Numerical Study of Wave Equation

Estudio Numérico de la Ecuación de Onda

Daniel Felipe López García

Juan Galvis (Orientador)

Universidad Nacional

Facultad de Ciencias

Proyecto Curricular de Matemticas

Bogotá D.C.

Diciembre de 2014

Contents

Chapter 1

Wave Equation

A main idea in several physical phenomena and engineering applications (as the antennas) is the wave propagation. Depending on the model, we can classify the partial differential equation in three groups; *acoustic equations* model mechanical waves in fluids, *elastic equations* describe the propagation of mechanical waves in solids and *Maxwell equations* model the electromagnetic waves. In this work we are going to emphasize on elastic equation, however, in this chapter we review some ideas about the three models.

1.1 Acoustic Wave

The propagation of a mechanical wave is different is different than that of an electromagnetic waves. The first one can be considered as vibrations of air (or whatever medium where it propagates) characterized by a sequence of expansions and compressions in time and space while the second one can propagate itself without medium (i.e., vacuum). Given a medium with density ρ (in this case is a solid medium), let u be the acoustic pressure field. This is a function defined on $\mathbb{R}^d \times \mathbb{R}^+$ with values in $\mathbb{R}$ (where $d = 2, 3$). From the Euler equations we derive the *acoustics equation* as follows,

$$\frac{1}{\kappa(\mathbf{x})}\frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} - \nabla\cdot\left(\frac{1}{\rho(\mathbf{x})}\nabla u(\mathbf{x},t)\right) = f(\mathbf{x},t) \tag{1.1}$$

where ∇ is the gradient operator, and ρ and κ are positive functions of the spatial variable $\mathbf{x}$. Note the $\nabla\cdot$ is the divergence operator.

It is possible to write this equation as a first-order system by using the vector velocity $\boldsymbol{v}$. We have

$$\rho(\mathbf{x})\frac{\partial v(\mathbf{x},t)}{\partial t} = \nabla u(\mathbf{x},t).$$

Then, if we replace this expression in the equation (1.1), we can write

$$\frac{1}{\kappa(\mathbf{x})}\frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} - \nabla\cdot\left(\frac{\partial v(\mathbf{x},t)}{\partial t}\right) = f(\mathbf{x},t)$$

and integrating in t, we set the system

$$\begin{cases} \frac{1}{\kappa(\mathbf{x})}\frac{\partial u(\mathbf{x},t)}{\partial t} = \nabla\cdot\boldsymbol{v}(\mathbf{x},t) + F(\mathbf{x},t), \\ \rho(\mathbf{x})\frac{\partial v(\mathbf{x},t)}{\partial t} = \nabla u(\mathbf{x},t), \end{cases} \tag{1.2}$$

where $F(\mathbf{x}, t) = \int_0^t f(\mathbf{x}, \tau) d\tau$.

There is a relation in terms of sound velocity in the position $\mathbf{x}$ for $\kappa(\mathbf{x})$ and $\rho(\mathbf{x})$ that is

$$c(\mathbf{x}) = \sqrt{\frac{\kappa(\mathbf{x})}{\rho(\mathbf{x})}}.$$

For the system to be viewed as well-posed problem is necessary write

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) \quad \text{and} \quad \frac{\partial u(\mathbf{x}, 0)}{t} = u_1(\mathbf{x})$$

where u_0, u_1 are functions defined on $\mathbb{R}^b \times \mathbb{R}^+$ with values in $\mathbb{R}$. These conditions are add to the equation (1.1), and for the equation (1.2) are posed in terms of $\boldsymbol{v}$.

1.1.1 Acoustic Wave Equation In One Dimension

The general wave equation must satisfy the following conditions:

1. Provide solution for waves propagating from the left or from right, i.e., it admits solutions of the form $y = f(x - vt)$ and $g = (x + vt)$, where f, g is an arbitrary function, v is the propagation velocity, t is a time variable and x, y are space variables.

2. The equation should admit that two or more signals propagate simultaneously, i.e., $y = f(x - vt) + g(x + vt)$ is a solution.

Assuming solution has the form $y = f(x - vt)$ and let $s = x - vt$. If its second space derivative is calculated, we have

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial x},$$

since $\frac{\partial s}{\partial x} = 1$,

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial s} = f'(s)$$

is obtained. Furthermore, we see that

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial (f'(s))}{\partial s}\frac{\partial s}{\partial x} = f''(s).$$

Now, its time derivative is

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial t} = -vf'(s).$$

Then, we have,

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial (-vf'(s))}{\partial t} = -v\frac{\partial (f'(s))}{\partial s}\frac{\partial s}{\partial t} = v^2 f''(s).$$

Finally, if the two equation are combined we obtain that

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}. \tag{1.3}$$

1.2 The Maxwell Equations

The Maxwell equations describe the electromagnetic wave propagation. In these equations there are relations among electric field $\mathbf{E}$, magnetic field $\mathbf{H}$, electric induction $\mathbf{D}$ and magnetic induction $\mathbf{B}$. The equations of electromagnetism can be posed as (Cohen, 2002)

$$\frac{\partial \mathbf{D}(\mathbf{x},t)}{\partial t} - \nabla \times \mathbf{H}(\mathbf{x},t) = -\mathbf{J}(\mathbf{x},t), \tag{1.4}$$

$$\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t} + \nabla \times \mathbf{E}(\mathbf{x},t) = 0, \tag{1.5}$$

$$\mathbf{D} = \varepsilon(\mathbf{x})\mathbf{E}, \tag{1.6}$$

and

$$\mathbf{B} = \mu(\mathbf{x})\mathbf{H}, \tag{1.7}$$

where ε and μ are dielectric permittivity and magnetic permeability, respectively. These parameters are symmetric and positive definite matrices depending on space. In general ε and μ model anisotropic media. The isotropic case is obtained when $\varepsilon = aI_3$ and $\mu = bI_3$, were a and b are strictly positive scalar functions and I_3 is the identity matrix of length three. The vector $\mathbf{J}$ is the current density.

Furthermore, the electric and magnetic inductions satisfy,

$$\mathrm{div}\mathbf{D} = \rho \tag{1.8}$$

and

$$\mathrm{div}\mathbf{B} = 0 \tag{1.9}$$

where ρ is the charge density.

The second-order form of the Maxwell is obtained by combining (1.4) and (1.5), as follows,

$$\varepsilon(\mathbf{x})\frac{\partial^2 \mathbf{E}(\mathbf{x},t)}{\partial t^2} + \nabla \times (\mu^{-1}(\mathbf{x}) \times \mathbf{E}(\mathbf{x},t)) = -\mathbf{j}(\mathbf{x},t), \tag{1.10}$$

and

$$\mu(\mathbf{x})\frac{\partial^2 \mathbf{H}(\mathbf{x},t)}{\partial t^2} + \nabla \times (\varepsilon^{-1}(\mathbf{x}) \times \mathbf{H}(\mathbf{x},t)) = \mathbf{J}'(\mathbf{x},t), \tag{1.11}$$

where $\mathbf{j} = \frac{\partial \mathbf{J}}{\partial t}$ and $\mathbf{J}' = \nabla \times (\varepsilon^{-1}\mathbf{J})$.
When we consider an isotropic media (vacuum for instance), and in addition, if we are far enough from the source, it is possible to approximate $\mathbf{J} = 0$ and $\rho = 0$. If the fact that $\nabla \times \nabla\mathbf{A} = \nabla(div\mathbf{A}) - \nabla \times \mathbf{A}$ and the equations (1.8), (1.9) are used, the equations (1.10) and (1.11) can be written as

$$\frac{\partial^2 \mathbf{E}(\mathbf{x},t)}{\partial t^2} - \frac{1}{\varepsilon\mu}\Delta\mathbf{E}(\mathbf{x},t) = 0,$$

and

$$\frac{\partial^2 \mathbf{H}(\mathbf{x},t)}{\partial t^2} - \frac{1}{\varepsilon\mu}\Delta\mathbf{H}(\mathbf{x},t) = 0,$$

Respectively.

The last set of equations show that each component of $\mathbf{E}$ and $\mathbf{H}$ satisfies the wave equation with velocity equal to $\frac{1}{\sqrt{\varepsilon\mu}}$. The light velocity c in a non-homogeneous isotropic medium is defined by $c^2(\mathbf{x})\varepsilon(\mathbf{x})\mu(\mathbf{x}) = 1$.

1.3 Elastic Wave

This work focuses on elastic wave propagation , i.e., wave that propagates in a solid medium. For this reason we emphasize this model. In the Appendix A present some review about tensors and the stress tensor.

1.3.1 General Formulation

Let $\boldsymbol{v}$ in $\mathbb{R}^d$ (with $d = 1, 2, 3$) be the *displacement vector* and $\boldsymbol{\sigma}$ the stress tensor for a elastic medium. Following (Cohen, 2002) we write the general formulation of the elastic system in a non-homogeneous, anisotropic medium, as

$$\rho(\boldsymbol{x})\frac{\partial^2 \boldsymbol{v}(\boldsymbol{x},t)}{\partial t^2} - \mathbf{div}\boldsymbol{\sigma}(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x},t) \tag{1.12}$$

and

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \boldsymbol{C}(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x},t), \tag{1.13}$$

where $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_d)$ and $\boldsymbol{\sigma}_i = (\sigma_{i1}, \ldots, \sigma_{id})^T$, i.e.,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma 12 & \ldots & \sigma 1d \\ \sigma_{21} & \sigma 22 & \ldots & \sigma 2d \\ \vdots & \vdots & & \vdots \\ \sigma_{d1} & \sigma d2 & \ldots & \sigma dd \\ . \end{pmatrix}.$$

Also we have $\mathbf{div}\boldsymbol{\sigma} = (\nabla \cdot \boldsymbol{\sigma}_1, \ldots, \nabla \cdot \boldsymbol{\sigma}_d)^T$ and $\varepsilon(\boldsymbol{v}) = (\varepsilon_{ij}\boldsymbol{v})$, where

$$\varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad \text{and} \quad (\boldsymbol{C}\varepsilon)_{ij} = \sum_{k=1}^{d}\sum_{l=1}^{d} C_{ijkl}\varepsilon_{kl}.$$

Furthermore, $\boldsymbol{C}$ is a cyclic symmetric tensor, i.e., $C_{ijkl} = C_{klij} = C_{jikl}$. The symmetry of $\boldsymbol{C}$ implies the symmetry of $\boldsymbol{\sigma}$. The system has initial condition,

$$\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}) \ \text{ and } \ \frac{\partial \boldsymbol{v}(\boldsymbol{x},0)}{\partial t} = \boldsymbol{v}_1(\boldsymbol{x}).$$

1.3.2 Isotropic Case

An isotropic medium is a medium that has uniformity of properties in all orientations. In this case we write $\boldsymbol{C}$ as

$$C_{ijkl} = \lambda\delta_{ij}\delta kl + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where δ_{jl} is the Kronecker's delta, i.e., $\delta_{jl} = \begin{cases} 0 \text{ if } j \neq l \\ 1 \text{ if } j = l \end{cases}$. Then the stress tensor $\boldsymbol{\sigma}$ takes the form

$$\sigma_{ij} = \lambda\delta_{ij}\sum_{k=1}^{d}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

where λ and μ are Lamé 's coefficients.

Thus, the equation (1.12) and (1.13) can be written as

$$\rho(\boldsymbol{x})\frac{\partial^2 \boldsymbol{v}(\boldsymbol{x},t)}{\partial t^2} - \mathbf{div}\boldsymbol{\sigma}(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x},t). \tag{1.14}$$

The stress tensor $\boldsymbol{\sigma}$ is written as follow

$$\boldsymbol{\sigma}_D = (\sigma_{ii})_i^d = \mathcal{A}(\boldsymbol{x})\boldsymbol{v},$$

and

$$\boldsymbol{\sigma}_{ND} = \left(\sigma_{ij}\right)_{1|\leq i<j\leq d} = \mathcal{B}(\boldsymbol{x})\boldsymbol{v}.$$

Here, $\boldsymbol{\sigma}_D$ represents the diagonal part of $\boldsymbol{\sigma}$, and $\boldsymbol{\sigma}_{ND}$ represents off-diagonal part of $\boldsymbol{\sigma}$. Note that $\boldsymbol{\sigma}_{ND}$ is defined using only the upper part to the diagonal because $\boldsymbol{\sigma}$ is symmetric.

Below we write $\mathcal{A}$ and $\mathcal{B}$ for $d = 2, 3$, i.e., 2-dimensions and 3-dimensions.

2-D case. Let $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{v} = (v_1, v_2,)$ and $\boldsymbol{f} = (f_1, f_2)$, then,

$$\mathcal{A} = \begin{pmatrix} (\lambda+2\mu)\frac{\partial}{\partial x_1} & \lambda\frac{\partial}{\partial x_2} \\ \lambda\frac{\partial}{\partial x_1} & (\lambda+2\mu)\frac{\partial}{\partial x_2} \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} \mu\frac{\partial}{\partial x_2} & \mu\frac{\partial}{\partial x_1} \end{pmatrix}.$$

Thus the equation (1.14) is written as

$$\rho\frac{\partial^2 v_1}{\partial t^2} - \frac{\partial}{\partial x_1}\left((\lambda+2\mu)\frac{\partial v_1}{\partial x_1} + \lambda\frac{\partial v_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) = f_1 \tag{1.15}$$

or equivalently

$$\rho\frac{\partial^2 v_2}{\partial t^2} - \frac{\partial}{\partial x_1}\left(\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) - \frac{\partial}{\partial x_2}\left(\lambda\frac{\partial v_1}{\partial x_1} + (\lambda+2\mu)\frac{\partial v_2}{\partial x_2}\right) = f_2. \tag{1.16}$$

3-D case. Let $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$ and $\boldsymbol{f} = (f_1, f_2, f_3)$, then,

$$\mathcal{A} = \begin{pmatrix} (\lambda+2\mu)\frac{\partial}{\partial x_1} & \lambda\frac{\partial}{\partial x_2} & \lambda\frac{\partial}{\partial x_3} \\ \lambda\frac{\partial}{\partial x_1} & (\lambda+2\mu)\frac{\partial}{\partial x_2} & \lambda\frac{\partial}{\partial x_3} \\ \lambda\frac{\partial}{\partial x_1} & \lambda\frac{\partial}{\partial x_2} & (\lambda+2\mu)\frac{\partial}{\partial x_3} \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} \mu\frac{\partial}{\partial x_2} & \mu\frac{\partial}{\partial x_1} & 0 \\ \mu\frac{\partial}{\partial x_3} & 0 & \mu\frac{\partial}{\partial x_1} \\ 0 & \mu\frac{\partial}{\partial x_3} & \mu\frac{\partial}{\partial x_2} \end{pmatrix}$$

Equation (1.14) is written as

$$\rho\frac{\partial^2 v_1}{\partial t^2} - \frac{\partial}{\partial x_1}\left((\lambda+2\mu)\frac{\partial v_1}{\partial x_1} + \lambda\frac{\partial v_2}{\partial x_2} + \lambda\frac{\partial v_3}{\partial x_3}\right) - \frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) - \frac{\partial}{\partial x_3}\left(\mu\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right)\right) = f_1$$

for the first component. For the second component we have

$$\rho\frac{\partial^2 v_2}{\partial t^2} - \frac{\partial}{\partial x_1}\left(\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) - \frac{\partial}{\partial x_2}\left(\lambda\frac{\partial v_1}{\partial x_1} + (\lambda+2\mu)\frac{\partial v_2}{\partial x_2} + \lambda\frac{\partial v_3}{\partial x_3}\right) - \frac{\partial}{\partial x_3}\left(\mu\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right)\right) = f_2$$

and finally for third component

$$\rho\frac{\partial^2 v_3}{\partial t^3}-\frac{\partial}{\partial x_1}\left(\mu\left(\frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1}\right)\right)-\frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial v_3}{\partial x_2}+\frac{\partial v_2}{\partial x_3}\right)\right)-\frac{\partial}{\partial x_3}\left(\lambda\frac{\partial v_1}{\partial x_1}+\lambda\frac{\partial v_2}{\partial x_2}+(\lambda+2\mu)\frac{\partial v_3}{\partial x_3}\right)=f_3.$$

In the homogeneous case, the last equation can be written as

$$\rho\frac{\partial^2 \boldsymbol{v}}{\partial t^2}=\mu\Delta\boldsymbol{v}+(\lambda+\mu)\nabla\,(\nabla\cdot\boldsymbol{v}), \tag{1.17}$$

where $\Delta\boldsymbol{v}=(\Delta v_j)_{j=1,2,3}$.

We can decompose the displacement vector $\boldsymbol{v}$, as $\boldsymbol{v}=\nabla\boldsymbol{\varphi}+\nabla\times\boldsymbol{\psi}$. Replacing this expression into the equation (1.17), we obtain

$$\rho\frac{\partial^2}{\partial t^2}(\nabla\boldsymbol{\varphi}+\nabla\times\boldsymbol{\psi})=\mu\Delta(\nabla\boldsymbol{\varphi}+\nabla\times\boldsymbol{\psi})+(\lambda+\mu)\nabla\,(\nabla\cdot(\nabla\boldsymbol{\varphi}+\nabla\times\boldsymbol{\psi})). \tag{1.18}$$

Considering that $\nabla\cdot\nabla\boldsymbol{\varphi}=\Delta\varphi$ and $\nabla\cdot(\nabla\times\boldsymbol{\psi})=0$, and grouping the terms $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$; the equation (1.18) is written as

$$\nabla\left(\rho\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}-(\lambda+2\mu)\Delta\boldsymbol{\varphi}\right)+\nabla\times\left(\rho\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}-\mu\Delta\boldsymbol{\psi}\right)=0.$$

Note that $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$, both satisfy one wave equation with a different velocity. This representation motivates the next definitions.
The P-wave, or pressure wave, corresponds to the propagation of a displacement parallel to the direction of propagation, whose velocity is

$$V_P=\sqrt{\frac{\lambda+2\mu}{\rho}}.$$

And the S-wave, or shear wave, corresponds to the propagation of a distortion in a plane orthogonal to the direction of propagation, whose velocity is

$$V_S=\sqrt{\frac{\mu}{\rho}}.$$

Note that $2V_S^2\le V_P^2$, so the P-waves are the firsts to arrive. In the below example we use the values V_P in different elements to create the velocity model. In real measurements the elastic wave are detected with sensor, which can identify between P and S depending of measure tech.

Chapter 2

Solution Of The Wave Equation

In this chapter we resolve the initial value problem wave equation for $\mathbb{R}^d, (d = 1, 2, 3)$. Next, we show the existence and uniqueness of initial boundary value problem wave equation via Hille- Yosida theorem. Finally we develop the weak solution for wave equation.

2.1 Classical Solutions

In order to resolve the initial value problem wave equation in $\mathbb{R}^d, (d = 1, 2, 3)$. First, we review the hyperbolic classification. Next, we solve the problem for one dimension. Following, we do the solution for three dimensions and finally we solve the problem for two dimensions.

2.1.1 Hyperbolic Classification

There is a classification the of second order semilinear (nonlinear equation with linear principal part) partial differential equation PDE, which is motivated by the conic sections.

