Bull. Sci. math. 170 (2021) 102998

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

The monodromy problem for hyperelliptic curves

Daniel López Garcia

Instituto de Matematica Pura e Aplicada (IMPA), Estrada Dona Castorina 110, Rio de Janeiro, 22460-320, RJ, Brazil

A R T I C L E I N F O

Article history:
Received 19 December 2020
Available online 26 May 2021

Keywords:
Foliations
Lefschetz fibrations
Monodromy action
Vanishing cycles

A B S T R A C T

We study the Dynkin diagrams associated to the monodromy of direct sums of polynomials. The *monodromy problem* asks under which conditions on a polynomial, the monodromy of a vanishing cycle generates the whole homology of a regular fiber. We consider the case $y^4 + g(x)$, which is a generalization of the results of Christopher and Mardešić about the monodromy problem for hyperelliptic curves. Moreover, we solve the monodromy problem for direct sums of fourth degree polynomials.

1. Introduction

The interest on polynomial foliations in $\mathbb{C}^2$ arises as an approach to the Hilbert Sixteen Problem [16,21]. These foliations are given by 1-forms $\omega = P(x,y)dy - Q(x,y)dx$, where P and Q are polynomials. The classical notation for the foliation associated to the form ω is $\mathcal{F}(\omega)$. In this context, a point $p \in \mathbb{C}^2$ is a singularity of $\mathcal{F}(\omega)$ if $P(p) = Q(p) = 0$. We say that the singularity p is a **center singularity** if there is a local chart such that p is mapped to $0 \in \mathbb{C}^2$, and a Morse function $f : (\mathbb{C}^2, 0) \to \mathbb{C}$ with fibers tangent to the leaves of $\mathcal{F}(\omega)$. The degree of a foliation $\mathcal{F}(\omega)$ is the greatest degree of the polynomials

E-mail address: daflopez@impa.br.

https://doi.org/10.1016/j.bulsci.2021.102998

P and Q, and the space of foliations of degree d is denoted by $\mathcal{F}(d)$. The closure of the set of foliations in $\mathcal{F}(d)$ with at least one center is denoted by $\mathcal{M}(d)$.

It is known that $\mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$ (e.g. [19, §6.1] [16]). The problem of describing its irreducible components is formulated by Lins Neto [20]. In [12], Y. Ilyashenko proves that the space of Hamiltonian foliation $\mathcal{F}(df)$, where f is a polynomial of degree $d+1$, is an irreducible component of $\mathcal{M}(d)$. In [17], H. Movasati considers the logarithmic foliations $\mathcal{F}\left(\sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}\right)$, with $f_i \in \mathbb{C}[x,y]_{\leq d_i}$ and $\lambda_i \in \mathbb{C}^*$. He proves that the set of logarithmic foliations form an irreducible component of $\mathcal{M}(d)$, where $d = \sum_{i=1}^{s} d_i - 1$. Moreover, in [23], Y. Zare works with pullback foliations $\mathcal{F}(P^*\omega)$, where $P = (R,S) : \mathbb{C}^2 \to \mathbb{C}^2$ is a generic morphism with $R, S \in \mathbb{C}[x,y]_{\leq d_1}$, and ω is a 1-form of degree d_2. Zare shows that they form an irreducible component of $\mathcal{M}(d_1(d_2+1)-1)$.

The main idea in the proofs of these assertions is to choose a particular polynomial f and consider deformations $df + \varepsilon\omega_1$ in $\mathcal{M}(d)$. Then, it is necessary to study the vanishing of the abelian integrals $\int_\delta \omega_1$, where δ is a homological 1-cycle in a regular fiber of f. This integral is zero on the vanishing cycle associated to the center singularity. If the monodromy action on this cycle generates the whole vector space $H_1(f^{-1}(b), \mathbb{Q})$ for a regular value b, then the deformation is relatively exact to df.

The condition that the vanishing of the integral $\int_\delta \omega_1$ implies that ω_1 is relatively exact to df, is known as the (*)-*property* (It was introduced by J.P. Françoise in [9]). The results of L. Gavrilov in [10], show that if we provided that the integral is zero over any cycle in a regular fiber, then ω_1 is relatively exact. Therefore, if the subspace generated by monodromy action on the vanishing cycle δ is the whole space $H_1(f^{-1}(b), \mathbb{Q})$, then the $(*) - $*property* is satisfied. This gives rise to the next natural problem, which is summarized by C. Christopher and P. Mardešić in [4] as follows.

Monodromy problem. Under which conditions on f is the $\mathbb{Q}$-subspace of $H_1(f^{-1}(b), \mathbb{Q})$ generated by the images of a vanishing cycle of a Morse point under monodromy equal to the whole of $H_1(f^{-1}(b), \mathbb{Q})$?

Furthermore, they show a characterization of the vanishing cycles associated to a Morse point in hyperelliptic curves given by $y^2 + g(x)$, depending on whether g is decomposable (Theorem 4.7). This case is closely related with the 0-dimensional monodromy problem; by using the definition of Abelian integrals of dimension zero in [11]. For example, if we think in the Dynkin diagram associated with $y^2 + g(x)$ and the one associated with $g(x)$, then we see that they coincide. However, the Dynkin diagram for $y^3 + g(x)$ is a bit more complicated. Moreover, in the case $y^4 + g(x)$ there is always a pullback associated to $y \to y^2$, thus the Dynkin diagram is expected to reflect this fact. For these two cases, we prove the following two theorems.

Theorem 1.1. *Let g be a polynomial with real critical points, and degree d such that $4 \nmid d$ and $d \leq 100$. Consider the polynomial $f(x,y) = y^4 + g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the homology $H_1(f^{-1}(t), \mathbb{Q})$.*
2. *The polynomial g is decomposable (i.e., $g = g_2 \circ g_1$), and $\pi_*\delta(t)$ is homotopic to zero in $\{y^4 + g_2(z) = t\}$, where $\pi(x, y) = (g_1(x), y) = (z, y)$. Or, the cycle $\pi_*\delta(t)$ is homotopic to zero in $\{z^2 + g(x) = t\}$, where $\pi(x, y) = (x, y^2) = (x, z)$.*

Theorem 1.2. *Let g be a polynomial with real critical points, and degree d such that, $3 \nmid d$ and $d \leq 100$. Consider the polynomial $f(x, y) = y^3 + g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the homology $H_1(f^{-1}(t), \mathbb{Q})$.*
2. *The polynomial g is decomposable (i.e., $f = g_2 \circ g_1$), and $\pi_*\delta(t)$ is homotopic to zero in $\{y^3 + g_2(z) = t\}$, where $\pi(x, y) = (g_1(x), y) = (z, y)$.*

Some parts in the proof are done numerically using computer, thus we have the restriction $d \leq 100$ in the degree of the polynomial g.

The monodromy problem for polynomials of degree 4, on the other hand, is very interesting, because the classification of the irreducible components of $\mathcal{M}(3)$ is still an open problem. In fact, the only case which has a complete classification is $\mathcal{M}(2)$ (see [8][3, p. 601]). For polynomials $f(x, y) = h(y) + g(x)$ where $\deg(h) = \deg(g) = 4$, we determine in the Theorem 5.4, a relation between the subspaces of $H_1(f^{-1}(b), \mathbb{Q})$ generated by the monodromy action on the vanishing cycles, and the property of f being decomposable. In oder to do that, we provide an explicit description of the space of parameters of the polynomials $h(y)+g(x)$ which satisfies some conditions in the critical values.

Organization. In section 2, we provide some definitions in Picard-Lefschetz theory and describe the Dynkin diagram for direct sum of polynomials in two variable. In section 3, we analyze the particular case, in which there is only one critical value. For this case, we compute the vector space generated by the monodromy action on the vanishing cycles. In section 4, we prove Theorems 1.1 and 1.2. Finally, in Section 5, we solve the monodromy problem for polynomials $h(x) + g(y)$ with $\deg(h) = \deg(g) = 4$; in this case there is another pullback to be considered, associated with the map $(x, y) \to (xy, x + y)$. For this reason, we do not know a geometrical characterization of some of vanishing cycles which do not generate the whole $H_1(f^{-1}(b), \mathbb{Q})$.

Acknowledgments. I thank Hossein Movasati for suggesting to study the action of monodromy and for introducing me to the center problem. I thank Lubomir Gavrilov for hosting me at the University of Toulouse during a short visit and for his helpful discussions. I thank the reviewer for pointing out that the examples in this paper are generally non-hyperelliptic curves. For example, the generic curve of genus 3, $\Gamma_t = \{y^4 + g(x) = t\}$ where $\deg(g) = 4$, is non-hyperelliptic curve. Namely, the canonical map

$$\Gamma_t \to \mathbb{P}^2$$

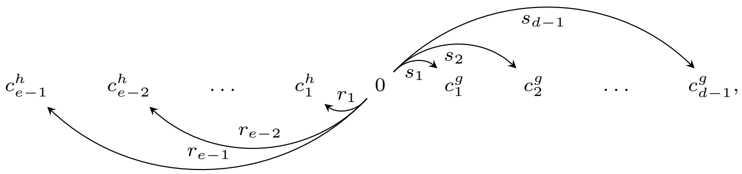

Fig. 1. A distinguished set of paths from b to the critical values of h and g.

$$[x : y : 1] \to \left[x\frac{dx}{y^3} : y\frac{dx}{y^3} : \frac{dx}{y^3}\right] = [x : y : 1]$$

is an embedding. Despite this, the title of the article refers to the case of hyperelliptic curves, since we were inspired in [4].

2. Lefschetz fibrations and monodromy action on direct sum of polynomials

Let $f \in \mathbb{C}[x, y]$ with the set of critical values C and a regular value b. Suppose that the origin is an isolated critical point of the highest-grade homogeneous piece of f. Hence, the Milnor number μ of f is finite, and there are **vanishing cycles** $\delta_1, \delta_2, \dots, \delta_\mu$ associated to the critical values, such that they generate freely the 1-homology of the fiber $X_b := f^{-1}(b)$, i.e. $H_1(X_b, \mathbb{Z}) = span\{\delta_i\}_{i=1}^{\mu}$ (see [2, Chs. 1,2], [18, §7.5], [13]). Moreover, there is an action $\pi_1(\mathbb{C} \setminus C) \times H_1(X_b) \xrightarrow{mon} H_1(X_b)$ called the **monodromy action** given by local trivialization of $f^{-1}(\gamma)$, where γ is any loop in $\mathbb{C} \setminus C$, see [18, §6.3]. A homological cycle in X_b such that its orbit by the monodromy action generates the whole homology group $H_1(X_b, \mathbb{Z})$ is called **simple cycle**. This definition of simple cycle was introduced in [16], and it is different from the definition of simple cycle used in [4,11].

Let $f(x, y) = h(y) + g(x)$ be a polynomial with real coefficients, such that the critical points of h and g are reals. By considering a deformation of f, we can suppose that f is a Morse function and all its critical values are different pairwise. Furthermore, by doing a translation we can suppose that the critical values of g are positive, the critical values of h are negatives, and $b = 0$. Let c^h_i and c^g_j be the critical values of h and g, respectively.

Consider the paths r_i and s_j, from 0 to c^h_i and c^g_j, respectively. Furthermore, the paths are without self-intersection, and they intersect each others only in 0. Also, $(s_1, s_2, \dots, s_{d-1}, r_1, r_2, \dots, r_{e-1})$ near 0 is the anticlockwise direction, as in Fig. 1. Moreover, we have chosen the enumeration of the paths such that $0 < c^g_1 < c^g_2 < \dots < c^g_{d-1}$ and $c^h_{e-1} < c^h_{e-2} < \dots < c^h_1 < 0$.

Let $C_g = \{(x, g(x)) \mid x \in \mathbb{R}\}$ be the real curve associated to g. For each $j = 1, \dots, d-1$, let p_j be the critical point associated to c^g_j, and we define a real number ε_j as follows. If p_j is a minimum, then we take $c^g_j < \varepsilon_j < c^g_{j+1}$, otherwise we take $c^g_{j-1} < \varepsilon_j < c^g_j$. Consider the real line $L_j = \{(x, \varepsilon_j) \mid x \in \mathbb{R}\}$, thus there are two points in $L_j \cap C_g$, such that, they go to (p_j, c^g_j) when ε_j goes to c^g_j. Thus, we define the 0-vanishing cycle $\sigma_j \in H_0(g^{-1}(\varepsilon_j), \mathbb{Z})$, as the formal sum of these two points, with coefficients 1 and -1.

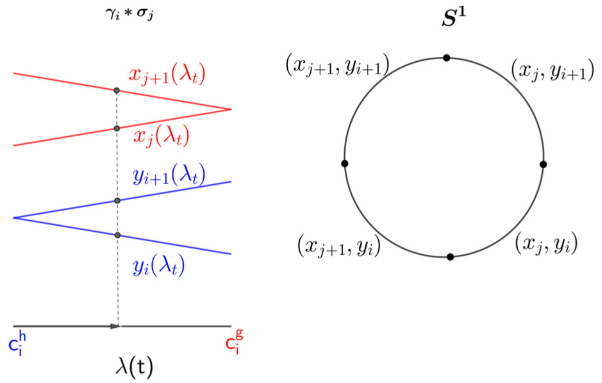

Fig. 2. Join cycle $\gamma_j * \sigma_i$ as S^1.

Note that these vanishing cycles are *simple cycles* according to the definition in [4,11], however, they are not simple cycles according to our definition.

By taking a simple path from 0 to ε_j, without encircling or passing through critical values, we can consider σ_j in $H_0(g^{-1}(0), \mathbb{Z})$. Note, that it is possible to choose a path from 0 to ε_j and compose it with the segment $\overline{\varepsilon_j c_j^g}$, such that this path is homotopic equivalent to s_j. Therefore, we can suppose that the cycle σ_j vanishes along the path s_j. Analogously, we define the 0-vanishing cycle $\gamma_i \in H_0(h^{-1}(0), \mathbb{Z})$ for the polynomial h.

