + g|^{p-1}\, dm \leq ||f||_p \Big[\int_E |f + g|^{(p-1)q}\, dm\Big]^{1/q} =& ||f||_p \Big[\int_E |f + g|^p\, dm\Big]^{1/q}\\ =& ||f||_p ||f + g||_p^{p/q}\end{aligned}$$

For the justification of the use of the inequality, we must show that each is integrand are elements of L^p and L^q. Note that

$$\int_E \Big[|f + g|^{p-1}\Big]^q\, dm = \int_E |f + g|^{pq-q}\, dm = \int_E |f + g|^p\, dm < \infty$$

We now see that, in parallel,

$$\int_E |g|\, |f + g|^{p-1}\, dm \leq ||g||_p ||f + g||_p^{p/q}$$

and therefore

$$||f + g||_p^p \leq \Big[||f||_p + ||g||_p\Big]\, ||f + g||_p^{p/q}$$

Dividing both sides by $||f+g||_p^{p/q}$,

$$||f+g||_q^{p-p/q} \leq ||f||_p + ||g||_p \quad \text{and} \quad ||f+g||_p \leq ||f||_p + ||g||_p$$

now we seen how for $p \geq 1$, $L^p(E)$ is a normed vector space. Next class, we show that, in addition to this, $L^p(E)$ is complete. $\square$

4 Lecture 17 (May 15th)

Theorem. Take $1 \leq p < \infty$. Then $L^p(E)$ is complete.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $L^p(E)$. Then there is a subsequence $\{f_{n_k}\}$ such that

$$||f_{n_{k+1}} - f_{n_k}||_p < \frac{1}{k^2}$$

If so, we'll show that

(i) f_{n_k} converges pointwise almost everywhere on E

(ii) If $f(x) = \lim_{n\to\infty} f_{n_k}(x)$ almost everywhere then $f_n \to f$ in $L^p(E)$

(*STEP* 1) We know that $f \in L^+(E)$ and $\int_E f\, dm < \infty$ then $m(\{x \in E \,|\, f(x) = \infty\}) = 0$. Notice how

$$\sum_{i=1}^{k-1} (f_{n_{i+1}}(x) - f_{n_i}(x)) = f_{n_k}(x) - f_{n_1}(x)$$

Now define $g_n(x) = \sum_{k=1}^{n} [f_{n_{k+1}}(x) - f_{n_k}(x)]$ and $g(x) = \sum_{k=1}^{\infty} [f_{n_{k+1}}(x) - f_{n_k}(x)]$. Then by the Minkowski inequality,

$$||g_n||_p \leq \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < 2$$

As $g_n \nearrow g$, we see that $g_n^p \nearrow g^p$, and due to the monotone convergence theorem,

$$\int_E |g|^p\, dm = \lim_{n\to\infty} \int_E |g_n|^p\, dm < 2^p$$

In otherwords, $m(\{x \in E \,|\, g(x) = \infty\}) = 0$ and it converges almost everywhere on E which implies that, by defintiion,

$$\sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

converges almost everywhere on E. Since

$$f_{n_k}(x) = f_{n_1}(x) + \sum_{i=1}^{k-1} (f_{n_{i+1}}(x) - f_{n_i}(x))$$

We see that $\lim_{k\to\infty} f_{n_k}(x)$ converges pointwise almost everywhere on E.

($STEP$ 2) Now define $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ where it converges and 0 elsewhere. We'll show that $f \in L^p(E)$ and $\lim_{n\to\infty} ||f_n - f||_p = 0$. Notice how for some N, if $n \geq N$ then

$$\int_E |f - f_n|^p\, dm \leq \liminf_{k\to\infty} \int_E |f_{n_k} - f_n|^p\, dm < \varepsilon^p$$

that is, if $n \geq N$ then $||f - f_n||_p < \varepsilon$. Additionally, for $n \geq N$, f can be expressed as

$$f = \underbrace{f - f_n}_{\in L^p} + \underbrace{f_n}_{\in L^p}$$

proving that $f \in L^p$. □

Definition. $f \in L^\infty(E)$ provided that there is $M \geq 0$ such that $|f(x)| \leq M$ almost everywhere on E. The infimum of such M is denoted by $||f||_\infty$. If so, $f \in L^\infty(E)$ is called essentially bounded and measurable on E. $||f||_\infty$ is called the essential supremum of f on E.

Remark. (i) $|f(x)| \leq ||f||_\infty$ almost everywhere on E

(ii) If the zero vector of $L^\infty(E)$ is defined by zero function almost everywhere then $L^\infty(E)$ is a Banach space

Definition. $l^p(\boldsymbol{N})$ is the space $L^p(\boldsymbol{N})$ with respect to the counting measure λ. We show that elements in $l^p(\boldsymbol{N})$ are bounded sequences. For $f \in l^1(\boldsymbol{N}, \lambda)$ define

$$f_n(k) = \begin{cases} |f(k)| & 1 \leq k \leq n \\ 0 & n < k \end{cases}$$

such that $f_n(k)$ is a simple function in $L^+(\boldsymbol{N})$ with $f_n \nearrow |f|$. By MCT,

$$\int_{\boldsymbol{N}} |f|\, d\lambda = \lim_{n\to\infty} \int f_n(k)\, d\lambda = \lim_{n\to\infty} \sum_{k=1}^{n} |f(k)| = \sum_{k=1}^{\infty} |f(k)|$$

A sequence $\{a_n\}$ ($a_n = f(n)$) therefore belongs to $l^p(\boldsymbol{N})$ if

$$||\{a_n\}||_p = \Big(\sum_{n=1}^{\infty} |a_n|^p \Big)^{1/p} < \infty$$

and $|a_n|^p$ converges absolutely. As an extension, $\{a_n\} \in l^\infty(\boldsymbol{N})$ if

$$||\{a_n\}||_\infty = \sup_{n\in\boldsymbol{N}} |a_n| < \infty$$

and is a bounded sequence.