Let

$$a(x, y)\frac{\partial^2 u}{\partial x^2} + 2b(x, y)\frac{\partial^2 u}{\partial x \partial y} + c(x, y)\frac{\partial^2 u}{\partial y^2} \tag{2.1}$$

be the principal part of a second order semilinear PDE, then we have the next classification (Iorio, De Magalhães, 2001).

Definition 2.1.1. *Equation (2.1)*

1. ***Elliptic*** *at* (x_0, y_0) *if* $b^2(x_0, y_0) - a(x_0, y_0)c(x_0, y_0) < 0$.
2. ***Parabolic*** *at* (x_0, y_0) *if* $b^2(x_0, y_0) - a(x_0, y_0)c(x_0, y_0) = 0$.
3. ***Hyperbolic*** *at* (x_0, y_0) *if* $b^2(x_0, y_0) - a(x_0, y_0)c(x_0, y_0) > 0$.

Note that if we consider the nonhomogeneous source wave equation in 1-D with constant medium velocity c,

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f(t, x)$$

we have $a(t,x) = 1$, $b(t,x) = 0$, $c(t,x) = -c^2$. Then $b^2(x_0, y_0) - a(x_0, y_0)c(x_0, y_0) = c^2 > 0$, and then, this equation is hyperbolic in the domain of f. For this reason we refer to the wave equation as a hyperbolic equation.

Consider the equation

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) \tag{2.2}$$

and put $p = u_x$ and $q = u_y$. The equation (2.2) is written as

$$ap_x + 2bp_y + cq_y = f(x,y,u,p,q). \tag{2.3}$$

Assume $u \in C^2(\mathbb{R}^2)$. Then, it is clear that $u_{yx} - u_{xy} = p_y - q_x = 0$. In addition, we introduce $x = x(s)$ and $y = y(s)$. Multiplying $p_y - q_x$ by a function $\lambda = \lambda(s)$ and adding to equation (2.3), we have

$$ap_x + (2b + \lambda)p_y - \lambda q_x + cq_y = f(x,y,u,p,q). \tag{2.4}$$

If $\frac{dx}{ds} = 1$ and $\frac{dy}{ds} = \frac{2b+\lambda}{a} = \frac{-c}{\lambda}$, the *characteristic curves method* says that

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} := \mu. \tag{2.5}$$

Then the equation (2.4) is written as

$$a\mu^2 - 2b\mu + c = f.$$

If we consider the homogeneous equation

$$a\mu^2 - 2b\mu + c = 0 \tag{2.6}$$

and for the hyperbolic case, where $b^2 - ac > 0$, there exist two families of real curves satisfying (2.5), where μ is solution of (2.6). As an example let us consider the equation

$$\frac{\partial^2 u}{\partial^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \quad (t,x) \in \mathbb{R}^2.$$

For this, the equation (2.6) is written as

$$\mu^2 - c^2 = 0,$$

and thus $\mu_1 = c$ and $\mu_2 = -c$. Hence $\frac{dx}{dt} = \mu_1 = c$ and $\frac{dx}{dt} = \mu_2 = -c$, where c (the medium velocity) is considered constant. Then we obtain

$$x = ct + k_1 \quad \text{and} \quad x = ct - k_2$$

where k_1 and k_2 are constants.

2.1.2 Solution of the Wave Equation in $\mathbb{R}$

Consider the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in} \quad \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & x \in \mathbb{R} \end{cases} \tag{2.7}$$

where the functions g and h are known. The aim is to express the function u in terms of g and h.

Note that this equation can be factorized as,

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = u_{tt} - u_{xx} = 0. \tag{2.8}$$

Defining

$$v(x,t) := \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u(x,t),$$

and if u is C^2 (in this case $u_{xt} = u_{tx}$), the equation (2.8) is written as

$$v_t(x,t) + cv_x(x,t) = 0.$$

This is a transport equation with constants coefficients (Evans, 1997, 18), then

$$v(x,t) = \alpha(x - ct)$$

for $\alpha(x) := v(x,0)$. Thus we have

$$u_t(x,t) - cu_x(t,x) = a(x - ct).$$

This is a nonhomogeneous transport equation, then

$$u(x,t) = g(x + ct) + \int_0^t \alpha(x - (s-t)c - cs)ds = g(x+t) + \int_0^t \alpha(x + 2t - 2cs))ds =$$
$$g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \alpha(s)ds.$$

Also $\alpha(x) = v(x,0) = u_t(x,0) - cu_x(x,0) = h(x) - cg_x(x)$, then

$$u(x,t) = g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s) - cg_s(s)ds.$$

Finally, using the fundamental theorem of calculus, we have the *d'Alembert's formula* for the solution of (2.7)

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds. \tag{2.9}$$

2.1.3 Spherical Mean

In this section we consider more than one space dimension, i.e., $n \geq 2$. Let u in $C^m(\mathbb{R}^n \times [0,\infty))$ where $m \geq 2$, we want to solve the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u(0,\boldsymbol{x}) = g(\boldsymbol{x}), \quad u_t(\boldsymbol{x},0) = h(\boldsymbol{x}) \quad \text{for all } x \in \mathbb{R}^n. \end{cases} \tag{2.10}$$

The goal is to obtain an expression for u in terms of g and h. The plan is to study the average of u over certain spheres. We start with the definition of average of a function over the ball and over the sphere.

Definition 2.1.2. *Let f be a integrable function in $\mathbb{R}^n$ (or in a suitable open in $\mathbb{R}^n$), then the equation*

$$⨍_{B(\boldsymbol{x},r)} f dy = \frac{1}{\alpha(n)r^n} \int_{B(\boldsymbol{x},r)} f dy \tag{2.11}$$

is called the average of f over the ball $B(\boldsymbol{x},r)$. And the equation

$$⨍_{\partial B(\boldsymbol{x},r)} f dS = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(\boldsymbol{x},r)} f dS \tag{2.12}$$

is called the average of f over the sphere $\partial B(\boldsymbol{x},r)$.

In the previous definition

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} = \begin{cases} \frac{\pi^{n/2}}{(n-1)!} & \text{if } n \text{ is even,} \\ \frac{\pi^{n/2}}{\sqrt{\pi}\frac{n!!}{2^{(n+1)/2}}} & \text{if } n \text{ is odd,} \end{cases}$$

is the volume of unit ball $B(0,1)$ in $\mathbb{R}^n$. And $n\alpha(n)$ is the surface area of unit sphere $\partial B(0,1)$ in $\mathbb{R}^n$. Remember that

$$n!! = \begin{cases} 1 & \text{if } n \leq 0 \\ (n-2)!! & \text{if } n > 0 \end{cases}$$

We follow the notation in (Evans, 1997). Let $\boldsymbol{x} \in \mathbb{R}^n$ be fixed, $t > 0$ and $r > 0$, then we put,

$$U(x;r,t) := ⨍_{\partial B(x,r)} u(y,t)dS(y), \tag{2.13}$$

the average of $u(.,t)$ over the sphere $\partial B(x,r)$. Similarly, for fixed x

$$G(x;r) := ⨍_{\partial B(x,r)} g(y)dS(y) \quad \text{and} \quad H(x;r) := ⨍_{\partial B(x,r)} u(y)dS(y).$$

Hereafter we consider U as a function of r and t. The next theorem helps us to solve the system (2.10) in terms of U.

Theorem 2.1.1. *(Euler-Poisson-Darboux equation).*
Fix $x \in \mathbb{R}^n$ and let u satisfy (2.10). Then $U \in C^m(\mathbb{R}_+ \times [0,\infty))$ and

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 \quad \textit{in } \mathbb{R}_+ \times (0,\infty) \\ U = G, \quad U_t = H \quad \textit{on } \mathbb{R}_+ \times \{t=0\}. \end{cases} \tag{2.14}$$

Proof. Note that

$$\int_{\partial B(0,r)} u(\boldsymbol{y})dS(\boldsymbol{y}) = r^{n-1}\int_{\partial B(0,1)} u(r\boldsymbol{z})dS(\boldsymbol{z}).$$

In fact, we use the substitution $\boldsymbol{y} = r\boldsymbol{z}$, i.e., $y_i = rz_i$, $i = 1, 2, \ldots, n$. Then $|\frac{\partial(y_1,\ldots,\partial_n)}{\partial(z_1\ldots,z_n)}| = r^n$. With this result and a translation of the origin we have

$$U(\boldsymbol{x}; r, t) = \fint_{\partial B(\boldsymbol{x},r)} u(\boldsymbol{y}, t)dS(\boldsymbol{y}) = \fint_{\partial B(0,1)} u(\boldsymbol{x} + r\boldsymbol{z}, t)dS(\boldsymbol{z}).$$

Now we have, by a rescaling transformation,

$$U_r(\boldsymbol{x}; r, t) = \fint_{\partial B(0,1)} \nabla u(\boldsymbol{x} + r\boldsymbol{z}, t)\cdot \boldsymbol{z}dS(\boldsymbol{z}) = \fint_{\partial B(\boldsymbol{x},r)} \nabla u(\boldsymbol{y}, t)\cdot \frac{\boldsymbol{y}-\boldsymbol{x}}{r}dS(\boldsymbol{y}).$$

Note that $\frac{\boldsymbol{y}-\boldsymbol{x}}{r} := \eta$ is a normal vector to sphere with center $\boldsymbol{x}$ and radius r at the point $\boldsymbol{y}$. Then, from the Green's theorem we have,

$$U_r(\boldsymbol{x}; r, t) = \fint_{\partial B(\boldsymbol{x},r)} \nabla u(\boldsymbol{y}, t)\cdot \eta dS(\boldsymbol{y}) = \fint_{B(\boldsymbol{x},r)} \mathrm{div}(\nabla u(\boldsymbol{y}, t))d\boldsymbol{y}.$$

By the definition of average over the ball in (2.11), and since $\mathrm{div}(\nabla) = \Delta$, we can write

$$U_r(\boldsymbol{x}; r, t) = \frac{1}{n\alpha(n)r^{n-1}}\int_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y}, t)d\boldsymbol{y} = \frac{r}{n}\fint_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y}, t)d\boldsymbol{y}. \tag{2.15}$$

In addition, let $\zeta : \mathbb{R}^n \to \mathbb{B}$ be a summable function. From polar coordinates theorem in (Evans, 1997, pag. 628) we have that for all $r > 0$ it holds,

$$\int_{B(x_0,r)} \zeta d\boldsymbol{x} = \int_0^r \left(\int_{\partial B(\boldsymbol{x}_0,r)} \zeta dS\right) dr.$$

In particular (2.15) gives

$$U_r(\boldsymbol{x}; r, t) = \frac{1}{n\alpha(n)r^{n-1}}\left(\int_0^r \left(\int_{\partial B(\boldsymbol{x},k)} \Delta u(\boldsymbol{y}, t)dS(\boldsymbol{y})\right) dk\right)$$

for $r > 0$. Also, $\int_{B(\boldsymbol{x}_0,r)} f d\boldsymbol{x} = \int_0^r \left(\int_{\partial B(\boldsymbol{x}_0,k)} f dS\right) dk$. With this idea in mind and following the equation (2.15), we write

$$(r^{n-1}U_r)_r = \frac{d}{dr}(r^{n-1}U_r(\boldsymbol{x}; r, t)) = \frac{1}{n\alpha(n)}\frac{d}{dr}\left(\int_0^r \left(\int_{\partial B(\boldsymbol{x},k)} \Delta u(\boldsymbol{y}, t)dS(\boldsymbol{y})\right) dk\right). \tag{2.16}$$

Then, by recalling that the surface are of sphere $\partial B(\boldsymbol{x}, r)$ is $n\alpha(n)r^n$ we obtain

$$\frac{n-1}{r}U_r(\boldsymbol{x}; r, t) + U_{rr}(\boldsymbol{x}; r, t) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y}, t)dS(\boldsymbol{y}) = \fint_{\partial B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y}, t)dS(\boldsymbol{y}),$$

and the using (2.15) again to replace U_r, we have,

$$U_{rr}(\boldsymbol{x}; r, t) = \fint_{\partial B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y}, t) dS(\boldsymbol{y}) - \frac{n-1}{n} \fint_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y}, t) d\boldsymbol{y}. \tag{2.17}$$

Finally, note that, by taking derivative with respect to t we obtain

$$r^{n-1} U_{tt} = r^{n-1} \fint_{\partial B(\boldsymbol{x},r)} u_{tt} dS = \frac{1}{n\alpha(n)} \int_{\partial B(\boldsymbol{x},r)} u_{tt} dS$$

and by using an argument similar to the one (2.16) we obtain

$$r^{n-1} U_{tt} = \frac{d}{dr}\left(\frac{1}{n\alpha(n)} \int_{B(\boldsymbol{x},r)} u_{tt}(\boldsymbol{y}) dS(\boldsymbol{y})\right) = \frac{d}{dr}\left(\frac{1}{n\alpha(n)} \int_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y})(\boldsymbol{y}) dS(\boldsymbol{y})\right) = (r^{n-1} U_r)_r \tag{2.18}$$

from the equations (2.16) and (2.18) we deduce the (2.14). □

The next theorem gives us the relation between the average over the sphere $U(\boldsymbol{x}; r, t)$ and the actual value $u(\boldsymbol{x}, t)$.

Theorem 2.1.2. *Let u in $C^m(\mathbb{R}^n \times [0, \infty))$, where $m \in \mathbb{Z}$ and $m \geq 2$. Let $\boldsymbol{x} \in \mathbb{R}^n$ and $U(\boldsymbol{x}; r, t)$ as in (2.13),.Then*

$$\lim_{r \to 0} U(\boldsymbol{x}; r, t) = u(\boldsymbol{x}, t)$$

Proof. Note that we can write the integrals over the unit sphere, so that

$$U(\boldsymbol{x}; r, t) = \fint_{\partial B(\boldsymbol{x},r)} u(\boldsymbol{y}, t) dS(\boldsymbol{y}) = \fint_{\partial B(0,1)} u(\boldsymbol{x} + rz, t) dS(z).$$

Since u is of class C^m, we can interchange the limit and the integral to get

$$\lim_{r \to 0} U(\boldsymbol{x}; r, t) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \lim_{r \to 0} (u(\boldsymbol{x} + rz, t) dS(z)) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(\boldsymbol{x} + \lim_{r \to 0}(rz), t) dS(z).$$

Since $u(\boldsymbol{x} + \lim_{r \to 0}(rz), t) = u(\boldsymbol{x}, t)$, and this do not depend on z, we can write

$$\lim_{r \to 0} U(\boldsymbol{x}; r, t) = u(\boldsymbol{x}, t) \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} dS = u(\boldsymbol{x}, t).$$

□

2.1.4 Solution for $n = 3$

Let us consider $n = 3$ and let $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ and assume that u satisfies (2.10). We denote

$$\bar{U} := rU \quad , \quad \bar{G} := rG \quad \text{and} \quad \bar{H} := rH. \tag{2.19}$$

Note that

$$\bar{U}_{tt} = r\left(U_{rr} + \frac{2}{r}U_r\right) = rU_r r + 2U_r = (U + rU_r)_r = \bar{U}_{rr}. \tag{2.20}$$

We now solve the problem

$$\begin{cases} \bar{U}_{tt} - \bar{U}_{rr} = 0 \quad \text{in} \quad \mathbb{R}_+ \times (0, \infty) \\ \bar{U} = \bar{G}, \quad \bar{U}_t = \bar{H} \quad \text{on} \quad \mathbb{R}_+ \times \{t = 0\} \\ \bar{U} = 0 \quad \text{on} \quad \{r = 0\} \times (0, \infty). \end{cases} \tag{2.21}$$

The system in (2.21) can be viewed as in Section 2.1.2 that is a $1 - D$ wave equation. From equation (2.9) we have that the solution is

$$\bar{U}(\boldsymbol{x}; r, t) = \frac{1}{2}\left[\bar{G}(r + t) - \bar{G}(t - r)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \bar{H}(y)dy.$$

Now from Theorem (2.1.2) we have

$$u(\boldsymbol{x}, t) = \lim_{r\to 0} \frac{U(\boldsymbol{x}; r, t)}{r} = \lim_{r\to 0}\left(\frac{\bar{G}(r + t) - \bar{G}(t - r)}{r} + \frac{1}{2r}\int_{-r+t}^{r+t} \bar{H}(y)dy\right) = \bar{G}'(t) + \bar{H}(t). \tag{2.22}$$

We also have

$$\bar{G}'(t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B(\boldsymbol{x},t)} g(\boldsymbol{y})dS(\boldsymbol{y})\right) = \frac{\partial}{\partial t}\left(t \fint_{\partial B(\boldsymbol{0},1)} g(\boldsymbol{x} + t\boldsymbol{z})dS(z)\right)$$

doing the derivative of product

$$\bar{G}'(t) = \fint_{\partial B(\boldsymbol{x},t)} g(\boldsymbol{y})dS(\boldsymbol{y}) + t\fint_{\partial B(0,1)} \nabla g(\boldsymbol{x} + t\boldsymbol{z}) \cdot \boldsymbol{z}dS(\boldsymbol{y})$$

By rescaling, we have

$$\bar{G}'(t) = \fint_{\partial B(\boldsymbol{x},t)} g(\boldsymbol{y})dS(\boldsymbol{y}) + t\fint_{\partial B(\boldsymbol{x},t)} \nabla g(\boldsymbol{y}) \cdot \left(\frac{\boldsymbol{y} - \boldsymbol{x}}{t}\right)dS(\boldsymbol{y}) = \fint_{\partial B(\boldsymbol{x},t)} g(\boldsymbol{y}) + \nabla g(\boldsymbol{y}) \cdot (\boldsymbol{y} - \boldsymbol{x})\, dS(\boldsymbol{y}). \tag{2.23}$$

Finally, by inserting (**??**) into (2.22) we set

$$u(\boldsymbol{x}, t) = \fint_{\partial B(\boldsymbol{x},t)} g(\boldsymbol{y}) + \nabla g(\boldsymbol{y}) \cdot (\boldsymbol{y} - \boldsymbol{x})\, dS(\boldsymbol{y}) + t\fint_{\partial B(\boldsymbol{x},t)} th(\boldsymbol{y})dS(\boldsymbol{y}) \tag{2.24}$$

$$= \fint_{\partial B(\boldsymbol{x},t)} th(\boldsymbol{y}) + g(\boldsymbol{y}) + \nabla g(\boldsymbol{y}) \cdot (\boldsymbol{y} - \boldsymbol{x})\, dS(\boldsymbol{y}), \tag{2.25}$$

where $\boldsymbol{x} \in \mathbb{R}^3$ and $t > 0$. This is the *Kirchhoff's formula* for the solution of the problem (2.10) in three dimensions.

2.1.5 Solution for $n = 2$

For $n = 2$, it is not possible write the equation (2.20), because in this case $\bar{U_{tt}}$ would be $r\left(U_{rr} + \frac{1}{r}U_r\right) \neq \bar{U}_{rr}$. Then we can not transform the system (2.10) to one dimension. The methodology for two dimensions starts with the setting of a new function of three variables which represents the function defined with two variables.