Consider the path $\lambda = s_j r_i^{-1}$, starting in c_i^h and ending in c_j^g, as in [18, §7.9], we define the **join cycle**

$$\gamma_i * \sigma_j := \bigcup_{t\in[0,1]} \gamma_i(\lambda_t) \times \sigma_j(\lambda_t),$$

where $\gamma_i(\lambda_t) \in H_0(h^{-1}(\lambda_t), \mathbb{Z})$ and $\sigma_j(\lambda_t) \in H_0(g^{-1}(\lambda_t), \mathbb{Z})$. The join cycle $\gamma_i * \sigma_j$ is homeomorphic to a circle S^1, the Fig. 2 shows this construction. On the other hand, note that the join cycle $\gamma_i * \sigma_j$ is a vanishing cycle of the fibration given by f along the path $s_j r_i^{-1} + c_i^h$. Therefore, the join cycles generate the homology $H_1(f^{-1}(b), \mathbb{Z})$. Next, we compute the intersection of two join cycles. The local formula for the intersection form of two vanishing cycles is due to A.M. Gabrielov (see [2, Thm 2.11]). Its reproduction in the global context of tame polynomials is done in [18, §7.10]. Since the particular case of 1-dimension fibers is simple, we reproduce the proof in the next proposition.

Proposition 2.1. *Let $\gamma_i * \sigma_j$ and $\gamma_{i'} * \sigma_{j'}$ be two join cycles along the paths $\lambda := s_j r_i^{-1}$ and $\lambda' := s_{j'} r_{i'}^{-1}$, respectively. Then*

$$\langle \gamma_i * \sigma_j, \gamma_{i'} * \sigma_{j'} \rangle = \begin{cases} sgn(j'-j)\langle \sigma_j, \sigma_{j'} \rangle & \text{if} \quad i = i' \text{ and } j \neq j' \\ sgn(i'-i)\langle \gamma_i, \gamma_{i'} \rangle & \text{if} \quad j = j' \text{ and } i \neq i' \\ sgn(i'-i)\langle \gamma_i, \gamma_{i'} \rangle \langle \sigma_j, \sigma_{j'} \rangle & \text{if} \quad (i'-i)(j'-j) > 0 \\ 0 & \text{if} \quad (i'-i)(j'-j) < 0. \end{cases} \tag{2.1}$$

Proof. Suppose that the paths intersect each other transversally in b. The join cycle $\gamma_i * \sigma_j$ intersects $\gamma_{i'} * \sigma_{j'}$ at one point if the 0-cycles intersect each other, and at zero points otherwise. The orientation of the intersection of the join cycles is given by $d\lambda \wedge d\lambda'$ times the sign of the intersection of the 0-cycles. Moreover, we consider as positive orientation the canonical orientation of $\mathbb{C}$ given by $dz_1 \wedge dz_2$ where $z = z_1 + \sqrt{-1}dz_2$.

Suppose that $i = i'$, thus the path λ and λ' intersect transversally in positive direction if $j' > j$. The intersection of the 0-cycles only depends on the intersection $\langle \sigma_j, \sigma_{j'} \rangle$. Analogously, for $j = j'$. When $(i'-i)(j'-j) > 0$ the intersection of λ and λ' is transversal again, with positive direction if $i' > i$, and the intersection of the 0-cycles is $\langle \gamma_i, \gamma_{i'} \rangle \langle \sigma_j, \sigma_{j'} \rangle$. Finally, if $(i'-i)(j'-j) < 0$, the paths do not intersect transversally. Furthermore, after doing a homotopy we can suppose that the path λ and λ' do not have intersection points, therefore the intersection of the join cycles is zero. $\square$

Next, we present a combinatorial way of representing the intersection form, which is described in [2, §2.8] and [18, §7.10].

Definition 2.2. The **Dynkin diagram** of $f(x, y) = h(y) + g(y)$, is a directed graph where the vertices are the vanishing cycles in a regular fiber. The vertices v_i and v_j are joined by an edge with multiplicity $|\langle v_i, v_j \rangle|$. If $\langle v_i, v_j \rangle > 0$, then the direction goes from v_i to v_j.

We can also relate a vertex in a Dynkin diagram with the critical value associated to the vanishing cycle. In order to define the Dynkin diagram of the polynomial $f(x, y) = h(y) + g(x)$, we consider a deformation $\tilde{f}$ such that the critical values are different pairwise. Although the Dynkin diagram for f and $\tilde{f}$ are equals (the same vertices and edge), the Dynkin diagram associated to f has relations among the vertices according to the equalities of the critical values. From the previous choice of paths r_i and s_j, we have rules in the Dynkin diagram which establish the possibilities to relate the critical values:

1. The Dynkin diagram associated to $f(x, y) = h(y) + g(x)$, can be thought as a two-dimensional array, where the rows are the critical values $c_i^h + c_j^g$ for a fixed i and $j = 1, \dots, d-1$. Thus, if two critical values of $f(x, y)$ in the same row of the Dynkin diagram are equals, then for the columns of these critical values there are equalities in the rows. This is obviously because if $c_i^h + c_j^g = c_i^h + c_l^g$ then $c_j^g = c_l^g$, consequently $c_k^h + c_j^g = c_k^h + c_l^g$ for all $k = 1 \dots e-1$. This happens in an analogous way for the columns.
2. If $c_i^h + c_j^g = c_k^h + c_l^g$ and additionally $i < k$, $j > l$ then $c_i^h = c_k^h$ and $c_j^g = c_l^g$. This follows from the choice of distinguished paths because $i < k$ implies that $c_i^h \geq c_k^h$ and $j > l$ implies $c_j^h \geq c_l^h$ then $c_i^h + c_j^g \geq c_k^h + c_l^g$ since $c_i^h + c_j^g = c_k^h + c_l^g$ then the inequalities actually are equalities.

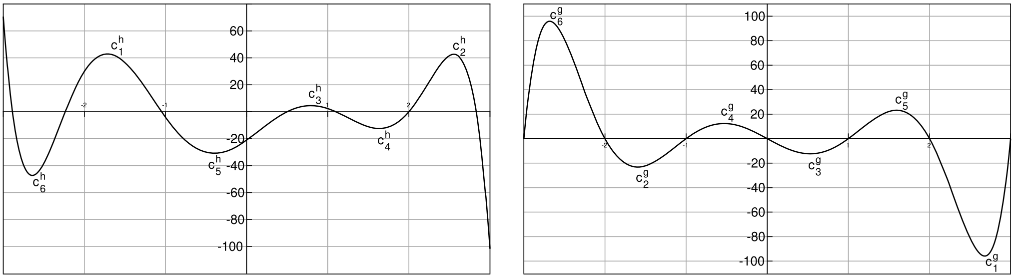

Fig. 3. Real part of the polynomials $h(y) = -(y + \frac{5\sqrt{3}}{3})(y + \sqrt{5})(y + \frac{\pi}{3})(y - \frac{1}{2})(y - \ln(3))(y - 2)(y - 2\sqrt{2})$ and $g(x) = (x + 3)(x + 2)(x + 1)x(x - 1)(x - 2)(x - 3)$, with its real critical values. On the left is $h(y)$ and on the right $g(x)$.

Similarly, a Dynkin diagram in dimension 0 consists of vertex which are the vanishing cycles associated to a polynomial, and dashed edges representing an intersection of -1, in this case the edges do not have direction. Note that the vanishing cycles associated to the critical values c^g_i with $i \leq [d/2]$ can only intersect vanishing cycles associated to critical values c^g_j with $j \geq [d/2]$, and similarly for the critical values c^h_k, for example, see the Fig. 3.

The 0-dimensional Dynkin diagrams associated to h and g of the Fig. 3 are:

$$\gamma_2 \dashrightarrow\!\!\!\!\!\! - \gamma_4 \;\text{--}\; \gamma_3 \;\text{--}\; \gamma_5 \;\text{--}\; \gamma_1 \;\text{--}\; \gamma_6, \qquad \sigma_1 \;\text{--}\; \sigma_5 \;\text{--}\; \sigma_3 \;\text{--}\; \sigma_4 \;\text{--}\; \sigma_2 \;\text{--}\; \sigma_6.$$

Thus, using (2.1) we get the next Dynkin diagram for $f(x, y) = h(y) + g(x)$ (in terms of critical values),

$$\begin{array}{ccccccccccc}
c^h_2 + c^g_1 & \leftarrow & c^h_4 + c^g_1 & \rightarrow & c^h_3 + c^g_1 & \leftarrow & c^h_5 + c^g_1 & \rightarrow & c^h_1 + c^g_1 & \leftarrow & c^h_6 + c^g_1 \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
c^h_2 + c^g_5 & \leftarrow & c^h_4 + c^g_5 & \rightarrow & c^h_3 + c^g_5 & \leftarrow & c^h_5 + c^g_5 & \rightarrow & c^h_1 + c^g_5 & \leftarrow & c^h_6 + c^g_5 \\
\downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow \\
c^h_2 + c^g_3 & \leftarrow & c^h_4 + c^g_3 & \rightarrow & c^h_3 + c^g_3 & \leftarrow & c^h_5 + c^g_3 & \rightarrow & c^h_1 + c^g_3 & \leftarrow & c^h_6 + c^g_3 \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
c^h_2 + c^g_4 & \leftarrow & c^h_4 + c^g_4 & \rightarrow & c^h_3 + c^g_4 & \leftarrow & c^h_5 + c^g_4 & \rightarrow & c^h_1 + c^g_4 & \leftarrow & c^h_6 + c^g_4 \\
\downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow \\
c^h_2 + c^g_2 & \leftarrow & c^h_4 + c^g_2 & \rightarrow & c^h_3 + c^g_2 & \leftarrow & c^h_5 + c^g_2 & \rightarrow & c^h_1 + c^g_2 & \leftarrow & c^h_6 + c^g_2 \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
c^h_2 + c^g_6 & \leftarrow & c^h_4 + c^g_6 & \rightarrow & c^h_3 + c^g_6 & \leftarrow & c^h_5 + c^g_6 & \rightarrow & c^h_1 + c^g_6 & \leftarrow & c^h_6 + c^g_6.
\end{array} \tag{2.2}$$

The **Picard-Lefschetz formula** gives us an explicit computation of the monodromy of a cycle δ, around a critical value $c_{ij} := c^h_i + c^g_j$. Namely, it is

$$\mathrm{Mon}_{c_{ij}}(\delta) = \delta - \sum_k \langle \delta, \delta_k \rangle \delta_k, \tag{2.3}$$

where k runs through all the join cycles in the singularities of $f^{-1}(c_{ij})$ (see [18, §6.6]). Therefore, in order to compute the monodromy of the fibration given by the polynomial $f(x,y) = h(y) + g(x)$, we just need to handle combinatorial aspects of Dynkin diagrams. In the remainder of the text, we denote by $\mathbf{Mon}(\delta)$, the subspace generated by the orbit of δ under the monodromy action.

3. Monodromy for direct sum of polynomials with one critical value

In this section, we provide the monodromy matrix around 0 for the polynomial $f(x,y) = y^e + x^d$, with $e = 2,3,4$. For simplicity, we denote by δ_i^j the vanishing cycles in the row i and the column j. Thus we have the Dynkin diagram for $e = 2,3,4$,

$$
\begin{tikzcd}
\delta_1^1 \arrow[r] & \delta_1^2 & \delta_1^3 \arrow[l] \arrow[r] & \delta_1^4 & \delta_1^5 \arrow[l] \arrow[r] & \delta_1^6 & \cdots \arrow[l] & \delta_1^{d-1}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\delta_1^1 \arrow[r] \arrow[d] & \delta_1^2 \arrow[d] & \delta_1^3 \arrow[l] \arrow[r] \arrow[d] & \delta_1^4 \arrow[d] & \delta_1^5 \arrow[l] \arrow[r] \arrow[d] & \delta_1^6 \arrow[d] & \cdots \arrow[l] & \delta_1^{d-1} \arrow[d] \\
\delta_2^1 \arrow[r] & \delta_2^2 \arrow[ul] \arrow[ur] & \delta_2^3 \arrow[l] \arrow[r] & \delta_2^4 \arrow[ul] \arrow[ur] & \delta_2^5 \arrow[l] \arrow[r] & \delta_2^6 \arrow[ul] & \cdots \arrow[l] & \delta_2^{d-1}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\delta_1^1 \arrow[r] \arrow[d] & \delta_1^2 \arrow[d] & \delta_1^3 \arrow[l] \arrow[r] \arrow[d] & \delta_1^4 \arrow[d] & \delta_1^5 \arrow[l] \arrow[r] \arrow[d] & \delta_1^6 \arrow[d] & \cdots \arrow[l] & \delta_1^{d-1} \arrow[d] \\
\delta_2^1 \arrow[r] & \delta_2^2 \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \delta_2^3 \arrow[l] \arrow[r] & \delta_2^4 \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \delta_2^5 \arrow[l] \arrow[r] & \delta_2^6 \arrow[ul] \arrow[dl] & \cdots \arrow[l] & \delta_2^{d-1} \\
\delta_3^1 \arrow[r] \arrow[u] & \delta_3^2 \arrow[u] & \delta_3^3 \arrow[l] \arrow[r] \arrow[u] & \delta_3^4 \arrow[u] & \delta_3^5 \arrow[l] \arrow[r] \arrow[u] & \delta_3^6 \arrow[u] & \cdots \arrow[l] & \delta_3^{d-1}, \arrow[u]
\end{tikzcd}
\tag{3.1}
$$

respectively. By Proposition 2.1, the intersection matrix, in the ordered vector basis $\delta_1^1, \dots, \delta_{e-1}^1, \delta_1^2, \dots, \delta_{e-1}^2, \dots, \delta_1^{d-1}, \dots, \delta_{e-1}^{d-1}$, for these Dynkin diagrams are

$$
\Psi_2 = \begin{pmatrix}
0 & -1 & 0 & 0 & \dots \\
1 & 0 & 1 & 0 & \dots \\
0 & -1 & 0 & -1 & \dots \\
0 & 0 & 1 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots &
\end{pmatrix},
\tag{3.2}
$$

$$
\Psi_3 = \begin{pmatrix}
0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & \dots \\
-1 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & \dots \\
0 & 0 & -1 & 1 & 0 & -1 & -1 & 1 & \dots \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & \dots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & \dots \\
0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix},
\tag{3.3}
$$

$$\Psi_4 = \begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.4}$$

The matrices are antisymmetric, and the superior diagonals are periodic sequences. For Ψ_2 the sequence is $(-1, 1, \dots)$. For Ψ_3 the sequence are $(-1, 0, -1, -1, -1, 0, \dots)$, $(-1, -1, 1, 1 \dots)$ and $(1, 0, 0, 0, \dots)$. For Ψ_4 the sequence are $(-1, 1, 0, \dots)$, $(0, 0, 1, 0, -1, 0, \dots)$, $(-1, -1, -1, 1, 1, 1, \dots)$ and $(-1, 0, 0, 0, 1, 0, \dots)$. From the Picard-Lefschetz formula (2.3), it follows that the monodromy matrices for $f(x, y) = y^e + x^d$ with $e = 2, 3, 4$, are

$$M_e = I_N - \Psi_e,$$

where I_N is the identity matrix of rank $N = (d - 1)(e - 1)$.

