1 Lecture 1 (March 4th)

Chapter 2 Modular Arithmetic

Remark. In abstract algebra, we learn algebraic structures such as rings (eg. $(\boldsymbol{Z}, +, \times)$) and groups (eg. $(\boldsymbol{Z}, +)$).

Remark. By convention we are going to denote the set of integers equipped with addition and multiplication by the triple $(\boldsymbol{Z}, +, \times)$.

Chapter 2.2 Congruence modulo n

Recall. Fix an integer $n > 0$, for example, $n = 5$. We can group integers (create a partition) that have the same remainder when divided by $n = 5$. This creates a congruence relation, denotable as "$7 = 12$" (In number theory, we would say that 7 and 12 are congruent mod 5). We let $\bar{a}$ denote the equivalence class of a with respect to the congruence modulo n.

Remark. Giving a partition on $\boldsymbol{Z}$ is equivalent to giving an equivalence relation on $\boldsymbol{Z}$. For example, we declare $6 \equiv_5 -4$. To summarize, $\bar{a} = \{b \mid b \equiv_n a\}$ or $[a]$. There is a mathematical reason why we prefer the former.

2 Lecture 2 (March 6th)

Last class, we have learnt $\boldsymbol{Z}/n\boldsymbol{Z}$ as a set. In this lecture, we will learn algebraic structures on $\boldsymbol{Z}/n\boldsymbol{Z}$.

Definition. (2.1) Let a and b be integers. We say that a is congruent to b modulo n if $a - b = nk$ for some $k \in \boldsymbol{Z}$. In this case, we write $a \equiv_n b$ (or $a \equiv b \pmod{n}$).

Remark. (2.2) $a \equiv_n b$ if and only if a and b have the same remainder after division by n.

Remark. $\equiv_n$ is an equivalence relation on $\boldsymbol{Z}$. Accordingly, $\bar{a} = \{b \in \boldsymbol{Z} \mid b \equiv_n a\}$.

Proof. (i) $a \equiv_n a$

(ii) $a \equiv_n b \implies b \equiv_n a$

(iii) $a \equiv_n b, b \equiv_n c \implies a \equiv_n c$

□

Definition. (2.5) We denote by $\boldsymbol{Z}/n\boldsymbol{Z}$ the set of congruence classes modulo n.

Remark. (i) $\bar{a} = \bar{b} \in \boldsymbol{Z}/n\boldsymbol{Z}$ if and only if $a \equiv_n b$

(ii) If $\bar{a} \cap \bar{b} \neq \emptyset$, then $\bar{a} = \bar{b}$

(iii) $\boldsymbol{Z} = \bar{0} \coprod \bar{1} \coprod \ldots \coprod \overline{n-1}$

Lastly, we also use $\boldsymbol{Z}/n\boldsymbol{Z}$ instead of $\boldsymbol{Z}_n$.

Remark. (2.9)

(i) $\boldsymbol{Z}/n\boldsymbol{Z}$ is a finite set having exactly n elements (how about $n = 0$?)

(ii) $\boldsymbol{Z}/0\boldsymbol{Z} = \boldsymbol{Z}$

Chapter 2.3 Algebra in $\boldsymbol{Z}/n\boldsymbol{Z}$

We want to define $+$ and $\cdot$ on $\boldsymbol{Z}/n\boldsymbol{Z}$. For example, $n = 5$ and we have $\boldsymbol{Z}/5\boldsymbol{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. Can we simply define $\bar{2} + \bar{3} = \overline{5}$? However, in process of formulating addition, we bump into the problem that we can add different representatives every time. In other words, we don't know whether "+" is well-defined! Let's phrase this differently. Let $\bar{a} = \bar{b}$ and $\bar{c} = \bar{d}$. Then we want $\overline{a+c} = \overline{b+d}$.

Lemma. (2.9) Let a, b, c, d be in $\boldsymbol{Z}$. If $a \equiv_n b$ and $c \equiv_n d$ then $a+c \equiv_n b+d$ and $ac \equiv_n bd$.

3 Lecture 3 (March 11th)

Last time, we dealt with the well-definedness of $+$ and $\cdot$ on $\boldsymbol{Z}$ / $n\boldsymbol{Z}$.
Lemma. (2.9) Let $n > 0$ be an integer and let a, b, c, and d be integers. If $a \equiv_n c$ and $b \equiv_n d$, then $a + b \equiv_n c + d$ and $a \cdot b \equiv_n c \cdot d$. A start of a proof would be by considering $(a + b) - (c + d) = (a - c) + (b - d)$.

Lemma. (2.13) Let a, b, and c in $\boldsymbol{Z}$. Then,

1. $(\bar{a} + \bar{b}) + \bar{c} = \bar{a} + (\bar{b} + \bar{c})$
2. $\bar{a} + \bar{0} = \bar{a} = \bar{0} + \bar{a}$
3. For each $\bar{a} \in \boldsymbol{Z}/n\boldsymbol{Z}$, there exists $\bar{b} \in \boldsymbol{Z}/n\boldsymbol{Z}$ such that $\bar{a} + \bar{b} = \bar{0} = \bar{b} + \bar{a}$
4. $\bar{a} + \bar{b} = \bar{b} + \bar{a}$
5. $\bar{a}(\bar{b}\bar{c}) = (\bar{a}\bar{b})\bar{c}$
6. $\bar{a}\bar{1} = \bar{a} = \bar{1}\bar{a}$
7. $\bar{a}(\bar{b} + \bar{c}) = \bar{a}\bar{b} + \bar{a}\bar{c}$
8. $(\bar{a} + \bar{b})\bar{c} = \bar{a}\bar{c} + \bar{b}\bar{c}$
9. $\bar{a}\bar{b} = \bar{b}\bar{a}$

The first three imply that $\boldsymbol{Z}$ is a group and four implies that it is abelian also. From five to eight, the properties tells us that group is a ring and the ninth tells us that it is a commutative one.

Proof. For the first property, we have

$$\begin{aligned}(\bar{a}+\bar{b})+\bar{c} =& \overline{a+b}+\bar{c}\\ =& \overline{(a+b)+c}\\ =& \overline{a+(b+c)}\\ =& \bar{a}+\overline{b+c}\\ =& \bar{a}+(\bar{b}+\bar{c})\end{aligned}$$

□

Remark. Unlike in $\boldsymbol{Z}$, $\bar{2}\cdot\bar{3}=\bar{6}=\bar{0}$ in $\boldsymbol{Z}/6\boldsymbol{Z}$. Note that $\bar{2}\neq\bar{0}$ in $\boldsymbol{Z}/6\boldsymbol{Z}$. Like so, two non-zero numbers can multiply to become zero in $\boldsymbol{Z}/n\boldsymbol{Z}$.

Theorem. (2.15) Let n be an integer greater than 1. Then the following are equivalent.

1. The integer n is a prime number.
2. Let a and b be in $\boldsymbol{Z}$. If $\bar{a}\bar{b}=\bar{0}$, then $\bar{a}=\bar{0}$ or $\bar{b}=\bar{0}$.
3. For all $\bar{a}\neq\bar{0}$ in $\boldsymbol{Z}/n\boldsymbol{Z}$, $\bar{a}$ has a multiplicative inverse.

Proof. We will prove that $2 \implies 3$. Let $\bar{a}\neq\bar{0}$ be an element of $\boldsymbol{Z}/n\boldsymbol{Z}$. Consider the subset of $\boldsymbol{Z}/n\boldsymbol{Z}$ consisting $\{\bar{a}\bar{0},\bar{a}\bar{1},\ldots,\bar{a}\overline{n-1}\}$. We claim that if $\bar{a}\bar{i}=\bar{a}\bar{j}$ for $0\leq i,j\leq n-1$, then $i=j$. Consequently, $\{\bar{a}\bar{0},\ldots,\bar{a}\overline{n-1}\}=\boldsymbol{Z}/n\boldsymbol{Z}$. In particular, $\bar{1}=\bar{a}\bar{b}$ for some $\bar{b}\in\boldsymbol{Z}/n\boldsymbol{Z}$. □

4 Lecture 4 (March 13th)

Last class, we have learned some properties of $\boldsymbol{Z}/n\boldsymbol{Z}$. Today, we will learn about rings.

Proposition. (2.16) Let n be an integer greater than 1. Then $\bar{a}\in\boldsymbol{Z}/n\boldsymbol{Z}$ has a multiplicative inverse if and only if $(a,n)=1$.

Proof. We know the following

$$1=(a,n)=ax+ny$$

for some $x,y\in\boldsymbol{Z}$. □

Theorem. (2.18) (Fermat's little theorem) Let p be a prime number and let a be an integer. Then $\bar{a}^p = \bar{a}$. In fact, $\bar{a}^{p-1} = \bar{1}$ for $\bar{a} \neq \bar{0}$. The proof is up to you.

Chapter 3 Rings

Chapter 3.1 Definition & Examples

Definition. (3.1) A ring is a set R equipped with two binary operations (a function $R \times R \to R$), an addition $+$ and a multiplication $\cdot$ which satisfies the following.

(i) $(a + b) + c = a + (b + c)$

(ii) There exists an element $0 \in R$ such that for every $a \in R$, $a + 0 = a = 0 + a$

(iii) For each a, there exists an a' such that $a + a' = 0 = a' + a$

(iv) $a + b = b + a$

(v) $(ab)c = a(bc)$

(vi) There exists an element $1 \in R$ such that for all $a \in R$, $a \cdot 1 = a = 1 \cdot a$

(vii) $a(b + c) = a \cdot b + a \cdot c$

(viii) $(a + b)c = ac + bc$

Example. Some examples of groups are

(i) $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$ are rings

(ii) $\boldsymbol{Z}/n\boldsymbol{Z}$ is a ring

(iii) $5\boldsymbol{Z}$ is not a ring as it has no multiplicative identity

(iv) $\boldsymbol{Z}^{\geq 0} = \{m \in \boldsymbol{Z} \mid m \geq 0\}$ is not a ring

(v) $(M_{n\times n}(\boldsymbol{R}), +, \cdot)$

(vi) $\boldsymbol{R}[x]$

Chapter 3.2 Basic Properties

Proposition. (3.14) The additive and multiplicative identities are unique.

Proof. Suppose there O and O' are two additive identities. Then,

$$O = O + O' = O'$$

□

Proposition. (3.15) The additive inverse is unique.

Proof. Let a be an element of R. Assume that both b and c are additive inverses of a.

$$c = O + c = (b + a) + c = b + O = b$$

□

Remark. (Notation)

(i) $a \cdot b = ab$

(ii) $a + a + \ldots + a = na$ and $a \cdot a \cdot \ldots \cdot a = a^n$

(iii) $a^0 = 1$ by convention

(iv) For $n > 0$, $(-n)a = (-a) + \ldots + (-a)$

Proposition. (3.17) Let R be a ring. If $a + c = b + c$, then $a = b$.

5 Lecture 5 (March 18th)

Last class, we have dealt with the basic properties of rings. Today, we learn more about rings.

Recall. Notation-wise, we have noted that

(i) $a \cdot b = ab$

(ii) a^n for $n > 0$ and a^0 is defined as 1.

(iii) na for $n \in \boldsymbol{Z}$

Proposition. (3.14, 15, 16) The uniqueness of 0, 1, and $-a$.

Proposition. (3.17) Let R be a ring. If $a + c = b + c$ then $a = b$.