Definition. Let $\mathbf{x} = \{a_n\}$ be a sequence. $\mathbf{x} \in l^2(\boldsymbol{N})$ if $||\mathbf{x}||^2 = \sum_{k=1}^{\infty} |a_n|^2 < \infty$. Consider the sequence $\{e_n\} \subset l^2(\boldsymbol{N})$ (an orthonormal basis for $l^2(\boldsymbol{N})$) then

$$\begin{cases} ||e_n|| = 1 \\ ||e_j - e_k|| = \sqrt{2} \quad j \neq k \end{cases}$$

this implies that $\{e_n\}$ is a bounded sequence in $l^2(\boldsymbol{N})$ but that there is no subsequence that is a Cauchy sequence. To summerise, $\{e_n\}$ is a

(i) Bounded sequence with no convergence subsequence in $l^2(\boldsymbol{N})$

(ii) Bounded infinite set with no limit point

(iii) Closed and bounded but not compact

Example. (Unbounded linear operators) Consider $X : C^1([0,1]))$ and $Y : C([0,\pi])$ both with the uniform norm

$$||f|| = \max_{[0,1]} |f(x)|$$

Let $T : X \to Y$ be defined by $Tf = f'$ which is linear. Take

$$f_n(x) = x^n \quad \text{and} \quad g_n(x) = \sin(nx)$$

Then, $f_n, g_n \in X$ with $||f_n|| = 1$ and $||g_n|| = 1$ for all n. Note that $T(f_n)(x) = nx^{n-1}$ so that $||Tf_n|| = n$ and $T(g_n)(x) = n\cos nx$ so that $T(g_n)(0) = n$. Therefore, there is no $M > 0$ so that

$$n = ||Tf_n|| \leq M||f_n|| = M$$

nor

$$n = ||Tg_n|| \leq M||g_n|| = M$$

for all $n \in \boldsymbol{N}$.

Definition. (Separable) A normed vector space X is called separable if X has a countable dense subset.

Remark. For $1 \leq p < \infty$, $L^p(E)$ is separable but $L^\infty(E)$ is not separable.

Proof. Suppose that $\{f_n\}$ is a countable dense subset of $L^\infty([a,b])$. Then, for every $a < x < b$ there is $f_{n(x)}$ such that

$$||\mathbf{1}_{[a,x]} - f_{n(x)}||_\infty < \frac{1}{2}$$

Suppose $a < x < y < b$ satisfies $f_{n(x)} = f_{n(y)}$. Then,

$$\begin{aligned}1 =&||\mathbf{1}_{[a,x]} - \mathbf{1}_{[a,y]}||_\infty \\ =&||\mathbf{1}_{[a,x]} - f_{n(x)} + f_{n(y)} - \mathbf{1}_{[a,y]}||_\infty \\ \leq&||\mathbf{1}_{[a,x]} - f_{n(x)}||_\infty + ||f_{n(y)} - \mathbf{1}_{[a,y]}||_\infty \\ <&\frac{1}{2} + \frac{1}{2} < 1\end{aligned}$$

which is a contradiction. We thus found that all dense subsets of $L^\infty(E)$ are uncountable.

□

Remark. We define $C_c^\infty(E)$ as the space of smooth functions with a compact support inside E. By compact support we mean the closure of the set where $f(x) \neq 0$.

$$\text{supp}(f) = \overline{\{x \in E \mid f(x) \neq 0\}}$$

We remark that $C_c^\infty(E)$ is dense in $L^p(E)$ for $1 \leq p < \infty$.

5 Lecture 18 (May 20th)

Remark. Let $f : E \to \boldsymbol{R}$ be a measurable function. We have established the following.

(i) For $p > 0$, if $\int_E |f|^p \, dm < \infty$ then $f \in L^p(E)$.

(ii) If there is $M \geq 0$ such that $|f(x)| \leq M$ almost everywhere on E, then $f \in L^\infty(E)$.

(iii) If $1 \leq p \leq \infty$ then $L^p(E)$ is a Banach space with $|| \cdot ||_p$ norm.

(iv) If $0 < p < 1$ then $L^p(E)$ is a complete metric space with $d(f, g) = \int_E |f - g|^p \, dm$.

Theorem. If $m(E) < \infty$ and $1 < p < q < \infty$ then $L^\infty(E) \subset L^q(E) \subset L^p(E) \subset L^1(E)$.

Proof. If $||f||_\infty < \infty$ then $\int_E |f|^p \, dm \leq \int_E ||f||_\infty^p \, dm = ||f||_\infty^p \cdot m(E) < \infty$ such that $f \in L^p(E)$. Meanwhile, if $p < q$ and $f \in L^q(E)$,

$$\begin{aligned}\int_E |f|^p \, dm = \int_E |f|^p \cdot 1 \, dm \leq& \Big[\int_E (|f|^p)^{q/p}\Big]^{p/q} \Big[\int_E 1 \, dm\Big]^c \\ =&||f||_q^p \cdot m(E)^c < \infty\end{aligned}$$

by taking $p = q/p$ and using Hölder's inequality. □

Theorem. If $m(E) < \infty$ and $f \in L^\infty(E)$ then $\lim_{p\to\infty} ||f||_p = ||f||_\infty$

Proof.
$$\int_E |f|^p \, dm \leq \int_E ||f||_\infty^p \, dm = ||f||_\infty^p \cdot m(E)$$
so that
$$\left(\int_E |f|^p \, dm\right)^{1/p} \leq ||f||_\infty \cdot m(E)^{1/p} \quad \text{and} \quad \limsup_{p\to\infty} ||f||_p \leq ||f||_\infty$$
where the measure goes to 1 as $p \to \infty$. Take any $\varepsilon > 0$, we'll show that
$$\liminf_{p\to\infty} ||f||_p \geq ||f||_\infty - \varepsilon$$
Define
$$E_\varepsilon = \{x \in E \,|\, |f(x)| > ||f||_\infty - \varepsilon\}$$
Then, $0 < m(E_\varepsilon) \leq m(E) < \infty$ and
$$\begin{aligned} ||f||_p = \left(\int_E |f|^p \, dm\right)^{1/p} = \left(\int_{E_\varepsilon} |f|^p \, dm\right)^{1/p} &\geq \left(\int_{E_\varepsilon} (||f||_\infty - \varepsilon)^p\right)^{1/p} \\ &= \Big[[||f||_\infty - \varepsilon]^p m(E_\varepsilon)\Big]^{1/p} = (||f||_\infty - \varepsilon) m(E_\varepsilon)^{1/p} \end{aligned}$$
where $m(E_\varepsilon)^{1/p}$ again goes to 1 as $p \to \infty$. Thus,
$$\liminf_{p\to\infty} ||f||_p \geq ||f||_\infty - \varepsilon$$
□

Theorem. Let $1 < p < q < r < \infty$. If $f \in L^p(E)$ and $f \in L^r(E)$, then $f \in L^q(E)$. Furthermore, $||f||_q < \max(||f||_p, ||f||_r)$.