For a vector v and a matrix $M \in \mathcal{M}_N(\mathbb{R})$, the Krylov subspace is the vector space generated by the vectors $M^l v$ where $l = 0, 2, \dots, N - 1$. Therefore, by taking M as one of the monodromy matrices M_2, M_3 or M_4, and $v = v_k$ a vector of the canonical basis of $\mathbb{R}^N$, the Krylov subspace is

$$\mathbf{Mon}\left(\delta^{\lfloor \frac{k}{e} \rfloor}_{\bmod_e(k)}\right) \tag{3.5}$$

for the fibration $y^e + x^d$. In the next proposition we provide the vanishing cycles that are in (3.5).

Proposition 3.1. *For the polynomial $y^2 + x^d$, the vanishing cycles in the subspace $Mon(\delta_i^j)$ are*

Vanishing cycle δ_1^j	*Vanishing cycles δ_1^l in $Mon(\delta_1^j)$*
$gcd(d, j) = r$	$l = rn$ *with* $n = 1, \dots, \frac{d}{r} - 1$.

For the polynomial $y^3 + x^d$ with $d \leq 100$ and $3 \nmid d$, the vanishing cycles in the subspace $Mon(\delta_i^j)$ are

Vanishing cycle δ_i^j	*Vanishing cycles* δ_m^l *in* $Mon(\delta_i^j)$
$i=1,2$ *and* $gcd(d,j)=r$	$m=1,2$ *and* $l=rn$ *with* $n=1,\dots,\frac{d}{r}-1$.

When $3\mid d$*, the number of different eigenvalues is less than* $2(d-1)$*. For the polynomial* y^4+x^d *with* $d\le 100$ *and* $4\nmid d$*, the vanishing cycles in the subspace* $Mon(\delta_i^j)$ *are*

Vanishing cycle δ_i^j	*Vanishing cycles* δ_m^l *in* $Mon(\delta_i^j)$
$i=1,3$ *and* $gcd(d,j)=r$	$m=1,2,3$ *and* $l=rn$ *with* $n=1,\dots,\frac{d}{r}-1$
$i=2$ *and* $gcd(d,j)=r$	$m=2$ *and* $l=rn$ *with* $n=1,\dots,\frac{d}{r}-1$.

When $4\mid d$*, the number of different eigenvalues is less than* $3(d-1)$*.*

Proof. Let M be one of the matrices M_2, M_3 or M_4, and $v:=v_k=(0,0,\dots,0,1,0\dots,0)$ for $k=1,\dots,(d-1)(e-1)$. The corresponding vanishing cycle to v_k is δ_a^b, with $a=\operatorname{mod}_e(k)$ and $b=\lfloor\frac{k}{e}\rfloor$. Since the matrices Ψ_e are skew-symmetric, then M is a normal matrix, consequently it is diagonalizable. Hence, its eigenvectors u_j are a basis for $\mathbb{R}^N$. Then we can write $v=\sum_j r_j u_j$ for scalars r_j. Let λ_j be the eigenvalue associated to u_j, thus we have

$$M^l v=\sum_{j=1}^{N} r_j\lambda_j^l u_j, \qquad \text{where } N=(d-1)(e-1),$$

and the matrix $\{v, Mv, M^2v,\dots,M^nv\}$ is

$$\begin{pmatrix} r_1u_1 & r_2u_2 & \dots & r_Nu_n\end{pmatrix}\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{N-1}\\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{N-1}\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & \lambda_N & \lambda_N^2 & \dots & \lambda_N^{N-1}\end{pmatrix}.$$

The matrix in the right, is the Vandermonde matrix with determinant $\prod_{i<j}(\lambda_j-\lambda_i)$. Hence, if the eigenvalues are different, then the Krylov subspace is the span of the vectors u_l such that $r_l\neq 0$.

For $e=2$, it is possible to show that the matrix M_2 is similar to a tridiagonal matrix with main diagonal of 1s, first diagonal below of -1s, and first diagonal above of 1s. The change of basis is given by the diagonal matrix, whose diagonal is $(-1,1,1,-1,-1,1,1,\dots)$. Hence, there is a known closed form for the eigenvalues and eigenvectors of the matrix M_2 (see [15,22,6]). Namely, the eigenvalues are given by

$$\lambda_j=1+2\sqrt{-1}\cos\left(\frac{j\pi}{d}\right), \qquad \text{whit } j=1,\dots,d-1.$$

If the vector $u_j=\left(u_j^{(1)},u_j^{(2)},\dots,u_j^{(d-1)}\right)^T$ is the eigenvector associated to λ_j, then the k-th coordinate satisfies

$$u_j^{(k)} = (\sqrt{-1})^{k-1}\sqrt{\frac{2}{d}}\sin\left(\frac{kj\pi}{d}\right).$$

If we denote $U = [u_1 \quad u_2 \quad \cdots \quad u_{d-1}]$ the matrix whose columns are the eigenvectors, then $UU^* = Id_{d-1}$. Hence, if we want to know which eigenvectors are used in the representation of δ_1^l, it is enough to note which terms in the row l of U are zero. This happens when $\frac{jl}{d} \in \mathbb{Z}$. Furthermore, the Krylov space of δ_1^l is contained in the Krylov space of $\delta_1^{l'}$, provided that the j's such that $\frac{jl'}{d} \in \mathbb{Z}$, satisfy $\frac{jl}{d} \in \mathbb{Z}$. It is equivalent to $gcd(d, l') \mid gcd(d, l)$.

For $e = 3, 4$, we do not know a close form for the eigenvalues. However, for given values of d, on a computer we can compute explicitly the eigenvalues, and a basis for the subspace generated by these eigenvectors. Next, we determine which vectors of the canonical base $\mathbb{R}^N$ are in this subspace. If $e = 3$ and $3 \mid d$, then the number of different eigenvalues is less than N. The same is true for $e = 4$ and $4 \mid d$. In other cases the number of different eigenvalues is N. The reader can use the functions written in MATLAB, MonMatrix and VanCycleSub,[1] for a numerical supplement of this proof (see Appendix B). $\square$

Remark 3.2. The condition $3 \nmid d$ in the case M_3 may be related with the fact that $y^3 + x^d$ is a pullback with the map $(x, y) \to (x^{\frac{d}{3}}, y)$. Analogously, the condition $4 \nmid d$ in the case M_4 associated to $y^4 + x^d$.

On the other hand, in general monodromy matrices are not diagonalizable. For example, the monodromy matrices of the mirror quintic Calabi-Yau threefold (see [7,14]).

4. Monodromy problem for $y^4 + g(x)$

Let $g \in \mathbb{R}[x]_{\leq d}$ be a polynomial with real critical points. Consider the polynomial $f(x, y) := y^4 + g(x)$ which has critical values equal to the critical values of g. Recall, in some cases we relate the vertices in the Dynkin diagram to the critical values associated with the vanishing cycles. Thus, we denote by C_j the critical value in the column j from left to right in the Dynkin diagram, and δ_i^j to the vanishing cycle in the row i over C_j. For example, if we suppose that d is even and C_1 is a local maximum, then the Dynkin diagram looks like

$$\begin{array}{ccccccccccccccc}
C_1 & \rightarrow & C_2 & \leftarrow & C_3 & \rightarrow & C_4 & \leftarrow & C_5 & \rightarrow & C_6 & \leftarrow \cdots \rightarrow & C_{d-2} & \leftarrow & C_{d-1} \\
\downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & & \downarrow & \nearrow & \downarrow \\
C_1 & \rightarrow & C_2 & \leftarrow & C_3 & \rightarrow & C_4 & \leftarrow & C_5 & \rightarrow & C_6 & \leftarrow \cdots \rightarrow & C_{d-2} & \leftarrow & C_{d-1} \\
\uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & & \uparrow & \searrow & \uparrow \\
C_1 & \rightarrow & C_2 & \leftarrow & C_3 & \rightarrow & C_4 & \leftarrow & C_5 & \rightarrow & C_6 & \leftarrow \cdots \rightarrow & C_{d-2} & \leftarrow & C_{d-1}.
\end{array} \tag{4.1}$$

[1] https://github.com/danfelmath/Intersection-matrix-for-polynomials-with-1-crit-value.git.

Definition 4.1. We say that the Dynkin diagram of $y^e + g(x)$ with $g \in \mathbb{C}[x]_{\leq d}$ has **horizontal symmetry** if there exists integer $r > 1$ such that for any j with g.c.d$(j, d) = r$ the critical values satisfy

$$C_{j-k} = C_{j+k} \text{ where } k = 1, \dots, r-1.$$

The vanishing cycles $\delta_i^{l\cdot r}$ with $l = 1, \dots, \frac{d}{r} - 1$ are called **vanishing cycles with horizontal symmetry**. We can define the **vertical symmetry** analogously. For the Dynkin diagram (4.1) the cycles δ_2^j are **vanishing cycles with vertical symmetry**.

From a direct computation in the Dynkin diagram (4.1) and Picard-Lefschetz formula, we observe that only the terms

$$\delta_i^{j-k} + \delta_i^{j+k} \text{ with g.c.d}(j, d) = r$$

and the cycles with horizontal symmetry appear in the subspace generated by the monodromy action on a cycle with horizontal symmetry. This happens in an analogous way for the vertical symmetry. Therefore, the subspace generated by the monodromy action on a cycle with horizontal symmetry or vertical symmetry is different to $H_1(f^{-1}(b), \mathbb{Q})$.

On the other hand, the definition of horizontal symmetry only depends on the relation among the critical values of g. Hence, for p, q integers greater than 1, there are cycles with horizontal symmetry in the Dynkin diagram associated with $y^p + g(x)$ if and only if there are in the Dynkin diagram associated with $y^q + g(x)$.

For the vertical symmetry, in the next lemma we provide a geometric characterization of the cycles δ_2^j with $j = 1, \dots, d-1$.

Lemma 4.2. *Consider the map* $\mathbb{C}^2 \xrightarrow{\pi} \mathbb{C}^2$, *given by* $\pi(x, y) = (x, y^2)$. *The cycles* $\delta_2^j \in H_1((y^4 + g(x))^{-1}(b))$ *for* $j = 1, \dots, d-1$, *are in the kernel of*

$$\pi_* : H_1((y^4 + g(x))^{-1}(b)) \to H_1((y^2 + g(x))^{-1}(b)).$$

Proof. Consider the perturbation $h_\varepsilon(y) := y^4 + \varepsilon(-y^2 + \frac{\varepsilon}{8})$ of y^4, where $\varepsilon \geq 0$. The roots of $h_\varepsilon(y)$ are $\frac{\pm 1}{2}\sqrt{\varepsilon(2 \pm \sqrt{2})}$. Therefore, the 0-cycle associated to δ_2^j, is

$$\gamma_1 = \left(\frac{1}{2}\sqrt{\varepsilon(2-\sqrt{2})}, 0\right) - \left(\frac{-1}{2}\sqrt{\varepsilon(2-\sqrt{2})}, 0\right)$$

(see Fig. 4). In the projection by y^2, these points are identified with $\left(\frac{1}{4}\varepsilon(2-\sqrt{2}), 0\right)$. Consequently, the image of the vanishing cycles $\delta_2^j = \gamma_1 * \sigma_j$ by the map π is trivial. $\square$

Note that the kernel of π_* is generated by the cycles

$$\gamma_1 * \sigma_j \quad \text{and} \quad (\gamma_3 - \gamma_2) * \sigma_j, \quad \text{for } j = 1, \dots d-1,$$

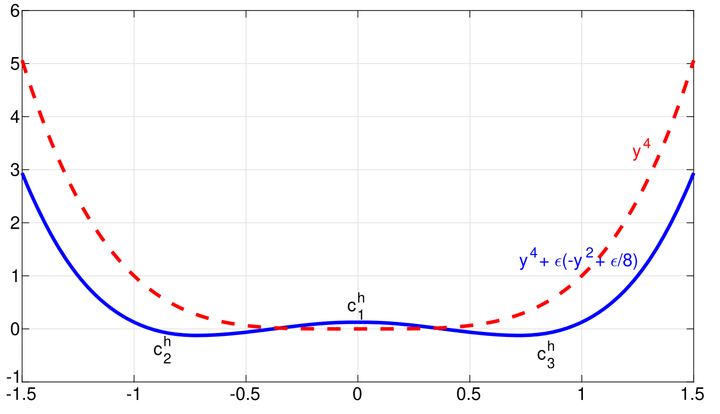

Fig. 4. Critical values for $y^4 + \varepsilon(-y^2 + \frac{\varepsilon}{8})$, a perturbation of y^4.

however, the first ones generate the others by monodromy.

We want to study the vanishing cycles in the subspace generated by the monodromy action when there is no horizontal symmetry in the Dynkin diagram. By using Lemma 4.3, we can reduce this analysis to the cases $\pi_1(\mathbb{P}^1 \setminus C) \equiv \mathbb{Z}$ and $\pi_1(\mathbb{P}^1 \setminus C) \equiv \mathbb{Z}^2$.