Corollary. (3.18) For every a in a ring R,

$$0a = 0 = a0$$

Proof.

$$0 + 0a = (0 + 0)a = 0a + 0a$$

□

Remark. There is no cancellation law for multiplication ($ac = dc$ does not imply that $a = d$).

Chapter 3.3 Special Types of Rings

Example. (3.19) Not every ring is commutative. For instance, consider $M_{2\times 2}(\boldsymbol{R})$.

Definition. (3.20) A ring is commutative if $ab = ba$ for all $a, b \in R$.

Definition. (3.22) Let a be an element of R. We say that a is a zero divisor if there exists a non-zero $b \in R$ such that $ab = 0$ or $ba = 0$.

Example. In $M_{2\times 2}(\boldsymbol{R})$,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

Observe that both matrices are zero-divisors.

Definition. (3.23) Let R be a commutative ring. We will say that R is an integral domain if $1 \neq 0$ and $ab = 0$ implies that $a = 0$ or $b = 0$.

Example. $\boldsymbol{Z}$ or equivalently $\boldsymbol{Z}/0\boldsymbol{Z}$ are integral domains whereas $\boldsymbol{Z}/1\boldsymbol{Z}$ is not as its multiplicative and additive identities are equal to each other.

Example. (3.26) Let $n > 1$ be an integer. Then $\boldsymbol{Z}/n\boldsymbol{Z}$ is an integral domain if and only if n is a prime.

Proposition. Let $a \in R$ be an element. If a is not a zero-divisor, then the multiplicative cancellation holds for a. That is, if $ab = ac$ or $ba = ca$, then $b = c$.

Proof. (3.26) If $ab = ac$, then $0 = a(b - c) = a(b + (-c))$. □

Definition. (3.27) If a has a multiplicative inverse (that is, there exists $b \in R$ such that $ab = 1 = ba$), then we say that a is invertible or a unit.

Remark. We denote the set of units in R by $R^{\times}$.

Proposition. (3.28) Let $n > 0$ be an integer.

$$(\boldsymbol{Z}/n\boldsymbol{Z})^{\times} = \{\bar{a} \mid (a, n) = 1\}$$

Definition. (3.27) We say that R is a field if

(i) R is commutative

(ii) $1 \neq 0$ in R

(iii) Every nonzero element is invertible

Remark. If R is a field, then $R^\times = R - \{0\}$.

Example. (3.30)

(i) $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$

(ii) $\boldsymbol{Z}/p\boldsymbol{Z}$ where p is a prime

Proposition. (3.31) Every field is an integral domain.

Proof. Yours! □

Remark. The conserve doesn't hold.

6 Lecture 6 (March 20th)

Last time, we have learned integral domains & fields. Today, we will learn Cartesian products and subrings.

Recall. (i) (3.22) $a \in R$ is a zero-divisor if there exists a $b \neq 0$ such that $ab = 0$ or $ba = 0$

(ii) (3.23) R is an integral domain if (1) R is commutative, (2) $1 \neq 0$, and (3) R has no nonzero zero-divisors

(iii) R has no nonzero zero-divisors

(iv) $a \in R$ is a unit for $ab = ba = 1$ for some $b \in R$

(v) R is a field if (1), (2), and every nonzero element is a unit

Remark. The fact that $1 = 0$ in R is equivalent to saying that R is the trivial ring $\{0\}$

Proposition. (3.28) When $n > 0$,

$$(\boldsymbol{Z}/n\boldsymbol{Z})^\times = \{\bar{a} \in \boldsymbol{Z}/n\boldsymbol{Z} \mid (a, n) = 1\}$$

The problem with this definition is that we don't know whether the condition $(a, n) = 1$ works for the entirety of $\bar{a}$. However, we know from number theory that if $a \equiv_n b$, then $(a, n) = (b, n)$.

Remark. We note that

$$\text{Fields} \subset M_{2\times 2}(\boldsymbol{R}), \boldsymbol{R}[x] \in \text{Integral Domains} \subset \boldsymbol{Z}/4\boldsymbol{Z} \in \text{Rings}$$

Proposition. (3.33) Let R be an integral domain having finitely many elements. Then R is a field.

Proof. The proof is similar to the proof of Fermat's little theorem. Set $R = \{a_1, a_2, \ldots, a_n\}$. It suffices to prove that We now prove that this is a field. Let's fix $a_i \neq 0$. If suffices to prove that $a_i \in R^\times$. Consider the subset of R $a_iR = \{a_i \cdot a_1, a_i \cdot a_2, \ldots, a_i \cdot a_n\}$. Since R is an integral domain, $a_iR = R$. Indeed, if $a_i \cdot a_j = a_i \cdot a_k$, $a_j = a_k$. Then, $1 = a_i \cdot a_j$ for some j. Consequently, every nonzero element in R has a multiplicative inverse. □

Chapter 4 The Category of Rings

Chapter 4.1 Cartesian Products

Definition. (4.1) Lets R and S be rings. The catesian product of R and S is the set $R \times S$ equipped with component-wise addition and multiplication.

Remark. In $R \times S$,

$$\begin{cases} (r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2) \\ (r_1, s_1) \cdot (r_2, s_2) = (r_1 \cdot r_2, s_1 \cdot s_2) \end{cases}$$

forms a ring. If it exists, the inverse of an element would look like (r^{-1}, s^{-1}). We remind ourselves that there exists projection functions (pr_1, pr_2) from $X \times Y$ to X and Y.

Example. (4.2, 4.3)

(i) $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z} \neq \boldsymbol{Z}/4\boldsymbol{Z}$

(ii) $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/3\boldsymbol{Z} \cong \boldsymbol{Z}/6\boldsymbol{Z}$

7 Lecture 7 (March 25th)

Last class, we have learned the properties of rings and subrings. Today, we will learn about ring homomorphisms.

Proposition. (3.33) A finite integral domain is a field

Definition. (4.1) Let R and S be rings. The set

$$(R \times S, (r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2), (r_1, s_1) \cdot (r_2, s_2) = (r_1 r_2, s_1 s_2))$$

is called the cartesian product of R and S.

Example. (4.2, 4.3)

(i) Comparing $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$ and $\boldsymbol{Z}/4\boldsymbol{Z}$ we find that these are very different sets. Adding identical elements in one results in the 0 whereas this isn't always the case for the other.

(ii) Comparing $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/3\boldsymbol{Z}$ and $\boldsymbol{Z}/6\boldsymbol{Z}$, we find that they are identical.

Definition. (4.5) Let S be a subset of a ring R. We say that S is a subring of R if

(i) $0, 1 \in S$

(ii) S is closed under $+$ and $\cdot$

(iii) $(S, +, \cdot)$ is a ring

Note that a subring is not only a ring of its own but also manifests the algebraic structure of the original ring.

Example. (4.6, 4.7, 4.8, 4.9, 4.11, 4.12, 4.13)

(i) $\boldsymbol{Z} \subset \boldsymbol{Q} \subset \boldsymbol{R} \subset \boldsymbol{C}$

(ii) $\boldsymbol{Z}/n\boldsymbol{Z} \subset \boldsymbol{Z}$

(iii) $\Delta_R = \{(r, r) \mid r \in R\} \subset R \times R$

(iv) $R \subset R[x]$

(v) $\boldsymbol{Z}[i] = \{m + in \mid m, n \in \boldsymbol{Z}\} \subset \boldsymbol{C}$

(vi)
$$\left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \;\middle|\; a, b \in R \right\} \subset M_{2\times 2}(\boldsymbol{R})$$

Proposition. (4.14) S is a subring of R if and only if ...

Chapter 4.3 Ring Homomorphisms

Chapter 4.4 Isomorphisms of Rings

Definition. (cf. definition (4.29)) Let R and S be rings, and let $\phi : R \to S$ be a function. We will say that ϕ is a ring isomorphism if it satisfies

(i) ϕ is bijective

(ii) ϕ preserves the ring operations, or $\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$ and $\phi(r_1 r_2) = \phi(r_1)\phi(r_2)$

Definition. (4.15) We will say that ϕ is a ring homomorphism if

(i) $\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$

(ii) $\phi(r_1 r_2) = \phi(r_1)\phi(r_2)$

8 Lecture 7 (March 27th)

Last time, we have learned about isomorphisms. Today, we will learn about homomorphisms.

Definition. (4.29) Let R and S be rings. A function $\phi : R \to S$ is said to be a isomorphism if

(i) ϕ is a bijection

(ii) $\phi(r_1, +r_2) = \phi(r_1) + \phi(r_2)$

(iii) $\phi(r_1 r_2) = \phi(r_1)\phi(r_2)$

We don't need conditions such as $\phi(0) = 0$, $\phi(-r) = -\phi(r)$, and $\phi(1) = 1$ as they are implied by the conditions above.

Definition. (4.15) Let R and S be rings. A function $\phi : R \to S$ is called a (ring) homomorphism if

(i) $\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$

(ii) $\phi(r_1 r_2) = \phi(r_1)\phi(r_2)$

(iii) $\phi(1) = 1$

As the function isn't bijective, there doesn't need to be a mapping of $\phi(1)$, and we require the third condition.

Example. The function $f : \boldsymbol{Z} \to \boldsymbol{Z} \times \boldsymbol{Z}$ defined by $n \mapsto (n, 0)$ is not a ring homomorphism.

Definition. (4.32) We will say that two rings R and S are isomorphic if there exists an isomorphism $\phi : R \to S$.

Remark. (4.33) The isomorphic relation is an equivalence relation.

Example. (4.38 - 4.43)

(i) $\boldsymbol{Z}/4\boldsymbol{Z}$ is not isomorphic to $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$

(ii) $\boldsymbol{Z}/6\boldsymbol{Z}$ is isomorphic to $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/3\boldsymbol{Z}$

(iii) The complex conjugation $\boldsymbol{C} \to \boldsymbol{C} : z \mapsto \bar{z}$ is an isomorphism

(iv) The function

$$\boldsymbol{C} \to \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;\middle|\; a, b \in \boldsymbol{R} \right\}$$

defined as

$$a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is an isomorphism.

(v) $R \to \Delta_R = \{(r, r) \mid r \in R\}$ defined as $r \mapsto (r, r)$ is an isomorphism

(vi) $R[x, y]$ and $(R[x])[y]$ is isomorphic

9 Lecture 9 (April 1st)

Last class, we have learned about isomorphisms and homomorphisms. This class, we learn about some properties of homomorphisms.

Definition. (4.29) A function $\phi : R \to S$ is called an isomorphism if it is a homomorphism and bijective.

Proposition. (4.16) Let $\phi : R \to S$ be a ring homomorphism. Then $\phi(0) = 0$

Proof. $\phi(0 + 0) = \phi(0) + \phi(0) = \phi(0) = \phi(0) + 0$ □

Example. (4.17 – 26)

(i) The unique function $0 : R \to 0$ is a homomorphism

(ii) $0 \to R : 0 \mapsto 0$ is not a ring homomorphism if R is nontrivial

(iii) $pr_1 : R \times S \to R$ and $pr_2 : R \times S \to S$ are ring homomorphisms

(iv) $\boldsymbol{Z} \to \boldsymbol{Z}/n\boldsymbol{Z} : m \mapsto \bar{m}$ is a ring homomorphism

(v) $\boldsymbol{Z} \to R : n \mapsto n \cdot 1$ is a ring homomorphism

(vi) $\boldsymbol{Z}/12\boldsymbol{Z} \to \boldsymbol{Z}/4\boldsymbol{Z} : \bar{n} \mapsto \bar{n}$ is a ring homomorphsim

(vii) Fix $r \in R$. $R[x] \to R : f(x) \mapsto f(r)$ is a ring homomorphism

(viii) $\boldsymbol{C} \to \boldsymbol{C} : a + bi \mapsto a - bi$ is a homomorphism

(ix) $\boldsymbol{Z} \to \boldsymbol{Z} : n \mapsto 2n$ is not a ring homomorphism

(x) $\boldsymbol{Z}/2\boldsymbol{Z} \to \boldsymbol{Z}$ there is no such ring homomorphism

(xi) $\det : M_{2\times 2}(R) \to R$ is not a ring homomorphism

We MUST check whether the function in (vi) is well defined.