Indeed, let $u \in C^2(\mathbb{R}^2 \times [0,\infty)$ solve (2.10) for $n = 2$. We define

$$\bar{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t). \tag{2.26}$$

The equation with initial-value (2.10) implies

$$\begin{cases} \bar{u}_{tt} - \Delta\bar{u} = 0 \quad \text{in } \mathbb{R}^3 \times (0,\infty) \\ \bar{u}(\boldsymbol{x}, 0) = \bar{g}(\boldsymbol{x}), \quad \bar{u}_t(\boldsymbol{x}, 0) = \bar{h}(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^3, \end{cases} \tag{2.27}$$

where $\bar{g}(x_1, x_2, x_3) := g(x_1, x_2)$ and $\bar{h}(x_1, x_2, x_3) := h(x_1, x_2)$.

We write $x = (x_1, x_2) \in \mathbb{R}^2$ and $\bar{x} = (x_1, x_2, 0) \in \mathbb{R}^3$. Then from equation (2.24) applied to (2.10) we obtain

$$u(x,t) = \bar{u}(\bar{x}, t) = \frac{\partial}{\partial t}\left(t \fint_{\partial\bar{B}(\bar{x},t)} \bar{g}d\bar{S}\right) + t \fint_{\partial\bar{B}(\bar{x},t)} \bar{h}d\bar{S} \tag{2.28}$$

where $\bar{B}(\bar{x}, t)$ denotes the ball in $\mathbb{R}^3$ with center in $\bar{x}$ and radius t>0, and $d\bar{S}$ denotes the two dimensional surface measure on $\partial\bar{B}(\bar{x}, t)$.

Note that if $\bar{y} = (y_1, y_2, y_3) = (y, y_3)$ and $\bar{y} \in \partial\bar{B}(\bar{x}, t)$, we can write $y_3 := \varphi((y_1, y_2)) = \sqrt{t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2} = \sqrt{t^2 - |y - x|^2}$, thus $\bar{y} = (y_1, y_2, \varphi((y_1, y_2)))$ and

$$d\bar{S}(y, y_3) = \|T_{y_1} \times T_{y_2}\|dy$$

where T_{y_1} and T_{y_2} are

$$T_{y_1} = \frac{\partial\bar{y}}{\partial y_1} = \left(1, 0, \frac{-(y_1 - x_1)}{\sqrt{t^2 - |y - x|^2}}\right) \quad \text{and} \quad T_{y_2} = \frac{\partial\bar{y}}{\partial y_1} = \left(0, 1, \frac{-(y_2 - x_2)}{\sqrt{t^2 - |y - x|^2}}\right).$$

Then

$$\|T_{y_1} \times T_{y_2}\| = \sqrt{1 + \frac{(y_1 - x_1)^2}{t^2 - |y - x|^2} + \frac{(y_2 - x_2)^2}{t^2 - |y - x|^2}} = \frac{t}{\sqrt{t^2 - |y - x|^2}}.$$

Thus

$$d\bar{S}(y, y_3) = \frac{1}{\sqrt{t^2 - |y - x|^2}}dy.$$

With this in mind we can write

$$\fint_{\partial\bar{B}(\bar{x},t)} \bar{g}d\bar{S} = \frac{1}{4\pi t^2}\int_{\partial\bar{B}(\bar{x},t)} \bar{g}d\bar{S} = \frac{2t}{4\pi t^2}\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |y - x|^2}}dy.$$

The factor 2 appeared because we split the integral over the sphere as the sum of two integrals each of them in half sphere. Then

$$\fint_{\partial\bar{B}(\bar{x},t)} \bar{g}d\bar{S} = \frac{1}{2\pi t}\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy = \frac{t}{2}\fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy.$$

Consequently equation (2.28) becomes

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2\fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy\right) + \frac{t^2}{2}\fint_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}dy. \tag{2.29}$$

Note that if $y = x + tz$, then $\frac{\partial(y_1,y_2)}{\partial(z_1,z_2)} = t^2$ and we have

$$\begin{aligned}\fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy &= \frac{1}{\pi t^2}\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy = \frac{1}{\pi t^2}\int_{B(0,1)} \frac{g(x+tz)}{\sqrt{t^2-|tz|^2}}\frac{\partial(y_1,y_2)}{\partial(z_1,z_2)}dz \\ &= \frac{1}{\pi t^2}\int_{B(0,1)} \frac{g(x+tz)}{t\sqrt{1^2-|z|^2}}t^2dz = \frac{1}{t}\int_{B(0,1)} \frac{g(x+tz)}{\sqrt{1^2-|z|^2}}dz.\end{aligned}$$

Thus, we see that

$$t^2\fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy = t\int_{B(0,1)} \frac{g(x+tz)}{\sqrt{1^2-|z|^2}}dz,$$

and therefore,

$$\frac{\partial}{\partial t}\left(t^2\fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy\right) = \frac{\partial}{\partial t}\left(t\int_{B(0,1)} \frac{g(x+tz)}{\sqrt{1^2-|z|^2}}dz\right).$$

Doing the derivative of product we set

$$\frac{\partial}{\partial t}\left(t^2\fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy\right) = \int_{B(0,1)} \frac{g(x+tz)}{\sqrt{1^2-|z|^2}}dz + t\int_{B(0,1)} \frac{\nabla g(x+tz)\cdot z}{\sqrt{1^2-|z|^2}}dz.$$

By rescaling

$$\frac{\partial}{\partial t}\left(t^2\fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy\right) = t\fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy + t\fint_{B(x,t)} \frac{\nabla g(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}dy.$$

Equation (2.29) becomes

$$u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{tg(y)+t^2h(y)+t\nabla g(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}dy \tag{2.30}$$

for $x \in \mathbb{R}^2$, $t > 0$. This is the *Poisson's formula*.

For general integer n, the treatment is similar to $n = 2$ and $n = 3$ depending if n is odd or even. See (Evans, 1997, pag. 74-81).

Remark: The dependence domain in $\mathbb{R}^3$ for the point $(\boldsymbol{x}, t)$ is the sphere $\partial B(x,t)$. For a fixed point x_0 in $\mathbb{R}^3$ the influence point is $C(x_0) = \{(\boldsymbol{x}, t) \in \mathbb{R}^3 \times [0,\infty) : |\boldsymbol{x} - \boldsymbol{x_0} = t|\}$.

In $\mathbb{R}^2$ the solution $u(x,t)$ depend on the values of h and g on the $\bar{B}(x,t)$. This ball is the dependence domain. For a fixed point x_0 in $\mathbb{R}^2$ the influence domain is $C(x_0) = \{(\boldsymbol{x}, t) \in \mathbb{R}^3 \times [0,\infty) : |\boldsymbol{x} - \boldsymbol{x_0} \leq t|\}$.

For this reason in $\mathbb{R}^3$ the wave has effect just in the instant t, while that in $\mathbb{R}^2$ the wave affect the space in every instant less to t.

2.1.6 Non-homogeneous Problem

In this section we work out the initial value problem for the non-homogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\boldsymbol{x}, 0) = 0, \quad u_t(\boldsymbol{x}, 0) = 0, & \text{for all } \boldsymbol{x} \in \mathbb{R}^n. \end{cases} \tag{2.31}$$

Following the Duhamel's principle (Evans, 1997, pag. 81, 82), we define $u = (x, t; s)$ to be solution of

$$\begin{cases} u_{tt}(\cdot; s) - \Delta u(\cdot; s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ u(\cdot; s) = 0, \quad u_t(\cdot; s) = f(\cdot; s) & \text{on } \mathbb{R}^n \times \{t = s\}. \end{cases} \tag{2.32}$$

And we set

$$u(x,t) := \int_0^t u(x,t;s)ds \quad (x \in \mathbb{R}^n, t \geq 0). \tag{2.33}$$

Duhamel's principle leads to the solution of (2.31) as follows.

Theorem 2.1.3. *Assume $n \geq 2$ and $f \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n) \times [0,\infty)$. Define u by (2.33). Then*

1. $u \in C^2(\mathbb{R}^n \times [0,\infty))$,
2. $u_{tt} - \Delta u = f$ *in* $\mathbb{R}^n \times (0,\infty)$,
3. $\displaystyle\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n, t>0}} u(x,t) = 0, \quad \lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n, t>0}} u_t(x,t) = 0$ *for each point* $x^0 \in \mathbb{R}^n$.

Proof. For a proof of the first clause see (Evans, 1997).

Now we prove the second statement. Note

$$u_t(x,t) = u(x,t;t) + \int_0^t u_t(x,t;s)ds = \int_0^t u_t(x,t;s)ds,$$

and thus deriving with respect to t we have,

$$u_{tt}(x,t) = u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)ds = f(x,t) + \int_0^t u_{tt}(x,t;s)ds.$$

Moreover, we can compute the Laplacian as,

$$\Delta u(x,t) = \int_0^t \Delta u(x,t;s)ds = \int_0^t u_{tt}(x,t;s)ds.$$

Thus, we obtain,

$$u_{tt}(x,t) - \Delta u(x,t) = f(x,t) \quad \text{for } x \in \mathbb{R}^n \quad \text{and } t > 0.$$

In this cases, $u(x,0) = u_t(x,0) = 0$.

$\square$

2.2 The Hille-Yosida Theorem Applied to Wave Equation

In this section we present a way to guarantee the existence and uniqueness for the wave equation with initial values

$$\begin{cases} u_{tt} + \beta u_t - \Delta u + \lambda u = 0 \quad \text{in} \quad \Omega \times (0,\infty) \\ u_{|\Gamma} = 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) \quad x \in \Omega \end{cases}$$

knowing properties of the differential operator

$$A = \begin{pmatrix} 0 & -I \\ -\Delta + \lambda & \beta \end{pmatrix}.$$

It is important because, we can work with a bounded domain Ω instead of $\mathbb{R}^n$ (as in previous sections).

We start with the definition of *monotone operators*. Next we recall the Hille-Yosida Theorem which is used to obtain existence of solutions of the heat equation. Finally we apply Hille-Yosida theorem to get existence of solutions for the wave equation.

Throughout this section H denotes a Hilbert space.

2.2.1 Maximal Monotone Operators

In this section we introduce the definition of the monotone operators and its fundamental properties.

Definition 2.2.1. *An lineal unbounded operator $A : D(A) \subset H \to H$, where $D(A)$ is the domain of A, is said to be* ***monotone*** *if it satisfies*

$$(Av, v) \geq 0 \quad \textit{for all } v \in D(A).$$

It is called ***maximal monotone*** *if, in addition, $R(I + A) = H$, i.e., if*

$$\textit{For each} \quad f \in H \quad \textit{there exists} \quad u \in D(A) \quad \textit{such that} \quad u + Au = f.$$

The next theorem is a corollary of Hahn-Banach theorem, and we use it for the proof of the proposition 2.2.1 below. (Brezis, 2011).

Theorem 2.2.1. *Let E be a Banach's space and $F \subset E$ a vectorial subspace such that $\bar{F} \neq E$, then exist $f \in E^*$ such that $(f, x) = 0$ for all $x \in F$ and $f \neq 0$.*

Proof. Let be x_0 in $E - \bar{F}$. The set $\{x_0\}$ is compact and the set $\bar{F}$ is closed, also both are convexes (remember that F is a vectorial space), disjuncts and non-empties, thus for the Hahn-Banach theorem (second geometric form) there is $f \in E^*$ and $\alpha > 0$ such that

$$f(x) < \alpha \quad \forall x \in \bar{F}$$

and

$$f(x_0) > \alpha$$

As $\lambda x \in \bar{F}$ for all $\lambda \in \mathbb{R}$, then $f(\lambda x) = \lambda f(x) < \alpha$ thus $f(x) = 0$ for all $x \in \bar{F}$, (particularly $\forall x \in F$) and $f \neq 0$. □

The next preposition gives us special conditions about the operator A

Proposition 2.2.1. *Let A be an monotone maximal operator, then*

1. *The domain of A, $D(A)$ is dense in H.*
2. *The operator A is an closed operator.*
3. *For every $\lambda > 0$, $I + \lambda A$ is bijective from $D(A)$ onto H, and $(I + \lambda A)^{-1}$ is a bounded operator and $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$.*

Proof. 1. Given $f \in H$ such that $(f, v) = 0$ for all $v \in D(A)$, how A is a monotone maximal operator, exist $w_0 \in D(A)$ such that

$$w_0 + Aw_0 = f$$

also

$$0 = (f, w_0) = (w_0 + Aw_0, w_0) = \|w_0\|^2 + (Aw_0, w_0) \geq \|w_0\|^2,$$

therefore $w_0 = 0$ then $f = 0$. We use the corollary (2.2.1) to conclude that $D(A)$ is dense in H.

2. First we see that given any $f \in H$, exists a only $u \in D(A)$ such that

$$u + Au = f.$$

If there exist $u_0 \in D(A)$ such that

$$u_0 + Au_0 = f,$$

then, subtracting the last two equations,

$$(u-u_0)+A(u-u_0)=0$$

and doing the inner product with $(u-u_0)$ and using that A is monotone, we deduce

$$0=(u-u_0+A(u-u_0),u-u_0)=\|u-u_0\|^2+(A(u-u_0),(u-u_0))\geq\|u-u_0\|^2$$

then $u=u_0$.

Now note that, by using that $f=u+Au$ and the properties of inner product,

$$(f,u)=(u+Au,u)=\|u\|^2+(Au,u)\geq\|u\|^2$$

then

$$\|f\|^2=\sup_u\frac{\|(f,u)\|^2}{\|u\|^2}\geq\|u\|^2.$$

We conclude that $\|u\|\leq\|f\|$. Therefore, the mapping $f\to u$ denoted by $(I+A)^{-1}$ is a bounded operator from H onto itself and $\|(I+A)^{-1}\|_{\mathcal{L}(H)}\leq 1$.

To see that A is a closed operator we consider a sequence $(u_n)\subset D(A)$ such that $u_n\to u$ and $Au_n\to f$, then we expect that $u\in D(A)$ and $Au=f$. In fact, note that

$$u_n+Au_n\longrightarrow u+f$$

also as the operator $(I+A)$ is a bijection from $D(A)$ onto $D(A)$, we can write $u_n\in D(A)$ as composition of this operator with its inverse i.e.,

$$u_n=(I+A)^{-1}(I+A)=(I+A)^{-1}(u_n+Au_n)$$

then, as $(I+A)^{-1}$ is a bounded operator, the mapping is continuous,

$$u_n=(I+A^{-1})(u_n+Au_n)\longrightarrow(I+A)^{-1}(u+f).$$

Thus,

$$u=(I+A)^{-1}(u+f)$$

i.e., $u\in D(A)$ and $u+Au=u+f$.

3. For $\lambda = 1$ is done. Now we will prove that if $R(I + \lambda_0 A) = H$ for some $\lambda_0 > 0$ then for every $\lambda > \lambda_0/2$, we have $R(I + \lambda A) = H$.

   Analogously to the arguments above, for every $f \in H$, there exists a unique $u \in D(A)$ such that $u + \lambda_0 Au = f$. Also the mapping $f \to u$ denoted by $(I + \lambda_0 A)^{-1}$ is a bounded operator and $\|(I + \lambda_0 A)^{-1}\|_{\mathcal{L}(H)} \leq 1$.

   Note that if we want to show $u + \lambda Au = f$, this is the same that writing

$$\frac{\lambda_0}{\lambda}u + \frac{\lambda_0}{\lambda}\lambda Au = \frac{\lambda_0}{\lambda}f$$

   thus, adding and subtracting u

$$u + \lambda_0 Au = \frac{\lambda_0}{\lambda}f + \left(1 - \frac{\lambda_0}{\lambda}\right)u$$

   so, computing the inverse operator $(I + \lambda_0 A)^{-1}$,

$$u = (I + \lambda_0 A)^{-1}\left(\frac{\lambda_0}{\lambda}f + \left(1 - \frac{\lambda_0}{\lambda}\right)u\right).$$

   Since $\|(I + \lambda_0 A)^{-1}\|_{\mathcal{L}(H)} \leq 1$, if $|1 - \frac{\lambda_0}{\lambda}| < 1$ then the mapping $(I + \lambda_0 A)^{-1}\left(\frac{\lambda_0}{\lambda}f + \left(1 - \frac{\lambda_0}{\lambda}\right)u\right)$ is contractive, i.e., if $\frac{\lambda_0}{2} < \lambda$, this mapping is contractive, then by the Banach contraction theorem exists $u \in D(A)$ such that

$$u = (I + \lambda_0 A)^{-1}\left(\frac{\lambda_0}{\lambda}f + \left(1 - \frac{\lambda_0}{\lambda}\right)u\right).$$

   Since is true for $\lambda_0 = 1$, will be true for $\lambda > 1/2$. Following with $\lambda_0 = 1/2$, will be true for $\lambda > 1/4$, and so on. We deduce that is true for $\lambda > 0$.

□

The next definition introduces the concept of the resolvent of an operator. This is very important for spectral theory, however in this text we do not do emphasize in this topic.

Definition 2.2.2. *Let A be a maximal monotone operator. For every $\lambda > 0$ the operator*

$$J_\lambda = (I + \lambda A)^{-1}$$

is called the resolvent of A. Additionally

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$$

is the Yosida's regularization of A.

Remember that $\|J_\lambda\|_{\mathcal{L}(H)} \leq 1$. Below we write important properties of A_λ and J_λ

Proposition 2.2.2. *Let A be a maximal monotone operator. Then*

1. $A_\lambda v = A(J_\lambda v)$ *for all* $v \in H$ *and for all* $\lambda > 0$.
2. $A_\lambda v = J_\lambda(Av)$ *for all* $v \in D(A)$ *and for all* $\lambda > 0$.
3. $|A_\lambda v| \leq |Av|$ *for all* $v \in D(A)$ *and for all* $\lambda > 0$.
4. $\lim_{\lambda \to 0} J_\lambda v = v$ *for all* $v \in H$.
5. $\lim_{\lambda \to 0} A_\lambda v = Av$ *for all* $v \in D(A)$.
6. $(A_\lambda v, v) \geq 0$ *for all* $v \in H$ *and for all* $\lambda > 0$.
7. $|A_\lambda v| \leq (1/\lambda)|v|$ *for all* $v \in H$ *and for all* $\lambda > 0$.

For a proof see (Brezis, 2011, 182, 183).

The next theorem shows that $A_\lambda : H \to H$ is a Lipschitz map. This fact is used in the next section.

Theorem 2.2.2. *Let* $A_\lambda : H \to H$ *is a Lipschitz map, i.e., there is a constant L such that*

$$\|A_\lambda u - A_\lambda v\| \leq L\|u - v\| \quad \textit{for all } u, v \in H$$

Proof. Note that

$$\|A_\lambda u - A_\lambda v\| = \frac{1}{\lambda}\|(u - v) - J_\lambda(u - v)\|$$

by the triangle inequality an, as $\|J_\lambda\|_{\mathcal{L}(H)} \leq 1$, we deduce,

$$\|A_\lambda u - A_\lambda v\| \leq \frac{1}{\lambda}\left(\|u - v\| + \|J_\lambda\|\|u - v\|\right) \leq \frac{1}{\lambda}(1 + 1)\|u - v\| = L\|u - v\|.$$

where $L = \frac{2}{\lambda}$. □

It is clear that $-A_\lambda$ is Lipschitz map too.