Lemma 4.3. *Let g be a polynomial of degree d, and let G_1 and G_g be the monodromy groups associated to $y^4 + x^d$ and $y^4 + g(x)$, respectively. For any $v \in V_g := H_1((y^4 + g)^{-1}(b))$, the subspaces generated by the orbits satisfy $\langle G_1 \cdot v\rangle \subset \langle G_g \cdot v\rangle$. Besides, if C has more than one element, then there exists a group of two elements $G_2 < Aut(V_g)$ such that $\langle G_2 \cdot v\rangle \subset \langle G_g \cdot v\rangle$.*

Proof. Any element in G_g can be written as a matrix $I_N - A_j$, where I_N is the identity matrix and A_j is constructed by putting rows of zeros in the matrix Ψ_4 of equation (3.4). Thus, we have that $M_4 = \sum_{A_j \in G_g} I_N - A_j + (1 - |G_g|)I_N$. Consequently, $M_4^k v \in \langle G_g \cdot v\rangle$ for $k \in \mathbb{Z}$.

In order to construct G_2 it is enough to divide the elements of G_g in two groups, and define two matrices as the sum of the matrices in these groups. Note that these sums correspond with identifications of some critical values in the Dynkin diagram. □

The next proposition follows from the Proposition 3.1 and Lemma 4.3.

Proposition 4.4. *Let g be a polynomial of degree d, where $d \le 100$ and $4 \nmid d$. If the Dynkin diagram of $f(x, y) = y^4 + g(x)$ does not have horizontal symmetry, then the subspace of $H_1(f^{-1}(b), \mathbb{Q})$ generated by the orbit of a vanishing cycle δ_i^j contains all the vanishing cycles in the row i. Moreover, if i is 1 or 3, then the generated submodule is the whole space $H_1(f^{-1}(b), \mathbb{Q})$.*

Proof. The restrictions on the degree d are due to the Proposition 3.1. By Proposition 3.1 the result it is true for $gcd(j, d) = 1$ and $g(x) = x^d$, then by Lemma 4.3 is true for any

$g \in \mathbb{C}[x]_{\leq d}$. If $gcd(j, d) = r > 1$, then $\mathrm{Mon}(\delta_i^j)$ contains the vanishing cycles δ_i^l with $l = rn$ and $n = 1, \ldots, \frac{d}{r} - 1$. Since the rows do not have horizontal symmetry, then d is prime or there are at least two different critical values. However if d is prime, then $r = 1$. Hence, by using Lemma 4.3 we suppose that there are two critical values A and B. Thus, it is enough to consider a initial vanishing cycle $v := \delta_2^j$ where the critical values C_{j-l} and C_{l+j} are equal for $l = 1, \ldots, k-1 < r-1$. Also, the critical values C_{j-k} and C_{j+k} are different. We can suppose that the Dynkin diagram looks like

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \cdots & C_{j-k-1} \arrow[l] \arrow[r] \arrow[d] & C_{j-k} \arrow[d] & C_{j-k+1} \arrow[l] \arrow[r] \arrow[d] & \cdots & A \arrow[l] \arrow[r] \arrow[d] & A \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & \cdots & C_{k-1+j} \arrow[l] \arrow[r] \arrow[d] & C_{k+j} \arrow[d] & * \arrow[l] \arrow[r] \arrow[d] & \cdots & * \arrow[l] \arrow[d] \\
* \arrow[r] & \cdots & D \arrow[l] \arrow[r] & B \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & A \arrow[l] \arrow[r] & \cdots & A \arrow[l] \arrow[r] & A \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & A \arrow[l] \arrow[r] & \cdots & A \arrow[l] \arrow[r] & A \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & * \arrow[l] \arrow[r] & \cdots & * \arrow[l] \\
* \arrow[r] \arrow[u] & \cdots & D \arrow[l] \arrow[r] \arrow[u] & B \arrow[u] & A \arrow[l] \arrow[r] \arrow[u] & \cdots & A \arrow[l] \arrow[r] \arrow[u] & A \arrow[u] & A \arrow[l] \arrow[r] \arrow[u] & \cdots & A \arrow[l] \arrow[r] \arrow[u] & A \arrow[u] & * \arrow[l] \arrow[r] \arrow[u] & \cdots & * \arrow[l] \arrow[u]
\end{tikzcd}.$$

where D can be A or B, and $*$ means that it does not matter what value it is. We denote by Mon_A and Mon_B the monodromy action around the critical values A and B, respectively. By doing $\mathrm{Mon}_B(\mathrm{Mon}_A)^{k-1}(v)$, we get a linear combination of cycles in the column k. In fact we have one of the next possibilities

$$(m_2 m_1)^s v, m_1(m_2 m_1)^s v, (m_1 m_2)^s v, m_2(m_1 m_2)^s v, \ s \in \mathbb{N},$$

where

$$m_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \ m_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the matrices are in the basis $\delta_1^{j-k}, \delta_2^{j-k}, \delta_3^{j-k}$. Hence, we generate the linear combination $w := m\delta_1^{j-k} + n\delta_2^{j-k} + m\delta_3^{j-k}$ where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$. If $D = B$, taking $\mathrm{Mon}_B \mathrm{Mon}_A(w)$, we get

$$(n - 3m)\delta_1^{j-k} + (2m - n)\delta_2^{j-k} + (n - 3m)\delta_3^{j-k}, \text{ or}$$
$$(n - m)\delta_1^{j-k} + (2m - 3n)\delta_2^{j-k} + (n - m)\delta_3^{j-k}.$$

Any of the linear combinations in the previous equations and w generate the vanishing cycle δ_2^{j-k}. If $D = A$, considering $\mathrm{Mon}_B(w)$ and w we can generate the cycle δ_2^{j-k}. If $gcd(j - k, d) = 1$, then the results follow from Proposition 3.1. If $gcd(j - k, d) = r'$, then we repeat the previous analysis with r' instead of r, thus the proof follows from $r' < r$. $\square$

The next propositions are proved using analogous arguments as in the proof of Proposition 4.4. For this reason, in their proof we only indicate the corresponding matrices m_1 and m_2.

Proposition 4.5. *Let g be a polynomial of degree d, where $d \leq 100$ and $3 \nmid d$. If the Dynkin diagram of $f(x,y) = y^3 + g(x)$ does not have horizontal symmetry, then the subspace generated by the orbit of a vanishing cycle δ_i^j is the whole space $H_1(f^{-1}(b), \mathbb{Q})$.*

Proof. Consider the matrices

$$m_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \; m_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

thus considering the initial vanishing cycle $v := \delta_2^j$, we get a vector $w = m\delta_1^j + n\delta_2^j$ where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$. $\square$

Proposition 4.6. *If the Dynkin diagram of $f(x,y) = y^2 + g(x)$ does not have horizontal symmetry, then the subspace generated by the orbit of a vanishing cycle δ_i^j is the whole space $H_1(f^{-1}(b), \mathbb{Q})$.*

Proof. In this case the matrices are $m_1 = -1$ and $m_2 = 1$. $\square$

The next theorem is the main result in [4], it is a solution of the monodromy problem for hyperelliptic curves $y^2+g(x)$. We will use it, in order to solve the monodromy problem for $y^3 + g(x)$ and $y^4 + g(x)$. Although, this theorem holds for $g \in \mathbb{C}[x]$, we are interested in the case of g being a real polynomial with real critical points.

Theorem 4.7 *(C. Christopher and P. Mardešić, 2008). Let $f(x,y) = y^2 + g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the whole homology $H_1(f^{-1}(b), \mathbb{Q})$.*
2. *The polynomial g is decomposable (i.e., $g = g_2 \circ g_1$), and $\pi_*\delta(t)$ is homotopic to zero in $\{y^2 + g_2(z) = t\}$, where $\pi(x,y) = (g_1(x), y) = (z,y)$.*

From this theorem, we can show that the condition of g being decomposable as $g = g_2 \circ g_1$, is equivalent to the Dynkin diagram associated to $y^e + g(x)$ has horizontal symmetry, with $e > 1$. In fact, as we mentioned above, the horizontal symmetry condition only depends on g.

Proposition 4.8. *The next assertions are equivalents.*

1. *The polynomial can be written as $g = g_2 \circ g_1$, where g_1, g_2 are polynomials such that* $\deg(g_1), \deg(g_2) > 1$.
2. *The Dynkin diagram associated to $y^e + g(x)$ has horizontal symmetry, for some $e > 1$ (and hence for all $e > 1$).*

Proof. From Proposition 4.6, follows that the condition 2 implies that there are vanishing cycles for the fibration $f(x, y) = y^2 + g(x)$, such that they do not generate the whole $H_1(f^{-1}(b))$. Thus, by Theorem 4.7, we conclude that 2 implies 1. The other implication follows by a direct computation on a Dynkin diagram similar to (4.1), but with $e - 1$ rows. $\square$

Although the horizontal symmetry is just a condition on the polynomial g, this proposition allows to extend the result in the Theorem 4.7 to the fibrations defined by $y^4 + g(x)$ and $y^3 + g(x)$. The non trivial part for this generalization is due to Propositions 4.4 and 4.5. However, since Proposition 3.1 is numerically proven, we have restrictions in the degree of g.

Theorem 4.9. *Let g be a polynomial with real critical points, and degree d such that, $4 \nmid d$ and $d \leq 100$. Consider the polynomial $f(x, y) = y^4 + g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the homology $H_1(f^{-1}(t), \mathbb{Q})$.*
2. *The polynomial g is decomposable (i.e., $g = g_2 \circ g_1$), and $\pi_*\delta(t)$ is homotopic to zero in $\{y^4 + g_2(z) = t\}$, where $\pi(x, y) = (g_1(x), y) = (z, y)$. Or, the cycle $\pi_*\delta(t)$ is homotopic to zero in $\{z^2 + g(x) = t\}$, where $\pi(x, y) = (x, y^2) = (x, z)$.*

Proof. The restrictions on the degree d are due to the Proposition 3.1, which is used in Proposition 4.4. Let $\delta_i^j := \delta(t)$ be a vanishing cycle. If the monodromy of δ_i^j does not generate the homology $H_1(f^{-1}(t), \mathbb{Q})$, then considering the contrapositive of Proposition 4.4, we have the next possibilities: The index i is 2 or the cycle δ_i^j has horizontal symmetry. If $i = 2$, then by the Lemma 4.2, we have that $\pi_*\delta_2^j$ is trivial, where $\pi(x, y) = (x, y^2)$. If δ_i^j has horizontal symmetry, then by using Proposition 4.8 we conclude that $g = g_2 \circ g_1$. Furthermore, δ_i^j is in correspondence with a cycle with horizontal symmetry in the Dynkin diagram of $y^2 + g(x)$. Consequently, δ_i^j is in the kernel of π_*, where $\pi(x, y) = (g_1(x), y)$.

On the other hand, if the condition 2 is true, then the vanishing cycle δ_i^j has vertical or horizontal symmetry. In any of these cases, it follows by direct computation in the Dynkin diagram (4.1), that the subspace generated by the orbit of the monodromy action on δ_i^j is different to $H_1(f^{-1}(t), \mathbb{Q})$. $\square$

We have an analogous result for degree $y^3 + g(x)$. Note that in this case there are not cycles with vertical symmetry.

Theorem 4.10. *Let g be a polynomial with real critical points, and degree d such that, $3 \nmid d$ and $d \leq 100$. Consider the polynomial $f(x, y) = y^3 + g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the homology $H_1(f^{-1}(t), \mathbb{Q})$.*

2. *The polynomial g is decomposable (i.e., $f = g_2 \circ g_1$), and $\pi_*\delta(t)$ is homotopic to zero in $\{y^3 + g_2(z) = t\}$, where $\pi(x, y) = (g_1(x), y)) = (z, y)$.*

Proof. The restrictions on the degree d are due to the Proposition 3.1, which is used in Proposition 4.5. Let $\delta_i^j := \delta(t)$ be a vanishing cycle. If the monodromy of δ_i^j does not generate the homology $H_1(f^{-1}(t), \mathbb{Q})$, then by Proposition 4.5, the cycle δ_i^j has horizontal symmetry. Hence, using Proposition 4.8 we conclude that $g = g_2 \circ g_1$. $\square$

5. Monodromy problem for 4th degree polynomials $h(y) + g(x)$

Consider $f(x, y) = h(y) + g(x)$ where $h \in \mathbb{R}[y]_{\leq 4}$ and $g \in \mathbb{R}[x]_{\leq 4}$, and b is a regular value. Moreover, we suppose that the critical points of h and g are reals. The aim of this section is to compute the part of the homology $H_1(f^{-1}(b))$ generated by the action of the monodromy. From the equation (2.1) it follows that the 1-dimensional Dynkin diagram depends on the 0-dimensional Dynkin diagrams of h and g. Let $\gamma_i \in H_0(h^{-1}(b), \mathbb{Z})$ and $\sigma_i \in H_0(g^{-1}(b), \mathbb{Z})$ be the 0-cycles, where $i = 1, 2, 3$. Thus, using the enumeration of vanishing cycles, indicated in § 2, we have the next three cases,

$\gamma_1 \dashrightarrow \gamma_3 \dashrightarrow \gamma_2$, $\sigma_2 \dashrightarrow \sigma_1 \dashrightarrow \sigma_3$ resulting in:

$$\begin{tikzcd}
\gamma_1 * \sigma_2 \arrow[d] & \gamma_3 * \sigma_2 \arrow[l] \arrow[r] \arrow[d] & \gamma_2 * \sigma_2 \arrow[d] \\
\gamma_1 * \sigma_1 \arrow[ur] \arrow[dr] & \gamma_3 * \sigma_1 \arrow[l] \arrow[r] & \gamma_2 * \sigma_1 \arrow[ul] \arrow[dl] \\
\gamma_1 * \sigma_3 \arrow[u] & \gamma_3 * \sigma_3 \arrow[l] \arrow[r] \arrow[u] & \gamma_2 * \sigma_3 \arrow[u]
\end{tikzcd}$$

$\gamma_1 \dashrightarrow \gamma_3 \dashrightarrow \gamma_2$, $\sigma_1 \dashrightarrow \sigma_3 \dashrightarrow \sigma_2$ resulting in:

$$\begin{tikzcd}
\gamma_1 * \sigma_1 \arrow[dr] & \gamma_3 * \sigma_1 \arrow[l] \arrow[r] & \gamma_2 * \sigma_1 \arrow[dl] \\
\gamma_1 * \sigma_3 \arrow[u] \arrow[d] & \gamma_3 * \sigma_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \gamma_2 * \sigma_3 \arrow[u] \arrow[d] \\
\gamma_1 * \sigma_2 \arrow[ur] & \gamma_3 * \sigma_2 \arrow[l] \arrow[r] & \gamma_2 * \sigma_2 \arrow[ul]
\end{tikzcd}$$

$\gamma_2 \dashrightarrow \gamma_1 \dashrightarrow \gamma_3$, $\sigma_2 \dashrightarrow \sigma_1 \dashrightarrow \sigma_3$ resulting in:

$$\begin{tikzcd}
\gamma_2 * \sigma_2 \arrow[r] \arrow[d] & \gamma_1 * \sigma_2 \arrow[d] & \gamma_3 * \sigma_2 \arrow[l] \arrow[d] \\
\gamma_2 * \sigma_1 \arrow[r] & \gamma_1 * \sigma_1 \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \gamma_3 * \sigma_1 \arrow[l] \\
\gamma_2 * \sigma_3 \arrow[r] \arrow[u] & \gamma_1 * \sigma_3 \arrow[u] & \gamma_3 * \sigma_3 . \arrow[l] \arrow[u]
\end{tikzcd}$$

If we consider $-f$ instead of f, the two last Dynkin diagrams coincide. Hence, we only focus in the first two 1-dimensional Dynkin diagrams.

