Instituto Nacional de Matemática Pura e Aplicada

Master's Thesis

Lie groupoid Morphisms inducing isomorphisms at the infinitesimal level

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Abstract.

We study which maps between Lie groupoids induce an isomorphism of Lie algebroids. We give a characterization of such a map through its kernel; the kernel has to be a normal groupoid discrete in fibers and has to be a closed submanifold. Conversely, if we have a subgroupoid with some conditions on the differential and algebraic structures, then the projection to the quotient groupoid induces an isomorphism between the Lie algebroids. We discuss how this theorem relates with classic theorems Lie 1 and Lie 2, and somehow simplifies and improves them. We present the examples of Lie groups and of pair groupoids, that are related to covering spaces. Then we study Poisson manifolds, a version of our theorem for symplectic groupoids, and as an example, we derive the classification of Lagrangian fibrations.

Resumo.
Nesta dissertação estuda-se que mapas entre grupóides de Lie induzem isomorfismos de algebróides de Lie. Estes mapas são caraterizados por seu núcleo; este núcleo tem que ser um subgrupóide normal e discreto nas fibras; ele também tem que ser uma subvariedade fechada. No outro sentido, se tem-se um subgrupóide tendo uma estrutura diferenciável e algébrica satisfazendo certas condições, sua projeção ao grupóide quociente induz um isomorfismos entre os algebróides. Discute-se como este teorema se relaciona com os teoremas clássicos Lie 1 e Lie 2, e de algum modo simplificá-los e melhorá-los. Apresentam-se os exemplos de grupos de Lie e grupóides de pares, que estão relacionados aos espaços de recobrimento. Depois, estudam-se variedades de Poisson, a versão para grupóides simpléticos do nosso teorema, e como exemplo, encontra-se a classificação de fibrações Lagrangianas.

Contents

Chapter 1

Introduction

Background. Several geometries can be understood in the framework of Lie groupoids, for example the Lie groups, vector bundles, group bundles, principal bundles and fundamental groupoids. A first idea of a groupoid, as generalization of Lie group, could be a fibration with a set of symmetries in the fibers, the fibration would be the source map and symmetries those given by right multiplication. A concise definition for groupoids is a category in which every arrow is invertible.

Lie algebroids can be defined indepently of Lie groupoids, like Lie algebras with respect to Lie groups, and they appear naturally in some geometries like Poisson geometry and regular foliations.

There is a functor from Lie groupoids to Lie algebroids, it is the differentiation map, which encodes global information to infinitesimal information. As in the case of Lie groups, this map loses global information, i.e. its inverse map ,"the integration", cannot be defined. There are three Lie theorems which relate a Lie groupoid with its Lie algebroid; the first is about the existence of a "bigger" groupoid which has the same Lie algebroid, the second is about integrating a morphism between the Lie algebroids given Lie groupoids, and the third is about the integration of any Lie algebroid. The last one is not true in general and the proof is not easy, this is not studied in this work. The first and second theorems are widely treated in the literature. In this work we propose a innovative vision which could help to simplify and improve Lie 1 and Lie 2 theorems.

For the examples of Lie algebroids given by a regular foliation, it is possible to defined the integration. However, the integration may not be a Hausdorff space, this case is not studied in this work. The integration of a Poisson manifold is not always possible, but when exists, the integration is a symplectic groupoid, which has an additional compatible symplectic structure.

Description. The main question that we address is: What maps of Lie groupoids induce isomorphisms of Lie algebroids? This question is significant in particular cases as Lie groups and related to problems such as the classification of Langrangian fibrations. Also this question can be useful to understand Lie 1 for Lie Groupoids with Hausdorff space, and the question implies Lie 2 through an argument we will describe.

The answer obtained, under the hypothesis of "source-coneected", is a characterization of these map through its kernel; the kernel has to be a normal groupoid discrete in fibers and has to be a closed embedded submanifold. Conversely, if we have a subgroupoid with some differential and algebraic conditions, then the projection to the quotient Lie groupoid induces an isomorphism between the Lie algebroids.

We present two interesting examples. The case of Lie groups and the case of pair groupoids; in these cases it is possible to relate the answer of the question with the theory of covering maps. For Lie groups, the maps are related with the covering maps of the arrows space in the groupoid, while the relation for pair groupoids is with the covering map of the objects space.

For symplectic groupoids, the answer of the question has an additional condition on the kernel; it has to be a Lagrangian submanifold. We study the Lagragian fibrations, which locally can be identified with a group bundle, that can be characterized through the main question worked here. However, a Lagrangian fibration is not a Lie groupoid (group bundle) in general, and the global obstruction is related with the existence of a global section.

Organization. In section 2 we review fundamentals concepts about Lie groupoids, in section 3 we give the definition and basic properties about Lie algebroids. The section 4 is the main part of the document, we present the relation between groupoids and algebroids, the Lie theorems and the principal theorem of our work, besides the examples of Lie groups and pair groupoids. The section 5 focuses on the symplectic case, we introduce the Poisson manifolds and the relation with Lie algebroids, the principal theorem for symplectic groupoids, and the example of Lagrangian fibrations.

Chapter 2

Lie groupoids

In this chapter we give some definitions and basic properties about the Lie groupois that are necessary for the development of the text. Then we show some examples and we introduce the definiton of an action of a Lie groupoid on a moment map. For a more extensive treatment of this subject the reader can consult [MM03], [CF06], [Mac05], [dH13].

2.1 Definitions and basic properties

Definition 2.1.1. A groupoid G consists of two sets, a set of arrows G_1 and a set of objects M, and five maps relating G_1 and M, that we describe below. The *source* map $s : G_1 \to M$ and the *target* map $t : G_1 \to M$; for any arrow such that $s(g) = x$ and $t(g) = y$ we write $g : x \to y$. The *multiplication* map

$$m : G_1 \times_M G_1 \to G_1 \quad (x \xrightarrow{g} y, y \xrightarrow{h} z) \to x \xrightarrow{hg} z$$

where the subset $G_1 \times_M G_1 \subset G_1 \times G_1$ are the composable arrows. The *unit* map and the *inverse* map

$$u : M \to G_1 \quad x \to (x \xrightarrow{u_x} x), \quad i : G_1 \to G_1 \quad (x \xrightarrow{g} y) \to (y \xrightarrow{g^{-1}} x)$$

These map satisfy the axioms $k(hg) = (kh)g$, $u(t(g))g = g = gu(s(g))$, $i(g)g = u(s(g))$ and $gi(g) = u(t(g))$ for any $k, h, g \in G_1$ and $s(k) = t(h)$ and $s(h) = t(g)$.

We use the notation $G \rightrightarrows M$ to refer to the groupoid with arrows G and objects M, the five maps are implicit. Sometimes, we just write G to denote the groupoid, by an abuse of notation.

Endowing the groupoid with an extra smooth structure we get the next definition.

Definition 2.1.2. A Lie groupoid G is a groupoid with a smooth structure on M and G, such that the five map are smooth, and the source and target are submersions.

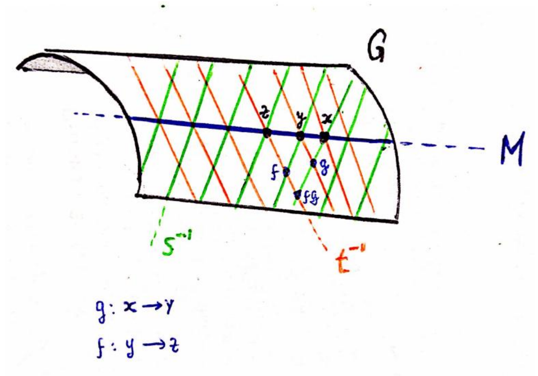

Figure 2.1: Group with the fibers of s and t.

For a Lie groupoid the subset $G \times_M G \to G \times G$ is the pullback in the diagram

$$\begin{tikzcd} G \times_M G \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & G \arrow[d, "t"] \\ G \arrow[r, "s"'] & M \end{tikzcd}$$

since s is submersion, then $G \times_M G$ is a submanifold of $G \times G$.

Definition 2.1.3. Let $G \rightrightarrows M$ and $H \rightrightarrows N$ be Lie groupoids. A morphism from G to H consist of a smooth map $F : G \to H$ and a smooth map $f : M \to N$, which are compatible with the groupoid structure.

A bisection of the Lie groupoid $G \rightrightarrows M$ is an embedded submanifold $B \subset G$ such that the restrictions $s_B : s|_B$ and $t_B := t|_B$ are open embeddings. For any $g \in G$ there exists a bisection B such that $g \in B$ ([dH13]).

Next we define the orbits and isotropies of a Lie groupoid. Let $G \rightrightarrows M$ be a Lie groupoid. Given x in M, its orbit is the set of all points where you can go from x using arrows in G, i.e. $O_x := t(s^{-1}(x)) \subset M$, and its isotropy is the set of arrows which start and end at x, i.e. $G_x := \{g \in G \mid s(g) = x = t(g)\}$.

We use the notation $G(x, y)$ to refer to the set of arrows with source x and target y. The next theorem is about the relation between the orbits and isotropies and the smooth structure.

Proposition 2.1.4. *Let G be a Lie groupoid, and let $x, y \in M$.*

1. *$G(x,y)$ is a closed submanifold of G_1*
2. *G_x is a Lie Group*
3. *The orbit of G passing through x, O_x, is an immersed submanifold of M.*
4. *$t|_{s^{-1}(x)} : s^{-1}(x) \to O_x$ is a principal G_x-bundle.*

Proof. We will show that the map $t_x := t|s^{-1}(x)$ has constant rank. Given $g, h \in s^{-1}(x)$, consider a bisection B such that $hg^{-1} \in B$, then we have the diagram

$$\begin{tikzcd} s^{-1}(x) \cap W_g \arrow[r, "L_B"] \arrow[d, "t_x"'] & s^{-1}(x) \cap W_h \arrow[d, "t_x"] \\ t_x(W_g) \arrow[r, "t_B \circ s_B^{-1}"'] & t_x(W_h) \end{tikzcd}$$

where L_B is left multiplication by the bisection, and where W_g and W_h are a neighborhoods of g, h, respectively in G such that the diagram commutes, i.e. $t_x L_B = (t_B \circ s_B^{-1}) t_x$. Since L_B and $(t_B \circ s_B^{-1})$ are diffeomorphisms then the rank of t_x in g is the same in h, therefore t_x has constant rank, thus $G(x,y)$ is a embedded submanifold of G. G_x is a embedded submanifold, and the group structure is inherited of the multiplication map, hence G_x is a Lie group. The action defined as

$$s^{-1}(x) \times G_x \to s^{-1}(x) \quad (x \xrightarrow{g} y, x \xrightarrow{h} x) \to (x \xrightarrow{gh} y)$$

is free and proper, then $O_x = s^{-1}(x)/G_x$ is an immersed submanifold of M. □

2.2 Examples

Example 2.2.1. Consider M a manifold. For any $x \in M$ consider the arrow $u_x : x \to x$, and set $G_1 := \bigcup_{x \in M} u_x = M$. This is the *unit groupoid* associated to M.

Given $x \in M$, its orbit is $O_x = \{x\}$ and its isotropy is $G_x = \{u_x\}$.

Example 2.2.2. Let M be a manifold. The Lie groupoid $Pair(M)$ is defined as $M \times M \rightrightarrows M$, where the source map is the projection in the first coordinate, the target map is the projection in the second coordinate, the multiplication map is given by $m((x,y),(y,z)) = (x,z)$, and the unit map is the diagonal $M \to M \times M$.

For any $x \in M$ its orbit is M and its isotropy is $\{(x,x)\}$.

Example 2.2.3. (Group bundle) Let G be a Lie group. Consider $G_1 = G$ and M a point. Then $G \rightrightarrows \{*\}$ is a Lie groupoid. A Lie groupoid where $s = t = \pi$ is, by definition, a Lie group bundle, any fiber is a Lie group (not necessarily the same in each fiber).

Given a Lie group bundle $G \rightrightarrows M$, for $x \in M$ its orbit is $O_x = \{x\}$ and its isotropy is the whole s-fiber $s^{-1}(x)$.

Example 2.2.4. Let G be a Lie group acting smoothly on a manifold M. The action groupoid is $G \ltimes M$, where $(G \ltimes M)_1 = G \times M$. The source is the projection on M and the target is given by the action map. The multiplication is defined by

$$m((h,y),(g,x)) = (hg,x)$$

Note that $s(h,y) = y$ and $t(g,x) = g * x$, then $y = g * x$, also $t(hg,x) = hg * x = h * y = t(h,y)$.

Given $x \in M$, its orbit and its isotropy are in correspondence with the orbit and the isotropy of x with respect to the action of G on M.

Example 2.2.5. (Gauge groupoid) Given $p : P \to M$ a principal G-bundle, the gauge groupoid over M has as manifold of arrows the orbit space of $P \times P$ by the diagonal action, i.e $P \times P/G \rightrightarrows M$.

For x, y, y', z with $p(y) = p(y') = a$, the five maps are defined as $s([x,y]) = p(x)$, $t([x,y]) = p(y)$, $m([x,y],[y',z]) = [x,z]$, $u(a) = [x,x]$, $i([x,y]) = [y,x]$.

Note that the quotient map $P \times P \to P \times P/G$ induces a morphism of groupoids between the pair groupoid and the gauge groupoid.

Since p is surjective then for any $m \in M$, its orbits is M. Its isotropy is $\{[x,gx] \mid p(x) = m \text{ and } g \in G\}$ because the action acts transitively on the fibers, but since the action acts freely on the fibers this group is isomorphic to G.

Example 2.2.6. Consider $E \to M$ a vector bundle. The Lie groupoid $GL(E)$ over M has as arrows from $x \in M$ to $y \in M$ the linear isomorphisms from E_x to E_y.

For $x \in M$ its orbits is M, and its isotropy is the group of automorphism of E_x.

Example 2.2.7. (The fundamental groupoid) Let M be a manifold. The groupoid $\Pi(M)$ over M has as arrows from $x \in M$ to $y \in M$ the homotopy classes of paths (relative to end-points) in M from x to y. The multiplication is induced by the concatenation of paths.

A description of the smooth structure can be found in [MM03] or [dH13].

If M is connected then for $x \in M$ its orbit is M, and its isotropy is the fundamental group of M.