2.2.2 Solution of the Evolution Problem With First Order in Time

In this section we work about the existence and uniqueness of the evolution problem

$$\frac{du}{dt} + Au = 0$$

on $[0, \infty)$, and with initial value $u(0) = u_0$.

We write a classic theorem of ordinary partial equation, the **Cauchy, Lipschitz, Picard** theorem. For a proof see (Brezis, 2011, 184, 185).

Theorem 2.2.3. *Let E be a Banach space and let $F : E \to E$ be a Lipschitz map, then given any $u_0 \in E$, there exists a unique solution $u \in C^1([0,\infty); E)$ of the problem*

$$\begin{cases} \frac{du}{dt}(t) - Fu(t) = 0 \quad on \quad [0,\infty), \\ u(0) = u_0. \end{cases} \tag{2.34}$$

Corollary. Given $u_0 \in D(A)$, there exists a unique solution $u_\lambda \in C^1([0,\infty); H)$ of the problem

$$\begin{cases} \frac{du_\lambda}{dt} + A_\lambda u_\lambda = 0 \quad \text{on} \quad [0,\infty), \\ u_\lambda(0) = u_0. \end{cases} \tag{2.35}$$

Proof. It is a immediate consequence of the theorem (2.2.2) and the Cauchy, Lipschitz, Picard theorem. □

Now we are ready for write the **Hille-Yosida** theorem.

Theorem 2.2.4. ***(Hille-Yosida)**. Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0,\infty); H) \cap C([0,\infty); D(A))$$

satisfying

$$\begin{cases} \frac{du}{dt} + Au = 0 \quad on \quad [0,\infty), \\ u(0) = u_0. \end{cases} \tag{2.36}$$

Moreover,

$$|u(t)| \leq |u_0| \quad and \quad |\frac{du}{dt}(t)| = |Au(t)| \leq |Au_0| \quad for\ all\ t \geq 0$$

Proof. In this proof we follow (Brezis, 2011, 185-190). This proof is divided into six step.

Step 1 (*Uniqueness*). Let u and $\bar{u}$ solutions of (2.36), then we have

$$\left(\frac{d}{dt}(u-\bar{u}), (u-\bar{u})\right) = -(A(u-\bar{u}), u-\bar{u}) \leq 0.$$

Note that if $\varphi \in C^1([0,\infty); H)$ then $|\varphi|^2 \in C^1([0,\infty); \mathbb{R})$ and $\frac{d}{dt}|\varphi|^2 = 2\left(\frac{d\varphi}{dt}, \varphi\right)$. In fact, if $\varphi(t) \in H$, $\varphi(t) := \varphi = \sum \varphi_k b_k$ where (b_k) is a base in H, then $\frac{d}{dt}|\varphi|^2 = \frac{d}{dt}(\varphi,\varphi) = \frac{d}{dt}\sum \varphi_k^2 = \sum \frac{d}{dt}\varphi_k^2 = 2\sum \varphi_k \frac{d\varphi_k}{dt} = 2(\frac{d\varphi}{dt}, \varphi)$.

Hence

$$\frac{1}{2}\frac{d}{dt}|u(t)-\bar{u}(t)|^2 = \left(\frac{d}{dt}(u(t)-\bar{u}(t), u(t)-\bar{u}(t))\right).$$

Then $\frac{d}{dt}|u(t)-\bar{u}(t)|^2 \leq 0$. Therefore the function $t \to |u(t)-\bar{u}(t)|$ is nonincreasing on $[0,\infty)$. Since $|u(0) - \bar{u}(0)| = |u_0 - u_0| = 0$, it follows that

$$|u(t) - \bar{u}(t)| = 0 \quad \text{for all } t \geq 0.$$

This implies that $u(t) = \bar{u}(t)$ for all $t \geq 0$.

Step 2. In this step we prove that if $w \in C^1([0,\infty]; H)$ is a function satisfying

$$\frac{dw}{dt} + A_\lambda w = 0 \quad \text{in } [0,\infty), \tag{2.37}$$

then the functions $t \to |w(t)|$ and $t \to |\frac{dw}{dt}(t)| = |A_\lambda w(t)|$ are nonincresing on $[0,\infty)$.
In fact, we can write

$$\left(\frac{dw}{dt}, w\right) + (A_\lambda w, w) = 0.$$

By proposition (2.2.2), 6., we have $(A_\lambda w, w) \geq 0$, thus $\left(\frac{dw}{dt}, w\right) \leq 0$, i.e.,

$$\frac{1}{2}\frac{d}{dt}|w|^2 \leq 0.$$

Then, the function $|w(t)|$ is nonincreasing. Moreover, since A_λ is a linear bounded operator, we deduce by induction from (2.37) that $w \in C^\infty([0,\infty); H)$ and

$$\frac{d}{dt}\left(\frac{dw}{dt}\right) + A_\lambda\left(\frac{dw}{dt}\right) = 0.$$

Analogously to preceding fact, applying to $\frac{dw}{dt}$, we have $|\frac{dw}{dt}(t)|$ is nonincreasing.

Now, let u_λ be the solution of the problem (the existence of u_λ is due to previous corollary).

$$\begin{cases} \frac{du_\lambda}{dt} + A_\lambda u_\lambda = 0 \quad \text{on} \quad [0,\infty), \\ u_\lambda(0) = u_0 \in D(A). \end{cases} \tag{2.38}$$

By the above fact we deduce

$$|u_\lambda(t)| \leq |u_0| \quad \text{for all } t \geq 0, \quad \text{for all } \lambda > 0 \tag{2.39}$$

and

$$|\frac{du_\lambda}{dt}(t)| = |A_\lambda u_\lambda(t)| \leq |Au_0| \quad \text{for all } t \geq 0, \quad \text{for all } \lambda > 0. \tag{2.40}$$

The last inequality uses (2.2.2), 3.

Step 3. In this step we prove that for every $t \geq 0$, $u_\lambda(t) \to u(t)$ as $\lambda \to 0$. Moreover we prove that the converge is uniform on every bounded interval $[0,T]$.
For every $\lambda, \mu > 0$ we write

$$\frac{du_\lambda}{dt} - \frac{du_\mu}{dt} + A_\lambda u_\lambda - A_\mu u_\mu = 0.$$

In order to see that for a fixed t, $\{A_\lambda\}$ is a Cauchy sequence as $\lambda \to 0$. Now for a fixed t, we do the inner product with $(u_\lambda(t) - u_\mu(t))$

$$\left(\frac{d}{dt}(u_\lambda(t) - u_\mu(t)), (u_\lambda(t) - u_\mu(t))\right) + (A_\lambda u_\lambda(t) - A_\mu u_\mu(t), (u_\lambda(t) - u_\mu(t))) = 0.$$

So, $\frac{d}{dt}|\varphi|^2 = 2\left(\frac{d\varphi}{dt}, \varphi\right)$, we write

$$\frac{1}{2}\frac{d}{dt}|u_\lambda(t) - u_\mu(t)|^2 + (A_\lambda u_\lambda(t) - A_\mu u_\mu(t), (u_\lambda(t) - u_\mu(t))) = 0. \tag{2.41}$$

Note that, (dropping t for simplicity). We have

$$(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) = (A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - J_\lambda u_\lambda - (u_\mu - J_\mu u_\mu) + J_\lambda u_\lambda - J_\mu u_\mu)$$

adding and subtracting the term $J_\lambda u_\lambda + J_\mu u_\mu$. As $\lambda A_\lambda = I - J_\lambda$ by definition, $A_\lambda v = AJ_\lambda v$ (by preposition (2.2.2), 1.) and by using the linearity of inner product, we have

$$\begin{aligned}(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) &= (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) + (A(J_\lambda u_\lambda - J_\mu u_\mu), J_\lambda u_\lambda - J_\mu u_\mu)\\ &\geq (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu),\end{aligned}$$

the last inequality is due to A is maximal. Then, the equation (2.41) can be written as

$$\frac{1}{2}\frac{d}{dt}|u_\lambda - u_\mu|^2 = -(A_\lambda u_\lambda - A_\mu u_\mu, (u_\lambda - u_\mu)) \leq -(A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu)$$

By using the linearity of inner product, we have

$$-(A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) = \lambda(A_\mu u_\mu, A_\lambda u_\lambda) + \mu(A_\mu u_\mu, A_\lambda u_\lambda) - \lambda|A_\lambda u_\lambda|^2 - \mu|A_\mu u_\mu|^2$$

this satisfies

$$\lambda(A_\mu u_\mu, A_\lambda u_\lambda) + \mu(A_\mu u_\mu, A_\lambda u_\lambda) - \lambda|A_\lambda u_\lambda|^2 - \mu|A_\mu u_\mu|^2 \leq (\lambda + \mu)|(A_\mu u_\mu, A_\lambda u_\lambda)| \leq (\lambda + \mu)(|A_\lambda u_\lambda|^2 + |A_\mu u_\mu|^2)$$

then

$$\frac{1}{2}\frac{d}{dt}|u_\lambda - u_\mu|^2 \leq (\lambda + \mu)(|A_\lambda u_\lambda|^2 + |A_\mu u_\mu|^2)$$

and by the equation (2.40), we conclude

$$\frac{1}{2}\frac{d}{dt}|u_\lambda - u_\mu|^2 =\leq 2(\lambda + \mu)|Au_0|^2,$$

therefore

$$|u_\lambda(t) - u_\mu(t)|^2 \leq 2\sqrt{(\lambda + \mu)t}|Au_0|. \tag{2.42}$$

It follows that for every fixed $t \geq$, $u_\lambda(t)$ is a Cauchy sequence as $\lambda \to 0$, also since H is complete this sequence have a unique limit $u(t)$.

Moreover, for $t \in [0, T]$, and passing to the limit in the equation (2.42) as $\mu \to 0$, we have

$$|u(t) - u_\lambda(t)| \leq 2\sqrt{\lambda t}|Au_0| \leq 2\sqrt{\lambda T}|Au_0|.$$

Thus, we obtain that $u(t)$ converge to 0 as $\lambda \to 0$. Therefore, the converge is uniform in t on every bounded interval $[0, T]$. Also, since every u_λ is continuous, then $u \in C([0, \infty); H)$.

Step 4. We assume in addition, that $u_0 \in D(A^2)$, i.e., $u_0 \in D(A)$ and $Au_0 \in D(A)$. We prove that $\frac{du_\lambda}{dt}(t)$ converges, to some limit as $\lambda \to 0$. Moreover, the converge is uniform on every bounded interval $[0, T]$.

Put $v_\lambda = \frac{du_\lambda}{dt}$, and $\frac{dv_\lambda}{dt} + A_\lambda v_\lambda = 0$ because A_λ is a linear bounded operator. We follow the same argument as in step 3. Observe that

$$\frac{1}{2}\frac{d}{dt}|v_\lambda - v_\mu|^2 \le (\lambda + \mu)(|A_\lambda v_\lambda|^2 + |A_\mu v_\mu|^2),$$

and note that, by step 2,

$$|A_\lambda v_\lambda(t)| \le |A_\lambda v_\lambda(0)| = |A_\lambda A_\lambda u(0)| = |A_\lambda^2 u(0)|.$$

Analogously

$$|A_\mu v_\mu(t)| \le |A_\lambda^2 u(0)|.$$

Also, since $Au_0 \in D(A)$, by using proposition (2.2.2) 1. and 2., we have

$$J_\lambda A J_\lambda A u_0 = J_\lambda J_\lambda A A u_0 = J_\lambda^2 A^2 u_0.$$

Then, since $\|J_\lambda\|_{\mathcal{L}(H)} \le 1$, we conclude that

$$|A_\lambda A_\lambda u_0| \le |A^2 u_0|.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}|v_\lambda - v_\mu|^2 \le 2(\lambda + \mu)|A^2 u_0|^2$$

Similar to in step 3, we conclude that $v_\lambda(t) = \frac{du_\lambda}{dt}(t)$ converges (as $\lambda \to 0$) to some limit. Moreover, the converge is uniform in every bounded interval.

Step 5. Assuming that $u_0 \in D(A^2)$, we prove that u is solution of

$$\begin{cases} \frac{du}{dt} + Au = 0 \quad \text{on} \quad [0, \infty), \\ u(0) = u_0. \end{cases}$$

By the steps 3 and 4, for every $0 < T < \infty$, we have $\lim\limits_{\lambda \to} u_\lambda(t) = u(t)$, uniformly in $[0, T]$ and $\frac{du_\lambda}{dt}(t)$ converges, as $\lambda \to 0$, uniformly in $[0, T]$. It follows that $u \in C^1([0, \infty); H)$ and

$$\lim_{\lambda \to 0} \frac{du_\lambda}{dt}(t) = \frac{du}{dt}(t) \quad \text{uniformly on } [0, T].$$

Note that $\lim\limits_{\lambda \to 0} J_\lambda u_\lambda(y) = u(t)$. In fact we can write

$$|u(t) - J_\lambda u_\lambda(t)| = |u(t) - J_\lambda u(t) + J_\lambda u(t) - J_\lambda u_\lambda(t)| \le |u(t) - J_\lambda u(t)| + |J_\lambda u(t) - J_\lambda u_\lambda(t)|$$

by factoring J_λ, and by using operator norm definition

$$|u(t) - J_\lambda u_\lambda(t)| \le |u(t) - J_\lambda u(t)| + \|J_\lambda\|_{\mathcal{L}(H)} |u(t) - u_\lambda(t)|$$

also, since $\|J_\lambda\|_{\mathcal{L}(H)} \le 1$

$$|u(t) - J_\lambda u_\lambda(t)| \le |u(t) - J_\lambda u(t)| + |u(t) - u_\lambda(t)|$$

and by the proposition (2.2.2), 4. and the step 3, we deduce

$$\lim_{\lambda\to 0}|u(t)-J_\lambda u_\lambda(t)|\leq \lim_{\lambda\to 0}|u(t)-J_\lambda u(t)|+|u(t)-u_\lambda(t)|=0$$

Rewriting equation (2.38) as

$$\frac{du_\lambda}{dt}(t)+A(J_\lambda u_\lambda(t))=0$$

and using the fact that A has a closed graph we deduce that

$$\lim_{\lambda\to 0}\left(\frac{du_\lambda}{dt}(t)+A(J_\lambda u_\lambda(t))\right)=\lim_{\lambda\to 0}\frac{du_\lambda}{dt}(t)+\lim_{\lambda\to 0}A(J_\lambda u_\lambda(t))=\frac{du}{dt}(t)+Au(t)=0$$

and $u\in D(A)$, for all $t\geq 0$.

Since $u\in C^1([0,\infty);H)$, the function $t\to Au(t)$ is continuous form $[0,\infty)$ into H and thus $u\in C([0,\infty);D(A))$. Hence, u is solution of (2.36).
Moreover by the equations (2.39) and (2.40) we have

$$|u(t)|\leq|u_0|\quad \text{for all } t\geq 0 \tag{2.43}$$

and

$$|\frac{du}{dt}(u)|=|Au(t)|\leq|Au_0|\quad \text{for all } t\geq 0. \tag{2.44}$$

Step 6. In order to finish the proof, we prove that $D(A^2)$ is dense in $D(A)$, i.e., let $u_0\in D(A)$, then for all $\varepsilon>0$ exists $\bar{u_0}$ such that $|u_0-\bar{u_0}|<\varepsilon$ and $|Au_0-A\bar{u_0}|<\varepsilon$.

Set $\bar{u_0}=J_\lambda u_0$ for some appropriate $\lambda>0$. We have by the proposition (2.2.1)

$$\bar{u_0}\in D(A)$$

and also

$$u_0=\bar{u_0}+\lambda A\bar{u_0}$$

thus $A\bar{u_0}=\frac{1}{\lambda}(u_0-\bar{u_0})$, then $A\bar{u_0}\in D(A)$, i.e., $\bar{u_0}\in D(A^2)$.

By the proposition (2.2.2).4, we have

$$\lim_{\lambda\to 0}|\bar{u_0}-u_0|=\lim_{\lambda\to 0}|J_\lambda u_0-u_0|=0,$$

and

$$\lim_{\lambda\to 0}|A\bar{u_0}-Au_0|=\lim_{\lambda\to 0}|AJ_\lambda u_0-Au_0|=\lim_{\lambda\to 0}|J_\lambda Au_0-Au_0|=0.$$

We choose $\lambda>0$ small enough depending of ε such that $|u_0-\overline{u_0}|=|u_0-J_\lambda u_0|<\varepsilon$ and $|Au_0-A\overline{u_0}|=|Au_0-AJ_\lambda u_0|<\varepsilon$.

Now, given $u_0\in D(A)$, using the last fact ($D(A^2)$ is dense in $D(A)$) we construct a sequence (u_{0_n}) in $D(A^2)$ such that $u_{0,n}\to u_0$ and $Au_{0,n}\to Au_0$. By the step 5, we know that there exists a solution u_n of

$$\begin{cases}\frac{du_n}{dt}+Au_n=0 \quad \text{on} \quad [0,\infty),\\ u_n(0)=u_{0,n},\end{cases} \tag{2.45}$$

and there exists a solution u_m of

$$\begin{cases} \frac{du_m}{dt} + Au_m = 0 \quad \text{on} \quad [0,\infty), \\ u_m(0) = u_{0,m}, \end{cases}$$

subtracting these problems we set

$$\begin{cases} \frac{d}{dt}(u_n - u_m) + A(u_n - u_m) = 0 \quad \text{on} \quad [0,\infty), \\ (u_n - u_m)(0) = u_{0,n} - u_{0,m}. \end{cases}$$

For all $t \geq 0$ we have, by equations (2.43) and (2.44)

$$\lim_{m,n\to\infty} |u_n(t) - u_m(t)| \leq \lim_{m,n\to\infty} |u_{0,n} - u_{0,m}| = 0$$

and

$$\lim_{m,n\to\infty} |\frac{du_n}{dt}(t) - \frac{du_m}{dt}(t)| \leq \lim_{m,n\to\infty} |Au_{0,n} - Au_{0,m}| = 0.$$

Then, we conclude that

$$u_n(t) \to u(t) \quad \text{uniformly on } [0,\infty)$$

and

$$\frac{du_n}{dt}(t) \to \frac{du}{dt}(t) \quad \text{uniformly on } [0,\infty)$$

with u in $C^1([0,\infty); H)$. Passing to the limit in problem (2.45) and using the fact that A is a closed operator, we deduce that $u(t) \in D(A)$ and u is solution for the problem with initial value (2.36). □

2.2.3 Hille-Yosida Theorem Applied to Wave Equation

In this section we want to know if the initial value problem of wave equation

$$\begin{cases} u_{tt} + \beta u_t - \Delta u + \lambda u = 0 \quad \text{in} \quad \Omega \times (0,\infty) \\ u_{|\Gamma} = 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) \quad x \in \Omega \end{cases} \tag{2.46}$$

has a unique solution. We start this section introducing the notion of Sóbolev spaces, because these are the spaces where the solution u is.