Proof. Let $q = \lambda p + (1-\lambda) r$ for some $\lambda \in (0,1)$. Then,
$$\begin{aligned} \int_E |f|^q \, dm = \int_E |f|^{\lambda p} \cdot |f|^{(1-\lambda)r} \, dm &\leq \left(\int_E \left[|f|^{\lambda p}\right]^{1/\lambda} dm\right)^\lambda \left(\int_E \left[|f|^{(1-\lambda)r}\right]^{1/(1-\lambda)} dm\right)^{1-\lambda} \\ &= ||f||_p^{\lambda p} ||f||_r^{(1-\lambda)r} \leq [\max(||f||_p, ||f||_r)]^{\lambda p + (1-\lambda) r} = [\max(||f||_p, ||f||_r)]^p \end{aligned}$$
□

Remark. For $p > 0$,
$$\int_0^1 \frac{1}{x^p} \, dx < \infty \quad \text{iff} \quad 0 < p < 1 \quad \text{and} \quad \int_1^\infty \frac{1}{x^p} < \infty \quad \text{iff} \quad 1 < p$$

Example. Given $a > 0$, find f on $(0, \infty) = E$ such that $f \in L^p((0,\infty))$ if and only if $p \in (a, b)$.

Proof. Simply take

$$f(x) = \begin{cases} x^{-1/b} & x \in (0,1) \\ x^{-1/a} & x \in (1,\infty) \end{cases} = x^{-1/b}\mathbf{1}_{(0,1)} + x^{-1/a}\mathbf{1}_{(1,\infty)}$$

$\square$

Example. Note how

$$\int_1^\infty \frac{1}{[x(1+\ln x)^2]^p}\,dx = \int_0^\infty \frac{e^{t(1-p)}}{(1+t)^{2p}}\,dt < \infty$$

if and only if $p \geq 1$, where we substituted $t = \ln x$. Also,

$$\int_0^1 \frac{1}{[x(1-\ln x)^2]^p}\,dx = \int_\infty^0 \frac{e^{t(p-1)}}{[1+t]^{2p}}\,(-dt) = \int_0^\infty \frac{e^{t(p-1)}}{(1+t)^{2p}}\,dt < \infty$$

if and only if $0 < p \leq 1$,with the substitution being $t = -\ln x$.

Example. Given $a > 0$, find f on $(0,\infty) = E$ such that $f \in L^p((0,\infty))$ if and only if $p \in [a,b]$ $(0 < a < b)$.

Proof. Simply take this time

$$f(x) = \left[x(1-\ln x)^2\right]^{-1/b}\mathbf{1}_{(0,1)} + \left[x(1+\ln x)^2\right]^{-1/a}\mathbf{1}_{[0,\infty)}$$

$\square$

Example. For $\alpha \in (0,\infty)$, find f so that $f \in L^p((0,\infty))$ if and only if $p = \alpha$. The f is given by the expression above with $\alpha = a = b$. We now find that there is a function that is $L^p((0,\infty))$ only when p is a given number. There is no subset relation when the measure of the space is infinite!

Theorem. For $1 < p < \infty$ and $1/p + 1/q = 1$, let $T : L^p(E) \to \boldsymbol{R}$ be defined by

$$T(f) = \int_E fg\,dm$$

for some $g \in L^q(E)$. Then, T is a bounded linear functional on $L^p(E)$ with $||T|| = ||g||_p$.

Remark. First of all,

$$T(\alpha_1 f_1 + \alpha_2 f_2) = \int_E (\alpha_1 f_1 + \alpha_2 f_2) g\,dm = \alpha_1 \int_E f_1 g\,dm + \alpha_2 \int_E f_2 g\,dm = \alpha_1 T(f_1) + \alpha_2 T(f_2)$$

and by Holder's inequality,

$$|T(f)| = \Big| \int_E fg\, dm \Big| \leq ||f||_p ||g||_q$$

so that $||T|| \leq ||g||_q$. We now want to prove that $||T|| \geq ||g||_q$. Let

$$f = \frac{|g|^q}{g}$$

we can easily see that $f \in L^p(E)$ (given $g \neq 0$ and if $g = 0$ we define $f = 0$). Now see that $fg = |g|^q$ and

$$T(f) = \int_E fg\, dm = \int_E |g|^q\, dm = ||g||_q^q = ||g||_q ||g||_q^{q-1} = ||f||_p ||g||_q$$

given the fact that

$$||g||_q^{q-1} = \Big(\int_E |g|^q\, dm\Big)^{(q-1)/q} = \Big(\int_E |f|^p\, dm\Big)^{1/p} = ||f||_p$$

We now see how

$$\Big| T\Big(\frac{f}{||f||_p}\Big) \Big| \geq ||g||_q$$

As the norm of an operator is defined by the supremum of which the left hand side is an element of,

$$||T|| \geq \Big| T\Big(\frac{f}{||f||_p}\Big) \Big| \geq ||g||_q$$

Finally, due to the previous remark, $||T|| = ||g||_q$.

Corollary. Let $T : L^1(E) \to \boldsymbol{R}$ be defined by

$$T(f) = \int_E fg\, dm$$

for some $g \in L^\infty(E)$. T is a bounded linear functional on $L^1(E)$ with $||T|| = ||g||_\infty$.

6 Lecture 19 (May 22nd)

Recall. We have seen how integral transforms of the form

$$T(f) = \int_E fg\, dm$$

are bounded linear operators with a norm of $||T|| = ||g||_q$. We see that, suprisingly, all bounded linear operators can be realised to be of this form.

Theorem. (Riesz representation theorem) If $1 \leq p < \infty$ and T is a bounded linear functional on $L^p(\mu)$, there is a unique $g \in L^q(\mu)$ (where q is the conjugate exponent of p) such that

$$T(f) = \int fg\,d\mu$$

for all $f \in L^p(\mu)$ and $||T|| = ||g||_q$. In particular, if μ is a σ-finite measure on X and T is a bounded linear functional on $L^1(\mu)$ then there is a unique $g \in L^\infty(\mu)$ such that

$$T(f) = \int_X fg\,d\mu$$

for all $f \in L^1(\mu)$ and $||T|| = ||g||_\infty$.