2.3 Actions of a Lie groupoid

Let $G \rightrightarrows M$ be a Lie groupoid and $\mu : N \to M$ a map (sometimes μ is called *momentum map*). Let $G \times_\mu N := \{(g,p) \mid s(g) = \mu(p)\}$, the pullback of $\mu : N \to M$ along the source map s.

Definition 2.3.1. An action of the groupoid $(G \rightrightarrows M)$ on the map $(\mu : N \to M)$ is a map $\hat{F} : G \times_\mu N \to N$, $\hat{F}(g,p) = g.p$, such that

- $\mu(g.p) = t(g)$ for any $(g,p) \in G \times_\mu N$.
- $g.(h.p) = (g.h).p$ for any $(h,p) \in G \times_\mu N$ and $s(g) = t(h)$.
- $u_{\mu(p)}.p = p$ for any $p \in N$.

The action is denoted $(G \rightrightarrows M) \curvearrowright^{\hat{F}} (\mu : N \to M)$

We discuss now another approach to this definition, maybe more "natural", but that involves infinite dimensional manifolds. We sketch it here, to provide some intuition, but we avoid technical aspects.

Consider the set

$$Diff\ (\mu : N \to M) := \bigcup_{x,y \in M} \{\alpha : \mu^{-1}(x) \to \mu^{-1}(y) \mid \alpha \text{ is a diffeomorphism}\}$$

We can endow $Diff(\mu : N \to M)$ with a groupoid structure on M:

- $s_\mu(\alpha : \mu^{-1}(x) \to \mu^{-1}(y)) = x$
- $t_\mu(\alpha : \mu^{-1}(x) \to \mu^{-1}(y)) = y$
- $m_\mu(\alpha : \mu^{-1}(x) \to \mu^{-1}(y), \beta : \mu^{-1}(y) \to \mu^{-1}(z)) = \beta\alpha : \mu^{-1}(x) \to \mu^{-1}(z)$
- $u_\mu(x) = Id_{\mu^{-1}(x)}$
- $i_\mu(\alpha : \mu^{-1}(x) \to \mu^{-1}(y)) = \alpha^{-1}$

Now, we could define an action of the groupoid $(G \rightrightarrows M) \curvearrowright (\mu : N \to M)$ as a morphism of groupoids,

$$\begin{aligned} \Psi : (G \rightrightarrows M) &\longrightarrow Diff\ (\mu : N \to M) \\ (g : x \to y) &\longrightarrow \alpha_g : \mu^{-1}(x) \to \mu^{-1}(y) \end{aligned}$$

corresponding with a map $\Phi : G \times_\mu N \to N$, $\Phi(g,p) = g.p$, by the exponential rule. Note that

- Since $\alpha_g : \mu^{-1}(s_\mu(g)) \to \mu^{-1}(t_\mu(g))$ we conclude $\mu(g.p) = \mu(\alpha_g(p)) = t_\mu(g)$.
- $g.(h.p) = \alpha_g(\alpha_h(p)) = (\alpha_g\alpha_h)(p) = (g.h).p$.
- $u_\mu(\mu(p))(p) = Id_{\mu^{-1}(\mu(p))}(p) = p$.

The smoothness of the map Φ should correspond with that of the map Ψ. The problem with this approach is that the differential structure on the groupoid $Diff\ (\mu : N \to M)$ should be of infinite dimensions.

Chapter 3

Lie Algebroids

In this chapter we focus on Lie algebroids, their definitions and some examples. Finally, we present the definition of an action of a Lie algebroid. The topics in this section are based on the references [MM03], [CF06], [Mac05].

3.1 Definitions

A Lie algebroid over a manifold M is a vector bundle $A \Rightarrow M$ with a bundle map, the anchor, $\rho : A \to TM$ and a Lie bracket $[,]_A$ on the sections of A satisfying the Leibniz indentity

$$[X, fY]_A = f[X,Y] + \hat{\rho}X(f)Y$$

where $\hat{\rho} : \Gamma(A) \to \mathfrak{X}(M)$ is the induced map by ρ.

Definition 3.1.1. Let $A_1 \Rightarrow M_1$ and $A_2 \Rightarrow M_2$ be two Lie algebroids. A morphism from A_1 to A_2 consists of a bundle map $F : A_1 \to A_2$ and a smooth map $f : M_1 \to M_2$ such that (i) the diagram

$$\begin{tikzcd} A_1 \arrow[r, "F"] \arrow[d, "\rho_1"'] & A_2 \arrow[d, "\rho_2"] \\ TM_1 \arrow[r, "Df"'] & TM_2 \end{tikzcd}$$

commutes, and (ii) the Lie bracket is preserved. To make (ii) precise, given α and β in $\Gamma(A_1)$, we can write $F(\alpha) = \sum_i a_i f^*(\alpha_i)$ and $F(\beta) = \sum_j b_j f^*(\beta_j)$ with a_i, b_j in $C^\infty(M_1)$ and α_i, β_j in $\Gamma(A_2)$ then the condition means

$$F([\alpha,\beta]_{\mathcal{G}_1}) = \sum_{i,j} a_i b_j [\alpha_i, \beta_j]_{\mathcal{G}_2} + \sum_j \hat{\rho}_1(b_j) f^*(\beta_j) - \sum_i \hat{\rho}_2 a_i f^*(\alpha_i).$$

We define isotropies and orbits for Lie algebroids similarly to what is done for Lie groupoids. Given $A \Rightarrow M$ a Lie algebroid, the isotropy of at the point $x \in M$ is the kernel of the anchor map at x, and the tangent to the orbit is the image of the anchor map at x.

Proposition 3.1.2. *Let $A \Rightarrow M$ a Lie algebroid, and $x \in M$.*

1. *The isotropy at x is a Lie algebra.*
2. *The image of the anchor map defines a singular foliation.*

The proof of item 1 is a consequence of the proposition 4.2.1 and the proof of item 2 can be found in [DZ00].

3.2 Examples

Example 3.2.1. Let $\mathfrak{g}$ be a Lie algebra, consider the Lie algebroid $\mathfrak{g} \Rightarrow \{*\}$, where the anchor is the zero map, and the bracket is the bracket of $\mathfrak{g}$. More generally, let $E \to M$ be a vector bundle with a smoothly varying Lie algebra structure on the fibers. Then it is a Lie algebroid with anchor zero. Such an example is called a Lie algebra bundle. They integrate to Lie group bundles.

Example 3.2.2. Let M be a manifold. The tangent space is an algebroid $TM \Rightarrow M$ with anchor map equal to the identity and Lie bracket the same of $\mathfrak{X}(M)$.

As generalization, we consider a regular foliation $\mathcal{F}$ of the manifold M. Then $T\mathcal{F}$ is an integrable sub-bundle of TM, i.e. it is closed by the Lie bracket. Then $T\mathcal{F}$ is a Lie algebroid over M with anchor map given by the inclusion in TM and Lie bracket the restriction of the Lie bracket from TM.

A Lie algebroid with regular anchor has kernel a Lie algebra bundle and image a regular foliation. Thus, in that case, we can think of a Lie algebroid as a combination of these two examples.

3.3 Action of a Lie algebroid

The definition is a generalization of that of an action of an algebra on a manifold, the morphism between algebras. Let $(A \Rightarrow M, [,], \rho)$ be a Lie algebroid. We say that $A \Rightarrow M$ acts on the moment map $\mu : N \to M$ if there is a Lie algebra morphism $F : \Gamma(A) \to \mathfrak{X}(N)$ from the Lie algebra of sections of A to the Lie algebra of vector fields on N, which satisfies the following conditions

- For $f \in C^\infty(M)$ and α a section of A, $F(f\alpha) = (\mu^* f)F(\alpha)$.

- For all $y \in N$, $\rho\alpha(\mu(y)) = \mu_*(F(\alpha)(y))$. This is equivalent to the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[d, Rightarrow] & TM & TN \arrow[l, "\mu_*"'] \arrow[d] \\
M \arrow[u, bend left, "\alpha"] \arrow[ur, "\rho\alpha"] & & N \arrow[ll, "\mu"] \arrow[u, bend right, "F(\alpha)"']
\end{tikzcd}$$

This definition can be found for instance in [DZ00]. It may seem not very intuitive a priori. Next we develop an heuristic discussion on it, in parallel to what we did for action of Lie groupoids, trying to rephrase it as an algebroid morphism.

Previously we consider a groupoid $G = Diff(\mu : N \to M)$ of symmetries of the moment map. Even though we have not put a differentiable structure on it (it should be infinite dimensional), we will try to understand what is its infinitesimal counterpart, its associated Lie algebroid. In G, we have

$$s_\mu^{-1}(p) = \{\mu^{-1}(p) \to \mu^{-1}(q) \text{ diffeomorphism} \mid q \in M\}$$

We want to define $(A_\mu)_p$ in a way that represents $T_{u(p)} s_\mu^{-1}(p)$. A vector on this vector space should be a sections of $TN|_{\mu^{-1}(p)}$ that is projectable, i.e. $D\mu_p X_p$ is constant. Then we can take $A_G = \coprod_p (A_\mu)_p \to M$ and define the projection $\mu_* : A_G \to TM$ as an anchor map. This set of data should be thought of as an infinite dimensional Lie algebroid. Then, an action $(A \Rightarrow M, \rho, [,]_A) \curvearrowright (\mu : N \to M)$ would be the same as a morphism of Lie algebroids $F : A \to A_\mu$ (over the identity of M), between $A \Rightarrow M$ and $A_\mu \Rightarrow M$.

$$\begin{tikzcd}
A_1 \arrow[rr, "F"] \arrow[dr, "\rho"'] & & A_G \arrow[dl, "\mu_*"] \\
& TM &
\end{tikzcd}$$

Chapter 4

Lie theory

In this chapter we study the main topic of the work. We start connecting the Lie groupoids with the Lie algebroids through the derivative functor. In the Section 4.3 we introduce the main theorem which relates the morphisms of Lie groupoids that induce isomorphisms at the infinitesimal level with the subgropoids under some conditions. Finally, we conclude with some application of the theorem, the example of Lie group and the pair groupoid. For some reference about topological and differential results we suggest [Hat02], [Lim06], [Bou67], [Lee13], [dH13].

4.1 The Lie algebroid of a Lie groupoid

Similarly to the case of Lie groups and their associated Lie algebras, given by the right invariant vector fields, we can associate a Lie algebroid to a Lie groupoid via a generalization of right invariant vector fields.

Definition 4.1.1. We say that a vector field X on G is a right invariant vector field if

- It is tangent to the fibers of s, i.e. for any $g : x \to y$ in G, $X_g \in T_g(s^{-1}(x)) = \ker(ds_g)$.
- For any $h, g \in G$ such that $s(g) = t(h)$, $X_{gh} = dR_h(X_g)$, where R_h is the right multiplication by the arrow h.

We will denote these vector fields as $\mathfrak{X}^s_{inv}(G)$.

The right invariant vector fields are involute, given X, Y right invariant vector fields, we have

$$dR_h([X,Y]_g) = [dR_h(X), dR_h(Y)]_{gh} = [X,Y]_{gh},$$

then the right invariant vector fields are a sub-algebra of the vector field of G (with the Lie brackets).

Note that for any $g : x \to y$ in G we have

$$dt(X_g) = dt(X_{u(y)q}) = dt(dR_g X_{u(y)}) = d(t \circ R_g)(X_{u(y)}) = dt(X_{u(y)})$$

therefore, the right invariant vector fields are projectable along the target map to M. Hence the map $dt : \mathfrak{X}^s_{inv}(G) \to \Gamma(M)$ is a Lie algebra homomorphism.

Since $X_g = dR_g(X_{u(t(g))})$ then any $X \in \mathfrak{X}^s_{inv}(G)$ is uniquely determined by its restriction to the set of units $\{u(x) \mid x \in M\}$. Hence we have an isomorphism of vector space

$$\mathfrak{X}^s_{inv}(G) \to \Gamma(A)$$

where $A = \bigsqcup_{x \in M} A_x$ and $A_x := T_{u(x)}(s^{-1}(x))$.

There is a unique Lie algebra structure on $\Gamma(A)$ such that the isomorphism of vector space is also an isomorphism of Lie algebras ([MM03]). Then $\Gamma(A)$ inherit the Lie bracket of $\Gamma(\mathfrak{X}(G))$. And the map $dt : \mathfrak{X}^s_{inv}(G) \simeq A \to \Gamma(M)$ is the anchor. We denote X^R the right invariant vector field associated to $X \in \Gamma(A)$. Then $[X, Y]^R = [X^R, Y^R]$ and for $f \in C^\infty(M)$ we have $(fX)^R = (f \circ t)X^R$.

Finally note that for $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$ we have

$$\begin{aligned}
[X, fY]^R &= [X^R, (fY)^R] \\
&= [X^R, (f \circ t)Y^R] \\
&= (f \circ t)[X^R, Y^R] + X^R(f \circ)tY^R \\
&= f([X, Y])^R + (dt(X^L)fY)^R \\
&= (f[X, Y])^R + (dt(X)fY)^R.
\end{aligned}$$

therefore the Leibniz rule holds.

We will discuss now the theorems Lie 1 and Lie 2. For a proof see [MM03]. A Lie groupoid $G \rightrightarrows M$ is said *source simply-connected* if the fibers of the source map are simply-connected. Source connected Lie groupoids are defined analogously.

The first says that for any Lie groupoid it is possible to find a source simply-connected groupoid that is somehow universal.

Theorem 4.1.2 (Lie 1)**.** *For any Lie groupoid $G \rightrightarrows M$ there exist a source-simply-conneted Lie groupoid $\hat{G} \rightrightarrows M$, possibly non-Hausdorff, and morphism of Lie groupoids $\hat{G} \to G$ which induces an isomorphism of the Lie algebroids.*

The construction of $\hat{G}$ is by taking the space of homotopy classes of paths in any s–fiber with a fixed initial point, i.e the universal cover of the s-fiber.

This space defines a Lie groupoid over G, then by taking quotient with the action of G we obtain a Lie groupoid over M.

We will present an example of a Lie groupoid such that its s–source simply connected "covering Lie groupoid" is non-Hausdorff.