Example 5.1. Consider the polynomials $h(y) = -y^4 + 9y^2$ and $g(x) = -x^4 + 16x^2 + 8x$. In Fig. 5, we show the real part of these polynomials with the critical values indexed according to §2. Let $f_1(x, y) = h(y) + g(x)$, thus the Dynkin diagram associated to f_1 is the first one.

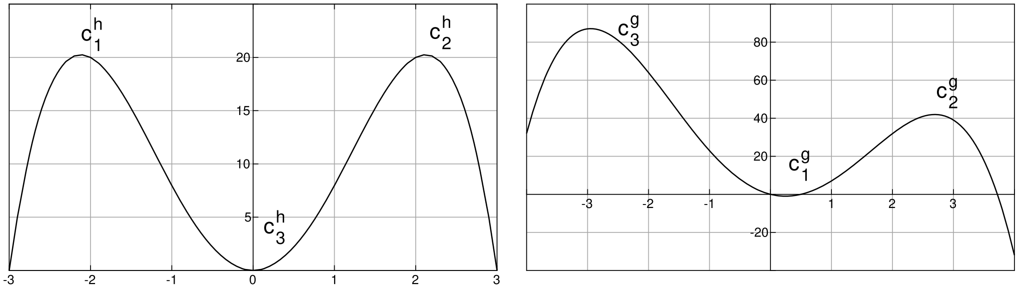

Fig. 5. Real part of the polynomials $h(y) = -y^4 + 9y^2$ and $g(x) = -x^4 + 16x^2 + 8x$, with its critical values. On the left is $h(y)$ and on the right $g(x)$.

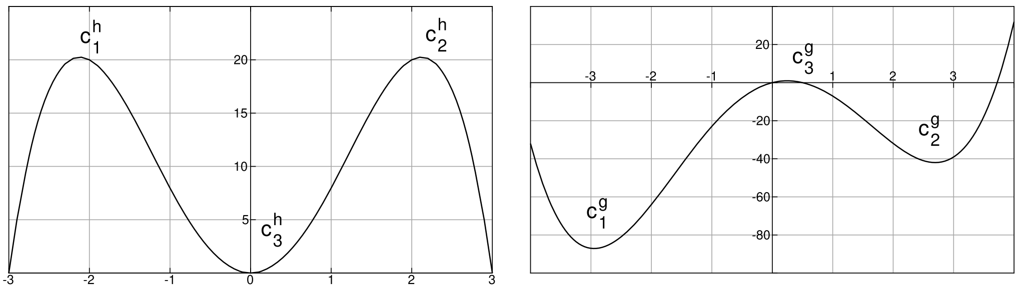

Fig. 6. Real part of the polynomials $h(y) = -y^4 + 9y^2$ and $g(x) = x^4 - 16x^2 - 8x$, with its critical values. On the left is $h(y)$ and on the right $g(x)$.

Example 5.2. Consider the polynomials $h(y) = -y^4 + 9y^2$ and $g(x) = x^4 - 16x^2 - 8x$. In Fig. 6, we present the real part of these polynomials with the critical values indexed according to §2. Let $f_2(x, y) = h(y) + g(x)$, thus the Dynkin diagram associated to f_2 is the second.

In Figs. 7 and 8, we present the real part of the fibration $f_1(x, y) = t$ and $f_2(x, y) = t$, respectively. Note that the maximum corresponds with the addition of the maximum of h and g, analogously for the minimum. The others critical points are known as saddles points.

We denote the critical values by

$$a_1 = c_1^h + c_1^g \ , \ a_2 = c_1^h + c_2^g \ , \ a_3 = c_1^h + c_3^g$$
$$a_4 = c_2^h + c_1^g \ , \ a_5 = c_2^h + c_2^g \ , \ a_6 = c_2^h + c_3^g$$
$$a_7 = c_3^h + c_1^g \ , \ a_8 = c_3^h + c_2^g \ , \ a_9 = c_3^h + c_3^g.$$

If we consider the contour lines associated to the Fig. 7, then we obtain a drawing in the plane which represent the vertex in the Dynkin diagram associated to the polynomial f_1. In fact, the correspondence between the Dynkin diagram and the curve in the plane is shown in [1]. In Fig. 9, we present the contour lines of the real part of the polynomial

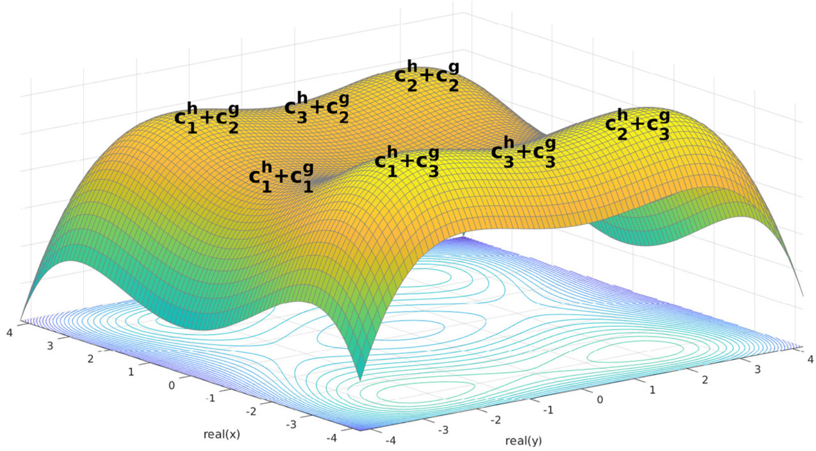

Fig. 7. Real part of graph defined by $f_1(x, y) = -y^4 + 9y^2 - x^4 + 16x^2 + 8x = t$, with its critical values.

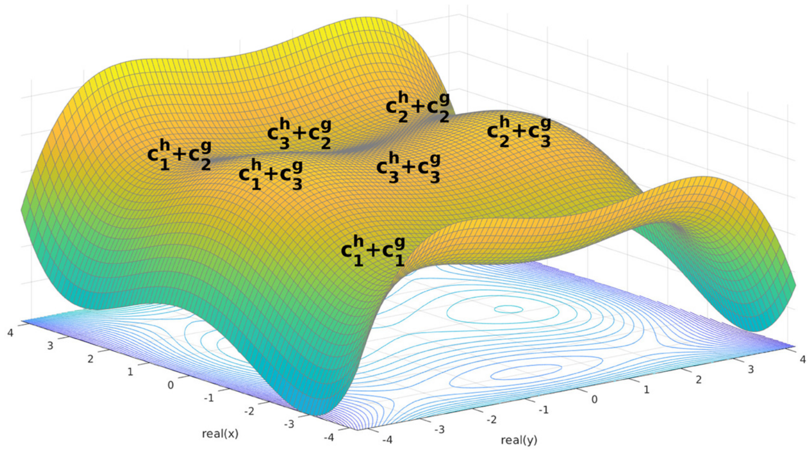

Fig. 8. Real part of graph defined by $f_2(x, y) = -y^4 + 9y^2 + x^4 - 16x^2 - 8x = t$, with its critical values.

$f_1(x, y)$. The critical values a_2, a_3, a_5, a_6 and a_7 correspond with ovals contained in the real part of the foliation defined by df_1.

Analogously, the contour lines of the Fig. 8 give us a drawing in the plane which represent the vertex in the Dynkin diagram associated to f_2. In Fig. 10, we present the contour lines of the real part of the polynomial $f_2(x, y)$. In this case, the critical values a_3, a_6, a_7 and a_8 correspond with ovals contained in the real part of the foliation defined by df_2.

Let α_i be the cycle which vanishes in the critical point corresponding to the critical value a_i for $i = 1, ..., 9$. For simplicity in the notation we call the possible critical values as $a, b, c, d, e, f, g, h, i$ if all are different and we are removing from right to left as soon as the critical values are repeated. Moreover, we indicate with "*" on the right, the vanishing cycles which is not contained in the real plane (or its associated critical value).

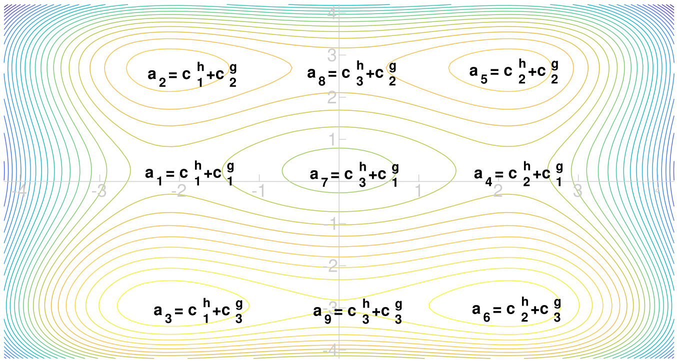

Fig. 9. Contour lines of the real part of $f_1(x, y) = -y^4 + 9y^2 - x^4 + 16x^2 + 8x = t$.

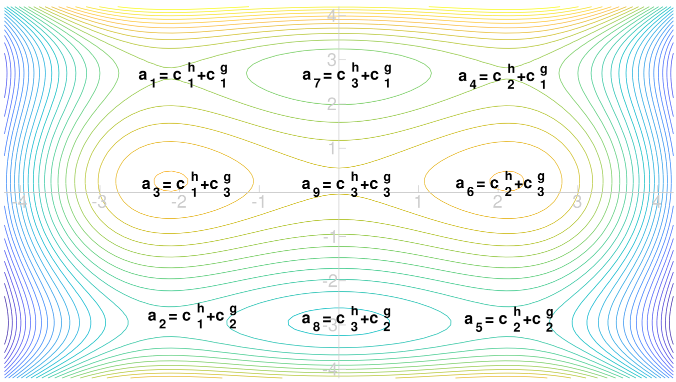

Fig. 10. Contour lines of the real part of $f_1(x, y) = -y^4 + 9y^2 + x^4 - 16x^2 - 8x = t$.

For instance, the polynomial $f_1(x, y)$ of the Example 5.1 satisfies that $a_1 = a_4, a_2 = a_5, a_3 = a_6$ and the other critical values are different, therefore its next Dynkin diagram is

$$\begin{array}{ccccc} b & \longleftarrow & e^* & \longrightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \longleftarrow & d & \longrightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \longleftarrow & f^* & \longrightarrow & c, \end{array}$$

Similarly, the polynomial $f_2(x, y)$ of the Example 5.2, has Dynkin diagram

$$\begin{array}{ccccc} a^* & \longleftarrow & d & \longrightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \longleftarrow & e^* & \longrightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \longleftarrow & f & \longrightarrow & b^*. \end{array}$$

The subspace of the homology $H_1(f^{-1}(b))$ generated by the monodromy action on a vanishing cycle α_i is denoted $\mathrm{Mon}(\alpha_i)$ $i = 1, \ldots, 9$. We will compute the monodromy for any α_i depending on the number of different critical values.

For the Dynkin diagram

$$\begin{tikzcd} a_2 \arrow[d] & a_8^* \arrow[l] \arrow[r] \arrow[d] & a_5 \arrow[d] \\ a_1^* \arrow[ur] \arrow[dr] & a_7 \arrow[l] \arrow[r] & a_4^* \arrow[ul] \arrow[dl] \\ a_3 \arrow[u] & a_9^* \arrow[l] \arrow[r] \arrow[u] & a_6, \arrow[u] \end{tikzcd} \tag{5.1}$$

when there is one critical value the ranks of the subspaces are: $\mathrm{rank}(\mathrm{Mon}(\alpha_i)) = 5$ for $i \neq 7$ and $\mathrm{rank}(\mathrm{Mon}(\alpha_7)) = 3$. For more than one critical value, in Table 1 we present the cases where the vanishing cycles are not simple cycles.

For the Dynkin diagram

$$\begin{tikzcd} a_1^* \arrow[dr] & a_7 \arrow[l] \arrow[r] & a_4^* \arrow[dl] \\ a_3 \arrow[u] \arrow[d] & a_9^* \arrow[l] \arrow[r] \arrow[u] \arrow[d] & a_6 \arrow[u] \arrow[d] \\ a_2^* \arrow[ur] & a_8 \arrow[l] \arrow[r] & a_5^*, \arrow[ul] \end{tikzcd} \tag{5.2}$$

when there is one critical value the ranks of the subspaces are: $\mathrm{rank}(\mathrm{Mon}(\alpha_i)) = 5$ for $i \neq 9$ and $\mathrm{rank}(\mathrm{Mon}(\alpha_9)) = 3$. For more than one critical value, in Table 2 we present the cases where the vanishing cycles are not simple cycles.

In Tables 1 and 2, the first column is the number of different critical values. In the second column are written the vanishing cycles which are not simple cycles. Right in front of any non simple vanishing cycle α_i, in the third column, it is a basis for the subspace $\mathrm{Mon}(\alpha_i)$. Note that there are vanishing cycles which generate the same subspace, then they are on the same line in the second column. In fourth column are the corresponding Dynkin diagram. Finally, in the last column, we add information about an equivalence class, which is explained below. This last column together with the Theorem 5.4 gives us examples of polynomials that satisfy these diagrams.

