Instituto Nacional de Matemática Pura e Aplicada

On the Orbits of Monodromy Action

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to my mother Gladys,
and my brothers Camilo and Pedro.

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Introduction

We present below an introduction to the study of monodromy theory in a general way. In order to avoid overloading in this introductory text, the results in this work are described in the introductions of the chapter 3 and 4. The chapter 3 corresponds with the article [LG20a], and the chapter 4 corresponds with the article [LG20b].

One of the most interesting questions in mathematics is about the passage from the local to the global. Monodromy measures the differences among extensions of a local object through different paths around a singularity. For example, if we consider an open $U \subset \mathbb{P}^1$, a holomorphic function f defined on a disc contained in U and a path $\gamma : [0,1] \to U$, then the holomorphic continuation of f along γ depends on the homotopic class of the path. It is called the monodromy theorem (see [Bal08, Thm. 67]). Moreover, for a system of linear differential equations

$$\frac{d\Psi}{dz} = D(z)\Psi, \text{ which } \Psi = \Psi(z), \tag{1}$$

where $D(z) \in \mathbb{C}^{n\times n}$ is holomorphic in $\mathbb{P}^1$ except for the points $C := \{c_1, c_2, \dots c_\mu\}$, the solutions of (1) are holomorphic far away from C. Therefore, a fundamental solution of the differential equation can be continued along any path in $\mathbb{P}^1 \setminus C$, and this continuation only depends on the homotopy class of the path.

Let $b \in \mathbb{P}^1 \setminus C$ and consider a loop $\gamma : [0,1] \to (\mathbb{P}^1 \setminus C, b)$ starting at b. For a fundamental solution $y(z)$ of (1), its continuation along γ gives us another solution $\tilde{y}(z)$ of the differential equation. They are related by a invertible matrix

$$\tilde{y}(z) = M_\gamma y(z),$$

which only depends on the homotopy class of γ. Hence, this map defines a representation

$$\begin{aligned} \mathrm{Mon} : \pi_1(\mathbb{P}^1 \setminus C) &\to \mathrm{GL}(n,\mathbb{C}) \\ [\gamma] &\to M_\gamma, \end{aligned}$$

which is called the *monodromy* of the differential equation (1). Let γ_i be a loop in $\mathbb{P}^1 \setminus C$ starting in b and circling once around c_i for $i = 1, \dots, \mu$. Note that the composition $\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_\mu$ is homotopic to a contractible loop, thus the product $M_{\gamma_1} M_{\gamma_2} \cdots M_{\gamma_\mu}$ is the identity matrix. Furthermore, the image of Mon is generated by the matrices $M_{\gamma_1}, \dots, M_{\gamma_\mu}$, it is called the *monodromy group* of (1).

As it says in the beginning of [Zol06], the origins of the monodromy theory are in the Riemann-Hilbert problem: To find a linear differential equation with given singular points and a monodromy group. This problem was presented by Hilbert at 1900 at the second congress of mathematics ICM, in his famous problems (possibly many of the deeper questions in mathematics are born here). Nowadays it is known as Hilbert's 21^{st} problem, which was originally written as: *Proof of the existence of linear differential equations having a prescribed monodromic group*, [Hil02]. The question is closely related to the Riemann ideas about the global construction of a function from given local analytic data, as Hilbert explicitly mentioned [Bot20]. The Riemann-Hilbert problem may be understood as: To give a linear system having only regular singularities for a given set of singularities and a monodromy group, which was proved by Plemelj in 1908 [Ple08]. Moreover, if the expected system have to be a Fuchsian system on $\mathbb{P}^1$, that means that the singularities are poles of order one, then the problem has negative answer as showed by Bolibrukh in 1989 [Bol89].

The solutions of some differential equations with regular singular points have geometric interpretation. For example, if we consider

$$(1-z)\theta^2 y - z\theta y - \frac{z}{4}y = 0, \quad \text{where } \theta = z\frac{d}{dz},$$

then we find the following solutions

$$\int_{\delta_1} \frac{dx}{y} \text{ and } \int_{\delta_2} \frac{dx}{y},$$

where δ_1 and δ_2 are two non trivial different classes in the homology group of the elliptic curve $E_z = \{y^2 = x(x-1)(x-z)\}$ for $z \neq 0, 1$. This kind of differential equation are called *Picard-Fuchs* equations. Hence, the study of monodromy is tightly related with the study of integrals over cycles in algebraic varieties.

Motivated by the relation between the behavior of these integrals with the domain of integration, H. Poincaré in his treatise *Analysis Situs* (1894) began the investigation of the topology of algebraic varieties (the foundation of algebraic topology). The study of E. Picard (1906) about double integrals with cycles in hypersurfaces inside $\mathbb{C}^3$, and later the work of S. Lefschetz (1924) as a deep understanding of the books of Picard and a systematic study of the topology of algebraic varieties, yield the Picard-Lefschetz formula. This formula determines the variation of the topology of a family of algebraic varieties when the parameter goes around a critical value.

For an analytic map between germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated critical point 0, J. Milnor in [Mil68] defines a fibration (*Milnor fibration*), and he computes the homology groups of a regular fiber intersecting a small ball. He shows that a regular fiber is homotopic to a bouquet of $n-1$ spheres. In [Lam81], K. Lamotke gives a reformulation of the Lefschetz's work by using a modern language of algebraic topology. Lamotke considers a compact complex n-dimensional manifold Y and an analytic map $f : Y \to \mathbb{P}^1$ with non degenerated critical values $c_1, c_2, \ldots, c_r$. Then he shows that the relative homology of the pair (Y_+, Y_b) satisfies

$$H_k(Y_+, Y_b) = \begin{cases} 0, & \text{if } k \neq n. \\ \mathbb{Z}^r, & \text{if } k = n. \end{cases}$$

where Y_+ is the preimage of a hemisphere of $\mathbb{P}^1$ containing the critical values of f, and Y_b is a regular fiber.

The generators of $H_n(Y_+, Y_b)$ are called *Lefschetz thimbles*, and its boundary are called *vanishing cycles*. In a small ball the situation is the same as in the Milnor fibration, thus the fiber looks locally like the tangent space of a $n-1$ sphere. Therefore, the vanishing cycles are $n-1$ spheres and they generate the $H_{n-1}(Y_b)$. There is a vanishing cycle δ_i associated to any critical value c_i. Moreover, the Picard- Lefschetz formula, for the computation of the monodromy on a cycle $\delta \in H_{n-1}(Y_b)$ around c_i, is written as

$$\mathrm{Mon}_i(\delta) = \delta + (-1)^{\frac{n(n+1)}{2}} \langle \delta, \delta_i \rangle \delta_i,$$

in which $\langle , \rangle$ refers to the intersection number between two cycles. In the book [AVGZ88], by V. I. Arnold, A. N. Varchenko and S. M. Gusein-Zade, there is a systematic study of the local singularities. They consider a germ of a holomorphic map and define an equivalent Milnor fibration [Zol06, §5]. Besides, they compute the monodromy groups of several polynomials and introduce the *Dynkin diagrams* in order to describe in a combinatorial way the intersection form, associated to different kind of singularities. Recently, in [Mov17a], H. Movasati introduces the *tame polynomials*, in order to do a link between the robust study of local singularities in [AVGZ88] and the global behavior of fibrations, studied by Picard and Lefschetz. The works of K. Lamotke and H. Movasati have as motivation the study of the Hodge theory from the original point of view of Picard and Lefschetz, instead of the methods in the Hodge theory's of harmonic differential forms or sheaf theory and spectral sequences.

On the other hand, there is a more algebraic approach of the monodromy theory , by considering resolution of the singularities of algebraic varieties given by H. Hironaka's theorem. This approach is based on the cohomology theory of coherent sheaves developed by J. Leray, A. Grothendieck, P. Deligne and others. The main result is the monodromy theorem [Zol06, Thm 4.71.], which provides information about the eigenvalues and the dimensions of Jordan cells of the monodromy operator. Finally, it is worth to mention that there is an extremely complete work on monodromy, that is the aforementioned book by H. Żołądek [Zol06].

The ideas of Picard-Lefschetz theory applied to symplectic geometry emerge with some works by S. Donaldson. He proves the existence of topological Lefschetz pencils for compact symplectic manifolds and points out the Lagrangian nature of vanishing cycles. As Donaldson said, *"These results, along with various further extensions, give a means of translating many questions in symplectic topology into questions about the monodromy of the pencil"* [Don99]. The main topics on symplectic Picard-Lefschetz theory are systematically studied in the book of P. Seidel [Sei08]. In addition to the Lagrangian nature of the Lefschetz thimbles, the monodromy acting by symplectomorphisms plays an important role in the study of Lagrangian intersections in a Lefschetz fibration. Furthermore, the vanishing cycles yield a subcategory of the Fukaya category of a regular fiber, and it has information of the Fukaya category of the whole space [Sei08, §18].

The study of monodromy in fibrations given by polynomials $F \in \mathbb{C}[x,y]$ has application in the Hilbert's 16 problem. Consider a first order deformation of dF,

$$dF + \varepsilon\omega, \quad \text{where} \quad \omega = P(x,y)dx + Q(x,y)dy$$

and $P, Q \in \mathbb{C}[x,y]$. If the integral

$$I(t) = \int_{\delta(t)} \omega, \qquad \text{with } \delta(t) \subset F^{-1}(t) \text{ being a family of cycles,}$$

is not identically zero, then the limit cycles correspond with the zeros of $I(t)$. The weak 16th Hilbert problem [Arn88, p. 313] is to find the least upper bound $Z(d)$ of the number of zeros of $I(t)$ for a fixed degree d and for all possible $F, P, Q \in \mathbb{C}[x,y]_{\leq d}$. If the integral $I(t) \equiv 0$, then it does not provide any information about the number of limit cycles. Therefore, conditions on F, P, Q under which the integral $I(t)$ vanishes, is an interesting issue.

If $\delta(t)$, with $0 \le t < 1$, is a family of cycles around a Morse singular point p of F, then the cycle $\delta := \delta(0)$ is a vanishing cycle. Furthermore, if p is still a Morse singularity of the foliation given by $dF + \varepsilon\omega$, for ε small enough, then the integral $I(t)$ around the vanishing δ is zero. In [Ily69], Y. Il'yashenko proves that for a generic F, the existence of such p implies that ω is relatively exact. The proof is by studying the subspace generated by the orbit of a vanishing cycle under the monodromy action, in fact, he proves that for a generic polynomial the monodromy action is transitive. In other words, the Hamiltonian foliations of degree d are an irreducible component of the space of polynomial foliation in $\mathbb{C}^2$ of degree d with a center singularity. Analogous results for the logarithmic foliations by H. Movasati [Mov04b], and for the pullback foliations by Y. Zare [Zar17] are obtained by computing the orbit of the vanishing cycles by the monodromy action.

In this work, we applied the monodromy theory to the study of two different kinds of problems. Problem 1: which homology classes in a symplectic manifold could be realized by Lagrangian submanifolds. This question is studied in [SW01] and [LW12], for symplectic manifolds of dimension 4. Here, we approach this question for the family of mirror quintic Calabi-Yau threefolds (mirror quintic for short). By using Picard-Fuchs equation it is possible to give the explicit matrices for the monodromy action in the mirror quintic [DM06, CYYE08]. Moreover, there are two cycles in the mirror quintic associated to singular points of a fibration, which can be described in coordinates [CdlOGP91]. These cycles are a Lagrangian 3-sphere and a Lagrangian 3-torus. We want to compute the orbit by monodromy action of these cycles.

Problem 2: It is the monodromy problem, arising from [CM10]. That is, under what conditions on a polynomial $f \in \mathbb{C}[x,y]$ is the $\mathbb{Q}$-subspace of $H_1(f^{-1}(b),\mathbb{Q})$ generated by the image of a vanishing cycle of a Morse point under monodromy equal to the whole of $H_1(f^{-1}(b),\mathbb{Q})$. We solve this problem for $y^4 + p(x)$, with some restrictions on the degree of the polynomial p, and for polynomials $h(y) + g(x)$ with h, g being polynomials of degree 4. The first kind of polynomials is a generalization of the hyperelliptic curve studied in [CM10], and the second one is very interesting because would have applications in the center problem for degree 4, which is still an open problem.

Organization. The present manuscript is divided in three chapters and one appendix. In order to make the thesis as self-contained as possible, in the Chapter 1 we include some definitions and basic results. These preliminaries are in symplectic geometry, in Picard-Lefschetz theory, and in Picard-Fuchs equations. Also we include a classic example of the monodromy action on elliptic curves. The Chapter 2 is devoted to the study of homology classes supported in Lagrangian submanifolds in the mirror quintic Calabi-Yau threefold. In this chapter we give explicit matrices for the monodromy action in the mirror quintic and we compute the orbits of two cycles which are realized by Lagrangian submanifolds. In Chapter 3, we study the monodromy problem for some families of polynomials. The first family is a generalization of the results in hyperelliptic curves and the second is about direct sum of polynomials of degree 4. In Appendix A we state some additional or complementaries topics. Those are some of the codes used in the numerical computations, and some approaches applied to the center problem in foliations theory.

Chapter 1

Preliminaries

1.1 Preliminaries in Symplectic Geometry

Since an important part of the results of this thesis are in symplectic geometry, we want to collect some definition and basic results for a quick reference throughout the document. Furthermore, we study the symplectic form associated to a projective variety, which is the Fubini-Study form ω_{FS}. Also, we observe how ω_{FS} can be obtained as a symplectic quotient. These results help us to show some properties about a cycle in the mirror quintic, which appear in chapter 2.

Let V be a vector space with a skew symmetric, non-degenerated, bi-linear map $\Omega : V \times V \to \mathbb{R}$, the pair (V, Ω) is called a **symplectic vector space**. Given $W \subset V$ a subspace, the symplectic complement of W is

$$W^{\Omega} = \{u \in V \mid \Omega(u,v) = 0 \ \forall v \in W\}.$$

- If $W \subset W^{\Omega}$ then is called a **isotropic subspace**. Note that this is equivalent to $\Omega|_W \equiv 0$.
- If $W^{\Omega} \subset W$ then is called a **coisotropic subspace**.
- If $W = W^{\Omega}$ then is called a **Lagrangian subspace**. This is equivalent to being an isotropic subspace with half of dimension of V.
- If $W \cap W^{\Omega} = \{0\}$ the subspace is called **symplectic**.

The generalization of symplectic vector space to smooth manifolds is the next definition.

Definition 1.1.1. Let M be a smooth manifold, and ω a two form on M such that is closed and non-degenerated. The pair (M, ω) is called a **symplectic manifold**.

It easy to note that the dimension of M has to be even and ω^n is a nowhere-vanishing volume form, therefore M is orientable. Also, since ω is non-degenerated, the map $\omega^{\#} : TM \to T^*M$ such that $\omega^{\#}(u) = \omega(u,\cdot)$ is an isomorphism. The definitions of isotropic, coisotropic, Lagrangian and symplectic subspace are extended to submanifolds in a natural way, for example the submanifold $L \hookrightarrow M$ is a Lagrangian submanifold if for any point $x \in L$ the restriction $\omega(x)|_{T_xL} \equiv 0$ and $\dim(L) = \frac{1}{2}\dim(M)$.

Example 1.1.2. For a smooth manifold M with local coordinates $(x_1, ...x_n)$, there is a canonical symplectic form defined on T^*M which in local coordinates $(x_1, ...x_n, \xi_1, ...\xi_n)$ is written as $\omega_{can} = \sum_{k=1}^{n} dx_k \wedge d\xi_k$. The zero section $M \hookrightarrow T^*M$ is a Lagrangian submanifold.

The previous example is the most important, because the Darboux's theorem says that locally any symplectic manifold looks like $(T^*\mathbb{R}^n, \omega_{can})$.

Definition 1.1.3. Given two symplectic manifolds (M_1, ω_1) and (M, ω_2), they are equivalents if there is a diffeomorphism $F : M_1 \to M_2$ such that it preserves the symplectic structure i.e. $F^*\omega_2 = \omega_1$, this map is called a **symplectomorphism**.

The group of symplectomorphism is denoted by Symp(M, ω) and is a subset of Diff(M). At infinitesimal level, the vector fields whose flows are symplectomorphisms are called **symplectic vector field** which are defined as

$$\mathfrak{X}_\omega(M) := \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X\omega = 0\},$$

where $\mathfrak{X}(M)$ is the set of vector fields in M. The **Hamiltonian vector fields** $\mathfrak{X}_{Ham}(M)$ are vector fields X_H on M such that $\imath_{X_H}\omega = dH$ for some $H \in C^\infty(M)$. Note that $\mathcal{L}_{X_H}\omega = d\imath_{X_H}\omega + \imath_{X_H}d\omega = d\imath_{X_H}\omega = d(dH = 0)$, then $\mathfrak{X}_{Ham}(M) \subset \mathfrak{X}_\omega(M)$. The **Hamiltonian symplectomorphism** are the flows associated to the Hamiltonian vector fields.

1.1.1 Symplectic Reduction

Here we present some results about action of Lie groups on momentum maps, and reduction of the orbifolds which inherits a symplectic form. Our main example is $S^1 \curvearrowright \mathbb{C}^{n+1}$

Let G be a Lie group with unit e, and $\mathfrak{g}$ its Lie algebra. G acts on itself by conjugation, that is $I_g(h) := ghg^{-1}$. This action induces an action on $\mathfrak{g}$ which is called the **adjoint action** $Ad : G \to \mathrm{GL}(\mathfrak{g})$. That is defined as $Ad_g := (dI_g)_e : \mathfrak{g} \to \mathfrak{g}$.

Also induces an action on $\mathfrak{g}^*$ which is called **coadjoint action** $Ad^* : G \to \mathrm{GL}(\mathfrak{g}^*)$. That is defined as

$$\langle Ad_g^*(\alpha), v\rangle = \langle \alpha, Ad_{g^{-1}}(v)\rangle,$$

where $\langle , \rangle : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$ denotes the natural pairing.

Now, suppose that G acts on a manifold M via Ψ. For $v \in \mathfrak{g}$, the **infinitesimal generator** v_m is defined as $v_m(x) = \frac{d}{dt}\big|_{t=0} \Psi_{exp(tv)}(x)$, where $exp : \mathfrak{g} \to G$ is the exponential map.

Definition 1.1.4. Let (M, ω) be a symplectic manifold and G a Lie group with Lie algebra $\mathfrak{g}$. An action by symplectomorphism $G \xrightarrow{\Psi}$ Symp(M) is called **Hamiltonian**, if there exist a map $\mu : M \to \mathfrak{g}^*$ such that $\imath_{v_M}\omega = d(\langle \mu, v\rangle)$ and μ is G-equivariant, that is the next diagram commutes

$$\begin{tikzcd} M \arrow[r, "\Psi_g"] \arrow[d, "\mu"'] & M \arrow[d, "\mu"] \\ \mathfrak{g}^* \arrow[r, "Ad_g^*"'] & \mathfrak{g}^* \end{tikzcd}$$

The map μ is called the **momentum map** and (M, ω, G, μ) is called **Hamiltonian G-space**.

Example 1.1.5. Let (M, ω) be $\mathbb{C}$ with the canonical symplectic form $\omega_{can} = \frac{i}{2}dz \wedge d\bar{z}$. Consider the action $S^1 \curvearrowright \mathbb{C}$ given by $\theta \cdot z \to e^{i\theta}z$ and the map $\mu(z) = \frac{-\|z\|^2+1}{2}$. Since S^1 is an abelian group, then Ad^* is the identity map, also is clear that μ is invariant by the action. Therefore, μ is S^1-equivariant.

For $v \in \mathbb{R}$, we have that the real vector field associated to $\frac{d}{dt}\big|_{t=0}(e^{itv} \cdot z)$ is $iv(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}})$. Hence, $\imath_{v_M}\omega = -\frac{v}{2}(zd\bar{z} + \bar{z}dz) = d(\mu(z)v)$. Thus, $S^1 \curvearrowright M$ is a Hamiltonian action.

Consider two Hamiltonian actions $G \curvearrowright (M_j, \omega_j)$ and momentum maps $\mu_i : M_i \to \mathfrak{g}^*$, with $j = 1, 2$. Then the diagonal action $G \curvearrowright (M_1 \times M_2, \omega_1 \times \omega_2)$ given by $g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2)$ is Hamiltonian, with momentum map $\mu(x_1, x_2) = \mu_1(x_1) + \mu_2(x_2)$. Thus, we have the next example:

Example 1.1.6. The action of $S^1 \curvearrowright (\mathbb{C}^{n+1}, \omega_{can})$ given by the diagonal action $e^{i\theta} \cdot (z_1, \ldots, z_{n+1}) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_{n+1})$ is Hamiltonian, with momentum map $\mu(z_1, \ldots, z_{n+1}) = \frac{-\sum_{k=1}^{n+1} \|z_k\|^2 + 1}{2}$.

The next theorem state that the orbifold of a regular set have an symplectic structure, for a proof see for example [MD17, §5] or [dS06, §23].

Theorem 1.1.7 (Marsden-Weinstein-Meyer)**.** *Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Suppose that 0 is a regular value of μ and G acts freely on $\mu^{-1}(0)$. Then the quotient $M_{red} = \mu^{-1}(0)/G$ is a manifold and there exists a unique symplectic form ω_{red} on M_{red} such that $\imath^* \omega = \pi^* \omega_{red}$, where the maps $\imath$ and π are the inclusion and the quotient map, respectively.*

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, "\imath"] \arrow[d, "\pi"] & M \\ M_{red} & \end{tikzcd}$$

Example 1.1.8. For the Hamiltonian S^1-space $(\mathbb{C}^{n+1}, \omega_{can}, S^1, \mu)$ from example 1.1.6, we have that $\mu^{-1}(0) = \{z \in \mathbb{C}^{n+1} \mid \|z\|^2 = 1\} = S^{2n+1}$ and $\mu^{-1}(0)/S^1 = \mathbb{P}^n$. In the next section we show that ω_{red} is the Fubini-Study form.

1.1.2 Fubini Study form

A **complex manifold** is a manifold M and a linear map $J : TM \to TM$ such that $J^2 = -1$ and for local charts $U_\alpha \xrightarrow{\phi_\alpha} V_\alpha \subset \mathbb{C}^n$ the diagram commutes

$$\begin{tikzcd} TU_\alpha \arrow[r, "J"] \arrow[d, "d\phi_\alpha"'] & TU_\alpha \arrow[d, "d\phi_\alpha"] \\ TV_\alpha \arrow[r, "J_0"] & TV_\alpha \end{tikzcd}$$

where J_0 is the canonical complex structure in $\mathbb{C}^n$. Note that the diagram commutes if and only if $\phi_\alpha \in T^{10}(M)$, and this equivalent to $\bar{\partial}\phi_\alpha = 0$, this implies that the transition maps are holomorphic. If ω is a symplectic form on M, we say that (M, J, ω) is a **Kälher** manifold if J and ω are compatible, i.e. the map $g : TM \times TM \to \mathbb{R}$ defined as $g(u, v) := \omega(u, Jv)$ is a Riemannian metric.

We are interested in projective manifolds $X \hookrightarrow \mathbb{P}^N$ which have a canonical complex structure given by restriction of the complex structure on $\mathbb{P}^N$. There is a symplectic form ω_{FS} on $\mathbb{P}^N$ which is compatible with its complex structure, this form is called the **Fubini-study** form, we will define it below. The cohomology class of ω_{FS} is the Poincaré dual of the homological class $[\mathbb{P}^{n-1}]$ where $\mathbb{P}^n = \mathbb{P}^{n-1} \cup \mathbb{C}^n$. Moreover, it is not difficult to see that a complex submanifold of a Kälher manifold inherits a Kälher structure, then for $X \hookrightarrow \mathbb{P}^N$ there is a symplectic structure given by pullback of the Fubini-study form via the inclusion map.

For coordinates $[z_1 : z_2 : \ldots : z_{n+1}]$ in $\mathbb{P}^n$, in the affine chart $z_{n+1} = 1$ we have that the form given by $\omega_{FS} = \frac{i}{2}\partial\bar{\partial}log(|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 + 1)$ defines a real symplectic form, see for example [MD17, dS06]. Thus we have

$$\omega_{FS} = \frac{i}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{(1+\langle z,z\rangle)\delta_{jk} - \bar{z}_j z_k}{(1+\langle z,z\rangle)^2}dz_j \wedge d\bar{z}_k,$$

where $\langle z,z\rangle = \sum z_i\bar{z}_i$ and δ_{jk} is 1 if $j = k$ and 0 if $j \neq k$. Its matrix associated, in coordinates $(z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_n, \bar{z}_n)$ is

$$\begin{bmatrix} 0 & W \\ -W & 0 \end{bmatrix}$$

where

$$W = \begin{bmatrix} 1+\langle z,z\rangle - \bar{z}_1 z_1 & -\bar{z}_1 z_2 & \ldots & -\bar{z}_1 z_n \\ -\bar{z}_2 z_1 & 1+\langle z,z\rangle - \bar{z}_2 z_2 & \ldots & -\bar{z}_2 z_n \\ \vdots & & \ldots & \vdots \\ -\bar{z}_n z_1 & -\bar{z}_n z_2 & \ldots & 1+\langle z,z\rangle - \bar{z}_n z_n \end{bmatrix}$$

Let $pr : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the quotient map given by the equivalence relation

$$[z_1 : \ldots : z_{n+1}] \equiv [\lambda z_1 : \ldots : \lambda z_{n+1}] \text{ for } \lambda \in \mathbb{C}^*,$$

so the pullback via pr of the Fubini study form in $\mathbb{C}^{n+1}$ is

$$\rho_{FS} := pr^*\omega_{FS} = \frac{i}{2}\sum_{j=1}^{n+1}\sum_{k=1}^{n+1}\frac{\langle z,z\rangle\delta_{jk} - \bar{z}_j z_k}{\langle z,z\rangle^2}dz_j \wedge d\bar{z}_k,$$

here $z = (z_1, \ldots, z_{n+1})$. We will see that this form agrees with the canonical symplectic form in S^{2n+1}. Consider the map $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$, given by $\mu(z) = \frac{-||z||^2+1}{2}$. It is the momentum map for the action $S^1 \curvearrowright \mathbb{C}^{n+1}$ defined as $\theta(z_1, \ldots z_{n+1}) = (e^{i\theta}z_1, \ldots, e^{i\theta}z_{n+1})$.

As in example 1.1.8, $\mu^{-1}(0) = \{z \in \mathbb{C}^{n+1} \mid ||z||^2 = 1\} = S^{2n+1}$, and $\mu^{-1}(0)/S^1 = \mathbb{P}^n$. Thus, we have the next commutative diagram

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, "\imath"] \arrow[d, "\pi"'] & \mathbb{C}^{n+1} \setminus \{0\} \arrow[dl, "pr"] \\ \mathbb{P}^n & \end{tikzcd}$$

where $\imath$ is the inclusion map, and π is the quotient map by the action of S^1. Note that $\bar{z}_{n+1}z_{n+1} = 1 - z_1\bar{z}_1 - z_2\bar{z}_2 \ldots - z_n\bar{z}_n$. Suppose that $\bar{z}_{n+1} \neq 0$, then

$$dz_{n+1} = \frac{-\bar{z}_1 dz_1 - z_1 d\bar{z}_1 - \ldots - \bar{z}_n dz_n - z_n d\bar{z}_n - z_{n+1}d\bar{z}_{n+1}}{\bar{z}_{n+1}},$$

for simplicity in the computation we will consider the case $n = 1$. The general case is analogous. So,

$$\begin{aligned} \frac{2}{i}\imath^* pr^*\omega_{FS} &= (1 - z_1\bar{z}_1)dz_1 \wedge d\bar{z}_1 - \bar{z}_1 z_2 dz_1 \wedge d\bar{z}_2 - \frac{\bar{z}_2 z_1}{\bar{z}_2}(-\bar{z}_1 dz_1 \wedge d\bar{z}_1 - z_2 d\bar{z}_2 \wedge d\bar{z}_1) \\ &+ \frac{z_1\bar{z}_1}{\bar{z}_2}(-\bar{z}_1 dz_1 \wedge d\bar{z}_2 - z_1 d\bar{z}_1 \wedge d\bar{z}_2) = dz_1 \wedge d\bar{z}_1 - \frac{1}{\bar{z}_2}(\bar{z}_1 dz_1 \wedge d\bar{z}_2 + z_1 d\bar{z}_1 \wedge d\bar{z}_2) \\ &= dz_1 \wedge d\bar{z}_1 - \frac{1}{\bar{z}_2}(\bar{z}_1 dz_1 + z_1 d\bar{z}_1) \wedge d\bar{z}_2 = dz_1 \wedge d\bar{z}_1 - \frac{1}{\bar{z}_2}d(z_1\bar{z}_1) \wedge d\bar{z}_2 \\ &= dz_1 \wedge d\bar{z}_1 - \frac{1}{\bar{z}_2}d(1 - z_2\bar{z}_2) \wedge d\bar{z}_2 = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2. \end{aligned}$$

Note that the Fubini- Study form is the symplectic form obtained by symplectic reduction in the Hamiltonian S^1-space $(\mathbb{C}^n \setminus \{0\}, \omega_{can}, S^1, \mu)$ from the example 1.1.6. By theorem 1.1.7, we have that $\pi^*\omega_{red} = \imath^*\omega_{can}$, thus $\pi^*\omega_{red} = \imath^* pr^* \omega_{FS} = \pi^*\omega_{FS}$, hence $\omega_{red} = \omega_{FS}$.

Remark 1.1.9. In the coordinates $(|z_j|, \theta_j)$, where $z_j = |z_j| e^{i\theta}$, thus $\theta = -i\ln(\frac{z_j}{|z_j|})$. We have $d|z_j|^2 = \bar{z}_j dz + z d\bar{z}_j$ and $d\theta_j = i\left(\frac{d|z_j|}{|z_j|} - \frac{dz_j}{z_j}\right)$, where $d|z| = \frac{1}{2|z|}(\bar{z}dz + zd\bar{z})$. Hence $\frac{1}{2}(\sum_j d|z_j|^2 \wedge d\theta_j) = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j = \omega_{can}$.

1.2 Picard-Lefschetz theory

From now on, all our manifolds are complex manifolds. In this section we will write the definition of Lefschetz fibration, and we remark how in our particular case of projective manifold Y there is a natural symplectic form on Y such that the restriction to the fibers remains symplectic.

1.2.1 Monodromy action

A **Lefschetz fibration** on an complex manifold Y is a surjective analytic map $f : Y \to \mathbb{P}^1$ whose critical points are isolated and contained in distinct fibers. Also for any critical point p there is a Morse chart, i.e. that for z in a neighborhood of p, $f(z) = z_1^2 + ... + z_n^2$.