Example 4.1.3. Let M be $\mathbb{R}^3 - (0,0,0)$. Consider the arrows space as

$$G = \{((u_1,u_2,u_3),(v_1,v_2,v_3)) \in M \times M \mid u_3 = v_3\}$$

For $u = (u_1,u_2,u_3)$ with $u_3 \neq 0$, $G(u,-)$ is isomorphic to $\mathbb{R}^2$, then the universal cover space of $G(u,-)$ is $\mathbb{R}^2$. However for $u = (u_1,u_2,0)$, $G(u,-)$ is not simply connected (it is homotopically equivalent to S^1), then its universal cover space is not $\mathbb{R}^2$ (it is like a spiral).

Consider the paths

$$\gamma_1^s(t) = ((\cos(t),\sin(t),s),(1,0,s)), \quad \gamma_2^s(t) = ((\cos(t),-\sin(t),s),(1,0,s))$$

if we consider the homotopy equivalence class then $[\gamma_1^s] = [\gamma_2^s]$ for $s \neq 0$, and $[\gamma_1^s] \neq [\gamma_2^s]$ for $s = 0$.

Note that $\lim_{s\to 0}[\gamma_1^s * \gamma_1^s] \neq 0$, however $[\gamma_1^s * \gamma_1^s] = 0$ for $s \neq 0$, then $\hat{G}$ is non-Hausdorff.

The second theorem establishes that any morphism of Lie algebroids can be integrated to a unique morphism of Lie groupoids.

Theorem 4.1.4 (Lie 2)**.** *Let $(G \rightrightarrows M)$ and $(H \rightrightarrows N)$ be Lie groupoids, and $(A_G \Rightarrow M)$, $(A_H \Rightarrow N)$ their Lie algebroids. Suppose that G is source simply-connected. If $\Phi' : A_G \to A_H$ is a morphism of Lie algebroids over $\phi : M \to N$, then there is a unique Lie groupoid morphism $\Phi : G \to H$ such that $Lie(\Phi) = \Phi'$.*

Later, as a consequence of our main theorem, we will give a sketch of the proof for the Lie 2 theorem. We close this section showing some examples that suggest a way to ensure the existence of a source simply connected cover Lie groupoid which turns out to be a Hausdorff topological space.

4.2 Relation betweem orbits and isotropies for Lie groupoids and Lie algebroids.

In the next propositions we will show that the derivative map form $G \rightrightarrows M$ to $A \Rightarrow M$ behaves well with respect to the orbits and isotropies.

Proposition 4.2.1. *Let $G \rightrightarrows M$ be a Lie groupoid and $A \Rightarrow M$ its Lie algebroid, then $Lie(G)_x = Lie(G_x)$ (the isotropy of the Lie algebroid at x is the same as the Lie algebra of the isotropy group at x).*

Proof. We can define both tangent spaces in term of velocities of curve as follow

$$Lie(G)_x = \ker(dt_{u_x}) \cap A_x = \ker(dt_{u_x}) \cap T_{u_x}(s^{-1}(x))$$

and

$$Lie(G_x) = T_{u_x}G_x = T_{u_x}(t^{-1}(x)) \cap T_{u_x}(s^{-1}(x))$$

since $t^{-1}(x)$ is a submanifold of G, then $\ker(dt_{u_x}) = T_{u_x}t^{-1}(x)$, therefore $Lie(G)_x = Lie(G_x)$. □

Now, we prove something similar for the orbits.

Proposition 4.2.2. *Let $G \rightrightarrows M$ be a Lie groupoid and $A \Rightarrow M$ its Lie algebroid. Then $T_xO_x = dt_{u_x}(A_x)$.*

Proof. By definion,

$$dt_{u_x}(A_x) = dt_{u_x}(T_{u_x}(s^{-1}(x)))$$

on the other hand

$$T_xO_x = T_x(t(s^{-1}(x)))$$

by the chain rule we can conclude the proof. □

Motivated by the notion of transitive group action, we say that $G \rightrightarrows M$ is transitive if for $x \in M$ the orbit at x is M, and $A \Rightarrow M$ is transitive if $dt_{u_x}|_{A_x}$ is surjective for all x.

Proposition 4.2.3. *Let $G \rightrightarrows M$ be a Lie groupoid and $A \Rightarrow M$ its Lie algebroid, M connected, then $G \rightrightarrows M$ is transitive if and only if $A \Rightarrow M$ is transitive.*

Proof. Since the map $t|_{s^{-1}(x)} : s^{-1}(x) \to O_x$ is submersion, if $O_x = M$, then $dt_{u_x}(A_x) = T_xM$.

Conversely if $dt_{u_x}|_{A_x}$ is surjective, then by local form of submersion theorem $t|_{s^{-1}(x)}$ is a open map, then O_x is open in M, also the complement of O_x is open (is the union of the orbits disjoints to O_x), since M is connected, then $O_x = M$. □

4.3 The main problem and the main theorem

The next definition is the natural extension of the definition of normal group, and the reason for introduce it is the same, to construct quotient groupoids.

Definition 4.3.1. Normal subgroupoid. Let $G \rightrightarrows M$ be a groupoid. A subgroupoid $K < G$ is normal if it is a group bundle ($s = t$) and for any $h \in K$, the arrow $g^{-1}hg$ is in K for any $g \in G$.

For the groupoid $G \rightrightarrows M$ and the normal subgroupoid K, we can define a new groupoid $G/K \rightrightarrows M$, where the groupoid maps are induced by the quotient map (then it naturally is a morphism of groupoids), i.e., if $\pi : G \to G/K$ is the map such that $\pi(g) = [g]$ (where the equivalence relation is $f \in [g]$ iff $f = hg$ for some $h \in K$), then

1. $s_\pi[g] = s(g)$, $t_\pi[g] = t(g)$.
2. $m_\pi([g],[f]) = [g][f] = m(g,f)$.
3. $i[g] = [i(g)]$.
4. The unit map in G/K is $[u_x]$.

With this notion of quotient groupoid, we want to characterize the morphisms between Lie groupoids which induce an isomorphism between Lie algebroids. Given a Lie groupoid $G \rightrightarrows M$, the aim is to show that there is a one to one correspondence between the morphims of groupoids $\varphi : G \to H$ with φ' isomorphism in the Lie algebroid level, and the subgroupoids of G with special properties. The next proposition tell us about these "special " properties.

Proposition 4.3.2. *Let $G \rightrightarrows M$ and $H \rightrightarrows M$ be Lie groupoids (Hausdorff), G source-fiber connected and $\varphi : G \to H$ a morphism of Lie groupoids such that it yields an isomorphism between the Lie algebroids of G and H. Then the subgroupoid $K := \ker(\varphi)$ satisfies*

- *K is a closed embedded submanifold of G.*
- *K is a normal subgroupoid discrete in $s-$fibers.*

Proof. • We are using the notation s_G and s_H for the source map of G and H, respectively. Given $g : x \to y \in G$, we have that the next diagram

$$\begin{tikzcd} s_G^{-1}(x) \subset G \arrow[r, "\varphi"] \arrow[d, "s_G"'] & s_H^{-1}(x) \subset H \arrow[d, "s_H"] \\ M \arrow[r, "id_M"'] & M \end{tikzcd}$$

induces a short exact sequence

$$\begin{tikzcd} 0 \arrow[r, "1_G"] & T_g s_G^{-1}(x) \arrow[r, "2_G"] \arrow[d, "(D\varphi|_{s_G^{-1}(x)})_g"'] & T_g G \arrow[r, "3_G"] \arrow[d, "(D\varphi)_g"'] & T_x M \arrow[r, "4_G"] \arrow[d, "id"'] & 0 \\ 0 \arrow[r, "1_H"'] & T_{\varphi(g)} s_H^{-1}(x) \arrow[r, "2_H"'] & T_{\varphi(g)} H \arrow[r, "3_H"'] & T_x M \arrow[r, "4_H"'] & 0 \end{tikzcd}$$

note that 2_G and 2_H are injective because the source maps are submersions (then the fibers $s_G^{-1}(x)$ and $s_H^{-1}(x)$ are submanifolds of G and H, respectively). Also, $T_g s_G^{-1} = \ker(ds_G)_g$ and $T_{\varphi(g)} s_H^{-1} = \ker(ds_H)_{\varphi(g)}$, therefore $Im(2_G) = \ker(3_G)$ and $Im(2_H) = \ker(3_H)$. Finally, since s_G and s_H are submersions then the maps 3_G and 3_H are surjective. Hence the horizontal maps form short exact sequences.

Since, $(D\varphi|_{s_G^{-1}(x)})_g$ is an isomorphism (by hypothesis) by five lemma we conclude that $D\varphi_g$ is an isomorphism. Therefore by inverse function theorem there exist neighborhoods of g and $\varphi(g)$, $U \subset G$, $W \subset H$, respectively, such that the map $\varphi|U : U \to W$ is a diffeomorphism.

Consider the open $V := s_G(U)$, then by shirking the opens U, V, W we can assume that the next diagram commutes

$$\begin{tikzcd} K \cap U \subset G \arrow[dr, "s_G"'] & & W \subset H \arrow[ll, "\varphi^{-1}"'] \\ & V \subset M \arrow[ur, "u_H"'] & \end{tikzcd}$$

Thus, $\ker \varphi \cap U$ is the image of the smooth section $\varphi^{-1} \circ u_H|V$.

For closed condition, consider a sequence $g_n : x_n \to y_n$ in K which converges to $g : x \to y$ in G. Consider U a neighbourhood of g as above, for any n we have

$$\varphi(g_n) = (u_H) \circ (s_G(g_n)) = (u_H)_{x_n}$$

since φ, u_H and S_G are continuous then by taking the limit we have $\varphi(g) = (u_H)_x$, then $g = \varphi^{-1}(u_H)_x$, hence $g \in K$, since G is **Hausdorff** we conclude that K is closed.

- Initially note that K is a group bundle; for $g : x \to y \in K$, $s_H(\varphi(g)) = s_H(u_H)_x = t_H(\varphi(g))$ and because φ preserves the source and target maps we have $s_G(g) = s_H(\varphi(g)) = t_G(g)$.

  For $f : x \to y$ in G, and $h : y \to y$ in K, note that $\varphi(f^{-1}hf) = \varphi(f^{-1})\varphi(h)\varphi(f) = \varphi(f^{-1})(u_H)_y\varphi(f) = (u_H)_x$. Therefore K is a normal subgroupoid.

  K is discrete in fibers: Given $x \in M$, the sets G_x and H_x are Lie groups, we denote $K_x := G_x \cap K$. Consider the action of G_x on $\varphi(s_G^{-1}(x))$

$$\begin{aligned} \alpha : G_x \times \varphi(s_G^{-1}(x)) &\longrightarrow \varphi(s_G^{-1}(x)) \\ (g : x \to x, \varphi(f) : x \to y) &\longrightarrow \alpha(g, \varphi(f)) = m(\varphi(f), \varphi(g)) \end{aligned}$$

  note that K_x is the isotropy subgroup of the group G_x for the value $(u_H)_x$, i.e.

$$K_x = (G_x)_{(u_H)_x} = \{g : x \to x \mid \varphi(g) = m((u_H)_x, \varphi(g)) = (u_H)_x\}$$

  by lemma 2.1.4 K_x is an embedded submanifold of G_x, and by lemma 4.4.2, if $\dim(K_x) > 0$ then there exists a non-zero vector $v \in T_{(u_H)_x}(G_x)_{(u_H)_x}$, and a curve $\gamma : \mathbb{R} \to G_x$ such that $\gamma(0) = (u_G)_x$, $\gamma'(0) = v$ and $\phi(\gamma(t)) = (u_H)_x$ for any $t \in \mathbb{R}$. Therefore

$$0 = \frac{d}{dt}_{t=0} \varphi(\gamma(t)) = (D\varphi|_{s_G^{-1}(x)})_{(u_G)_x}(v)$$

  since $D\varphi$ restricted to fiber is an isomorphism, then $v = 0$, hence $\dim(K_x) = 0$. Thus, K_x is discrete.

$\square$

It is clear that the morphism of groupoids φ restricts to morphism of groups

$$\psi := \varphi|_{G_x} : G_x \to H_x.$$

note that $K_x = \ker(\psi)$, then K_x is normal subgroup of G_x , in fact $H_x \cong G_x/K_x$, since G is source fiber connected, by lemma 4.4.3, K_x is in the center of G_x.

The next proposition is the converse of the previous. [HM90]

Proposition 4.3.3. *Let $G \rightrightarrows M$ be a Lie groupoid, and K a subgroupoid of G such that*

- *K is a closed embedded submanifold of G.*
- *K is a normal subgroupoid discrete in fibers.*

then G/K is a Lie groupoid, such that the canonical map $G \to G/K$ is a morphism of groupoids which induces a isomorphism in its Lie algebroids.

Proof. The first condition is necessary to ensure the differentiable structure in the quotient G/K. We want to write G/K as G/R where $R \subset G \times G$ is a equivalence relation, after we will show that R is a closed embedded submanifold for use the *Godement criterion* [dH13].

Consider the pullback

$$\begin{tikzcd} K \times_M G \arrow[r] \arrow[d] & G \arrow[d, "t"] \\ K \arrow[r, "s|_K"'] & M \end{tikzcd}$$

since s is a submersion then the map $K \times_M G \to K \times G$ is a closed embedding. Also the map

$$\begin{aligned} \alpha : K \times_M G &\longrightarrow \alpha(K \times_M G) \\ (k : y \to y, g : x \to y) &\longrightarrow (m(k, g), g) \end{aligned}$$

is a diffeomorphism, and we define the equivalence relation $R := \alpha(K \times_M G)$. Then we have the next diagram

$$R \cong K \times_M G \to K \times G \to G \times G$$

where the last map (inclusion) is a closed embedding, because K is a closed embedded submanifold. In conclusion $R \subset G \times G$ is a closed embedded submanifold. The map $\pi_2|R : R \to M$, is defined as $\pi_2(k, g) = g$, then is a submersion; therefore, by Godement criterion $G/R = G/K$ inherits a unique manifold structure that makes the projecting $G \to G/R$ a submersion, see [dH13]. The structure of groupoid follows of K is normal subgroupoid. The maps in the quotient groupoid G/K are submersion because π is submersion (and the morphism of groupoids commutes with the maps of G which are submersions).