Definition. If μ is a counting measure on $\boldsymbol{N}$, we define three vector spaces.

(i) $l^p(\boldsymbol{N})$ the set of sequence $\mathbf{x} = (x_1, x_2, \ldots)$ with the norm

$$||\mathbf{x}||_p = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p}$$

for $p \geq 1$.

(ii) $l^\infty(\boldsymbol{N})$ is the set of all bounded sequences with the supremum norm.

(iii) $C_0(\boldsymbol{N})$ the sequence $\mathbf{x}$ which converges to zero with the supremum norm.

Definition. A linear map $T : l^1(\boldsymbol{N}) \to \boldsymbol{R}$ is defined by

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n$$

where $\mathbf{y} = (y_1, y_2, \ldots) \in l^\infty(\boldsymbol{N})$. We now try to show that the norm of $\mathbf{y}$ is the norm of the linear operator. Observe that

$$|T(\mathbf{x})| \leq ||y||_\infty \sum_{n=1}^{\infty} |x_n| = ||\mathbf{x}||_1 ||\mathbf{y}||_\infty$$

so that $||T|| \leq ||\mathbf{y}||_\infty$. To show that $||T|| = ||\mathbf{y}||_\infty$, suppose $y = \{y_n\}$. Then we put

$$\mathbf{x} = \frac{|y_k|}{y_k}\mathbf{e}_k \quad \text{for} \quad y_k \neq 0 \quad \text{implying} \quad ||\mathbf{x}||_1 = 1$$

and

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n = |y_k|$$

This tells us that

$$|y_k| \leq ||T||$$

for every $k \in \boldsymbol{N}$ and $||T|| \geq ||\mathbf{y}||_\infty$.

Theorem. (Riesz representation theorem for $l^1(\boldsymbol{N})$) If T is a bounded linear functional on $l^1(\boldsymbol{N})$ then we'll show $\mathbf{y} = (y_1, y_2, \ldots) \in l^\infty(\boldsymbol{N})$ such that

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n$$

with $||T|| = ||\mathbf{y}||_\infty$.

Proof. To do that just define $T(\mathbf{e}_k) = y_k$. Then by continuity and linearity of T,

$$T(\mathbf{x}) = T\Big(\sum_{k=1}^{\infty} x_k \mathbf{e}_k\Big) = \sum_{k=1}^{\infty} T(x_k \mathbf{e}_k) = \sum_{k=1}^{\infty} T(x_k \mathbf{e}_k) = \sum_{k=1}^{\infty} x_k T(\mathbf{e}_k) = \sum_{k=1}^{\infty} x_k y_k$$

To prove that uniqueness, do this again, and we'll find

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} x_n z_n$$

and by putting $\mathbf{x} = \mathbf{e}_k$ we find $y_k = z_k$. $\square$

Remark. We have previously shown that, in the limited case of $p = 1$,

$$l^p(\boldsymbol{N})^* = l^q(\boldsymbol{N})$$

for $1 \leq p < \infty$ and $1/p + 1/q = 1$. Also, for the Lesbegue measure,

$$L^p(E)^* = L^q(E)$$

for $1/p + 1/q = 1$ with $1 \leq p < \infty$. We now show that

$$C_0(\boldsymbol{N})^* = l^1(\boldsymbol{N})$$

and also that

$$C_0(E)^* = M(E)$$

Notice that, importantly, the dual of $C_0(\boldsymbol{N})$ is $l^1(\boldsymbol{N})$ while the dual of $l^1(\boldsymbol{N})$ is $l^\infty(\boldsymbol{N})$. The dual of a dual is not itself!

Theorem. Recall that $C_0(\boldsymbol{N})$ is defined as the set of infinite sequences that converge to 0 provided the supremum norm. If $T : C_0(\boldsymbol{N}) \to \boldsymbol{R}$ is defined by

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n$$

for some $\mathbf{y} = (y_1, \ldots, y_n, \ldots) \in l^1(\boldsymbol{N})$, then T is continuous and linear with $||T|| = ||\mathbf{y}||_1$.

Conversely, if T is a bounded linear functional, then there is a unique $\mathbf{y} \in l^1(\mathbf{N})$ such that

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n$$

with $||T|| = ||\mathbf{y}||_1$. The latter converse follows from just defining $T(e_k) = y_k$.

Proof. T is obviously linear.

$$|T(x)| = \Big|\sum_{n=1}^{\infty} x_n y_n\Big| \leq \sum_{n=1}^{\infty} |x_n|\,|y_n| \leq ||\mathbf{x}||_\infty ||\mathbf{y}||_1$$

This implies that $||T|| \leq ||y||_1$. Conversely, for each n, we define $\mathbf{x}_n \in C_0(\boldsymbol{N})$ as $\mathbf{x}_n = \{x_{n,k}\}_{k=1}^{\infty}$ where

$$x_{n,k} = \begin{cases} \dfrac{|y_k|}{y_k} & \text{if} \quad 1 \leq k \leq n \quad \text{and} \quad y_k \neq 0 \\ 0 & \text{if} \quad k > n \quad \text{and} \quad y_k = 0 \end{cases}$$

Notice how $\lim_{k\to\infty} x_{n,k} = 0$. Then $||\mathbf{x}_n||_\infty = 1$ for each n, and

$$T(\mathbf{x}_n) = \sum_{k=1}^{\infty} x_{n,k} y_k = \sum_{k=1}^{n} |y_k|$$

Thus, $\sum_{k=1}^{n} |y_k| \leq ||T||$ for every $n \in \boldsymbol{N}$. Therefore, $||T|| \geq ||y||_1$. For the converse, we simply define $T(\mathbf{e}_k) = y_k$. Then by the continuity and linearity, $T(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n$. □

Example. We see another example where which a sequence that is bounded fails to have a subsequence that converges. Define $f_n(x) = \sin nx$ for $x \in [0, 2\pi]$. This implies that $|f_n(x)| \leq 1$ for all $x \in [0, 2\pi]$ and for all $n \in \boldsymbol{N}$. Suppose $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges pointwise on $[0, 2\pi]$. Then, $\lim_{k\to\infty} \sin n_k x = \lim_{k\to\infty} \sin n_{k+1} x$, that is,

$$\lim_{k\to\infty} (\sin n_{k+1} x - \sin n_k x) = 0$$

for all $x \in [0, 1]$. Or, equivalently, $\lim_{k\to\infty} (\sin n_{k+1} x - \sin n_k x)^2 = 0$. By LDCT,

$$\lim_{k\leftarrow\infty} \int_0^{2\pi} (\sin n_{k+1} x - \sin n_k x)^2 = 0$$

When this is actually computed, we have 2π for all k. To elaborate further, observe that the above is equal to

$$\int_0^{2\pi} (\sin^2 n_{k+1} x + \sin^2 n_k x - 2 \sin n_{k+1} x \sin n_{kx})\, dx$$

and that the first two terms become π each and the last term vanishes.