For $Y \hookrightarrow \mathbb{P}^N$ a projective manifold, we have a natural symplectic form ω given by the pullback of the Fubini study form in $\mathbb{P}^N$, i.e. $\omega = i^*\omega_{FS}$ where i is the inclusion map. By taking f holomorphic we have that the fibers of f over the regulars values are complex submanifolds of Y, then the restriction to the fiber $\omega|_{f^{-1}(b)}$ is symplectic. From here we will consider $X := Y_b \hookrightarrow Y$ a regular fiber and we denote C the set of critical values of f. By using a symplectic version of the Ehresmann lemma we will show that the fibers over regular values of the Lefschetz fibration $f : Y \to \mathbb{P}^1$ are symplectomorphic.

Let $f : E \to B$ be a smooth map and $(E, \mathcal{F})$ the foliation associated to the fibers of f. The kernel of the map Df is denoted by $T\mathcal{F}$ and is called the vertical bundle of f or the tangent space of the foliation $\mathcal{F}$.

Proposition 1.2.1. *Let (E, ω) be a symplectic manifold and B be a connected manifold. Consider $f : E \to B$ a proper surjective map with a finite set of critical values C, such that ω is symplectic at every regular fiber of f. Then the regular fibers are symplectomorphic.*

Proof. Using ω we can decompose the tangent bundle TE, over the set of regular values, as a direct sum of a vertical bundle VE and a horizontal bundle $HE := (VE)^\omega$. Here, the vertical space V_eE is the space of vectors tangent to the fibers of f and the horizontal space H_eE is its symplectic complement, i.e.

$$H_eE := \{u \in T_eE \mid \omega_z(u, v) = 0 \text{ for all } v \in V_eE\},$$

for all $e \in E$ regular point. This split of TE is well-defined since the restriction of ω to the fibers is symplectic. Let $b \in B$ be a regular value and $U \subset B \setminus C$ be a neighborhood of b. We take a vector field W defined on U, without singularities. Since f is a submersion on U, the map f_* is an isomorphism between H_eE and $T_{f(e)}B$ for all $f(e) \in U$. Thus, we can define a vector field V, as the preimage of W to any point in the fibers on U.

For $z \in E_b$ there is a neighborhood $U_z \subset U$ of z and a positive number ε_z such that the flow of V is defined at least in $(-\varepsilon_z, \varepsilon_z) \times U_z$. By the compactness of E_b we can cover it with finite U_z, and choose ε as the minimum of ε_z. Thus we define a flow $\theta : (-\varepsilon, \varepsilon) \times U' \to E$, where U' is a neighborhood of E_b. For t close enough to 0, we define a diffeomorphism φ from the fiber E_b to the fiber E_{b_t} as

$$\varphi_t(z) = \theta(t, z) \quad \text{for} \quad z \in E_b \text{ and } t \in (-\varepsilon, \varepsilon).$$

In order to show that φ_t preserves the symplectic form at the fibers is enough to show that $\frac{d}{d\tau}\big|_{\tau=t} \varphi_\tau^* \omega_b = 0$ for $t \in I$, where ω_b is the form ω restricted to the fiber E_b. This follows noting that

$$\frac{d}{d\tau}\bigg|_{\tau=t} \varphi_\tau^* \omega_b = \varphi_t^*(\mathcal{L}_V \omega_b) = \varphi_t^*(d\imath_V \omega_b + \imath_V d\omega_b) = \varphi_t^*(d\imath_V \omega_b),$$

and that $\imath_V \omega_b = 0$ since V is in HE. Consequently $\varphi_t^* \omega_b = \omega_b$, for any $t \in (-\varepsilon, \varepsilon)$. Because B is connected, we conclude the proof. □

Let $\gamma : [0,1] \to B \setminus C$ be a simple path. We denote by $P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ the symplectomorphism given by the lifting of γ as in the previous proposition.

Corollary 1.2.2. *Let Y be a projective manifold and $f : Y \to \mathbb{P}^1$ be a Lefschetz fibration with critical values C. For a simple path $\gamma : [0,1] \to \mathbb{P}^1 \setminus C$, the map P_γ is a symplectomorphism.*

From now on, we will denote by $X := Y_b \hookrightarrow Y$ a regular fiber of f. Since these fibers are symplectomorphic we simply denote any symplectic fiber by (X, ω_X). Thus, we have a map $\pi_1(\mathbb{P}^1 \setminus C) \to \mathrm{Symp}(X, \omega_X)$ which descends to homology, inducing the so called *monodromy action* $\pi_1(\mathbb{P}^1 \setminus C) \curvearrowright H_*(X, \mathbb{Z})$ given by $(\gamma, \delta) \to (P_\gamma)_* \delta$. Suppose that $\dim_{\mathbb{C}}(\mathrm{Y}) = \mathrm{n}$, by Lefschetz hyperplane theorem [Lam81, Mov17a], we are interested in the $(n-1)$-homology of X.

Definition 1.2.3. The monodromy action is the action of $\pi_1(\mathbb{P}^1 \setminus C)$ on $H_{n-1}(X, \mathbb{Z})$, i.e. the map

$$\mathrm{Mon} : \pi_1(\mathbb{P}^1 \setminus C) \times H_{n-1}(X, \mathbb{Z}) \to H_{n-1}(X, \mathbb{Z})$$

given by $(\gamma, \delta) \to (P_\gamma)_* \delta$.

Remark 1.2.4. In fact, if γ is a null homotopic loop in $\mathbb{P}^1 \setminus C$, then P_γ is a Hamiltonian symplectomorphism of $Y_{\gamma(0)}$ (see [Sei08]). Thus, the monodromy map comes from a map $\pi_1(\mathbb{P}^1 \setminus C) \to \mathrm{Symp}(\mathrm{X}, \omega_\mathrm{X}) / \mathfrak{X}_{\mathrm{Ham}}(\mathrm{X}, \omega_\mathrm{X})$, which descends to homology.

1.2.2 Vanishing cycles

Let Y a projective manifold and a regular fiber X, in this section we define vanishing thimble and vanishing cycles for a Lefschetz fibration $f : Y \to \mathbb{P}^1$ with critical values C. Let $\gamma : [0,1] \to \mathbb{P}^1$ be a simple path such that $\gamma(1) \in C$ and $\gamma(t) \in \mathbb{P}^1 \setminus C$ for $t \in [0,1)$. Let z_0 be the critical point in $f^{-1}(\gamma(1))$.

Definition 1.2.5. The set of points

$$V_\gamma = \{z \in f^{-1}(Im(\gamma)) \mid \lim_{t \to 1} P_\gamma(t)(z) = z_0\}$$

is called **Lefschetz thimble** and **vanishing cycle** δ_γ is the intersection of V_γ with the fiber $f^{-1}(\gamma(0))$.

In the next proposition we show that the Lefschetz thimbles are Lagrangian spheres of (Y,ω), and the vanishing cycles are Lagrangian spheres of (X,ω_X).

Proposition 1.2.6. *The Lefschetz thimble V_γ is a Lagrangian submanifold of (Y,ω) and the vanishing cycle δ_γ is a Lagrangian sphere of (X,ω_X).*

Proof. In a compact neighborhood U of p we can suppose that $f(z) = f(p) + z_1^2 + \cdots + z_n^2$ and that γ is a real curve in $\mathbb{C}$ with $\gamma(1) = 0$, and $\gamma(0) > 0$. Let $H : U \to \mathbb{R}$ be the map given by $H(z) = \mathrm{Re}(f(z))$. The Hamiltonian vector field X_H is horizontal because H is constant in the fibers of f and $\omega(X_H, V) = dH(V) = 0$ for any vertical vector field V. Since JV is also vertical then ∇H is horizontal. On the other hand $-\nabla H$ projects to $\frac{\partial}{\partial x}$ and so V_γ is the unstable set of p .

By lemma 1.2.8, we see that H is a Morse function with index n. Using the unstable manifold theorem [BH04, Thm. 4.2] we conclude that V_γ is a n-ball inside Y. To see that V_γ is isotropic, consider $u, v \in T_z V_\gamma$ for any $z \in V_\gamma$. Since the horizontal component of V_γ is one dimensional, we have $\omega_z(u,v) = \omega_X(z)(u_v, v_v)$, where u_v and v_v are the vertical components of u and v. As the fibers over $\gamma(t)$ with $t \in [0,1)$ are symplectomorphic via φ_t, we have that

$$\omega_X(z)(u_v, v_v) = \omega_X(z(t))(u_v(t), v_v(t))$$

where $z(t) = \varphi_t(z)$, $u_v(t) = (\varphi_t(z))_* u_v$ and $v_v(t) = (\varphi_t(z))_* v_v$. In the limit the tangent space is a point, then by continuity we can conclude the result. $\square$

Below, we present the fundamental theorem of Picard-Lefschetz theory. It says that the monodromy action is a Dehn-twist transformation, for a proof see for example [Lam81], [Mov17a, §6.6]. In this case we consider a more general situation for a Lefschetz fibration. Suppose that Y is a projective manifold and $f : Y \to \mathbb{P}^1$ a sujerctive analytic map with non-degenerate critical points. Let $c_i \in C$ be a critical value of f and $\delta \in H_{n-1}(X,\mathbb{Z})$, where X is a regular fiber . We denote $\mathrm{Mon}_i(\delta)$ the monodromy action by a simple loop in $\mathbb{P}^1 \setminus C$ around c_i in counterclockwise direction.

Theorem 1.2.7 (Picard-Lefschetz formula)**.** *The monodromy of a cycle δ around the critical value c_i can be written as*

$$Mon_i(\delta) = \delta + (-1)^{\frac{n(n+1)}{2}} \sum_j \langle \delta, \delta_j \rangle \delta_j,$$

where j runs through the vanishing cycles δ_j on c_i.

In order to prove the proposition 1.2.6 we have used the next technical result in lemma 1.2.8. Let $f : \mathbb{C}^n \to \mathbb{C}$ be the function given by $f(z_1, z_2, ..., z_n) = z_1^2 + ... + z_n^2$, and $\omega = i/2 \sum_{k=1}^n \alpha_k(z,\bar{z}) dz_k \wedge d\bar{z}_k$ a symplectic form in $\mathbb{C}^n$, where $\alpha_k(z,\bar{z}) \in \mathbb{R}^*$. In real coordinates $(x_1, y_1, ..., x_n, y_n)$ where $z_k = x_k + iy_k$, we have $f(x,y) = (\sum_k (x_k^2 - y_k^2), 2\sum_k x_k y_k)$ and $\omega = \sum_{k=1}^n \alpha_k dx_k \wedge dy_k$.

Lemma 1.2.8. *Let $H : \mathbb{C}^n \to \mathbb{R}$ be the function given by $H(z) = Im(f(z))$. The point $(0,\ldots,0) \in \mathbb{C}^n$ is a hyperbolic critical point of the Hamiltonian vector field X_H of the function H. Moreover there are the same quantity of positive eigenvalues than negative eigenvalues.*

Proof. Let $h:\mathbb{C}\to\mathbb{R}$ be the function such that $h(x+iy)=y$, thus $dh=(0,1)$.
Moreover $H=h\circ f$, consequently $dH=dh\circ df$. Since

$$df_{(x,y)}=\begin{pmatrix}2x_1 & -2y_1 & \dots. & 2x_n & -2y_n\\ 2y_1 & 2x_1 & \dots. & 2y_n & 2x_n\end{pmatrix}$$

then $dH_{(x,y)}=2\sum_{k=1}^n y_k dx_k + x_k dy_k$. The vector field X_H by definition is the only vector field such that $\imath_{X_H}\omega=dH$ (because ω is non-degenerated). If we write $X_H=\sum_k a_k\frac{\partial}{\partial x_k}+\sum_k b_k\frac{\partial}{\partial y_k}$, then $\imath_{X_H}\omega=\sum_k \alpha_k a_k dy_k-\alpha_k b_k dx_k$, consequently

$$a_k=2x_k\alpha_k^{-1}\text{ and }b_k=-2y_k\alpha_k^{-1},$$

therefore the Jacobian matrix of X_H in the point $0\in\mathbb{C}^n$ is

$$2\begin{pmatrix}\alpha_1^{-1}(0) & 0 & 0 & \dots & 0\\ 0 & -\alpha_1^{-1}(0) & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & \alpha_n^{-1}(0) & 0\\ 0 & 0 & \dots & 0 & -\alpha_n^{-1}(0)\end{pmatrix}$$

with n eigenvalues $\lambda_k^+=2\alpha_k^{-1}(0)>0$ and n eigenvalues $\lambda_k^-=-2\alpha_k^{-1}(0)<0$. □

We want to finish this section showing that in the regular hypersurface in $\mathbb{P}^n$ with n even, any homology $n-1$ class could be represented by a linear combination of Lagrangian spheres. Suppose that the origin is an isolated critical point of the highest-grade homogeneous piece of f. The $(n-1)$-homology group of the fiber over b is generated by the vanishing cycles, see [Lam81], [Mov17a, §7.4]. As a consequence we can prove the next proposition.

Proposition 1.2.9. *Let $F\in\mathbb{C}[z_0,\dots,z_n]$ be a homogeneous polynomial with n even. Suppose that F defines a smooth variety X in $\mathbb{P}^n$. Then, any homology class $\delta\in H_{n-1}(X,\mathbb{Z})$ can be written as a finite sum $\delta=\sum_j a_j\delta_j$, where $a_j\in\mathbb{Z}$ and δ_j is supported in a Lagrangian $(n-1)$-sphere.*

Proof. Consider a hyperplane that intersects transversally X, and let Z be its intersection. We can suppose that the hyperplane section is $Z=X\cap\{z_0=0\}$. Let $f\in\mathbb{C}[z_1,\dots,z_n]$ be the polynomial $F(1,z_1,\dots,z_n)$ and we define the affine variety $U:=X\setminus Z=\{(z_1,\dots,z_n)\in\mathbb{C}^n\mid f(z_1,\dots,z_n)=0\}$. The pair (X,U) induces the exact sequence in homology

$$\dots\to H_n(X,U)\to H_{n-1}(U)\to H_{n-1}(X)\to H_{n-1}(X,U)\to\dots,$$

where the map $H_k(U)\to H_k(X)$ comes from the inclusion $U\subset X$. By Leray-Thom-Gysin isomorphism we have $H_k(X,U)\simeq H_{k-2}(Z)$. By Lefschetz hyperplane section theorem we know that $H_k(Z)\simeq H_k(\mathbb{P}^{n-2})$ if $k\neq n-2$, see [Mov17a, §5.4]. Since n is even we have that $H_{n-3}(Z)=0$, then the map

$$H_{n-1}(U)\to H_{n-1}(X)\to H_{n-3}(Z)=0$$

is surjective. The vanishing cycles associated to the fibration $f:\mathbb{C}^n\to\mathbb{C}$ generate the homology group $H_{n-1}(U)$, and they are supported in Lagrangian spheres of U. □

1.3 Picard-Fuchs equations

Computing monodromy using topological methods is in practice very hard. There is an alternative in which one first compute Picard-Fuchs equations. The solutions of these differential equation are a basis of a vectorial space which isomorphic to the module of the homology group of a regular fiber.

Consider an ordinary differential equation over the field of rational functions in z,

$$y^{(n)} + p_1(z)y^{(n-1)} + \ldots + p_{n-1}(z)y' + p_n(z)y = 0, \text{ for } i = 1, \ldots, n \ , \ p_i(z) \in \mathbb{C}(z). \tag{1.1}$$

A point $z_0 \in \mathbb{P}^1$ is **singular** if for some $j = 1, \ldots, n$, $p_j(z)$ has a pole in z_0. A singular point is **regular** if for any j, the limit $\lim_{z \to z_0}(z - z_0)^j p_j(z)$ exists and is finite. The point ∞ is a regular singularity if $\lim_{z\to\infty} z^j p_j(z)$ exists and is finite. An ordinary differential equation over $\mathbb{C}(z)$ whose critical points are all regular is called **Fuchsian**.

Example 1.3.1. The equation $y^{(2)} + \left(\frac{1}{z} + \frac{1}{z-1}\right) y' - \frac{1}{4z(1-z)}y = 0$ has regular singularities in 0 and 1. This equation is the normalization of the equation

$$(1-z)\theta^2 y - z\theta y - \frac{z}{4}y = 0, \text{ where } \theta = z\frac{d}{dz},$$

whose solution are the periods of the family of elliptic curves $E_z : y^2 = x(x-1)(x-z)$.

As in the previous example, for a compact Calabi-Yau manifold X of dimension $2n$ we can define the periods. By definition, X has a holomorphic volume form which is a nowhere-zero section of the canonical bundle. Thus any holomorphic section in $H^0(X, \Omega^n)$ is the volume form times a holomorphic function, and because X is compact then the function is constant. Hence $\dim(H^0(X, \Omega^n)) = 1$.

For a family in 1 parameter of compact Calabi-Yau $n-$manifolds, that is X_z with $z \in \mathbb{P}^1$, we choose the only holomorphic $n-$form (up to multiplication by a scalar) and set the integral

$$\int_{\delta_z} \eta_z, \qquad \text{where} \qquad \delta_z \in H_n(X_z)$$

they are holomorphic function which are called **periods**. The Gauss-Manin connections theory states that there is an Fuchsian differential equation whose solutions are the periods $\int_{\delta_z} \eta_z$. This differential equation is called **Picard-Fuchs differential equation**.

Example 1.3.2. The differential equation

$$(\theta^4 - z(\theta + A)(\theta + 1 - A)(\theta + B)(\theta + 1 - B))y = 0 \tag{1.2}$$

for 14 values of (A, B) corresponds with Picard-Fuchs equation of Calabi-Yau threefolds, and the singularities are in $z = 0, 1, \infty$. X_z is the *B-model* of mirror symmetry in which we have listed the *A-model* in table 2.2.

1.3.1 Monodromy of Picard Fuchs differential equation

If z_0 is a regular singularity, then a local solution near of z_0 can be written as

$$y = \sum_{k=0}^{\infty} a_k (z - z_0)^{k+\lambda}.$$

By replacing this solution in the differential equation and considering the coefficient that multiplies a_0 we get the **indicial equation**.

Example 1.3.3. By expanding the equation 1.2 we get

$$(P_0(z)\theta^4 + P_1(z)\theta^3 + P_2(z)\theta^2 + P_3(z)\theta + P_4(z))y = 0,$$

where $P_0(z) = 1 - z$, $P_1(z) = -2z$, $P_2(z) = -z(A(1-A) + B(1-B) + 1)$, $P_3(z) = -z(A(1-A) + B(1-B))$ and $P_4(z) = -zA(1-A)B(1-B)$.

Consider the solution around to 0, that is $y = \sum_{k=0} a_k z^{k+\beta}$. Note that $\theta^j y = \sum_{k=0}^{\infty}(k+\lambda)^j a_k z^{k+\beta}$. Thus by replacing y in the differential equation, we get

$$\sum_{k=0}^{\infty} a_k z^{k+\lambda}(P_0(z)(k+\lambda)^4 + P_1(z)(k+\lambda)^3 + P_2(z)(k+\lambda)^2 + P_3(z)(k+\lambda) + P_4(z)) = 0$$

The polynomial $P_0(z)$ is the only polynomial of the coefficients which has a degree 0 component, then by doing a shift in the index of the sum we obtain

$$\sum_{k=0}^{\infty} a_k z^{k+\lambda}(k+\lambda)^4 - \sum_{k=1}^{\infty} a_{k-1} z^{k+\lambda}((k-1+\lambda)^4 + d_{k-1}) = 0$$

where $d_k = \frac{1}{z}(P_1(z)(k+\lambda)^3 + P_2(z)(k+\lambda)^2 + P_3(z)(k+\lambda) + P_4(z))$. Because the power of z are independents, it is necessary that each part of the sum be zero. And since $a_0 \neq 0$, with $k = 0$ we set $(\lambda)^4 = 0$. This is the indicial equation for 1.2 in the singularity $z_0 = 0$.

The next lemma give us an algorithm to compute the roots of the indicial equation.

Lemma 1.3.4. *Let $z_0 \in \mathbb{C}$ be a regular singularity of the equation 1.1. Set $a_j = \lim_{z\to z_0}(z-z_0)^j p_j$. for $j = 1, \dots, n$ and $a_0 = 1$. Then the indicial equation at z_0 is given by*

$$\sum_{k=0}^{n-1}\left(\prod_{l=0}^{n-(k+1)}(\lambda - l)\right)a_k + a_n = 0$$

. When ∞ is a regular singularity, set $a_j = \lim_{z\to\infty} z^j p_j(z)$ for $j = 1, \dots, n$. Then the indicial equation at ∞ is given by

$$\sum_{k=0}^{n-1}\left(\prod_{l=0}^{n-(k+1)}(\lambda + l)\right)(-1)^k a_k + (-1)^n a_n = 0$$

.

Example 1.3.5. The equation 1.2 can be rewritten as

$$\left(\frac{d^4}{dz^4} + p_1(z)\frac{d^3}{dz^3} + p_2(z)\frac{d^2}{dz^2} + p_3(z)\frac{d}{dz} + p_4(z)\right)y = 0,$$

where $p_1(z) = \frac{8z-6}{z(z-1)}$, $p_2(z) = \frac{z(A-A^2+B-B^2+14)-7}{z^2(z-1)}$, $p_3(z) = \frac{2z(A-A^2+B-B^2+2)-1}{z^3(z-1)}$ and
$p_4(z) = \frac{AB(A-1)(B-1)}{z^3(z-1)}$.

Set $a_j = \lim_{z\to 1}(z-1)^j p_j(z)$, then $a_1 = 2$ and $a_2 = a_3 = a_4 = 0$. By lemma 1.3.4 we have that the indicial equation at 1 is: $\lambda(\lambda-1)^2(\lambda-2) = 0$.

Now, by taking $a_j = \lim_{z\to\infty} z^j p_j(z)$ we have $a_1 = 8$, $a_2 = A - A^2 + B - B^2 + 14$, $a_3 = 2(A - A^2 + B - B^2 + 2)$ and $a_4 = AB(A-1)(B-1)$. By lemma 1.3.4 we have that the indicial equation at ∞ is:

$$(\lambda - A)(\lambda - 1 + A)(\lambda - B)(\lambda - 1 + B) = 0.$$

With the roots of the indicial equation we can generates the monodromy matrix as in [CYYE08]. Let z_0 a regular singularity of the differential equation 1.1. If the indicial equation has different solutions $\lambda_1\,\lambda_2,\ldots,\lambda_n$ and $\lambda_i-\lambda_j\notin\mathbb{Z}$ for $i\neq j$, then a basis of the solutions if given by

$$y_j=(z-z_0)^{\lambda_j}f_j(z),\ \text{with } j=1,\ldots,n$$

where $f_j(z)$ is a holomorphic function near to z_0 and have non-vanishing constant terms.

Because f_j is holomorphic, the analytic continuation along a small closed curve circling z_0 does not modify it. For the other factor we have the next claim.

Claim 1.3.6. The action on $(z-z_0)^{\lambda_j}$ of circling z_0 once in the counterclockwise direction is

$$(z-z_0)^{\lambda_j}\to(z-z_0)^{\lambda_j}e^{2\pi i\lambda_j}.$$

Proof. Without loss of generality we suppose that $z_0=0$. The function $f(z)=z^\lambda$ in Taylor series around of w_l is given by

$$f_l(z):=\sum_{k=0}^{\infty}\binom{\lambda}{k}w_l^{\lambda-k}(z-w_l)^k$$

with radius of convergence $r=\lim_{k\to\infty}\mid\frac{\binom{\lambda}{k}}{\binom{\lambda}{k+1}}w_lw\mid=\mid w_l\mid$. Set $w_l=e^{2\pi il/N}$ with $l=0,1,\ldots N-1$ and N some positive integer.

The path $\gamma(t)=e^{2\pi it}$ goes through the points w_l and circle 0 once in the counterclockwise direction. The analytic extension of f can be compute by comparing $f_0(z)$ and $f_N(z)$. Thus we have $f_N(z)=e^{2\pi i\lambda}f_0(z)$. □

Hence, in this case the monodromy action with respect to the basis $\{y_j\}_{j=1}^n$ is

$$\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}\to\begin{pmatrix}e^{2\pi i\lambda_1}&0&\ldots&0\\0&e^{2\pi i\lambda_2}&\ldots&0\\\vdots&\vdots&&\vdots\\0&0&\ldots&e^{2\pi i\lambda_n}\end{pmatrix}\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}$$

If the indicial equation has solutions $\lambda_1,\lambda_2,\ldots,\lambda_r$ with multiplicities $d_1,d_2,\ldots,d_r$, and for $i\neq j$, $\lambda_i-\lambda_j\notin\mathbb{Z}$. Then for any $j=1,\ldots,r$, the functions

$$\begin{aligned}
y_{j,0}&=(z-z_0)^{\lambda_j}f_{j,0}\\
y_{j,1}&=(z-z_0)^{\lambda_j}f_{j,0}\log(z-z_0)+(z-z_0)^{\lambda_j}f_{j,1}\\
&\vdots\\
y_{j,k}&=(z-z_0)^{\lambda_j}\sum_{l=0}^{k}\frac{1}{l!}f_{j,k-l}\log^l(z-z_0)\\
&\vdots\\
y_{j,d_j-1}&=(z-z_0)^{\lambda_j}\sum_{l=0}^{d_j-1}\frac{1}{l!}f_{j,d_j-1-l}\log^l(z-z_0)
\end{aligned}$$

are linearly independent solutions. Where $f_{j,k}$ are holomorphic functions near to z_0, and satisfy $f_{j,0}(z_0)=1$ and $f_{j,k}(z_0)=0$ for $k\neq 0$. As in claim 1.3.6, it is easy to show that the monodromy action on $\log(z-z_0)$ of circling z_0 once in the counterclockwise direction is

$$\log(z-z_0)\to\log(z-z_0)+2\pi i,$$

just consider the Taylor series around to w_l given by $f_j(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} w_l^{-k}(z-w_l)^k + \log(w_l)$.

Hence the monodromy action on the solution $y_{j,k}$ is given by

$$(z-z_0)^{\lambda_j} \sum_{l=0}^{k} \frac{1}{l!} f_{j,k-l} \log^l(z-z_0) \to e^{2\pi i \lambda_j}(z-z_0)^{\lambda_j} \sum_{m=0}^{k} \frac{(2\pi i)^m}{m!} \sum_{l=0}^{k-m} \frac{1}{l!} f_{j,k-m-l} \log^l(z-z_0)$$

in other words,

$$y_{j,k} \to e^{2\pi i \lambda_j} \sum_{m=0}^{k} \frac{(2\pi i)^m}{m!} y_{j,k-m}.$$

Therefore, the monodromy matrix block associated with the solutions $\{y_{j,k}\}_{k=0}^{d_j-1}$ satisfies

$$\begin{pmatrix} y_{j,0} \\ y_{j,1} \\ y_{j,2} \\ \vdots \\ y_{j,k} \\ \vdots \\ y_{j,d_j-1} \end{pmatrix} \to e^{2\pi i \lambda_j} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 2\pi i & 1 & 0 & \dots & 0 & \dots & 0 \\ \frac{(2\pi i)}{2} & 2\pi i & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \frac{(2\pi i)^k}{k!} & \frac{(2\pi i)^{k-1}}{(k-1)!} & \frac{(2\pi i)^{k-2}}{(k-2)!} & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \frac{(2\pi i)^{d_j-1}}{(d_j-1)!} & \frac{(2\pi i)^{d_j-2}}{(d_j-2)!} & \frac{(2\pi i)^{d_j-3}}{(d_j-3)!} & \dots & \frac{(2\pi i)^{d_j-k-1}}{(d_j-k-1)!} & \dots & 1 \end{pmatrix} \begin{pmatrix} y_{j,0} \\ y_{j,1} \\ y_{j,2} \\ \vdots \\ y_{j,k} \\ \vdots \\ y_{j,d_j-1} \end{pmatrix}$$

If the solutions of the indicial equation at z_0 satisfy $\lambda_i - \lambda_j \in \mathbb{Z}$, there are many possibilities for the monodromy matrix associated to the basis $\{y_{j,k}\}$ [CYYE08], for this reason we do not consider this case.

Example 1.3.7. We compute the monodromy matrix for the differential equation 1.2. For $z_0 = 0$, we know that the solution of the indicial equation is $\lambda^4 = 0$. By using the basis $y_k(z) := y_{1,k}(z)/(2\pi i)^k$ with $k = 0, 1, 2, 3$ we have that the monodromy action is given by

$$\begin{pmatrix} y_4 \\ y_3 \\ y_2 \\ y_1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1/2 & 1/6 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_4 \\ y_3 \\ y_2 \\ y_1 \end{pmatrix}$$

The monodromy matrix in the previous example is in a local basis, in the next chapter we see a numerical method to compute the matrix of change of basis.

Frobenius Basis

Here we compute the solutions of the Picard-Fuchs equation of Calabi-Yau threefold $(\theta^4 - z(\theta + A)(\theta + 1 - A)(\theta + B)(\theta + 1 - B))y = 0$.

This equation is equivalent to

$$\left(z^3(z-1)\frac{d^4}{dz^4} + z^2(8z-6)\frac{d^3}{dz^3} + z(z(C+14)-7)\frac{d^2}{dz^2} + (2z(C+2)-1)\frac{d}{dz} + D\right)y = 0 \tag{1.3}$$

where $C := A - A^2 + B - B^2$ and $D := AB(A-1)(B-1)$.

Solution at $z_0 = 0$. Because $z_0 = 0$ is a regular singularity of the equation 1.3, we have that a solution can be written as

$$y = \sum_{k=0}^{\infty} d_k z^{k+s},$$

for some $s \in \mathbb{N}$ (e.g. [Inc27, Ch. 16]).

Then $y^{(j)} = \sum_{k=0}^{\infty} d_k(k+s)(k+s-1)+\ldots(k+s-j+1)z^{k-j}$. By replacing in 1.3, we get

$$\sum_{k=0}^{\infty} d_k z^k (k^4 + k^3(4s+2) + k^2(6s^2+6s+C+1) + k(4s^3+6s^2+2(C+1)s+C)$$
$$+ s^4 + 2s^3 + (C+1)s^2 + Cs + D) - \sum_{k=0}^{\infty} d_k z^{k-1}(k+s)^4 = 0,$$

that is equivalent to

$$\sum_{k=1}^{\infty} d_{k-1} z^{k-1}((k-1)^4 + (k-1)^3(4s+2) + (k-1)^2(6s^2+6s+C+1)$$
$$+ (k-1)(4s^3+6s^2+2(C+1)s+C) + s^4 + 2s^3 + (C+1)s^2 + Cs + D) = \sum_{k=0}^{\infty} d_k z^{k-1}(k+s)^4.$$

We can consider d_k as a function of the parameter s. Thus, the recursion form is

$$d_k(s) = d_{k-1}(s)\frac{k^4 + k^3(4s-2) + k^2(6s^2-6s+C+1) + k(4s^3-6s^2+2(C+1)s-C) + s^4 - 2s^3 + (C+1)s^2 - Cs + D}{(k+s)^4}$$

for $k > 0$ and $d_0(s) \neq 0$.