Let $f(x,y) = h(y) + g(y)$ be a polynomial of degree 4, we consider the vector space $V_f = H_1(f^{-1}(b), \mathbb{Q})$ with the basis given by the vanishing cycles $\{\alpha_i\}_{i=1,\ldots,9}$. Let $G_f = \pi_1(\mathbb{C} \setminus \{a_1, \ldots, a_9\})$ be a free group acting on V_f by monodromy. For polynomials f, f', we relate f and f' if there is a permutation φ of the set $\{\alpha_i\}_{i=1,\ldots,9}$, such that

$$\mathrm{span}(G_f \cdot (\varphi(\alpha_i)) = \varphi(\mathrm{span}(G_{f'} \cdot \alpha_i)), \text{ for } i = 1, \ldots, 9.$$

Table 1
Monodromy for $h(x)+g(y)\in\mathbb{R}[x,y]_{d\leq 4}$ and Dynkin diagram (5.1).

# critical values	α_i	$\mathrm{Mon}(\alpha_i)$	Dynkin diagram of $f(x,y)=h(x)+g(y)$	$[f]$
2	α_1^*,α_4^* α_8^*,α_9^* α_7	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$ $\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$ $\langle\alpha_7,\alpha_1^*+\alpha_4^*,\alpha_8^*+\alpha_9^*,\alpha_2+\alpha_3+\alpha_5+\alpha_6\rangle$	$\begin{array}{ccccc} a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & b^* & \rightarrow & a \end{array}$ $\begin{array}{ccccc} a & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & b & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & a^* & \rightarrow & a \end{array}$	$\mathfrak{D}_3$
2	$\alpha_7,\alpha_8^*,\alpha_9^*$	$\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$	$\begin{array}{ccccc} a & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & b^* & \rightarrow & b \end{array}$	$\mathfrak{D}_2$
2	$\alpha_1^*,\alpha_4^*,\alpha_7$	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$	$\begin{array}{ccccc} b & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & a \end{array}$	$\mathfrak{D}_2$
3	α_1^*,α_4^* α_2,α_6 α_3,α_5 α_8^*,α_9^* α_7	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$ $\langle\alpha_2,\alpha_6,\alpha_7,\alpha_1^*-\alpha_8^*,\alpha_4^*-\alpha_9^*,\alpha_3+\alpha_5,\alpha_1^*+\alpha_4^*+\alpha_8^*+\alpha_9^*\rangle$ $\langle\alpha_3,\alpha_5,\alpha_7,\alpha_1^*-\alpha_9^*,\alpha_4^*-\alpha_8^*,\alpha_2+\alpha_6,\alpha_1^*+\alpha_4^*+\alpha_8^*+\alpha_9^*\rangle$ $\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$ $\langle\alpha_7,\alpha_1^*+\alpha_4^*,\alpha_8^*+\alpha_9^*,\alpha_2+\alpha_3+\alpha_5+\alpha_6\rangle$	$\begin{array}{ccccc} b & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & b \end{array}$	$\mathfrak{D}_4$
3	$\alpha_7,\alpha_8^*,\alpha_9^*$	$\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$	$\begin{array}{ccccc} b & \leftarrow & b^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & c^* & \rightarrow & c \end{array}$ $\begin{array}{ccccc} a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \end{array}$	$\mathfrak{D}_2$
3	$\alpha_1^*,\alpha_4^*,\alpha_7$	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$	$\begin{array}{ccccc} a & \leftarrow & c^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & c^* & \rightarrow & b \end{array}$ $\begin{array}{ccccc} b & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & c^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & a \end{array}$	$\mathfrak{D}_2$
4	α_1^*,α_4^* α_8^*,α_9^* α_7	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$ $\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$ $\langle\alpha_7,\alpha_1^*+\alpha_4^*,\alpha_8^*+\alpha_9^*,\alpha_2+\alpha_3+\alpha_5+\alpha_6\rangle$	$\begin{array}{ccccc} a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ c^* & \leftarrow & d & \rightarrow & c^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & b^* & \rightarrow & a \end{array}$	$\mathfrak{D}_3$

Table 1 (*continued*)

# critical values	α_i	$\mathrm{Mon}(\alpha_i)$	Dynkin diagram of $f(x,y)=h(x)+g(y)$	$[f]$
4	$\alpha_7, \alpha_8^*, \alpha_9^*$	$\langle \alpha_7, \alpha_8^*, \alpha_9^*, \alpha_1^*+\alpha_4^*, \alpha_2+\alpha_5, \alpha_3+\alpha_6 \rangle$	$\begin{array}{ccccc} a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & d^* & \rightarrow & c \end{array} \quad \begin{array}{ccccc} b & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & b^* & \rightarrow & c \end{array}$	$\mathfrak{O}_2$
4	$\alpha_1^*, \alpha_4^*, \alpha_7$	$\langle \alpha_1^*, \alpha_4^*, \alpha_7, \alpha_2+\alpha_3, \alpha_5+\alpha_6, \alpha_8^*+\alpha_9^* \rangle$	$\begin{array}{ccccc} b & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ d^* & \leftarrow & c & \rightarrow & c^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & a \end{array} \quad \begin{array}{ccccc} c & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & b \end{array}$	$\mathfrak{O}_2$
5	$\alpha_7, \alpha_8^*, \alpha_9^*$	$\langle \alpha_7, \alpha_8, \alpha_9, \alpha_1^*+\alpha_4^*, \alpha_2+\alpha_5, \alpha_3+\alpha_6 \rangle$	$\begin{array}{ccccc} b & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & e^* & \rightarrow & c \end{array} \quad \begin{array}{ccccc} b & \leftarrow & e^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \end{array} \quad \begin{array}{ccccc} a & \leftarrow & e^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & d & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \end{array}$	$\mathfrak{O}_2$
5	$\alpha_1^*, \alpha_4^*, \alpha_7$	$\langle \alpha_1^*, \alpha_4^*, \alpha_7, \alpha_2+\alpha_3, \alpha_5+\alpha_6, \alpha_8^*+\alpha_9^* \rangle$	$\begin{array}{ccccc} c & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ e^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & b \end{array} \quad \begin{array}{ccccc} c & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & e^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & b \end{array} \quad \begin{array}{ccccc} c & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & e^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & b^* & \rightarrow & a \end{array}$	$\mathfrak{O}_2$
6	$\alpha_7, \alpha_8^*, \alpha_9^*$	$\langle \alpha_7, \alpha_8^*, \alpha_9^*, \alpha_1^*+\alpha_4^*, \alpha_2+\alpha_5, \alpha_3+\alpha_6 \rangle$	$\begin{array}{ccccc} b & \leftarrow & e^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & f^* & \rightarrow & c \end{array}$	$\mathfrak{O}_2$
6	$\alpha_1^*, \alpha_4^*, \alpha_7$	$\langle \alpha_1^*, \alpha_4^*, \alpha_7, \alpha_2+\alpha_3, \alpha_5+\alpha_6, \alpha_8^*+\alpha_9^* \rangle$	$\begin{array}{ccccc} a & \leftarrow & c^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ d^* & \leftarrow & f & \rightarrow & e^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & c^* & \rightarrow & b \end{array}$	$\mathfrak{O}_2$

Table 2
Monodromy for $h(x) + g(y) \in \mathbb{R}[x, y]_{d\leq 4}$ and Dynkin diagram (5.2).

# critical values	α_i	$\mathrm{Mon}(\alpha_i)$	Dynkin diagram of $f(x, y =)h(x) + g(y)$	$[f]$
2	α_3, α_6 α_7, α_8 α_9^*	$\langle \alpha_3, \alpha_6, \alpha_9^*, \alpha_1^* + \alpha_2^*, \alpha_7 + \alpha_8, \alpha_4^* + \alpha_5^* \rangle$ $\langle \alpha_7, \alpha_8, \alpha_9^*, \alpha_1^* + \alpha_4^*, \alpha_2^* + \alpha_5^*, \alpha_3 + \alpha_6 \rangle$ $\langle \alpha_9^*, \alpha_3 + \alpha_6, \alpha_7 + \alpha_8, \alpha_1^* + \alpha_2^* + \alpha_4^* + \alpha_5^* \rangle$	$\begin{array}{ccccc} a^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & b^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & a & \rightarrow & a^* \end{array}$ $\begin{array}{ccccc} a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \end{array}$	$\mathfrak{O}_3$
2	$\alpha_7, \alpha_8, \alpha_9^*$	$\langle \alpha_7, \alpha_8, \alpha_9^*, \alpha_1^* + \alpha_4^*, \alpha_2^* + \alpha_5^*, \alpha_3 + \alpha_6 \rangle$	$\begin{array}{ccccc} b^* & \leftarrow & b & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & a & \rightarrow & a^* \end{array}$	$\mathfrak{O}_2$
2	$\alpha_3, \alpha_6, \alpha_9^*$	$\langle \alpha_3, \alpha_6, \alpha_9^*, \alpha_1^* + \alpha_2^*, \alpha_4^* + \alpha_5^*, \alpha_7 + \alpha_8 \rangle$	$\begin{array}{ccccc} b^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & a^* \end{array}$	$\mathfrak{O}_2$
3	α_3, α_6 α_7, α_8 α_9^*	$\langle \alpha_3, \alpha_6, \alpha_9^*, \alpha_1^* + \alpha_2^*, \alpha_7 + \alpha_8, \alpha_4^* + \alpha_5^* \rangle$ $\langle \alpha_7, \alpha_8, \alpha_9^*, \alpha_1^* + \alpha_4^*, \alpha_2^* + \alpha_5^*, \alpha_3 + \alpha_6 \rangle$ $\langle \alpha_9^*, \alpha_3 + \alpha_6, \alpha_7 + \alpha_8, \alpha_1^* + \alpha_2^* + \alpha_4^* + \alpha_5^* \rangle$	$\begin{array}{ccccc} a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \end{array}$	$\mathfrak{O}_3$
3	$\alpha_7, \alpha_8, \alpha_9^*$	$\langle \alpha_7, \alpha_8, \alpha_9^*, \alpha_1^* + \alpha_4^*, \alpha_2^* + \alpha_5^*, \alpha_3 + \alpha_6 \rangle$	$\begin{array}{ccccc} a^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & c^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & b & \rightarrow & b^* \end{array}$ $\begin{array}{ccccc} a^* & \leftarrow & c & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & b^* \end{array}$	$\mathfrak{O}_2$
3	$\alpha_3, \alpha_6, \alpha_9^*$	$\langle \alpha_3, \alpha_6, \alpha_9^*, \alpha_1^* + \alpha_2^*, \alpha_4^* + \alpha_5^*, \alpha_7 + \alpha_8 \rangle$	$\begin{array}{ccccc} a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & c^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \end{array}$ $\begin{array}{ccccc} a^* & \leftarrow & b & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & b^* \end{array}$	$\mathfrak{O}_2$
4	α_3, α_6 α_7, α_8 α_9^*	$\langle \alpha_3, \alpha_6, \alpha_9^*, \alpha_1^* + \alpha_2^*, \alpha_7 + \alpha_8, \alpha_4^* + \alpha_5^* \rangle$ $\langle \alpha_7, \alpha_8, \alpha_9^*, \alpha_1^* + \alpha_4^*, \alpha_2^* + \alpha_5^*, \alpha_3 + \alpha_6 \rangle$ $\langle \alpha_9^*, \alpha_3 + \alpha_6, \alpha_7 + \alpha_8, \alpha_1^* + \alpha_2^* + \alpha_4^* + \alpha_5^* \rangle$	$\begin{array}{ccccc} a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & d^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \end{array}$	$\mathfrak{O}_3$

Table 2 (*continued*)

# critical values	α_i	$\mathrm{Mon}(\alpha_i)$	Dynkin diagram of $f(x, y =)h(x) + g(y)$	$[f]$
4	$\alpha_7, \alpha_8, \alpha_9^*$	$\langle \alpha_7, \alpha_8, \alpha_9^*, \alpha_1^* + \alpha_4^*, \alpha_2^* + \alpha_5^*, \alpha_3 + \alpha_6 \rangle$	$\begin{array}{ccccc} c^* & \leftarrow & d & \rightarrow & c^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \end{array}$ $\begin{array}{ccccc} a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & b^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & b^* \end{array}$	$\mathfrak{O}_2$
4	$\alpha_3, \alpha_6, \alpha_9^*$	$\langle \alpha_3, \alpha_6, \alpha_9^*, \alpha_1^* + \alpha_2^*, \alpha_4^* + \alpha_5^*, \alpha_7 + \alpha_8 \rangle$	$\begin{array}{ccccc} b^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & c^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & a^* \end{array}$ $\begin{array}{ccccc} a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \end{array}$	$\mathfrak{O}_2$
5	$\alpha_7, \alpha_8, \alpha_9^*$	$\langle \alpha_7, \alpha_8, \alpha_9^*, \alpha_1^* + \alpha_4^*, \alpha_2^* + \alpha_5^*, \alpha_3 + \alpha_6 \rangle$	$\begin{array}{ccccc} a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & e^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & b^* \end{array}$ $\begin{array}{ccccc} a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & e & \rightarrow & b^* \end{array}$ $\begin{array}{ccccc} b^* & \leftarrow & d & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & e & \rightarrow & a^* \end{array}$	$\mathfrak{O}_2$
5	$\alpha_3, \alpha_6, \alpha_9^*$	$\langle \alpha_3, \alpha_6, \alpha_9^*, \alpha_1^* + \alpha_2^*, \alpha_4^* + \alpha_5^*, \alpha_7 + \alpha_8 \rangle$	$\begin{array}{ccccc} a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & e^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \end{array}$ $\begin{array}{ccccc} a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & a^* & \rightarrow & e \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \end{array}$ $\begin{array}{ccccc} b^* & \leftarrow & c & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & a^* & \rightarrow & e \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & c & \rightarrow & a^* \end{array}$	$\mathfrak{O}_2$
6	$\alpha_7, \alpha_8, \alpha_9^*$	$\langle \alpha_7, \alpha_8, \alpha_9^*, \alpha_1^* + \alpha_4^*, \alpha_2^* + \alpha_5^*, \alpha_3 + \alpha_6 \rangle$	$\begin{array}{ccccc} a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & e^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & f & \rightarrow & b^* \end{array}$	$\mathfrak{O}_2$
6	$\alpha_3, \alpha_6, \alpha_9^*$	$\langle \alpha_3, \alpha_6, \alpha_9^*, \alpha_1^* + \alpha_2^*, \alpha_4^* + \alpha_5^*, \alpha_7 + \alpha_8 \rangle$	$\begin{array}{ccccc} a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & e^* & \rightarrow & f \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \end{array}$	$\mathfrak{O}_2$