Corollary. (4.31) Let $\phi : R \to S$ be a ring homomorphism. Then ϕ is an isomorphism if and only if there exists a ring homomorphism $\psi : S \to R$ such that $\psi \circ \phi = \mathrm{id}_R$ and $\phi \circ \psi = \mathrm{id}_S$.

Chapter 5 Canonical Decomposition, Quotients, and Isomorphism Theorems

Chapter 5.1 Rings: Canonical Decomposition I

Remark. Any function can be written as a composition of a surjection and an injection.

Proposition. (5.1) Let $\phi : R \to S$ be a homomorphism. Then the image of ϕ (denoted as Im ϕ) is a subring of S.

Remark. (5.2) If R' is a subring of R, then $f(R')$ is a subring of S.

Chapter 5.2 Kernels and Ideals

Definition. (5.3) Let $\phi : R \to S$ be a homomorphism. The kernal of ϕ is the subset $\{r \in R \mid \phi(r) = 0\} \subset R$ and will be denoted by Ker ϕ. Note that Ker $\phi = \phi^{-1}(\{0\})$.

Example. (5.4, 5.5)

(i) Let n be a nonnegative integer. Then the kernel of the homomorphism $\boldsymbol{Z} \to \boldsymbol{Z}/n\boldsymbol{Z}$ (given by $m \mapsto \bar{m}$) is $n\boldsymbol{Z}$

(ii) Let $\text{ev}_0 : R[x] \to R$ be the homomorphism defined by the evaluation at 0. Then, Ker ev_0 is the set of all polynomials with no constant terms.

Proposition. (5.6) The set (Ker $\phi, +$) satisfies the four ring axioms.

10 Lecture 10 (April 3rd)

Last time, we learned homomorphisms and kernals. Today, we will learn ideals and quotient rings.

Recall. Let $\phi : R \to S$ be a ring homomorphism.

Proposition. (5.1) Im $\phi = \phi(R) \subset S$ is a subring.

Remark. (5.2) If $R' \subset R$ is a subring, then $\phi(R') \subset S$ is also a subring.

Definition. (5.3) Ker $\phi = \phi^{-1}(\{0\}) = \{r \in R \mid \phi(r) = 0\}$ is called the kernal of ϕ.

Proposition. (5.6) (Ker $\phi, +$), closed under addition, satisfies the ring properties (i) through (iv). That is, it is a abelian group.

Proposition. (5.17) For all $a \in$ Ker ϕ and all $r \in R$ both ra and ar belong to Ker ϕ.

Definition. (5.8) Let R be a ring and I be a subset of R. I is an ideal if it is:

(i) Closed under addition

(ii) The additive identity is in I ($0 \in I$)

(iii) (Absorption property) For all $a \in I$ and $r \in R$, ar and ra are in I

Remark. (i) If $a \in I$, then $(-1) \cdot a = -a \in I$

(ii) If I is nonempty, then the condition that $0 \in I$ is redundant for I to be an ideal

Example. (5.10 – 15)

(i) Ker ϕ is an ideal of R

(ii) 0 and R are ideals of R

(iii) $\boldsymbol{Z} \subset \boldsymbol{Q}$ is not an ideal

(iv) $m\boldsymbol{Z} \subset \boldsymbol{Z}$ is an ideal for all $m \in \boldsymbol{Z}$

(v) The set of all polynomials $f(x, y)$ in $\boldsymbol{C}[x, y]$ that have no constant term is an ideal.

Proposition. (5.16) Let R be a commutative ring and let $r \in R$ be an element. Then the subset

$$(a) = \{ra \mid r \in R\}$$

is an ideal of R.

Definition. (5.17) Let R be a commutative ring and let $a \in R$ be an element. We say that (a) is a principle ideal generated by a.

Remark. Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring R. Then the subset $(a_1, \ldots, a_r) = \{r_1a_1 + r_2a_2 + \ldots + r_na_n \mid r_i \in R\}$ is an ideal and called the ideal generated by $a_1, \ldots, a_n$.

Chapter 5.3 Quotient Rings

The following diagram shows what we are trying to do.

$$\begin{array}{rcl} \boldsymbol{Z} & \longleftrightarrow & R \\ n\boldsymbol{Z} & \longleftrightarrow & I \\ \boldsymbol{Z}/n\boldsymbol{Z} & \longleftrightarrow & R/I \end{array}$$

Alike how we partitioned the integers using the relationship of multiples of integers, we are going to partition a ring using the relation that elements are in identical ideals.

Definition. (5.19) Let R be a ring and I be an ideal of R. We define a relation $\sim_I$ on R by declaring that $a \sim_I b$ if and only if $b - a \in I$. We say that a is congruent to b modulo the ideal I.

Proposition. (5.20) The relation $\sim_I$ is an equivalence relation.

Remark. Let $R = \boldsymbol{Z}$ and $I = n\boldsymbol{Z}$ then

$$a \sim_I b \qquad \Longleftrightarrow \qquad a \equiv_n b$$

Remark. For each $a \in R$, the equivalence class a can be described as follows:

$$[a] = \{r \in R \mid a \sim_I r\} = \{a + i \mid i \in I\} = a + I$$

For example,

$$\bar{2} = 2 + 5\boldsymbol{Z}$$

Definition. (5.22) We call $\bar{a}$ the coset of a modulo I. We will denote by R/I the set of all cosets and call it the quotient of R modulo I.

11 Lecture 11 (April 8th)

Last class, we have learned kernals and ideals. Today, we will learn quotient rings and isomorphism theorems.

Definition. (5.3) Let $\phi : R \to S$ be a ring homomorphism.

$$\ker \phi = \{r \in R \mid \phi(r) = 0\} = \phi^{-1}(\{0\})$$

Definition. (5.8) Let I be a subset of a ring R. Then I is said to be an ideal if

(i) It is closed under $+$ and additive inverses

(ii) $0 \in I$

(iii) (Absorbtion property) $ar = ra \in I$ for all $r \in R$ and all $a \in I$

Proposition. (5.16) If R is commutative, then $\ker \phi$ is an ideal of R.

Definition. (5.19) Let R be a commutative ring and I be an ideal of R. $a \sim_I b$ if and only if $b - a \in I$. Then, we say "a is congruent to b modulo I".

Remark. (i) Here, $\sim_I$ is an equivalence relation

(ii) The equivalence class of $a \in R$

$$[a] = \bar{a} + I = \{a + i \mid i \in I\} \subset R$$

is called the (left) coset of a modulo I.

Definition. (5.22) R/I (the set of all cosets $\{\bar{a} \mid a \in R\}$) is called the quotient of R modulo I.

Remark. We can give a ring structure to $R/I = \{\bar{a} \mid a \in R\}$by defining

$$\begin{cases} R/I \times R/I \to R/I : (\bar{a}, \bar{b}) \mapsto \overline{a+b} \\ R/I \times R/I \to R/I : (\bar{a}, \bar{b}) \mapsto \overline{ab} \end{cases}$$

Theorem. (5.26) Let R be a (commutative) ring and let $I \subset R$ be an ideal. Then $(R/I, +, \cdot)$ is a ring.

Proof. We first show well-definedness of $+$ and $\cdot$. Then, we can show the eight ring properties. □

Example. (5.27 - 34)

(i) $\boldsymbol{Z}/n\boldsymbol{Z}$

(ii) $R/R = \{\bar{a} \mid a \in R\} = \{R\} \neq R$ In this case, $\bar{a} = R$ for every $a \in R$.

(iii) If R is commutative, then R/I is also commutative ($\bar{a} + \bar{b} = \overline{a+b} = \overline{b+a} = \bar{b} + \bar{a}$).

(iv) R/I is not necessarily an integral domain, even if R is an integral domain. For example, $\boldsymbol{Z}/4\boldsymbol{Z}$ is not an integral domain, since $\bar{2} \cdot \bar{2} = \bar{0}$.

(v) Consider $R[x]$ / (x). $\overline{f(x)} = \overline{1 + 2x + 3x^2} = \overline{1} + \overline{2} \cdot \overline{x} + \overline{3}(\overline{x})^2$.

(vi) For a commutative ring R, $R[x]$ / $(x - r) \cong R$

(vii) $\boldsymbol{R}[x]$ / $(x^2+1) \cong \boldsymbol{C}$. For $f(x) \in \boldsymbol{R}[x]$, $f(x) = (x^2+1)q(x)+ax+b$ and $\overline{f(x)} = \overline{ax+b}$. The function would be $\overline{f(x)} \to ai + b$.

12 Lecture 12 (April 10th)

Last time we have learned about quotient rings. Today, we learn about the first isomorphism theorems.

Chapter 5.3 Rings: Canonical Decomposition II

Proposition. (5.35) Let R be a ring and I be an indeal of R. Then the natural projection $\pi : R \to R/I$ given by $r \mapsto \overline{r}$ is a surjective ring homomorphism with $\ker \pi = I$.

Proof. Almost yours!

$$\begin{aligned} r \in \ker I &\iff \pi(r) = \overline{r} = \overline{0} \\ &\iff r = r - 0 \in I \end{aligned}$$

□

Theorem. (5.37 5.38) (The 1st isomorphism theorem) Let $\phi : R \to S$ be a ring homomorphism. Then

(i) The function $\overline{\phi} : R \,/\, \ker \phi \to S$ given by the rule $\overline{\phi}(\overline{r}) = \phi(r)$ is a well-defined ring homomorphism

(ii) $\overline{\phi}$ is an injective ring homomorphism

(iii) $\operatorname{Im} \overline{\phi} = \operatorname{Im} \phi$

In particular, $\overline{\phi}$ induces an isomorphism $R \,/\, \ker \phi \to \operatorname{Im} \phi$.

Remark. (i) For the projection function $\pi : R \to R \,/\, I$, applying the above, we have $R \,/\, \ker \pi \cong \operatorname{Im} \pi$.

(ii) We recall that a function can be decomposed into a surjection and an injection. Likewise, a homomorphism can be decomposed into a projection, isomorphism, and surjection.

$$\begin{array}{ccc} R & \xrightarrow{\quad\varphi\quad} & S \\ {\scriptstyle\pi}\downarrow & & \uparrow \\ R/\ker\varphi & \xrightarrow[\sim]{} & \operatorname{Im}\varphi \end{array}$$

(iii) In linear algebra

$$\begin{aligned} L : V \to W &\implies V \,/\, \ker L \cong \operatorname{Im} L \\ &\implies \dim V - \dim \ker L = \dim V \,/\, \ker L = \dim \operatorname{Im} L = \operatorname{rank} L \\ &\implies \dim L = \dim \ker L + \operatorname{rank} L \end{aligned}$$

Proof. The steps are as follows.