Definition. Simply put, weak* convergence is pointwise convergence (for a sequence of bounded linear operators). For $T_n \in X^*$ and $T \in X^*$, $T_n \to T$ weak* in X^* provided that $\lim_{n\to\infty} T_n(x) = T(x)$ for every $x \in X$.

Theorem. (Arzela-Ascoli theorem) (Important!) Let X be a separable normed vector space. Then every bounded sequence in X^* has a weak* convergent subsequence.

Corollary. The following is an application for the above theorem in $L^p(E) = L^q(E)^*$. For $1 \leq p < \infty$, let $f_n \in L^p(E)$ with $||f_n||_p \leq M$ for all $n \in \boldsymbol{N}$. Then $\{f_n\}$ is bounded in $L^q(E)^*$. Then there is $f \in L^p(X)$ and a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ weak*. Notice that by the Riesz-representation theorem, there is a bounded linear functional

$$T(g) = \int_E gf\, dm$$

for $g \in L^q(E)$ so that

$$\lim_{k\to\infty} \int_E f_{n_k} g\, dm = \lim_{k\to\infty} T_{n_k}(g) = T(g) = \int_E fg\, dm$$

for all $g \in L^q(E)$.

7 Lecture 20 (May 27th)

Definition. A separable metric space X is a space that has a countable dense subset.

Definition. For $f_n : X \to \boldsymbol{R}$, $\{f_n\}$ is pointwise bounded on X provided that for every $x \in X$ there is $M_x > 0$ such that $|f_n(x)| \leq M_x$ for all $n \in \boldsymbol{N}$.

Example. Let X, Y be normed vector spaces. Let $T_n : X \to Y$ be a bounded linear map such that $||T_n|| \leq M$ for all $n \in \boldsymbol{N}$. Then for every $x \in X$, we have

$$||T_n(x)|| \leq ||T_n||\, ||x|| \leq M||x||$$

Thus $\{T_n\}$ is pointwise bounded on X.

Definition. $\{f_n\}$ is equicontinuous on X provided that for every $\varepsilon > 0$, there is $\delta > 0$ such that if $x, y \in X$ satisfies $d(x, y) < \delta$, then $|f_n(x) - f_n(y)| < \varepsilon$ for all $n \in \boldsymbol{N}$.

Example. Consider $f_n : \boldsymbol{R} \to \boldsymbol{R}$ where $f_n = nx$. As

$$|f_n(x) - f_n(y)| = n|x - y|$$

and each f_n is uniformly continuous on $\boldsymbol{R}$. However, $\{f_n\}$ is not equicontinuous on $\boldsymbol{R}$.

Example. The function $f_n(x) = \sin nx$ is not equicontinuous on $[0, 2\pi]$.

Example. Let $T_n : X \to Y$ be norm bounded, that is, $||T_n|| \leq M$ for all $n \in \boldsymbol{N}$. This implies that

$$||T_n(x) - T_n(y)|| = ||T_n(x-y)|| \leq M||x-y||$$

so that $\{T_n\}$ is equicontinuous on X.

Theorem. (Arzela-Ascoli theorem) Consider a sequence in X^*, that is, $f_n : X \to \boldsymbol{R}$ for $n \in \boldsymbol{N}$ where X is a separable metric space. Take $\{f_n\}$ to be pointwise bounded and equicontinuous on X. Then, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges uniformly on all compact subsets of X.

Proof. Let $E = \{x_1, x_2, \ldots\}$ be a countable dense subset of X.

$$\begin{array}{cccc}
f_{1,1}, & f_{2,1} & f_{3,1}, & \cdots \\
f_{1,2}, & f_{2,2} & f_{3,2}, & \cdots \\
 & \vdots & & \\
f_{1,k}, & f_{2,k} & f_{3,k}, & \cdots
\end{array}$$

(*STEP* 1) We construct a subsequence of $\{f_n\}$ which converges at every $x_k \in E$. This requires only pointwise boundedness. Note that $\{f_n(x_1)\}$ is a bounded sequence on $\boldsymbol{R}$, so that there is a subsequence $\{f_{n,1}\}$ of $\{f_n\}$ which converges at x_1. Likewise, $\{f_{n,1}(x_2)\}$ is a bounded sequence in $\boldsymbol{R}$ so that there is a subsequence $\{f_{n,2}\}$ of $\{f_{n,1}\}$ converges at x_2 (it also converges at x_1). Continuing, we find $\{f_{n,k}\}$ which converges at $\{x_{k+1}\}$ and also $x_1, x_2, \ldots, x_k$. Now, take the diagonal sequence $\{f_{n,n}\}_{n=1}^{\infty}$. Then we'll show that $\lim_{n\to\infty} f_{n,n}(x_k)$ converges at every $x_k \in E$. Take any $x_k \in E$ then $\{f_{n,n}\}_{n=k}^{\infty}$ is a subsetquence of $\{f_{n,k}\}_{n=k}^{\infty}$.

Define $g_n = f_{n,n}$. Let K be any compact subset of X. We'll show that $\{g_n\}$ converges uniformly on K. Equivalently, we'll show that the sequence $\{g_n\}$ is uniformly Cauchy on K. Take any $\varepsilon > 0$, we will find $N \in \boldsymbol{N}$ such that if $n, m > N$ then $|g_m(x) - g_n(x)| < \varepsilon$ for all $x \in K$.

(*STEP* 2) We show that the above subsequence converges uniformly on all compact subsets of X. This requires only equicontinuity. By equicontinuity of $\{g_n\}$, there is $\delta > 0$ such that if $d(p,q) < \delta$, then $|g_n(p) - g_n(q)| < \varepsilon/3$ for all $n \in \boldsymbol{N}$. Notice that

$$\mathcal{F} = \left\{ B\Big(x, \frac{\delta}{2}\Big) \,\Big|\, x \in K \right\}$$

is an open cover of K so that is a finite subcover $\{B_1, B_2, \ldots, B_M\}$ all with radius $\delta/2$. Recall that E was a countable dense subset in X such that there is a subsequence of $\{f_n\}$ that converges. Since $E = \{x_n\}$ is dense in X, for every $1 \leq k \leq M$, there is $x_k \in E \cap B_k$.