Let us introduce the suggestive notation v_{11}, v_{12}, v_{13}, $v_{21}, v_{22}, v_{23}, v_{31}, v_{32}, v_{33}$ as a basis for V_f. Then, we compute the equivalence classes $[f]$ of the polynomials in $\mathbb{R}[x]_{\leq 4} \oplus \mathbb{R}[y]_{\leq 4}$ with real critical points. From Tables 1 and 2, we conclude that there are 5 equivalence classes of these polynomials, they are

- $\mathfrak{O}_0$:
  $\mathrm{span}(G_f \cdot v_{ij}) = V_f$, for $i, j = 1, 2, 3$.
- $\mathfrak{O}_1$: In this cases G_f is a free group generated by a matrix M, and
  $\mathrm{span}(G_f \cdot v_{ij}) = \langle M^k v_{ij} \rangle$ with $k = 0, \dots, 4$ and $(i, j) \neq (2, 2)$.
  $\mathrm{span}(G_f \cdot v_{22}) = \langle M^k v_{ij} \rangle$ with $k = 0, \dots, 2$.
- $\mathfrak{O}_2$:
  $\mathrm{span}(G_f \cdot v_{21}) = \mathrm{span}(G_f \cdot v_{22}) = \mathrm{span}(G_f \cdot v_{23}) = \langle v_{21}, v_{22}, v_{23}, v_{11} + v_{31}, v_{12} + v_{32}, v_{13} + v_{33} \rangle$.
  $\mathrm{span}(G_f \cdot v_{ij}) = V_f$, in other cases.
- $\mathfrak{O}_3$:
  $\mathrm{span}(G_f \cdot v_{21}) = \mathrm{span}(G_f \cdot v_{23}) = \langle v_{21}, v_{22}, v_{23}, v_{11} + v_{31}, v_{12} + v_{32}, v_{13} + v_{33}. \rangle$.
  $\mathrm{span}(G_f \cdot v_{12}) = \mathrm{span}(G_f \cdot v_{32}) = \langle v_{12}, v_{22}, v_{32}, v_{11} + v_{13}, v_{21} + v_{23}, v_{31} + v_{33} \rangle$.
  $\mathrm{span}(G_f \cdot v_{22}) = \langle v_{22}, v_{12} + v_{32}, v_{21} + v_{23}, v_{11} + v_{13} + v_{31} + v_{33} \rangle$.
  $\mathrm{span}(G_f \cdot v_{ij}) = V_f$, in other cases.
- $\mathfrak{O}_4$:
  $\mathrm{span}(G_f \cdot v_{21}) = \mathrm{span}(G_f \cdot v_{23}) = \langle v_{21}, v_{22}, v_{23}, v_{11} + v_{31}, v_{12} + v_{32}, v_{13} + v_{33}. \rangle$.
  $\mathrm{span}(G_f \cdot v_{12}) = \mathrm{span}(G_f \cdot v_{32}) = \langle v_{12}, v_{22}, v_{32}, v_{11} + v_{13}, v_{21} + v_{23}, v_{31} + v_{33} \rangle$.
  $\mathrm{span}(G_f \cdot v_{11}) = \mathrm{span}(G_f \cdot v_{33}) = \langle v_{11}, v_{22}, v_{33}, v_{12} - v_{21}, v_{23} - v_{32}, v_{13} + v_{31}, v_{12} + v_{21} + v_{23} + v_{32} \rangle$.
  $\mathrm{span}(G_f \cdot v_{13}) = \mathrm{span}(G_f \cdot v_{31}) = \langle v_{13}, v_{22}, v_{31}, v_{21} - v_{32}, v_{12} - v_{23}, v_{11} + v_{33}, v_{12} + v_{21} + v_{23} + v_{32} \rangle$.
  $\mathrm{span}(G_f \cdot v_{22}) = \langle v_{22}, v_{12} + v_{32}, v_{21} + v_{23}, v_{11} + v_{13} + v_{31} + v_{33} \rangle$.

These equivalence classes of polynomial $f(x, y) = h(x) + g(y)$ in terms of the subspaces generated by the orbit of monodromy action, can be written in terms of the polynomials h and g. That is showed in the Theorem 5.4. We also consider polynomials up to linear transformation, because these do not change the monodromy action. For a polynomial $h \in \mathbb{C}[x]_{\leq d}$ and a partition $(d_1, d_2, \dots, d_M)$ of $d - 1$, we say that h has **critical values degree** $(d_1, d_2, \dots, d_M)$ if it has M different critical values $c_1, c_2, \dots, c_M$ and for any $i = 1 \dots, M$ there are d_i critical points over c_i, counted with multiplicity. The next lemma is proved in appendix A, and we give an algorithm to compute the ideals.

Lemma 5.3. *Given a positive integer d and a partition $(d_1, d_2, \dots, d_M)$ of $d - 1$, the set of polynomials in $\mathbb{C}[x]_{\leq d}$ with critical values degree $(d_1, d_2, \dots, d_M)$ is an algebraic subvariety of $\mathbb{C}[x]_{\leq d}$. Its ideal associated is denoted by $I_{(d_1, d_2, \dots, d_M)}$.*

For example for $d = 4$, we consider $h(x) := x^4 + r_3x^3 + r_2x^2 + r_1x + r_0$. Since translations in the abscissa and in the ordinate do not change the monodromy action, we can suppose that $h(x) := x^4 + r_2x^2 + r_1x$. Thus, we have

$$\begin{aligned} I_{(3,0)} &= \langle r_1, r_2\rangle, \\ I_{(2,1)} &= \langle r_1\rangle \cap \langle 27r_1^2 + 8r_2^3\rangle =: \langle r_1\rangle \cap \langle H\rangle, \\ I_{(1,1,1)} &= 0. \end{aligned}$$

Another interesting example, is when we consider the polynomials $h(x) = x^4+r_2x^2+r_1x$, $g(y) = y^4+s_2y^2+s_1y$ and we suppose that the critical values of h are equal to the critical values of g. In this case, we have a subvariety of $\mathbb{C}[x, y]_{\leq d}$ given by the ideal

$$\langle r_2 - s_2, r_1 - s_1\rangle \cap \langle r_2 + s_2, r_1^2 + s_1^2\rangle \cap \langle r_2 - s_2, r_1 + s_1\rangle \cap \langle s_2, s_1, r_2, r_1\rangle.$$

Theorem 5.4. *Let $f(x, y) = h(x)+g(y)$, where $h \in \mathbb{R}[x]_{\leq 4}$ and $g \in \mathbb{R}[y]_{\leq 4}$ are polynomials with real critical points. There is a characterization of the equivalence class of f in terms of h, g as follows,*

1. $[f] \in \mathfrak{O}_1$ *iff* $f(x, y) = x^4 + y^4$.
2. $[f] \in \mathfrak{O}_2$ *iff* $f(x, y) = (h_2 \circ h_1)(x) + g(y)$, *where* $h_1, h_2 \in \mathbb{R}[x]_{\leq 2}$ *and* g *is not decomposable.*
3. $[f] \in \mathfrak{O}_3$ *iff* $f(x, y) = (h_2\circ h_1)(x)+(g_2\circ g_1)(y)$, *where* $h_1, h_2 \in \mathbb{R}[x]_{\leq 2}$, $g_1, g_2 \in \mathbb{R}[y]_{\leq 2}$.
4. $[f] \in \mathfrak{O}_4$ *iff* $f(x, y) = (h_2 \circ h_1)(x) + (h_2 \circ h_1)(\pm y)$, *where* $h_1, h_2 \in \mathbb{R}[x]_{\leq 2}$.
5. $[f] \in \mathfrak{O}_0$ *iff* $h(x)$ *and* $g(y)$ *are not decomposable.*

Proof. It is easy to show that the conditions on h and g are sufficient conditions. Following, we show that they are necessary conditions. Since we consider polynomials up to linear transformation, we can suppose $h(x) = x^4 + r_2x^2 + r_1x$, $g(y) = y^4 + s_2y^2 + s_1y$. Thus, for h and g be decomposable polynomials it is necessary that $r_1 = 0$ and $s_1 = 0$, respectively.

1. For $[f] \in \mathfrak{O}_1$, the polynomial $f(x, y)$ has a critical value, thus h and g have only one critical value. Since $h \in I_{(3,0)}$, then $h(x) = x^4$, analogously for g.
2. When $[f] \in \mathfrak{O}_2$ we have the next possibilities: If the Dynkin diagram is (5.1), then the critical values satisfy $a_1 = a_4, a_2 = a_5, a_3 = a_6$ or $a_2 = a_3, a_5 = a_6, a_8 = a_9$. If the Dynkin diagram is (5.2), then the critical values satisfy $a_1 = a_4, a_2 = a_5, a_3 = a_6$ or $a_1 = a_2, a_4 = a_5, a_7 = a_8$. The first conditions in both Dynkin diagrams imply that $c_1^h = c_2^h$, the others conditions imply $c_2^g = c_3^g$ and $c_1^g = c_3^g$, respectively. Without loss of generality we consider $c_1^h = c_2^h$, then $h \in I_{(2,1)}$, and recall that the 0-dimensional Dynkin diagram for h in this case is $\gamma_1 \cdots \gamma_3 \cdots \gamma_2$.
   Furthermore, the discriminant of $h'(x)$ is equal to $-16H$, thus $\mathbf{V}(H)$ corresponds to the polynomials with at most 2 critical points. Hence, the polynomials in

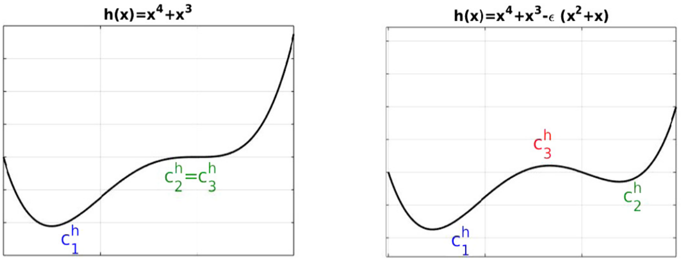

Fig. 11. In the left a polynomial $g \in \mathbb{R}[y]_{\leq 4}$ with two critical values and two critical points. In the right a perturbation of g which separates the critical values. In both cases the enumeration is done according to section 2. The vanishing cycle associated to c_1^h and c_3^h always intersects.

$\mathbf{V}(H) \setminus \mathbf{V}(I_{(3,0)})$ are polynomials that have two different critical values and two critical points, and it is not the case of $c_1^h = c_2^h$ and $c_1^h \neq c_3^h$ see the Fig. 11. Therefore, the polynomials in $\mathfrak{O}_2$ satisfy that $h(x) = x^4 + r_2 x^2$. Then $h(x) = h_2(h_1(x))$ where $h_1(x) = x^2$ and $h_2(x) = x(x + r_2)$.

3. If $[f] \in \mathfrak{O}_3$, then the conditions $c_1^h = c_2^h$ and $c_1^g = c_2^g$, with 0-dimensional Dynkin diagram $\delta_1 \cdots \delta_3 \cdots \delta_2$ associated to g, are satisfied simultaneously (or $c_2^g = c_3^g$, with 0-dimensional Dynkin diagram $\delta_2 \cdots \delta_1 \cdots \delta_3$ associated to g). Hence, analogously to the previous case we have $f(x, y) = h_2(h_1(x)) + g_2(g_1(y))$ where $h_1(x) = x^2$, $h_2(x) = x(x + r_2)$, $g_1(y) = y^2$, $g_2(y) = y(y + s_2)$.
4. For $[f] \in \mathfrak{O}_4$, the Dynkin diagram is (5.1) and the critical values of f satisfy the relations $a_1 = a_9$, $a_2 = a_6$ and $a_4 = a_8$, that means $c_1^h + c_1^g = c_3^h + c_3^g$, $c_1^h + c_2^g = c_2^h + c_3^g$ and $c_2^h + c_1^g = c_3^h + c_2^g$. Thus $c_1^h = c_3^g + k$, $c_2^h = c_2^g + k$ and $c_3^h = c_1^g + k$, where $k = c_3^h - c_1^g$. Hence, by doing a translation, we can suppose that the critical values of the polynomial h are equals to the critical values of g. Therefore, $g(x) = h(\pm x)$. On the other hand, the conditions $a_1 = a_4$, $a_2 = a_5$, $a_3 = a_6$, imply that h is decomposable.
5. When $[f] \in \mathfrak{O}_0$, the critical values of h are different or $h \in \mathbf{V}(H) \setminus \mathbf{V}(I_{(3,0)})$. Therefore, $r_1 \neq 0$. Analogously for g, we conclude that $s_1 \neq 0$. □

Remark. Similar to Theorem 4.7, if $[f] \in \mathfrak{O}_2$, then the vanishing cycles v_{21}, v_{22}, v_{23} are in the kernel of the map

$$H_1(f^{-1}(b), \mathbb{Q}) \to H_1(\tilde{f}^{-1}(b), \mathbb{Q}), \qquad \text{where } \tilde{f} = h_2(x) + g(y)$$

coming from the map $(x, y) \to (h_1(x), y)$. If $[f] \in \mathfrak{O}_3$, then the vanishing cycles v_{21}, v_{22}, v_{23} are as before, and the vanishing cycles v_{12}, v_{22}, v_{32} are in the kernel of the map

$$H_1(f^{-1}(b), \mathbb{Q}) \to H_1(\hat{f}^{-1}(b), \mathbb{Q}), \qquad \text{where } \hat{f} = h(x) + g_2(y)$$

coming from the map $(x, y) \to (x, g_1(y))$. The other not simple vanishing cycles appear with the symmetry $h(x) = g(\pm x)$. Therefore, they may be related with the pullback

$$\begin{aligned} \mathbb{C}^2 \longrightarrow \qquad \mathbb{C}^2 \qquad &\longrightarrow \mathbb{C} \\ (x,y) \to (x+y, xy) &\to \check{f}(\check{x},\check{y}) \end{aligned}$$

where $\check{x} = x + y$, $\check{y} = xy$ and some $\check{f} \in \mathbb{R}[\check{x},\check{y}]_{\leq 4}$. So far we do not know a geometrical characterization for these vanishing cycles.