Since the indicial equation is $(\lambda)^4 = 0$, the solutions for 1.3 at $z_0 = 0$ are

$$y_1 = \lim_{s\to 0} \sum_{k_0}^{\infty} d_k(s) z^{k+s} = f_{1,0}$$
$$y_2 = \lim_{s\to 0} \frac{d}{ds}\left(\sum_{k_0}^{\infty} d_k(s) z^{k+s}\right) = f_{1,0}\log(z) + f_{1,1}$$
$$y_3 = \frac{1}{2!}\lim_{s\to 0} \frac{d^2}{ds^2}\left(\sum_{k_0}^{\infty} d_k(s) z^{k+s}\right) = \frac{1}{2} f_{1,0}\log^2(z) + f_{1,1}\log(z) + f_{1,2}$$
$$y_4 = \frac{1}{3!}\lim_{s\to 0} \frac{d^3}{ds^3}\left(\sum_{k_0}^{\infty} d_k(s) z^{k+s}\right) = \frac{1}{6} f_{1,0}\log^3(z) + \frac{1}{2} f_{1,1}\log^2(z) + f_{1,2}\log(z) + f_{1,3}$$

Where

$$f_{1,0}(z) = \sum_{k=0}^{\infty} d_k z^k \text{ with } d_0 = 1\ ,\ d_k := d_k(0) = d_{k-1}\frac{Ck(k-1) + (k-1)^2k^2 + D}{k^4} \text{ for } k > 0$$
$$f_{1,1}(z) = \sum_{k=0}^{\infty}\left(\left.\frac{d(d_k(s))}{ds}\right|_{s=0}\right) z^k$$
$$f_{1,2}(z) = \frac{1}{2}\sum_{k=0}^{\infty}\left(\left.\frac{d^2(d_k(s))}{ds^2}\right|_{s=0}\right) z^k$$
$$f_{1,3}(z) = \frac{1}{6}\sum_{k=0}^{\infty}\left(\left.\frac{d^3(d_k(s))}{ds^3}\right|_{s=0}\right) z^k.$$

Example 1.3.8. For $A = \frac{1}{5}$, $B = \frac{2}{5}$, we have $C = \frac{2}{5}$ and $D = \frac{24}{625}$, hence the holomorphic functions which appear in the solutions at $z_0 = 0$ look like

$$\begin{aligned}
f_{10} &= 1 + \frac{24}{625}z + \frac{60}{5167}z^2 + \frac{308}{55893}z^3 + \frac{55}{17167}z^4 + \frac{11}{5259}z^5 + \mathcal{O}(z^6),\\
f_{11} &= \frac{154}{625}z + \frac{151}{1820}z^2 + \frac{251}{6135}z^3 + \frac{95}{3916}z^4 + \frac{157}{9795}z^5 + \mathcal{O}(z^6),\\
f_{12} &= \frac{23}{125}z + \frac{269}{2497}z^2 + \frac{142}{2299}z^3 + \frac{453}{11519}z^4 + \frac{88}{3249}z^5 + \mathcal{O}(z^6),\\
f_{13} &= -\frac{46}{125}z - \frac{312}{3695}z^2 - \frac{429}{15151}z^3 - \frac{73}{6163}z^4 - \frac{29}{5149}z^5 + \mathcal{O}(z^6).
\end{aligned}$$

Solutions at $z_0 = 1$. Since $z_0 = 1$ is a regular singularity of the equation 1.3, a solution can be written as

$$y = \sum_{k=0}^{\infty} d_k (z-1)^{k+s}.$$

In this case the indicial equation is $\lambda(\lambda-1)^2(\lambda-2)$, hence, it is necessary to consider solutions for roots that differ by integers (see [Inc27, Ch. 16]). The recursive form is

$$d_k(s) = \frac{d_{k-3}(s)K_3(s) + d_{k-2}(s)K_2(s) + d_{k-1}(s)K_1(s)}{K_0(s)}$$

where

$$K_0(s) = -(k+s)(k+s-1)^2(k+s-2)$$

$$K_1(s) = 3(k+s-1)(k+s-2)(k+s-3)(k+s-4) + 12(k+s-1)(k+s-2)(k+s-3) + (C+7)(k+s-1)(k+s-2)$$

$$K_2(s) = 3(k+s-2)(k+s-3)(k+s-4)(k+s-5) + 18(k+s-2)(k+s-3)(k+s-4) + (2C+21)(k+s-2)(k+s-3) + (2C+3)(k+s-2)$$

$$K_3(s) = (k+s-3)(k+s-4)(k+s-5)(k+s-6) + 8(k+s-3)(k+s-4)(k+s-5) + (C+14)(k+s-3)(k+s-4) + (2C+4)(k+s-3) + D.$$

Example 1.3.9. Using *formal_ sol* from the software Maple, we compute formal solutions for $A = \frac{1}{5}$, $B = \frac{2}{5}$, we have $C = \frac{2}{5}$ and $D = \frac{24}{625}$. These solutions are,

$$\begin{aligned}
\tilde{y}_1 &= -\frac{1}{2} + \frac{37}{60} + (z-1) - \frac{2309}{3600}(z-1)^2 + \frac{322579}{540000}(z-1)^3 - \frac{17855219}{32400000}(z-1)^4 + \frac{413505701}{810000000}(z-1)^5 + \mathcal{O}((z-1)^6),\\
\tilde{y}_2 &= \frac{23}{360}(z-1)^3 - \frac{6397}{60000}(z-1)^4 + \frac{333323}{2500000}(z-1)^5 + \mathcal{O}((z-1)^6) + \log(z-1)\left(\tilde{y}_3 - \frac{8}{15}\tilde{y}_4\right),\\
\tilde{y}_3 &= -(z-1) + \frac{37}{30}(z-1)^2 - \frac{271}{225}(z-1)^3 + \frac{76771}{67500}(z-1)^4 - \frac{450874}{421875}(z-1)^5 + \frac{31832413}{31640625}(z-1)^6 + \mathcal{O}((z-1)^7),\\
\tilde{y}_4 &= (z-1)^2 - \frac{37}{30}(z-1)^3 + \frac{2309}{1800}(z-1)^4 - \frac{286471}{225000}(z-1)^5 + \frac{41932661}{33750000}(z-1)^6 + \mathcal{O}((z-1)^7).
\end{aligned}$$

For these solutions $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$, the monodromy matrix is given by

$$\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\\ \tilde{y}_3\\ \tilde{y}_4\end{pmatrix} \to \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 2\pi\sqrt{-1} & -2\pi\sqrt{-1}\frac{8}{15}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\\ \tilde{y}_3\\ \tilde{y}_4\end{pmatrix}$$

Following the approach in [CYYE08], we can compute numerically a matrix of change of basis from $(\tilde{y}_k)_{k=1}^4$ to $(y_{5-k})_{k=1}^4$. The idea is to consider truncated series from Examples 1.3.8

and 1.3.9, and evaluate them at a common point of the convergence domain. In fact, if A is a matrix of change if basis, then it satisfies

$$\begin{pmatrix} y_4 & y_4' & y_4^{(2)} & y_4^{(3)} \\ y_3 & y_3' & y_3^{(2)} & y_3^{(3)} \\ y_2 & y_2' & y_2^{(2)} & y_2^{(3)} \\ y_1 & y_1' & y_1^{(2)} & y_1^{(3)} \end{pmatrix} = A \begin{pmatrix} \tilde{y}_1 & \tilde{y}_1' & \tilde{y}_1^{(2)} & \tilde{y}_1^{(3)} \\ \tilde{y}_2 & \tilde{y}_2' & \tilde{y}_2^{(2)} & \tilde{y}_2^{(3)} \\ \tilde{y}_3 & \tilde{y}_3' & \tilde{y}_3^{(2)} & \tilde{y}_3^{(3)} \\ \tilde{y}_4 & \tilde{y}_4' & \tilde{y}_4^{(2)} & \tilde{y}_4^{(3)} \end{pmatrix}.$$

Therefore, we can also consider the truncated series for the derivatives and evaluate them in $z = 1/2$. In order to compute the matrix A, we solve the previous system for this numerical approximation. For 60th order in the truncated series, we have

$$A = \begin{pmatrix} \frac{-2624}{21}i & \frac{1243}{67} & \frac{757}{148}i & \frac{-128}{359} \\ \frac{-1280}{29} - \frac{4891}{21}i & -\frac{3235}{116} - \frac{4692}{83}i & \frac{374}{21} + \frac{423}{130}i & -\frac{28}{163} + \frac{203}{445}i \\ \frac{5947}{20}i & -\frac{1427}{37} & -\frac{1597}{169}i & -\frac{325}{842} \\ \frac{8499}{16}i & -\frac{1273}{33} & -\frac{1354}{191}i & -\frac{248}{249}. \end{pmatrix}$$

Thus, the numerical monodromy matrix around $z_0 = 1$ in the basis $(y_{5-k})_{k=1}^4$ is

$$\begin{pmatrix} 1 - \frac{37}{716}i & \frac{64}{967} & \frac{25}{1199}i & -\frac{5}{9362} \\ \frac{462}{229} & 1 + \frac{385}{149}i & -\frac{232}{285} & -\frac{25}{1199}i \\ \frac{1031}{161}i & -\frac{1017}{124} & 1 - \frac{385}{149}i & \frac{64}{967} \\ -5 & -\frac{1031}{161}i & \frac{462}{229} & 1 + \frac{37}{716}i \end{pmatrix}.$$

The monodromy matrix around to $z_0 = 1$ in [CYYE08] is

$$\begin{pmatrix} 1+a & 0 & \frac{ab}{d} & \frac{a^2}{d} \\ -b & 1 & -\frac{b^2}{d} & -\frac{ab}{d} \\ 0 & 0 & 1 & 0 \\ -d & 0 & -b & 1-a \end{pmatrix}$$

where a, b, d are topological invariants associated to the mirror quintic Calabi-Yau threefold. Hence, the numerical computation suggests that $a = \frac{-37}{716}i$, $b = -\frac{462}{229}$, $d = 5$. Finally, we remark that by conjugating with the matrix

$$\begin{pmatrix} -\frac{b}{d} & -\frac{a}{d} & 0 & -\frac{1}{d} \\ -\frac{1}{2} & \frac{b}{d} & \frac{1}{d} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

the monodromy matrices in [DM06] are obtained.

1.4 Classic Example of Monodromy action: Elliptic curve on $\mathbb{C}$

For $t = (t_2, t_3) \in \mathbb{C}^2 \setminus \{\Delta = 0\}$ where $\Delta := t_2^3 - 27t_3^2$, consider the family of elliptic curves E_t on $\mathbb{C}$ given by

$$E_t := \{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - t_2 x - t_3\}.$$

Let $E = E_b$ be a fixed elliptic curve on $\mathbb{C}$ with $b \in \mathbb{C}^2 \setminus \Delta = 0$. The curve E can be holomorphically embedded in $\mathbb{P}^2$, thus the pullback of the Fubini-Study form via the inclusion map gives a symplectic structure (E, ω) . Since E is a curve of genus 1 minus one point, $H_1(E, \mathbb{Z})$ is a free group generated by two elements i.e. $H_1(E, \mathbb{Z}) = \mathbb{Z}\delta_1 + \mathbb{Z}\delta_2$. The set of automorphism$H_1(E, \mathbb{Z})$ which preserve the intersection map $\langle , \rangle$ is $SL(2, \mathbb{Z})$.

For a fixed $t_2 \neq 0$, and t_3 varying in $\mathbb{C}$, the points where Δ vanishes are $c_1 = \sqrt{\frac{t_2^3}{27}}$ and $c_2 = -\sqrt{\frac{t_2^3}{27}}$. Therefore, the function $F : \mathbb{C}^2 \to \mathbb{C} \setminus \{c_1, c_2\}$ given by $F = y^2 - 4x^3 + t_2 x$ is a Lefschetz fibration. We will compute the monodromy [Mov17a]

$$h : \pi_1(\mathbb{C} \setminus \{c_1, c_2\}, b) \to \operatorname{Aut}(H_1(E, \mathbb{Z}), \langle , \rangle)$$

We suppose that the cycle δ_i is associated to the critical value c_i for $i = 1, 2$. By using the Picard-Lefschetz formula, we get for a loop around of c_1

$$h(\delta_1) = \delta_1 + \langle \delta_1, \delta_1 \rangle \delta_1, \quad h(\delta_2) = \delta_2 + \langle \delta_2, \delta_1 \rangle \delta_1$$

and for a loop around to c_2

$$h(\delta_1) = \delta_1 + \langle \delta_1, \delta_2 \rangle \delta_2, \quad h(\delta_2) = \delta_2 + \langle \delta_2, \delta_2 \rangle \delta_2$$

It is possible to choose δ_1 and δ_2 such that $\langle \delta_1, \delta_2 \rangle = 1$, so since the intersection map is skew-symmetric for n even, then we write the previous equations as

$$h\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \text{ and } h\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}.$$

Set

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Proposition 1.4.1. *$SL(2, \mathbb{Z})$ is generated by the matrix A and B. Moreover $SL(2, Z)$ is splitting as $\mathbb{Z} * \mathbb{Z}_3$.*

Proof. Let $l = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, and consider $C = A^{-1}B^{-1}A^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Note that

$$Cl = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \text{ and } B^k l = \begin{pmatrix} a + kc & b + kd \\ c & d \end{pmatrix}.$$

We claim that there is a sequence $D = CB^{n_s} \cdots CB^{n_1}$, such that $Dl = \begin{pmatrix} \lambda & m \\ 0 & \lambda \end{pmatrix}$, where $\lambda = \pm 1$, and $m \in \mathbb{Z}$. If $c \neq 0$, then we can suppose that $|c| < |a|$ (in other case consider Cl instead l), thus it is possible to write $a = cn_1 + r_1$ where $0 < |r_1| < |c|$. Hence, we have

$$B^{-n_1} l = \begin{pmatrix} r_1 & \lambda_1 \\ c & d \end{pmatrix},$$

where $\lambda_1 \in \mathbb{Z}$. By doing induction, we can compute until to obtain $r_s = 0$. Because this product of matrices is in $SL(2,\mathbb{Z})$, we conclude that $\lambda_s = \pm 1$. Therefore $Dl = B^m$ or $Dl = -B^{-m}$. Since $D^{-1} \in \operatorname{span}\{A, B\}$, we conclude that l is in the group generated by A and B.

By using the ping-pong lemma , it is possible to show that $SL(2,\mathbb{Z}) = \mathbb{Z} * \mathbb{Z}_3$. As in [BT14], by setting $R = (AB)^2$ and $T = B^3$. □

Consequently the monodromy h in this case is a surjective map.

Proposition 1.4.2. *If $\delta \in H_1(E,\mathbb{Z})$ is a primitive cycle, then is possible to complete to a base $\delta_1, \delta_2 \in H_1(E,\mathbb{Z})$, such that $\langle \delta_1, \delta_2 \rangle = 1$.*

Proof. Let d_1, d_2 be a basis for $H_1(E,\mathbb{Z})$ such that $\langle d_1, d_2 \rangle = 1$. Since δ is primitive, then we can write $\delta = ad_1 + bd_2$ where $a, b \in \mathbb{Z}$ and $g.c.d(a,b) = 1$. By Bézout's identity it is possible to find $-c, d$ such that $ad - cb = 1$.

Consider the matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, and the cycles $\delta_1 = \delta = ad_1 + bd_2$ and $\delta_2 = cd_1 + dd_2$; then

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = P \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

is the desired base. □

In dimension 2 the question about which part of the homology is supported in Lagrangian submanifolds is trivial, because any 1-dimension submanifold in a symplectic manifold of dimension 2 is Lagrangian. However we can refine the question to which cycles in homology are the Lagrangian spheres given by the vanishing cycles.

Theorem 1.4.3. *Let $\delta \in H_1(E,\mathbb{Z})$ be a primitive cycle, then δ is supported in a vanishing cycle.*

Proof. Let L_1 and L_2 be vanishing cycles of E. Let d_1, d_2 be its homological classes. We can suppose that $\langle d_1, d_2 \rangle = 1$ (By normalization of the intersection map).

By the proposition 1.4.2 it is possible to write $\begin{pmatrix} \delta \\ * \end{pmatrix} = P \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ for $P \in \operatorname{Aut}(H_1(E,\mathbb{Z}))$, and since the monodromy map $h : \pi_1(\mathbb{C} \smallsetminus \{c_1, c_2\}, b) \to \operatorname{Aut}(H_1(E,\mathbb{Z}), \langle , \rangle)$ is surjective, we can write $\delta = h(\gamma) d_1$ for some path $\gamma \in \mathbb{C} \smallsetminus \{c_1, c_2\}$. Thus δ is also a vanishing cycle. □

Chapter 2

Homology supported in Lagrangian submanifolds in quintic Calabi-Yau threefold

2.1 Introduction

Given a symplectic manifold (X, ω) of dimension $2n$ there are homology classes in $H_n(X, \mathbb{Q})$ which may be represented by Lagrangian cycles. In [SW01], the authors define Lagrangian cycles as cycles in a symplectic 4-manifold, whose two-simplices are given by C^1 Lagrangian maps and a *Lagrangian homology class* is a homology class which can be represented by a Lagrangian cycle. In that article, they show a characterization of the Lagrangian homology classes in terms of the minimizers of an area functional. Moreover, they show for a compact Kälher 4-manifold (X, ω, J) and a homology class $\alpha \in H_2(X, \mathbb{Z})$, that α is a Lagrangian homology class if and only if $[\omega](\alpha) = 0$. If the Chern class $c_1(X)$ also annihilates α, then α can be represented by an immersed Lagrangian surface (not necessarily embedded).

The question about which part of the homology is supported in Lagrangian submanifolds, can be refined a little more if we look for Lagrangian spheres. In [LW12] for a (X, ω, J) Kälher 4-manifold with Kodaira dimension $-\infty$, i.e. for rational or ruled surfaces it is shown that the class $\alpha \in H_2(X, \mathbb{Z})$ is represented by a Lagrangian sphere if and only if $[\omega](\alpha) = 0$, $c_1(X)(\alpha) = 0$, $\alpha^2 = -2$ and α is represented by a smooth sphere. For 4-manifolds, the dimension of the 2-cycles allows us to relate the property of being represented by Lagrangian cycles with the vanishing of the *periods* $\int_\alpha \omega$ and $\int_\alpha c_1(X)$. For higher dimension manifolds this pairing is not well-defined, hence we do not have a natural generalization of the previous results. Despite of this, it is possible to show that in any regular hypersurface of $\mathbb{P}^n$ with n even, all $(n-1)$-cycles can be written as a linear combination of cycles supported in Lagrangian spheres, see Proposition 1.2.9.

A more interesting question for $n = 4$, is to ask not only which homology classes are generated by Lagrangian spheres but which ones are supported in Lagrangian spheres. In this article we consider a family $\tilde{X}_\varphi$ of mirror quintic Calabi-Yau threefolds and study some classes in $H_3(\tilde{X}_\varphi, \mathbb{Z})$ which are supported in Lagrangian 3-spheres and Lagrangian 3-tori. This family is constructed as follows. Consider the Dwork family X_φ in $\mathbb{P}^4$ given by the locus of the polynomial

$$p_\varphi := \varphi z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4 = 0,$$

with critical values in $\varphi = 0, 1, \infty$. For every $\varphi \neq 0, 1, \infty$, $\tilde{X}_\varphi$ is obtained as a desingularization of the quotient of X_φ by the action of a finite group, see §2.2 and [CdlOGP91, CYYE08, DM06]. The rank of the free group $H_3(\tilde{X}_\varphi,\mathbb{Z})$ is four and hence it is isomorphic to $\mathbb{Z}^4$ after choosing a basis. In this basis the homology class $\delta_2 = (0\ 1\ 0\ 0)$ is represented by a torus associated to the singularity of X_φ when $\varphi \to 0$ and the class $\delta_4 = (0\ 0\ 0\ 1)$ is represented by a sphere S^3 associated to the singularity of X_φ when $\varphi \to 1$. As in [CdlOGP91] we give an explicit description of these two cycles in §2.3, and furthermore we show that these cycles are Lagrangian submanifolds of $\tilde{X}_\varphi$.

The monodromy action of the family is given by symplectomorphisms at each regular fiber. It is possible to determine two matrices M_0 and M_1 such that the monodromy action over $H_3(\tilde{X}_\varphi,\mathbb{Z})$ corresponds (with respect to the basis mentioned above) to the free subgroup of $Sp(4,\mathbb{Z})$ generated by M_0 and M_1, see §2.2. Therefore, the orbit of δ_2 and δ_4 by the action of $M_0 * M_1$ are homology classes which can be represented by Lagrangian submanifolds. Our main result is about $H_3(\tilde{X}_\varphi,\mathbb{Z}_p)$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for some primes p, and it is summarized in the following theorem.

Theorem 2.1.1. *For the mirror quintic Calabi-Yau threefold $\tilde{X} := \tilde{X}_\varphi$ with $\varphi \neq 0,1,\infty$, the homology classes*

$$(0\ 0\ 1\ 1),\ (0\ 1\ 0\ 0),\ (0\ 1\ 0\ 1),\ (1\ 0\ 0\ 1),\ (1\ 0\ 1\ 1) \in H_3(\tilde{X},\mathbb{Z}_2) \tag{2.1}$$

$$\begin{gathered}(0\ 1\ 0\ 0),(0\ 1\ 0\ 1),(0\ 1\ 0\ 2),(0\ 1\ 0\ 3),(0\ 1\ 0\ 4),\\ (0\ 1\ 1\ 0),(0\ 1\ 1\ 1),(0\ 1\ 1\ 2),(0\ 1\ 1\ 3),(0\ 1\ 1\ 4),\\ (0\ 1\ 2\ 0),(0\ 1\ 2\ 1),(0\ 1\ 2\ 2),(0\ 1\ 2\ 3),(0\ 1\ 2\ 4),\\ (0\ 1\ 3\ 0),(0\ 1\ 3\ 1),(0\ 1\ 3\ 2),(0\ 1\ 3\ 3),(0\ 1\ 3\ 4),\\ (0\ 1\ 4\ 0),(0\ 1\ 4\ 1),(0\ 1\ 4\ 2),(0\ 1\ 4\ 3),(0\ 1\ 4\ 4) \in H_3(\tilde{X},\mathbb{Z}_5)\end{gathered} \tag{2.2}$$

are represented by Lagrangian 3-tori. The homology classes

$$\begin{gathered}(0\ 0\ 0\ 1),\ (0\ 0\ 1\ 0),(0\ 1\ 1\ 0),(0\ 1\ 1\ 1),(1\ 0\ 0\ 0),\\ (1\ 0\ 1\ 0),(1\ 1\ 0\ 0),(1\ 1\ 0\ 1),(1\ 1\ 1\ 0),(1\ 1\ 1\ 1) \in H_3(\tilde{X},\mathbb{Z}_2)\end{gathered} \tag{2.3}$$

$$(0\ 0\ 0\ 1),\ (0\ 0\ 1\ 1),\ (0\ 0\ 2\ 1),\ (0\ 0\ 3\ 1),\ (0\ 0\ 4\ 1)\ \in H_3(\tilde{X},\mathbb{Z}_5) \tag{2.4}$$

are represented by Lagrangian 3-spheres. For $p = 3,7,11,13,17,19,23$, any homology class in $H_3(\tilde{X},\mathbb{Z}_p)$ different from $(0\ 0\ 0\ 0)$ can be represented by Lagrangian 3-tori and by Lagrangian 3-spheres.

In general for a manifold M, a class $\delta \in H_k(M,\mathbb{Z})$ is called *primitive* if there is no $m \in \mathbb{Z}$ and $\delta' \in H_k(M,\mathbb{Z})$ such that $\delta = m\delta'$. We believe that for any prime different to 2 and 5, all classes in $H_3(\tilde{X},\mathbb{Z}_p)$ different to $(0\ 0\ 0\ 0)$ can be represented by Lagrangian 3-tori and by a Lagrangian 3-spheres. This is a consequence of the following conjecture.

Conjecture 2.1.2. *Let δ be a primitive class in $H_3(\tilde{X},\mathbb{Z})$. If $mod_2(\delta)$ is a homology class in the list (2.1) and $mod_5(\delta)$ is a homology class in the list (2.2), then δ is represented by a Lagrangian 3-torus. If $mod_2(\delta)$ is a homology class in the list (2.3) and $mod_5(\delta)$ is a homology class in the list (2.4), then δ is represented by a Lagrangian 3-sphere.*

We have analogous results for other 14 examples of Calabi-Yau threefolds which appear in Table 2.1. However, in these cases we do not know if the vectors δ_2 and δ_4 have Lagrangian submanifolds associated as in the Dwork family case.

2.2 Monodromy action on mirror quintic threefolds

In this section we recall the definition of a mirror quintic Calabi-Yau threefold and its monodromy action coming from the Picard-Fuchs equations. We also list the monodromy action of other 14 examples of Calabi-Yau threefolds. For a more detailed description, the reader is referred to [CdlOGP91, DM06, Mov17b, NU12].

The family of hypersurfaces in $\mathbb{P}^4$ given by a generic polynomial of degree 5 is denoted $\mathbb{P}^4[5]$. The elements of $\mathbb{P}^4[5]$ are quintic Calabi-Yau threefolds, with Hodge numbers $h^{1,1} = 1$ and $h^{2,1} = 101$. Let $\{X_\varphi\}_\varphi$ be the one-parameter family of hypersurfaces in $\mathbb{P}^4$ given by

$$p_\varphi = \varphi z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4, \quad \varphi \neq 0, 1. \tag{2.5}$$

Consider the finite group

$$G = \{(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{C}^5 \mid \xi_i^5 = 1 \ , \ \xi_0\xi_1\xi_2\xi_3\xi_4 = 1\}$$

acting on $\mathbb{P}^4$, as $(\xi_0, \xi_1, ..., \xi_4) \cdot [z_0 : z_1 : \cdots : z_4] = [\xi_0 z_0 : \cdots : \xi_4 z_4]$. It is known that the action of G is free away from the curves $C_{ijk} := \{z_i^5 + z_j^5 + z_k^5 = 0,\ z_l = 0 \text{ for all } l \neq i,j,k\}$ for $0 \leq i < j < k \leq 4$ (see [Mor93]). The *mirror quintic Calabi-Yau threefold*, mirror quintic for short, is the variety $\tilde{X}_\varphi$ obtained after resolving the orbifolds singularities of the quotient X_φ/G. The manifold $\tilde{X}_\varphi$, has Hodge numbers $h^{1,1} = 101$ and $h^{2,1} = 1$ and Betti number $b_3 = 4$. In terms of the mirror symmetry $\mathbb{P}^4[5]$ is called the *A-model* and $\{\tilde{X}_\varphi\}_\varphi$ the *B-model* (see for example [Kon95]).

The variety $\tilde{X}_\varphi$ has a holomorphic 3 form η that vanishes nowhere. Moreover, $H^{3,0}$ is spanned by η. The *periods* of η are functions $\int_\Delta \eta$, where the homology class $\delta = [\Delta] \in H_3(\tilde{X}_\varphi, \mathbb{Z})$ is supported in the submanifold Δ. The fourth-order linear differential equation

$$\left(\theta^4 - \varphi\left(\theta + \frac{1}{5}\right)\left(\theta + \frac{2}{5}\right)\left(\theta + \frac{3}{5}\right)\left(\theta + \frac{4}{5}\right)\right)y = 0, \quad \theta = \varphi\frac{\partial}{\partial\varphi}$$

is called Picard-Fuchs equations, and its solutions are the periods of η.

The Picard-Fuchs ODE has 3 regular singular points $\varphi = 0, 1, \infty$. The analytic continuation of this ODE, gives us the monodromy operators M_0, M_1, M_∞. Since the monodromy is a representation $\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to Sp(4)$, we have the relation $M_0M_1M_\infty = Id$. There exits a basis such that the monodromy operators in this basis are written as

$$M_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

see for example [BT14, CYYE08, DM06].

The matrices M_0 and M_1 are conjugated to the matrices

$$T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

appearing in [CYYE08, DM06], via the matrix $P = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 5 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. Thus $P^{-1}T_iP = M_i$, $i = 0,1$.

In [CdlOGP91], the matrices for the monodromy are

$$S_\infty = \begin{pmatrix} 51 & 90 & -25 & 0 \\ 0 & 1 & 0 & 0 \\ 100 & 175 & -49 & 0 \\ -75 & -125 & 35 & 1 \end{pmatrix} \text{ and } S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

these matrices are associated to the equation

$$p_\psi = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4,$$

with singularities at $\psi^5 = 1, \infty$. The change of variable $\psi = \varphi^{\frac{-1}{5}}$ gives us the family defined by the equation (2.5). Moreover, the matrix M_0^5 is conjugated to S_∞. In fact with the matrix

$$M = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.6}$$

we obtain the equations $M^{-1}S_1M = M_1$ and $M^{-1}S_\infty M = M_0^5$.

More generally, it is known that the differential equation

$$(\theta^4 - \varphi(\theta + A)(\theta + 1 - A)(\theta + B)(\theta + 1 - B))y = 0, \quad \theta = \varphi\frac{\partial}{\partial\varphi} \tag{2.7}$$

corresponds to the Picard-Fuchs equation of a mirror Calabi-Yau threefold for 14 values of (A, B), and the singularities are in $\varphi = 0, 1, \infty$. We have listed the *A-model* of these 14 examples in Table 2.1.

(d,k)	A	B	A-model of equation (2.7)
$(5,5)$	1/5	2/5	$X(5) \subset \mathbb{P}^4$
$(2,4)$	1/8	3/8	$X(8) \subset \mathbb{P}^4(1,1,1,1,4)$
$(1,4)$	1/12	5/12	$X(2,12) \subset \mathbb{P}^5(1,1,1,1,4,6)$
$(16,8)$	1/2	1/2	$X(2,2,2,2) \subset \mathbb{P}^7$
$(12,7)$	1/3	1/2	$X(2,2,3) \subset \mathbb{P}^6$
$(8,6)$	1/4	1/2	$X(2,4) \subset \mathbb{P}^5$
$(4,5)$	1/6	1/2	$X(2,6) \subset \mathbb{P}^5(1,1,1,1,1,3)$
$(2,3)$	1/4	1/3	$X(4,6) \subset \mathbb{P}^5(1,1,1,2,2,3)$
$(1,2)$	1/6	1/6	$X(6,6) \subset \mathbb{P}^5(1,1,2,2,3,3)$
$(6,5)$	1/6	1/4	$X(3,4) \subset \mathbb{P}^5(1,1,1,1,1,2)$
$(3,4)$	1/6	1/3	$X(6) \subset \mathbb{P}^4(1,1,1,1,2)$
$(1,3)$	1/10	3/10	$X(5) \subset \mathbb{P}^4(1,1,1,2,5)$
$(4,4)$	1/4	1/4	$X(4,4) \subset \mathbb{P}^5(1,1,1,1,2,2)$
$(9,6)$	1/3	1/3	$X(3,3) \subset \mathbb{P}^5$

Table 2.1: Fourteen values for equation 2.7 with the corresponding Calabi-Yau threefold.