Then there is $N \in \boldsymbol{N}$ such that if $n, m > N$ then

$$|g_n(x_k) - g_m(x_k)| < \frac{\varepsilon}{3}$$

for $1 \leq k \leq M$. In sum, if $n, m > N$ for some N and $x \in K$, then $x \in B_j$ for some $1 \leq j \leq M$ so that $d(x, y) < \delta$ if $y \in B_j$. We finally see that,

$$\begin{aligned} |g_n(x) - g_m(x)| =& |g_n(x) - g_n(p_j) + g_n(p_j) - g_m(p_j) + g_m(p_j) - g_m(x)| \\ \leq& |g_n(x) - g_n(p_j)| + |g_n(p_j) - g_m(p_j)| + |g_m(p_j) - g_m(x)| \\ <& \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

□

Remark. Let X be a normed vector space, and let X^* be the dual space $\mathcal{L}(X, \boldsymbol{R})$. There are three types of convergences in this book.

(i) Norm convergence, both in X and X^*

(ii) Weak convergence in X

(iii) Weak* convergence in X^*

We now learn weak convergence in X.

Definition. Let $x_n \in X$ and $x \in X$. $x_n \to x$ weakly in X if

$$\lim_{n \to \infty} T(x_n) = T(x)$$

for every $T \in X^*$. On the other hand, $T_n \to T$ weak* if

$$\lim_{n \to \infty} T_n(x) = T(x)$$

for every $x \in X$. The latter is simply pointwise convergence in X.

Remark. For $1 \leq p < \infty$ and $1/p + 1/q = 1$, by the Riesz representation theorem, $(L^p)^* = L^q$ and $(L^q)^* = L^p$. Let $f_n \in L^p(E)$. $f_n \to f$ weakly in $L^p(E)$ means that

$$\lim_{n \to \infty} \int_E f_n g \, dm = \int_E f g \, dm$$

for every $g \in L^q(E)$. Notice how this is the same as $f_n \to f$ weak* in $(L^q(E))^*$.

weak convergence in $L^p(E)$ = weak* convergence in $(L^q(E))^*$!

Corollary. Let X be a separable normed vector space. Every bounded sequence in X^* has a weak* convergent subsequence.

Proof. If $\{T_n\}$ is a bounded sequence in X^* ($T_n : X \to \boldsymbol{R}$, $||T_n|| \leq M$) then $\{T_n\}$ is a pointwise bounded and equicontinuous. Then, as every singleton set is compact, for every $x \in X$, there exists a subsequence $T_{n_k}(x)$ that converges. Define $T : X \to \boldsymbol{R}$ as $T(x) = \lim_{k\to\infty} T_{n_k}(x)$. Then, $T \in X^*$ and $\{T_n\}$ converges weak*. $\square$

Corollary. Assume $1 \leq p < \infty$ and $||f_n||_p < \infty$. A bounded linear operator $T_n : L^q(E) \to \boldsymbol{R}$ is defined as

$$T_n(g) = \int_E f_n g \, dm$$

where $||T_n|| = ||f_n||_p$. By the previous corollary, T_n has a weak* convergent subsequence where

$$\lim_{n\to\infty} T_{n_k}(g) = T(g)$$

for all $g \in L^q$. Then,

$$\lim_{k\to\infty} \int_E f_{n_k} g \, dm = \int_E fg \, dm$$

for some $f \in L^p(E)$. Therefore, if there is a sequence of $f_n \in L^p(E)$, there exists a subsequence f_{n_k} and $f \in L^p(E)$ such that

$$\lim_{k\to\infty} \int_E f_{n_k} g \, dm = \int_E fg \, dm$$

for all $g \in L^q(E)$.

8 Lecture 21 (May 29th)

Theorem. Take X to be a separable Banach space. Let $T_n \in X^*$ with $||T_n|| \leq M$ for all n. There is $\{T_{n_k}\}$ such that $\lim_{k\to\infty} T_{n_k}(x)$ converges for every $x \in X$. If we define $T : X \to \boldsymbol{R}$ by $T(x) = \lim_{k\to\infty} T_{n_k}(x)$ then obviously, T is linear in X and for $||x|| = 1$,

$$|T(x)| = \lim_{k\to\infty} |T_{n_k}(x)| \leq M$$

so that $||T|| \leq M$. Therefore, $T \in X^*$.

Theorem. (Fubini theorem) Let $f(x, y)$ be measurable on $E \times F$. If

$$\int_F \int_E |f(x, y)| \, dm(x) \, dm(y) < \infty$$

or

$$\int_E \int_F |f(x, y)| \, dm(y) \, dm(x) < \infty$$

(meaning that $f \in L^1(E \times F, m \times m)$), then

$$\int_E \int_F f(x, y) \, dm(y) \, dm(x) = \int_F \int_E f(x, y) \, dm(x) \, dm(y)$$

If $f(x,y) \geq 0$, the result satisfies also.

Theorem. Take $f \in L^p(E)$, $1 \leq p \leq \infty$. An integral operator is defined as

$$Tf(x) = \int_E K(x,y)f(y)\,dm(y)$$

If there is $c > 0$ such that

$$\sup_{x\in E}\int_E |K(x,y)|\,dm(y) \leq c \quad \text{and} \quad \sup_{y\in E}\int |K(x,y)|\,dm(x) < c$$

then $||Tf||_p \leq c||f||_p$ for all $f \in L^p(E)$.