Definition 2.2.3. *The Sóbolev space $W^{m,p}$ is a vectorial space of functions, which is subset of L^p. This space is formed by the functions classes whose derivatives up to order m are in L^p. Let Ω be a domain subset of $\mathbb{R}^n$ then*

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) \;\; | \;\; D^\alpha f \in L^p(\Omega), \forall \alpha \in \mathbb{N}^n : |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \leq m\}$$

where $D^\alpha f$ is multi-index notation for partial derivatives.

For $p = 2$, is used the notation

$$H^m(\Omega) = W^{m,2}(\Omega)$$

this is space is a Hilbert space equipped with the scalar product

$$(u,v)_{H^m} = \sum_{0\leq|\alpha|\leq m} (D^\alpha u, D^\alpha v)_{L^2}$$

(Brezis, 2011, 271).

And finally, the space $H_0^1(\Omega)$ is a closure of the functions of class C^∞ with compact support in H^1, i.e., $\overline{H^1(\Omega) \cap C_0^\infty(\Omega)}$.

Set $U := (u, u_t)^T$ in $X := H_0^1(\Omega) \times L^2(\Omega)$. We write the equation in (2.46) as $U_t + AU = 0$, where

$$A = \begin{pmatrix} 0 & -I \\ -\Delta + \lambda & \beta \end{pmatrix}$$

in fact

$$U_t + AU = \begin{pmatrix} u_t \\ u_{tt} \end{pmatrix} + \begin{pmatrix} -u_t \\ \beta u_t - \Delta u + \lambda u \end{pmatrix} = \begin{pmatrix} 0 \\ u_{tt} + \beta u_t - \Delta u + \lambda u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

and

$$U(0) = \begin{pmatrix} u(0,x) \\ u_t(0,x) \end{pmatrix} = \begin{pmatrix} g(x) \\ h(x) \end{pmatrix}$$

The domain of A is then given by

$$D(A) = \{U \in X, AU \in X\}$$

this is $u \in H_0^1(\Omega)$, $u_t \in H_0^1(\Omega)$ and $u_{tt} \in L^2(\Omega)$, then

$$D(A) = \{U \in X, AU \in X\} = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$$

Applying the result of the *Hille-Yosida* theorem, the initial value problem of wave equation (2.46) has unique solution if A is a maximal monotone operator.

2.3 Weak Solution of Wave Equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded open and $0 < T < \infty$. Considering hyperbolic initial/boundary-value problem

$$\begin{cases} u_{tt} + Lu = f \quad \text{in} \quad \Omega_T \\ u_{|\Gamma} = 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) \quad \forall x \in \Omega \end{cases} \tag{2.47}$$

where $\Omega_T := \Omega \times (0, T]$, $f : \Omega_T \to \mathbb{R}$, $g, h : \Omega \to \mathbb{R}$ and $u : \overline{\Omega_T} \to \mathbb{R}$. We have used the notation $u = u(x,t)$. The operator L is defined as

$$Lu(x,t) = -\sum_{i,j=1}^{n} (a^{ij}(x,t)u(x,t)_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x,t)u(x,t)_{x_i} + c(t,x)u(x,t) \tag{2.48}$$

this is the divergence form of L. The non-divergence form is

$$Lu(x,t) = -\sum_{i,j=1}^{n} a^{ij}(x,t)u(x,t)_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t)u(x,t)_{x_i} + c(x,t)u(x,t). \tag{2.49}$$

The operator L is said uniformly hyperbolic if there exist a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \theta(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)$$

for all $(x,t) \in \Omega_T$ and $\xi \in \mathbb{R}^n$ (Evans, 1997, 377, 378).

If $a^{ij} = \delta_{ij}$, the bidimensional delta function Kronecker, $b^i = c = 0$, for all $(i, j = 1, 2, \ldots, n)$, the $L = -\Delta$ (the Laplacian). In general second order hyperbolic PDE models wave transmission in heterogeneous non-isotropic media.

Now, we suppose

$$a^{ij}, b^i, c \in C^1(\overline{\Omega}_T) \quad \text{for } (i, j = 1, 2, \ldots, n),$$
$$f \in L^2(\Omega_T),$$
$$g \in H_0^1(\Omega), \quad h \in L^2(\Omega),$$

and

$$a^{ij} = a^{ji}.$$

Set $\boldsymbol{u} : [0,T] \to H_0^1(\Omega)$ as

$$[\boldsymbol{u}(t)](x) := u(x,t) \quad \text{for } x \in \Omega \quad \text{and } 0 \leq t \leq T,$$

similarly $\boldsymbol{f} : [0,T] \to L^2(\Omega)$ as

$$[\boldsymbol{f}(t)](x) := f(x,t) \quad \text{for } x \in \Omega \quad \text{and } 0 \leq t \leq T.$$

Fixing any function $v \in H_0^1(\Omega)$, and if we multiply the expression $u_{tt} + Lu = f$ by v and integrating we have

$$\int_\Omega u_{tt}(x,t)v(x)dx + \int_\Omega Lu(x,t)v(x)dx = \int_\Omega f(x,t)vdx. \tag{2.50}$$

We can write

$$\int_\Omega u_{tt}(x,t)v(x)dx = \int_\Omega (\boldsymbol{u}(t)_{tt}v)(x)dx = (\boldsymbol{u}(t)_{tt}, v)_{L^2(\Omega)}$$

$(\quad,\quad)_{L^2(\Omega)}$ denotes the inner product in $L^2(\Omega)$.

If we set $A = [a^{ij}]$, we have

$$\operatorname{div}[v(x)A\nabla u(x,t)] = \nabla v(x) \cdot A\nabla u(x,t) + v(x)\operatorname{div}[A\nabla u(x,t)]$$

then

$$-\int_\Omega \sum_{i,j=1}^{n} (a^{ij}(x,t)u(x,t)_{x_i})_{x_j} v(x)dx = -\int_\Omega \operatorname{div}[v(x)A\nabla u(x,t)]dx + \int_\Omega \nabla v(x) \cdot A\nabla u(x,t)dx$$

by Green theorem we have

$$\int_\Omega \operatorname{div}[v(x)A\nabla u(x,t)]dx = \int_{\partial\Omega} v(x)(A\nabla u(x,t) \cdot \eta)dx$$

where η is a normal vector in $\partial\Omega$, and as v has compact support

$$\int_\Omega \operatorname{div}[v(x)A\nabla u(x,t)]dx = \int_{\partial\Omega} v(x)(A\nabla u(x,t)\cdot\eta)dx = 0.$$

Then

$$\begin{aligned}\int_\Omega Lu(x,t)v(x)\,\mathrm{dx} &= -\int_\Omega \sum_{i,j=1}^n (a^{ij}(x,t)u(t,x)_{x_i})_{x_j}v(x)\,\mathrm{dx} + \int_\Omega \sum_{i=1}^n b^i(x,t)u_{x_i}v(x)\,\mathrm{dx} + \int_\Omega c(x,t)u(x,t)v(x)\,\mathrm{dx}\\ &= \int_\Omega \nabla v(x)\cdot A\nabla u(x,t)\,\mathrm{dx} + \int_\Omega \sum_{i=1}^n b^i(x,t)u_{x_i}v(x)\,\mathrm{dx} + \int_\Omega c(x,t)u(x,t)v(x)\,\mathrm{dx}\\ &= \int_\Omega \nabla v(x)\cdot A\nabla[\boldsymbol{u}(t)](x)\,\mathrm{dx} + \int_\Omega \sum_{i=1}^n b^i(t,x)[\boldsymbol{u}(t)](x)_{x_i}v(x)\,\mathrm{dx} + \int_\Omega c(t,x)[\boldsymbol{u}(t)](x)v(x)\,\mathrm{dx}.\end{aligned}$$

We introduce the time-dependent bilinear form

$$B[u,v;t] = \int_\Omega \sum_{i,j=1}^n a^{ij}(t,\cdot)u_{x_i}v_{x_j} + \sum_{i=1}^n b^i(t,\cdot)u_{x_i}v + c(t,\cdot)uv$$

for $u,v\in H_0^1(\Omega)$ and $0\le t\le T$. Then $\int_\Omega Lu(x,t)v(x)dx = B[\boldsymbol{u}(t),v;t]$. Finally we deduce that the equation (2.50) is equivalent to

$$(\boldsymbol{u}(t)_{tt},v)_{L^2(\Omega)} + B[\boldsymbol{u}(t),v;t] = (\boldsymbol{f}(t),v)_{L^2(\Omega)}.$$

It motivates the next definition (Evans, 1997, 379, 380).

Definition 2.3.1. *The function* $\boldsymbol{u}$*, such that* $\boldsymbol{u}\in L^2([0,T];H_0^1(\Omega))$*,* $\boldsymbol{u}_t\in L^2([0,T];L^2(\Omega))$*,* $\boldsymbol{u}_{tt}\in L^2([0,T];(H_0^1(\Omega))^*)$*; is a weak solution of the hyperbolic initial/boundary-value problem (2.47) whenever*

1. $(\boldsymbol{u}(t)_{tt},v)_{L^2(\Omega)} + B[(\boldsymbol{u},v)(x);t] = (\boldsymbol{f}(t),v)_{L^2(\Omega)}$. *For each* $v\in H_0^1(\Omega)$.
2. $\boldsymbol{u}(0)=g$, $\boldsymbol{u}_t(0)=h$.

For $L=-\Delta$

$$B[\boldsymbol{u}(t),v;t] = \int_\Omega (\nabla\boldsymbol{u}(t)\cdot\nabla v)(x)dx.$$

2.3.1 Galerkin Approximations, by Eigenvectors

The existence of the weak solution is not proved in this book, for a complete proof you can see (Evans, 1997, 380-390). However we will construct a approximate solution of the hyperbolic initial/boundary-value problem

$$\begin{cases} u_{tt}+Lu=f \quad \text{in} \quad \Omega_T\\ u_{|\Gamma}=0\\ u(x,0)=g(x), \quad u_t(x,0)=h(x) \quad \forall x\in\Omega\end{cases} \tag{2.51}$$

by solving the problem in a finite dimensional space. The *Galerkin's method* (via eigenvectors) begins with a selection of smooth functions w_k $(k=1,2,\ldots)$ such that

$$\{w_k\}_{k=1}^\infty \quad \text{in an orthogonal basis of } H_0^1(\Omega)$$

and

$$\{w_k\}_{k=1}^{\infty} \quad \text{in an orthonormal basis of } L^2(\Omega).$$

For a fix positive integer N, we set

$$\boldsymbol{u}_N(t) := \sum_{k=1}^{N} d_N^k(t) w_k \tag{2.52}$$

the aim is that

$$d_N^k(0) = (g, w_k)_{L^2(\Omega)}, \tag{2.53}$$

$$\left(d_N^k(0)\right)_t = (h, w_k)_{L^2(\Omega)} \tag{2.54}$$

since

$$\boldsymbol{u}_N(0) = \sum_{k=1}^{N} d_N^k(0) w_k = \sum_{k=1}^{N} (g, w_k) w_k$$

and

$$(\boldsymbol{u}_N(0))_t = \sum_{k=1}^{N} (d_N^k(0))_t w_k = \sum_{k=1}^{N} (h, w_k) w_k$$

also we pretend that

$$((\boldsymbol{u}_N(t))_{tt}, w_k)_{L^2(\Omega)} + B[(\boldsymbol{u}_N, w_k)(x); t] = (\boldsymbol{f}(t), w_k)_{L^2(\Omega)} \quad \text{for } 0 \le t \le T \quad \text{and } (k = 1, 2, \ldots, N). \tag{2.55}$$

The next theorem allows to construct a approximate solution $\boldsymbol{u}_N$ satisfying the equations (2.53)-(2.55).

Theorem 2.3.1. *For each integer $N = 1, 2, \ldots$, there exists a unique function $\boldsymbol{u}_N$ of the form (2.52), satisfying equations (2.53)-(2.55).*

Proof. Let $\boldsymbol{u}_m$ be given by (2.52), employing that $\{w_k\}_{k=1}^{\infty}$ is a orthonormal basis, we deduce that

$$((\boldsymbol{u}_N(t))_{tt}, w_k)_{L^2(\Omega)} = \left(\sum_{l=1}^{N} (d_N^k(t))_{tt} w_l, w_k \right)_{L^2(\Omega)} = \sum_{l=1}^{N} (d_N^k(t))_{tt} (w_l, w_k)_{L^2(\Omega)} = (d_N^k(t))_{tt}$$

Furthermore we have

$$B[\boldsymbol{u}_N(t), w_k; t] = \int_\Omega \sum_{i,j=1}^{n} a^{ij}(t, \cdot)(\boldsymbol{u}_N(t))_{x_i}(w_k)_{x_j} + \sum_{i=1}^{n} b^i(t, \cdot)(\boldsymbol{u}_N(t))_{x_i} w_k + c(t, \cdot)\boldsymbol{u}_N(t) w_k$$

Note that

$$\int_\Omega \sum_{i,j=1}^{n} a^{ij}(t, \cdot)(\boldsymbol{u}_N(t))_{x_i}(w_k)_{x_j} = \int_\Omega \sum_{i,j=1}^{n} a^{ij}(t, \cdot) \left(\sum_{l=1}^{N} d_N^l(t) w_l \right)_{x_i} (w_k)_{x_j} = \sum_{l=1}^{N} d_N^l(t) \int_\Omega \sum_{i,j=1}^{n} a^{ij}(t, \cdot)(w_l)_{x_i}(w_k)_{x_j},$$

similarly

$$\int_\Omega \sum_{i=1}^{n} b^i(t, \cdot)(\boldsymbol{u}_N(t))_{x_i} w_k = \int_\Omega \sum_{i=1}^{n} b^i(t, \cdot) \left(\sum_{l=1}^{N} d_N^l(t) w_l \right)_{x_i} w_k = \sum_{l=1}^{N} d_N^l(t) \int_\Omega \sum_{i=1}^{n} b^i(t, \cdot)(w_l)_{x_i}(w_k)$$

and

$$\int_\Omega c(t,\cdot)\boldsymbol{u}_N(t)v = \int_\Omega c(t,\cdot)\left(\sum_{l=1}^N d_N^l(t)w_l\right)v = \sum_{l=1}^N d_N^l(t)\int_\Omega c(t,\cdot)w_l w_k.$$

If we define $e^{kl}(t) := B[w_l, w_k; t]$ with $(k, l = 1, 2, \ldots, N)$ then

$$\begin{aligned} B[\boldsymbol{u}_m(t), w_k; t] &= \sum_{l=1}^N d_N^l(t)\left(\int_\Omega \sum_{i,j=1}^n a^{ij}(t,\cdot)(w_l)_{x_i}(w_k)_{x_j} + \sum_{i=1}^n b^i(t,\cdot)(w_l)_{x_i}(w_k) + c(t,\cdot)w_l w_k\right) \\ &= \sum_{l=1}^N d_N^l(t)B[w_l, w_k; t] = \sum_{l=1}^N d_N^l(t)e^{kl}(t). \end{aligned}$$

We also write

$$f^k(t) := (\boldsymbol{f}(t), w_k)_{L^2(\Omega)}.$$

The equation (2.55) becomes in

$$(d_N^k(t))_{tt} + \sum_{l=1}^N d_N^l(t)e^{kl}(t) = f^k(t) \quad \text{for } 0 \le t \le T \quad \text{and } (k = 1, 2, \ldots, N)$$

subject to the conditions (2.53) and (2.54). This system of ordinary differential equations has a unique solution function of class 2, $\boldsymbol{d}_N(t) = (d_N^1(t), \ldots, d_N^N(t))$ according to standard theory for ordinary differential equations. □

Chapter 3

Examples

3.1 Elastic Wave in Homogeneous, Isotropic Media Using Finite Element Method. Working in Matlab

The main idea of this method, as with another numerical methods, is to transform a continuous problem, with infinitely many degrees of freedom in a discrete problem or a equations system with a finite number of unknowns which should be resolved using a computer (Johnson, 1987).

In the finite element method (FEM) the discretization is to use the Galerkin's method in order to change the continuous problem to a problem in a space of finite dimension, as we show below.

Finally, we want to focus this example in a particular application. We will use a elastic wave for identify the subsurface. The idea is analyze the reflected wave; the waves are reflected when there is a change of media.T

3.1.1 Boundary Conditions in the Real Solution

The method used for determining positions of medium changes in the subsurface of the earth has some peculiarities with respect to the wave equation studied before. This difference is in boundary conditions. On the one hand with the real data the space Ω_∞ is considered as space determined by the half-plane $z < 0$ (or very close to this, because the wave travels through the earth until it is fully attenuated), i.e.,

$$\Omega_\infty = \{(x,z) : -\infty < x < \infty, z < 0\},$$

On the other hand, in the simulation by FEM the space Ω is a rectangle in $\mathbb{R}^2$, i.e. $\Omega = (a,b) \times (d,0) \subset \mathbb{R}^2$. Let Γ be the boundary of the rectangle. Put $\Gamma = A \cup B \cup C \cup D$, where,

$$A = \{(x,z) : a < x < c, z = 0\}, \quad D = \{(x,z) : a < x < c, z = d\}$$

are the horizontal boundaries and

$$B = \{(x,z) : x = a, d < z < 0\}, \quad C = \{(x,z) : x = b, d < z < 0\},$$

are the vertical boundaries.

Finally, Let E be defined as the complement in Ω_∞ of $\Omega \cup \Gamma$, i.e.,

$$\Omega_\infty = \Omega \cup \Gamma \cup E$$

It is necessary to determine conditions on Γ to solve the problem in Ω. Note that we do not know the behavior of the wave in the complement E and we want to avoid reflections on Γ (which do not correspond to reality model) (Becerra, 2011).

Non-reflecting Boundary Conditions, Reynolds's Method

If in the whole domain Ω_∞ is satisfied the equation

$$c^2\nabla^2 u = u_{tt} \tag{3.1}$$

and since $\Gamma \subset \Omega_\infty$, in particular the equation (3.1) also holds on Γ.

An approach that can taken is that the direction of u on the border is normal only to Γ. Thus, the formulation of (3.1) in B, C, D, is reduced to a differential equation unidimensional.
On B, C we have that,

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)\phi = 0, \tag{3.2}$$

and on$D$ we have that,

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2}\right)\phi = 0. \tag{3.3}$$

Note that on A there is a Dirichlet's boundary conditions equal to zero. The operator in these equations, can be written as

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = L_1 L_2.$$

Then, the equations above can be written as,

$$L_1 L_2 \phi = 0$$

and

$$L_2 L_1 \phi = 0$$

respectively.
To model non-reflective boundaries $L_2\phi = 0$ or $L_1\phi = 0$ is chosen. Particularly On B we have

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\phi = 0.$$

On C we choose.

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\phi = 0.$$

On D we have

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial z}\right)\phi = 0.$$

. See for instance (Becerra, 2011).