Also, by the before diagram we have that $\dim(G) + \dim(K) = \dim(M) + \dim(K \times_M G)$, and since $\dim(K) = \dim(M)$, then $\dim(G) = \dim(K \times_M G)$.

Now by consider the diagram

$$\begin{tikzcd} K \times_M G \arrow[r] \arrow[d] & G \arrow[d, "\pi_1"] \\ G \arrow[r, "\pi_2"'] & G/K \end{tikzcd}$$

which is a good pullback in the sense of [dH13]. We have $\dim(G/K) = 2\dim(G) - \dim(K \times_M G) = \dim(G)$. And since π_1 is a submersion we conclude that for any $x \in M$, $T_{u(x)}G \simeq T_{[u(x)]}G/K$. As before, we have that the next horizontal sequence are short exact

$$\begin{tikzcd} 0 \arrow[r] & (A_G)|_x \arrow[r] \arrow[d, "(D\pi|_{s_G^{-1}(x)})_{u(x)}"'] & T_{u(x)}G \arrow[r] \arrow[d, "(D\pi)_{u(x)}"'] & T_xM \arrow[r] \arrow[d, "id"'] & 0 \\ 0 \arrow[r] & (A_{G/K})|_x \arrow[r] & T_{[u(x)]}G/K \arrow[r] & T_xM \arrow[r] & 0 \end{tikzcd}$$

and by using the five lemma, because $(D\pi)_{u(x)}$ is an isomorphism, we conclude that $(A_G)_x \cong (A_{G/K})_x$ for any $x \in M$, therefore the Lie algebroids are isomorphic.

□

Analogously to the isomorphism theorem for groups, it is possible to show that the image of a morphism of Lie groupoids is isomorphic to the coimage (when the coimage has a Lie groupoid structure).

Corollary 4.3.4. *Let $(G \rightrightarrows M) \xrightarrow{\varphi} (H \rightrightarrows M)$ be a morphism of Lie groupoids with φ' and isomorphism of Lie algebroids. Let $K := \ker\varphi$. Then the groupoid G/K and H are isomorphic.*

Proof. We define

$$\hat{\varphi} : (G/K \rightrightarrows M) \to (H \rightrightarrows M), \quad \hat{\varphi}[g] = \varphi(g)$$

it is well define because for $h \in K$ and $t(h) = s(g)$, we have $\hat{\varphi}[gh] = \varphi(gh) = \varphi(g)\varphi(h) = \varphi(g)u_{s(g)} = \varphi(g)$.

By construction the map $\hat{\varphi}$ is injective, and since ϕ and π are a local diffeormophisms then $\hat{\varphi}$ is a diffeomorphism. It is clear that the groupoid maps commute with $\hat{\varphi}$. □

Theorem 4.3.5. *Given a Lie groupoid $G \rightrightarrows M$, the morphisms between Lie groupoids from G with base M , which induces an isomorphism at the level of the Lie algebroids are in correspondence with the normal subgroupoid of G, such that are discrete in the $s-$fibers and are closed embedded submanifolds of G.*

Proof. Given a morphism of Lie groupoids $(G \rightrightarrows M) \xrightarrow{\varphi} (H \rightrightarrows M)$ such that φ' is an isomorphism of Lie algebroids, by proposition 4.3.2 the kernel $K := \ker(\varphi)$ is a closed, embedded submanifold of G, also is a normal subgroupoid discrete in the $s-$ fibers.

Conversely, if K is a closed embedded submanifold of G, such that $K \rightrightarrows M$ is a normal subgroupoid of $(G \rightrightarrows M)$, such that is discrete in the $s-$fibers, then by proposition 4.3.3, G/K inherits a groupoid Lie structure, such that the projection $G \xrightarrow{\varphi} G/K$ is a morphism of Lie groupoids with φ' isomorphism.

If $(G \rightrightarrows M) \xrightarrow{\varphi} (H \rightrightarrows M)$ is a morphism of groupoids then by the corollary 4.3.4, $G/\ker\varphi \simeq H$, thus the correspondence is well defined

□

Corollary 4.3.6. *Let $G \rightrightarrows M$ be a Lie groupoid that is source connected. Then G is isomorphic to $\hat{G}/H$, where $\hat{G}$ is the source-simply connected groupoid over M from Lie 1. Also H is a group bundle, discrete in source fibers, closed, and embedded submanifold.*

Proof. It follows from the previous theorem and from the Lie 1 theorem for groupoids 4.1.2, however it is possible that $\hat{G}$ is non-Hausdorff. □

4.4 Example: Case of a Lie group

Consider an action of a Lie group G on a manifold M, $\phi : G \times M \to M$. Given $x \in M$, the isotropy subgroup of x, is denoted G_x, i.e $G_x := \{g \in G \mid \phi(g, x) = x\}$. The next lemmas will be useful later.

The following lemma is about the differentiable structure of G_x respect to the differentiable structure of G.

Lemma 4.4.1. *The subgroup G_x is an embedded submanifold of G.*

Proof. It is a consequence of theorem 2.1.4

□

Now, if the dimension of G_x is positive then there is a curve contained in the isotropy subgroup, this is,

Lemma 4.4.2. *If $\dim(G_x) > 0$, there exist $\gamma : \mathbb{R} \to G$, such that for ant $t \in \mathbb{R}$, $\gamma(t) \in G_x$, i.e. $\phi(\gamma(t), x) = \gamma(t).x = x$.*

Proof. If $\dim(G_x) > 0$ there exist a non-zero vector $v \in T_eG_x$ (where e is the identity of G). Consider the vector field $X \in \mathfrak{X}(M)$, such that

$$X(y) = D\phi_{(e,y)}(v, 0)$$

note that $D\phi_(e, y) : T_eG \times T_yM \to T_yM$ is the natural projection, then $X(y) = D\phi_{(e,y)}(v, 0) = 0$. In other hand, if we consider the curve $\gamma : \mathbb{R} \to G$, defined as $\gamma(t) = exp(tv)$, satisfies $\gamma(0) = e$ and $\gamma'(0) = v$, therefore

$$0 = X(x) = D\phi_{(e,x)}(v, 0) = \frac{d}{dt}_{t=0}(\phi(\gamma(t), x))$$

therefore $\phi(\gamma(t), x)$ is constant, then $\phi(\gamma(t), x) = \phi(e, x) = x$ for $t \in \mathbb{R}$. □

The next lemma is a fact about the normal and discrete subgroup of the a connected Lie group.

Lemma 4.4.3. *Let G be a connected Lie group. If H is a discrete, normal subgroup of G, then it lies in the center of G.*

Proof. For $h \in H$, because H is normal then $ghg^{-1} \in H$. The map $G \to H$ such that g is send to ghg^{-1} is continuous and since G is connected then for any g, the elements ghg^{-1} and $h = ehe^{-1}$ are in the same component, by using the fact that H is discrete we conclude that $ghg^{-1} = h$. □

Proposition 4.4.4. *Let G be a Lie group, and $\mathcal{G}$ its Lie algebra. If $\mathcal{H}$ is a subalgebra of $\mathcal{G}$, then there exists a connected Lie subgroup of G whose Lie algebra is $\mathcal{H}$*

Proof. For any $x \in G$ set

$$D_x = \{X_x \mid X \in \mathcal{H}\} \subset T_xG$$

and consider $D = \bigcup_{x \in G} D_x \subset TG$. Since any X in $\mathcal{H}$ is left-invariant vector field, for $x, y \in G$ the map $d(L_{yx^{-1}})$ is an isomorphism from D_x to D_y therefore for any x we assume that D_x has dimention k.

Any basis $(X_1, ..., X_k)$ for $\mathcal{H}$ is a global smooth frame for D; then for $X, Y \in D$, we have $[X, Y] = [f^iX_i, g^jX_j] = f^ig^j[X_i, X_j] + f^i(X_ig^j)X_j - g^j(X_jf^i)X_i$ which is in $\mathcal{H}$ because $[X_i, X_j] \in \mathcal{H}$, thus D is a involutive distribution on G.

Let $\mathcal{F}$ be the foliation determined by D. For $x \in G$, let $\mathcal{F}_x$ be the leaf of $\mathcal{F}$ which contains x. Since D is left- invariant we have that for $x, y \in G$

$$L_x(\mathcal{F}_y) = \mathcal{F}_{xy}$$

Let $H = \mathcal{F}_e$ be the leaf containing the identity e. We claim that H is a Lie group. For the group structure note that for $x, y \in H$

$$xy = L_x(y) \in L_x(H) = L_x(\mathcal{F}_e) = \mathcal{F}_x = H$$

and

$$x^{-1} = x^{-1}e \in L_{-x}(\mathcal{F}_e) = L_{-x}(\mathcal{F}_x) = \mathcal{F}_e = H$$

Being that $H \times H$ is a submanifold of $G \times G$, the multiplication map restricted to $m : H \times H \to G$ is smooth. Since H is a initial submanifold (because is a foliation), then $m : H \times H \to H$ is smooth. Analogously for the inversion map.

Finally note that $T_eH = D_e = \{X_e \mid X \in \mathcal{H}\}$, then $Lie(H) = \mathcal{H}$.

□

We will prove the second Lie theorem for Lie group by using the theorem 4.3.5.

Proposition 4.4.5. *(Lie 2 for Lie groups) Let G and H be Lie groups and G simply connected, and $\mathcal{G}$ and $\mathcal{H}$ its Lie algebras, respectively. If $\varphi : \mathcal{G} \to \mathcal{H}$ is a homomorphism between algebras, then there exists a homomorphism Φ of G into H with $d\Phi_e = \varphi$.*

Proof. Given $\mathcal{J} = \{(X, \varphi(X)) \in \mathcal{G} \oplus \mathcal{H} | X \in \mathcal{G}\}$, which is sub-algebra, because φ is a homormorphism between algebras, thus $(X, \phi(X)) + (Y, \phi(Y)) = (X + Y, \phi(X) + \phi(Y)) = (X + Y, \phi(X + Y))$ and $[(X, \phi(X)), (Y, \phi(Y))] = ([X, Y], \phi([X, Y]))$ for $X, Y \in \mathcal{G}$.

By proposition (4.4.4) consider J a connected subgroup associated to $\mathcal{J}$. Now consider

$$\begin{tikzcd}
 & J \subset G \times H \arrow[dl, "p_G"'] \arrow[dr, "p_H"] & \\
G \arrow[rr, dashed, "\Phi"] & & H
\end{tikzcd}$$

where p_G and p_H are the restriction to J of the projection of $G \times H$ to G and H. Note that the differential of p_G is the map $Dp_G : \mathcal{J} \to \mathcal{G}$ such that $Dp_G(X, \varphi(X)) = X$, then the Lie algebras $\mathcal{G}$ and $\mathcal{J}$ are isomorphic. By theorem 4.3.5 we know that the kernel K of the morphism p_G is a discrete normal subgroup, then we have that $p_G : J \to G$ is a covering map, since G is simply connected there exists a inverse map for p_G, therefore p_G is an isomorphism, so we define the morphism

$$\Phi : G \xrightarrow{p_H \circ p_G^{-1}} H$$

□

Corollary 4.4.6. *Let G and H be simply connected Lie groups. If its Lie algebras are isomorphic, then G and H are isomorphic.*

Proposition 4.4.7. *Let G be connected. There exists a simply conneted Lie group $\hat{G}$, the universal covering group of G, that admits a smooth covering map $\pi : \hat{G} \to G$ that is also a Lie group homomorphism.*

Proof. Consider the universal cover of G, $\pi : \hat{G} \to G$. Is easy to see that $\pi \times \pi : \hat{G} \times \hat{G} \to G \times G$ is a universal cover. Due to $\hat{G}$ is simply connected, there exist a unique continuos map lift $\hat{m}$ such that the next diagram commutes,

$$\begin{tikzcd}
 & & \hat{G} \arrow[d, "\pi"] \\
\hat{G} \times \hat{G} \arrow[r, "\pi \times \pi"'] \arrow[urr, dashed, "\hat{m}"] & G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

□

Proposition 4.4.8. *Let G be a Lie group with Lie algebra $\mathcal{G}$. If a connected Lie group G_1 has associated an Lie algebra isomorphic to $\mathcal{G}$, then G_1 is isomorphic to $\hat{G}/D$ where $\hat{G}$ is the universal covering group of G and D is a discrete normal subgroup of G.*

Proof. By proposition 4.4.7, the universal covering group of G_1, $\pi_1 \coloneqq \hat{G}_1 \to G_1$ is a homomorphism groups, also is surjective. Hence $G_1 \cong \hat{G}_1/\ker(\pi_1)$.

Note that $D_1 \coloneqq \ker(\pi_1)$ is a discrete normal subgroup because $D_1 = \pi_1^{-1}(e)$.

Since the Lie algebra associated to the universal covering group $\hat{G}$ is the same that the algebra of G then $\hat{G}_1$ and $\hat{G}$ have isomorphic algebras, therefore by corollary (4.4.6), there exists an isomorphism $\Phi : \hat{G}_1 \to \hat{G}$, thus $G_1 \cong \hat{G}/\Phi(D_1)$. □

Remark. Analogously to the proof of the Lie 2 theorem for Lie groups (proposition 4.4.5), it is possible to show that the Lie 2 theorem for groupoids 4.1.4 can be deduced from our theorem 4.3.5. We present a sketch of the proof.

Theorem 4.4.9 (Lie 2)**.** *Let $(G \rightrightarrows M)$ and $(H \rightrightarrows N)$ be Lie groupoids, and $(A_G \Rightarrow M)$, $(A_H \Rightarrow N)$ their Lie algebroids. Suppose that G is source simply-connected. If $\Phi' : A_G \to A_H$ is a morphism of Lie algebroids over $\phi : M \to N$, then there is a unique Lie groupoid morphism $\Phi : G \to H$ such that Lie$(\Phi) = \Phi'$.*

Proof. (Sketch)

- Φ' is a morphism of Lie algebroids if and only if its graph $A_\Phi := \{(X, \Phi'(X)) \in A_G \oplus A_H \mid X \in A_G\}$ is a subalgebroid of the Lie algebroid $(A_G \oplus A_H \Rightarrow (M \times N))$ with anchor (ρ_G, ρ_H).
- Let be $J \rightrightarrows (M \times N)$ the Lie groupoid which integrates the Lie algebroid $A_{\Phi'} \Rightarrow (M \times N)$; this is a subgroupoid of the groupoid $(G \times H \rightrightarrows M \times N)$. Also the canonical projection on G and H are morphism of Lie groupoids.
- Let $x \in M$, we know that the map

$$(p_G)|_{J((x,-),(\phi(x),-))} : J(x,-) \to G(x,-)$$

is locally a difeomorphism because its derivative is an isomorphism, also by theorem 4.3.5, its kernel is discrete, therefore this is a cover maping space; since $G(x,-)$ is simply connected, then this map is an isomorphism.