Proof. For $p = 1$,

$$\begin{aligned}
||Tf||_1 =& \int_E \Big| \int_E K(x,y)f(y)\,dm(y)\Big|\,dm(x) \\
\leq& \int_E\int_E |K(x,y)|\,|f(y)|\,dm(y)\,dm(x) \\
\leq& c\int_E |f(y)|\,dm(y)
\end{aligned}$$

The case is identical for $p = \infty$. For $1 < p < \infty$,

$$\begin{aligned}
|Tf(x)| \leq \int_E |K(x,y)|\,|f(y)|\,dm(y) =& \int_E |K(x,y)|^{1/p+1/q}|f(y)|\,dm(y) \\
\leq& \Big[\int_E |K(x,y)|\,dm(y)\Big]^{1/q}\Big[\int_E |K(x,y)|\,|f(y)|^p\,dm(y)\Big]^{1/p} \\
\leq& c^{1/q}\Big[\int_E |K(x,y)|\,|f(y)|^p\,dm(y)\Big]^{1/p}
\end{aligned}$$

So that

$$\begin{aligned}
\int_E |Tf(x)|^p\,dm(x) \leq& c^{p/q}\int_E\int_E |K(x,y)|\,|f(y)|^p\,dm(y)\,dm(x) \\
=& c^{p/q}\int_E |f(y)|^p\int_E |K(x,y)|\,dm(x)\,dm(y) \\
\leq& c^{(p+q)/q}\int_E |f(y)|^p\,dm(y)
\end{aligned}$$

this implies that

$$||Tf||_p \leq c||f||_p$$

□

Definition. Let f, g be measurable functions on $\boldsymbol{R}$. We define $f * g$ (convolution) by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy = (g * f)(x)$$

if it exists (note how it is commutative).

Theorem. If $f \in L^1(\boldsymbol{R})$ and $g \in L^p(\boldsymbol{R})$ for $1 \leq p \leq \infty$, then $||f * g||_p \leq ||f||_1 ||g||_p$.

Proof. Define $K(x-y) = f(x-y)$ in the previous theorem. $\square$

Remark. Consider

$$g(t) = \begin{cases} \exp\left(-\dfrac{1}{1-t^2}\right) & -1 < t < 1 \\ 0 & |t| \geq 1 \end{cases}$$

which is C^∞ with a support $[-1, 1]$. Define

$$\phi(x) = \frac{g(x)}{\int_{-1}^{1} g(t)\,dt}$$

then $\phi \in C^\infty(\boldsymbol{R})$ with support $[-1, 1]$ with $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$. Then

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right)$$

for $\varepsilon > 0$ supported $[-\varepsilon, \varepsilon]$ with

$$\int_{-\infty}^{\infty} \phi_\varepsilon(x)\,dx = 1$$

If $f \in L^p(\boldsymbol{R})$ with compact support then $f * \phi_\varepsilon \to f$ in L^p for $1 < p < \infty$ as $\varepsilon \to 0$. The convolution $f * \phi_\varepsilon \in C_c^\infty(\boldsymbol{R})$ and $C_c^\infty(\boldsymbol{R})$ is dense in $L^p(\boldsymbol{R})$ $(1 \leq p < \infty)$.

Theorem. (Schur) If there is a non-negative measurable function h on E such that

$$\int_E |K(x,y)|\, h(y)^q\, dm(y) \leq c_1 h(x)^q$$

and

$$\int_E |K(x,y)|\, h(y)^p\, dm(x) \leq c_2 h(y)^p$$

then $||Tf||_p \leq c_1^{1/q} c_2^{1/p} ||f||_p$ for $1 < p < \infty$.

Proof. We provide a hint.

$$|Tf(x)| \leq \int_E |K(x,y)|\, h(y) h(y)^{-1} |f(y)|\, dm(y)$$

$\square$

Definition. Let H be a vector space over $\boldsymbol{R}$. If there is a function $\langle \cdot, \cdot \rangle : H \times H \to \boldsymbol{R}$ satifying

(i) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in H$

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for $x, y \in H$, $\alpha \in \boldsymbol{R}$

(iii) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$ for $x, y, z \in H$

(iv) $\langle x, x \rangle \geq 0$ for all $x \in H$ and $\langle x, x \rangle = 0$ if and only if $x = \mathbf{0}$.

Then, H is called an inner product space over $\boldsymbol{R}$. If we define $||x|| = \langle x, x \rangle^{1/2}$, we can show that $||x||$ is a norm.

Example. (i) $\boldsymbol{R}^n$ with the inner product $\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$

(ii) $L^2(\mu)$ with the inner product

$$\langle f, g \rangle = \int_X f g \, d\mu$$

(iii) $C([0,1])$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x) \, dx$$

Theorem. (Cauchy-Schwarz Inequality) For $x, y \in H$, $|\langle x, y \rangle| \leq ||x|| \, ||y||$

Proof. For every $t \in \boldsymbol{R}$, $||x - ty||^2 \geq 0$ and

$$||x - ty||^2 = \langle x - ty, x - ty \rangle = ||x||^2 - 2\langle x, y \rangle t + ||y||^2 t^2$$

If $y \neq 0$, then $D/4 \leq 0$ where $D/4 = \langle x, y \rangle^2 - ||x||^2 \, ||y||^2$. $\square$

Corollary. $||x + y|| \leq ||x|| + ||y||$ for $x, y \in \boldsymbol{R}$.

Proof.

$$\begin{aligned} ||x + y||^2 = \langle x + y, x + y \rangle =& ||x||^2 + 2\langle x, y \rangle + ||y||^2 \\ \leq& ||x||^2 + 2||x|| \, ||y|| + ||y||^2 \\ =& (||x|| + ||y||)^2 \end{aligned}$$

We now have shown that $\langle x, x \rangle^{1/2}$ indeed satisfies a norm. $\square$

Definition. An inner product spcae H is called a Hilbert space if it is complete with respect to the norm which is induced by the inner product.

Theorem. An Hilbert space is a Banach space.

Example. $C([0,1])$ is a Banach space and an inner product space but is not an Hilbert space with respect to the norm

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$$

Technically speaking, an Hilbert space is a Banach space, not with the inner product, but satisfying the paralleogram law.

9 Lecture 22 (June 5th)

Theorem. (Parallelogram law) If $x, y \in H$, then

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$$

Proposition. If M is a subspace of H, then $\bar{M}$ is a closed subspace of H.

Definition. (Convex subset) A convex subset of H is a subset where if $x, y \in E$ and $\lambda \in (0,1)$, we have $\lambda x + (1-\lambda)y \in E$.

Proposition. If E is a convex subset of H, then $x + E = \{x + y \,|\, y \in E\}$ is also convex.

Definition. ($x^\perp$) For $x \in H$, we define $x^\perp = \{y \in H \,|\, \langle x, y\rangle = 0\}$. This is a subspace of H. If we define $T : H \to \boldsymbol{R}$ by $T(y) = \langle x, y\rangle$, by the Cauchy Schwarz inequality, $|T(y)| \leq ||x||\,||y||$. That is, T is continuous and $x^\perp = \{y \in H \,|\, T(y) = 0\}$ is a closed subspace of H.