(∗) Well-definedness of $\overline{\phi} : R \,/\, \ker \phi \to S : \overline{r} \mapsto \phi(r)$. We claim that if $\overline{r_1} = \overline{r_2}$, then $\phi(r_1) = \phi(r_2)$. Note that $r_1 - r_2 \in \ker \phi$. Then,

$$0 = \phi(r_1 - r_2) = \phi(r_1) + \phi(-r_2) = \phi(r_1) - \phi(r_2)$$

(i) Let $\overline{\phi}$ be a ring homomorphism. Prove + separately.

$$\overline{\phi}(\overline{a} \cdot \overline{b}) = \overline{\phi}(\overline{ab}) = \overline{\phi(ab)} = \overline{\phi(a)\phi(b)} = \overline{\phi(a)} \cdot \overline{\phi(b)} = \overline{\phi}(\overline{a}) \cdot \overline{\phi}(\overline{b})$$

(ii) We now prove that $\overline{\phi}$ is injective. Suppose that $\overline{r} \in \ker \overline{\phi}$. We want to prove that $\overline{r} = \overline{0}$. By the assumption,

$$0 = \overline{\phi}(\overline{r}) = \phi(r)$$

that is, $r \in \ker \phi$, which completes the proof.

(iii) $\operatorname{Im} \overline{\phi} = \operatorname{Im} \phi$

□

Example. (5.40, 5.41)

(i) Let R be a ring and $r \in R$. Consider the evaluation homomorphism

$$\mathrm{ev}_r : R[x] \to R : f(x) \mapsto f(r)$$

Note that $\ker \mathrm{ev}_r = (x - r)$. Applying the 1st isomorphism theorem,

$$R[x] \,/\, (x - r) = R[x] \,/\, \ker \mathrm{ev}_r \to \operatorname{Im} \mathrm{ev}_r = R$$

13 Lecture 13 (April 15th)

Last time we have learned

Theorem. (5.37, 5.38) Let $\phi : R \to S$ be a ring homomorphism. Then:

(i) The induced map $\overline{\phi} : R \,/\, \ker \phi \to S$ is defined by $\overline{\phi}(\overline{r}) = \phi(r)$ is a well-defined ring homomorphism

(ii) $\overline{\phi}$ is injective

(iii) $\operatorname{Im} \overline{\phi} = \operatorname{Im} \phi$

In particular, there is an isomorphism of rings $\overline{\phi} : R \,/\, \ker \phi \to \operatorname{Im} \phi$.

Example. (i) For each $r \in R$ there is an isomorphism

$$R[x] \,/\, (x - r) \to R$$

(ii) There is an isomorphism of rings

$$\boldsymbol{R}[x] \,/\, (x^2 + 1) \to \boldsymbol{C}$$

Chapter 5.6 The Chinese Remainder Theorem

Example. Let's examine whether we can solve the following system of congruences.

$$\begin{cases} x \equiv_3 2 \\ x \equiv_7 2 \\ x \equiv_8 5 \end{cases}$$

We attempt to generalize this from $\boldsymbol{Z}$ to R.

Definition. (5.43) Let R be a ring and let I and J be ideals of R. The sum $(I + J)$ of I and J is defined to be

$$\{a + b \mid a \in I, b \in J\}$$

The product (IJ) of I and J is defined to be

$$\Big\{ \sum_{i=1}^{n} a_i b_i \mid a_i \in I, b_i \in J \Big\}$$

Remark. (i) $I + J$ and IJ are ideals of R

(ii) $I + J$ is the smallest ideal containing both I and J

(iii) $IJ \subset I \cap J \subset I$ or $J \subset I + J$

(iv) In $R[x]$, set $I = J = (x)$. Then $IJ = (x^2) \not\subset (x) = I \cap J$

(v) In $R[x]$, let $I = (x, 2)$. Then $x^2 + 4$ cannot be written as a product of two elements of I. Moreover, $I^2 \not\subset I$

Example. In $\boldsymbol{Z}$,

(i) $(a) + (b) = (\gcd(a, b))$

(ii) $(a) \cap (b) = (\operatorname{lcd}(a, b))$

(iii) $(a) \cdot (b) = (ab)$

Theorem. (5.52) (Chinese Remainder Theorem) Let R be a commutative ring, and let I and J be ideals of R. If $I + J = R$, then

$$R \,/\, IJ \cong R \,/\, I \times R \,/\, J$$

Corollary. (5.53) Let $n_1, \ldots, n_r$ be pairwise relatively prime positive integers. Let $N = n_1 n_2 \ldots n_r$. Then $\boldsymbol{Z}/n\boldsymbol{Z}$ is isomorphic to $\boldsymbol{Z}/n_1\boldsymbol{Z} \times \boldsymbol{Z}/n_2\boldsymbol{Z} \times \ldots \times \boldsymbol{Z}/n_r\boldsymbol{Z}$.

Remark. Going back to solving a system of congruences, $(3, 7) = (7, 8) = (3, 8) = 1$ implied that there existed a unique solution in modulo $3 \cdot 7 \cdot 8$.

$$\boldsymbol{Z}/3 \cdot 7 \cdot 8 \cong \boldsymbol{Z}/3\boldsymbol{Z} \times \boldsymbol{Z}/7\boldsymbol{Z} \times \boldsymbol{Z}/8\boldsymbol{Z}$$

Proposition. (5.48) Let R be a ring, let I and J be ideals of R. Assume that $I + J = R$. Then the homomorphism $\pi : R \to R/I \times R/J : r \mapsto (\bar{r}, \bar{r})$ is surjective.

Proof. For a given $(\bar{a}, \bar{b}) \in R/I \times R/J$, we can find a $x \in R$ such that $\pi(x) = (\bar{a}, \bar{b}) \in R/I \times R/J$. Let $x - a = i \in I$ and $x - b = j \in J$. □

14 Lecture 14 (April 19th)

Theorem. (5.57) (Third Isomorphism Theorem) Let R be a ring and let I be an ideal of R. Let J be an ideal containing I ($I \subset J$). Then $\bar{J} = J/I$ is an ideal of R/I, and all ideals of R/I may be realised in this way. Furthermore, the natural map $R/I \to R/J$ (given by $\bar{r} \mapsto \bar{r}$) induces an isomorphism

$$R/I \Big/ J/I \cong R/J$$

Proof. We start by proving that $\bar{J}$ is an ideal.

(1) If J containing I is an ideal of R,

$$\bar{J} = J/I = \{\bar{j} \mid j \in J\} \lhd R/I$$

is an ideal of R/I as it is the image of an ideal under a surjective homomorphism $\pi : R \to R/I$.

(2) There is a bijection between the set of ideals of R containing I and the set of ideals of R/I.

(2-1) Let $\bar{J}$ be an ideal of R/I, that is, $\bar{J} \lhd R/I$. This implies that $I \subset \pi^{-1}(\bar{J}) \lhd R$. To show that $I \subset \pi^{-1}(\bar{J})$, let $i \in I$. We want to verify that $\pi(i) \in J$. Note that $\pi(i) = \bar{i} = \bar{0} \in J$.

(2-2) We have seen that all ideals containing I have images that are ideals of R/I and the converse. Lastly notice that $J \mapsto \bar{J}$ and $\bar{J} \mapsto \pi^{-1}(J)$ are inverses of each other.

(3) Note that there is a well-defined ring homomorphism $p : R/I \to R/J : \bar{r} \mapsto \bar{r}$ ($\bar{r} = \bar{s} \in R/I \implies \bar{r} = \bar{s} \in R/J$).

(3-1) p is surjective.

(3-2) $r \in \ker p \iff p(\bar{r}) = \bar{r} = \bar{0} \in R/J \iff r - 0 = r \in J$. So, $\ker p = J/I$.

(3-3) Using the 1st isomorphism theorem, we see that

$$R/I \Big/ \ker p = R/I \Big/ J/I \cong R/J$$

□

Example. (i) Consider the ideals of $\boldsymbol{Z}/6\boldsymbol{Z}$, which are also the ideals of $\boldsymbol{Z} = \{(n) \mid n \in \boldsymbol{Z}\}$ containing $6\boldsymbol{Z} = (6)$. They are $\{(1), (2), (3), (6)\}$, and

$$\boldsymbol{Z}/6\boldsymbol{Z} = \{\overline{(1)}, \overline{(2)}, \overline{(3)}, \overline{(6)}\}$$

Chapter 6 Integral Domains

Remark. There are some key properties of $\boldsymbol{Z}$, such as

(i) (Division algorithm) $a = bq + r$ which generalises to Euclidean domains (EDs)

(ii) (Every ideal is principle) $(n) = I \lhd \boldsymbol{Z}$ which generalises to principle ideal domains (PIDs)

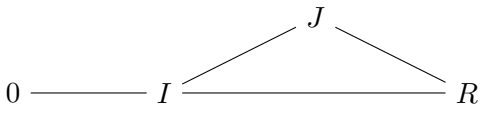
(iii) (There exists an unique factorisation) $n = p_1 p_2 \dots p_r$ which generalises to qnique factorisation domains (UFDs)

Chapter 6.1 Prime and Maximal Ideals

Definition. (6.4, 6.9) Let R be commutative ring, and I be an ideal of R.

(i) A prime ideal I satisfies $I \neq R$ (I is proper) and if a and b are in R such that $ab \in I$, then $a \in I$ or $b \in I$.

(ii) A maximal ideal I satisfies $I \neq R$ and if J is an ideal containing I, then $J = I$ or $J = R$



Remark. (i) If p is a prime number, then $(p) \subset \boldsymbol{Z}$ is a prime ideal.

(ii) $0 \subset \boldsymbol{Z}$ is a prime ideal.

Proposition. (6.3) Every maximal ideal is a prime ideal.

Proof. Let I be a maximal ideal of R. Since I is maximal, $I \neq R$. Suppose now that $ab \in I$ but a is not in I. In this case, $I + (a)$ is an ideal properly containing I, so that $I + (a) = R$ by the maximality of I. In particular, $1 = i + ca$ for some $c \in R$. Then $b = b \cdot 1 = b(i + ca) = bi + cab \in I$, because $ab \in I$. □

Theorem. (6.1) Let R be a commutative ring, and I be an ideal of R. Then I is a prime ideal if and only if R/I is an integral domain.

Theorem. (6.2) Let R be a commutative ring, and I be an ideal of R. Then I is a maximal ideal if and only if R/I is a field.

15 Lecture 15 (May 6th)

Chapter 6.2 Primes and Irreducibles

Definition. (6.15) Let R be a commutative ring. Let a and b be elements of R. We say that b divides a and write $b|a$ if $a \in (b)$.

Definition. (6.18) Let R be an integral domain. Then $p \in R$ is a prime element if it is nonzero, nonunit, and if $p|ab$, then $p|a$ or $p|b$.

Example. In $\boldsymbol{Z}$, a prime number is a prime element.

Lemma. Let R be an integral domain and let p be a nonzero element of R. Then p is a prime element if and only if (p) is a prime ideal of R.

Definition. (6.20) Let R be an integral domain and let q be an element of R. We say that q is irreducible if it is nonunit and if $q = ab$, then a or b is a unit.

Remark. An irreducible element is nonzero.

Theorem. (6.23) Let R be an integral domain. Then every prime element is irreducible.

Proof. Let $p \in R$ be a prime element. Note that p is not a unit. Assume that $p = ab$. Then, $p|a$ or $p|b$ because $p|ab$. If p divides a, then $a = pc$ for some $c \in R$. Therefore, $p = ab = pcb$. This implies that b is a unit. □

Example. (6.21) In $\boldsymbol{C}[x, y]/(y^2 - x^3)$, the element $\bar{x}$ is irreducible, but not prime.