- For $g \in G(x,-)$, we define $\Psi(x) = p_H \circ (p_G)|_{J(x,-)}$.

$$\begin{tikzcd}
 & J((x,-),(\phi(x),-) \arrow[dl, "p_G"'] \arrow[dr, "p_H"] & \\
G(x,-) \arrow[ur, bend left, "(p_G|_{J(x,-)})^{-1}"] \arrow[rr, dashed, "\Phi|_{G(x,-)}"'] & & H(\phi(x),-)
\end{tikzcd}$$

□

4.5 Example: Case of a pair groupoid

Consider a morphism of Lie groupoids $\varphi : (G \rightrightarrows M) \to (H \to M)$, such that φ induces an isomorphism at the level of the algebroigs. Moreover we suppose that G is a transitive Lie groupoid, and G has discrete isotropy. Note that by the theorem 4.2.1 we have that $\ker(dt_G|_{A_x}) = 0$ and by the theorem 4.2.2 we have $dt_G(A_x) = T_xM$, then the anchor map induces an isomorphism between A_G and TM. Conversely if the anchor map induces an isomorphism then the Lie groupoid G has to have discrete isotropy and must be transitive.

Since the Lie algebroid of H is isomorphic to the algebroid of G, then H is transitive and has discrete isotropy.

By theorem 4.3.5 we know that the kernel of φ is a closed embedded submanifold of G, and it is a normal subrgroupoid discrete in the s-fibers; here, we will show that given any Lie groupoid $G \rightrightarrows M$ with Lie algerboid isomorphic to TM, then the pair groupoid $M \times M$ is isomorphic to the coimage of $G \xrightarrow{\varphi} M \times M$. We will to show some lemmas that are true in the case of transitive groupoids.

Lemma 4.5.1. *Let $(G \rightrightarrows M) \xrightarrow{\varphi} (H \rightrightarrows M)$ a morphism of Lie groupoids such that $\varphi|_{s^{-1}(x)} = \psi$ for some fixed $x_0 \in M$; if G is transitive then φ is determined by ψ.*

Proof. Let be $g : x \to y \in G$, since G is transitive, there exists $h : x_0 \to x \in G$.

Then $\varphi(g) = \varphi(ghh^{-1}) = \varphi(gh)\varphi(h^{-1}) = \psi(gh)\psi(h)^{-1}$.

Note that this definition does not depend on h chosen. Consider $k : x_0 \to x$, then $\varphi(g) = \psi(gk)\psi(k)^{-1} = \psi(ghh^{-1}k)\psi(k)^{-1} = \psi(gh)\psi(h)^{-1}$.

□

Lemma 4.5.2. *Let $(G \rightrightarrows M) \xrightarrow{\varphi} (H \rightrightarrows M)$ a morphism of Lie groupoids, for $x, y \in M$, if there exists $x \xrightarrow{g} y \in G$, then K_x the kernel of φ restricted to $s^{-1}(x)$ is isomorphic to K_y, the kernel restricted to the fiber $s^{-1}(y)$, via the conjugation map with g.*

Proof. Given $x \xrightarrow{h} x \in K_x$, note that

$$\varphi(ghg^{-1}) = u_y \text{ iff } \varphi(g)\varphi(h)\varphi(g)^{-1} = u_y \text{ iff } \varphi(g)\varphi(h) = \varphi(g) \text{ iff } \varphi(h) = u_x$$

□

For G transitive in the last lemma, there is a only one K_x, except by isomorphisms. The next lemma says that for a transitive groupoid the isotropy is locally trivial.

In the next lemma we will use the fact of that the inverse of source map is a embedded submanifold, and the target map restricted to this submanifold is a surjective submersion, for take a local section of the target map restricted to a fiber of source map.

Lemma 4.5.3. *Let $(G \rightrightarrows M) \xrightarrow{\varphi} (H \rightrightarrows M)$ a morphism of Lie groupoids; if G transitive then the isotropy of G, $\bigcup_{x \in M} s^{-1}(x) \cap t^{-1}(x)$, is locally trivial.*

Proof. Let $x \in M$, consider a local section of the target map $\sigma : U \to t^{-1}(U)|_{s^{-1}(x)}$, with $x \in U$. We denote the isotropy restricted to U as $G_u := s^{-1}(U) \cap t^{-1}(U)$.

Consider the map

$$\psi : G_U \to G_x \times U, \quad (y \xrightarrow{k} y) \to (\sigma(y)^{-1}k\sigma(y), y),$$

this map define a local trivialization. □

The next propositions show a correspondence between source connected, transitive Lie groupoids with discrete isotropy and regular covering spaces.

Proposition 4.5.4. *The transitive Lie groupoids with connected source fibers correspond with principal bundles.*

Proof. Given $G \rightrightarrows M$ a transitive Lie groupoids with connected source fibers, by theorem 2.1.4, we have that $t|_{s^{-1}(x)} : s^{-1}(x) \to M$ is a principal G_x-bundle.

Now, given $P \to M$ a principal G-bundel, by the example 2.2.5, we associate to the gauge groupoid $P \times P/G \rightrightarrows M$.

Note that the map $s^{-1}(x) \times s^{-1}(x) \to G$, such that $(g,h) \to gh^{-1}$ induces an isomorphism $s^{-1}(x) \times s^{-1}(x)/G_x \simeq G$, because if (g_1, h_1) and (g_2, h_2) are in $s^{-1}(x) \times s^{-1}(x)$ such that $g_1 h_1^{-1} = g_2 h_2^{-1}$, then $g_1 = g_2 h_2^{-1} h_1$ and $h_2^{-1} h_1 \in G_x$, then $gauge(s^{-1}(x)) = G$.

Thus the correspondence is well defined.

□

The following proposition relates the principal bundle whose group is discrete, with the regular cover map (see [Hat02]):

Proposition 4.5.5. *Let $p : (E, e) \to (X, x)$ be a cover map, E connected. It is a regular cover if and only if is a principal deck(E)-bundle, where the group deck(E) is the set of homeomorphisms $f : E \to E$ such that $p = p \circ f$.*

Then we have the correspondence

$$\left\{ \begin{array}{c} \text{Source connected, transitive} \\ \text{Lie Groupoids} \\ \text{with discrete isotropy} \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{Principal } G\text{-bundles} \\ P \text{ connected} \\ G \text{ discrete} \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{Regular} \\ \text{covering space} \end{array} \right\}$$

Proposition 4.5.6. *Let $G \rightrightarrows M$ and $H \rightrightarrows M$ be source connected Lie groupoids with Lie algebroids isomorphic to $TM \Rightarrow M$ (or equivalently they are transitive with discrete isotropy). Suppose that for some $x \in M$, $\pi_1(s_G^{-1}(x)) \subset \pi_1(s_H^{-1}(x))$, then there exists a morphism $G \xrightarrow{\varphi} H$, which integrates the isomorphism between algerboids.*

Proof. By the propositions 4.5.4 and 4.5.5 we have that $t_G : s_G^{-1}(x) \to M$ and $t_H : s_H^{-1}(x) \to M$ are regular cover spaces, with fiber G_x and H_x, respectively.

Since $t_G^{\#}(\pi_1(s_G^{-1}(x))) \subset t_H^{\#}(\pi_1(s_H^{-1}(x)))$ then there there exist a homomorphism $\psi : s_G^{-1}(x) \to s_H^{-1}(x)$ which commutes with the target maps

$$\begin{tikzcd} s_G^{-1}(x) \arrow[rr, "\psi"] \arrow[dr, "t_G"'] & & s_H^{-1}(x) \arrow[dl, "t_H"] \\ & M & \end{tikzcd}$$

By lemma 4.5.1, we can define $\varphi : G \to H$ via ψ. Let K be the kernerl of φ, since $K_x = \ker(\psi) \subset G_x$, by using the lemma 4.5.2 then K is discrete in fibers. By lemma 4.5.3 we conclude that K is a closed embedded submanifold of G, then by using the theorem 4.3.5 the morphism φ induces an isomorphism between the algebroids. □

For the next corollaries we assume M connected.

Corollary 4.5.7. *Let $G \rightrightarrows M$ be a Lie groupoid, with Lie algebroid isomorphic to $TM \Rightarrow M$, then there exist a morphism $(G \rightrightarrows M) \xrightarrow{\varphi} (M \times M \rightrightarrows M)$, such that the pair groupoid is isomorphic to the coimage of φ.*

Corollary 4.5.8. *Given $N \trianglelefteq \pi_1(M)$ a normal subgroup, and $x \in M$, then there exists a Lie groupoid $H \rightrightarrows M$ with Lie algebroid isomorphic to $TM \Rightarrow M$, such that $t^{\#}(s^{-1}(x)) = N$.*

Proof. There exists a covering map $p : M' \to M$, with M' connected and $\hat{x} \in M'$ such that $p^{\#}(M', \hat{x}) = N$. Then by using the proposition 4.5.5 and the example 2.2.5 we define the groupoid $H = M' \times M'/deck(P)$, which satifies the conditions. □

Remark. The previous examples, the Lie group and the pair groupoid, have a s−source simply connected cover groupoid which is Hausdorff. This motivates the next definition; let $G \rightrightarrows M$ be a Lie groupoid. We call $\pi_1(G) \to M$ to the group bundle where any fiber is the fundamental group of the fibers of the source map of G.

For the Lie group and the pair groupoid, $\pi_1(G)$ is a local trivial bundle, however for the example 4.1.3, $G \rightrightarrows M$ where $M = \mathbb{R}^3 - (0,0,0)$ and

$$G = \{((u_1, u_2, u_3), (v_1, v_2, v_3)) \in M \times M \mid u_3 = v_3\}$$

$\pi_1(G) \to M$ is isomorphic to $(\mathbb{R}^- \times \{0\}) \cup (0 \times \mathbb{Z}) \cup (\mathbb{R}^+ \times \{0\})$ which is not local trivial bundle.

With this idea in mind, we make the following question to be treated elsewhere: Given a Lie groupoid $G \rightrightarrows M$, when can we insure that the source simply connected cover groupoid is Hausdorff? Does it depend on if the group bundle $\pi_1(G) \to M$ is local trivial?

Chapter 5

Applications to symplectic geometry

Here we extend the above results to the symplectic case. We discuss symplectic Lie groupoids and Poisson manifolds. Then, we state the symplectic version of the main theorem. We conclude this chapter with an example about Lagrangian fibrations. More about symplectic theory, Poisson manifolds and Lagrangian fibrations can be found in [dS06], [dSW99], [DZ00], [Vai94], [Fuc82], [Dui80], [Sep12].

5.1 Symplectic Groupoids

We will present the concept of symplectic groupoid, which is a groupoid with an additional symplectic structure.

Definition 5.1.1. A symplectic groupoid is a Lie groupoid $G \rightrightarrows M$ endowed with ω a symplectic form on G ($\omega \in \Omega^2(G)$, $d\omega = 0$ and ω is non-degenerate) such that the form is multiplicative. This means that, considering the maps

$$\begin{tikzcd} & G \\ G \times_M G \arrow[ur, "pr_1"] \arrow[r, "m"] \arrow[dr, "pr_2"'] & G \\ & G \end{tikzcd}$$

where pr_1 and pr_2 are the canonical projections, the following equation holds:

$$m^*\omega = pr_1^*\omega + pr_2^*\omega. \tag{5.1.1}$$

This definition is equivalent to the one given in the following proposition.

Proposition 5.1.2. *Let $G \rightrightarrows M$ be a Lie groupoid. A symplectic form ω on G is multiplicative if and only if the graph of the multiplication $\gamma_m = \{(g,h,m(g,h))\}$ is a Lagrangian submanifold of $G \times G \times \hat{G}$.*

Proof. For $f = (g,h,gh) \in \gamma_m$ and for any $v,u \in T_f\gamma_m$, we can write

$$v = v_1 + v_2 + Dm_{(g,h)}(v_1,v_2) \quad u = u_1 + u_2 + Dm_{(g,h)}(u_1,u_2)$$

where $v_1, u_1 \in T_gG$, and $v_2, u_2 \in T_hG$, then

$$\begin{aligned} \omega_f^{G\times G\times\hat{G}}(v,u) &= 0 \\ \omega_g(v_1,u_1) + \omega_h(v_2,u_2) - \omega_{gh}(Dm_{(g,h)}(v_1,v_2), Dm_{(g,h)}(u_1,u_2)) &= \\ pr_1^*\omega_f(v,u) + pr_2^*\omega_f(v,u) - m^*\omega_f(v,u) & \end{aligned}$$

therefore (5.1.1) is true if and only if $\omega_f^{G\times G\times\hat{G}}(v,u) = 0$, but the last equality is the definition of the isotropic submanifold.

The dimension of γ_m is the dimension of $G \times_M G$, and $\dim(G \times_M G) = 2\dim(G) - \dim(M)$. Let $2n = \dim(G)$, and $k = \dim(M)$, we claim that $k = n$. In fact note that for any $x \in M$, $T_{u_x}M \subset T_{u_x}M^\omega$ (considering M in G via the unit map), because for $X, Y \in T_{u_x}M$ we have

$$\begin{aligned} (Dm)_{(u_x,u_x)}(X,X) &= X \\ (Dpr_1)_{(u_x,u_x)}(X,X) &= X \\ (Dpr_2)_{(u_x,u_x)}(X,X) &= X \end{aligned}$$

then

$$m^*\omega_{(u_x,u_x)}((X,X),(Y,Y)) = \omega_{u_x}(X,Y)$$

in the other hand

$$(pr_1^*\omega)_{(u_x,u_x)}(X,X),(Y,Y) + (pr_2^*\omega)_{(u_x,u_x)}(X,X),(Y,Y) = 2\omega_{u_x}(X,Y),$$

since the form is multiplicative we conclude that $\omega_{u_x}(X,Y) = 0$.