Definition. ($M^\perp$) If M is a subspace of H, we define

$$M^\perp = \{y \in H \,|\, \langle x, y\rangle = 0\} = \bigcap_{x \in M} x^\perp$$

which implies that $M^\perp$ is a closed subspace of H.

Theorem. If E is a non-empty closed convex subset of a Hilbert space H, then E has a unique element of smallest norm.

Proof. Let $\delta = \inf\{||x|| \,|\, x \in E\}$. We'll show that there is a unique $x_0 \in E$ with $||x_0|| = \delta$.

Let $x, y \in E$. Then, $x/2, y/2 \in H$. Apply the parallelogram law and we have

$$\Big|\Big|\frac{x}{2}+\frac{y}{2}\Big|\Big|^2+\Big|\Big|\frac{x}{2}-\frac{y}{2}\Big|\Big|^2=2\Big(\Big|\Big|\frac{x}{2}\Big|\Big|^2+\Big|\Big|\frac{y}{2}\Big|\Big|^2\Big)$$

so that

$$||x-y||^2=2||x||^2+2||y||^2-4\Big|\Big|\frac{x}{2}+\frac{y}{2}\Big|\Big|^2$$

for $x, y \in E$. By the convexity of E, the last term is in E. We then know that

$$||x-y||^2 \leq 2(||x||^2+||y||^2)-4\delta^2$$

If $||x|| = ||y|| = \delta$, then $||x-y||^2 \leq 0$ such that $x = y$. This tells us that x_0 is unique if it exists.

By definition of δ, there is a sequence $y_n \in E$ such that $\lim_{n\to\infty} ||y_n|| = \delta$. Put y_n, y_m in the inequality above and we have

$$||y_n-y_m||^2 \leq 2(||y_n||^2+||y_m||^2)-4\delta^2$$

so that $\{y_n\}$ is a Cauchy sequence in $E \subset H$. Since H is complete, there is $x_0 \in H$ such that $\lim_{n\to\infty} ||y_n - x_0|| = 0$. Since E is closed, $x_0 \in E$. Finally,

$$\lim_{n\to\infty} ||x_0|| = \lim_{n\to\infty} ||y_n|| = \delta$$

since $||\cdot||$ is continuous. $\square$

Example. In $L^1([0,1])$, define

$$E=\Big\{f \in L^1([0,1]) \,\Big|\, \int_0^1 f(x)\,dx = 1\Big\}$$

Notice that the set is convex as the integral of $g = \lambda f + (1-\lambda)h$ is 1. In addition to this, E is closed. To see this, we ask whether for $f_n \in E$ and $\lim_{n\to\infty} ||f_n - f||_1 = 0$, $\int_0^1 f\,dx = 1$. this is true, as

$$\Big|\int_0^1 f_n\,dx-\int_0^1 f\,dx\Big| \leq \int_0^1 |f_n-f|\,dx \to 0$$

Together, we now know that the set is a closed convex subset of $L^1([0,1])$ with infinitely many elements with the smallest norm.

Example. Consider $f_n \in C([0,1])$ with the uniform norm. Define

$$E=\Big\{f \in C([0,1]) \,\Big|\, \int_0^{1/2} f(x)\,dx-\int_{1/2}^1 f(x)\,dx = 1\Big\}$$

Obviously, this is a convex set. By LDCT, E is closed. The infinum of the norm is 1, while there is no single function that has this norm.

Theorem. (Orthogonal decomposition) (Big theorem) If M is a closed subspace of a Hilbert space H then every $x \in H$ can be uniquely expressed as $x = P_x + Q_x$ where $P_x \in M$ and $Q_x \in M^\perp$. Indeed, $P \in \mathcal{L}(H, M)$ and $Q \in \mathcal{L}(H, M^\perp)$ are norm 1 linear operators with

$$||x||^2 = ||P_x||^2 + ||Q_x||^2$$

The key idea behind this theorem is that $x + M$ is a closed convex subset of H.

Proof. $x + M = \{x + y \,|\, y \in M\}$ is a non-empty closed convex subset of H. Define Q_x as the unique element of $x + M$ with the smallest norm. Then define $P_x = x - Q_x$, and by definition, $P_x \in M$.

We have to show that $Q_x \in M^\perp$ and that the decomposition is unique. The latter part is simple, as if we take $x = p + q = p' + q'$, $x - x' = y' - y \in M \cap M^\perp = \{0\}$. To show that $Q_x \in M^\perp$, we need to show that $\langle Q_x, y\rangle = 0$ for all $y \in M$ or that $\langle Q_x, y\rangle = 0$ for all $y \in M$ with $||y|| = 1$.

Take any $y \in M$ with $||y|| = 1$. Then for every $\alpha \in \boldsymbol{R}$, $Q_x - \alpha y \in x + M$ so that

$$||Q_x||^2 \leq ||Q_x - \alpha y||^2 = ||Q_x||^2 - 2\alpha\langle Q_x, y\rangle + |\alpha|^2$$

for all $\alpha \in \boldsymbol{R}$. Put $\alpha = \langle Q_x, y\rangle$ to get

$$||Q_x||^2 \leq ||Q_x||^2 - \langle Q_x, y\rangle^2$$

or that $\langle Q_x, y\rangle^2 \leq 0$ and $\langle Q_x, y\rangle = 0$. □

Corollary. If M is a closed subspace of H with $M \neq H$, then there is $x \in M^\perp$ with $||x|| = 1$.

Theorem. (Riesz-representation theorem) If T is a bouned linear functional on a Hilbert space H, then there is a unique element $y \in H$ such that $T(x) = \langle x, y\rangle$ for all $x \in H$.

Proof. If $T(x) = 0$ for all $x \in H$ then put $y = 0$. If $T(x) \neq 0$ for some $x \in H$,

$$M = \{x \in H \,|\, T(x) = 0\}$$

is a closed subspace of H so that there is $z \in M^\perp$ with $||z|| = 1$. Put $y = T(z)z$ and the proof is over. Take any $x \in H$ and put $u = T(x)z - T(z)x$ and $T(u) = 0$ so that $u \in M$. Hence $\langle u, y\rangle = 0$ as

$$\langle T(x)z - T(z)x, T(z)z\rangle = T(x) - \langle x, T(z)z\rangle$$

□