Chapter 6.3 Euclidean Domains and Principle Ideal Domains

Definition. (6.24) Let R be an integral domain. We say that R is a Euclidean domain if there exists a function

$$\nu : R \setminus \{0\} \to \boldsymbol{Z}^{\geq 0}$$

satisfying the following property: for every $a \in R$ and every nonzero $b \in R$, there exists q and r in R such that

$$a = bq + r$$

where $r = 0$ or $\nu(r) < \nu(b)$. The function ν is called a Euclidean valuation.

Example. (i) Consider $(\boldsymbol{Z}, |\cdot| : \boldsymbol{Z} \setminus \{0\} \to \boldsymbol{Z}^{\geq 0})$. This implies that $\boldsymbol{Z}$ is an Euclidean domain.

(ii) Let K be a field. Consider $(K[x], \deg : K[x] \setminus \{0\} \to \boldsymbol{Z}^{\geq 0})$. This implies that $K[x]$ is a Euclidean domain.

Definition. (6.25) Let R be an integral domain. We say that R is a principle ideal domain or PID, if every ideal in R is principle. That is, if I is an ideal of R, then $I = (r)$ for some $r \in R$.

Theorem. (6.26) Every Euclidean domain is a PID.

Proof. Let R be a ED. Let I be an ideal of R. If $I = 0$, then $I = (0)$. Otherwise, consider the set $\{\nu(a) : a \in I \setminus \{0\}\}$. Since this set is nonempty, it contains the smallest element by the well-ordering principle b for $\boldsymbol{Z}^{\geq 0}$. We claim that $I = (b)$. Since $b \in I$, $(b) \subset I$. So it suffices to show that $I \subset (b)$. Let $a \in I$. Then $a = bq + r$, where $r = 0$ or $\nu(r) < \nu(b)$. If $\nu(r) < \nu(b)$, then $r \in I \setminus \{0\}$. This contradicts to the fact that b is the smallest element of the set $\{\nu(a) \,|\, a \in I \setminus \{0\}\}$. Therefore, $r = 0$, and $a = bq \in (b)$. □

Example. (6.27, 28)

(i) $\boldsymbol{Z}$ is a PID

(ii) Every field is a PID (the ideals of K are $0 = (0)$ and $K = (1)$)

(iii) If K is a field, then $K[x]$ is a ED, hence a PID

(iv) $\boldsymbol{Z}[x]$ is not a PID. The ideal $(2, x)$ is not principle.

Proposition. (6.30) Let R be a PID. Let I be a nonzero prime ideal. Then I is a maximal ideal.

Proof. Suppose we are given an ideal J such that $I \subset J \subset R$. Since R is a PID, we can set $I = (a)$ and $J = (b)$. Then $a = bc$ for some $c \in R$. Note that $a \neq 0$. Since $I = (a)$ is a prime ideal, a is a prime element. Therefore, $a|b$ or $a|c$. If $a|b$, then $I = J$ and if $a|c$, $b = (1) = R$. □

Proposition. (6.32) Let R be a PID. Then it satisfies the ascending chain condition on principle ideals: if $I_1 \subset I_2 \subset \ldots$ is a chain of ideals in R, then there is an index m such that $I_m = I_{m+1} = I_{m+2} = \ldots$ and so on.

Proof. Let I be the union of the ideals I_k (verify that I is an ideal of R). Since R is a PID, $I = (r)$ for some $r \in R$. In particular, $r \in I_m$ for some $m \geq 1$. This implies that

$I_m = I_{m+i}$ for every $i \geq 0$.

$$I_m \subset I = \bigcup I_k = (r) \subset I_m$$

□

Chapter 6.4 Principle Ideal Domains and Unique Factorisation Domains

Definition. (6.37) Let R be an integral domain. Then R is a unique factorization domain or UFD if it satisfies the following condition: suppose that $a \in R$ is nonzero and nonunit. Then there exists finitely many irreducible elements $q_1, q_2, \ldots, q_r$ such that $a = q_1 q_2 \ldots q_r$. Moreover, this factorisation is unique in the senes that if

$$a = q_1 q_2 \ldots q_r = p_1 p_2 \ldots p_s$$

where all p_i and q_j are irreducible, then $s = r$ and after reordering factors we have $(p_i) = (q_i)$ for all i.

Proposition. (6.33) Let R be a PID and let $q \in R$ be an irreducible element. Then (q) is a maximal ideal.

Proof. Since q is nonunit, $(q) \neq R$. Let I be an ideal of R such that $(q) \subset I \subset R$. Since R is a PID, $I = (a)$ for some $a \in R$. Then $q = ab$ for some $b \in R$. □

Lemma. Let R be a PID. Let a be a nonzero and nonunit element of R. If (a) is maximal, then a is irreducible.

Proof. Let $a = bc$. Then $(a) \subset (b) \subset R$. As (a) is maximal, $(b) = (1) = R$ or $(b) = (a)$ and either b or c are units. □

Theorem. (6.35) Every PID is a UFD.

Proof. Let R be a PID. We make the following observation: let a be a non-zero and non-unit element of R. If a is not irreducible, then there exists an irreducible element $q \in R$ such that $q|a$. By lemma, (a) is not maximal. That is, we can find $(a) \subsetneq (a_1) \subsetneq R$. If (a_1) is maximal, then a_1 is irreducible by lemma. Otherwise, we can find $(a_1) \subsetneq (a_2) \subsetneq R$.

For existence, let a be a nonzero nonunit element of R. If a is irreducible, then we're done. Otherwise, we can find an irreducible factor q_1 of a by observation, so that $a = q_1 a_1$. Note that a_1 is nonzero and nonunit (in particular, $(a) \subsetneq (a_1) \subsetneq R$). If a_1 is irreducible, then we're done. Otherwise, we can find an irreducible factor q_2, so that

$$a_1 = q_2 a_2$$

Try uniqueness on your own. □

Remark. (6.39) Not every UFD is a PID. For example, $\boldsymbol{Z}[x]$ and $\boldsymbol{C}[x, y]$.

$$\text{ED} \subset \text{PID} \subset \text{UFD} \subset \text{ID}$$

16 Lecture 16 (May 17th)

Chapter 6.5 The Field of Fractions of an Integral Domain

Remark. In this chapter, we try to generalise the construction of the field of integers $\boldsymbol{Q}$ of rational numbers from the ring $\boldsymbol{Z}$ of integers.

Lemma. (6.42) Let R be an integral domain. Let $\hat{F}$ be the set defined as follows:

$$\{(n, d) \in R \times R \,|\, d \neq 0\}$$

We define the relation on $\hat{F}$ as follows:

$$(n_1, d_1) \sim (n_2, d_2) \iff n_1 d_2 = n_2 d_1$$

The relation $\sim$ on $\hat{F}$ is an equivalence relation.

Proof. We prove reflexivity, symmetry, and transistivity.

(i) (Reflexivity) $(r, s) \sim (r, s)$

(ii) (Symmetry) $(r_1, s_1) \sim (r_2, s_2)$ implies that $(r_2, s_2) \sim (r_1, s_1)$

(iii) (Transistivity) $(r_1, s_1) \sim (r_2, s_2)$ and $(r_2, s_2) \sim (r_3, s_3) \implies (r_1, s_2) \sim (r_3, s_3)$. For transistivity, we want to show that $r_1 s_3 = r_3 s_1$. Note that

$$\begin{aligned}(r_1 s_3)s_2 =& r_1(s_3 s_2) = r_1(s_2 s_3) = r_1(s_2 s_3) = (r_1 s_2)s_3 = (r_2 s_1)s_3 \\ =& r_2(s_1 s_3) = r_2(s_3 s_1) = (r_2 s_3)s_1 = (r_3 s_2)s_1 = r_3(s_1 s_2) = (r_3 s_1)s_2\end{aligned}$$

Since s_2 is nonzero and R is an integral domain, $r_1 s_2 = r_3 s_1$.

□

Definition. Let R be an integral domain. The field of fractions of R is the quotient $F = \hat{F}/\sim$ endowed with the following addition and multiplication.

$$\overline{(r_1, s_1)} + \overline{(r_2, s_2)} = \overline{(r_1 s_2 + r_2 s_1, s_1 s_2)}$$

and

$$\overline{(r_1, s_1)} \cdot \overline{(r_2, s_2)} = \overline{(r_1 r_2, s_1 s_2)}$$

Remark. The addition and multiplication are well-defined! In other words, $\overline{(r_1, s_1)} = \overline{(r_1', s_1')}$ and $\overline{(r_2, s_2)} = \overline{(r_2', s_2')}$ implies that $\overline{(r_1 r_2, s_1 s_2)} = \overline{(r_1' r_2', s_1' s_2')}$. We first do multiplication. Suppose that $(r_1, s_1) \sim (r_1', s_1')$ and $(r_2, s_2) \sim (r_2', s_2')$. We wish to show that $(r_1 r_2, s_1 s_2) \sim (r_1' r_2', s_1' s_2')$. We see that

$$\begin{aligned}(r_1 r_2)(s_1' s_2') &= (r_1 s_1')(r_2 s_2') \\ &= (r_1' s_1)(r_2 s_2') \\ &= (r_1' s_1)(r_2' s_2) \\ &= (r_1' r_2')(s_1 s_2)\end{aligned}$$

Proposition. $(F, +, \cdot)$ forms a field. For the proof, we need to verify the following:

(i) $(F, +, \cdot)$ forms a ring

(ii) The multiplcation on F is commutative

(iii) $(F, +, \cdot)$ is nontrivial

(iv) If $\overline{(r, s)}$ is nonzero in F, it has a multiplicative inverse.

Proof. For (ii), we see that

$$\overline{(r_1, s_1)} \cdot \overline{(r_2, s_2)} = \overline{(r_1 r_2, s_1 s_2)} = \overline{(r_2 r_1, s_2 s_1)} = \overline{(r_2, s_2)} \cdot \overline{(r_1, s_1)}$$

For (iii), we claim that $\overline{(0, 1)} \neq \overline{(1, 1)}$. Suppose not. Then $0 \cdot 1 = 1 \cdot 1$ which is a contradiction because R is an integral domain.

For (iv), assume that $\overline{(r, s)} \neq \overline{(0, 1)}$ in F. Then $r \neq 0$ and $\overline{(s, r)}$ is a multiplcative inverse of $\overline{(r, s)}$. Indeed,

$$\overline{(r, s)} \cdot \overline{(s, r)} = \overline{(s, r)} \cdot \overline{(r, s)} = \overline{(sr, rs)} = \overline{(rs, sr)} = \overline{(1, 1)}$$

□

Remark. (i) Just like there is an injective ring homomorphism $\boldsymbol{Z} \to \boldsymbol{Q}$ by sending n to $n/1$, there exists an injective ring homomorphism

$$i : R \to F : r \mapsto \overline{(r, 1)}$$

Where (1) i is a ring homomorphism, and (2) we have

$$\begin{aligned} r \in \ker i &\iff \overline{(r,1)} = \overline{(0,1)} \\ &\iff r \cdot 1 = 0 \cdot 1 \end{aligned}$$

in R.

Theorem. (6.45) Let R be an integral domain. Let F be the field of fractions of R and let $i : R \to F$ denote the ring homomorphism given by $r \mapsto \overline{(r,1)}$. Let K be a field and let $i_K : R \to K$ be an injective ring homomorphism. Then there exists a unique injective ring homomorphism $j : F \to K$ such that $i_K = j \circ i$.