The notation $X(d_1, d_2, \ldots, d_l) \subset \mathbb{P}^n(w_1, w_2, \ldots, w_n)$ denotes a complete intersection of l hypersurfaces of degrees $d_1, d_2, \ldots, d_l$ in the weighted projective space with weight $(w_1, w_2, \ldots, w_n)$, see for example [CYYE08]. For these cases the monodromy matrices correspond to the same M_1 as before and

$$M_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix}.$$

2.3 Lagrangian sphere and Lagrangian torus in mirror quintic threefold

In the basis of homology used in [CdlOGP91] there are two homology classes which are supported on Lagrangian submanifolds. We observe that one class is realized by a Lagrangian 3-sphere and the other by a Lagrangian 3-torus.

Consider the mirror quintic Calabi-Yau threefold $\tilde{X}_\psi$ associated to the equation

$$p_\psi = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4,$$

with singularities in $\psi = 1, \infty$. Let η be the holomorphic form on $\tilde{X}_\psi$. The basis of the matrices S_∞ and S_0 are the periods $\int_{\Delta_k} \eta$ with $k = 1, 2, 3, 4$. The cycle Δ_2 is a torus associated with the degeneration of the manifold as ψ goes to ∞, see [CdlOGP91, §3]. In coordinates it can be described as

$$\Delta_2 = \{[1 : z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^4 \mid |z_1| = |z_2| = |z_3| = r \text{ and } z_4 \text{ given by } p_\psi = 0 \text{ when } \psi \to \infty\} \tag{2.8}$$

for $r > 0$ small enough, and z_4 is defined as the branch of the solution $p_\psi(z) = 0$ which tends to zero as $\psi \to \infty$. The cycle Δ_2 does not intersect the curves C_{ijk}, and so its quotient by the group G is again a torus.

Proposition 2.3.1. *The cycle Δ_2 is a Lagrangian submanifold of (X_ψ, ω), where ω is the symplectic form given by the pullback of the Fubini-Study form.*

Proof. Consider the Hamiltonian S^1-space $(\mathbb{C}^5 \setminus \{0\}, \omega_{can}, S^1, \mu)$, where $\mu(z) = \frac{-\|z\|^2+1}{2}$. By Marsden-Weinstein-Meyer theorem, there exist a symplectic form in the reduction $\mu^{-1}(0)/S^1 = \mathbb{P}^4$, and in this case it corresponds to the Fubini-Study form ω_{FS} (see the Example 1.1.8). Furthermore if we denote the reduction by

$$\begin{tikzcd} \mu^{-1}(0) = S^9 \arrow[r, "\imath"] \arrow[d, "\pi"'] & \mathbb{C}^5 \setminus \{0\} \arrow[dl, "pr"] \\ \mathbb{P}^4 & \end{tikzcd}$$

the reduced form satisfies $\pi^* \omega_{FS} = \imath^* \omega_{can}$. The canonical form can be written as $\omega_{can} = \frac{1}{2} \sum_j d|z_j|^2 \wedge d\theta_j$. Therefore, for $\epsilon > 0$ small enough, the set

$$T := \{(z_0, z_1, z_2, z_3, z_4) \in \mathbb{C}^5 \mid |z_0| = \epsilon,\ |z_1| = |z_2| = |z_3| = r,\ |z_4|^2 = 1 - \epsilon^2 - 3r^2\} \subset S^9,$$

is a Lagrangian submanifold of $(\mathbb{C}^5, \omega_{can})$. Besides, Δ_2 is the intersection of X_ψ with the projection of T to $\mathbb{P}^4$. Consequently, the tangent space of Δ_2 is contained in the tangent space of $\pi(T)$. Since $0 = (\pi^* \omega_{FS})|_T = (\omega_{FS})|_{\pi(T)}$, we conclude that $(\omega_{FS})|_{\Delta_2} = 0$. $\square$

The cycle Δ_4 is associated with the degeneration of the manifold when ψ goes to 1 [CdlOGP91, §3]. In coordinates can be described as

$$\Delta_4 = \{[1 : z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^4 \mid z_1, z_2, z_3 \text{ reals and } z_4 \text{ given by } p_\psi = 0 \text{ when } \psi \to 1\} \tag{2.9}$$

where z_4 is defined as the branch of of the solution of $p_\psi(z) = 0$ which is an S^3 when $\psi \to 1$. Follows from the next proposition that Δ_4 is an Lagrangian sphere S^3.

Proposition 2.3.2. *The cycle Δ_4 is a vanishing cycle.*

Proof. In the chart $z_0 = 1$ consider the function $f : \mathbb{C}^4 \to \mathbb{C}$ given by $f(z_1, \dots, z_4) = p_\psi(1, z_1, \dots, z_4)$. The critical points of f, $(\xi^{k_1}\psi, \xi^{k_2}\psi, \xi^{k_3}\psi, \xi^{k_4}\psi)$ where $\xi = e^{\frac{2\pi i}{5}}$, $k_i = 1, \dots, 5$ and $5 | \sum_{j=1}^4 k_j$ are non degenerated. After doing the quotient by the finite group G, these critical points are identified with $(\psi, \dots, \psi)$.

For real $\psi > 1$ close enough to 1 and by taking $z_j = x_j + iy_j$ we have that the map can be locally defined as

$$f(z_1, ..., z_4) = (1 - \psi^5) + \sum x_j^2 - \sum y_j^2 + 2i \prod x_j y_j,$$

and so the vanishing cycle δ_γ in Proposition 1.2.6 is the sphere $(\psi^5 - 1) = \sum x_j^2$. $\square$

Let $\delta_1, \delta_2, \delta_3, \delta_4$ be the basis on which the matrices S_1 and S_∞ are written. Consider the isomorphism between $span\{\delta_i\}_{i=1}^4$ and $\mathbb{R}^4$ with the canonical basis, given by $\sum_{i=1}^4 n_i \delta_i \to (n_1, n_2, n_3, n_4)$. Thus, the monodromy acting on a vector $\delta = \sum_{i=1}^4 n_i \delta_i$ corresponds to

$$S_j(\delta) = (n_1\ n_2\ n_3\ n_4) S_j \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} \quad \text{with } j = 1, \infty.$$

From [CdlOGP91] and Picard-Lefschetz formula, we know that the monodromy matrices satisfy $S_\infty \delta_2 = \delta_2$ and $S_1 \delta_2 = \delta_2 + \delta_4$. Therefore $[\Delta_2] = \delta_2 \equiv [0\ 1\ 0\ 0]$ and $[\Delta_4] = \delta_4 \equiv [0\ 0\ 0\ 1]$.

2.4 Orbits for δ_2 and δ_4

Let H be the subgroup of $Sp(4)$, generated by M_0 and M_1. Moreover, the vectors $\delta_2 = (0\ 1\ 0\ 0)$ and $\delta_4 = (0\ 0\ 0\ 1)$ are invariants by the change of basis M defined in (2.6). In this section we compute the orbit of δ_2 and δ_4 by the action of H in $\mathbb{Z}_p$, for some prime numbers p.

For the mirror quintic $\tilde{X}$, any element in $H_3(\tilde{X}, \mathbb{Z})$ which is in the orbit $H \cdot \delta_4$ is a homology class supported in a Lagrangian 3-sphere, and any element in the orbit $H \cdot \delta_2$ a homology class supported in a Lagrangian 3-torus. So far we have not computed the orbits in $\mathbb{Z}$. However, considering $H_3(\tilde{X}, \mathbb{Z}_p)$ for some primes p, it is possible to compute the orbits. The next lemma helps us to reduce the possible words appearing in H mod $p\mathbb{Z}$.

Lemma 2.4.1.

$$\bmod_p(M_0^p) = Id_4,\ p \neq 2, 3,\ \ mod_2(M_0^4) = Id_4, mod_3(M_0^9) = Id_4,$$
$$\bmod_p(M_1^p) = Id_4 \quad \forall\ prime\ p.$$

Proof. Computing the power of theses matrix, we have

$$M_0^m = \begin{pmatrix} 1 & m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ dm & a_m & 1 & 0 \\ b_m & c_m & -m & 1 \end{pmatrix} \text{ and } M_1^m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a_m = \frac{d}{2}m(m+1)$, $b_m = \frac{d}{2}m(1-m)$ and $c_m = \frac{d}{6}m(1-m^2)-km$. Thus, it is enough to show that $p|a_p$, $p|b_p$ and $p|c_p$. However, this is immediate because $2|(p+1)$, $2|(1-p)$ and $6|(1-p^2)$, for $p \neq 2,3$ prime. □

Let v be the vector δ_2 (or δ_4) and we denote by orb_p the list of the vector in the orbit of v modulo p. Firstly, we start with $orb_p = \{mod_p(v)\}$ and we compute the vectors mod $_p(wM_1^j M_0^i)$ for $j = 0, \dots, p$, $i = 0, \dots p$ and $w \in orb_p$. If these vectors are not in orb_p, we add them to orb_p. This step is repeated with the new orb_p, if there are not new vectors then the orbit is complete. This process is finite because we have at most p^4 different vectors in $(\mathbb{Z}/p\mathbb{Z})^4$. We summarize the algorithm [1] to compute the orbit of v modulo p as follows,

```
Input: v, M0, M1, p
Output: Orb_p
Orb_p := mod_p(v)
norm := 1
while norm > 0 do
    W := Orb_p; L := length(Orb_p)
    l := 1; c := 1
    while l ≤ L do
        j:=0
        while j ≤ p do
            i := 0
            while i ≤ p do
                v_aux := mod_p(W(l) * (M1^j M0^i))
                if v_aux ∉ Orb_p then
                    Orb_p(L + c) := v_aux
                    c := c + 1; i := i + 1
                else
                    i := i + 1
                end
            end
            j : j + 1
        end
        l := l + 1
    end
    norm := length(Orb_p) − length(W)
end
```

Proof of theorem 2.1.1. Consider the free group H when $d = k = 5$. Given p, we denote the orbit of δ_2 and δ_4 modulo p as $orb_p(\delta_2)$ and $orb_p(\delta_4)$, respectively. By using the previous algorithm we have,

$$orb_2(\delta_2) = \{(0\ 0\ 1\ 1),\ (0\ 1\ 0\ 0),\ (0\ 1\ 0\ 1),\ (1\ 0\ 0\ 1),\ (1\ 0\ 1\ 1)\}.$$

$$\begin{aligned} orb_2(\delta_4) = \{&(0\ 0\ 0\ 1),\ (0\ 0\ 1\ 0), (0\ 1\ 1\ 0), (0\ 1\ 1\ 1), (1\ 0\ 0\ 0), \\ &(1\ 0\ 1\ 0), (1\ 1\ 0\ 0), (1\ 1\ 0\ 1), (1\ 1\ 1\ 0), (1\ 1\ 1\ 1)\}. \end{aligned}$$

$$\begin{aligned} orb_5(\delta_2) = \{&(0\ 1\ 0\ 0), (0\ 1\ 0\ 1), (0\ 1\ 0\ 2), (0\ 1\ 0\ 3), (0\ 1\ 0\ 4), \\ &(0\ 1\ 1\ 0), (0\ 1\ 1\ 1), (0\ 1\ 1\ 2), (0\ 1\ 1\ 3), (0\ 1\ 1\ 4), \\ &(0\ 1\ 2\ 0), (0\ 1\ 2\ 1), (0\ 1\ 2\ 2), (0\ 1\ 2\ 3), (0\ 1\ 2\ 4), \\ &(0\ 1\ 3\ 0), (0\ 1\ 3\ 1), (0\ 1\ 3\ 2), (0\ 1\ 3\ 3), (0\ 1\ 3\ 4), \\ &(0\ 1\ 4\ 0), (0\ 1\ 4\ 1), (0\ 1\ 4\ 2), (0\ 1\ 4\ 3), (0\ 1\ 4\ 4)\}. \end{aligned}$$

[1] We have written a MATLAB code for the computation. It is available in https://github.com/danfelmath/mirrorquintic.git

$$orb_5(\delta_4) = \{(0\ 0\ 0\ 1),\ (0\ 0\ 1\ 1),\ (0\ 0\ 2\ 1),\ (0\ 0\ 3\ 1),\ (0\ 0\ 4\ 1)\}.$$

$$orb_p(\delta_2) = orb_p(\delta_4) = (\mathbb{Z}/p\mathbb{Z})^4 \smallsetminus (0\ 0\ 0\ 0),\ \text{for } p = 3, 7, 11, 13, 17, 19, 23.$$

$\square$

From the map $H_3(\tilde{X}, \mathbb{Z}) \xrightarrow{mod_p} H_3(\tilde{X}, \mathbb{Z}_p)$ we have that if $\delta \in H_3(\tilde{X}, \mathbb{Z})$ is a primitive class and it is in the orbit of δ_2 (or δ_4), then $mod_p(\delta) \in orb_p(\delta_2)$ (or $mod_p(\delta_4) \in orb_p(\delta_4)$) for all p. We think that the converse should be true; that is the Conjecture 2.1.2.

For the other examples of quintic threefolds appearing in Table 2.1, we have analogous results. However, in this case we do not know if the vectors $\delta_2 = (0\ 1\ 0\ 0)$ and $\delta_4 = (0\ 0\ 0\ 1)$ are really supported in a Lagrangian submanifold. In Table 2.2 we present the orbits of the vectors δ_2 and δ_4 modulo p for the fourteen cases of (d, k). If the orbit is $(\mathbb{Z}/p\mathbb{Z})^4 \smallsetminus (0\ 0\ 0\ 0)$ we call it *complete*. The orbits for the vector δ_2 are presented in Table 2.3 and the orbits for the vector δ_4 are presented in Table 2.4.

(d,k)	Prime	Orbit
$(5,5)$	$p=5$	$(0001),(0011),(0021),(0031),(0041),(0100),(0101),(0102),(0103),(0104),(0110),(0111),(0112),(0113),(0114),(0120),(0121),(0122),(0123),(0124),(0130),(0131),(0132),(0133),(0134),(0140),(0141),(0142),(0143),(0144)$
	$p=2,3,7,11,13,17,19,23$	Complete
$(2,4)$	$p=2$	$(0001),(0011),(0100),(0101),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(1,4)$	$p=2,3,5,7,11,13,17,19,23$	Complete
$(16,8)$	$p=2$	$(0001),(0011),(0100),(0101),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(12,7)$	$p=2$	$(0001),(0011),(0100),(0101),(0110),(0111)$
	$p=3$	$(0001),(0002),(0021),(0022),(0100),(0101),(0102),(0110),(0111),(0112),(0210),(0211),(0212),(0220),(0221),(0222)$
	$p=5,7,11,13,17,19,23$	Complete
$(8,6)$	$p=2$	$(0001),(0011),(0100),(0101),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(4,5)$	$p=2$	$(0001),(0011),(0100),(0101),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(2,3)$	$p=2$	$(0001),(0011),(0100),(0101),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(1,2)$	$p=2,3,5,7,11,13,17,19,23$	Complete
$(6,5)$	$p=2$	$(0001),(0011),(0100),(0101),(0110),(0111)$
	$p=3$	$(0001),(0002),(0011),(0012),(0100),(0101),(0102),(0120),(0121),(0122),(0210),(0211),(0212),(0220),(0221),(0222)$
	$p=5,7,11,13,17,19,23$	Complete
$(3,4)$	$p=3$	$(0001),(0002),(0021),(0022),(0100),(0101),(0102),(0110),(0111),(0112),(0210),(0211),(0212),(0220),(0221),(0222)$
	$p=2,5,7,11,13,17,19,23$	Complete
$(1,3)$	$p=2,3,5,7,11,13,17,19,23$	Complete
$(4,4)$	$p=2$	$(0001),(0011),(0100),(0101),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(9,6)$	$p=3$	$(0001),(0011),(0021),(0100),(0101),(0102),(0110),(0111),(0112),(0120),(0121),(0122)$
	$p=2,5,7,11,13,17,19,23$	Complete

Table 2.2: Orbit of vectors δ_2 and δ_4 by the monodromy action for the fourteen mirror Calabi-Yau threefolds.

(d,k)	Prime	Orbit
$(5,5)$	$p=2$	$(0011),(0100),(0101),(1001),(1011)$
	$p=5$	$(0100),(0101),(0102),(0103),(0104),(0110),(0111),(0112),(0113),(0114),(0120),(0121),(0122),(0123),(0124),(0130),(0131),(0132),(0133),(0134),(0140),(0141),(0142),(0143),(0144)$
	$p=3,7,11,13,17,19,23$	Complete
$(2,4)$	$p=2$	$(0100),(0101)(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(1,4)$	$p=2$	$(0010),(0100),(0101),(0110),(0111),(1001),(1010),(1110),(1111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(16,8)$	$p=2$	$(0100),(0101),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(12,7)$	$p=2$	$(0011),(0100),(0101)$
	$p=3$	$(0021),(0022),(0100),(0101),(0102),(0210),(0211),(0212)$
	$p=5,7,11,13,17,19,23$	Complete
$(8,6)$	$p=2$	$(0100),(0101),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(4,5)$	$p=2$	$(0011),(0100),(0101)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(2,3)$	$p=2$	$(0011),(0100),(0101)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(1,2)$	$p=2$	$(0010),(0100),(0101),(0110),(0111),(1001),(1010),(1110),(1111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(6,5)$	$p=2$	$(0011),(0100),(0101)$
	$p=3$	$(0011),(0012),(0100),(0101),(0102),(0220),(0221),(0222)$
	$p=5,7,11,13,17,19,23$	Complete
$(3,4)$	$p=2$	$(0010),(0100),(0101),(0110),(0111),(1001),(1010),(1110),(1111)$
	$p=3$	$(0021),(0022),(0100),(0101),(0102),(0210),(0211),(0212)$
	$p=5,7,11,13,17,19,23$	Complete
$(1,3)$	$p=2$	$(0011),(0100),(0101),(1001),(1011)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(4,4)$	$p=2$	$(0100),(0101),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(9,6)$	$p=2$	$(0010),(0100),(0101),(0110),(0111),(1001),(1010),(1110),(1111)$
	$p=3$	$(0100),(0101),(0102),(0110),(0111),(0112),(0120),(0121),(0122)$
	$p=5,7,11,13,17,19,23$	Complete

Table 2.3: Orbit of vector δ_2 by the monodromy action for the fourteen mirror Calabi-Yau threefolds.

(d,k)	Prime	Orbit
$(5,5)$	$p=2$	$(0001),(0010),(0110),(0111),(1000),(1010),(1100),(1101),(1110),(1111)$
	$p=5$	$(0001),(0011),(0021),(0031),(0041)$
	$p=3,7,11,13,17,19,23$	Complete
$(2,4)$	$p=2$	$(0001),(0011)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(1,4)$	$p=2$	$(0001),(0011),(1000),(1011),(1100),(1101)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(16,8)$	$p=2$	$(0001),(0011)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(12,7)$	$p=2$	$(0001),(0110),(0111)$
	$p=3$	$(0001),(0002),(0110),(0111),(0112),(0220),(0221),(0222)$
	$p=5,7,11,13,17,19,23$	Complete
$(8,6)$	$p=2$	$(0001),(0011)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(4,5)$	$p=2$	$(0001),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(2,3)$	$p=2$	$(0001),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(1,2)$	$p=2$	$(0001),(0011),(1000),(1011),(1100),(1101)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(6,5)$	$p=2$	$(0001),(0110),(0111)$
	$p=3$	$(0001),(0002),(0120),(0121),(0122),(0210),(0211),(0212)$
	$p=5,7,11,13,17,19,23$	Complete
$(3,4)$	$p=2$	$(0001),(0011),(1000),(1011),(1100),(1101)$
	$p=3$	$(0001),(0002),(0110),(0111),(0112),(0220),(0221),(0222)$
	$p=5,7,11,13,17,19,23$	Complete
$(1,3)$	$p=2$	$(0001),(0010),(0110),(0111),(1000),(1010),(1100),(1101),(1110),(1111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(4,4)$	$p=2$	$(0001),(0011)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(9,6)$	$p=2$	$(0001),(0011),(1000),(1011),(1100),(1101),$
	$p=3$	$(0001),(0011),(0021)$
	$p=5,7,11,13,17,19,23$	Complete

Table 2.4: Orbit of vector δ_4 by the monodromy action for the fourteen mirror Calabi-Yau threefolds.

Chapter 3

The Monodromy problem for hyperelliptic curves

3.1 Introduction

The interest on polynomial foliations in $\mathbb{C}^2$ arises as an approach to the Hilbert Sixteen Problem [Mov04a, Rou98]. These foliations are given by 1-forms $\omega = P(x,y)dy - Q(x,y)dx$, where P and Q are polynomials. The classical notation for the foliation associated to the form ω is $\mathcal{F}(\omega)$. In this context, a point $p \in \mathbb{C}^2$ is a singularity of $\mathcal{F}(\omega)$ if $P(p) = Q(p) = 0$. We say that the singularity p is a **center singularity** if there is a local chart such that p is mapped to $0 \in \mathbb{C}^2$, and a Morse function $f : (\mathbb{C}^2, 0) \to \mathbb{C}$ with fibers tangent to the leaves of $\mathcal{F}(\omega)$. The degree of a foliation $\mathcal{F}(\omega)$ is the greatest degree of the polynomials P and Q, and the space of foliations of degree d is denoted by $\mathcal{F}(d)$. The closure of the set of foliations in $\mathcal{F}(d)$ with at least one center is denote by $\mathcal{M}(d)$.

It is known that $\mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$ (e.g. [Net07, §6.1] [Mov04a]). The problem of describing its irreducible components is formulated by Lins Neto [Net14]. In [Ily69], Y. Ilyashenko proves that the space of Hamiltonian foliation $\mathcal{F}(df)$, where f is a polynomial of degree $d+1$, is an irreducible component of $\mathcal{M}(d)$. In [Mov04b], H. Movasati considers the logarithmic foliations $\mathcal{F}\left(\sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}\right)$, with $f_i \in \mathbb{C}[x,y]_{\leq d_i}$ and $\lambda_i \in \mathbb{C}^*$. He proves that the set of logarithmic foliations form an irreducible component of $\mathcal{M}(d)$, where $d = \sum_{i=1}^{s} d_i - 1$. Moreover, in [Zar17], Y. Zare works with pullback foliations $\mathcal{F}(P^*\omega)$, where $P = (R,S) : \mathbb{C}^2 \to \mathbb{C}^2$ is a generic morphism with $R, S \in \mathbb{C}[x,y]_{\leq d_1}$, and ω is a 1-form of degree d_2. Zare shows that they form an irreducible component of $\mathcal{M}(d_1(d_2+1) - 1)$.

The main idea in the proofs of these assertions is to choose a particular polynomial f and consider deformations $df + \varepsilon\omega_1$ in $\mathcal{M}(d)$. Then, it is necessary to study the vanishing of the abelian integrals $\int_\delta \omega_1$, where δ is a homological 1-cycle in a regular fiber of f. This integral is zero on the vanishing cycle associated to the center singularity. If the monodromy action on this cycle generates the whole vector space $H_1(f^{-1}(b), \mathbb{Q})$ for a regular value b, then the deformation is relatively exact to df.

The condition that the vanishing of the integral $\int_\delta \omega_1$ implies that ω_1 is relatively exact to df, is known as the (*)-*property* (It was introduced by J.P. Françoise in [Fra96]). The results of L. Gavrilov in [Gav98], show that if we provided that the integral is zero over any cycle in a regular fiber, then ω_1 is relatively exact. Therefore, if the subspace generated by monodromy action on the vanishing cycle δ is the whole space $H_1(f^{-1}(b), \mathbb{Q})$, then the $(*) - $*property* is satisfied. This gives rise to the next natural problem, which is summarized by C. Christopher and P. Mardešić in [CM10] as follows.

Monodromy problem. Under which conditions on f is the $\mathbb{Q}$-subspace of $H_1(f^{-1}(b),\mathbb{Q})$ generated by the image of a vanishing cycle of a Morse point under monodromy equal to the whole of $H_1(f^{-1}(b),\mathbb{Q})$?

Furthermore, they show a characterization of the vanishing cycles associated to a Morse point in hyperelliptic curves given by $y^2+g(x)$, depending on whether g is decomposable (Theorem 3.4.7). This case is closely related with the 0-dimensional monodromy problem; by using the definition of Abelian integrals of dimension zero ([GM07]). For example, if we think in the Dynkin diagram associated with $y^2+g(x)$ and the one associated with $g(x)$, then we see that they coincide. However, the Dynkin diagram for $y^3+g(x)$ is a bit more complicated. Moreover, in the case $y^4+g(x)$ there is always a pullback associated to $y\to y^2$, thus the Dynkin is expected to reflect this fact. For these two cases, we prove the following two theorems.

Theorem 3.1.1. *Let g be a polynomial with real critical points, and degree d such that, $4\nmid d$ and $d\leq 100$. Consider the polynomial $f(x,y)=y^4+g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the homology $H_1(f^{-1}(t),\mathbb{Q})$.*
2. *The polynomial g is decomposable (i.e., $g=g_2\circ g_1$), and $\pi_*\delta(t)$ is homotopic to zero in $\{y^4+g_2(z)=t\}$, where $\pi(x,y)=(g_1(x),y)=(z,y)$. Or, the cycle $\pi_*\delta(t)$ is homotopic to zero in $\{z^2+g(x)=t\}$, where $\pi(x,y)=(x,y^2)=(x,z)$.*

Theorem 3.1.2. *Let g be a polynomial with real critical points, and degree d such that, $3\nmid d$ and $d\leq 100$. Consider the polynomial $f(x,y)=y^3+g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the homology $H_1(f^{-1}(t),\mathbb{Q})$.*
2. *The polynomial g is decomposable (i.e., $f=g_2\circ g_1$), and $\pi_*\delta(t)$ is homotopic to zero in $\{y^3+g_2(z)=t\}$, where $\pi(x,y)=(g_1(x),y)=(z,y)$.*

Some parts in the proof are done numerically using computer, thus we have the restriction $d\leq 100$ in the degree of the polynomial g.

The monodromy problem for polynomials of degree 4, on the other hand, is very interesting, because the classification of the irreducible components of $\mathcal{M}(3)$ is still an open problem. In fact, the only case which has a complete classification is $\mathcal{M}(2)$ (see [Dul08][CN96, p. 601]). For polynomials $f(x,y)=h(y)+g(x)$ where $\deg(h)=\deg(g)=4$, we determine in the Theorem 3.5.4, a relation between the subspaces of $H_1(f^{-1}(b),\mathbb{Q})$ generated by the monodromy action of the vanishing cycles, and the property of f being decomposable . In oder to do that, we provide an explicit description of the space of parameters of the polynomials $h(y)+g(y)$ which satisfies some conditions in the critical values.

Organization. In section 2, we provide some definitions in Picard-Lefschetz theory and describe the Dynkin diagram for direct sum of polynomials in two variable. In section 3, we analyze the particular case, in which there is only one critical value. For this case, we compute the vector space generated by the monodromy action on the vanishing cycles. In section 4, we prove Theorems 3.1.1 and 3.1.2. Finally, in Section 5, we solve the monodromy problem for polynomials $h(x)+g(y)$ with $\deg(h)=\deg(g)=4$, in this case there is another pullback to be considered, associated to the map $(x,y)\to(xy,x+y)$. For this reason, we do not know a geometrical characterization of some of vanishing cycles which do not generate the whole $H_1(f^{-1}(b),\mathbb{Q})$.

3.2 Lefschetz fibrations and monodromy action on direct sum of polynomials

Let $f\in\mathbb{C}[x,y]$ with the set of critical values C and a regular value b. Suppose that the origin is an isolated critical point of the highest-grade homogeneous piece of f. Hence, the Milnor number μ of f is finite, and there are **vanishing cycles** $\delta_1,\delta_2,\dots,\delta_\mu$ associated to the critical values, such that they generate freely the 1-homology of the fiber $X_b := f^{-1}(b)$, i.e. $H_1(X_b,\mathbb{Z}) = span\{\delta_i\}_{i=1}^{\mu}$ (see [AVGZ88, Chs. 1,2],[Mov17a, §7.5],[Lam81]). Moreover, there is an action $\pi_1(\mathbb{C}\setminus C)\times H_1(X_b)\xrightarrow{mon} H_1(X_b)$ called the **monodromy action** given by local trivialization of $f^{-1}(\gamma)$, where γ is any loop in $\mathbb{C}\setminus C$, see [Mov17a, §6.3]. A homological cycle in X_b such that its orbit by the monodromy action generates the whole homology group $H_1(X_b,\mathbb{Z})$ is called **simple cycle**. This definition of simple cycle was introduced in [Mov04a], and it is different from the definition of simple cycle used in [CM10, GM07].

Let $f(x,y)=h(y)+g(x)$ be a polynomial with real coefficients, such that the critical points of h and g are reals. By considering a deformation of f, we can suppose that f is a Morse function and all its critical values are different pairwise. Furthermore, by doing a translation we can suppose that the critical values of g are positive, the critical values of h are negatives, and $b=0$. Let c_i^h and c_j^g be the critical values of h and g, respectively.

Consider the paths r_i and s_j, from 0 to c_i^h and c_j^g, respectively. Furthermore, the paths are without self-intersection, and they intersect each others only in 0. Also, $(s_1,s_2,\dots,s_{d-1},r_1,r_2,\dots r_{e-1})$ near 0 is the anticlockwise direction, as in Figure 3.1. Moreover, we have chosen the enumeration of the paths such that $0<c_1^g<c_2^g<\dots<c_{d-1}^g$ and $c_{e-1}^h<c_{e-2}^h<\dots<c_1^h<0$.

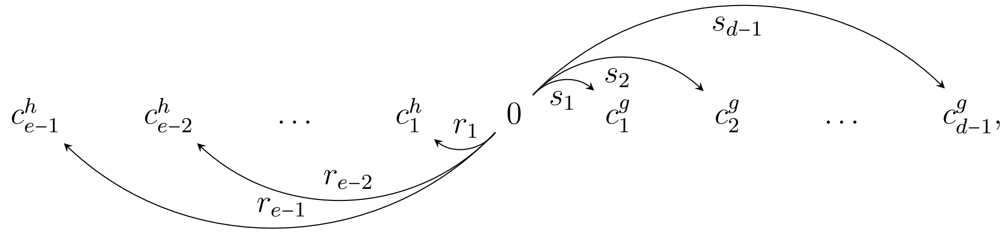

Figure 3.1: A distinguished set of paths from b to the critical values of h and g

Let $C_g=\{(x,g(x)) \mid x\in\mathbb{R}\}$ be the real curve associated to g. For each $j=1,\dots,d-1$, let p_j be the critical point associated to c_j^g, and we define a real number ε_j as follows. If p_j is a minimum, then we take $c_j^g<\varepsilon_j<c_{j+1}^g$, otherwise we take $c_{j-1}^g<\varepsilon_j<c_j^g$. Consider the real line $L_j=\{(x,\varepsilon_j) \mid x\in\mathbb{R}\}$, thus there are two points in $L_j\cap C_g$, such that, they go to (p_j,c_j^g) when ε_j goes to c_j^g. Thus, we define the 0-vanishing cycle $\sigma_j\in H_0(g^{-1}(\varepsilon_j),\mathbb{Z})$, as the formal sum of

these two points, with coefficients 1 and -1. Note that these vanishing cycles, are *simple cycles* according to the definition in [CM10, GM07], however, they are not simple cycles according to our definition.