Also we have $T^t_{u_x}G \subset (T^s_{u_x}G)^\omega$; analogously consider $Y \in T^t_{u_x}G$ and any $X \in T^s_{u_x}G$. We can think X like the velocity of a curve in $s^{-1}(x)$, this is, $\gamma : I \to G$ such that $\gamma'(0) = X$, and $s(\gamma(t)) = x$. Then we can compose $\gamma(t)$

and the constant map in x u_x, i.e. $m(u_x, \gamma(t)) = \gamma(t)$ for any $t \in I$, then we have

$$(Dm)_{(u_x,u_x)}(0,X) = X$$
$$(Dpr_1)_{(u_x,u_x)}(0,X) = 0$$
$$(Dpr_2)_{(u_x,u_x)}(0,X) = X$$

For Y similarly,

$$(Dm)_{(u_x,u_x)}(Y,0) = Y$$
$$(Dpr_1)_{(u_x,u_x)}(Y,0) = Y$$
$$(Dpr_2)_{(u_x,u_x)}(Y,0) = 0$$

thus we have

$$m^*\omega_{(u_x,u_x)}((0,X),(Y,0)) = \omega_{u_x}(X,Y)$$

and

$$(pr_1^*\omega)_{(u_x,u_x)}(0,X),(Y,0)+(pr_2^*\omega)_{(u_x,u_x)}(0,X),(Y,0) = \omega_{u_x}(0,Y)+\omega_{u_x}(X,0) = 0$$

therefore $\omega_{u_x}(X,Y) = 0$. Since $x \in M$, $T_{u_x}M \subset T_{u_x}M^{\omega}$, then $k \leq 2n-k$ and from $T^t_{u_x}G \subset (T^s_{u_x}G)^{\omega}$ we have $2n-k \leq k$, hence $k = n$. In conclusion $\dim(G \times_M G) = 2\dim(G) - \dim(M) = 4n - n = 3n = 1/2\dim(G \times G \times \hat{G})$. Therefore $\gamma_m = graph(m)$ is a Lagrangian submanifold. □

Example 5.1.3. Let (M,ω) be a symplectic manifold. Consider the groupoid $G = M \times M \rightrightarrows M$, the pair groupoid with the symplectic form on G, $\omega \oplus (-\omega)$.

For $((x,y),(y,z)) \in G \times G$, the linear map

$$m_* : T_{(x,y)}G \times T_{(y,z)}G \simeq T_xM \oplus T_yM \oplus T_yM \oplus T_zM \to T_{(x,z)}G \simeq T_xM \oplus T_zM$$
$$(u_1,u_2,u_2,u_4) \to (u_1,u_4)$$

then

$$\begin{aligned} m^*(\omega \oplus (-\omega))((v_1,v_2,v_2,v_4),(u_1,u_2,u_2,u_4)) &= \\ (\omega \oplus (-\omega))(m_*(v_1,v_2,v_2,v_4), m_*(u_1,u_2,u_2,u_4)) &= \\ \omega(v_1,u_1) - \omega(v_4,u_4) &= \\ \omega(v_1,u_1) - \omega(v_2,u_2) + \omega(v_2,u_2) - \omega(v_4,u_4) &= \\ pr_1^*(\omega \oplus (-\omega))((v_1,v_2,v_2,v_4),(u_1,u_2,u_2,u_4)) &+ \\ pr_2^*(\omega \oplus (-\omega))((v_1,v_2,v_2,v_4),(u_1,u_2,u_2,u_4)) & \end{aligned}$$

thus, $\omega \oplus (-\omega)$ is a multiplicative form, therefore the pair groupoid with this form is a symplectic groupoid.

Example 5.1.4. Let M be a manifold, consider the groupoid $(T^*M, \omega_{can}) \rightrightarrows M$, i.e. the cotangent space with the canonical symplectic form (in local coordinates (q^i, p_i) $\omega_{can} = \sum dq^i \wedge dp_i$), and the structure of group bundle with $(\mathbb{R}^n, +)$ in any fiber. Note that the tanget space for any point $(x, v) \in T^*M$ is $\mathbb{R}^n$, the multiplication map (the sum in $\mathbb{R}^n$) acts linearly for the vector i.e.

$$\begin{aligned} m^*\omega_{can}((u^1, u^2), (v^1, v^2)) &= \omega_{can}((u^1_{q^i}, u^1_{p_i} + u^2_{p_i}), (v^1_{q^i}, v^1_{p_i} + v^2_{p_i})) \\ &= \omega_{can}(u^1_{q^i}, v^1_{p_i}) + \omega_{can}(u^1_{p_i}, v^1_{q^i}) \\ &+ \omega_{can}(u^1_{q^i}, v^2_{p_i}) + \omega_{can}(u^2_{p_i}, v^1_{q^i}) \\ &= pr_1\omega_{can}((u^1, u^2), (v^1, v^2)) + pr_2\omega_{can}((u^1, u^2), (v^1, v^2)) \end{aligned}$$

in conclusion, the canonical symplectic form is a multipicative form.

5.2 Poisson manifolds

We start with the definition of a Poisson manifold ([DZ00]]).

Definition 5.2.1. Le M be a smooth manifold, and a bilinear, antysimmetric map $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, such that

- The Leibniz rule is satisfied, i.e. $\{f, gh\} = \{f, g\}h + g\{f, h\}$.
- The Jacobi identity is verified i.e. $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

for all $f, g, h \in C^\infty(M)$. The pair $(M, \{,\})$ is a Poisson manifold.

Let us show two natural example.

Example 5.2.2. Let M be a manifold and consider $\{\} = 0$. This is the trivial Poisson bracket.

Example 5.2.3. Consider a symplectic manifold (M, ω). Given $f, g \in C^\infty(M)$, consider its Hamiltonian vector fields X_f and X_g (the only vector field such that $i_{X_f}\omega = -df$). Then the bracket given by $\{f, g\} = \omega(X_f, X_g) = -X_g(f) = X_f(g)$, is a Poisson bracket. The Leibniz rule is just by definition, and the Jacobi identity follows of the Cartan's formula for the differential of a k-form.

And immediate result of the definition is,

Lemma 5.2.4. *Given a smooth Poisson manifold $(M, \{,\}_\omega)$, the map*

$$\begin{aligned} (C^\infty(M), \{,\}_\omega) &\to (\mathfrak{X}(M), [,]) \\ f &\to X_f \end{aligned}$$

is a morphism of algebras where $(\mathfrak{X}(M), [,])$, is the vector fields on M with the Lie bracket.

Proof. For $f, g, h, \in C^\infty(M)$, we have

$$\begin{aligned}[X_f, X_g]h &= X_f(X_g h) - X_g(X_f h) \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\} = \{\{f, g\}, h\} = X_{\{f,g\}}h\end{aligned}$$

since this is true for any $h \in C^\infty(M)$, then $[X_f, X_g] = X_{\{f,g\}}$. □

We can define a Poisson bracket as bi-vector of the manifold M. A bi-vector Π is an element of $\Lambda^2 TM$ (this space is a vector bundle over M whose fibers on any $x \in M$ are $\Lambda^2(T_x M)$, the exterior product of two copies of $T_x M$). Then, given a bi-vectorw Π such that $[\Pi, \Pi] = 0$ (where $[,]$ is the Schouten - Nijenhuis bracket), we can define $\{f, g\} = \Pi(df, dg)$. With this idea of Poisson manifold we can think in another example of Lie algebroid.

Let $(M, \{,\})$ be a Poisson manifold. Let $\Pi \in \Lambda^2(TM)$ be the bivector associated to Poisson bracket, $\{f, g\} = \Pi(df, dg)$ (in local coordinates $\{f, g\} = \sum_{i,j} \Pi^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$). Then the map

$$\begin{aligned}\Pi^\# : T^*M &\to TM \\ \alpha &\to i_\alpha \Pi = \Pi(\alpha, \cdot)\end{aligned}$$

is the anchor.

And the Lie bracket in T^*M is defined as,

$$\begin{aligned}[\alpha, \beta] &= \mathcal{L}_{\Pi^\# \alpha}\beta - \mathcal{L}_{\Pi^\# \beta}\alpha - d(\Pi(\alpha, \beta)) \\ &= i_{\Pi^\# \alpha} d\beta - i_{\Pi^\# \beta} d\alpha - d(\Pi(\alpha, \beta)).\end{aligned}$$

This example was introduced by ([Fuc82]).

Remark: Under additional conditions in the Poisson structure we can get a one to one correspondence. Let $\pi : E \to M$ be a vector bundle. Let g be a function on M, then the function $g \circ \pi$ on E is called a *basic function*; in others words a basic function is a function on E which is constant in the fibers. A *fiber-wise linear function* on E is a function whose restriction to each fiber is linear.

Given the dual bundle $E^* \to M$ is possible to identify the fiber-wise linear functions on E^* with the sections of E. Consider $\alpha \in \Gamma(E)$ and define f_α a fiber-wise linear function on E^* as $f_\alpha(x, \xi) = \xi(\alpha(x))$. And the basic function of E^* can be identify we the function on M.

Definition 5.2.5. Let $\pi : E \to M$ be a vector bundle with Poisson structure on E, such that

1. The Poisson bracket of two basic function is zero.

2. The Poisson bracket is closed for fiber-wise linear functions.

3. The Poisson bracket of a basic function and a fiber-wise linear function is a basic function.

then, E is called fiber-wise linear Poisson structure. See ([DZ00]).

Given a $A \Rightarrow M$ an algebroid with anchor ρ and Lie bracket $[,]$. Consider the dual bundle $A^* \to M$, and define the Poisson bracket $\{,\}$, in the basic function on A^* (the function on M) and the fiber-wise linear function on A^* (section of E) as,

1. $\{f, g\} = 0$ for $f, g \in C^\infty(M)$.

2. $\{\alpha, \beta\} = [\alpha, \beta]$ for $\alpha, \beta \in \Gamma(A)$.

3. $\{\alpha, f\} = \rho\alpha(f)$ for $\alpha \in \Gamma(A)$, and $f \in C^\infty(M)$

This definition can be extend to any pair of functions in $\mathcal{G}^*$ by using Leibniz rule and linearity, see [DZ00].

Imagine that we have a symplectic manifold (M, ω) and a map from this manifold and another manifold B, and we want to know when $C^\infty(B)$ is a subalgebra of $(C^\infty(M), \{,\}_\omega)$, where $\{,\}_\omega$ is the Poisson bracket induced by the symplectic form. Note that we are taking $C^\infty(B)$ as subset of $C^\infty(M)$ by using the pull-back map.

Theorem 5.2.6. *(Liberman theorem). Let (M, ω) be a symplectic manifold, and $\pi : M \to B$ a surjective submersion, with connected fibers. Then $C^\infty(B)$ is a subalgebra of $(C^\infty(M), \{,\}_\omega)$ if and only if $(\ker(d\pi))^\omega$ is integrable.*

Proof. Let $f \in C^\infty(B)$. Consider a vertical vector field i.e. $Y \in \ker d\pi$. Note that $\omega(X_{\pi^* f}, Y) = (i_{X_{\pi^* f}}\omega)(Y) = d(\pi^* f)(Y) = 0$, then $X_{\pi^* f} \in (\ker(d\pi))^\omega$.

Note that the dimension of $(\ker(d\pi))^\omega$ is the complement of the dimension of the fiber, then its dimension is the dimension of B. In other hand by using $f_i \in C^\infty(B)$ as independent coordinates maps we can get as many $X_{\pi^f_i}$ linearly independent vector fields as the dimension of B. Thus by dimension $\{X_{\pi^* f}\} = (\ker(d\pi))^\omega$.

By Lemma (5.2.4) we have $[X_{\pi^* g}, X_{\pi^* g}] = X_{\{\pi^* f, \pi^* g\}}$. If $C^\infty(B)$ is a subalgebra, $\{\pi^* f, \pi^* g\} = \pi^* h$ for $f, g, h \in C^\infty(B)$, then $[X_{\pi^* g}, X_{\pi^* g}] = X_{\pi^* h} \in (\ker(d\pi))^\omega$.

Conversely, if $(\ker(d\pi))^\omega$ is integrable then $[X_{\pi^* f}, X_{\pi^* g}]$ is in $(\ker(d\pi))^\omega$, and

$$0 = \omega(V, [X_{\pi^* f}, X_{\pi^* g}]) = \omega(V, X_{\{\pi^* f, \pi^* g\}}) = d\{\pi^* f, \pi^* g\}(V) = V(\{\pi^* f, \pi^* g\})$$

for every vertical vector field V. It follows that $\{\pi^* f, \pi^* g\}$ is locally constant when restricted to the fibers. Since the fibers are connected, it has to be constant on the fibers, therefore it is a pullback function $\pi^* h$. $\square$

5.3 Symplectic version of the main theorem

Now we want to characterize the morphisms between symplectic groupoids which induce isomorphisms in the infinitesimal level. The first question is, if $(G, \omega_G) \rightrightarrows M$ and $(H, \omega_H) \rightrightarrows M$ are symplectic groupoids with $\varphi : G \to H$ morphism of symplectic groupoids which induces an isomorphism between the algebroids, then what can we say about $K = \ker(\phi)$?

From the theorem 4.3.5 we know that K is a closed submanifold of G and is a normal subgroupoid, such that is discrete in the s−fibers. With the adittional symplectic structure K also is a Lagrangian submanifold.

Proposition 5.3.1. *Let $(G, \omega_G) \rightrightarrows M \xrightarrow{\varphi} (H, \omega_H) \rightrightarrows M$ a morphism of symplectic Lie groupoids which induces an isomorphism between the algebroids, then its kernel $K := \ker(\varphi)$ is a Lagrangian submanifold.*

Proof. Given $h \in K$, from the proof of the proposition 4.3.2, we can consider a neighborhood U of h in G and a neigborhood $V \subset M$ of $s_G(h)$, such that $K \cap U$ is diffeomorphic to V, since $u_H(V)$ is a lagrangian submanifold in H and φ is a simplectomorphism, then $K \cap U$ is a lagrangian submanifold of G. $\square$

The converse of this result is summarized in the next proposition

Proposition 5.3.2. *Given $(G, \omega) \rightrightarrows M$ a symplectic Lie groupoid, and a K normal subgroupoid, discrete en fibers, closed embedded submanifold with the additional condition of being Lagrangian, then the cannonical projection $G \to G/K$ is a morphism of symplectic groupoids which induces an isomorphism in the Lie algebroids.*

Proof. By the proposition 4.3.3, we know that G/K is a Lie groupoid and the map $\pi : G \to G/K$ is a morphism of groupoids which induces an isomorphism in its Lie algebroids, then we need to define a mulitplicative symplectic form in G/K such that π is a symplectomorphism.

