Differential Equations in Electromagnetism

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I. INTEGRATING FORMS OVER SPACE I

In electrostatics (chapter 2), we often encounter integrals of vector-valued functions which we integrate over a particular space contextually representative of a charge distribution. In each problem, we thus chronologically ought to find (1) the particular vector function $\boldsymbol{F}$, (2) parameter(s) and their boundary conditions that sufficiently represent the space (charge distribution) we're integrating over, and (3) the anti-derivative of the integral that we'll substitute the boundary values to. Steps (1) and (2) together are at minimum at a similar level of difficulty compared to step (3), and thus require equally careful consideration.

II. INTEGRATING FORMS OVER SPACE II

Gauss's law, a statement about the divergence of the electric field, along with the *generalised Stokes' theorem*, paves a lee-way in deriving the *magnitude* of the electric field over a equipotential surface. Thus, for symmetric (spherical, cylindrical, plane) charge distributions where the *direction* of the field is intuitive, solutions are easily attainable.

The divergence behaviour of electric fields described above is the cornerstone of many significant lemmas, including the electronic field behaviour near sheets (and therefore how a potential's, typically exterior, normal derivative acts like around boundaries) and the work done in assembling a charge distribution (and accordingly electrostatic pressure). To add, the inconsistency between taking discrete summations for potential and continuous integrals over space lies in the fact that it creates redundancies with particles brought to point singularities created by its own charge.

III. SOLVING DIFFERENTIAL EQUATIONS OVER SPACE

The analysis in electromagnetism make a quick turn from differential forms to differential equations as the topic of *potential* arises. The usefulness of potential comes directly from Maxwell's equations, where knowledge of the curl and divergence behaviour of the differential form $\boldsymbol{E}$ converts the problem of solving for a differential form into a problem of solving for a scalar field V. Without any magnetic field present (therefore not requiring any knowledge of the *Helmholtz decomposition* or *Hodge decomposition*), the uniqueness of V is guaranteed by *Poincaré's lemma* as the electric field is continuous for all open spheres $U \subset \mathbb{R}^3 \setminus \{0\}$. The problem effectively reduces from solving a whole system of partial differential equations to solving a single second order partial differential equation (*Poisson's equation*).

With an absence of the non-homogeneous term accounting for charge density, Poisson's equation reduces to *Laplace's equation* (the entirety of chapter 3). To solve this differential equation, we enforce boundary conditions of the first and second type to guarantee its uniqueness (*Dirichlet* and *Neumann*). Here, the PDE-equivalent of endpoints and first derivatives are points and (exterior) normal derivatives. As the domain would be 3-dimensional ($M \subset \mathbb{R}^3$), the boundary would be a surface, denoted as ∂M.

Laplace's equation can be intuitively understood as describing the extent that a point in the domain of V is a local extremum (it should be neither). For more formality, one can refer to the *maximum principle* and the *mean value property* of *harmonic functions*.

We are introduced to largely two ways of solving Laplace's equations: the *method of images* and *separation of variables*. The first method takes advantage of the uniqueness of the field, largely replying on intuition. The second method directly solves Laplace's equation using multiplicative separation, starting with a product of functions with single component dependencies as an ansatz.

A particular theory that is used in determining the coefficients and constants of the ansatz are *Fourier series* (particularly the *Fourier sine series* given that we normally set our origin to lie on grounded a grounded boundary, more aptly approximated as a odd potential function) and *Fourier-Legendre series*. Notable information about these expansions are the intervals that they are *orthogonal* ($(\phi_m, \phi_n) = 0$) on and their *square norms* (also known as the *generalised length*). For the Fourier sine series, we normally consider half-range expansions, arbitrarily suggesting symmetry about the origin. For a Fourier sine series defined on the interval $(-a, a)$, the square norm is known to be a, whereas for a Fourier-Legendre series defined on the interval $(-1, 1)$, the square norm of Legendre polynomial $P_n(x)$ is known to be $2n/2n + 1$.

IV. REFERENCES

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