Declaration of competing interest

There is no competing interest.

Appendix A. Algebraic space $I_{(d_1,d_2,\ldots,d_M)}$

In this section we show that the space of polynomials $f(x)$ of degree d with a given number of critical values is an algebraic subspace. Actually, we need other conditions in the cardinality of the critical values, it motivates the next definition.

Definition A.1. For an integer d and a partition $(d_1, d_2, \ldots, d_M)$ of $d-1$, we say that the polynomial $f(x) \in \mathbb{C}[x]_{\leq d}$ has **critical values degree** $(d_1, d_2, \ldots, d_M)$ if it has M different critical values $c_1, c_2, \ldots, c_M$ and for any $i = 1\ldots, M$ there are d_i critical points over c_i, counted with multiplicity.

We show that the condition of critical values degree for a polynomial can be given in terms of algebraic expressions.

Proof of Lemma 5.3. By definition of the *discriminant* Δ, see [18, §10.9], we know that ξ is a critical value of $f(x)$ if and only if $\Delta(f(x)-\xi) = 0$. For $f(x) = x^d + r_{d-1}x^{d-1} + \ldots + r_0$, we have that $\Delta_\xi(f) := \Delta(f-\xi)$ is a polynomial $\lambda(\xi) = \alpha_{d-1}\xi^{d-1} + \alpha_{d-2}\xi^{d-2} + \ldots + \alpha_0$. It defines the map

$$\begin{gathered} \mathbb{C}^d \xrightarrow{\Delta_\xi} \mathbb{C}^{d-1} \\ (r_{d-1}, \ldots, r_0) \to (\alpha_{d-2}, \ldots, \alpha_0). \end{gathered}$$

The polynomial $\lambda(\xi)$ can also be expressed in terms of the critical values $t_1, \ldots, t_{d-1}$ of f, as $\lambda(\xi) = (\xi - t_1)(\xi - t_2)\ldots(\xi - t_{d-1}) = (\xi - t_{i_1})^{d_1}(\xi - t_{i_2})^{d_2}\ldots(\xi - t_{i_M})^{d_M}$, where we have used the definition of critical values degree. Let $\mathbb{C}^{d-1} \xrightarrow{\varphi} \mathbb{C}^{d-1}$ be the map given by the Vieta's formula

$$\begin{aligned} &\mathbb{C}^{d-1} \xrightarrow{\varphi} \mathbb{C}^{d-1} \\ (t_1, t_2, \ldots, t_{d-1}) \xrightarrow{\varphi} (\varepsilon_1 &\sum_i t_i, \varepsilon_2 \sum_{i\neq j} t_i t_j, \ldots, \varepsilon_{d-1} t_1 t_2 \ldots t_{d-1}), \end{aligned}$$

where $\varepsilon_j = (-1)^j \alpha_{d-1}$. Hence, φ take the roots of a polynomial and gives the coefficients of the polynomial. Let V be a subvariety in the domain of φ given by M equations of the form $t_{i_1} = t_{i_2} \ldots = t_{i_{d_j}}$ with $j = 1, \ldots, M$. The subvariety V has the information of the critical values degree.

The closure of $\varphi(V)$ is a subvariety of $\mathbb{C}^{d-1}$, we denote it by W. The pullback of W by the map Δ_ξ is a subvariety in $\mathbb{C}^d$ in terms of the parameters r_k which is the closure of the space of polynomial in $\mathbb{C}[x]_{\leq d}$ with critical values degree $(d_1, d_2, \ldots d_M)$.

$$\begin{tikzcd} V \subset \mathbb{C}^{d-1} \arrow[r, "\varphi"] & W \subset \mathbb{C}^{d-1} \\ \mathbb{C}^d \arrow[ur, "\Delta_\xi"'] & \end{tikzcd}$$

□

In order to compute an explicit expression for W we use the implicitation algorithm [5, §3.3]. Let $v_1, v_2, ..., v_s$ be the polynomials which describes V, thus $v_i = v_i(t_1, \ldots, t_{d-1})$. Let $I \subset \mathbb{C}[t_1, \ldots, t_{d-1}, x_1, \ldots, x_{d-1}]$ be the ideal

$$I = \langle z_1 - \sum t_i, z_2 - \sum_{i \neq j} t_i t_j, ..., z_{d-1} - t_1 t_2 ... t_{d-1}, v_1, ..., v_s \rangle.$$

If G is a Groebner basis of I with respect to lexicographic order $t_1 > t_2 > ... > t_{d-1} > z_1 > ... > z_{d-1}$, then $G_z = G \cap \mathbb{C}[z]$ is a Groebner basis of the ideal $I_z := I \cap \mathbb{C}[z]$. Also $W := \mathbf{V}(I_z)$ is the smallest variety in $\mathbb{C}^{d-1}$ containing $\varphi(V)$.

Appendix B. Numerical supplementary items

In this section we provide the explanation of the codes used in the proof of Proposition 3.1. To start, it is necessary to get the MATLAB's functions MonMatrix and VanCycleSub. These are available in https://github.com/danfelmath/Intersection-matrix-for-polynomials-with-1-crit-value.git.

The function MonMatrix computes the monodromy matrix for the polynomial

$$f := y^e + x^d \tag{B.1}$$

in the basis described in §3. That is, for each by considering a perturbation of y^e and x^d, such that the critical values are different. Thus, we have a real curve similar to the Fig. 3. Moreover, we can suppose that the critical points induce the 0-Dynkin diagrams

$$\sigma_{l_d+1} \text{ -- } \sigma_1 \text{ -- } \sigma^1_{l_d+2} \text{ -- } \sigma_2 \text{ -- } \sigma_{l_d+3} \text{ -- } \sigma_3 \quad \cdots$$

$$\gamma_{l_e+1} \text{ -- } \gamma_1 \text{ -- } \gamma^1_{l_e+2} \text{ -- } \gamma_2 \text{ -- } \gamma_{l_e+3} \text{ -- } \gamma_3 \quad \cdots$$

where $l_d = \lfloor \frac{d-1}{2} \rfloor$ and $l_e = \lfloor \frac{e-1}{2} \rfloor$. The cycles σ_i with $i = 1, \ldots, d-1$ and γ_j with $j = 1, \ldots, e-1$ are the 0-dimensional vanishing cycles associated to x^d and y^e, respectively. The last vanishing cycle on the right in this Dynkin diagram is γ_{l_e} or γ^i_{e-1}, depending on whether e is odd or even, respectively (analogously for the last σ_i).

Then, we consider the basis given by the join cycles of the vanishing cycles $\gamma_j * \sigma_i$ where $j = 1, \ldots, e-1$ and $i = 1, \ldots, d-1$. Furthermore, we consider the orderings $\sigma_{l_e+1} > \gamma_1 > \sigma_{l_e+2} > \sigma_2 > \sigma_{l_e+3} > \sigma_3 > \cdots$, and $\gamma_{l_d+1} > \gamma_1 > \gamma_{l_d+2} > \gamma_2 > \gamma_{l_d+3} > \gamma_3 > \cdots$, for any i. The ordering for the join cycles is given by

$$\gamma_j * \sigma_i > \gamma_{j'} * \sigma_{i'}, \text{ if and only if, } \sigma_i > \sigma_{i'}, \text{ or } i = i' \text{ and } \gamma_j > \gamma_{j'}.$$

We use the ordered basis $(\gamma_j * \sigma_i, >)$, in order to write the intersection and monodromy matrices associated to the fibration given by (B.1). For example, let $f(x, y) = x^6 + y^4$, therefore the ordered basis is

$$\gamma_2 * \sigma_3, \gamma_1 * \sigma_3, \gamma_3 * \sigma_3, \gamma_2 * \sigma_1, \gamma_1 * \sigma_1, \ldots, \gamma_3 * \sigma_5.$$

In §3 we use the notation δ^j_i to denote the vanishing cycle in the row i and column j of the Dynkin diagram (3.1). Thus, $\delta^j_1 = \gamma_2 * \sigma_{\rho(j)}$, $\delta^j_2 = \gamma_1 * \sigma_{\rho(j)}$, $\delta^j_3 = \gamma_3 * \sigma_{\rho(j)}$, where ρ is the permutation $(1, 2, 3, 4, 5) \xrightarrow{\rho} (3, 1, 4, 2, 5)$.

In order to compute the intersection matrix by using the function MonMatrix, it is enough to write *Im=MonMatrix(m,p)*, where m is a vector whose coordinate corresponds to $m(1) = d$ and $m(2) = e$. The parameter p should be a integer number such that: If $p = 0$, then Im is the intersection matrix, else Im is the monodromy matrix. For the previous example, we get the matrix (3.4), by writing the lines

```
m=[6,4];
Im=MonMatrix(m, 0)
```

The function VanCycleSub computes the subspace spanned by the monodromy action of the fibration given by (B.1), acting on each vanishing cycle. In other words, this function computes the Krylov space of the monodromy matrix and each one of the vectors of the basis previously described. That is by computing the eigenvalues and eigenvector of the monodromy matrix as in the proof of Proposition 3.1. The usage of this function is as follows: *[Dim, Wout, Vout]=VanCycleSub(m)*, where the vector m represented again the degrees d, e.

The output *Dim* is the number of different eigenvalues of the monodromy matrix. Note that the dimension of the homology group $H_n(f^{-1}(b))$, is $N = (d-1)(e-1)$. If $Dim = N$, then the array *Wout* of size $N \times N \times N$, represents the Krylov space of each vanishing cycle. Thus, the columns of the matrix *Wout(:,:,j)* are a basis of the subspace generated by the monodromy action on the vector $e_j = (0 \cdots 0\ 1\ 0 \cdots 0)$. The vector e_j corresponds with the j-th joint cycle according to the previously defined order.

Finally, *Vout* is a matrix where the j-th column is a list of the vanishing cycles in the Krylov subspace of the vector e_j with the monodromy matrix. Actually, this list is the position given by the order, associated to these vanishing cycles. Note that these vanishing cycles correspond to the rows of *Wout(:,:,j)* with a single 1 and zeros in the others. Continuing the example,

```
m=[6,4];
[Dim, Wout,Vout]=VanCycleSub(m)
```

In this case the second column of $Vout$ is the list $(2, 5, 8, 11, 14)$, which are the positions associated to the vanishing cycles δ_2^k, with $k = 1, \ldots, 5$ (see Dynkin diagram (3.1)). The fifth column is the list $(5, 11)$; it is because the vanishing cycles associated to the position 5th and 11th are δ_2^2 and δ_2^4, respectively, and $gcd(d, 2) = 2$ (see Proposition 3.1).

References

[1] N. A'Campo, Le groupe de monodromie du déploiement des singularités isolées de courbes planes I, Math. Ann. (1975) 1–32.
[2] V.I. Arnold, A.N. Varchenko, S. Gusein-Zade, Singularities of Differentiable Maps: Volume II Monodromy and Asymptotic Integrals, vol. 83, Springer Science & Business Media, 1988.
[3] D. Cerveau, A.L. Neto, Irreducible components of the space of holomorphic foliations of degree two in $\mathbb{CP}(n), n \geq 3$, Ann. Math. (1996) 577–612.
[4] C. Christopher, P. Mardešić, The monodromy problem and the tangential center problem, Funct. Anal. Appl. 44 (1) (2010) 22–35.
[5] D. Cox, J. Little, D. O'Shea, Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, Springer Science & Business Media, 2013.
[6] C. Da Fonseca, V. Kowalenko, Eigenpairs of a family of tridiagonal matrices: three decades later, Acta Math. Hung. (2019) 1–14.
[7] C. Doran, J. Morgan, Mirror symmetry and integral variations of Hodge structure underlying one parameter families of Calabi-Yau threefolds, in: V, AMS/IP Studies in Advanced Mathematics, vol. 38, 2006.
[8] H. Dulac, Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un centre, vol. 32, Gauthier-Villars, 1908.
[9] J.P. Françoise, Successive derivatives of a first return map, application to the study of quadratic vector fields, Ergod. Theory Dyn. Syst. 16 (1) (1996) 87–96.
[10] L. Gavrilov, Petrov modules and zeros of Abelian integrals, Bull. Sci. Math. 122 (8) (1998) 571–584.
[11] L. Gavrilov, H. Movasati, The infinitesimal 16th Hilbert problem in dimension zero, Bull. Sci. Math. 131 (2007).
[12] Y. Ilyashenko, The origin of limit cycles under perturbation of the equation $dw/dz = -r_z/r_w$, where $r(z, w)$ is a polynomial, Mat. Sb. 120 (3) (1969) 360–373.
[13] K. Lamotke, The topology of complex projective varieties after S. Lefschetz, Topology 20 (1) (1981) 15–51.
[14] D. López Garcia, Homology supported in Lagrangian submanifolds in mirror quintic threefolds, Can. Math. Bull. (2020).
[15] L. Losonczi, Eigenvalues and eigenvectors of some tridiagonal matrices, Acta Math. Hung. 60 (3–4) (1992) 309–322.
[16] H. Movasati, Abelian integrals in holomorphic foliations, Rev. Mat. Iberoam. 20 (1) (2004) 183–204.
[17] H. Movasati, Center conditions: rigidity of logarithmic differential equations, J. Differ. Equ. 197 (1) (2004) 197–217.
[18] H. Movasati, A Course in Hodge Theory, with Emphasis on Multiple Integrals, http://w3.impa.br/~hossein/myarticles/hodgetheory.pdf, To be published by IP, Boston, 2017.
[19] A.L. Neto, Componentes irredutíveis dos espaços de folheações, Publicaçoes Matematicas do IMPA, 2007.
[20] A.L. Neto, Foliations with a Morse center, J. Singul. 9 (2014) 82–100.

[21] R. Roussarie, Bifurcation of Planar Vector Fields and Hilbert's Sixteenth Problem, vol. 164, Birkhäuser, 1998.
[22] W. Yueh, Eigenvalues of several tridiagonal matrices, Appl. Math. E-Notes 5 (66–74) (2005) 210–230.
[23] Y. Zare, Center conditions: pull back of differential equations, Trans. Am. Math. Soc. (2017).