The canonical form that we can define is the push forward of ω via π; however it does not have to be well defined. For g in G, let $[g]$ be its equivalence class in G/K, then we set

$$\hat{\omega}_{[g]}(\hat{u},\hat{v}) := \omega_g(u,v)$$

where $u, v \in T_gG$, and $\hat{u} = (D_g\pi)u$ and $\hat{v} = (D_g\pi)v$. We need to show that is well define, is to say do not depend on the choose of g and the vectors u, v.

In the proposition 4.3.3 we show that π_* is an isomorphism then for $\hat{u} \in T_{[g]}G/K$ there is only one $u \in T_gG$ such that $\hat{u} = (D_g\pi)u$.

Now, consider $(k,g) \in K \times_M G \subset G \times_M G$ and $u, v \in T_{kg}G$. We can take $(u_1,u_2) \in T_{(k,g)}K \times_M G$ and $(v_1,v_2) \in T_{(k,g)}K \times_M G$ such that $m_*(u_1,u_2) = u$ and $m_*(v_1,v_2) = v$ (because the map $K \times_M G \to G$ is submersion). Since the form is multiplicative we have

$$\begin{aligned} \omega_{kg}(u,v) &= (m^*\omega)_{(k,g)}((u_1,u_2),(v_1,v_2)) \\ &= (pr_1^*\omega)_{(k,g)}((u_1,u_2),(v_1,v_2)) + (pr_2^*\omega)_{(k,g)}((u_1,u_2),(v_1,v_2)) \\ &= \omega_k(u_1,v_1) + \omega_g(u_2,v_2) \end{aligned}$$

since K is Lagrangian submanifold then $\omega_k(u_1,v_1) = 0$, therefore

$$\omega_{kg}(u,v) = \omega_g(u_2,v_2).$$

i.e. ω is invariant by the action of K, thus $\hat{\omega}$ is well defined. □

Theorem 5.3.3. *Given a symplectic Lie groupoid $G \rightrightarrows M$, the morphisms between symplectic Lie groupoids on M from G, which induces an isomorphism at the level of the Lie algebroids are in correspondence with the normal subgroupoid of G, such that are discrete in the s–fibers and are closed, Lagrangian submanifolds of G.*

Proof. From the theorem 4.3.5 we know that for morphism of Lie groupoids which induces isomorphism at the level of Lie algebroids are in correspondence with the normal subgroupoids, discrete in fiber that are closed, embedded submanifold.

By proposition 5.3.1, we know that with the additional sympectic structure in the morphism, the kernel of morphism is Lagrangian submanifold.

Conversely, by proposition 5.3.2, if the kernel K satifies the condition of theorem 4.3.5 beside the condition of being Lagrangian, then there exists an structure of symplectic Lie groupoid in in G/K such that the projection map is a morphism of symplectic Lie groupoids which induces an isomorphism in the Lie algebroids. □

5.4 Example: Lagrangian fibrations

Consider in the theorem 5.2.6 the particular case when the Poisson bracket is the trivial $\{,\} = 0$; What can we say about the map $i : (M, \omega) \to B$? The answer to this question introduces the definition of Lagrangian Fibration.

Definition 5.4.1. A lagrangian fibration is a symplectic manifold (M, ω), and a surjective submersion $\pi : M \to B$ whose fibers are Lagrangian submanifolds of (M, ω)

Note that $\dim(M) = 2\dim(B)$, because $\dim(\pi^{-1}(b)) = \dim(M) - \dim(B) = 1/2\dim(M)$.

Example 5.4.2. A natural example is a cotangent bundle with the canonical symplectic form; besides the fibers, we have that the zero section is also Lagrangian. In fact if we have that α is a closed 1-form, then its image is a Lagrangian submanifold ([dS06]).

Example 5.4.3. Let $(G, \omega) \rightrightarrows M$ be a group bundle. Consider the source map, which is a surjective submersion; we showed that $s^{-1}(x) \subset (t^{-1}(x))^{\omega}$ and $t^{-1}(x) \subset (s^{-1}(x))^{\omega}$, since $s = t$ then $s^{-1}(x) = (s^{-1}(x))^{\omega}$, therefore $s : (G, \omega) \to M$ is a Lagrangian fibration.

Example 5.4.4. Consider the symplectic group bundle $(T^*\mathbb{R}^n, \omega_{can}) \rightrightarrows \mathbb{R}^n$, with source and target map the canonical projection, which is a Lagrangian fibration (the previous example). Consider the normal subgroupoid $\mathbb{R}^n \times \mathbb{Z}^n$, which is discrete in fibers, and is a Lagrangian submanifold, because the section are translations of the zero section. By the theorem 5.3.3, $(\mathbb{R}^n/\mathbb{Z}^n, \omega) \rightrightarrows \mathbb{R}^n$ is a symplectic group bundle with the same algebroid of $(T^*\mathbb{R}^n, \omega_{can}) \rightrightarrows \mathbb{R}^n$, thus $(\mathbb{R}^n/Z^n, \omega) \to \mathbb{R}^n$ is a lagrangian fibration (where ω is given by the theorem 5.3.3).

The next proposition (actually it is a corollary of the Theorem 5.2.6) relate the Lagrangian fibrations and trivial Poisson brackets.

Proposition 5.4.5. *If $\pi : M \to B$ is a Lagrangian fibration then B inherit the trivial Poisson bracket. Conversely, if $\pi : (M, \omega) \to B$ is a surjective submersion such that the Poisson bracket inherited on B is zero and* $\dim(M) = 2\dim(B)$*, then it is a Lagrangian fibration.*

Proof. Regarding the first implication, for $f, g \in C^{\infty}(B)$, we have $\{f, g\}_B = \{\pi^* f, \pi^* g\} = \omega(X_{\pi^* f}, X_{\pi^* g})$, this is equal to zero because $\{X_{\pi^* f}\} = (\ker(d\pi))^{\omega} = \ker(d\pi)$.

For the converse, since $0 = \{f, g\} = \omega(X_{\pi^* f}, X_{\pi^* g})$, then we conclude that $(\ker(d\pi))^{\omega} \subset \ker(d\pi)$. Also $\dim(\ker(d\pi)) = \frac{1}{2}\dim(M) = \dim(B) = \dim(\ker(d\pi))^{\omega})$.

$\square$

If we start with a trivial Poisson bracket $(B,0)$, there is a canonical Lagrangian fibration that induces it, the surjective submersion $pr : (T^*B, \omega_{can}) \to B$. Moreover, this is a symplectic groupoid integrating the algebroid. Now the question is, if $\pi : (M,\omega) \to B$ is a Lagrangian fibration, what is the relation with $pr : (T^*B, \omega_{can}) \to B$? Is it a symplectic groupoid? Or are there Lagrangian fibrations that do not come from integrations of the algebroid? The answer has two parts, a local characterization and a global obstruction.

The goal now is to generalize the idea of the Example 5.4.4 to any proper Lagrangian fibration $\pi : (M,\omega) \to B$ whose fibers are connected and is a surjective submersion.

Associated to the Poisson structure in $(B,0)$, ([dSW99]) 5.2, there is a Lie algebroid structure in $pr : T^*B \Rightarrow B$, which anchor and Lie brackets are equal to zero. We denote A_B the algeborid $(T^*B \Rightarrow B, 0, 0)$.

The first step is to show that the Lie algebroid A_B acts on the map $\pi : M \to B$. Given $\alpha \in \Omega^1(B)$, since ω is symplectic, we call $X_{\pi^*\alpha}$ the only vector field in TM, such that

$$i_{X_{\pi^*\alpha}}\omega = \omega(X_{\pi^*\alpha}, \cdot) = \pi^*\alpha$$

Consider the set $\mathfrak{X}(M \xrightarrow{\pi} B) := \{X \in \mathfrak{X}(M) \mid \exists Y \in \mathfrak{X}(B)\ ,\ X \backsim_\pi Y\}$, the vector fields on M which are projectable along π. This set is a subalgebra of $\mathfrak{X}(M)$, by the naturality of Lie bracket.

Set the map $F : \Gamma(T^*B) \to \mathfrak{X}(M \xrightarrow{\pi} B)$, such that $F(\alpha) = X_{\pi^*\alpha}$. We claim that this map deines a action of the algebroid A_B on the map $\pi : M \to B$.

For any α and β 1-form on B, and any $f \in C^\infty(B)$, F is linear $(F(\alpha+\beta) = F(\alpha) + F(\beta))$, and satisfies the condition $F(f\alpha) = (\pi^*f)F(\alpha)$ we have

$$\begin{aligned}
\omega(X_{\pi^*(\alpha+f\beta)}, \cdot) &= \pi^*(\alpha + f\beta) \\
&= \pi^*\alpha + (\pi^*f)\pi^*\beta \\
&= \omega(X_{\pi^*\alpha} + (\pi^*f)X_{\pi^*\beta}, \cdot),
\end{aligned}$$

therefore $F(\alpha + f\beta) = X_{\pi^*(\alpha+f\beta)} = X_{\pi^*\alpha} + (\pi^*f)X_{\pi^*\beta} = F(\alpha) + (\pi^*f)F(\beta)$.

The next lemma give us the condition for the diagram in the definition of action of Lie algebroid commutes.

Lemma 5.4.6. *For any $\alpha \in \Omega^1(B)$, is satisfied that $\pi_* X_{\pi^*\alpha} = 0$.*

Proof. In a local coordinates $(x_1, ..., x_n)$ for a neighborhood $U \subset B$ we have $\alpha = \sum_{i=1}^n \alpha_i dx^i$, where α_i are real smooth function on U. Note that $X_{\pi^*\alpha} = \sum_{i=1}^n \pi^*\alpha_i X_{\pi^*dx^i}$. Hence is enough to show that result is held for the vector fields X_{π^*df}, for any $f \in C^\infty(U)$.

Let Y be a vector field in the kernel of π_*. Since $f \circ \pi$ is constant in the fibres of π, $Y(f\circ\pi) = 0$, therefore

$$\omega(X_{\pi^* df}, Y) = (\pi^* df)(Y) = Y(f \circ \pi) = 0$$

thus $X_{\pi^* df} \in (\ker(\pi_*))^\omega$, and since as the fibration is lagrangian $\ker(\pi_*) = (\ker(\pi_*))^\omega$.

□

Follows from the lemma we have $\pi_* F(\alpha) = 0 = \rho(\alpha)$.

Finally, we have to show that $[F(\alpha), F(\beta)] = F([\alpha, \beta]) = 0$. In a local coordinates and by Leibniz rule we have

$$\begin{aligned}
&[X_{\pi^*\alpha}, X_{\pi^*\beta}] \\
&= \sum_{i,j=1}^{n} [(\pi^*\alpha_i) X_{\pi^* dx^i}, (\pi^*\beta_j) X_{\pi^* dx^j}] \\
&= \sum_{i,j=1}^{n} (\pi^*\alpha_i)(\pi^*\beta_j)[X_{\pi^* dx^i}, X_{\pi^* dx^j}] + ((\pi^*\alpha_i) X_{\pi^* dx^i}(\pi^*\beta_j)) X_{\pi^* dx^j} - ((\pi^*\beta_j) X_{\pi^* dx^j}(\pi^*\alpha_i)) X_{\pi^* dx^i} \\
&= \sum_{i,j=1}^{n} (\pi^*\alpha_i)(\pi^*\beta_j)[X_{\pi^* dx^i}, X_{\pi^* dx^j}] + ((\pi^*\alpha_i)(\pi_* X_{\pi^* dx^i})\beta_j) X_{\pi^* dx^j} - ((\pi^*\beta_j)(\pi_* X_{\pi^* dx^j})\alpha_i) X_{\pi^* dx^i} \\
&= \sum_{i,j=1}^{n} (\pi^*\alpha_i)(\pi^*\beta_j)[X_{\pi^* dx^i}, X_{\pi^* dx^j}]
\end{aligned}$$

The last equality follows from the lemma 5.4.6. Thus is enough to show that $[X_{\pi^* df}, X_{\pi^* dg}] = 0$ for any f, g in $C^\infty(B)$, but we have that ω^b preserves the brackets, i.e. $[X_{\pi^* df}, X_{\pi^* dg}] = X_{\{\pi^* f, \pi^* g\}_\omega} = X_{\pi^* 0} = 0$. Therefore

$$A_B \rightsquigarrow^F (\pi : M \to B).$$

Remark: Note that for any $b \in B$ and any $m \in \pi^{-1}(b)$, $F(T_b^* B) \subset \ker(D\pi_m)$ (lemma 5.4.6), also since ω^b is a isomorphism $\dim(T_b^* B) = \dim(F(T_b^* B)) = \dim(B) = \dim(\ker(D\pi_m))$, then $F(T_b^* B) = \ker(D\pi_m)$.

Consider the groupoid $T^* B \rightrightarrows B$, where the source and target map are the canonical projection on B, the multiplication map is given by fiber-wise addition of covectors, the inverse map is given by taking the negative covectors and the unit map is the zero section.

Proposition 5.4.7. *The Lie algebroid associated to the groupoid $T^* B \rightrightarrows B$ is A_B.*

Proof. It clear that $T_{u(b)}(s^{-1}(b)) = T_{u(b)}(T_b^* B) = T_b^* B$ then $Lie(T^* B \rightrightarrows B) = \bigsqcup_{b \in B} T_{u(b)}(s^{-1}(b)) = T^* B$.