By taking a simple path from 0 to ε_j, without encircling or passing through critical values, we can consider σ_j in $H_0(g^{-1}(0),\mathbb{Z})$. Note, that it is possible to chose a path from 0 to ε_j and compose it with the segment $\overline{\varepsilon_j c_j^g}$, such this path is homotopic equivalent to s_j. Therefore, we can suppose that the cycle σ_j vanishes along the path s_j. Analogously for the polynomial h, we define the 0-vanishing cycle $\gamma_i \in H_0(h^{-1}(0),\mathbb{Z})$.

Consider the path $\lambda = s_j r_i^{-1}$, starting in c_i^h and ending in c_j^g, as in [Mov17a, §7.9], we define the **join cycle**

$$\gamma_i * \sigma_j := \bigcup_{t\in[0,1]} \gamma_i(\lambda_t)\times\sigma_j(\lambda_t),$$

where $\gamma_i(\lambda_t)\in H_0(h^{-1}(\lambda_t))$ and $\sigma_j(\lambda_t)\in H_0(g^{-1}(\lambda_t))$. The join cycle $\gamma_i*\sigma_j$ is homeomorphic to a circle S^1, the Figure 3.2 shows this construction. On the other hand, note that the join

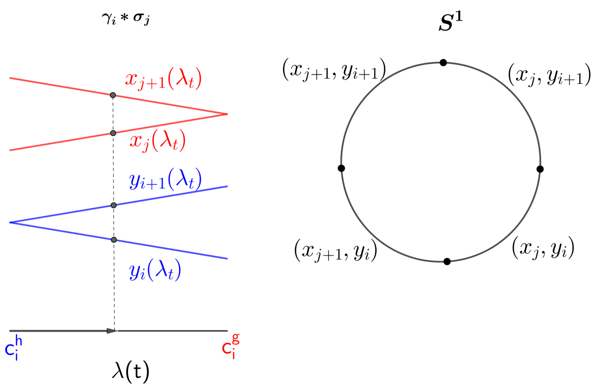

Figure 3.2: Join cycle $\gamma_j * \sigma_i$ as S^1

cycle $\gamma_i*\sigma_j$ is a vanishing cycle of the fibration given by f along the path $s_j r_i^{-1}+c_i^h$. Therefore, the join cycles generate the homology $H_1(f^{-1}(b),\mathbb{Z})$. Next, we compute the intersection of two join cycles. The local formula for the intersection form of two vanishing cycles is due to A.M. Gabrielov (see [AVGZ88, Thm 2.11]). Its reproduction in the global context of tame polynomials is done in [Mov17a, §7.10]. Since the particular case of 1-dimension fibers is simple, we reproduce the proof in the next proposition.

Proposition 3.2.1. *Let $\gamma_i*\sigma_j$ and $\gamma_{i'}*\sigma_{j'}$ be two join cycles along the paths $\lambda := s_j r_i^{-1}$ and $\lambda' := s_{j'} r_{i'}^{-1}$, respectively. Then*

$$\langle\gamma_i*\sigma_j,\gamma_{i'}*\sigma_{j'}\rangle = \begin{cases} sgn(j'-j)\langle\sigma_j,\sigma_{j'}\rangle & \text{if } i=i' \text{ and } j\neq j' \\ sgn(i'-i)\langle\gamma_i,\gamma_{i'}\rangle & \text{if } j=j' \text{ and } i\neq i' \\ sgn(i'-i)\langle\gamma_i,\gamma_{i'}\rangle\langle\sigma_j,\sigma_{j'}\rangle & \text{if } (i'-i)(j'-j)>0 \\ 0 & \text{if } (i'-i)(j'-j)<0. \end{cases} \tag{3.1}$$

Proof. Suppose that the paths intersect each other transversally in b. The join cycle $\gamma_i*\sigma_j$ intersects $\gamma_{i'}*\sigma_{j'}$ at one point if the 0-cycles intersect each other, and at zero points otherwise. The orientation of the intersection of the join cycles is given by $d\lambda\wedge d\lambda'$ times the sign of the intersection of the 0-cycles. Moreover, we consider as positive orientation the canonical orientation of $\mathbb{C}$ given by $dz_1\wedge dz_2$ where $z=z_1+\sqrt{-1}dz_2$.

Suppose that $i = i'$, thus the path λ and λ' intersect transversally in positive direction if $j' > j$. The intersection of the 0-cycles only depends on the intersection $\langle\sigma_j, \sigma_{j'}\rangle$. Analogously, for $j = j'$. When $(i' - i)(j' - j) > 0$ the intersection of λ and λ' is transversal again, with positive direction if $i' > i$, and the intersection of the 0-cycles is $\langle\gamma_i, \gamma_{i'}\rangle\langle\sigma_j, \sigma_{j'}\rangle$. Finally, if $(i' - i)(j' - j) < 0$, the paths do not intersect transversally. Furthermore, after doing a homotopy we can suppose that the path λ and λ' do not have intersection points, therefore the intersection of the join cycles is zero. □

Next, we present a combinatorial way of representing the intersection form, which is described in [AVGZ88, §2.8] and [Mov17a, §7.10].

Definition 3.2.2. The **Dynkin diagram** of $f(x, y) = h(y) + g(y)$, is a directed graph where the vertices are the vanishing cycles in a regular fiber. The vertices v_i and v_j are joined by an edge with multiplicity $|\langle v_i, v_j\rangle|$. If $\langle v_i, v_j\rangle > 0$, then the direction goes from v_i to v_j.

We can also relate the vertex in a Dynkin diagram with the critical value associated to the vanishing cycle. In order to define the Dynkin diagram of the polynomial $f(x, y) = h(y) + g(x)$, we consider a deformation $\tilde{f}$ such that the critical values are different pairwise. Although the Dynkin diagram for f and $\tilde{f}$ are equals (the same vertices and edge), the Dynkin diagram associated to f has relations among the vertices according to the equalities of the critical values. From the previous choice of paths r_i and s_j, we have rules in the Dynkin diagram which establish the possibilities to relate the critical values:

1. The Dynkin diagram associated to $f(x, y) = h(y) + g(x)$, can be thought as a two-dimensional array, where the rows are the critical values $c_i^h + c_j^g$ for a fixed i and $j = 1, \ldots, d-1$. Thus, if two critical values of $f(x, y)$ in the same row of the Dynkin diagram are equals, then for the columns of these critical values there are equalities in the rows. This is obviously because if $c_i^h + c_j^g = c_i^h + c_l^g$ then $c_j^g = c_l^g$, consequently $c_k^h + c_j^g = c_k^h + c_l^g$ for all $k = 1 \ldots e-1$. This happens in an analogous way for the columns.

2. If $c_i^h + c_j^g = c_k^h + c_l^g$ and additionally $i < k$, $j > l$ then $c_i^h = c_k^h$ and $c_j^g = c_l^g$. This follows from the choice of distinguished paths because $i < k$ implies that $c_i^h \geq c_k^h$ and $j > l$ implies $c_j^h \geq c_l^h$ then $c_i^h + c_j^g \geq c_k^h + c_l^g$ since $c_i^h + c_j^g = c_k^h + c_l^g$ then the inequalities actually are equalities.

Similarly, a Dynkin diagram in dimension 0 consists of vertex which are the vanishing cycles associated to a polynomial, and dashed edges representing an intersection of -1, in this case the edges do not have direction. Note that the vanishing cycles associated to the critical values c_i^g with $i \leq [d/2]$ can only intersect vanishing cycles associated to critical values c_j^g with $j \geq [d/2]$, and similarly for the critical values c_k^h, for example, see the figure 3.3.

The 0-dimensional Dynkin diagrams associated to h and g of the Figure 3.3 are:

$$\gamma_2 \text{ -- } \gamma_4 \text{ -- } \gamma_3 \text{ -- } \gamma_5 \text{ -- } \gamma_1 \text{ -- } \gamma_6 \quad , \quad \sigma_1 \text{ -- } \sigma_5 \text{ -- } \sigma_3 \text{ -- } \sigma_4 \text{ -- } \sigma_2 \text{ -- } \sigma_6,$$

thus, by using (3.1) we get the next Dynkin diagram for $f(x, y) = h(y) + g(x)$ (in terms of

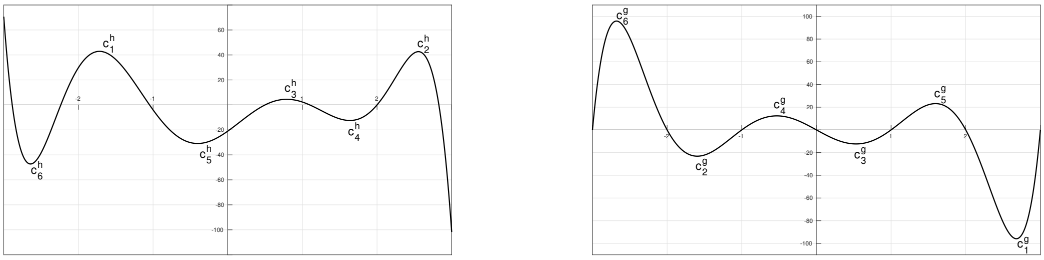

Figure 3.3: Real part of the polynomials $h(y) = -(y+\frac{5\sqrt{3}}{3})(y+\sqrt{5})(y+\frac{\pi}{3})(y-\frac{1}{2})(y-\ln(3))(y-2)(y-2\sqrt{2})$ and $g(x) = (x+3)(x+2)(x+1)x(x-1)(x-2)(x-3)$, with its real critical values. On the left is $h(y)$ and on the right $g(x)$.

critical values),

$$
\begin{array}{ccccccccccc}
c_2^h+c_1^g & \leftarrow & c_4^h+c_1^g & \rightarrow & c_3^h+c_1^g & \leftarrow & c_5^h+c_1^g & \rightarrow & c_1^h+c_1^g & \leftarrow & c_6^h+c_1^g \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
c_2^h+c_5^g & \leftarrow & c_4^h+c_5^g & \rightarrow & c_3^h+c_5^g & \leftarrow & c_5^h+c_5^g & \rightarrow & c_1^h+c_5^g & \leftarrow & c_6^h+c_5^g \\
\downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow \\
c_2^h+c_3^g & \leftarrow & c_4^h+c_3^g & \rightarrow & c_3^h+c_3^g & \leftarrow & c_5^h+c_3^g & \rightarrow & c_1^h+c_3^g & \leftarrow & c_6^h+c_3^g \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
c_2^h+c_4^g & \leftarrow & c_4^h+c_4^g & \rightarrow & c_3^h+c_4^g & \leftarrow & c_5^h+c_4^g & \rightarrow & c_1^h+c_4^g & \leftarrow & c_6^h+c_4^g \\
\downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow \\
c_2^h+c_2^g & \leftarrow & c_4^h+c_2^g & \rightarrow & c_3^h+c_2^g & \leftarrow & c_5^h+c_2^g & \rightarrow & c_1^h+c_2^g & \leftarrow & c_6^h+c_2^g \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
c_2^h+c_6^g & \leftarrow & c_4^h+c_6^g & \rightarrow & c_3^h+c_6^g & \leftarrow & c_5^h+c_6^g & \rightarrow & c_1^h+c_6^g & \leftarrow & c_6^h+c_6^g.
\end{array}
\tag{3.2}
$$

The **Picard-Lefschetz formula** give us an explicit computation of the monodromy of a cycle δ, around to a critical value $c_{ij} := c_i^h + c_j^g$. Namely, it is

$$\mathrm{Mon}_{c_{ij}}(\delta) = \delta - \sum_k \langle \delta, \delta_k \rangle \delta_k, \tag{3.3}$$

where k runs through all the join cycles in the singularities of $f^{-1}(c_{ij})$ (see [Mov17a, §6.6]). Therefore, in order to compute the monodromy of the fibration given by the polynomial $f(x,y) = h(y) + g(x)$, we just need to handle combinatorial aspects of Dynkin diagrams. In the remainder of the text, we denote as $\mathbf{Mon}(\delta)$, the subspace generated by the orbit of δ by monodromy action.

3.3 Monodromy for direct sum of polynomials with one critical value

In this section, we provide the monodromy matrix around 0 for the polynomial $f(x,y) = y^e + x^d$, with $e = 2,3,4$. For simplicity, we denote by δ_i^j the vanishing cycles in the row i and column j. Thus we have the Dynkin diagram for $e = 2,3,4$,

$$\delta_1^1 \longrightarrow \delta_1^2 \longleftarrow \delta_1^3 \longrightarrow \delta_1^4 \longleftarrow \delta_1^5 \longrightarrow \delta_1^6 \longleftarrow \cdots \qquad \delta_1^d$$

$$
\begin{tikzcd}
\delta_1^1 \arrow[r] \arrow[d] & \delta_1^2 \arrow[d] & \delta_1^3 \arrow[l] \arrow[r] \arrow[d] & \delta_1^4 \arrow[d] & \delta_1^5 \arrow[l] \arrow[r] \arrow[d] & \delta_1^6 \arrow[d] & \cdots \arrow[l] & \delta_1^d \arrow[d] \\
\delta_2^1 \arrow[r] & \delta_2^2 \arrow[ul] \arrow[ur] & \delta_2^3 \arrow[l] \arrow[r] & \delta_2^4 \arrow[ul] \arrow[ur] & \delta_2^5 \arrow[l] \arrow[r] & \delta_2^6 \arrow[ul] & \cdots \arrow[l] & \delta_2^d
\end{tikzcd}
$$

$$
\begin{tikzcd}
\delta_1^1 \arrow[r] \arrow[d] & \delta_1^2 \arrow[d] & \delta_1^3 \arrow[l] \arrow[r] \arrow[d] & \delta_1^4 \arrow[d] & \delta_1^5 \arrow[l] \arrow[r] \arrow[d] & \delta_1^6 \arrow[d] & \cdots \arrow[l] & \delta_1^d \arrow[d] \\
\delta_2^1 \arrow[r] & \delta_2^2 \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \delta_2^3 \arrow[l] \arrow[r] & \delta_2^4 \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \delta_2^5 \arrow[l] \arrow[r] & \delta_2^6 \arrow[ul] \arrow[dl] & \cdots \arrow[l] & \delta_2^d \\
\delta_3^1 \arrow[r] \arrow[u] & \delta_3^2 \arrow[u] & \delta_3^3 \arrow[l] \arrow[r] \arrow[u] & \delta_3^4 \arrow[u] & \delta_3^5 \arrow[l] \arrow[r] \arrow[u] & \delta_3^6 \arrow[u] & \cdots \arrow[l] & \delta_3^d \arrow[u]
\end{tikzcd}, \tag{3.4}
$$

respectively. By Proposition 3.2.1, the intersection matrix in the ordered vector basis $\delta_1^1, \ldots, \delta_e^1, \delta_1^2, \ldots, \delta_e^2, \ldots, \delta_1^d, \ldots, \delta_e^d$ for these Dynkin diagram are

$$
\Psi_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & \ldots \\ 0 & -1 & 0 & -1 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3.5}
$$

$$
\Psi_3 = \begin{pmatrix}
0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & \ldots \\
-1 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & \ldots \\
0 & 0 & -1 & 1 & 0 & -1 & -1 & 1 & \ldots \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & \ldots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & \ldots \\
0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix}, \tag{3.6}
$$

$$
\Psi_4 = \begin{pmatrix}
0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
-1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & \ldots \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & -1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix}. \tag{3.7}
$$

The matrices are antisymmetric, and the superior diagonals are periodic sequences. For Ψ_2 the sequence is $(-1, 1, \ldots)$. For Ψ_3 the sequence are $(-1, 0, -1, -1, -1, 0, \ldots)$, $(-1, -1, 1, 1 \ldots)$ and $(1, 0, 0, 0, \ldots)$. For Ψ_4 the sequence are $(-1, 1, 0, \ldots)$, $(0, 0, 1, 0, -1, 0, \ldots)$, $(-1, -1, -1, 1, 1, 1, \ldots)$

and $(-1,0,0,0,1,0,\ldots)$. From the Picard-Lefschetz formula (3.3), it follows that the monodromy matrices for $f(x,y)=y^e+x^d$ with $e=2,3,4$, are

$$M_e = I_N - \Psi_e,$$

where I_N is the identity matrix of rank $N=(d-1)(e-1)$.

For a vector v and a matrix $M\in\mathcal{M}_N(\mathbb{R})$, the Krylov subspace is the vectorial space generated by the vectors $M^l v$ where $l=0,2,\ldots,N-1$. Therefore, by taking M as one of the monodromy matrices M_2, M_3 or M_4, and $v=v_k$ a vector of the canonical basis of $\mathbb{R}^N$, the Krylov subspace is

$$\mathrm{Mon}\left(\delta^{\lfloor\frac{k}{e}\rfloor}_{\mathrm{mod}\ e(k)}\right) \tag{3.8}$$

for the fibration y^e+x^d. In the next proposition we provide the vanishing cycles that are in (3.8).

Proposition 3.3.1. *For the polynomial y^2+x^d, the vanishing cycles in the subspace Mon(δ_i^j) are*

Vanishing cycle δ_1^j	Vanishing cycles δ_1^l in Mon(δ_1^j)
$gcd(d,j)=r$	$l=rn$ with $n=1,\ldots,\frac{d}{r}-1$.

For the polynomial y^3+x^d with $d\le 100$ and $3\nmid d$, the vanishing cycles in the subspace Mon(δ_i^j) are

Vanishing cycle δ_i^j	Vanishing cycles δ_m^l in Mon(δ_i^j)
$i=1,2$ and $gcd(d,j)=r$	$m=1,2$ and $l=rn$ with $n=1,\ldots,\frac{d}{r}-1$.

When $3\mid d$, the number of different eigenvalues is less than $2(d-1)$. For the polynomial y^4+x^d with $d\le 100$ and $4\nmid d$, the vanishing cycles in the subspace Mon(δ_i^j) are

Vanishing cycle δ_i^j	Vanishing cycles δ_m^l in Mon(δ_i^j)
$i=1,3$ and $gcd(d,j)=r$	$m=1,2,3$ and $l=rn$ with $n=1,\ldots,\frac{d}{r}-1$
$i=2$ and $gcd(d,j)=r$	$m=2$ and $l=rn$ with $n=1,\ldots,\frac{d}{r}-1$.

When $4\mid d$, the number of different eigenvalues is less than $3(d-1)$.

Proof. Let M be one of the matrices M_2, M_3 or M_4, and $v:=v_k=(0,0,\ldots,0,1,0\ldots,0)$ for $k=1,\ldots,(d-1)(e-1)$. The corresponding vanishing cycle to v_k is δ_a^b, with $a=\ \mathrm{mod}\ _e(k)$ and $b=\lfloor\frac{k}{e}\rfloor$. Since the matrices Ψ_e are skew-symmetric, then M is a normal matrix, consequently it is diagonalizable. Hence, its eigenvectors u_j are a basis for $\mathbb{R}^N$. Then we can write $v=\sum_j r_j u_j$ for scalars r_j. Let λ_j be the eigenvalue associated to u_j, thus we have

$$M^l v = \sum_{j=1}^{N} r_j\lambda_j^l u_j, \qquad \text{where } N=(d-1)(e-1),$$

and the matrix $\{v, Mv, M^2v,\ldots,M^nv\}$ is

$$\begin{pmatrix} r_1u_1 & r_2u_2 & \ldots & r_Nu_n\end{pmatrix}\begin{pmatrix}1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{N-1}\\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{N-1}\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & \lambda_N & \lambda_N^2 & \ldots & \lambda_N^{N-1}\end{pmatrix}.$$

The matrix in the right, is the Vandermonde matrix with determinant $\prod_{i<j}(\lambda_j - \lambda_i)$. Hence, if the eigenvalues are different, then the Krylov subspace is the span of the vectors u_l such that $r_l \neq 0$.

For $e = 2$, it is possible to show that the matrix M_2 is similar to a tridiagonal matrix with main diagonal of 1s, first diagonal below of -1s, and first diagonal above of 1s. The change of basis is given by the diagonal matrix, whose diagonal is $(-1,1,1,-1,-1,1,1,\ldots)$. Hence, there is a known closed form for the eigenvalues and eigenvectors of the matrix M_2 (see [Los92, Yue05, DFK19]). Namely, the eigenvalues are given by

$$\lambda_j = 1 + 2\sqrt{-1}\cos\left(\frac{j\pi}{d}\right), \qquad \text{whit } j = 1,\ldots,d-1.$$

If the vector $u_j = \left(u_j^{(1)}, u_j^{(2)}, \ldots, u_j^{(d-1)}\right)^T$ is the eigenvector associated to λ_j, then the $k-$th coordinate satisfies

$$u_j^{(k)} = (\sqrt{-1})^{k-1}\sqrt{\frac{2}{d}}\sin\left(\frac{kj\pi}{d}\right).$$

If we denote $U = [u_1 \quad u_2 \quad \cdots \quad u_{d-1}]$ the matrix whose columns are the eigenvalues, then $UU^* = Id(d-1)$. Hence, if we want to know which eigenvectors are used in the representation of δ_1^l, it is enough to note which terms in the row l of U are zero. This happens when $\frac{jl}{d} \in \mathbb{Z}$. Furthermore, the Krylov space of δ_1^l is contained in the Krylov space of $\delta_1^{l'}$, provided that the j's such that $\frac{jl'}{d} \in \mathbb{Z}$, satisfy $\frac{jl}{d} \in \mathbb{Z}$. It is equivalent to $gcd(d,l') \mid gcd(d,l)$.

For $e = 3,4$, we do not know a close form for the eigenvalues. However, for given values of d, on a computer we can compute explicitly the eigenvalues, and a basis for the subspace generated by these eigenvectors. Next, we determine which vectors of the canonical base $\mathbb{R}^N$ are in this subspace. If $e = 3$ and $3 \mid d$, then the number of different eigenvalues is less than N. The same is true for $e = 4$ and $4 \mid d$. In other cases the number of different eigenvalues is N. The reader can use the functions written in MATLAB, MonMatrix and VanCycleSub [1], for a numerical supplement of this proof (see §A.4.2). □

Remark 3.3.2. The condition $3 \nmid d$ in the case M_3 may be related with the fact that $y^3 + x^d$ is a pullback with the map $(x,y) \to (x^{\frac{d}{3}}, y)$. Analogously, the condition $4 \nmid d$ in the case M_4 associated to $y^4 + x^d$.

On the other hand, in general a monodromy matrices is not diagonalizable. For example, the monodromy matrices of the mirror quintic Calabi-Yau threefold (see [DM06]).

3.4 Monodromy problem for $y^4 + g(x)$

Let $g \in \mathbb{R}[x]_{\leq d}$ be a polynomial with real critical points. Consider the polynomial $f(x,y) := y^e + g(x)$ which has critical values equal to the critical values of g. Recall, in some cases we relate the vertices in the Dynkin diagram to the critical values associated with the vanishing cycles. Thus, we denote by C_j the critical value in the column j from left to right in the Dynkin diagram, and δ_i^j to the vanishing cycle in the row i over C_j. For example, if we suppose that d

[1] https://github.com/danfelmath/Intersection-matrix-for-polynomials-with-1-crit-value.git

is even and C_1 is a local maximum, then the Dynkin diagram looks like

$$
\begin{array}{ccccccccccccccc}
C_1 & \to & C_2 & \leftarrow & C_3 & \to & C_4 & \leftarrow & C_5 & \to & C_6 & \leftarrow \cdots \to & C_{d-2} & \leftarrow & C_{d-1} \\
\downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & & \downarrow & \nearrow & \downarrow \\
C_1 & \to & C_2 & \leftarrow & C_3 & \to & C_4 & \leftarrow & C_5 & \to & C_6 & \leftarrow \cdots \to & C_{d-2} & \leftarrow & C_{d-1} \\
\uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & & \uparrow & \searrow & \uparrow \\
C_1 & \to & C_2 & \leftarrow & C_3 & \to & C_4 & \leftarrow & C_5 & \to & C_6 & \leftarrow \cdots \to & C_{d-2} & \leftarrow & C_{d-1}
\end{array}
. \tag{3.9}
$$

Definition 3.4.1. We say that the Dynkin diagram of $y^e + g(x)$ with $g \in \mathbb{C}[x]_{\leq d}$ has **horizontal symmetry** if there exits integer $r > 1$ such that for any j with $\text{g.c.d}(j,d) = r$ the critical values satisfy

$$C_{j-k} = C_{j+k} \text{ where } k = 1, \dots, r-1.$$

The vanishing cycles $\delta_i^{l \cdot r}$ with $l = 1, \dots, \frac{d}{r} - 1$ are called **vanishing cycles with horizontal symmetry**. We can define the **vertical symmetry** analogously.For the Dynkin diagram (3.9) the cycles δ_2^j are **vanishing cycles with vertical symmetry**.

From a direct computation in the Dynkin diagram (3.9)and Picard-Lefschetz formula, we observe that only the terms

$$\delta_i^{j-k} + \delta_i^{j+k} \text{ with } \text{g.c.d}(j,d) = r$$

and the cycles with horizontal symmetry appear in the subspace generated by the monodromy action on a cycle with horizontal symmetry. This happens in an analogous way for the vertical symmetry. Therefore, the subspace generated by the monodromy action on a cycle with horizontal symmetry or vertical symmetry is different to $H_1(f^{-1}(b), \mathbb{Q})$.

On the other hand, the definition of horizontal symmetry only depend on the relation among the critical values of g. Hence, for p, q integers greater than 1, there are cycles with horizontal symmetry in the Dynkin diagram associated to $y^p + g(x)$ if and only if there are in the Dynkin diagram associated to $y^q + g(x)$.

For the vertical symmetry, in the next lemma we provide a geometric characterization of the cycles δ_2^j with $j = 1, \dots, d-1$.

Lemma 3.4.2. *Consider the map $\mathbb{C}^2 \xrightarrow{\pi} \mathbb{C}^2$, given by $\pi(x,y) = (x, y^2)$. The cycles $\delta_2^j \in H_1((y^4 + g(x))^{-1}(b))$ for $j = 1, \dots, d-1$, are in the kernel of*

$$\pi_* : H_1((y^4 + g(x))^{-1}(b)) \to H_1((y^2 + g(x))^{-1}(b)).$$

Proof. Consider the perturbation $h_\varepsilon(y) := y^4 + \varepsilon(-y^2 + \frac{\varepsilon}{8})$ of y^4, where $\varepsilon \geq 0$. The roots of $h_\varepsilon(y)$ are $\frac{\pm 1}{2}\sqrt{\varepsilon(2 \pm \sqrt{2})}$. Therefore, the 0-cycle associated to δ_2^j, is

$$\gamma_1 = \left(\frac{1}{2}\sqrt{\varepsilon(2-\sqrt{2})}, 0\right) - \left(\frac{-1}{2}\sqrt{\varepsilon(2-\sqrt{2})}, 0\right)$$

(see Figure 3.4). In the projection by y^2, these points are identified with $\left(\frac{1}{4}\varepsilon(2-\sqrt{2}), 0\right)$. Consequently, the image of the vanishing cycles $\delta_2^j = \gamma_1 * \sigma_j$ by the map π is trivial. $\square$

Note that the kernel of π_* is generated by the cycles

$$\gamma_1 * \sigma_j \quad \text{and} \quad (\gamma_3 - \gamma_2) * \sigma_j, \quad \text{for } j = 1, \dots d-1,$$

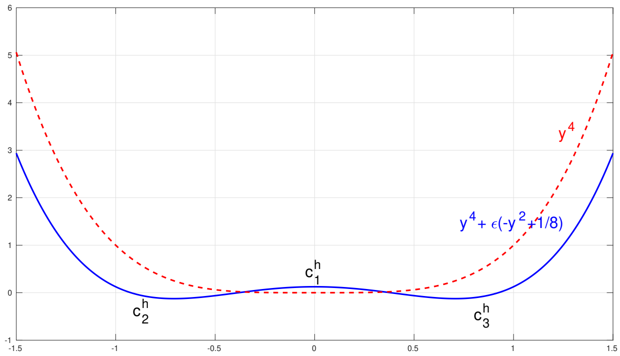

Figure 3.4: Critical values for $y^4 + \varepsilon(-y^2 + \frac{\varepsilon}{8})$, a perturbation of y^4.

however, the first ones generate the others by monodromy.

We want to study the vanishing cycles in the subspace generated by the monodromy action when there is no horizontal symmetry in the Dynkin diagram. By using Lemma 3.4.3, we can reduce this analysis to the cases $\pi_1(\mathbb{P}^1 \setminus C) \equiv \mathbb{Z}$ and $\pi_1(\mathbb{P}^1 \setminus C) \equiv \mathbb{Z}^2$.

Lemma 3.4.3. *Let g be a polynomial of degree d, and let G_1 and G_g be the monodromy groups associated to $y^4 + x^d$ and $y^4 + g(x)$, respectively. For any $v \in H_1((y^4+g)^{-1}(b), \mathbb{Z})$ the subspaces generated by the orbits satisfies $\langle G_1 \cdot v \rangle \subset \langle G_g \cdot v \rangle$. Besides, if C has more than one element, then there exist a group of two elements $G_2 < Aut(V_g)$ such that $\langle G_2 \cdot v \rangle \subset \langle G_g \cdot v \rangle$.*

Proof. Any element in G_g can be written as a matrix $I_N - A_j \in \mathcal{M}_{3(d-1)}(\mathbb{R})$. Moreover, this A_j is constructed by putting rows of zeros in the matrix Ψ_4 of equation (3.7). Thus, we have that $M_4 = \sum_{A_j \in G_g} I_N - A_j + (1 - |G_g|) I_N$. Consequently, $M_4^k v \in \langle G_g \cdot v \rangle$ for $k \in \mathbb{Z}$.

In order to construct G_2 is enough to divide in two groups the elements of G_g, and define two matrix as the sum of the matrices in these groups. Note that these sums correspond with identifications of some critical values in the Dynkin diagram. □

The next proposition follows from the proposition 3.3.1 and lemma 3.4.3.