3.1.2 Statement of the Problem with Reynolds's conditions

Strong formulation

We start with the elastic wave equation with initial values in Ω. This is known as the strong formulation, and we write as

$$(S)\begin{cases} \text{Find } u : \Omega \times (0,T) \to \mathbb{R} \text{ such that} \\ \frac{\partial^2 u}{\partial t^2} - div(k^2 \nabla u) = S, \\ u(x,z,0) = 0 \quad \text{for all } (x,z) \in \Omega, \\ u_t(x,z,0) = 0, \quad \text{for all } (x,z) \in \Omega, \\ u(x,z,t) = 0, \quad z \geq, 0 \\ \text{With Reynold's conditions over } \Gamma. \end{cases}$$

where $\Omega = (a,b) \times (d,0) \subset \mathbb{R}^2$, T is a fix value of time which depend on the wave duration. S is a source (e.g. a detonation), x and z are spatial variables; the horizontal and vertical coordinates respectively. t is a temporal variable and k is the speed of the media. The problem consists in find a function u in $L^2([0,T]; H^1(\Omega))$ that is solution of the differential equation raised.

Weak Formulation

Similarly to section (2.3). For any function v in $H^1(\Omega)$, we multiply the strong formulation by v and we integrate over Ω. In contrast, here we work with v without compact support since we want to establish special boundary conditions. Thus, we have the weak formulation as follow

$$(W)\begin{cases} \text{Find } u : \Omega \times (0,T) \to \mathbb{R} \text{ such that} \\ \int_\Omega u_{tt} v + \int_\Omega k^2 \nabla u \cdot \nabla v - \int_\Gamma k^2 (\nabla u \cdot \eta) v = \int_\Omega S\, v, \quad \text{for all } v \in H^1(\Omega), \\ u(x,z,0) = 0 \quad \text{for all } (x,z) \in \Omega, \\ u_t(x,z,0) = 0, \quad \text{for all } (x,z) \in \Omega, \\ u(x,z,t) = 0, \quad z \geq 0, \\ \text{With Reynold's conditions over } \Gamma. \end{cases}$$

where η is an unitary vector normal toΓ.

Note that boundary integral can be broke it as

$$\int_\Gamma k^2 (\nabla u \cdot \eta) v = \int_B k^2 (\nabla u \cdot \eta) v + \int_C k^2 (\nabla u \cdot \eta) v + \int_D k^2 (\nabla u \cdot \eta) v,$$

where we used condition on A ($z = 0$). Additionally we have used our geometry.

Also, note that for B, $\eta = -\hat{\mathbf{\imath}}$, for C, $\eta = \hat{\mathbf{\imath}}$ and for D, $\eta = -\hat{\mathbf{k}}$, as $\nabla = \frac{\partial}{\partial x}\hat{\mathbf{\imath}} + \frac{\partial}{\partial y}\hat{\mathbf{\jmath}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$, then we set this equation as,

$$\int_\Gamma k^2 (\nabla u \cdot \eta) v = \int_B k^2 \frac{-\partial u}{\partial x} v + \int_C k^2 \frac{\partial u}{\partial x} v + \int_D k^2 \frac{-\partial u}{\partial z} v.$$

With Reynolds's conitions, this expression reduces to

$$\int_\Gamma k^2 (\nabla u \cdot \eta) v = \int_B -k \frac{\partial u}{\partial t} v + \int_C -k \frac{\partial u}{\partial t} v + \int_D -k \frac{\partial u}{\partial t} v.$$

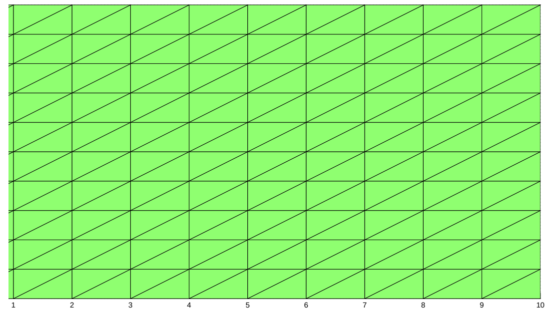

Figure 3.1: Triangulation used in wave equation simulation. The triangulation is done over a rectangle. Any triangle in this triangulation is right, and the length between two adjoint points is constant.

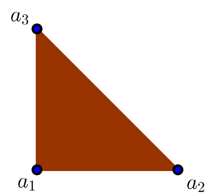

Figure 3.2: Triangle used in the triangulation. It is a right triangle with its vertex.

Replacing in the weak formulation, we have the formulation

$$(W)\begin{cases} \text{Find } u : \Omega \times (0,T) \to \mathbb{R} \text{ such that} \\ \int_\Omega u_{tt} v + \int_\Omega k^2 \nabla u \cdot \nabla v + \sum_{i=B,C,D} \int_i k u_t v = \int_\Omega S\, v, \quad \text{for all } v \in H^1(\Omega), \\ u(x,z,0) = 0 \quad \text{for all } (x,z) \in \Omega, \\ u_t(x,z,0) = 0, \quad \text{for all } (x,z) \in \Omega, \\ u(x,z,t) = 0, \quad z \geq 0. \end{cases}$$

Galerkin Formulation

As we wrote previously, the Galerkin formulation is a approximation of the weak formulation. We change the continuous space Ω for a discrete (and finite) space τ^h which is a triangulation over Ω. Also we change the space $H^1(\Omega)$, with infinite dimension, for the space $P^1(\tau^h)$ (polynomials of degree 1).

Triangulation τ^h For this paper we worked with a regular triangulation that covers the entire rectangular surface with right triangles, as shown in Figure 3.1. For a triangle, as Figure 3.2, the vertex $a_1 = (x_1, y_1)$, $a_2 = (x_2, y_2) = (x_1 + h, y_1)$, $a_3 = (x_3, y_3) = (x_1, y_1 + h)$ are considered, where h is the distance (constant) between two adjacent grid points.

Space of the finite elements
The space where the solution is considered is $P^1(\tau^h)$. The basis polynomials for every vertex (as triangulation is regular, we can generalize for every triangle of τ^h) are of the form,

for the vertex a_1

$$\varphi_1 = 1 - \frac{x - x_1}{h} - \frac{y - y_1}{h}, \tag{3.4}$$

for the vertex a_2

$$\varphi_2 = \frac{x - x_1}{h} \tag{3.5}$$

and for the vertex a_3

$$\varphi_3 = \frac{y - y_1}{h}. \tag{3.6}$$

Next, the finite element formulation is;

$$(G)\begin{cases} \text{Find } u : P^1(\tau^h) \times (0,T) \to \mathbb{R} \text{ such that} \\ \int_\Omega u_{tt} v + \int_\Omega k^2 \nabla u \cdot \nabla v + \sum_{i=B,C,D} \int_i k u_t v = \int_\Omega S\, v, \quad \text{for all } v \in P_0^1(\tau^h) \\ u(x,z,0) = 0 \quad \text{for all } (x,z) \in \Omega \\ u_t(x,z,0) = 0, \quad \text{for all } (x,z) \in \Omega \\ u(x,z,t) = 0, \quad z \geq 0. \end{cases}$$

Discrete Galerkin formulation

To switch from the continuous formulation on the time to a discrete formulation on the time we perform the Finite Difference approximation. Approximations for the derivatives with respect to time are obtained, as is shown below, by

$$\frac{\partial u}{\partial t} \approx \frac{u(x,z,t+\Delta t) - u(x,z,t)}{\Delta t}.$$

and

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u(x,z,t+2\Delta t) - 2u(x,z,t+\Delta t) + u(x,z,t)}{\Delta t^2}.$$

The following notation is used for the formulation

$$u_m = u(x,z,t+2\Delta t), \quad u_{m-1} = u(x,z,t+\Delta t) \quad \text{and} \quad u_{m-2} = u(x,z,t).$$

Now the discrete Galerkin formulation is proposed as,

$$(G_d)\begin{cases} \text{Find } u : P^1(\tau^h) \times (0,T) \to \mathbb{R} \text{ such that} \\ \frac{1}{\Delta t^2} \int_\Omega u_m v - \frac{2}{\Delta t^2} \int_\Omega u_{m-1} v + \frac{1}{\Delta t^2} \int_\Omega u_{m-2} v + \int_\Omega k^2 \nabla u_m \cdot \nabla v + \\ \sum_{i=B,C,D} \left(\frac{1}{\Delta t} \int_i k u_m v - \frac{1}{\Delta t} \int_i k u_{m-1} v \right) = \int_\Omega S\, v \quad \text{for all } v \in P^1(\tau^h) \\ . \end{cases}$$

The initial conditions and boundary conditions are the same that in the previous formulations.

Matrix Formulation

In this formulation we want to write the equations as matrix in order to convert the previous problem in a system of linear equations. We define the matrix $T = [T_{ij}]$, where T_{ij} is the number of the a_j vertex, of the triangle i.

Based on the equation (3.4), (3.5), (3.6), for every triangle we have,

$$\nabla\varphi_1 = \begin{bmatrix} \frac{-1}{h} & \frac{-1}{h} \end{bmatrix}^T, \quad \nabla\varphi_2 = \begin{bmatrix} \frac{1}{h} & 0 \end{bmatrix}^T, \quad \text{and} \quad \nabla\varphi_3 = \begin{bmatrix} 0 & \frac{1}{h} \end{bmatrix}^T.$$

Thus, for the n-th triangle we define, the local matrix $A^n = [A^n_{ij}]$, where

$$A^n_{ij} = \int_{\text{triangle } n} k^2 \nabla\varphi_i \cdot \nabla\varphi_j.$$

Since the area of the triangle is $\frac{h^2}{2}$, A^n is given by

$$A^n = k^2 \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}.$$

Since generally k is dependent of (x, z) , to simplify the calculation of the integral, the integral is aproximate with the average of k in the three vertex of the triangle; thus k is constant in every triangle (but is not constant in all grid).

Finally, we define the matrix $\boldsymbol{A} = [\boldsymbol{A}_{lk}]$. For a triangle n we set, the local matrix A^n as,

$$k^2 \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{T_{n1}T_{n1}} & \boldsymbol{A}_{T_{n1}T_{n2}} & \boldsymbol{A}_{T_{n1}T_{n3}} \\ \boldsymbol{A}_{T_{n2}T_{n1}} & \boldsymbol{A}_{T_{n2}T_{n2}} & \boldsymbol{A}_{T_{n2}T_{n3}} \\ \boldsymbol{A}_{T_{n3}T_{n1}} & \boldsymbol{A}_{T_{n3}T_{n2}} & \boldsymbol{A}_{T_{n3}T_{n3}} \end{pmatrix}. \tag{3.7}$$

It is clear that the subindex lk can be represented by several (at most two) triangles in τ^h, then $\boldsymbol{A}_{lk}$ is the sum of all values given by (3.7) for every triangle. Note that the size of $\boldsymbol{A}$ is the total number of vertex by the total number of vertex.

For the n-th triangle, the matrix $M^n = [M^n_{ij}]$ is defined by

$$M^n_{ij} = \int_{\text{triangle } n} \varphi_i \varphi_j$$

After we compute the integral on any triangle, we have,

$$M^n = \begin{pmatrix} h^2/12 & h^2/24 & h^2/24 \\ h^2/24 & h^2/12 & h^2/24 \\ h^2/24 & h^2/24 & h^2/12 \end{pmatrix}.$$

Similarly to the matrix $\boldsymbol{A}$, we have the matrix $\boldsymbol{M} = [\boldsymbol{M}_{lk}]$. For a triangle n we set, the local matrix M^n as,

$$M^n = \begin{pmatrix} h^2/12 & h^2/24 & h^2/24 \\ h^2/24 & h^2/12 & h^2/24 \\ h^2/24 & h^2/24 & h^2/12 \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_{T_{n1}T_{n1}} & \boldsymbol{M}_{T_{n1}T_{n2}} & \boldsymbol{M}_{T_{n1}T_{n3}} \\ \boldsymbol{M}_{T_{n2}T_{n1}} & \boldsymbol{M}_{T_{n2}T_{n2}} & \boldsymbol{M}_{T_{n2}T_{n3}} \\ \boldsymbol{M}_{T_{n3}T_{n1}} & \boldsymbol{M}_{T_{n3}T_{n2}} & \boldsymbol{M}_{T_{n3}T_{n3}} \end{pmatrix}. \tag{3.8}$$

Then $\boldsymbol{M}_{lk}$ is the sum of all values given by (3.8) for every triangle where the vertices l and k are.

Analogously the matrix $F^n = \left[F^n_{ij}\right]$ is defined, where

$$F^n_{ij} = \int_{\text{triangle } n, \text{ over } \Gamma} k\varphi_i\varphi_j.$$

In contrast to the matrix M, the integrals in F are performed only on the sides that are on the border. We set $\boldsymbol{F}$ similarly to $\boldsymbol{M}$. Note that the size of $\boldsymbol{A}$, $\boldsymbol{M}$ are $\boldsymbol{F}$ are the total number of vertex by the total number of vertex.

With these matrix, we use the fact that u and v are in the space $P^1(\tau^h)$ so are lineal combination of $\varphi_i^{(n)}$, with $i = 1, 2, 3$ (the superindex n indicate the triangle), hence

$$u = \sum_{n=1}^{N} \alpha_i^{(n)}\varphi_i^{(n)} \quad \text{with } (i = 1, 2, 3)$$

where N is the total number of the triangles and $\alpha_i^{(n)} \in \mathbb{R}$. Furthermore is enough that we work with $v = \sum_{n=1}^{N} \varphi_i^{(n)} \quad \text{with } (i = 1, 2, 3).$

With this in mind, the Galerkin formulation is proposed in matrix form as,

$$(M) \begin{cases} \text{Find } \alpha_m \text{ for every moment } m, \text{ such that} \\ \frac{1}{\Delta t^2}M\alpha_m + A\alpha_m + \frac{1}{\Delta t}F\alpha_m = b + \frac{2}{\Delta t^2}M\alpha_{m-1} - \frac{1}{\Delta t^2}M\alpha_{m-2} + \frac{1}{\Delta t}F\alpha_{m-1} \end{cases}$$

where $b = \left[\int S\varphi_i\right]$. And the initial conditions $\alpha_{-2} = \alpha_{-1} = 0$.

Hence the problem is reduced to iterate (for every instant) a linear system of the form,

$$\alpha_m = \left(b + \frac{2}{\Delta t^2}M\alpha_{m-1} - \frac{1}{\Delta t^2}M\alpha_{m-2} + \frac{1}{\Delta t}F\alpha_{m-1}\right)\left(\frac{1}{\Delta t^2}M + A + \frac{1}{\Delta t}F\right)^{-1}$$

(Johnson, 1987)

3.1.3 Velocity Model for Simulation

We simulate under the following scale,

$$1\textit{square} = 1 \text{ meter} = 1 \text{ m},$$

where one *square* refers to one square in the triangulation grid. In our triangulation ten squares are one unit. Also, we consider the time in milliseconds (ms).

If we consider a velocity V_r in $\frac{\text{m}}{\text{s}}$, then the correspond velocity in the simulation can be calculate, using cross-multiply, as

$$k = V_r\frac{\text{m}}{\text{s}} = V_r\frac{\textit{square}}{1000 \text{ ms}} = V_r(10^{-3})\frac{\textit{square}}{\text{ms}}.$$

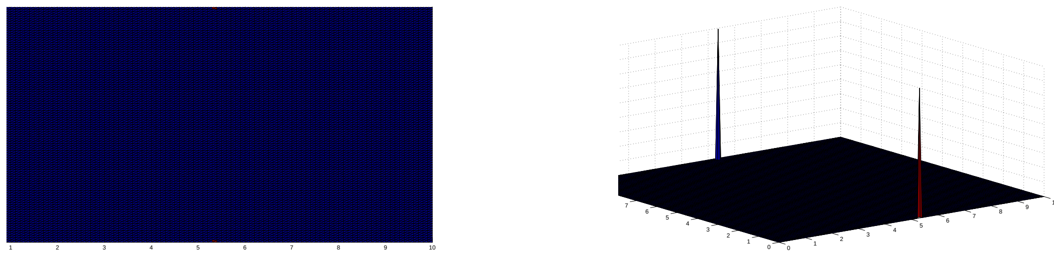

Figure 3.3: Velocity model with $k = 1$. The peaks are the position of the source and a sensor. The distance between these points is 100 *square*, i.e., 100 m. In the left is the image in 2D, and in the right is the image in 3D.

We introduce a correction factor of scale. Referring to Figure 3.3, we set a model velocity $k = 1$. The peaks are the position of the source and a sensor. The distance between these points is 100 *square*, i.e., 100 m. In the Figure 3.4, we observe that the arrival, measured in the source, time is 10.25 ms and the output, measured in the sensor, time is 0.75 ms, thus the delay time is 9.5 ms, different to 10 ms expected. For this reason, we redefine k as

$$k = 1.05 V_r (10^{-3}) \frac{square}{\text{ms}}. \tag{3.9}$$

where 1.05 is the correction factor.

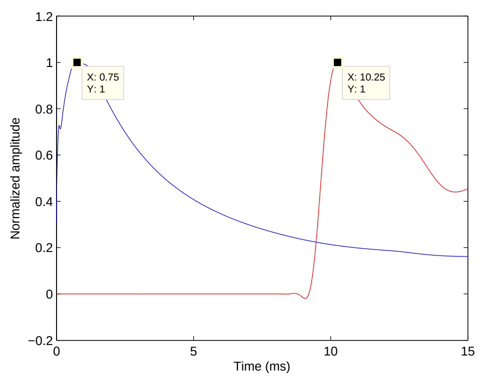

Figure 3.4: Time output (source) in blue, and arrival time (sensor) in red, for model velocity with $k = 1$. The horizontal axis is the time in ms, and the vertical axis is the normalized amplitude of the wave.

3.1.4 Results

Description of Physical Conditions and Results

We test the wave equation for velocity models in which a source is placed over a geological fault. The geological fault is modeled as velocity changes. We choose two representative velocities in the earth's subsurface, these are: Wet sand whose P-wave velocity is $k = 1700$ m/s and granite whose P-wave velocity is $k = 4700$ m/s. The velocity values was consulted in (Gary Mavko,).

The size of Ω is 100 m$\times$100 m, then we work with 100$\times$100 squares. The numbers of triangles is the double.

The source is over the entire surface. Also, the source is a high amplitude pulse (we just modify the amplitude in order to improve the simulation instead of following a real model). Below in the velocity models, the source is represented as a edge in the top.

For the reflection's graphs we have: The horizontal axis is the time in milliseconds, the vertical axis is the positions in x, for $z = 0$, in meters . Also, the color scale represent the normalized amplitude wave, where red is the highest and blue is the lowest.

We start with the velocity model in the Figure 3.5, this represent a space just with wet sand.

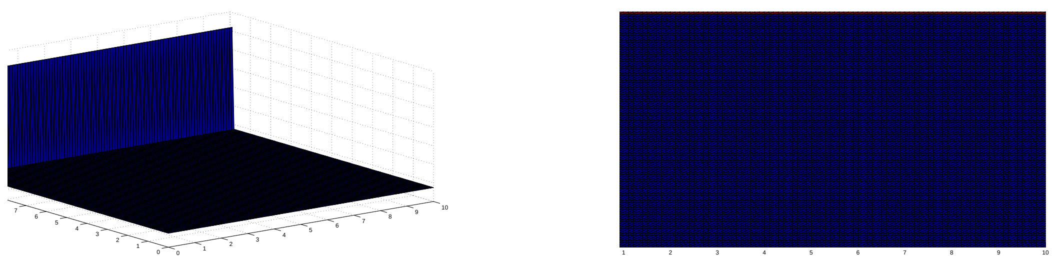

Figure 3.5: Velocity model for the first geological fault. In this case we use wet sand in the entire rectangle Ω.