The anchor map is $dt = 0$ because the right-invariant vector fields are tangents to the fibers of s, and for this groupoid $s = t$. Finally note that the

$R_h(g) = g + h$ then dR_h is the identity, thus the right invariant vector fields have a constant coefficients, and since

$$[X,Y] = \sum_{i,j}(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i})\frac{\partial}{\partial x_j}$$

then $[X,Y]^R = [X^R, Y^R] = 0$. □

Given a 1-form α on B with compact support, the flow of the vector field $X_{\pi^*\alpha} \in \mathfrak{X}(M)$ is denoted as φ_α^t. By the lemma (5.4.6) the vector field $X_{\pi^*\alpha}$ is tangent to the fibers of π, then the flow φ_α^t lies along the fiber for all $t \in \mathbb{R}$. Also, since $\pi : M \to B$ is a proper map, then the fibers are compacts, hence the flows are complete.

Lemma 5.4.8. *Given the 1-forms α and β on B, then its flows satisfy $\varphi_\alpha^t \varphi_\beta^s = \varphi_{t\alpha+s\beta}^1$.*

Proof. By notation, let $X := X_{\pi^*\alpha}$ and $Y := X_{\pi^*\beta}$. Consider the curve $\gamma(r) = \varphi_\alpha^{tr}\varphi_\beta^{rs}$, note that $\gamma(0) = id = \varphi_{t\alpha+s\beta}^0$. Furthermore

$$\frac{d}{dr}\gamma(r) = \frac{d}{dr}(\varphi_\alpha^{tr}(\varphi_\beta^{rs})) = tX + D\varphi_\alpha^{rt}(\frac{d}{dr}\varphi_\beta^{sr}) = tX + D\varphi_\alpha^{rt}(sY)$$

also, since $[X,Y] = 0$ we have

$$\begin{aligned} D\varphi_\alpha^{rt}(sY) &= D\varphi_\alpha^0(sY) + \int_0^{rt} \frac{d}{dr_1} D\varphi_\alpha^{r_1}(sY)dr_1 \\ &= sY + \int_0^{rt} D\varphi^{r_1}(\mathcal{L}_Y X)dr_1 = sY \end{aligned}$$

therefore $\gamma(r)$ is the flow of the vector field $tX + sY$, then by unicity of the solution $\varphi_\alpha^{tr}\varphi_\beta^{rs} = \varphi_{t\alpha+s\beta}^r$. With $r = 1$ we have the result. □

Consider the map $\hat{F} : T^*B \times_\pi M \to M$, such that $\hat{F}(\alpha, m) = \varphi_\alpha^1(m)$.

Claim 1. The map $\hat{F} : T^*B \times_\pi M \to M$ defined an action of the groupoid $T^*B \rightrightarrows B$ on $\pi : M \to B$.

Proof. For any $m \in \pi^{-1}(b)$,

- $\pi(\hat{F}(\alpha, m)) = \pi(\varphi_\alpha^1(m)) = b = t(\alpha)$.
- By using the lemma (5.4.8); $\alpha.(\beta.m) = \alpha.(\varphi_\beta^1(m)) = \varphi_\alpha^1(\varphi_\beta^1(m)) = \varphi_{\alpha+\beta}^1(m) = (\alpha.\beta).m$.
- $u(\pi(m)).m = u(b).m = \varphi_0^1(m) = m$.

□

Consider a covector $\alpha_b \in T^*_b B$, it is possible to construct a compact 1-form α such that $\alpha(b) = \alpha_b$ (via partition of unity). Since for any $m \in \pi^{-1}(b)$ the value of $X_{\pi^*\alpha}(m)$ depends only on α_b, because $X_{\pi^*\alpha}(m) = \omega^{\#}_m(\pi^*\alpha_b)$, (where $\omega^{\#}$ is the inverse of the isomorphism ω^b). Also since the vector field is tangent to the fiber of π, then is possible to define an action of the group $T^*_b B \curvearrowright \pi^{-1}(b)$, it is the morphism of Lie groups

$$\begin{aligned} T^*_b B &\to Diff(\pi^{-1}(b)) \\ \alpha_b &\to \hat{F}_b := \varphi^1_\alpha|_{\pi^{-1}(b)} \end{aligned}$$

it is clear that this action of Lie group is the restriction of the previous action $\hat{F}$ to the fibers of π. This action is transitive because the infinitesimal action F, satisfies, as we showed before, $F(T^*_b B) = \ker(D\pi_m) = T_m(\pi^{-1}(b))$.

Also, since the group $T^*_b B$ with the sum of covectors is abelian, for any m_1 and m_2 in the fiber $\pi^{-1}(b)$, the isotropy subgroups G_{m_1} and G_{m_2} are equals because if $\alpha_b.m_1 = m_1$, and by transitive of the action exists some β_b such that $\beta_b.m_1 = m_2$, then $\alpha_b.m_2 = \alpha_b.\beta_b.m_1 = \beta_b.\alpha_b.m_1 = \beta_b.m_1 = m_2$, we denote the isotropy group by Λ_b. Therefore the orbit space $\pi^{-1}(b)$ can be identified with the quotient of the group $T^*_b B$ with the isotropy, i.e. $\pi^{-1}(b) \simeq T^*B/\Lambda_b$. Note that this correspondence is by choosing a distinguished point and consider its orbit.

Remark. The notation of Λ_b was chosen to highlight the classical proof [Dui80]), where is defined the subgroup of *periods*

$$\begin{aligned} \Lambda_b &:= \{\alpha_b \in T^*_b B \mid \exists m \in \pi^{-1}(b) \text{ such that } \varphi^1_{\alpha_b}(m) = m\} \\ &= \{\alpha_b \in T^*_b B \mid \varphi^1_{\alpha_b} \text{ is the identity }\}. \end{aligned}$$

and the *period net*

$$\Lambda := \bigsqcup_{b \in B} \Lambda_b$$

In works as ([Dui80]), ([Sep12]) there are proofs for show that Λ_b is discrete, Λ is a closed Lagrangian submanifold of (T^*B, ω_{can}) and The quotient T^*B/Λ is a smooth manifold. We will show that these facts follow from the symplectic version of our theorem 5.3.3.

For $x \in B$, let $U \subset B$ be an open with $x \in U$, and a local section $(U) \xrightarrow{\sigma} \pi^{-1}(U)$. Then the composition of the action $\hat{F}$ restricted to $pr^{-1}(U)$, with the section induces a morphism of groupoids between $(T^*U \rightrightarrows U, \omega_{can})$ and the group bundle $\pi^{-1}(U) \rightrightarrows U$

$$(T^*U \rightrightarrows U, \omega_{can}) \xrightarrow{\xi := \hat{F} \circ \sigma} (\pi^{-1}(U) \rightrightarrows U, \omega)$$

Via the section, we choose the distinguished point in the identification of the fiber with its orbit also the section induces the map unit in the groupoid. Moreover, these algebroids are isomorphic because $F(T_b^*U) = T_m(\pi^{-1}(b))$, and by lemma 5.4.6 the anchor are equals to zero.

Therefore by using the theorem 5.3.3 we conclude that $\Lambda_U := \bigsqcup_{b\in U} \Lambda_b$, which is the kernel of the morphism τ, is discrete in fibers, closed embedded Lagrangian submanifold. Since this was proved for any $x \in B$, and the properties are locally, then the same conditions are satisfied by Λ.

Since the morphism τ, induces an isomorphism $\hat{\tau} : T^*U/\Lambda_U \to \pi^{-1}(U)$, (M, ω) is locally equivalent to a product of torus, however it is not true globally. The obstruction is because in general a proper lagrangian fibration is not a group bundle.

Globally obstruction for Lagrangian Fibration

In general we can not define a global group bundle structure for $\pi : M \rightrightarrows B$ such that the map $(T^*B \rightrightarrows B) \to (\pi : M \rightrightarrows B)$ is a morphism of groupoids because we do not have a global section which define the unit map. The next goal is to know when this is possible.

Let $U_i \xrightarrow{\sigma_i} \pi^{-1}(U_i)$ and $U_j \xrightarrow{\sigma_j} \pi^{-1}(U_j)$ for some $x \in U_i \cap U_j$ be two local section, consider the map

$$\hat{\xi}_{\sigma_j}^{-1} \circ \hat{\xi}_{\sigma_i} : T^*(U_i \cap U_j)/\Lambda \to T^*(U_i \cap U_j)/\Lambda.$$

Note that the section zero in the point x is carried to $\hat{\xi}_{\sigma_j}^{-1} \circ \hat{\xi}_{\sigma_i}(0_b) = \hat{\xi}_{\sigma_j}^{-1}(\hat{F}_b(\sigma_i(b))) = \hat{\xi}_{\sigma_j}^{-1}(\sigma_i(b))$. Then $\hat{\xi}_{\sigma_j}\hat{\xi}_{\sigma_i}^{-1}(\sigma_i(b)) = \sigma_j(b)$ for all $b \in U_i \cap U_j$, i.e. the map $\hat{\xi}_{\sigma_j}\hat{\xi}_{\sigma_i}^{-1}$ carries the unit map in the local groupoid restrictes to $U_i \cap U_j$ defined by σ_i to the unit map defined by σ_j, see the Figure 5.1.

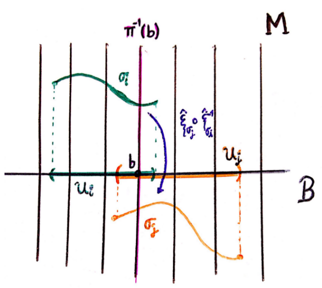

Figure 5.1: Local section $\sigma_i : U_i \to \pi^{-1}(U_i)$ and $\sigma_j : U_j \to \pi^{-1}(U_j)$

.

We can define the sections $s_{ji} = \hat{\xi}^{-1}_{\sigma_j}(\sigma_i) : U_i \cap U_j \to T^*(U_i \cap U_j)/\Lambda$, these sections satisfy

$$\hat{F}_{s_{ji}}(\sigma_j) = \hat{\xi}_{\sigma_j}(s_{ji}) = \hat{\xi}_{\sigma_j}\hat{\xi}^{-1}_{\sigma_j}(\sigma_i) = \sigma_i.$$

Let $\{U_i\}_{i\in I}$ be a cover of B such that the finite intersections are contractible, and consider the sheaf on B, $C^\infty(T^*B/\Lambda)$ of the smooth section of $T^*B/\Lambda \to B$. Then the map $s_{ji} \in C^\infty(T^*B/\Lambda)(U_i \cap U_j)$, also

$$s_{kj} + s_{ji} = \hat{\xi}^{-1}_{\sigma_k}(\sigma_j) + \hat{\xi}^{-1}_{\sigma_j}(\sigma_i) = \hat{\xi}^{-1}_{\sigma_k}(\sigma_i) = s_{ki}$$

then $(s_{ij})_{i,j\in I}$ defines a Čech 1-cocycle of $C^\infty(T^*B/\Lambda)$ over $\{U_i\}_{i\in I}$, then the equivalence relation $s_{ij} \sim s'_{ij}$ if and only if there exist a sections $h_i : U_i \to T^*U_i$ and $h_j : U_j \to T^*U_j$ such that $s_{ij} - s'_{ij} = h_i - h_j$ (in other words $s_{ij} - s'_{ij}$ is a 1-coboundary), define the first Čech cohomology of (T^*B/Λ) over $\{U_i\}_{i\in I}$, since every finite intersection is contractible, this is the first Čech cohomology $\check{\mathrm{H}}_1(B, C^\infty(T^*B/\Lambda))$.

Theorem 5.4.9. *Given a proper lagrangian fibration $\pi : (M,\omega) \to B$. If the periods net, Λ satifies that the first Čech cohomology of the sheaf of B on the section of $T^*B/\Lambda \to B$ is zero, i.e. $\check{H}_1(B, C^\infty(T^*B/\Lambda)) = 0$, then there exist a group bundle structure $M \rightrightarrows B$ and a morphisms of symplectic Lie groupoids $\xi_\sigma : (T^*B \rightrightarrows B, \omega_{can}) \to (M \rightrightarrows B, \omega)$.*

Proof. By following the ideas above, is enough to find a global section $\sigma : B \to M$. Consider a cover $\{Ui\}_{i\in I}$ of B with contractible finite intersections and local sections $\sigma_i : U_i \to \pi^{-1}(U_i)$; since $\check{\mathrm{H}}_1(B, \hat{\Gamma}(T^*B/\Lambda)) = 0$ then the Čech 1-cocycle $s_{ij} = h_i - h_j$ for some $h_i : U_i \to T^*U_i$ and $h_j : U_j \to T^*U_j$.

Now we know that $\sigma_i = \hat{F}_{s_{ji}}(\sigma_j) = \hat{F}_{h_i - h_j}(\sigma_j) = \hat{F}_{h_i}\hat{F}_{-h_j}(\sigma_j)$ then for any $b \in B$, take $\sigma(b) \coloneqq \hat{F}_{-h_i}((\sigma_i)(b))$. □

The next corollary relates previous theorem with the Lie theorem for groupoids presented in the section 4.1.

Corollary 5.4.10. *In the conditions of the previous theorem. For the group bundle $M \rightrightarrows B$, its source simply-connected Lie groupoid associated, as in the Theorem 4.1.2, is $T^*B \rightrightarrows B$.*

Proof. Any fiber of $T^*B \rightrightarrows B$ is a vector space, therefore any fiber is simply-connected. Also the map $\xi_\sigma : T^*B \to M$ is a morphism of groupoids. □

therefore both groupoids have the same Lie algebroid associated.

Recapitulation of the section

If we start with a manifold B with the trivial Poisson bracket, there exist a canonical symplectic manifold and a surjective submersion on B such the map induces a morphism of Poisson algebras, this is $pr : (T^*B, \omega_{can}) \to B$.

Any other symplectic manifold and proper surjective submersion $\pi : (M, \{,\}_\omega) \to (B, \{\} = 0)$ morphism of Poisson algebras, is a Lagrangian fibrations.

For the Lagrangian fibration $\pi : (M, \omega) \to B$ we can associate a periods net Λ which is a submanifold of T^*B, discrete in fibers. Locally, for a open $U \subset B$, we have that T^*U/Λ is diffeomorphic to $\pi^{-1}(U)$ and there is a group bundle structure $\pi^{-1}(U) \rightrightarrows U$, such that the diffeomorphism is a morphism of groupoids. This local map depend on a local section.

Globally, in general we cannot define diffeomorphism of T^*B/Λ and M, and this obstruction can be measured by the first Čech cohomology of T^*B/Λ on the sheaf of smooth section of $T^*B/\Lambda \to B$.

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