Proposition 3.4.4. *Let g be a polynomial of degree d, where $d \leq 100$ and $4 \nmid d$. If the Dynkin diagram of $f(x,y) = y^4 + g(x)$ does not have horizontal symmetry, then the subspace of $H_1(f^{-1}(b), \mathbb{Q})$ generated by the orbit of a vanishing cycle δ_i^j contains all the vanishing cycles in the row i. Moreover, if i is 1 or 3, then the submodule generated is whole space $H_1(f^{-1}(b), \mathbb{Q})$.*

Proof. The restrictions on the degree d are due to the Proposition 3.3.1. By Proposition 3.3.1 the result is true for $gcd(j,d) = 1$ and $g(x) = x^d$, then by Lemma 3.4.3 is true for any $g \in \mathbb{C}[x]_{\leq d}$. If $gcd(j,d) = r > 1$, then $\mathrm{Mon}(\delta_i^j)$ contains the vanishing cycles δ_i^l with $l = rn$ and $n = 1, \ldots, \frac{d}{r} - 1$. Since the rows do not have horizontal symmetry, then d is prime or there are at least two different critical values. However if d is prime, then $r = 1$. Hence, by using lemma 3.4.3 we suppose that there are two critical values A and B. Thus, it is enough to consider a initial vanishing cycle $v := \delta_2^j$ where the critical values C_{j-l} and C_{l+j} are equal for $l = 1, \ldots, k-1 < r-1$. Also, the critical values C_{j-k} and C_{j+k} are different. We can suppose that the Dynkin diagram looks like

$$
\begin{tikzcd}[column sep=small]
* \arrow[r] \arrow[d] & \cdots & C_{j-k-1} \arrow[l] \arrow[r] \arrow[d] & C_{j-k} \arrow[d] & C_{j-k+1} \arrow[l] \arrow[r] \arrow[d] & \cdots & A \arrow[l] \arrow[r] \arrow[d] & A \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & \cdots & C_{k-1+j} \arrow[l] \arrow[r] \arrow[d] & C_{k+j} \arrow[d] & * \arrow[l] \arrow[r] \arrow[d] & \cdots & * \arrow[l] \arrow[d] \\
* \arrow[r] & \cdots & D \arrow[l] \arrow[r] & B \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & A \arrow[l] \arrow[r] & \cdots & A \arrow[l] \arrow[r] & A \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & A \arrow[l] \arrow[r] & \cdots & A \arrow[l] \arrow[r] & A \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & * \arrow[l] \arrow[r] & \cdots & * \arrow[l] \\
* \arrow[r] \arrow[u] & \cdots & D \arrow[l] \arrow[r] \arrow[u] & B \arrow[u] & A \arrow[l] \arrow[r] \arrow[u] & \cdots & A \arrow[l] \arrow[r] \arrow[u] & A \arrow[u] & A \arrow[l] \arrow[r] \arrow[u] & \cdots & A \arrow[l] \arrow[r] \arrow[u] & A \arrow[u] & * \arrow[l] \arrow[r] \arrow[u] & \cdots & * \arrow[l] \arrow[u]
\end{tikzcd}
.
$$

where D can be A or B, and $*$ means that no matter what value it is. We denote by Mon_A and Mon_B the monodromy action around to the critical values A and B, respectively. By doing $\mathrm{Mon}_B(\mathrm{Mon}_A)^{k-1}(v)$, we get a linear combination of cycles in the column k. In fact we have one of the next possibilities

$$(m_2m_1)^s v, m_1(m_2m_1)^s v, (m_1m_2)^s v, m_2(m_1m_2)^s v, \ s \in \mathbb{N},$$

where

$$m_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the matrices are in the basis $\delta_1^{j-k}, \delta_2^{j-k}, \delta_3^{j-k}$. Hence, we generate the linear combination $w := m\delta_1^{j-k} + n\delta_2^{j-k} + m\delta_3^{j-k}$ where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$. If $D = B$, taking $\mathrm{Mon}_B\mathrm{Mon}_A(w)$, we get

$$(n-3m)\delta_1^{j-k} + (2m-n)\delta_2^{j-k} + (n-3m)\delta_3^{j-k}, \text{ or}$$

$$(n-m)\delta_1^{j-k} + (2m-3n)\delta_2^{j-k} + (n-m)\delta_3^{j-k}.$$

Any of the linear combinations in the previous equation and w generate the vanishing cycle δ_2^{j-k}. If $D = A$, considering $\mathrm{Mon}_B(w)$ and w we also generate the cycle δ_2^{j-k}. If $gcd(j-k, d) = 1$, then the results follows from proposition 3.3.1. If $gcd(j-k, d) = r'$, then we repeat the previous analysis with r' instead of r, thus the proof follows from $r' < r$. □

The next propositions are proved with analogous arguments as in the proof of Proposition 3.4.4. For this reason, in their proof we only indicate the corresponding matrices m_1 and m_2 .

Proposition 3.4.5. *Let g be a polynomial of degree d, where $d \leq 100$ and $3 \nmid d$. If the Dynkin diagram of $f(x,y) = y^3 + g(x)$ does not have horizontal symmetry, then the subspace generated by the orbit of a vanishing cycle δ_i^j is the whole space $H_1(f^{-1}(b), \mathbb{Q})$.*

Proof. Consider the matrices

$$m_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

thus considering the initial vanishing cycle $v := \delta_2^j$, we get a vector $w = m\delta_1^j + n\delta_2^j$ where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$. □

Proposition 3.4.6. *If the Dynkin diagram of $f(x,y) = y^2 + g(x)$ does not have horizontal symmetry, then the subspace generated by the orbit of a vanishing cycle δ_i^j is the whole space $H_1(f^{-1}(b), \mathbb{Q})$.*

Proof. In this case the matrices are $m_1 = -1$ and $m_2 = 1$. □

The next theorem is the main result in [CM10], it is a solution of the monodromy problem for hyperelliptic curves $y^2 + g(x)$. We will use it, in order to solve the monodromy problem for $y^3 + g(x)$ and $y^4 + g(x)$. Although, this theorem holds for $g \in \mathbb{C}[x]$, we are interested in the case of g being a real polynomial with real critical points.

Theorem 3.4.7 (C. Christopher and P. Mardešić, 2008). *Let $f(x,y) = y^2 + g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the whole homology $H_1(f^{-1}(b), \mathbb{Q})$.*
2. *The polynomial g is decomposable (i.e., $g = g_2 \circ g_1$), and $\pi_* \delta(t)$ is homotopic to zero in $\{y^2 + g_2(z) = t\}$, where $\pi(x,y) = (g_1(x), y) = (z, y)$.*

From this theorem, we can show that the condition of g being decomposable as $g = g_2 \circ g_1$, is equivalent to the Dynkin diagram associated to $y^e + g(x)$ has horizontal symmetry, with $e > 1$. In fact, as we mentioned above, the horizontal symmetry condition only depend on g.

Proposition 3.4.8. *The next assertions are equivalents.*

1. *The polynomial can be written as $g = g_2 \circ g_1$, where g_1, g_2 are polynomials such that* $\deg(g_1), \deg(g_2) > 1$.
2. *The Dynkin diagram associated to $y^e + g(x)$ has horizontal symmetry, for some $e > 1$ (and hence for all $e > 1$).*

Proof. From Proposition 3.4.6, follows that the condition 2 implies that there are vanishing cycles for the fibration $f(x,y) = y^2 + g(x)$, such that they do not generates the whole $H_1(f^{-1}(b))$. Thus, by the Theorem 3.4.7, we conclude that 2 implies 1. The other implication follows by a direct computation on a Dynkin diagram similar to 3.9, but with $e - 1$ rows. □

Although the horizontal symmetry is just a condition on the polynomial g, this proposition allows to extend the result in the Theorem 3.4.7 to the fibrations defined by $y^4 + g(x)$ and $y^3 + g(x)$. The non trivial part for this generalization are due to the Propositions 3.4.4 and 3.4.5. However, since Proposition 3.3.1 is numerically proven, we have restrictions in the degree of g.

Theorem 3.4.9. *Let g be a polynomial with real critical points, and degree d such that, $4 \nmid d$ and $d \leq 100$. Consider the polynomial $f(x,y) = y^4 + g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the homology $H_1(f^{-1}(t), \mathbb{Q})$.*
2. *The polynomial g is decomposable (i.e., $g = g_2 \circ g_1$), and $\pi_* \delta(t)$ is homotopic to zero in $\{y^4 + g_2(z) = t\}$, where $\pi(x,y) = (g_1(x), y) = (z, y)$. Or, the cycle $\pi_* \delta(t)$ is homotopic to zero in $\{z^2 + g(x) = t\}$, where $\pi(x,y) = (x, y^2) = (x, z)$.*

Proof. The restrictions on the degree d are due to the Proposition 3.3.1, which is used in proposition 3.4.4. Let $\delta_i^j := \delta(t)$ be a vanishing cycle. If the monodromy of δ_i^j does not generate the homology $H_1(f^{-1}(t), \mathbb{Q})$, then considering the contrapositive of Proposition 3.4.4, we have the next possibilities: The index i is 2 or the cycle δ_i^j has horizontal symmetry. If $i = 2$, then by the Lemma 3.4.2, we have that $\pi_* \delta_2^j$ is trivial, where $\pi(x,y) = (x, y^2)$. If δ_i^j has horizontal symmetry, then by using Proposition 3.4.8 we conclude that $g = g_2 \circ g_1$. Furthermore, δ_i^j is in correspondence with a cycle with horizontal symmetry in the Dynkin diagram of $y^2 + g(x)$. Consequently, δ_i^j is in the kernel of π_*, where $\pi(x,y) = (g_1(x), y)$.

On the other hand, if the condition 2 is true, then the vanishing cycle δ_i^j has vertical or horizontal symmetry. In any of these cases, it follows by direct computation in the Dynkin diagram 3.9, that the subspace generated by the orbit of the monodromy action on δ_i^j is different to $H_1(f^{-1}(t), \mathbb{Q})$. $\square$

We have an analogous result for degree $y^3 + g(x)$. Note that in this case there are not cycles with vertical symmetry.

Theorem 3.4.10. *Let g be a polynomial with real critical points, and degree d such that, $3 \nmid d$ and $d \leq 100$. Consider the polynomial $f(x,y) = y^3 + g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the homology $H_1(f^{-1}(t), \mathbb{Q})$.*
2. *The polynomial g is decomposable (i.e., $f = g_2 \circ g_1$), and $\pi_*\delta(t)$ is homotopic to zero in $\{y^3 + g_2(z) = t\}$, where $\pi(x,y) = (g_1(x), y)) = (z, y)$.*

Proof. The restrictions on the degree d are due to the Proposition 3.3.1, which is used in proposition 3.4.5. Let $\delta_i^j := \delta(t)$ be a vanishing cycle. If the monodromy of δ_i^j does not generate the homology $H_1(f^{-1}(t), \mathbb{Q})$, then by Proposition 3.4.5, the cycle δ_i^j has horizontal symmetry. Hence, by using Proposition 3.4.8 we conclude that $g = g_2 \circ g_1$. $\square$

3.5 Monodromy problem for 4th degree polynomials $h(y) + g(x)$

Consider $f(x,y) = h(y) + g(x)$ where $h \in \mathbb{R}[y]_{\leq 4}$ and $g \in \mathbb{R}[x]_{\leq 4}$, and b is a regular value. Moreover, we suppose that the critical points of h and g are reals. The aim of this section is to compute the part of the homology $H_1(f^{-1}(b))$ generated by the action of the monodromy. From the equation (3.1) it follows that the 1-dimensional Dynkin diagram depends on the 0-dimensional Dynkin diagrams of h and g. Let $\gamma_i \in H_0(h^{-1}(b), \mathbb{Z})$ and $\sigma_i \in H_0(g^{-1}(b), \mathbb{Z})$ be the 0-cycles, where $i = 1, 2, 3$. Thus, using the enumeration of vanishing cycles, indicated in § 3.2, we have the next three cases,

$\gamma_1 - \gamma_3 - \gamma_2$, $\sigma_2 - \sigma_1 - \sigma_3$ resulting in:

$$\begin{tikzcd}
\gamma_1 * \sigma_2 \arrow[d] & \gamma_3 * \sigma_2 \arrow[l] \arrow[r] \arrow[d] & \gamma_2 * \sigma_2 \arrow[d] \\
\gamma_1 * \sigma_1 \arrow[ur] \arrow[dr] & \gamma_3 * \sigma_1 \arrow[l] \arrow[r] & \gamma_2 * \sigma_1 \arrow[ul] \arrow[dl] \\
\gamma_1 * \sigma_3 \arrow[u] & \gamma_3 * \sigma_3 \arrow[l] \arrow[r] \arrow[u] & \gamma_2 * \sigma_3 \arrow[u]
\end{tikzcd}$$

$\gamma_1 - \gamma_3 - \gamma_2$, $\sigma_1 - \sigma_3 - \sigma_2$ resulting in:

$$\begin{tikzcd}
\gamma_1 * \sigma_1 \arrow[dr] & \gamma_3 * \sigma_1 \arrow[l] \arrow[r] & \gamma_2 * \sigma_1 \arrow[dl] \\
\gamma_1 * \sigma_3 \arrow[u] \arrow[d] & \gamma_3 * \sigma_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \gamma_2 * \sigma_3 \arrow[u] \arrow[d] \\
\gamma_1 * \sigma_2 \arrow[ur] & \gamma_3 * \sigma_2 \arrow[l] \arrow[r] & \gamma_2 * \sigma_2 \arrow[ul]
\end{tikzcd}$$

$$\gamma_2 \dashrightarrow \gamma_1 \dashrightarrow \gamma_3 \qquad , \qquad \sigma_2 \dashrightarrow \sigma_1 \dashrightarrow \sigma_3 \qquad \text{resulting in:}$$

$$\begin{tikzcd}
\gamma_2 * \sigma_2 \arrow[r] \arrow[d] & \gamma_1 * \sigma_2 \arrow[d] & \gamma_3 * \sigma_2 \arrow[l] \arrow[d] \\
\gamma_2 * \sigma_1 \arrow[r] & \gamma_1 * \sigma_1 \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \gamma_3 * \sigma_1 \arrow[l] \\
\gamma_2 * \sigma_3 \arrow[r] \arrow[u] & \gamma_1 * \sigma_3 \arrow[u] & \gamma_3 * \sigma_3 . \arrow[l] \arrow[u]
\end{tikzcd}$$

If we consider $-f$ instead of f, the two last Dynkin diagram coincide. Hence, we only focus in the first two 1-dimensional Dynkin diagrams.

Example 3.5.1. Consider the polynomials $h(y) = -y^4 + 9y^2$ and $g(x) = -x^4 + 16x^2 + 8x$. In Figure 3.5, we show the real part of this polynomials with the critical values indexed according to §3.2. Let $f_1(x, y) = h(y) + g(x)$, thus the Dynkin diagram associated to f_1 is the first one.

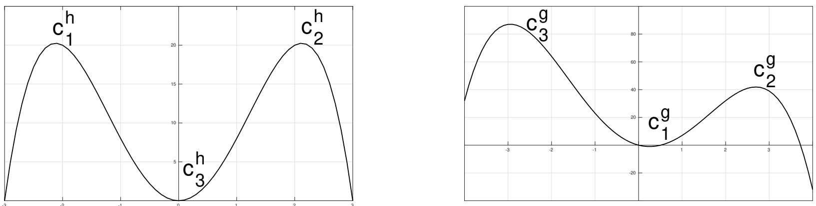

Figure 3.5: Real part of the polynomials $h(y) = -y^4 + 9y^2$ and $g(x) = -x^4 + 16x^2 + 8x$, with its critical values. On the left is $h(y)$ and on the right $g(x)$.

Example 3.5.2. Consider the polynomials $h(y) = -y^4+9y^2$ and $g(x) = x^4-16x^2-8x$. In Figure 3.6, we present the real part of this polynomials with the critical values indexed according to §3.2. Let $f_2(x, y) = h(y) + g(x)$, thus the Dynkin diagram associated to f_2 is the second.

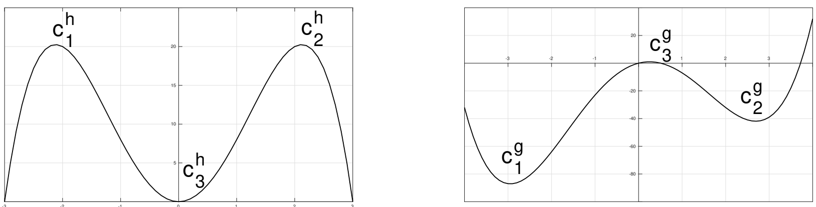

Figure 3.6: Real part of the polynomials $h(y) = -y^4 + 9y^2$ and $g(x) = x^4 - 16x^2 - 8x$, with its critical values. On the left is $h(y)$ and on the right $g(x)$.

In Figures 3.7 and 3.8, we present the real part of the fibration $f_1(x, y) = t$ and $f_2(x, y) = t$, respectively. Note that the maximum corresponds with the addition of the maximum of h and g, analogously for the minimum. The others critical points are known as saddles points. We denote the critical values as

$$a_1 = c^h_1 + c^g_1 \ , \ a_2 = c^h_1 + c^g_2 \ , \ a_3 = c^h_1 + c^g_3$$
$$a_4 = c^h_2 + c^g_1 \ , \ a_5 = c^h_2 + c^g_2 \ , \ a_6 = c^h_2 + c^g_3$$
$$a_7 = c^h_3 + c^g_1 \ , \ a_8 = c^h_3 + c^g_2 \ , \ a_9 = c^h_3 + c^g_3 .$$

If we consider the contour lines associated to the Figure 3.7, then we obtain a drawing in the plane which represent the vertex in the Dynkin diagram associated to the polynomial f_1. In

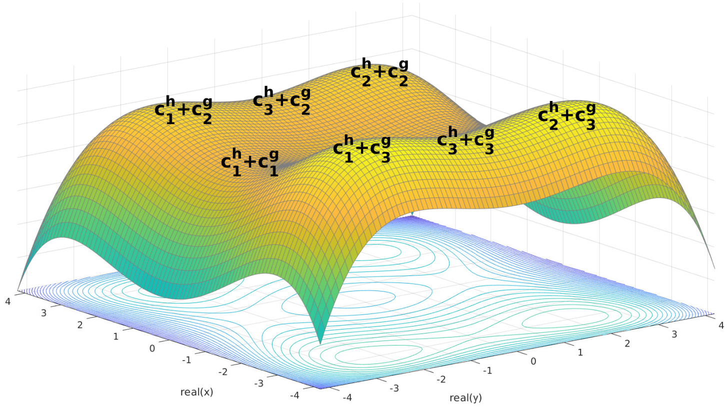

Figure 3.7: Real part of graph defined by $f_1(x,y) = -y^4 + 9y^2 - x^4 + 16x^2 + 8x = t$, with its critical values.

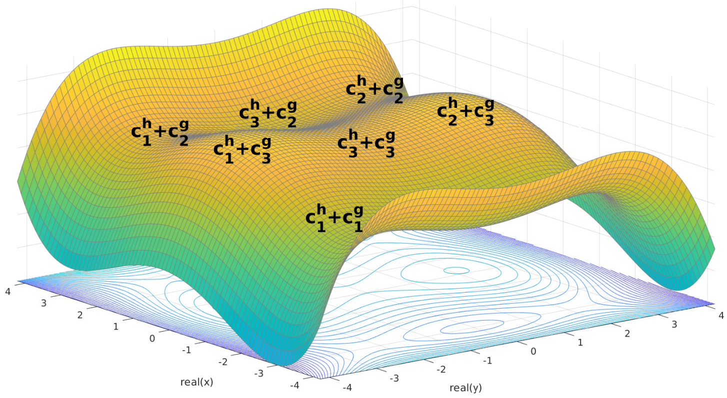

Figure 3.8: Real part of graph defined by $f_2(x,y) = -y^4 + 9y^2 + x^4 - 16x^2 - 8x = t$, with its critical values.

fact, the correspondence between the Dynkin diagram and the curve in the plane is shown in [A'C75]. In Figure 3.9, we present the contour lines of the real part of the polynomial $f_1(x,y)$. The critical values a_2, a_3, a_5, a_6 and a_7 correspond with ovals contained in the real part of the foliation defined by df_1.

Analogously, the contour lines of the Figure 3.8 give us a drawing in the plane which represent the vertex in the Dynkin diagram associated to f_2. In Figure 3.10, we present the contour lines of the real part of the polynomial $f_2(x,y)$. In this case, the critical values a_3, a_6, a_7 and a_8 correspond with ovals contained in the real part of the foliation defined by df_2.

Let α_i be the cycle which vanishes in the critical point corresponding to the critical value a_i for $i = 1, ..., 9$. For simplicity in the notation we call the possible critical values as $a, b, c, d, e, f, g, h, i$ if all are different and we are removing from right to left as soon as the critical values are repeated. Moreover, we indicate with "*" on the right, the vanishing cycles which is not contained in the real plane (or its associated critical value). For instance, the polynomial $f_1(x,y)$ of the Example 3.5.1 satisfies that $a_1 = a_4, a_2 = a_5, a_3 = a_6$ and the other critical values

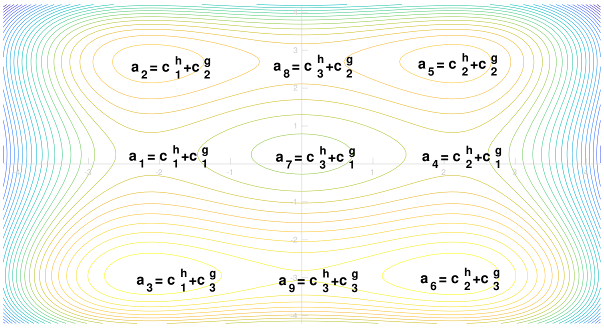

Figure 3.9: Contour lines of the real part of $f_1(x,y) = -y^4 + 9y^2 - x^4 + 16x^2 + 8x = t$.

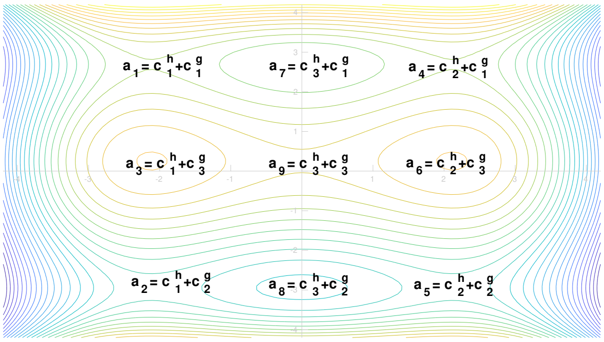

Figure 3.10: Contour lines of the real part of $f_1(x,y) = -y^4 + 9y^2 + x^4 - 16x^2 - 8x = t$.

are different, therefore its next Dynkin diagram is

$$\begin{tikzcd}
b \arrow[d] & e^* \arrow[l] \arrow[r] \arrow[d] & b \arrow[d] \\
a^* \arrow[ur] \arrow[dr] & d \arrow[l] \arrow[r] & a^* \arrow[ul] \arrow[dl] \\
c \arrow[u] & f^* \arrow[l] \arrow[r] \arrow[u] & c, \arrow[u]
\end{tikzcd}$$

Similarly, the polynomial $f_2(x,y)$ of the Example 3.5.2, has Dynkin diagram

$$\begin{tikzcd}
a^* \arrow[dr] & d \arrow[l] \arrow[r] & a^* \arrow[dl] \\
c \arrow[u] \arrow[d] & e^* \arrow[l] \arrow[r] \arrow[u] \arrow[d] & c \arrow[u] \arrow[d] \\
b^* \arrow[ur] & f \arrow[l] \arrow[r] & b^*. \arrow[ul]
\end{tikzcd}$$

The subspace of the homology $H_1(f^{-1}(b))$ generated by the monodromy action on a vanishing cycle α_i is denoted $\mathrm{Mon}(\alpha_i)$ $i = 1, \ldots, 9$. We will compute the monodromy for any α_i depending on the number of different critical values.

For the Dynkin diagram

$$\begin{tikzcd}
a_2 \arrow[d] & a_8^* \arrow[l] \arrow[r] \arrow[d] & a_5 \arrow[d] \\
a_1^* \arrow[ur] \arrow[dr] & a_7 \arrow[l] \arrow[r] & a_4^* \arrow[ul] \arrow[dl] \\
a_3 \arrow[u] & a_9^* \arrow[l] \arrow[r] \arrow[u] & a_6, \arrow[u]
\end{tikzcd} \tag{3.10}$$

when there is one critical value the ranks of the subspaces are: $\mathrm{rank}(\mathrm{Mon}(\alpha_i)) = 5$ for $i \neq 7$ and $\mathrm{rank}(\mathrm{Mon}(\alpha_7)) = 3$. For more than one critical value, in Table 3.1 we present the cases where the vanishing cycles are not simple cycles.

For the Dynkin diagram

$$\begin{tikzcd}
a_1^* \arrow[dr] & a_7 \arrow[l] \arrow[r] & a_4^* \arrow[dl] \\
a_3 \arrow[u] \arrow[d] & a_9^* \arrow[l] \arrow[r] \arrow[u] \arrow[d] & a_6 \arrow[u] \arrow[d] \\
a_2^* \arrow[ur] & a_8 \arrow[l] \arrow[r] & a_5^*, \arrow[ul]
\end{tikzcd} \tag{3.11}$$

when there is one critical value the ranks of the subspaces are: $\mathrm{rank}(\mathrm{Mon}(\alpha_i)) = 5$ for $i \neq 9$ and $\mathrm{rank}(\mathrm{Mon}(\alpha_9)) = 3$. For more than one critical value, in Table 3.2 we present the cases where the vanishing cycles are not simple cycles.

In Tables 3.1 and 3.2, the first column is the number of different critical values. In the second column are written the vanishing cycles which are not simple cycles. Right in front of any non simple vanishing cycle α_i, in the the third column, it is a basis for the subspace $\mathrm{Mon}(\alpha_i)$. Note that there are vanishing cycles which generate the same subspace, then they are on the same line in the second column. In fourth column are the corresponding Dynkin diagram. Finally, in the last column, we add information about an equivalence class, which is explained below. This last column together with the Theorem 3.5.4 give us examples of polynomials that satisfy these diagrams.

# critical values	α_i	$\mathrm{Mon}(\alpha_i)$	Dynkin diagram of $f(x,y)=h(x)+g(y)$	$[f]$
2	α_1^*,α_4^* α_8^*,α_9^* α_7	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$ $\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$ $\langle\alpha_7,\alpha_1^*+\alpha_4^*,\alpha_8^*+\alpha_9^*,\alpha_2+\alpha_3+\alpha_5+\alpha_6\rangle$	$\begin{smallmatrix} a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & b^* & \rightarrow & a \end{smallmatrix}$ $\begin{smallmatrix} a & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & b & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & a^* & \rightarrow & a \end{smallmatrix}$	$\mathfrak{O}_3$
2	$\alpha_7,\alpha_8^*,\alpha_9^*$	$\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$	$\begin{smallmatrix} a & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & b^* & \rightarrow & b \end{smallmatrix}$	$\mathfrak{O}_2$
2	$\alpha_1^*,\alpha_4^*,\alpha_7$	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$	$\begin{smallmatrix} b & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & a \end{smallmatrix}$	$\mathfrak{O}_2$
3	α_1^*,α_4^* α_2,α_6 α_3,α_5 α_8^*,α_9^* α_7	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$ $\langle\alpha_2,\alpha_6,\alpha_7,\alpha_1^*-\alpha_8^*,\alpha_4^*-\alpha_9^*,\alpha_3+\alpha_5,\alpha_1^*+\alpha_4^*+\alpha_8^*+\alpha_9^*\rangle$ $\langle\alpha_3,\alpha_5,\alpha_7,\alpha_1^*-\alpha_9^*,\alpha_4^*-\alpha_8^*,\alpha_2+\alpha_6,\alpha_1^*+\alpha_4^*+\alpha_8^*+\alpha_9^*\rangle$ $\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$ $\langle\alpha_7,\alpha_1^*+\alpha_4^*,\alpha_8^*+\alpha_9^*,\alpha_2+\alpha_3+\alpha_5+\alpha_6\rangle$	$\begin{smallmatrix} b & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & b \end{smallmatrix}$	$\mathfrak{O}_4$
3	$\alpha_7,\alpha_8^*,\alpha_9^*$	$\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$	$\begin{smallmatrix} b & \leftarrow & b^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & c^* & \rightarrow & c \end{smallmatrix}$ $\begin{smallmatrix} a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \end{smallmatrix}$	$\mathfrak{O}_2$
3	$\alpha_1^*,\alpha_4^*,\alpha_7$	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$	$\begin{smallmatrix} a & \leftarrow & c^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & c^* & \rightarrow & b \end{smallmatrix}$ $\begin{smallmatrix} b & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & c^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & a \end{smallmatrix}$	$\mathfrak{O}_2$
4	α_1^*,α_4^* α_8^*,α_9^* α_7	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$ $\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$ $\langle\alpha_7,\alpha_1^*+\alpha_4^*,\alpha_8^*+\alpha_9^*,\alpha_2+\alpha_3+\alpha_5+\alpha_6\rangle$	$\begin{smallmatrix} a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ c^* & \leftarrow & d & \rightarrow & c^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & b^* & \rightarrow & a \end{smallmatrix}$	$\mathfrak{O}_3$
4	$\alpha_7,\alpha_8^*,\alpha_9^*$	$\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$	$\begin{smallmatrix} a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & d^* & \rightarrow & c \end{smallmatrix}$ $\begin{smallmatrix} b & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & b^* & \rightarrow & c \end{smallmatrix}$	$\mathfrak{O}_2$
4	$\alpha_1^*,\alpha_4^*,\alpha_7$	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$	$\begin{smallmatrix} b & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ d^* & \leftarrow & c & \rightarrow & c^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & a \end{smallmatrix}$ $\begin{smallmatrix} c & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & b \end{smallmatrix}$	$\mathfrak{O}_2$
5	$\alpha_7,\alpha_8^*,\alpha_9^*$	$\langle\alpha_7,\alpha_8,\alpha_9,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$	$\begin{smallmatrix} b & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & e^* & \rightarrow & c \end{smallmatrix}$ $\begin{smallmatrix} b & \leftarrow & e^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \end{smallmatrix}$ $\begin{smallmatrix} a & \leftarrow & e^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & d & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \end{smallmatrix}$	$\mathfrak{O}_2$
5	$\alpha_1^*,\alpha_4^*,\alpha_7$	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$	$\begin{smallmatrix} c & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ e^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & b \end{smallmatrix}$ $\begin{smallmatrix} c & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & e^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & b \end{smallmatrix}$ $\begin{smallmatrix} c & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & e^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & b^* & \rightarrow & a \end{smallmatrix}$	$\mathfrak{O}_2$
6	$\alpha_7,\alpha_8^*,\alpha_9^*$	$\langle\alpha_7,\alpha_8^*,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2+\alpha_5,\alpha_3+\alpha_6\rangle$	$\begin{smallmatrix} b & \leftarrow & e^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & f^* & \rightarrow & c \end{smallmatrix}$	$\mathfrak{O}_2$
6	$\alpha_1^*,\alpha_4^*,\alpha_7$	$\langle\alpha_1^*,\alpha_4^*,\alpha_7,\alpha_2+\alpha_3,\alpha_5+\alpha_6,\alpha_8^*+\alpha_9^*\rangle$	$\begin{smallmatrix} a & \leftarrow & c^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ d^* & \leftarrow & f & \rightarrow & e^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & c^* & \rightarrow & b \end{smallmatrix}$	$\mathfrak{O}_2$

Table 3.1: Monodromy for $h(x)+g(y)\in\mathbb{R}[x,y]_{d\leq 4}$ and Dynkin diagram (3.10)