Proof. We claim that there exists a unique injective ring homomorphism j such that $j \circ i = i_K$. For existence, define $j : F \to K$ to be defined as $j(\overline{(r,s)}) = i_K(r) \cdot (i_K(s))^{-1}$. Then

$$\begin{aligned} j(\overline{(r,s)}) =& j(\overline{(r,1)} \cdot \overline{(1,s)}) \\ =& j(\overline{(r,1)}) \cdot j(\overline{(1,s)}) \\ =& j(i(r)) \cdot j(\overline{(s,1)}^{-1}) \\ =& j(i(r)) \cdot j(i(s)^{-1}) \\ =& (j \circ i)(r) \cdot ((j \circ i)(s))^{-1} \\ =& i_K(r) \cdot (i_K(s))^{-1} \end{aligned}$$

The remaining task is to verify that j is well defined, that it is a ring homomorphism, j is injective, and that $j \circ i = i_K$.

For uniqueness, suppose that $i'_K : F \to K$ is another injective ring homomorphism with $j' \circ = i_K$. We need to show that $j = j'$. □

17 Lecture 17 (May 8th)

Recall. (5.57) (The 3rd Isomorphism Theorem) Let R be a ring and $I \subset J$ be ideals of R. Then,

$$R/I \Big/ J/I \cong R/J$$

Example. (5.38) Observe that

$$\boldsymbol{Z}[x] \,/\, (2,x) \cong \boldsymbol{Z} \,/\, 2\boldsymbol{Z} \quad \text{as} \quad \boldsymbol{Z}[x] \,/\, (2,x) \cong \boldsymbol{Z}[x] \,/\, (x) \Big/ (2,x) \,/\, (x)$$

Example. Here are some points that are worth considering.

(1) Let R be a commutative ring. R is an integral domain if and only if 0 is a prime ideal.

(2-1) $0 \neq (p) \subset \boldsymbol{Z}$ is a prime ideal if and only if p is a prime number.

(2-2) $0 \subset \boldsymbol{Z}$ is a prime ideal.

(3-1) Is $(x) \subset \boldsymbol{Z}[x]$ a prime ideal? Well, $(a) \neq \boldsymbol{Z}[x]$ and $(x) \mid f(x) \cdot g(x)$ implies that $(x) \mid f(x)$ or $(x) \mid g(x)$. An equivalent question would be whether $\boldsymbol{Z}[x] / (x)$ is an integral domain. By the 1st isomorphism theorem, it is isomorphic to $\boldsymbol{Z}$ and this is true.

(3-2) Since $\boldsymbol{Z}[x] / (x)$ is not a field, (x) is not a maximal ideal of $\boldsymbol{Z}[x]$. Notice that there exists $(x, 2)$.

(4) $(y - x^2) \subset \boldsymbol{C}[x, y]$ is prime. Notice that the following is an integral domain

$$\boldsymbol{C}[x, y] / (y - x^2) \cong \boldsymbol{C}[x]$$

(5) Let $I \subset \boldsymbol{Z}$ be an ideal. I is maximal if and only if I is a nonzero prime ideal.

(6) Let K be a field. Then (x) is a maximal ideal in $K[x]$. However, (x) is not maximal in $K[x, y]$, as

$$(x) \subsetneq (x, y) \subsetneq K[x, y]$$

Note that $(x, y) \subset K[x, y]$ is maximal.

18 Lecture 18 (May 13th)

Remark. Last time, we learned about prime and maximal ideals. Today, we will be learning some basics on polynomials.

Chapter 7 Polynomial Rings and Factorization

Chapter 7.2 The Polynomial Rings with Coefficients in a Field

Definition. (7.1) Let R be a ring. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ be a nonzero polynomial with $a_n \neq 0$. Then the integer n is called the degree of f; a_n is the leading coefficient of f; a_0 is the constant term of f.

Theorem. (7.2) Let K be a field. Then $K[x]$ is a Euclidean domain.

Proposition. (7.3) Let R be a ring and let $f(x)$ and $g(x)$ be elements of $R[x]$ with $g(x) \neq 0$. Suppose that the leading coefficient of $g(x)$ is a unit. Then there exists $q(x)$ and $r(x)$ in $R[x]$ such that $f(x) = g(x) \cdot q(x) + r(x)$, where $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Proof. The proof is omitted for now. Loosely speaking, one we set the minimum order n for $f(x)$ such that the division algorithm doesn't work, we find that it doesn't work for $n-1$ either, which is a contradiction. □

Example. (7.5) R is a ring, $f(x) \in R[x]$. Then $(x-a)$ is a factor of $f(x)$ if and only if $f(a) = 0$. For the if direction, write $f(x) = (x-a) \cdot q(x) + r(x)$. Then $r(x)$ must be 0.

Definition. (7.6) Let R be a ring and $f(x) \in R[x]$ be a polynomial. We will say that $a \in R$ is a zero of $f(x)$ if $f(a) = 0$.

Corollary. (7.7) If K is a field, then $K[x]$ is a PID.

Corollary. (7.8) If K is a field, then $K[x]$ is a UFD.

Proposition. (7.9) Let R be an integral domain. Let $f(x)$ and $g(x)$ be polynomials over $R[x]$ with $g(x) \neq 0$. Suppose that the leading coefficient of $g(x)$ is a unit. Then the quotient $q(x)$ and the remainder $r(x)$ are unique.

Proof. See how

$$g_1(x) - g_2(x) = (q_1(x) - q_2(x))f(x) + (r_1(x) - r_2(x))$$

If $(r_1(x) - r_2(x)) \neq 0$, we find that neither of the left terms are zero and as $\deg f(x) > \deg r(x)$ the statement is a contradiction. □

Theorem. (7.11) Let K be a field and let $f(x)$ be a nonzero polynomial of degree n. Then $K[x]\,/\,(f(x))$ is an n-dimensional vector space over K.

Proof. We briefly sketch a proof. First note how

$$a \in K,\ \overline{g(x)} \in K[x]\,/\,(f(x)) \implies a \cdot \overline{g(x)} = \overline{a \cdot g(x)}$$

and that $K[x]\,/\,(f(x))$ is closed under scalar multiplication. We further note how the subset $\{\overline{x^0}, \overline{x^1}, \overline{x^2}, \ldots, \overline{x^{n-1}}\} \subset K[x]\,/\,(f(x))$ forms a basis for the K-vector space $K[x]\,/\,(f(x))$. See that for any $\overline{g(x)} \in K[x]\,/\,(f(x))$,

$$\overline{g(x)} = \overline{f(x)} \cdot \overline{q(x)} + \overline{r(x)} = \overline{r(x)} = \overline{b_0} \cdot \overline{1} + \ldots + \overline{b_{n-1}} \cdot \overline{x^{n-1}}$$

and that a polynomial can be expressed in terms of the basis. □

19 Lecture 19 (May 15th)

Last time we have learned some definitions about polynomials. Today, we will learn about irreducible polynomials.

Chapter 7.3 Irreducibility in Polynomial Rings

Example. (7.13) If K is a field and a is nonzero, then $ax + b$ is irreducible. Note that, first of all, it is a non-zero non-unit element ($ax + b \notin (K[x])^\times = K^\times = K$). Secondly, if $ax + b = f(x) \cdot g(x)$, we observe that $f(x)$ or $g(x)$ is in K, which would make it a unit.

Remark. We make two short remarks. Let R be an integral domain.

(i) $f(x), g(x) \in R[x]$ implies that $\deg(f(x) \cdot g(x)) = \deg f(x) + \deg g(x)$

(ii) $(R[x])^\times = R^\times$

Remark. The condition that K is a field is essential. Note how $2x \in \boldsymbol{Z}[x]$ is not irreducible (it is reducible) as it can be expressed as $2 \cdot x$ with both 2 and x being non-unit non-zero elements.

Proposition. (7.14) Let K be a field and let $f(x) \in K[x]$ be a polynomial of degree at least 2 ($\deg f(x) \geq 2$). If $f(x)$ has a zero, then $f(x)$ is reducible.

Proof. Assume that $f(a) = 0$ for some $a \in K$. We can write $f(x) = (x - a)g(x)$, where $\deg g(x) \geq 1$ ($\deg f(x) = \deg(x - a) + \deg g(x)$). $g(x)$ is not a unit and $(x - a)$ isn't either. Therefore, $f(x)$ is reducible. The converse isn't true, with examples such as $(x^2 + 1)^2 \in \boldsymbol{R}[x]$. □

Proposition. (7.15) Let K be a field and let $f(x) \in K[x]$ be a polynomial of degree 2 or 3. Then $f(x)$ is irreducible if and only if $f(x)$ has no zeros in K.

Proof. Thanks to the above proposition, it suffices to prove that if $f(x)$ has no zeros, $f(x)$ is irreducible. We prove by contraposition, that if $f(x)$ is reducible (it is trivially non-zero and non-unit), $f(x)$ has zeros in K. As $f(x)$ is reducible

$$f(x) = g(x) \cdot h(x)$$

for non-unit $g(x)$ and $h(x)$. Then, the degree of $g(x)$ and $h(x)$ are atleast 1, telling us that either one of them has to be 1 due to the fact that

$$\deg f(x) = \deg g(x) + \deg h(x)$$

Now, as $f(x)$ contains a linear factor $(ax + b)$ with $a \neq 0$, $f(x)$ has a zero. □

Remark. (7.16)

(i) The fact that the ring of choice is a field is critical. For example, $(2x-1)^2 \in \boldsymbol{Z}[x]$ is reducible, but has no zero in $\boldsymbol{Z}$.

(ii) We can't extend the argument to higher degrees. Take, for example, $x^4 - 4 = (x^2+2)(x^2-2) \in \boldsymbol{Q}[x]$. The polynomial is reducible and has no zeros in $\boldsymbol{Q}$.

Theorem. (7.17) (Fundamental theorem of algebra) Every non-constant polynomial over $\boldsymbol{C}$ has a zero in $\boldsymbol{C}$.

Corollary. A polynomial $f(x) \in \boldsymbol{C}[x]$ is irreducible if and only if its $\deg f(x) = 1$.

Proposition. (7.20) Let $f(x) \in \boldsymbol{R}[x]$ be a polynomial. Then $f(x)$ is irreducible if and only if

(i) $\deg f(x) = 1$

(ii) $\deg f(x) = 2$ and $f(x) = ax^2 + bx + c$ with $b^2 - 4ac < 0$

Proof. The if part should be trivial. We prove the only if part. For degrees higher than 3, Let $f(x) = c(x-a_1)(x-a_2)\dots(x-a_n) \in \boldsymbol{C}[x]$ and $f(x) \in \boldsymbol{R}[x] \subset \boldsymbol{C}[x]$. From here, if $a_i \in \boldsymbol{C}$, $f(a_i) = 0$ implies that $f(\bar{a}_i) = 0$. With this knowledge, $(x-a_i)(x-\bar{a}_i)$ is a polynomial with real coefficients that divides $f(x)$. □

20 Lecture 20 (May 20th)

Last time, we have learned about irreducible polynomials over $\boldsymbol{C}$ and $\boldsymbol{R}$. Today, we'll do $\boldsymbol{Q}$ and $\boldsymbol{Z}$.