The reflections for the velocity model in the Figure 3.5, are shown in the Figure 3.6. Note that for every time instant after of 100 ms the reflections in every position are almost equivalent, the curvature is due to the superposition effects of the source. Before of 100 ms the behavior is defined by the source effects.

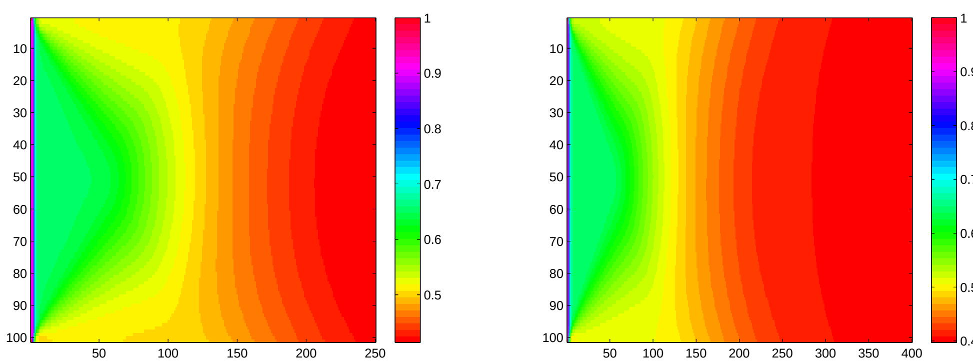

Figure 3.6: Reflections for velocity model for the first geological fault. In this case, the horizontal axis is the time in milliseconds. The vertical axis is the positions in x for $z = 0$ in meters. The color scale represent the normalized amplitude wave.

We show in the Figure 3.7, the wave in the time instant of 100 milliseconds for the first velocity model. Note that the wave moves equally in all directions. There are not reflections because there are not medium changes and also due to the special boundary conditions.
The second velocity model is presented in the Figure 3.8. This represent a granite segment with stairway shape in the wet sand. We show in the Figure 3.9 the reflections for the velocity model in the Figure 3.8.

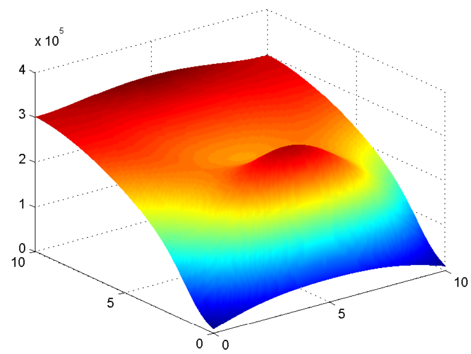

Figure 3.7: Wave in the time instant of 100 milliseconds for the first velocity model. The amplitude is not normalized.

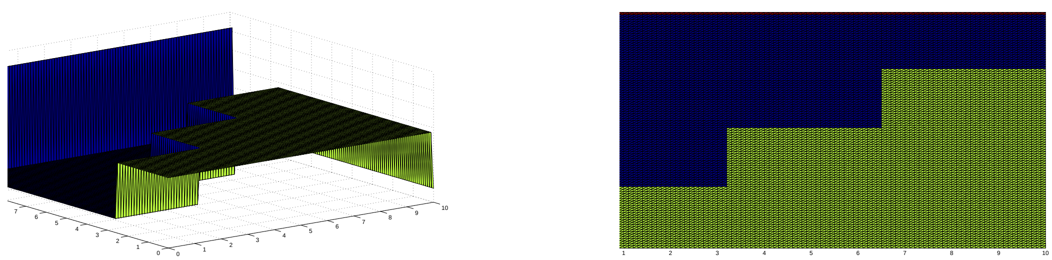

Figure 3.8: Velocity model for the second geological fault. In this case we use wet sand and a granite segment with stairway shape.

In contrast to the reflections for the first model, the traces for every time instant are not constant. We can identify the stairway shape in the reflections. Note that for a specific color, the arrival time for the segment between 66.6 meters and 100 meters is the less. It is coherent because this segment has more quantity of granite, whose P-wave velocity is major than the velocity in wet sand.

In the Figure 3.10, is presented the wave in the time instant of 150 milliseconds for the second velocity model. Note that the wave amplitude change with the velocity change. Also there are reflections wave due to the edge of the granite segment. We can observe clearly the stairway shape in the attenuate wave.

Finally we present the third velocity model in the Figue 3.11. It is a granite segment with hole shape in the wet sand.

The reflections for the third velocity model in the Figure 3.11, are shown in the Figure 3.12. We can identify the hole shape in the reflections. Although, in this graph we do not have information about the hole depth, the image suggests some approximation. In fact, there are methods for extract the depth information, these are known as *migration techniques*. The Figure 3.13 shows the wave in the time instant of 50 milliseconds for the third velocity model. It is clear how the hole generates a big reflection.

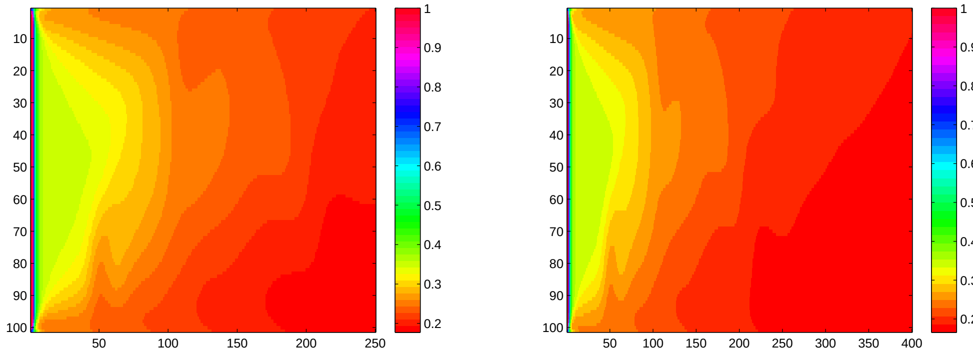

Figure 3.9: Reflections for velocity model for the second geological fault. In this case, the horizontal axis is the time in milliseconds. The vertical axis is the positions in x for $z = 0$ in meters. The color scale represent the normalized amplitude wave, where red is the highest and blue is the lowest.

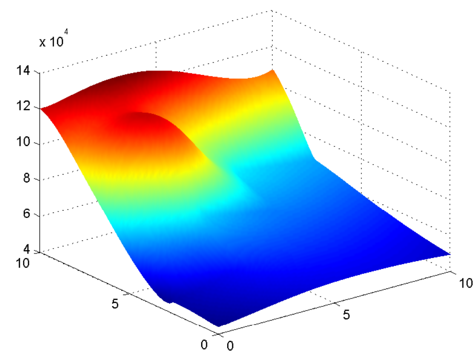

Figure 3.10: Wave in the time instant of 150 milliseconds for the second velocity model. The amplitude is not normalized.

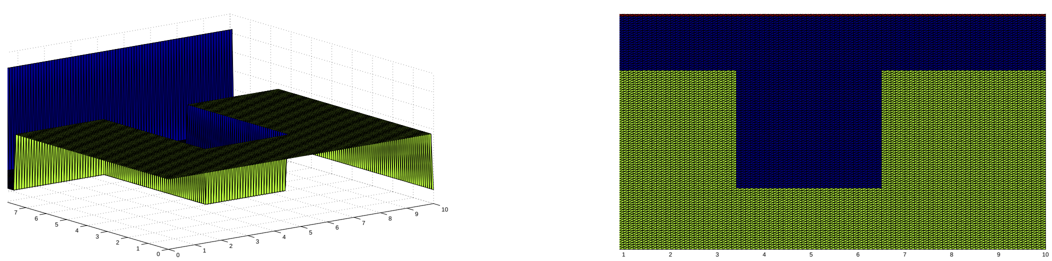

Figure 3.11: Velocity model for the third geological fault. In this case we use wet sand and a granite segment with hole shape .

3.2 Elastic Wave Equation in Three Dimensions Using FreeFem

In this section we present the behavior of the elastic wave in three dimensions. In this section we do not work the variables with units, as in the previous example, because we do not use this example in order to show a application but rather we want to present this example as academic instance.

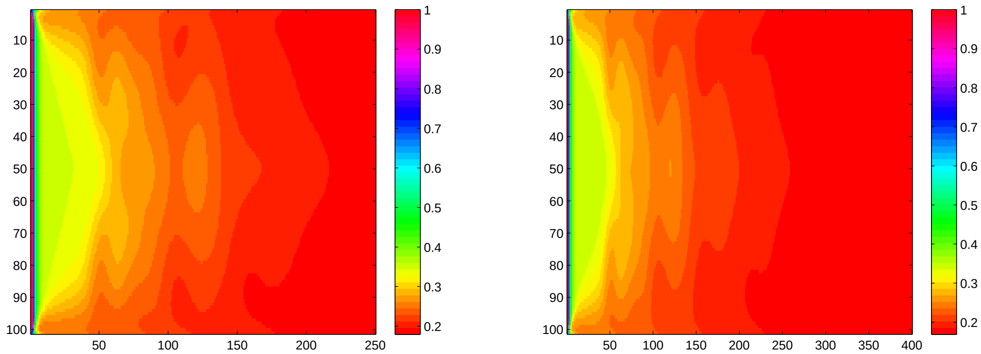

Figure 3.12: Reflections for velocity model for the third geological fault. In this case, the horizontal axis is the time in milliseconds. The vertical axis is the positions in x for $z = 0$ in meters. The color scale represent the normalized amplitude wave, where red is the highest and blue is the lowest.

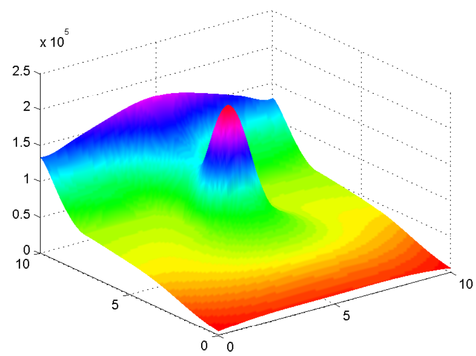

Figure 3.13: Wave in the time instant of 50 milliseconds for the third velocity model. The amplitude is not normalized.

Similarly to previous example, we deduce the weak formulation for the problem. However we change the space Ω. Here we consider Ω as a subset of $\mathbb{R}^3$, and it is define as follow,

$$\Omega = [0, 100] \times [0, 100] \times [0, 100],$$

it is a cube with size of 100. We consider $T \in \mathbb{R}$ large enough. Then we write the weak formulation for this example as,

$$(W) \begin{cases} \text{Find } u : \Omega \times (0, T) \to \mathbb{R} \text{ such that} \\ \int_\Omega u_{tt} v + \int_\Omega k^2 \nabla u \cdot \nabla v - \int_\Gamma k^2 (\nabla u \cdot \eta) v = \int_\Omega S\, v, \quad \text{for all } v \in H^1(\Omega), \\ u(x, y, z, 0) = 0 \quad \text{for all } (x, y, z) \in \Omega, \\ u_t(x, y, z, 0) = 0, \quad \text{for all } (x, y, z) \in \Omega, \\ u(x, y, z, t) = 0, \quad \text{on } \Gamma \quad \text{(Dirichlet's conditions equal to zero).} \end{cases}$$

The source is $S = e^{-((x-50)^2+(y-50)^2+(z-50)^2}\delta(t)$, i.e., it is a impulse in the initial time instant, which it localizes in the cube center.

In order to define the the time derivatives as finite differences, we use $\Delta t = 1$. Finally the grid of the cube have ten square for every side, and the triangulation is done with right triangles; there are two triangle in every square.

The first simulation is in a cube where the velocity is the same for every point. We take a picture for a fixed time instant, this is present in the Figure 3.14. Note as the wave travels in spherical way. For this velocity model, u has the same value for fixed distances, then we can conclude that us is a radial function, i.e., u is fixed for every rotation transformation. The second simulation is done in a velocity model presented in the

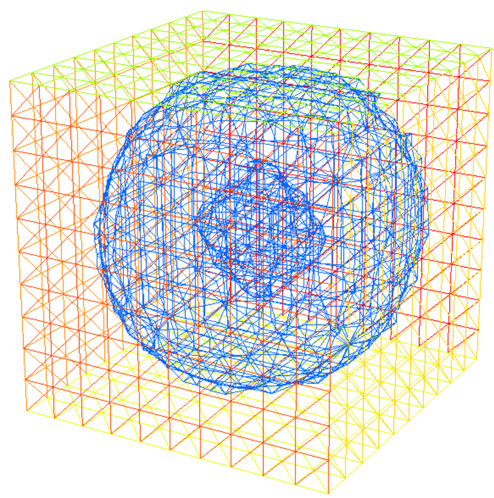

Figure 3.14: Wave in Three Dimensions for Homogeneous Isotropic Model Velocity, $k = 1$. The source is a impulse in the center of the cube. This plot is for a specific time instant. Note the symmetry due to the media.

Figure 3.15. In this model we have in top $k = 2$ and in the bottom $k = 1$. We take a picture for a fixed time instant in this model, this is presented in the Figure 3.16. In contrast with the Figure 3.14, it dose not have spherical symmetry. The function u travels quicker in the top than in the bottom as consequence of the difference of velocities.

We want to remark a important difference between the solutions in three dimensions and two dimensions. In the previous example in the Figure 3.7 we note that the final value for u in the source position is near to 3 instead of 1.785 (equivalent to 1700 m/s by the equation (3.9)). This residual value is because the solution was done for two dimensions. In the Figure 3.17 we present a picture of u for the first model (homogeneous) at a time instant away. Contrary to two dimensions, here there are not residual values, i.e., after that wave interacts with a point, this point returns to its initial behavior.

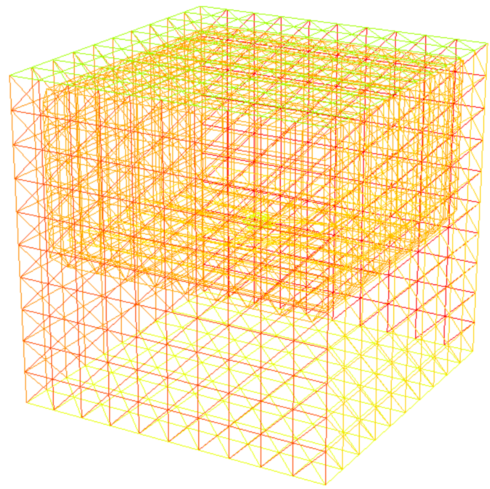

Figure 3.15: Velocity Model in Three Dimensions. On top $k = 2$, in the bottom $k = 1$.

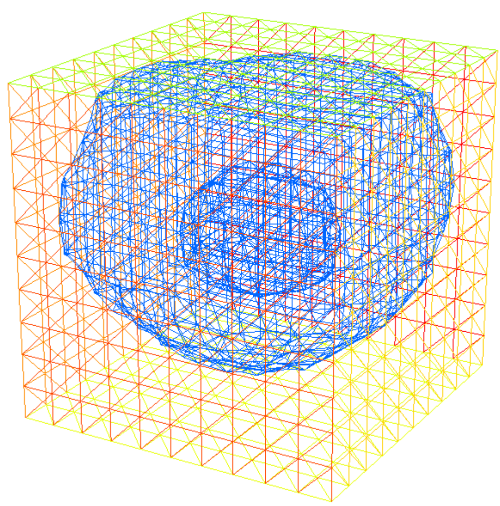

Figure 3.16: Wave in Three Dimensions for Nonhomogeneous Model Velocity in Figure 3.15. The source is a impulse in the center of the cube. This plot is for a specific time instant. Note as the wave travel quicker in the medium where the velocity is higher.

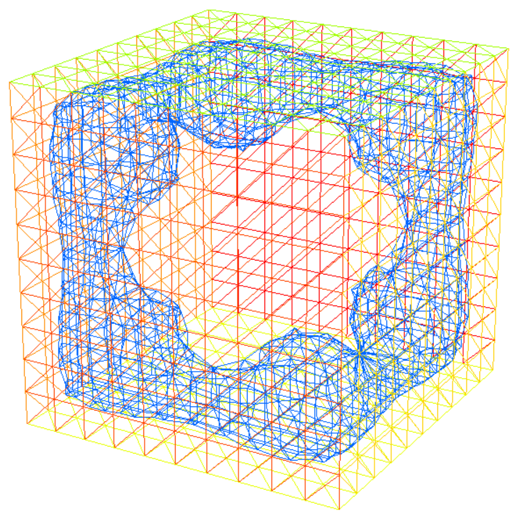

Figure 3.17: Wave in Three Dimensions for Homogeneous Isotropic Model Velocity, $k = 1$. The source is a impulse in the center of the cube. This plot is for a specific time instant high enough. In contrast wit the wave in two dimensions in three dimensions there is not recording of the wave.

Chapter 4

Conclusions and Future Studies

The classical solutions are developed in entire $\mathbb{R}^n$, therefore are not so useful in practical applications. In contrast the Hille-Yosida theorem and the existence of weak solutions can help us to ensure the existence and uniqueness of the solution for a bounded space.

The weak solution is not necessarily regular, this is very important because we can prove existence of solution. In addition, the Galerkin formulation gives a approximation of the weak solution, and this approximation can be computed in numerically.

Using an inverse problem related to the wave equation it is possible to identify the position of objects in the subsurface. Nevertheless, it is necessary to use a additional post-processing with the purpose of changing the time scale to deep scale and give better definition to shapes.

For the wave equation it is possible use the finite element method in order to approximate the solution, however this method present instability problems when the medium is non homogeneous and the space is not so regular as in the examples.

Appendices

Appendices A

Elastic Wave Preliminaries

In this section we present concepts of tensor product. For this we review the construction based on the *universal property (UP)*. Next we define the tensor product as a vector space. With this in mind the tensor space is presented and finally the Cauchy Tensor is obtained.

A.1 Universal Property

Referring to Figure (A.1). Let A be a set, and f and g be functions with domain A. We say that the diagram *commutes* if there exits a function $t : S \to X$ such that $g = t \circ f$.

Let $\mathcal{S}$ be a family of sets and $\mathcal{F} = \{g : A \to X | X \in \mathcal{S}\}$, we define a *measuring family* $\mathcal{H}$, as

$$\mathcal{H} = \{g : X \to Y | X, Y \in \mathcal{S}\}$$

We assume that $\mathcal{H}$ has the following structure:

1. The family $\mathcal{H}$ contains the identity function i_X for each X in $\mathcal{S}$.

2. The family $\mathcal{H}$ is closed under composition of functions.

3. For any $t \in \mathcal{H}$ and $f \in \mathcal{F}$ the composition $t \circ f$ is defined and belongs to $\mathcal{F}$.