# critical values	α_i	Mon(α_i)	Dynkin diagram of $f(x,y=)h(x)+g(y)$	$[f]$
2	α_3,α_6 α_7,α_8 α_9^*	$\langle \alpha_3,\alpha_6,\alpha_9^*,\alpha_1^*+\alpha_2^*,\alpha_7+\alpha_8,\alpha_4^*+\alpha_5^*\rangle$ $\langle \alpha_7,\alpha_8,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2^*+\alpha_5^*,\alpha_3+\alpha_6\rangle$ $\langle \alpha_9^*,\alpha_3+\alpha_6,\alpha_7+\alpha_8,\alpha_1^*+\alpha_2^*+\alpha_4^*+\alpha_5^*\rangle$	$\begin{smallmatrix} a^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & b^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & a & \rightarrow & a^* \end{smallmatrix}$ $\begin{smallmatrix} a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \end{smallmatrix}$	$\mathfrak{O}_3$
2	$\alpha_7,\alpha_8,\alpha_9^*$	$\langle \alpha_7,\alpha_8,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2^*+\alpha_5^*,\alpha_3+\alpha_6\rangle$	$\begin{smallmatrix} b^* & \leftarrow & b & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & a & \rightarrow & a^* \end{smallmatrix}$	$\mathfrak{O}_2$
2	$\alpha_3,\alpha_6,\alpha_9^*$	$\langle \alpha_3,\alpha_6,\alpha_9^*,\alpha_1^*+\alpha_2^*,\alpha_4^*+\alpha_5^*,\alpha_7+\alpha_8\rangle$	$\begin{smallmatrix} b^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & a^* \end{smallmatrix}$	$\mathfrak{O}_2$
3	α_3,α_6 α_7,α_8 α_9^*	$\langle \alpha_3,\alpha_6,\alpha_9^*,\alpha_1^*+\alpha_2^*,\alpha_7+\alpha_8,\alpha_4^*+\alpha_5^*\rangle$ $\langle \alpha_7,\alpha_8,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2^*+\alpha_5^*,\alpha_3+\alpha_6\rangle$ $\langle \alpha_9^*,\alpha_3+\alpha_6,\alpha_7+\alpha_8,\alpha_1^*+\alpha_2^*+\alpha_4^*+\alpha_5^*\rangle$	$\begin{smallmatrix} a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \end{smallmatrix}$	$\mathfrak{O}_3$
3	$\alpha_7,\alpha_8,\alpha_9^*$	$\langle \alpha_7,\alpha_8,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2^*+\alpha_5^*,\alpha_3+\alpha_6\rangle$	$\begin{smallmatrix} a^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & c^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & b & \rightarrow & b^* \end{smallmatrix}$ $\begin{smallmatrix} a^* & \leftarrow & c & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ b & \leftarrow & a^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & b^* \end{smallmatrix}$	$\mathfrak{O}_2$
3	$\alpha_3,\alpha_6,\alpha_9^*$	$\langle \alpha_3,\alpha_6,\alpha_9^*,\alpha_1^*+\alpha_2^*,\alpha_4^*+\alpha_5^*,\alpha_7+\alpha_8\rangle$	$\begin{smallmatrix} a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & c^* & \rightarrow & b \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \end{smallmatrix}$ $\begin{smallmatrix} a^* & \leftarrow & b & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & b^* \end{smallmatrix}$	$\mathfrak{O}_2$
4	α_3,α_6 α_7,α_8 α_9^*	$\langle \alpha_3,\alpha_6,\alpha_9^*,\alpha_1^*+\alpha_2^*,\alpha_7+\alpha_8,\alpha_4^*+\alpha_5^*\rangle$ $\langle \alpha_7,\alpha_8,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2^*+\alpha_5^*,\alpha_3+\alpha_6\rangle$ $\langle \alpha_9^*,\alpha_3+\alpha_6,\alpha_7+\alpha_8,\alpha_1^*+\alpha_2^*+\alpha_4^*+\alpha_5^*\rangle$	$\begin{smallmatrix} a^* & \leftarrow & b & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & d^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \end{smallmatrix}$	$\mathfrak{O}_3$
4	$\alpha_7,\alpha_8,\alpha_9^*$	$\langle \alpha_7,\alpha_8,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2^*+\alpha_5^*,\alpha_3+\alpha_6\rangle$	$\begin{smallmatrix} c^* & \leftarrow & d & \rightarrow & c^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ a & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & b & \rightarrow & a^* \end{smallmatrix}$ $\begin{smallmatrix} a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & b^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & b^* \end{smallmatrix}$	$\mathfrak{O}_2$
4	$\alpha_3,\alpha_6,\alpha_9^*$	$\langle \alpha_3,\alpha_6,\alpha_9^*,\alpha_1^*+\alpha_2^*,\alpha_4^*+\alpha_5^*,\alpha_7+\alpha_8\rangle$	$\begin{smallmatrix} b^* & \leftarrow & a & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & c^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & a^* \end{smallmatrix}$ $\begin{smallmatrix} a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & b^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \end{smallmatrix}$	$\mathfrak{O}_2$
5	$\alpha_7,\alpha_8,\alpha_9^*$	$\langle \alpha_7,\alpha_8,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2^*+\alpha_5^*,\alpha_3+\alpha_6\rangle$	$\begin{smallmatrix} a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & e^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & a & \rightarrow & b^* \end{smallmatrix}$ $\begin{smallmatrix} a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & e & \rightarrow & b^* \end{smallmatrix}$ $\begin{smallmatrix} b^* & \leftarrow & d & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & a^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & e & \rightarrow & a^* \end{smallmatrix}$	$\mathfrak{O}_2$
5	$\alpha_3,\alpha_6,\alpha_9^*$	$\langle \alpha_3,\alpha_6,\alpha_9^*,\alpha_1^*+\alpha_2^*,\alpha_4^*+\alpha_5^*,\alpha_7+\alpha_8\rangle$	$\begin{smallmatrix} a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & e^* & \rightarrow & a \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \end{smallmatrix}$ $\begin{smallmatrix} a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & a^* & \rightarrow & e \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \end{smallmatrix}$ $\begin{smallmatrix} b^* & \leftarrow & c & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & a^* & \rightarrow & e \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & c & \rightarrow & a^* \end{smallmatrix}$	$\mathfrak{O}_2$
6	$\alpha_7,\alpha_8,\alpha_9^*$	$\langle \alpha_7,\alpha_8,\alpha_9^*,\alpha_1^*+\alpha_4^*,\alpha_2^*+\alpha_5^*,\alpha_3+\alpha_6\rangle$	$\begin{smallmatrix} a^* & \leftarrow & d & \rightarrow & a^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ c & \leftarrow & e^* & \rightarrow & c \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ b^* & \leftarrow & f & \rightarrow & b^* \end{smallmatrix}$	$\mathfrak{O}_2$
6	$\alpha_3,\alpha_6,\alpha_9^*$	$\langle \alpha_3,\alpha_6,\alpha_9^*,\alpha_1^*+\alpha_2^*,\alpha_4^*+\alpha_5^*,\alpha_7+\alpha_8\rangle$	$\begin{smallmatrix} a^* & \leftarrow & c & \rightarrow & b^* \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ d & \leftarrow & e^* & \rightarrow & f \\ \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\ a^* & \leftarrow & c & \rightarrow & b^* \end{smallmatrix}$	$\mathfrak{O}_2$

Table 3.2: Monodromy for $h(x)+g(y)\in\mathbb{R}[x,y]_{d\leq 4}$ and Dynkin diagram (3.11)

Let $f(x,y) = h(y) + g(y)$ be a polynomial of degree 4, we consider the vector space $V_f = H_1(f^{-1}(b),\mathbb{Q})$ with the basis given by the vanishing cycles $\{\alpha_i\}_{i=1,\dots,9}$. Let $G_f = \pi_1(\mathbb{C} \setminus \{a_1,\dots,a_9\})$ be a free group acting on V_f by monodromy. For polynomials f, f', we relate f and f' if there is a permutation φ of the set $\{\alpha_i\}_{i=1,\dots,9}$, such that

$$\operatorname{span}(G_f \cdot (\varphi(\alpha_i)) = \varphi(\operatorname{span}(G_{f'} \cdot \alpha_i)), \text{ for } i = 1,\dots,9.$$

Let us introduce the suggestive notation $v_{11}, v_{12}, v_{13},\ v_{21}, v_{22}, v_{23}, v_{31}, v_{32}, v_{33}$ as a basis for V_f. Then, we compute the equivalence classes $[f]$ of the polynomials in $\mathbb{R}[x]_{\le 4} \oplus \mathbb{R}[y]_{\le 4}$ with real critical points. From Tables 3.1 and 3.2, we conclude that there are 5 equivalence classes of these polynomials, they are

- $\mathfrak{O}_0$:

  $\operatorname{span}(G_f \cdot v_{ij}) = V_f$, for $i,j = 1,2,3$.

- $\mathfrak{O}_1$: In this cases G_f is a free group generated by a matrix M, and

  $\operatorname{span}(G_f \cdot v_{ij}) = \langle M^k v_{ij}\rangle$ with $k = 0,\dots,4$ and $(i,j) \neq (2,2)$.

  $\operatorname{span}(G_f \cdot v_{22}) = \langle M^k v_{ij}\rangle$ with $k = 0,\dots,2$.

- $\mathfrak{O}_2$:

  $\operatorname{span}(G_f \cdot v_{21}) = \operatorname{span}(G_f \cdot v_{22}) = \operatorname{span}(G_f \cdot v_{23}) = \langle v_{21}, v_{22}, v_{23}, v_{11} + v_{31}, v_{12} + v_{32}, v_{13} + v_{33}\rangle$.

  $\operatorname{span}(G_f \cdot v_{ij}) = V_f$, in other cases.

- $\mathfrak{O}_3$:

  $\operatorname{span}(G_f \cdot v_{21}) = \operatorname{span}(G_f \cdot v_{23}) = \langle v_{21}, v_{22}, v_{23}, v_{11} + v_{31}, v_{12} + v_{32}, v_{13} + v_{33}.\rangle$.

  $\operatorname{span}(G_f \cdot v_{12}) = \operatorname{span}(G_f \cdot v_{32}) = \langle v_{12}, v_{22}, v_{32}, v_{11} + v_{13}, v_{21} + v_{23}, v_{31} + v_{33}\rangle$.

  $\operatorname{span}(G_f \cdot v_{22}) = \langle v_{22}, v_{12} + v_{32}, v_{21} + v_{23}, v_{11} + v_{13} + v_{31} + v_{33}\rangle$.

  $\operatorname{span}(G_f \cdot v_{ij}) = V_f$, in other cases.

- $\mathfrak{O}_4$:

  $\operatorname{span}(G_f \cdot v_{21}) = \operatorname{span}(G_f \cdot v_{23}) = \langle v_{21}, v_{22}, v_{23}, v_{11} + v_{31}, v_{12} + v_{32}, v_{13} + v_{33}.\rangle$.

  $\operatorname{span}(G_f \cdot v_{12}) = \operatorname{span}(G_f \cdot v_{32}) = \langle v_{12}, v_{22}, v_{32}, v_{11} + v_{13}, v_{21} + v_{23}, v_{31} + v_{33}\rangle$.

  $\operatorname{span}(G_f \cdot v_{11}) = \operatorname{span}(G_f \cdot v_{33}) = \langle v_{11}, v_{22}, v_{33}, v_{12} - v_{21}, v_{23} - v_{32}, v_{13} + v_{31}, v_{12} + v_{21} + v_{23} + v_{32}\rangle$.

  $\operatorname{span}(G_f \cdot v_{13}) = \operatorname{span}(G_f \cdot v_{31}) = \langle v_{13}, v_{22}, v_{31}, v_{21} - v_{32}, v_{12} - v_{23}, v_{11} + v_{33}, v_{12} + v_{21} + v_{23} + v_{32}\rangle$.

  $\operatorname{span}(G_f \cdot v_{22}) = \langle v_{22}, v_{12} + v_{32}, v_{21} + v_{23}, v_{11} + v_{13} + v_{31} + v_{33}\rangle$.

These equivalence classes of polynomial $f(x,y) = h(x)+g(y)$ in terms of the subspaces generated by the orbit of monodromy action, can be written in terms of the polynomials h and g. That is showed in the theorem 3.5.4. We also consider polynomials up to linear transformation, because these do not change the monodromy action. For a polynomial $h \in \mathbb{C}[x]_{\le d}$ and a partition $(d_1, d_2, \dots, d_M)$ of $d-1$, we say that h has **critical values degree** $(d_1, d_2, \dots, d_M)$ if it has M different critical values $c_1, c_2, \dots, c_M$ and for any $i = 1\dots, M$ there are d_i critical points over c_i, counted with multiplicity. The next lemma is proved in appendix A.1, and we give a algorithm to compute the ideals.

Lemma 3.5.3. *Given a positive integer d and a partition $(d_1, d_2, \dots, d_M)$ of $d-1$, the set of polynomials in $\mathbb{C}[x]_{\le d}$ with critical values degree $(d_1, d_2, \dots, d_M)$ is an algebraic subvariety of $\mathbb{C}[x]_{\le d}$, and its ideal associated is denoted $I_{(d_1,d_2,\dots,d_M)}$.*

For example fo $d = 4$, we consider $h(x) := x^4 + r_3x^3 + r_2x^2 + r_1x + r_0$. Since translations in the abscissa and in the ordinate do not change the monodromy action, we can suppose that $h(x) := x^4 + r_2x^2 + r_1x$. Thus, we have

$$\begin{aligned} I_{(3,0)} &= \langle r_1, r_2 \rangle, \\ I_{(2,1)} &= \langle r_1 \rangle \cap \langle 27r_1^2 + 8r_2^3 \rangle =: \langle r_1 \rangle \cap \langle H \rangle, \\ I_{(1,1,1)} &= 0. \end{aligned}$$

Another interesting example, is when we consider the polynomials $h(x) = x^4 + r_2x^2 + r_1x$, $g(y) = y^4 + s_2y^2 + s_1y$ and we suppose that the critical values of h are equal to the critical values of g. In this case, we have a subvariety of $\mathbb{C}[x,y]_{\leq d}$ given by the ideal

$$\langle r_2 - s_2, r_1 - s_1 \rangle \cap \langle r_2 + s_2, r_1^2 + s_1^2 \rangle \cap \langle r_2 - s_2, r_1 + s_1 \rangle \cap \langle s_2, s_1, r_2, r_1 \rangle.$$

Theorem 3.5.4. *Let $f(x,y) = h(x) + g(y)$, where $h \in \mathbb{R}[x]_{\leq 4}$ and $g \in \mathbb{R}[y]_{\leq 4}$ are polynomials with real critical points. There is a characterization of the equivalence class of f in terms of h, g as follows,*

1. *$[f] \in \mathfrak{O}_1$ iff $f(x,y) = x^4 + y^4$.*
2. *$[f] \in \mathfrak{O}_2$ iff $f(x,y) = (h_2 \circ h_1)(x) + g(y)$, where $h_1, h_2 \in \mathbb{R}[x]_{\leq 2}$ and g is not decomposable.*
3. *$[f] \in \mathfrak{O}_3$ iff $f(x,y) = (h_2 \circ h_1)(x) + (g_2 \circ g_1)(y)$, where $h_1, h_2 \in \mathbb{R}[x]_{\leq 2}$, $g_1, g_2 \in \mathbb{R}[y]_{\leq 2}$.*
4. *$[f] \in \mathfrak{O}_4$ iff $f(x,y) = (h_2 \circ h_1)(x) + (h_2 \circ h_1)(\pm y)$, where $h_1, h_2 \in \mathbb{R}[x]_{\leq 2}$.*
5. *$[f] \in \mathfrak{O}_0$ iff $h(x)$ and $g(y)$ are not decomposable.*

Proof. It is easy to show that the conditions on h and g are sufficient conditions. Following, we show that they are necessary conditions. Since we consider polynomials up to linear transformation, we can suppose $h(x) = x^4 + r_2x^2 + r_1x$, $g(y) = y^4 + s_2y^2 + s_1y$. Thus, for h and g be decomposable polynomials it is necessary that $r_1 = 0$ and $s_1 = 0$, respectively.

1. For $[f] \in \mathfrak{O}_1$, the polynomial $f(x,y)$ has a critical value, thus h and g have only one critical value. Since $h \in I_{(3,0)}$, then $h(x) = x^4$, analogously for g.

2. When $[f] \in \mathfrak{O}_2$ we have the next possibilities: If the Dynkin diagram is (3.10), then the critical values satisfy $a_1 = a_4, a_2 = a_5, a_3 = a_6$ or $a_2 = a_3, a_5 = a_6, a_8 = a_9$. If the Dynkin diagram is (3.11), then the critical values satisfy $a_1 = a_4, a_2 = a_5, a_3 = a_6$ or $a_1 = a_2, a_4 = a_5, a_7 = a_8$. The first conditions in both Dynkin diagrams implies that $c_1^h = c_2^h$, the others conditions implies $c_2^g = c_3^g$ and $c_1^g = c_3^g$, respectively. Without loss of generality we consider $c_1^h = c_2^h$, then $h \in I_{(2,1)}$, and recall that the 0-dimensional Dynkin diagram for h in this case is $\gamma_1 \cdots \gamma_3 \cdots \gamma_2$.

   Furthermore, the discriminant of $h'(x)$ is equal to $-16H$, thus $\mathbf{V}(H)$ corresponds to the polynomials with at most 2 critical points. Hence, the polynomials in $\mathbf{V}(H) \setminus \mathbf{V}(I_{(3,0)})$ are polynomials that have two different critical values and two critical points, and it is not the case of $c_1^h = c_2^h$ and $c_1^h \neq c_3^h$ see the Figure 3.11. Therefore, the polynomials in $\mathfrak{O}_2$ satisfy that $h(x) = x^4 + r_2x^2$. Then $h(x) = h_2(h_1(x))$ where $h_1(x) = x^2$ and $h_2(x) = x(x + r_2)$.

3. If $[f] \in \mathfrak{O}_3$, then the conditions $c_1^h = c_2^h$ and $c_1^g = c_2^g$, with 0-dimensional Dynkin diagram $\delta_1 \cdots \delta_3 \cdots \delta_2$ associated to g, are satisfied simultaneously (or $c_2^g = c_3^g$, with 0-dimensional Dynkin diagram $\delta_2 \cdots \delta_1 \cdots \delta_3$ associated to g). Hence, analogously to the previous case we have $f(x,y) = h_2(h_1(x)) + g_2(g_1(y))$ where $h_1(x) = x^2$, $h_2(x) = x(x+r_2)$, $g_1(y) = y^2$, $g_2(y) = y(y+s_2)$.

4. For $[f] \in \mathfrak{O}_4$, the Dynkin diagram is (3.10) and the critical values of f satisfies only the relations $a_1 = a_9$, $a_2 = a_6$ and $a_4 = a_8$, that means $c_1^h + c_1^g = c_3^h + c_3^g$, $c_1^h + c_2^g = c_2^h + c_3^g$ and $c_2^h + c_1^g = c_3^h + c_2^g$. Thus $c_1^h = c_3^g + k$, $c_2^h = c_2^g + k$ and $c_3^h = c_1^g + k$, where $k = c_3^h - c_1^g$. Hence, by doing a translation, we can suppose that the critical values of the polynomial h are equals to the critical values of g. Therefore, $g(x) = h(\pm x)$. On the other hand, the conditions $a_1 = a_4$, $a_2 = a_5$, $a_3 = a_6$, implies that h is decomposable.

5. When $[f] \in \mathfrak{O}_0$, the critical values of h are different or $h \in \mathbf{V}(H) \setminus \mathbf{V}(I_{(3,0)})$. Therefore, $r_1 \neq 0$. Analogously for g, we conclude that $s_1 \neq 0$.

□

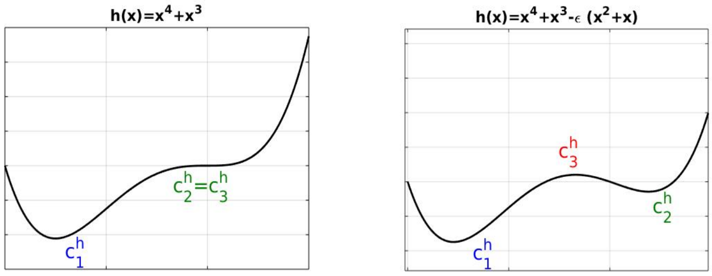

Figure 3.11: In the left a polynomial $g \in \mathbb{R}[y]_{\leq 4}$ with two critical values and two critical points. In the right a perturbation of g which separates the critical values. In both cases the enumeration is done according to section 3.2. The vanishing cycle associated to c_1^h and c_3^h always intersect.

Remark. Similar to Theorem 3.4.7, if $[f] \in \mathfrak{O}_2$, then the vanishing cycles v_{21}, v_{22}, v_{23} are in the kernel of the map

$$H_1(f^{-1}(b), \mathbb{Q}) \to H_1(\tilde{f}^{-1}(b), \mathbb{Q}), \qquad \text{where } \tilde{f} = h_2(x) + g(y)$$

coming from the map $(x,y) \to (h_1(x), y)$. If $[f] \in \mathfrak{O}_3$, then the vanishing cycles v_{21}, v_{22}, v_{23} are as before, and the vanishing cycles v_{12}, v_{22}, v_{32} are in the kernel of the map

$$H_1(f^{-1}(b), \mathbb{Q}) \to H_1(\hat{f}^{-1}(b), \mathbb{Q}), \qquad \text{where } \hat{f} = h(x) + g_2(y)$$

coming from the map $(x,y) \to (x, g_1(y))$. The other not simple vanishing cycles appear with the symmetry $h(x) = g(\pm x)$. Therefore, they are related with the pullback

$$\begin{array}{ccccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 & \longrightarrow & \mathbb{C} \\ (x,y) & \to & (x+y, xy) & \to & \check{f}(\check{x}, \check{y}) \end{array}$$

where $\check{x} = x + y$, $\check{y} = xy$ and some $\check{f} \in \mathbb{R}[\check{x}, \check{y}]_{\leq 4}$. So far we do not know a geometrical characterization for these vanishing cycles.

Appendix A

Appendixes

A.1 Algebraic space $I_{(d_1,d_2,\ldots,d_M)}$

In this section we show that the space of polynomials $h(x)+g(y)$ of degree d with a given number of critical values is an algebraic subspace. Actually, we need other conditions in the cardinality of the critical values, it motivates the next definition.

Definition A.1.1. For an integer d and a partition $(d_1,d_2,\ldots,d_M)$ of $D=(d-1)^2$, we say that the polynomial $f(x,y)=h(x)+g(y)\in\mathbb{C}[x,y]_{\le d}$ has **critical values degree** $(d_1,d_2,\ldots,d_M)$ if it has M different critical values $c_1,c_2,\ldots,c_M$ and for any $i=1\ldots,M$ there are d_i critical points over c_i, counted with multiplicity.

We show that the condition of critical values degree for a polynomial can be given in terms of algebraic expressions.

Proposition A.1.2. *Given a positive integer d and a partition $(d_1,d_2,\ldots,d_M)$ of $D=(d-1)^2$, the set of polynomials in $\mathbb{C}[x]_{\le d}\oplus\mathbb{C}[y]_{\le d}$ with critical values degree $(d_1,d_2,\ldots,d_M)$ is an algebraic subvariety of $\mathbb{C}[x,y]_{\le d}$*

Proof. The subspace of polynomials which can be written as $f(x,y)=h(x)+g(y)$ with $h\in\mathbb{C}[x]_{\le d}$ and $g\in\mathbb{C}[y]_{\le d}$ is an algebraic subspace of $\mathbb{C}[x,y]_{\le d}$, because is given by the zeros of the coordinates in the space of parameters corresponding to the terms x^py^q. Thus we only check that the condition on the number of critical values of f can be written as zeros of polynomials.

By definition of *discriminant* Δ, see [Mov17a, §10.9], we know that ξ is a critical value of $f(x,y)$ if and only if $\Delta(f(x,y)-\xi)=0$. For $f(x,y)=x^d+r_{d-1}x^{d-1}+\ldots+r_0+y^d+s_{d-1}y^{d-1}+\ldots+s_0$, we have that $\Delta_\xi(f):=\Delta(f-\xi)$ is a polynomial $\lambda(\xi)=\alpha_D\xi^D+\alpha_{D-1}\xi^{D-1}+\ldots+\alpha_0$, that define the map

$$\mathbb{C}^{2d}\xrightarrow{\Delta_\xi}\mathbb{C}^D$$
$$(r_{d-1},\ldots,r_0,s_{d-1},\ldots,s_0)\to(\alpha_{D-1},\ldots,\alpha_0).$$

The polynomial $\lambda(\xi)$ can also be expressed in terms of the critical values $t_1,\ldots,t_D$ of f, as $\lambda(\xi)=(\xi-t_1)(\xi-t_2)\ldots(\xi-t_D)=(\xi-t_{i_1})^{d_1}(\xi-t_{i_2})^{d_2}\ldots(\xi-t_{i_M})^{d_M}$, where we have used the definition of critical values degree. Let $\mathbb{C}^D\xrightarrow{\varphi}\mathbb{C}^D$ be the map given by the Vieta's formula

$$\mathbb{C}^D\xrightarrow{\varphi}\mathbb{C}^D$$
$$(t_1,t_2,\ldots,t_D)\xrightarrow{\varphi}(\varepsilon(1)\sum_i t_i,\varepsilon(2)\sum_{i\neq j}t_it_j,\ldots,\varepsilon(D)t_1t_2\cdots t_D),$$

where $\varepsilon(j) = (-1)^j \alpha_D$, thus φ take the roots of a polynomial and gives the coefficients of the polynomial. Let V be a subvariety in the domain of φ given by M equations of the form $t_{i_1} = t_{i_2} \ldots = t_{i_{d_j}}$ with $j = 1, \ldots, M$. The subvariety V has the information of the critical values degree. The closure of $\varphi(V)$ is a subvariety of $\mathbb{C}^D$, we denote it as W. The pullback of W by the map Δ_ξ is a subvariety in $\mathbb{C}^{2d}$ in terms of the parameters r_k, s_l which is the closure of the space of polynomial in $\mathbb{C}[x,y]_{\leq d}$ with critical values degree $(d_1, d_2, \ldots d_M)$.

$$\begin{tikzcd} V \subset \mathbb{C}^D \arrow[r, "\varphi"] & W \subset \mathbb{C}^D \\ \mathbb{C}^{2d} \arrow[ur, "\Delta_\xi"'] & \end{tikzcd}$$

□

Clearly, we have analogous results if we consider a polynomial in one variable, that is $f(x,y) = h(x)$. In order to compute an explicit expression for W we use the implicitation algorithm [CLO13, §3.3]. Let $v_1, v_2, ..., v_s$ be the polynomials which describes V, thus $v_i = v_i(t_1, \ldots, t_D)$. Let $I \subset \mathbb{C}[t_1, \ldots, t_D, x_1, \ldots, x_D]$ be the ideal

$$I = \langle x_1 - \sum t_i, x_2 - \sum_{i \neq j} t_i t_j, ..., x_D - t_1 t_2 ... t_D, v_1, ..., v_s \rangle.$$

If G if a Groebner basis of I with respect to lexicographic order $t_1 > t_2 > ... > t_D > x_1 > ... > x_D$, then $G_x = G \cap \mathbb{C}[x]$ is a Groebner basis of the ideal $I_x := I \cap \mathbb{C}[x]$. Also $W := \mathbf{V}(I_x)$ is the smallest variety in $\mathbb{C}^D$ containing $\varphi(V)$. In §A.4.4, we provides some codes in SINGULAR, in order to compute these ideals.

A.2 Polynomial Foliations in $\mathbb{C}^2$ with Center

An algebraic foliation $\mathcal{F}$ in $\mathbb{C}^2$ is given by a 1 form $\omega = P(x,y)dx + Q(x,y)dy$ where P, Q are polynomials. In order to do emphasis in the form we call the foliation as $\mathcal{F}(\omega)$. For P, Q coprime polynomials, the degree of the foliation $\mathcal{F}(\omega)$ is defined as the the maximum of the degrees of the polynomials P and Q. The set of points in $\mathbb{C}^2$ where ω vanishes are the singular points of $\mathcal{F}$. If in a neighborhood of a singular point p, the leaves of the foliation $\mathcal{F}$ can be written as the set levels of a Morse function, then the singularity is called **center**.

Given a natural number d, the space of foliation with degree less than or equal to d is denoted as $\mathcal{F}(d)$, and the closure of its subset of foliations with at least one center is denoted as $\mathcal{M}(d)$. In [CN96, Mov04a], $\mathcal{M}(d)$ shown to bean algebraic subset of $\mathcal{F}(d)$. Therefore, the problem of finding its irreducible components arises [Net07, Net14]. In [Ily69], Y. Ilyashenko prove that the spaces of Hamiltonian foliation $\mathcal{F}(dF)$ where F is a generic polynomial of degree $d+1$ is an irreducible component of $\mathcal{M}(d)$. More precisely, he proves the next theorem.

Theorem A.2.1 (Destruction of center). *Let F be a generic polynomial in $\mathbb{C}[x,y]$ and p a singular point of dF. If p is a persistent center for deformations $dF + t(A(x,y)dy + B(x,y)dx)$, then the form $A(x,y)dy + B(x,y)dx$ is exact.*

Let us recall the following classic definition.

Definition A.2.2. Let $\mathcal{F}$ be a foliation in $\mathbb{C}^2$. A polynomial 1-form ω is called relatively exact modulo F, if it is exact in any leaves of the foliation i.e. for any leaf L in $\mathcal{F}$, there exist a function on L such that

$$\omega|_L = df.$$

The next proposition characterizes the 1-form relatively exact modulo Hamiltonian foliation, it is proved in [Gav98].

Proposition A.2.3. *Let $F \in \mathbb{C}[x,y]$ be a polynomial with isolated critical points and connected fibers. Set the foliation $\mathcal{F} = \mathcal{F}(dF)$, then for any polynomial 1-form ω, the next three conditions are equivalents:*

1. *ω is relatively exact modulo $\mathcal{F}$.*
2. *$\int_\delta \omega = 0$ for any cycle δ in the leaves of $\mathcal{F}$ where the integral is well-defined.*
3. *There exist polynomials $A, B \in C[x,y]$ such that $\omega = dA + BdF$.*

We observe that the theorem A.2.1 can be generalize to a polynomials F in $\mathbb{C}[x,y]$ not necessarily generic, but with transitive action of monodromy on any vanishing cycle. The proof is essentially the same that in [Ily69]. However, we rewrite it here in order to emphasize the importance of understanding the monodromy action in the study of the irreducible components of $\mathcal{M}(d)$.

Proposition A.2.4. *Let $F \in \mathbb{C}[x,y]_{\le d+1}$ be a polynomial with isolated critical points, connected fibers. Let p be a center singularity of foliation $\mathcal{F} := \mathcal{F}(dF)$, which induces a simple cycle. If p is a persistent center for the foliation deformations $\mathcal{F}_\varepsilon$ given by $dF + \varepsilon\omega_1 \in \mathcal{M}(d)$ then the form ω_1 is exact.*

Proof. Let δ_t be a family of cycles in leaves L_t of the foliation $\mathcal{F}$. Set Σ a transversal section of the foliation in the point p, and suppose that it is parameterized by the image of F, i.e. $t = F(z)$ for $z \in \Sigma$. The return map of perturbed foliation is denoted as $h_\varepsilon(t)$, the Taylor expansion of the return map minus the identity is:

$$h_\varepsilon(t) - t = \varepsilon M_1(t) + \varepsilon^2 M_2 + (t) + h.o.t.$$

where the function $M_i(t)$ are called the Melnikov functions.