Chapter 7.4 Irreducibility in $\boldsymbol{Q}[x]$ and $\boldsymbol{Z}[x]$

Definition. (6.16) Let R be an integral domain and let a and b be elements in R. We will say that $g \in R$ is a greatest common divisor of a and b if

(i) $(a) + (b) \subset (g)$

(ii) If $(a) + (b) \subset (g')$ for $g' \in R$, $(g) \subset (g')$

Definition. (7.25) Let R be a UFD and $f(x)$ be a nonzero polynomial over R. Then, the content of $f(x)$ denoted $\text{cont}(f)$ is defined as the greatest common divisor of its coefficients. We say that $f(x) \in R[x]$ is primitive if its content is 1.

Remark. Let R be a UFD and $f(x) \neq 0$ be in $R[x]$. Then we can write $f(x) = \text{cont}(f) \cdot f_1(x)$ where $f_1(x) \in R[x]$ is primative.

21 Lecture 21 (May 22nd)

Today, we will learn about Gauss' lemma.

Remark. In a UFD R, if $p \in R$ is irreducible, p is prime.

Theorem. (Gauss' lemma) Let R be a UFD, and let $f(x)$ and $g(x)$ be nonzero polynomials over R. Then

$$\text{cont}(fg) = \text{cont}(f) \cdot \text{cont}(g)$$

In particular, if f and g are primitive, then so is $f \cdot g$.

Proof. It will suffice to show that if $f(x)$ and $g(x)$ are primative, then so are $f(x) \cdot g(x)$. Then, the theorem follows automatically from the fact that

$$\text{cont}\left(\text{cont}(f)f_1\right) = \text{cont}(f)\,\text{cont}(f_1)$$

Suppose that $f(x)g(x)$ is not primative. Then there exists an irreducible p in R such that the image of $f(x)g(x)$ under the ring homomorphism

$$\phi : R[x] \to \Big(R/(p)\Big)[x]$$

is zero. That is,

$$\overline{f(x) \cdot g(x)} = 0$$

This tells us that, as $R/(p)$ is an integral domain, one of the polynomials are zero. However, this implies that one of the polynomials has a coefficient that isn't 1, that is, one of the polynomials are not primative. This is a contradiction. □

Corollary. (7.26) Let R be a UFD and $f(x) \in R[x]$ be primative. Then $f(x)$ is irreducible in $R[x]$ if and only if $f(x)$ is irreducible in $Q[x]$, where $Q = \text{Frac}(R)$.

Corollary. (7.27) If R is a UFD, then $R[x]$ is also a UFD.

Remark. If R is a UFD, $R[x_1, \ldots, x_n]$ is also a UFD.

Remark. Let $f(x) \in Q[x]$ be a nonzero polynomial (R is a UFD and $Q = \text{Frac}(R)$). We can write

$$f(x) = \frac{a}{b} f_1(x)$$

where a and b are nonzero elements in R and $f_1(x) \in R[x]$ is primative polynomial over R. Remark that the irreducibility of Q is equivalent to the irreducibility of $f_1(x)$ over Q.

22 Lecture 22 (May 27th)

Last time, we have learned Gauss' lemma. Today, we will be learning irreducibility tests.

Recall. (Gauss' lemma) Let R be a UFD. For $f(x), g(x) \in R[x]$,

$$\text{cont}(f \cdot g) = \text{cont}(f) \cdot \text{cont}(g)$$

Corollary. (7.26) Let R be a UFD, $Q = \text{Frac}(R)$ and $f(x) \in R[x]$ be a primative polynomial. $f(x)$ is irreducible in $R[x]$ if and only if $f(x)$ is irreducible in $Q[x]$.

Proof. We do the only if part first. Suppose that $f(x) = g(x)h(x)$ in $Q[x]$. Write for $f(x) = g(x)h(x)$ for $g(x), h(x) \in Q[x]$. Write $g(x) = (a/b)g_1(x)$ and $h(x) = (c/d)h_1(x)$ where $a, b, c, d \in R \setminus \{0\}$, and $g_1(x)$ and $h_1(x)$ are primatives in $R[x]$. Then $bd \cdot f(x) = ac \cdot g_1(x)h_1(x)$. □

Corollary. (7.27) If R is a UFD, so is $R[x_1, x_2, \ldots, x_n]$.

Example. (i) $\boldsymbol{Z}[x_1, \ldots, x_n]$ is a UFD.

(ii) If K is a field, $K[x_1, \ldots, x_n]$ is a UFD.

Theorem. (7.22) For a UFD R, let $f(x) \in R[x]$. If $f(x) = g(x) \cdot h(x)$ in $Q[x]$, there exists $a, b \in Q$ such that $f(x) = (a \cdot g(x))(b \cdot h(x))$ where $a \cdot g(x)$ and $b \cdot h(x)$ are in $R[x]$. By Gauss' lemma, $u \cdot bd = ac$ for some $u \in R^\times$. Then $f(x) = u \cdot g_1(x) \cdot h_1(x)$. Since $f(x) \in R[x]$ is irreducible, either $g_1(x)$ and $h_1(x)$ is in $R^\times$. Consequently, either

$$g(x) = \frac{a}{b} g_1(x) \in Q^\times \quad \text{or} \quad h(x) = \frac{c}{d} h_1(x) \in Q^\times$$

Example. $R = \boldsymbol{Z}$ and $Q = \boldsymbol{Q}$. Test the irreducibility of

$$f(x) = \frac{4}{3} - \frac{6}{5}x^7$$

Chapter 7.4 Irreducibility Tests in $R[x]$ Where R is a UFD

Proposition. (7.29) (The rational root test) Let R be a UFD. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in R[x]$ be a polynomial of degree n. Let $c = p/q \in \boldsymbol{Q}$ be a zero of $f(x)$, where $p, q \in R$ with $(p, q) = 1$. Then $p | a_0$ and $q | a_n$.

Proof. Notice that

$$0 = f\Big(\frac{p}{q}\Big) = a_0 + a_1\Big(\frac{p}{q}\Big) + \ldots + a_n\Big(\frac{p}{q}\Big)^n$$

and multiplying q^n, we have

$$0 = a_0 q^d + (a_1 p q^{n-1} + \ldots + a_n p^n)$$

carefully examining the expression, we find that p divides a_0 and q divides a_n. □

Example. (7.30, 31, 32)

(i) $f(x) = 4x^3 + 2x^2 - 5x + 3$ is irreducible in $\boldsymbol{Q}[x]$. The possible roots are

$$\pm\Big(1, 3, \frac{1}{2}, \frac{3}{2}, \frac{1}{4}, \frac{3}{4}\Big)$$

however, none of these is a zero of $f(x)$.

(ii) $g(x) = 3 - 4x + 2x^2 + 4x^3$ is a reducible polynomial. This is bceause $g(-3/2) = 0$.

(iii) $h(x) = 1 - 3x^2 + x^4$ has no rational roots. It does not follow that $h(x)$ is irreducible. In fact, $h(x) = (-1 + x + x^2)(-1 - x + x^2)$, so $h(x)$ is reducible in $\boldsymbol{Q}[x]$.

Theorem. (7.35) (Eisenstein's criterion) Let R be a UFD and let $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in R[x]$ be a polynomial of degree n. Let $p \in R$ be an irreducible element. Suppose that

(i) $p \nmid a_n$

(ii) $p | a_0, a_1, \ldots, a_{n-1}$

(iii) $p^2 \nmid a_0$

Then $f(x)$ cannot be written as a product of two polynomials of positive degrees, and hence it is irreducible in $Q[x]$.

Example. Let $f(x) = 5 + 10x + x^4 \in \boldsymbol{Z}[x] \subset \boldsymbol{Q}[x]$. Take $p = 5$ which is a prime number in $\boldsymbol{Z}$. $5 \nmid 1$, $5 | 5, 10$ and $5^2 = 25 \nmid 5$. Altogether, they imply that $f(x)$ is irreducible in $\boldsymbol{Q}[x]$.

23 Lecture 23 (May 29th)

Last time, we learned about irreducibility tests including Eisenstein's criterion. Today we learn about some applications of the criterion and group theory.

Theorem. (7.35) (Eisenstein's criterion) Take R to be a UFD. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in R[x]$ be a polynomial of degree n. Let $p \in R$ be irreducible. If

(i) $p \nmid a_n$

(ii) $p \mid a_0, a_1, \ldots, a_{n-1}$

(iii) $p^2 \nmid a_0$

then $f(x)$ is irreducible in $Q[x]$.

Example. (i) $f(x) = x^4 + 10x + 5 \in \boldsymbol{Z}[x]$. By taking $p = 5$, we notice that $f(x)$ is irreducible over $\boldsymbol{Q}[x]$.

(ii) $f(x) = x^4 + 1 \in \boldsymbol{Z}[x]$. The function is irreducible if and only if $f(x+1) \in \boldsymbol{Z}[x]$ is irreducible. That is,

$$f(x+1) = x^4 + 4x^3 + 6x^2 + 4x + 2$$

is irreducible. Taking $p = 2$, we notice that $f(x)$ is irreducible over $\boldsymbol{Q}[x]$.

Corollary. (7.38) There are irreducible polynomials in $\boldsymbol{Q}[x]$ of arbitrary high degree.

Proof. For each $n \geq 1$, consider $x^n - 2 \in \boldsymbol{Q}[x]$ □

Example. (7.38) Let $p \in \boldsymbol{Z}$ be a prime number. Then the cyclotomic polynomial

$$\Phi_p(x) = 1 + x + x^2 + \ldots + x^{p-1}$$

is irreducible in $\boldsymbol{Q}[x]$. For this, we note that $\Phi_p(x)$ is rreducible over $\boldsymbol{Q}$ if and only if $\Phi_p(x+1)$ is irreducible over $\boldsymbol{Q}$. Since

$$\Phi_p(x) = \frac{x^p - 1}{x - 1}$$

we have

$$\Phi_p(x+1) = \frac{(x+1)^p - 1}{x} = \sum_{i=1} {}^pC_i x^{i-1} x^{i-1} = {}^pC_1 + {}^pC_2 x^2 + \ldots + {}^pC_p x^{p-1}$$

Using the fact that $p \mid {}^pC_i$ for $1 \leq i \leq p-1$, we can apply Eisenstein's criterion to deduce that $\Phi_p(x+1) \in \boldsymbol{Q}[x]$ is irreducible.

Remark. Read proposition (7.33) and example (7.34).

Chapter 11 Groups-Preliminaries

Chapter 11.1 Groups nad their Categories

Definition. (11.1) A group consists of a set G along with a binary operation on G such that

(i) (Associativity) $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ for all $g_1, g_2, g_3 \in G$

(ii) (Identity element) There exists $e \in G$ such that $g * e = e * g = g$ for all $g \in G$

(iii) (Inverse element) For each $g_1 \in G$, there exists $g_2 \in G$ such that $g_1 * g_2 = e$

Definition. For group G, we say that G is abelian if

$$g_1 * g_2 = g_2 * g_1$$

for all $g_1, g_2 \in G$.

Example. If $(R, +, \cdot)$ is a ring, then $(R, +)$ is an abelian group.

Remark. $(R, \cdot)$ is not necessarily a group.

Remark. (i) The identity e is unique

(ii) For each $g \in G$, the inverse of g is unique

(iii) $(g^{-1})^{-1} = g$

(iv) (Cancellation holds in groups) If $g_1 * h = g_2 * h$ or $h * g_1 = h * g_2$, then $g_1 = g_2$

Remark. In the future, we will write

$$g * h \to gh$$

and

$$g * g * g \to g^3$$

24 Lecture 24 (June 5th)

Last time, we learned the basics of group theory. Today, we'll learn basic properties of groups.