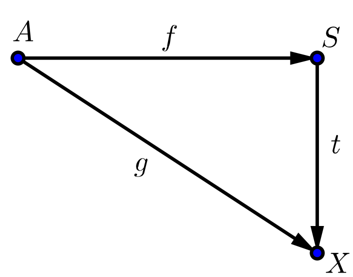

Figure A.1: A is a set, f and g are function with domain A. The diagram commutes, i.e., exists a function t, such that $g = t \circ f$.

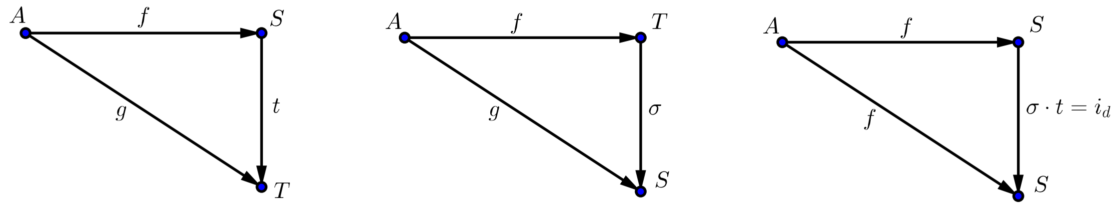

Figure A.2: Diagram, where $(S, f : A \to S)$ and $(T, g : A \to T)$ are universal pairs for $(\mathcal{F}, \mathcal{H})$.

The members of $\mathcal{H}$ is called *measuring functions*.

A pair $(S, f : A \to S)$ where $S \in \mathcal{S}$ and $f \in \mathcal{F}$ have the *universal property* for the family $\mathcal{F}$ as measured by $\mathcal{H}$ (or is a *universal pair* for $(\mathcal{F}, \mathcal{H})$) if for every function $g : A \to X$ in $\mathcal{F}$ there exists a unique $t : S \to X$ in $\mathcal{H}$ for which the diagram in the Figure A.1 commutes, i.e., $g = t \circ f$. The unique measuring function t is called the mediating morphism for g. (Marsden, Hughes, 1994).

Theorem A.1.1. *Let $(S, f : A \to S)$ and $(T, g : A \to T)$ be universal pairs for $(\mathcal{F}, \mathcal{H})$, then there exists a bijective measuring function $\mu \in \mathcal{H}$ for which $\mu S = T$. In fact the mediating morphism of f with respect to g and the mediating morphism of g with respect to f are isomorphisms.*

Proof. Referring to Figure A.2 . There exists mediating morphisms $t : S \to T$ and $\sigma : T \to S$ such that

$$g = t \circ f \quad \text{and} \quad f = \sigma \circ g.$$

Then we can write

$$g = (t \circ \sigma) \circ g \quad \text{and} \quad f = (\sigma \circ t) \circ .$$

Hence the identity map $i : S \to S$ and $\sigma \circ t : S \to S$ are mediating morphisms, and the uniqueness of mediating morphisms implies $\sigma \circ t = i$. Analogously $t \circ \sigma = i$, also t and σ are inverses of one another, making t the desired bijection. □

Let F be a field, we call $Vect(F)$ the family of all vector spaces over F. Let be U, V, W vectorial spaces in $Vect(F)$. We define the set of all bilinear functions from $U \times V$ to W as $hom_F(U, V; W)$, i.e.,

$$hom_F(U, V; W) = \{f : U \times V \to W | f \text{ is a bilinear map}, U, V, W \in Vect(F)\}.$$

See (Roman, 2007, 356-362).

A.2 Tensor Product

Let U and V be in $Vect(F)$. The definition of the tensor product $U \otimes V$ will be via universal property for bilinear functions, as measured by linearity. Referring to Figure A.3, we want to define a vector space T and a bilinear function $t : U \times V \to T$ such that the diagram commute for any bilinear function f with domain $U \times V$.

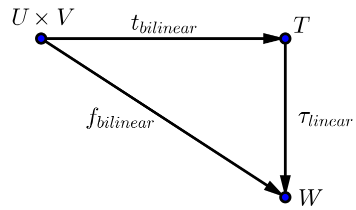

Figure A.3: Diagram, where $(T, t : U \times V \to T)$ is universal pair for bilinearity.

Definition A.2.1. *Let $U \times V$ be the cartesian product of two vector space over F. Let*

$$\mathcal{F} = \bigcup_{W} \{hom_F(U, V; W) | W \in Vect(F)\},$$

be the family of all bilinear maps from $U \times V$ to vector space W over F. The measuring family $\mathcal{H}$ is the family of all linear transformations.

A pair $(T, t : U \times V \to T)$ is ***universal for bilinearity*** *if it is universal for $(\mathcal{F}, \mathcal{H})$, that is, if for every bilinear map $f : U \times V \to W$, there exist a unique linear transformation $\tau : T \to W$ such that $f = \tau \circ t$, as in Figure A.3.*

The map τ is called the ***mediating morphism*** *for f.*

Now the tensor product via universal property is defined as follows.

Definition A.2.2. *Let U and V be vector spaces over a field F, any universal pair for bilinearity $(T, t : U \times V \to T)$ is called* ***tensor product*** *of U and V. The vector space T is denoted $U \otimes V$, from Theorem A.1.1, T is well defined. The map t is called* ***tensor map****, and the elements of $U \otimes V$ are called tensors. In addition*

$$u \otimes v = t(u, v)$$

for $u \in U$ and $v \in V$ (Roman, 2007, 362).

A.3 Tensor Spaces

Let V be a vector space of finite dimension over a field F. For p, q nonnegative integers the tensor product

$$T^p_q(V) = \underbrace{V \otimes V \otimes \ldots \otimes V}_{p \text{ factors}} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{q \text{ factors}} := V^{\otimes p} \otimes (V^*)^{\otimes q}$$

is called the space of **tensor of type** $\binom{p}{q}$. Here p is called contra-variant type and q is called covariant type. Note that if $p = q = 0$ then $T^p_q(V) = F$ (Roman, 2007).

In the case of reflexive spaces we have $V \approx V^{**}$ and then

$$T^p_q(V) = V^{\otimes p} \otimes (V^*)^{\otimes q} \approx ((V^*)^{\otimes p} \otimes (V^*)^{\otimes q})^* \approx hom_F((V^*)^{\times p} \times V^{\times q}, F)$$

where $V^{\times q} = \underbrace{V \times \ldots \times V}_{q\text{factors}}$. The last equivalence is illustrated in the Figure A.3. In other words, a tensor of type $\binom{p}{q}$ is a multilinear mapping

$$\mathbf{T} : (V^*)^{\times p} \times V^{\times q} \to F.$$

We denote $T^{A_1,A_2,\ldots,A_p}_{B_1,B_2,\ldots,B_q} = \mathbf{T}(\mathbf{E}^{A_1}, \mathbf{E}^{A_2}, \ldots, \mathbf{E}^{A_p}, \mathbf{E}_{B_1}, \mathbf{E}_{B_2}, \ldots, \mathbf{E}_{B_1})$. Where $\mathbf{E}_{B_i}$ is a base for V and $\mathbf{E}^{A_i}$ is a base for dual V^*.

Note that a tensor of type $\binom{p}{0}$ is

$$T^p(V) = T^p_0(V) = \underbrace{V \otimes V \otimes \ldots \otimes V}_{p \text{ factors}}.$$

Last equation is very important in order to define the stress tensor.

A.4 Riemannian Metric

Here we review the concept of *Riemannian metric* in order to define the *stress tensor* in a general way. We start with a previous definition about the *tangent spaces*. Because on this we define a inner product, and then the desired metric.

An immersion of a open domain $U \subset \mathbb{R}^m$ in $\mathbb{R}^n$ (with $m \leq n$) is a differentiable application $f : U \to \mathbb{R}^n$, such that $f'(x) : \mathbb{R}^m \to \mathbb{R}^n$ is a linear injective transformation.

A parameterization of class C^k and dimension m of a set $V \subset \mathbb{R}^n$ is an immersion $\varphi : V_0 \subset \mathbb{R}^m \to V$ of class C^k.

A set $M \subset \mathbb{R}^n$ is called a surface of dimension m and class C^k if for every point p in M there is a open $U \subset \mathbb{R}^n$, with $p \in U$, such that $V = U \cap M$, where V is the image of a parameterization $\varphi : V_0 \to V$. φ is of dimension m and class C^k. The set V is a open in M, this is called parameterized neighborhood of the point p.

Definition A.4.1. ***Tangent Spaces****: Let p a point of a surface M of dimension m and class C^k in $\mathbb{R}^n$. A tangent space to M in the point p, is a vector space $T_p(M) \subset \mathbb{R}^n$ such that, is the set of velocity vectors $v = \lambda'(0)$ of different trajectories $\lambda : (-\varepsilon, \varepsilon) \to M$ such that $\lambda(0) = p$. (Lima, 2004)*

Now we define an inner product on $T_x(\mathcal{S})$ special metric on this vector space.

Definition A.4.2. *A* ***Riemannian metric*** *on a surface $\mathcal{S}$ is a C^∞ tensor $\mathbf{g}$ of type $\binom{0}{2}$ such that for every $x \in \mathcal{S}$*

1. *The $\mathbf{g}(x)$ is symmetric; that is for every $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_x\mathcal{S}$, we have $\mathbf{g}(x)(\boldsymbol{w}_1, \boldsymbol{w}_2) = \mathbf{g}(x)(\boldsymbol{w}_2, \boldsymbol{w}_1)$.*

2. *The $\mathbf{g}(x)$ is positive-definite; i.e. $\mathbf{g}(x)(\boldsymbol{w}, \boldsymbol{w}) > 0$ for $\boldsymbol{w} \in T_x(\mathcal{S})$ nonzero.*

Note that symmetry of $\mathbf{g}$ means $g_{AB} = g_{BA}$

A.5 Stress Tensor

The concept of stress is very important in continuum mechanics. This expresses the interaction of a material with surrounding material in terms of surface contact forces. In (Marsden, Hughes, 1994, 1, 2) the authors present the stress principle of Cauchy as "Upon any smooth, closed, orientable surface $\mathcal{S}$, be it an imagined surface within the body or the bounding surface of the body itself, there exists an integrable field if traction vectors $\mathbf{t}_{\mathcal{S}}$ equipollent (same resultant and moment) to the action exerted by the matter exterior to $\mathcal{S}$ and contiguous to it on that interior to $\mathcal{S}$".

There exists a vector field $\mathbf{t}_{\mathcal{S}}(x, t, \mathbf{n})$ (or simply $\mathbf{t}(x, t, \mathbf{n})$ when $\mathcal{S}$ is clear from the context) depending on time t, the spatial point x, a unit vector $\mathbf{n}$, and implicitly the motion $\phi(x, t)$ itself. Physically, $\mathbf{t}$ *represents the force per unit area exerted on a surface element oriented with normal* $\boldsymbol{n}$. The $\mathbf{t}$ is called the Cauchy stress vector.

We introduce the concept *balance of momentum* in order to establish a relation between vector t and a tensor of type $\binom{2}{0}$.

Let $\mathcal{B}$ a simple body, i.e. an open set in $\mathbb{R}^3$. A *configuration* of $\mathcal{B}$ is a mapping $\phi : \mathcal{B} \to \mathcal{R}^3$ that is sufficiently smooth, orientation preserving, and invertible. Following the notation in (Marsden, Hughes, 1994). A point in $\mathcal{B}$ is denoted as $X = (X_1, X_2, X_3)$ and is called *material point*, while a point in $\mathbb{R}^3$ is denoted $x = (x_1, x_2, x_3)$ this is called *spacial point*.

A *motion* of $\mathcal{B}$ is a time-dependent family of configurations, written $x = \phi(X, t)$. The *material velocity* is defined by $\mathbf{V}(X, t) = \frac{\partial\phi(X,t)}{\partial t}$, and *spatial velocity* is $\mathbf{v} : \phi(\mathcal{B}) \to \mathbb{R}^3$, $\mathbf{v} = \mathbf{V} \circ \phi^{-1}$. The next definition is the general expression for balance, this is the *master balance law*.

Definition A.5.1. *Let $a(x, t)$, $b(x, t)$ be given scalar functions defined for $t \in \mathbb{R}$ (or an open interval in $\mathbb{R}$), $x \in \phi_t(\mathcal{B})$, and $\boldsymbol{c}(x, t)$ a given vector field on $\phi_t(\mathcal{B}$. We say that a, b, $\boldsymbol{x}$ satisfy the* ***master balance law*** *if, for every open $\mathcal{U} \subset \mathcal{B}$*

$$\frac{d}{dt}\int_{\phi(\mathcal{U})} a dv = \int_{\phi(\mathcal{U})} b dv + \int_{\partial\phi(\mathcal{U})} \langle \boldsymbol{c}, \boldsymbol{n} \rangle da$$

where $\boldsymbol{n}$ is the unit outward normal vector to $\partial\phi(\mathcal{U})$ and da is the area element on this surface.

Now, we write a particular case of this law, in terms of $\mathbf{t}(x, t, \mathbf{n})$, the mass density $\rho(x, t)$, $\mathbf{v}(x, t)$; it is the *balance of momentum*.

Definition A.5.2. *Given a simple body $\mathcal{B}$ in $\mathbb{R}^n$. Let $\phi(X, t)$ be the motion of $\mathcal{B}$ in $\mathbb{R}^n$ (ϕ is C^1). Let $\rho(x, t)$ be C^1, $\boldsymbol{t}(x, t, \boldsymbol{n})$ and $\boldsymbol{b}(x, t)$, the* balance of momentum *is satisfied if for every open $\mathcal{U}$ in $\mathcal{B}$*

$$\frac{d}{dt}\int_{\phi(\mathcal{U})} \rho \boldsymbol{v} dv = \int_{\phi(\mathcal{U})} \rho \boldsymbol{b} dv + \int_{\partial\phi(\mathcal{U})} \boldsymbol{t} da$$

where $\boldsymbol{t}$ is evaluated on the unit outward normal vector $\boldsymbol{n}$ to $\partial\phi(\mathcal{U})$ at a point x.

With the last definition is possible write the next theorem.

Theorem A.5.1. *Assume that balance of momentum holds, that $\phi(X, t)$ is C^1 and $\boldsymbol{t}(x, t, \boldsymbol{n})$ is a continuous function of its arguments. We consider x in a surface $\mathcal{S}$ (Normally $\mathcal{S} = \mathbb{R}^3$). Then there is a unique $\binom{2}{0}$ tensor field, denote $\boldsymbol{\sigma}$, depending only on x and t such that*

$$\boldsymbol{t}(x, t, \boldsymbol{n}) =< \boldsymbol{\sigma}(x, t), \boldsymbol{n} > .$$

In coordinates $\{x_a\}$ *on* $\mathcal{S}$*, the preceding equation reads*

$$t^a(x, t, \boldsymbol{n}) = \sigma^{ac}(x, t) g_{bc} n^b = \sigma^a_b n^b$$

For a proof see (Marsden, Hughes, 1994).

The associated tensor $\boldsymbol{\sigma}$ with components σ^a_b is called ***Cauchy stress tensor***.

Appendices B

Transport Equation

We consider the partial differential equation

$$u_t + b \cdot \nabla u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \tag{B.1}$$

where b is a fixed vector in $\mathbb{R}^n$ and $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is the unknown. We call, for convenience, $u = u(x, t)$, where $x \in \mathbb{R}^n$, x is a point in the space, and $t > 0$ denotes a typical time.

Here, we suppose that u is smooth enough. For (x, t) fixed, we call $z(s)$ to u over the line through (x, t) with the direction $(b, 1) \in \mathbb{R}^{n+1}$ with $s \in \mathbb{R}$, i.e., $z(s) = u((x, t) + s(b, 1)) = u(x + sb, t + s)$. Note that

$$\frac{dz}{ds}(s) = \nabla z(s) \cdot b + z_t(s) = \nabla u(x + sb, t + s) \cdot b + u_t(x + sb, t + s)$$

by the equation (B.1), we conclude $\frac{dz}{ds}(s) = \nabla u(x + sb, t + s) \cdot b + u_t(x + sb, t + s) = 0$. Thus, $z(s)$ is constant for s in $\mathbb{R}$. With this in mind we solve the next initial-value problem.

B.1 Initial-value Transport Problem

Let the problem

$$\begin{cases} u_t + b \cdot \nabla u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{B.2}$$

Where $b \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are known. The problem is to compute u. For (x, t) given, the line through (x, t) with direction $(b, 1)$, $(x + sb, t + s)$ with $(s \in \mathbb{R})$ hits the plane $\mathbb{R}^n \times \{t = 0\}$ when $s = -t$, at the point $(x - tb, 0)$. Since u is constant on the line and $u(x - tb, 0) = g(x - tb)$, we conclude

$$u(x, t) = g(x - tb) \quad (x \in \mathbb{R}^n, t \geq 0). \tag{B.3}$$

If (B.4) has a regular solution u, it must be given by (B.3). And conversely, if g is C^1, the u defined by (B.3) is indeed a solution of (B.4), (Evans, 1997, 28, 29).

B.2 Nonhomogeneous Problem

Now, we consider the problem

$$\begin{cases} u_t + b \cdot \nabla u = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{B.4}$$

Given $(x,t) \in \mathbb{R}^{n+1}$ fixed. If $z(s) = u(x+sb, t+s)$ for $s \in \mathbb{R}$, the

$$\frac{\partial z}{\partial s}(s) = \nabla u(x+sb, t+s) \cdot b + u_t(x+sb, t+s) = f(x+sb, t+s).$$

Note that

$$u(x,t) - g(x-tb) = z(0) - z(-t) = \int_{-t}^{0} \frac{\partial z}{\partial s}(s) ds$$

by fundamental theorem calculus and using the substitution $\zeta = t + s$, we have

$$u(x,t) - g(x-tb) = \int_{-t}^{0} f(x+sb, t+s) ds = \int_{0}^{t} f(x+(s-t)b, s) ds,$$

hence,

$$u(x,t) = g(x-tb) + \int_{0}^{t} f(x+(s-t)b, s) ds \quad (x \in \mathbb{R}^n, t \geq 0).$$

Bibliography

Brezis, H. (2011), *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York,USA, 1-48.

G, Margraver. (2003), *Numerical Methods of Exploration Seismology with algorithms in MATLAB*. University of Calgary.

S, Becerra. (2011), *Propagación de ondas sísmicas y migración*. Universidad Nacional de Colombia.

G, Cohen. (2002), *Higher-Order Numerical Methods for Transient Wave Equations*. CSpringer.

C, Johnson. (1987), *Numerical solution of partial differential equations by the finite element method*. Cambridge university press.

E, Lima (2004), *Análise Real*. Rio de Janeiro, Brazil.

E, Marsden. T, Hughes (1994), *Mathematical Foundations of Elasticity*. Dover Publicarions, Inc.

S. Roman (2007), *Advanced Linear Algebra, Third Edition*, Springer.

R. Iorio, V. De Magalhães (2001), *Fourier Analysis and Partial Differential Equations*, Cambrige University Press.

L. Evans (1997), *Partial Differential Equations*, American Mathematical Society.

Conceptual Overview of Rock and Fluid Factors that Impact Seismic Velocity and Impedance. https://pangea.stanford.edu/courses/gp262/Notes/8.SeismicVelocity.pdf