Remark. A group is called abelian if $g * h = h * g$.

(i) (Uniqueness of the identity element) e is unique

(ii) (Uniqueness of inverses) g^{-1} is unique

(iii) $(g^{-1})^{-1} = g$

(iv) (Cancellation law) $g_1 * h = g_2 * h$ implies that $g_1 = g_2$

(v) We use multiplicative notation

(vi) We use additive notation for abelian groups

Definition. (11.2) Let G and H be groups. A function $\phi : G \to H$ is a group homomorphism if it satisfies

$$\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$$

for all g_1 and g_2 in G.

Example. (The trivial group homomorphism) The function $G \to H : g \mapsto e$ is a group homomorphism, where 1 is the identity element.

Proposition. (11.3)

(i) The map $\mathrm{id} : G \to G$ is a group homomorphism

(ii) A composition of group homomorphisms is also a group homomorphism. That is, if $\phi : G \to H$ and $\psi : H \to K$ are group homomorphisms, then so is the composition $\psi \circ \phi : G \to K$ as

$$(\psi \circ \phi)(g_1 * g_2) = \psi(\phi(g_1) * \phi(g_2)) = \psi(\phi(g_1)) * \psi(\phi(g_2))$$

Definition. (11.5) Let $\phi : G \to H$ be a group homomorphism. We will say that ϕ is an isomorphism if there exists a group homomorphism $\psi : H \to G$ such that $\psi \circ \phi = \mathrm{id}_G$ and $\phi \circ \psi = \mathrm{id}_H$.

Remark. (i) ϕ is an isomorphism if and only if it is bijective

(ii) Being isomorphic is an equivalence relation

(iii) $\phi(e) = e$

Definition. Let A be a nonempty set. Then the pair

$$\Big(\text{the set of all bijections on } A,\ \text{their composition}\Big)$$

is called a symmetric group on A and is denoted S_A. Notationwise, if $|A| = n$, we will denote S_A by S_n.

Remark. If $|A| = |B|$, then S_A is isomorphic to S_B. Thus, we justify our notation from above.

Definition. (11.6) Let G be a group and H be a subset of G. We say that H is a subgroup of G if

(i) H is closed under the group operation

(ii) e is in H

(iii) For each $h \in H$, $h^{-1} \in H$

Example. (11.8, 11.9, 11.10)

(i) If H and K are subgroups of G, then $H \cap K$ is also a subgroup of G

(ii) Let $\phi : G \to G'$ be a group homomorphism. If H is a subgroup of G, then $\phi(H)$ is a subgroup of G'. In particular, $\mathrm{im}\,\phi = \phi(G)$ is a subgroup of G'. If H' is a subgroup

of G', then $\phi^{-1}(H')$ is a subgroup of G.

Definition. (11.49) Let $\phi : G \to H$ be a group homomorphism. Then the kernel of ϕ, denoted by $\ker \phi$, is the subgroup of $\phi^{-1}(\{1\})$.

Remark. If $\phi : R \to S$ is a ring homomorphism, then

$$\ker(\phi : (R, +, \cdot) \to (S, +, \cdot)) = \ker(\phi : (R, +) \to (S, +))$$

Example. (11.14, 11.15, 11.16)

(i) A singleton set is a group

(ii) If $(R, +, \cdot)$ is a ring, then $(R, +)$ is an abelian group

(iii) If V is a vector space, then $(V, +)$ is an abelian group

(iv) If R is a ring, then the set $R^\times$ of units in R is a group under multiplication. In particular, $GL_n(\boldsymbol{R}) = (M_{n\times n}(\boldsymbol{R}))^\times$ is a group with matrix multiplication

(v) The set of 2×2 real matrices whose determinants are 1 forms a subgroup of $GL_2(\boldsymbol{R})$

$$SL_2(\boldsymbol{R})$$

Lecture 25 (June 10th)

Last time, we went over some properties of groups. Today, we learn about the symmetric group S_n, normal subgroups $H \triangleleft G$, and cyclic groups. Next time, we will learn about the classification of finite abelian groups.

Example. (i) The general linear group over $\boldsymbol{R}$

$$\mathrm{GL}_n(\boldsymbol{R}) = \{M \in M_{n\times n}(\boldsymbol{R}) \mid \det M \neq 0\}$$

(ii) The special linear group over $\boldsymbol{R}$

$$\mathrm{SL}_2(\boldsymbol{R}) = \{M \in M_{2\times 2}(\boldsymbol{R}) \mid \det M = 1\}$$

(iii) The symmetric group

$$S_n = (\sigma, \circ)$$

where $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a bijection.

Definition. (11.32) (Dihedral group D_{2n})

Definition. (11.41) For each $r \geq 2$, an r-cycle $(a_1, a_2, \ldots, a_r)$ is a permutation cyclically permuting r elements, a_1 through a_r. That is,

$$\begin{cases} a_i \mapsto a_{i+1} & 1 \leq i \leq r-1 \\ a_r \mapsto a_1 & i = r \end{cases}$$

A transposition, meanwhile, is a 2-cycle. Observe how $(123) = (13)(12)$.

Lemma. (11.45)

$$(a_1 \ldots a_r) = (a_1 a_r)(a_1 a_{r-1}) \ldots (a_1 a_2)$$

Remark. Two cycles are disjoint if their nontrivial orbits are disjoints.

Proposition. (11.44) Every nontrivial permutation can be written as product of disjoint nontrivial cycles.

Theorem. Every permutation can be written as a product of transpositions. Furthermore, the partiy of the number of transpositions in each decomposition is the same.

Example. There is no unique decompostion of a permutation.

$$\begin{aligned} \sigma =& (123)(67) \\ =& (13)(12)(67) \\ =& (13)(12)(67)(52)(52) \end{aligned}$$

Definition. The sign of a permutation is defined as

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if even} \\ -1 & \text{if odd} \end{cases}$$

as in the number of transpositions that construct σ. We will call σ even if $\operatorname{sgn}(\sigma) = 1$ and odd if $\operatorname{sgn}(\sigma) = -1$. This is called the permutation's parity.

Chapter 11.4

Definition. We can define an equivalence relation on G by saying, for a subgroup H,

$$a \sim b \iff a^{-1}b \in H$$

The following set is called a left coset

$$aH = \{ah \,|\, h \in H\} = \bar{a}$$

The set of equivalence classes can be denoted

$$G/H = \{aH \mid a \in G\} = \{\bar{a} \mid a \in H\}$$

We notice that the definition of an ideal of a ring is exactly the defintion of a coset where the ring addition is identified as the group operation.

Definition. (11.52) We will say that H is a normal subgroup if $aH = Ha$ for all $a \in G$.

Remark. (11.53) H is a normal in G if and only if $aHa^{-1} \subset H$ for all $a \in G$.

25 Lecture 26 (June 12th)

Last time, we learned about the symmetric group S_n and normality. Today, we learn about cyclic groups and the classification of finite abelian groups.

Definition. (11.52) A subgroup H of G is normal if $g * H = H * g$ for all $g \in G$.

Remark. (11.53) H is normal in G if and only if $a * H * a^{-1} \subset H$ for all $a \in G$.

Theorem. (11.64) H is normal in G if and only if $\overline{a} * \overline{b} = \overline{a * b}$ is well-defined. Moreover, H is normal if and only if $(G/H, \cdot)$ forms a group.

Definition. (11.65) If G is a group and $H \subset G$ is a normal subgroup, $(G/H, \cdot)$ is called the quotient group of G modulo H.

Remark. If G is abelian, every subgroup of G is normal.

Proposition. (11.58) If $\phi : G \to H$ is a group homomorphism, then

$$\ker \phi = \phi^{-1}(\{e_H\})$$

is normal in G.

Example. (11.60) Let G be a group. The center is defined as

$$Z(G) = \{z \in G \mid gz = zg\}$$

for all $g \in G$. This group is a subgroup and also normal in G.

Theorem. (Isomorphism theorems) These will be on you.

Chapter 10 Abelian Groups

Definition. (11.11) Let G be a group and $S \subset G$ be a subset. The subgroup of G generated by S, denoted $\langle S \rangle$, is the smallest subgroup of G containing S.

Remark. $\langle S \rangle$ exists and is unique.

Definition. (10.8) Let G be a group. We will say that G is cyclic if there exists $g \in G$ such that $G = \langle g \rangle$.

Remark. (i) $\langle g \rangle$ must contain $\{g^n \mid n \in \boldsymbol{Z}\}$ including negative powers. Notice that this is also a subgroup, and thus

$$\langle g \rangle = \{g^n \mid n \in \boldsymbol{Z}\}$$

(ii) If G is cyclic, any element can be expressed as g^n for some $n \in \boldsymbol{Z}$. Notice that then

$$g^m \cdot g^n = g^{m+n} = g^{n+m} = g^n \cdot g^m$$

and that G must be abelian.

Example. Consider the two central examples.

(i) $(\boldsymbol{Z}, +) = \langle 1 \rangle = \langle -1 \rangle$

(ii) $(\boldsymbol{Z}/n\boldsymbol{Z}, +) = \langle \pm\bar{1} \rangle$

Proposition. Let G be a cyclic group. Then

$$\begin{cases} G \cong \boldsymbol{Z} & \text{if} \quad |G| = \infty \\ G \cong \boldsymbol{Z}/n\boldsymbol{Z} & \text{if} \quad |G| = n < \infty \end{cases}$$

Proof. Choose a generator g so that $G = \langle g \rangle$. Consider a group homomorphism

$$\phi : \boldsymbol{Z} \to G : m \mapsto g^n$$

By the 1st isomorphism theorem, $\boldsymbol{Z}/\ker\phi$ is isomorphic to $\operatorname{im}\phi = G$. Note that $\ker\phi = k\boldsymbol{Z}$ for some $k \in \boldsymbol{Z}$.

$$\boldsymbol{Z}/k\boldsymbol{Z} \cong G$$

This k should be exactly equal to the cardinality of the group, therefore

$$\boldsymbol{Z}/n\boldsymbol{Z} \cong G$$

□

Proposition. (10.10)

(i) Every subgroup of a cylic group is cylic

(ii) If G is a cyclic group of order $n < \infty$, then G has exactly one cyclic subgroup of order m for each positive divisor m of n

Chapter 10.3 The Classification Theorem

Theorem. (10.18) (Classification of finite abelian groups) Let G be a finite abelian group. Then there exists integers $1 < d_s < d_{s-1} < \ldots < d_1$ with $d_s|d_{s-1}|\ldots|d_2|d_1$ such that G is isomorphic to

$$\boldsymbol{Z}/d_s\boldsymbol{Z} \times \boldsymbol{Z}/d_{s-1}\boldsymbol{Z} \times \ldots \times \boldsymbol{Z}/d_1\boldsymbol{Z}$$

Furthermore, this decomposition is unique.

Remark. (i) $d_s d_{s-1} \ldots d_1 = n = |G|$

(ii) If p is a prime divisor of n, then $p|d_1$

Example. Let $n = 180 = 2^3 \cdot 3^2 \cdot 5$.

G	d_1	d_2	d_3
$\mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	$2, 3, 5$	2	
$\mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$2^2, 3, 5$	3	
$\mathbb{Z}/90\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$2, 3^2, 5$	$2, 3$	3
$\mathbb{Z}/180\mathbb{Z}$	$2^2, 3^2, 5$	2	

Table 1. Decomposition of $\mathbb{Z}/180\mathbb{Z}$