By Francoise recursion formula [Fra96, Mov04b] we have

$$M_1(t) = \int_{\delta_t} \omega_1,$$

since the center is persistent then $h_\varepsilon(t) = t$ for ε small enough, thus $\int_{\delta_t} \omega_1 = 0$. If we fixed a t_0 such that the leaf $L := L_{t_0}$ is not singular then the cycle $\delta_0 := \delta_{t_0}$ is a vanishing cycle of L.

The monodromy action on δ_0 around to a vanishing cycle δ_i yields an analytic continuation $\int_{\mathrm{Mon}_j(\delta_0)} \omega_1$ of the function $\int_{\delta_0} \omega_1 = 0$. Thus

$$\int_{\mathrm{Mon}_j(\delta_0)} \omega_1 = 0 \text{ for all } j, \text{ where } H_1(L,\mathbb{Q}) = \mathrm{span}\langle\delta_j\rangle_j.$$

Because the cycle δ_0 is simple then any other cycle in the leaf L can be generated as linear combination of monodromy actions on δ_0, hence the integral $\int_\delta \omega_1$ is zero for any $\delta \in H_1(L,\mathbb{Q})$. By the proposition A.2.3, we conclude that $\omega_1 = dA + BdF$ for $A, B \in \mathbb{C}[x,y]$. Since $deg(\omega_1) = d$, then $deg(B) = d - (deg(F) - 1) = 0$, thus $\omega_1 = d(A + BF)$. □

The next lemma will be useful in order to determine the limit of some components. It is inspired in a computation in[Net07].

Lemma A.2.5. *Given a foliation $\mathcal{F}$ on $\mathbb{C}^2$ defined by the 1-form*

$$\eta = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + d\left(\frac{g}{f_1^{s_1-1} f_2^{s_2-1} \dots f_r^{s_r-1}}\right),$$

with $\lambda_j \in \mathbb{C}^*$ and $s_j \geq 1$. There exits a family of logarithmic foliation $\mathcal{F}_t$, such that they converge to $\mathcal{F}$ when t goes to 0. This family is defined by the 1-forms

$$\eta_t = \sum_{j=1}^{r} \left(\lambda_j - \left(\frac{s_j - 1}{t}\right)\right) \frac{df_j}{f_j} + \frac{1}{t} \frac{d(f_1^{s_1-1} f_2^{s_2-1} \dots f_r^{s_r-1} + tg)}{f_1^{s_1-1} f_2^{s_2-1} \dots f_r^{s_r-1} + tg}.$$

Proof. Set $f := f_1^{s_1-1} f_2^{s_2-1} \cdots f_r^{s_r-1}$, $h := f_1 f_2 \cdots f_r$ and the 1-form $\Omega := f_1^{s_1} \cdots f_r^{s_r} \eta = hf\eta$ which defines the same foliation $\mathcal{F}$. Consider the family of one parameter $\Omega_t := \Omega + tgh\left(\sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j}\right)$ with $0 < t < \varepsilon$.

Note that $df = f\sum_{j=1}^{r}(s_j - 1)\frac{df_j}{f_j}$, thus

$$\begin{aligned}
\Omega &= hf \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + hdg - gh\frac{df}{f} \\
&= h\left(f\sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + dg - g\sum_{j=1}^{r}(s_j - 1)\frac{df_j}{f_j}\right).
\end{aligned}$$

Then,

$$\begin{aligned}
\Omega_t &= h\left((f + tg)\sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + dg - g\sum_{j=1}^{r}(s_j - 1)\frac{df_j}{f_j}\right) \\
&= h\left((f + tg)\sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + \frac{1}{t}d(f + tg) - \frac{1}{t}(f + tg)\sum_{j=1}^{r}(s_j - 1)\frac{df_j}{f_j}\right) \\
&= h(f + tg)\left(\sum_{j=1}^{r}\left(\lambda_j - \left(\frac{s_j - 1}{t}\right)\right)\frac{df_j}{f_j} + \frac{1}{t}\frac{d(f + tg)}{f + tg}\right).
\end{aligned}$$

Let be the logarithmic 1-form:

$$\eta_t = \frac{1}{h(f + tg)}\Omega_t = \sum_{j=1}^{r}\left(\lambda_j - \left(\frac{s_j - 1}{t}\right)\right)\frac{df_j}{f_j} + \frac{1}{t}\frac{d(f + tg)}{f + tg},$$

which define the family of foliations $\mathcal{F}_t$. Also in the limit $\mathcal{F}_t$ converges to the foliation define by Ω. ☐

The degree of the foliation defined by η_t in the previous lemma is $deg(\mathcal{F}_t) = \sum_{j=1}^{r} def(f_j) + deg(f_1^{s_1-1} f_2^{s_2-1} \cdots f_r^{s_r-1} + gt) - 1$.

In the following lemma we show that if a Foliation in $\mathbb{C}^2$ with finite critical values and infinite connected leaves has a polynomial first integral, then all first integral are polynomials.

Lemma A.2.6. *Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with finite number of critical values and infinite connected fibers. Consider the foliation $\mathcal{F} = \mathcal{F}(dF)$, if $G : \mathbb{C}^2 \to \mathbb{C}$ is a rational first integral for $\mathcal{F}$, then there exists a polynomial $p : \mathbb{C} \to \mathbb{C}$, such that the next diagram commutes*

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "F"] \arrow[dr, "G"'] & \mathbb{C} \arrow[d, "p"] \\ & \mathbb{C} \end{tikzcd}$$

consequently G is a polynomial.

Proof. Firstly we show that there are finite fibers which are not connected. We can define a rational extension of F to $\mathbb{P}^1 \to \mathbb{P}$, and by doing finite blow ups [Hir64, Kol05], we can set a map $M \xrightarrow{\hat{F}} \mathbb{C}$, with M a compact manifold and $\hat{K}$ a closed submanifold of M. Also there is a map $M \xrightarrow{\varphi} \mathbb{C}^2$ such that $\hat{F} := F \circ \varphi$ is a morphism, and a finite set of points $K := \varphi(\hat{K})$ of $\mathbb{C}^2$ such that $M \setminus \hat{K} \xrightarrow{\varphi} \mathbb{C}^2 \setminus K$ is an isomorphism. Since the critical values of F are finite, then there are finite critical values of $\hat{F}$, we call this set as $\hat{C}$. Thus, by using the fibration Ehresmann theorem we conclude that the fibers of $M \setminus \hat{K} \xrightarrow{\hat{F}} \mathbb{P}^1$ are isomorphic. Hence the fibers in $\mathbb{C}^2 \setminus \{\varphi(\hat{C}) \cup K\} \xrightarrow{F} \mathbb{C}$ are isomorphic, and since $\varphi(\hat{C}) \cup K$ is a finite set and there are infinite connected fibers by hypothesis, then we conclude that there are finite not connected fibers.

Let $C = \{c_1, c_2, \dots, c_p\} \subset \mathbb{C}$ be the values where the preimage of F is not connected. Thus the map $p : \mathbb{C} \setminus C \to \mathbb{C}$ given as $p(z) = G|_{F^{-1}(z)}$ is well defined because G is constant in the leaves of $\mathcal{F}$, also it is holomorphic. Consider $c \in C$, let X and Y be two connected components in $F^{-1}(c)$. Take a neighborhood U of c which does not contain another point of C, set x_n and y_n sequences of points such that $F(x_n) = F(y_n) = c_n \in U$, also suppose that x_n goes to a point $x \in X$ and y_n goes to a point $y \in Y$ when n goes to infinity. Thus by continuity of G we have $G(x) = \lim_{n\to\infty} G(x_n) = \lim_{n\to\infty} G(y_n) = g(y)$, so we can extend p to all $\mathbb{C}$. By using Riemann extension theorem, because p is locally bounded, we conclude that p is holomorphic in $\mathbb{C}$.

By taking a meromorphic extensions of F and G to $\mathbb{P}^1$ we can define a map $\mathbb{P}^1 \xrightarrow{\hat{p}} \mathbb{P}^1$ which extends p. The meromorphic function $\hat{p}$ on $\mathbb{P}^1$ is rational $\frac{P}{Q}$, (e.g. [DS13]). Since G restricts to the leaves of $\mathcal{F}$ takes finite values, then $\hat{p}$ takes the value infinity only in infinity. Thus in coordinates we have $\hat{p}([z_0 : z_1]) = \frac{P([z_0:z_1])}{z_0}$. Then $p(z) = \hat{p}(1, z) = P([1, z])$ is a polynomial. By construction it is clear that $G(z) = p(F(z))$. □

A.3 Components for $d = 3$

Set $f(x, y) = h(x) + g(y)$ where $h \in \mathbb{C}[x]_{\le 3}$ and $g \in \mathbb{C}[y]_{\le 3}$, and $b \in \mathbb{C}$ a regular value. The aim of this section is to compute the part of the homology $H_1(f^{-1}(b))$ generated by the action of the monodromy. Firstly we compute the irreducible components of the critical values degree for $d = 3$. Next, we compute the Dynkin diagram and the orbits of the action of the monodromy of any vanishing cycle.

Irreducible components of the critical values degree for $d = 3$

For $d = 3$, we have five partition of $D = (d-1)^2 = 4$. The Groebner basis in any case of these partitions are:

1. $D = 4 + 0$: With $t_1 = t_2 = t_3 = t_4$. In this case $G = \{x_3^4 - 256x_4^3, 8x_2x_4 - 3x_3^2, x_2x_3^2 - 96x_4^2, x_2^2 -$

$36x_4, 6x_1x_4 - x_2x_3, 9x_1x_3 - 4x_2^2, x_1x_2 - 6x_3, 3x_1^2 - 8x_2, 4t_4 - x_1, t_3 - t_4, t_2 - t_3, t_1 - t_3\}$, thus

$$\begin{aligned} G_x = \{&x_3^4 - 256x_4^3, 8x_2x_4 - 3x_3^2,\\ &x_2x_3^2 - 96x_4^2, x_2^2 - 36x_4,\\ &6x_1x_4 - x_2x_3,\\ &9x_1x_3 - 4x_2^2,\\ &x_1x_2 - 6x_3,\\ &3x_1^2 - 8x_2\} \end{aligned}$$

2. $D = 3 + 1$: With $t_1 = t_3 = t_4$. We have

$$\begin{aligned} G_x = \{&3x_2^4x_4 - x_2^3x_3^2 + 72x_2^2x_4^2 - 108x_2x_3^2x_4 + 27x_3^4 + 432x_4^3,\\ &3x_1x_3 - x_2^2 - 12x_4,\\ &9x_1x_2^2x_4 - 27x_1x_2x_3^2 + 108x_1x_4^2 + 8x_2^3x_3 + 27x_3^3,\\ &27x_1^2x_4 - 27x_1x_2x_3 + 8x_2^3 + 27x_3^2\} \end{aligned}$$

3. $D = 2 + 2$: With $t_1 = t_3$ and $t_2 = t_4$. Then,

$$\begin{aligned} G_x = \{&16x_2^2x_4^2 - 8x_2x_3^2x_4 + x_3^4 - 64x_4^3,\\ &8x_1x_4^2 - 4x_2x_3x_4 + x_3^3,\\ &2x_1x_3x_4 - 4x_2^2x_4 + x_2x_3^2 + 16x_4^2,\\ &x_1x_3^3 - 8x_2^3x_4 + 2x_2^2x_3^2 + 32x_2x_4^2 + 8x_3^2x_4,\\ &4x_1x_2x_4 - x_1x_3^2 - 8x_3x_4,\\ &x_1^2x_4 - x_3^2,\\ &x_1^2x_3 + 8x_1x_4 - 4x_2x_3,\\ &x_1^2x_2 + 2x_1x_3 - 4x_2^2 + 16x_4,\\ &x_1^3 - 4x_1x_2 + 8x_3\} \end{aligned}$$

4. $D = 2 + 1 + 1$: With $t_1 = t_2$. Thus,

$$\begin{aligned} G_x = \{&27x_1^4x_4^2 - 18x_1^3x_2x_3x_4 + 4x_1^3x_3^3 + 4x_1^2x_2^3x_4 - x_1^2x_2^2x_3^2 - 144x_1^2x_2x_4^2 + 6x_1^2x_3^2x_4\\ &+ 80x_1x_2^2x_3x_4 - 18x_1x_2x_3^3 + 192x_1x_3x_4^2 - 16x_2^4x_4 + 4x_2^3x_3^2 + 128x_2^2x_4^2\\ &- 144x_2x_3^2x_4 + 27x_3^4 - 256x_4^3\} \end{aligned}$$

5. $D = 1 + 1 + 1 + 1$: With $V = \mathbb{C}^4$. We get,

$$G_x = \{0\}$$

To count the components of the space of polynomials with a fixed critical values degree, we compute the primary decomposition of $\Delta_\xi^* G_x$ which is a radical ideal that describe the subvariety of these polynomials. For the previous cases we denote these radical ideal as $\sqrt{I_i}$, with $i = 1, \ldots, 5$ are the enumeration of any case. By a translation of ξ we can suppose that $r_0 = s_0 = 0$. The $\sqrt{I_i}$ for $d = 3$ are:

- $\sqrt{I_1} = \langle 3s_1 - s_2^2, 3r_1 - r_2^2 \rangle$
- $\sqrt{I_2} = \langle 3s_1 - s_2^2, 3r_1 - r_2^2 \rangle$
- $\sqrt{I_3} = \langle 3s_1 - s_2^2 \rangle \cap \langle 3r_1 - r_2^2 \rangle$
- $\sqrt{I_4} = \langle 3r_1 - r_2^2 - 3s_1 + s_2^2 \rangle \cap \langle 9r_1^2 - 6r_1r_2^2 + 9r_1s_1 - 3r_1s_2^2 + r_2^4 - 3r_2^2s_1 + r_2^2s_2^2 + 9s_1^2 - 6s_1s_2^2 + s_2^4 \rangle \cap \langle 3s_1 - s_2^2 \rangle \cap \langle 3r_1 - r_2^2 \rangle$.
- $\sqrt{I_5} = 0$.

Note that $\sqrt{I_1} = \sqrt{I_2}$, it is because in a join polynomial of degree 3 if three critical values are equal, the fourth is also equal. For $d = 3$ we have 4 components given by $\langle 3r_1 - r_2^2 - 3s_1 + s_2^2 \rangle$, $\langle 9r_1^2 - 6r_1r_2^2 + 9r_1s_1 - 3r_1s_2^2 + r_2^4 - 3r_2^2s_1 + r_2^2s_2^2 + 9s_1^2 - 6s_1s_2^2 + s_2^4 \rangle$, $\langle 3s_1 - s_2^2 \rangle$ and $\langle 3r_1 - r_2^2 \rangle$. However note that the first two polynomials are in $\langle 3s_1 - s_2^2, 3r_1 - r_2^2 \rangle$.

We remark that the equations $G := 3s_1 - s_2^2$, $H := 3r_1 - r_2^2$ are the discriminant of $g'(y)$ and $h'(x)$, respectively. With this notation we have the previous components are

- $\sqrt{I_1} = \langle G, H \rangle$
- $\sqrt{I_3} = \langle G \rangle \cap \langle H \rangle$
- $\sqrt{I_4} = \langle G - H \rangle \cap \langle G^2 + H^2 + GH \rangle \cap \langle G \rangle \cap \langle H \rangle$.
- $\sqrt{I_5} = 0$.

In Figure A.1 there is a sketch of how these components look like in $\mathbb{C}[x]_{\le 3} \oplus \mathbb{C}[y]_{\le 3} \subset \mathbb{C}[x,y]_{\le 3}$.

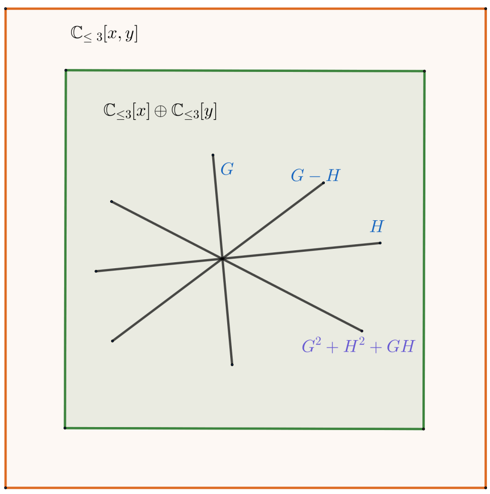

Figure A.1: Irreducible components of the Joint polynomials for degree 3.

Orbits of monodromy action

For degree 3 we have that the 0−dimensional homology of a regular fiber of h is generated by the vanishing cycles δ_1, δ_2 which vanish on the critical values c_h^1, c_h^2, respectively. Analogously, for the 0−dimensional homology of a regular fiber of g, generated by the vanishing cycles γ_1, γ_2. For this case the Dynkin diagram for dimension 0 are

$$\delta_1 - - - \delta_2 \text{ and } \gamma_1 - - - \gamma_2.$$

We denote the critical values of f as

$$a_1 = c_1^h + c_1^g \ , \ a_2 = c_1^h + c_2^g \ , \ a_3 = c_2^h + c_1^g \ , \ a_4 = c_2^h + c_2^g.$$

Set α_i the cycle which vanishes in the critical point corresponding to the critical value a_i for $i = 1,2,3,4$. By using the intersection formula 3.1 we compute the Dynkin diagram for f, it is:

$$\begin{array}{ccc} \delta_1 * \gamma_1 & \longrightarrow & \delta_2 * \gamma_1 \\ \downarrow & \nwarrow & \downarrow \\ \delta_1 * \gamma_2 & \longrightarrow & \delta_2 * \gamma_2 \end{array} \quad \text{or in terms of critical values} \quad \begin{array}{ccc} a_1 & \longrightarrow & a_3 \\ \downarrow & \nwarrow & \downarrow \\ a_2 & \longrightarrow & a_4 \end{array}$$

By using the rules in the section 3.2 about the critical values degree for $D = 4$, we note that the only possibilities are $D = 4 + 0$, $D = 2 + 1 + 1$ with $a_1 = a_4$, $D = 1 + 1 + 1 + 1$, and $D = 2 + 2$ with $a_1 = a_2$ and $a_3 = a_4$, or $a_1 = a_3$ and $a_2 = a_4$. For $D = 4 + 0$ we get that the subspaces generated by the action of monodromy on a vanishing cycles are.

$$\mathrm{Mon}(\alpha_1) = \mathrm{Mon}(\alpha_4) = \langle \alpha_1, \alpha_2 + \alpha_3, \alpha_4 \rangle,$$

$$\mathrm{Mon}(\alpha_2) = \langle \alpha_2, \alpha_4 - \alpha_1, \alpha_4 - \alpha_3 \rangle,$$

$$\mathrm{Mon}(\alpha_3) = \langle \alpha_3, \alpha_4 - \alpha_1, \alpha_4 - \alpha_2 \rangle.$$

In the others critical values degree the vanishing cycles are simple cycles.

Intersection of foliations with center with polynomials components

The following theorem is the main idea about the classification of irreducible components of the foliation space $\mathcal{M}(2)$, due to Dulac [Dul08]. In [CN96] there is a extension of this theorem to foliations in $\mathbb{P}^2$ of degree 2, with a center singularity. For the purpose of this work we consider the foliations in the theorem just as foliations in $\mathbb{C}^2$, and we work with the definition of degree of foliation as $max\{deg(P), deg(Q)\}$ where the form which defines the foliation is $P(x,y)dx + Q(x,y)dy$.

Theorem A.3.1 (Dulac)**.** *Let $\mathcal{F}$ be a foliation of degree less than or equal to 2 in $\mathbb{C}^2$, which has a Morse type singularity p. Then $\mathcal{F}$ can be represented by a closed 1-form η which can be obtained in one of the following forms:*

1. *$\eta = dq$ where q is a polynomials of degree 3.*
2. *$\eta = \sum_{j=1}^{3} \lambda_j \frac{dp_j}{p_j}$ where $\lambda_j \in \mathbb{C}^*$ and p_j is a polynomial of degree 1, $j = 1,2,3$.*
3. *$\eta = \sum_{j=1}^{2} \lambda_j \frac{dp_j}{p_j}$ where $\lambda_j \in \mathbb{C}^*$ for $j = 1,2$. Degree p_1 is 2 and degree of p_2 is 1.*
4. *$\eta = \sum_{j=1}^{2} \lambda_j \frac{dp_j}{p_j} + dq$ where $\lambda_j \in \mathbb{C}^*$ for $j = 1,2$. The three polynomials have degree 1.*
5. *$\eta = \sum_{j=1}^{2} \lambda_j \frac{dp_j}{p_j} + d\left(\frac{q}{p_1}\right)$ where $\lambda_j \in \mathbb{C}^*$ for $j = 1,2$. The three polynomials have degree 1.*

6. $\eta = \frac{dp}{p} + d\left(\frac{q}{p^2}\right)$, *degree of* p *is 1 and degree of* q *is 2.*

7. $\eta = \frac{dp}{p} + d\left(\frac{q}{p}\right)$, *degree of* p *is 1 and degree of* q *is 2.*

8. $\eta = \frac{dp}{p} + dq$, *degree of* p *is 1 and degree of* q *is 2.*

9. $\eta = \frac{dp}{p} + dq$, *degree of* p *is 2 and degree of* q *is 1.*

10. $\eta = 3\frac{df}{f} - 2\frac{dg}{g}$, *degree of* f *is 2 and degree of* g *is 3.*

    The polynomials p_j *and* p *are irreducible. The polynomial* f *and* g *are not any, below we will specify which are are.*

Remark: The polynomials f and g are generated as the orbit of the action $SL(2,\mathbb{C}) \curvearrowright \mathbb{C}^2$ on some polynomials f_0 and g_0. There are two possibilities for these pair: The firs one is with $f_0 = 1 + 2x + y^2$ and $g_0 = y(3x + y^2)$, in this case note that $3g_0df_0 - 2f_0dg_0 = 6y(x-1)(dx + ydy)$ thus the foliation has degree 1.

The other case is with $f_0 = 1 + 2x + 2y + ax^2 + 2xy + \frac{1}{a}y^2$ and

$$\begin{aligned} g_0 =& 1 + 3(x+y) + \frac{3}{2}\left((a+1)x^2 + \left(2 + a + \frac{1}{a}\right)xy + \left(\frac{1}{a} + 1\right)y^2\right) + \\ & \frac{1}{2}\left(a(a+1)x^3 + 3(a+1)x^2y + 3\left(\frac{1}{a} + 1\right)xy^2 + \frac{1}{a}\left(\frac{1}{a} + 1\right)y^3\right), \end{aligned}$$

with $a \neq 0$. Here we have

$$3g_0df_0 - 2f_0dg_0 = \frac{-3(a-1)^2}{a}\left((ax^2 + y - xy + 2y^2)dx + \frac{1}{a}(ax + 2ax^2 - axy + y^2)dy\right),$$

thus the foliation has degree 2.

Some of the previous components are in the limit of the other ones. In order to determine these components, we use the lemma A.2.5.

Corollary A.3.2. *There are 4 irreducible components of* $\mathcal{M}(2)$ *which are given by the forms:*

- $\eta = dq$ *where* q *is a polynomials of degree 3.*
- $\eta = \sum_{j=1}^{3} \lambda_j \frac{dp_j}{p_j}$ *where* $\lambda_j \in \mathbb{C}^*$ *and* p_j *is a polynomial of degree 1,* $j = 1,2,3$.
- $\eta = \lambda_1\frac{dp_1}{p_1} + \lambda_2\frac{dp_2}{p_2}$ *where* $\lambda_j \in \mathbb{C}^*$ *for* $j = 1,2$. *Degree* p_1 *is 2 and degree of* p_2 *is 1.*
- $\eta = 3\frac{df}{f} - 2\frac{dg}{g}$, *degree of* f *is 2 and degree of* g *is 3.*

The polynomials p_j, p, q, f, g *as in theorem A.3.1.*

Proof. From the classification A.3.1, we only have to check that the others cases are at the limit of the previous ones. By using the lemma A.2.5 is easy to note that in the theorem A.3.1 the cases 4, 5 are limits of families of foliations in the case 2: In the case 4, $\eta = \lambda_1\frac{dp_1}{p_1} + \lambda_2\frac{dp_2}{p_2} + dq$ with $deg(p_j) = deg(q) = 1$ we have that the family of 1-forms is $\eta_t = \lambda_1\frac{dp_1}{p_1} + \lambda_2\frac{dp_2}{p_2} + \frac{1}{t}\frac{d(1+tg)}{1+tg}$.

The case 5, $\eta = \sum_{j=1}^{2} \lambda_j \frac{dp_j}{p_j} + d(\frac{q}{p_1})$ with $deg(p_j) = deg(q) = 1$, has a family of 1-forms $\eta_t = (\lambda_1 - \frac{1}{t})\frac{dp_1}{p_1} + \lambda_2 \frac{dp_2}{p_2} + \frac{1}{t}\frac{d(p_1+tq)}{p_1+tq}$.

In the cases 6, 7, 8, 9 are limits of families of foliations in the case 3: The case 6, $\eta = \frac{dp}{p} + d(\frac{q}{p^2})$ with $deg(p) = 1$ and $deg(q) = 2$, we have $\eta_t = (1 - \frac{2}{t})\frac{dp}{p} + \frac{1}{t}\frac{d(p^2+tq)}{p^2+tq}$.

For the case 7, $\eta = \frac{dp}{p} + d(\frac{q}{p})$ with $deg(p) = 1$ and $deg(q) = 2$, we have $\eta_t = (1 - \frac{1}{t})\frac{dp}{p} + \frac{1}{t}\frac{d(p+tq)}{p+tq}$.

And for the cases 8,9, $\eta = \frac{dp}{p} + dq$ with $deg(p) = 1, 2$ and $deg(q) = 2, 1$, we have $\eta_t = \frac{dp}{p} + \frac{1}{t}\frac{d(1+tq)}{1+tq}$. □

From proposition A.2.4 we get the next result in $\mathcal{M}(2)$. Let $\mathcal{F}(d(h \oplus g))$ be the subspace of the Hamiltonian foliations $\mathcal{F}(dF)$ given by the join polynomials.

Corollary A.3.3. *The components of $\mathcal{M}(2)$ can intersect $\mathcal{F}(d(h \oplus g))$ only in the join polynomials of degree 3 with one critical value.*

Proof. Set the polynomial $f(x, y) = h(x) + g(y) \in \mathbb{C}[x]_{\le 3} \oplus \mathbb{C}[y]_{\le 3}$ by the previous section we know that the only case when the action the monodromy on the vanishing cycle of a regular fiber is not always transitive is when the critical values of f are equals, i.e. when $f \in \mathbf{V}(\sqrt{I_1})$. Thus, by using Proposition A.2.4, if $f \in \mathbf{V}(\sqrt{I_i})$ with $i = 3, 4, 5$, then perturbations of f in $\mathcal{M}(2)$ will be again in the Hamiltonian component $\mathcal{F}(dF)$. □

We finish this section by computing the intersection of the irreducible component of $\mathcal{M}(2)$ with $\mathbf{V}(\sqrt{I_1}) \subset \mathbb{C}[x]_{\le 3} \oplus \mathbb{C}[y]_{\le 3}$:

- If $\eta = \sum_{j=1}^{3} \lambda_j \frac{dp_j}{p_j}$ where $\lambda_j \in \mathbb{C}^*$, then a first integral is $(p_1^{\lambda_1} p_2^{\lambda_2} p_3^{\lambda_3})^n$, where $p_1 = a_2x + a_1y + a_0$, $p_2 = b_2x + b_1y + b_0$, $p_3 = c_2x + c_1y + c_0$. If $d\eta$ is a Hamiltonian foliation then λ_i is a rational number and n is the greatest common divisor of the λ_i's. For degree 2 in the foliation, we have that $(p_1^{\lambda_1} p_2^{\lambda_2} p_3^{\lambda_3})^n$ has degree equal to 3, thus $\lambda_i = 1$ and $n = 1$.

  We want to know when $p_1p_2p_3$ is a join polynomial, hence we look for the components when the coefficients of x^2y, xy^2, xy are 0. This conditions are $a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1 = 0$, $a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1 = 0$ and $a_0b_1c_2 + a_0b_2c_1 + a_1b_0c_2 + a_1b_2c_0 + a_2b_0c_1 + a_2b_1c_0 = 0$. Let J be the ideal generated by these polynomials. Since $p_1p_2p_3$ has to have degree 3, then to know if $d\eta \in \mathbf{V}(\sqrt{I_1})$ is enough to show that the variety associated to J is not contained in the variety associated to the ideal $\langle a_1b_1c_1, a_2b_2c_2\rangle$, which correspond to the condition of the coefficients of x^3 and y^3.

  The prime decomposition of $\sqrt{J}$ is,

  $$\begin{aligned}\sqrt{J} =& \langle c_0, c_1, c_2\rangle \cap \langle c_1, c_2, a_1b_2 + a_2b_1\rangle \cap \langle b_1, b_2, a_1c_2 + a_2c_1\rangle \cap \langle a_2, b_2, c_2\rangle \cap \langle a_1, a_2, b_1c_2 + b_2c_1\rangle \\ & \langle a_1, b_1, c_1\rangle \cap \langle b_0, b_1, b_2\rangle \cap \langle a_0, a_1, a_2\rangle \cap I,\end{aligned}$$

  where I is the ideal

  $$\begin{aligned}I =& \langle b_1^2c_2^2 + b_1b_2c_1c_2 + b_2^2c_1^2, a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1, a_1b_2^2c_1 - a_2b_1^2c_2, a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1, \\ & a_1^2c_2^2 + a_1a_2c_1c_2 + a_2^2c_1^2, a_1^2b_2^2 + a_1a_2b_1b_2 + a_2^2b_1^2, a_0b_2c_1 + a_1b_0c_2 + a_2b_1c_0, a_0b_1c_2 + a_1b_2c_0 + a_2b_0c_1, \\ & a_0a_2b_1c_1 - a_1^2b_0c_2 - a_1^2b_2c_0 - a_1a_2b_0c_1 - a_1a_2b_1c_0, a_0^2b_2^2c_2^2 - a_0a_2b_0b_2c_2^2 - a_0a_2b_2^2c_0c_2 + a_2^2b_0^2c_2^2 \\ & - a_2^2b_0b_2c_0c_2 + a_2^2b_2^2c_0^2, a_0^2b_1^2c_1^2 - a_0a_1b_0b_1c_1^2 - a_0a_1b_1^2c_0c_1 + a_1^2b_0^2c_1^2 - a_1^2b_0b_1c_0c_1 + a_1^2b_1^2c_0^2\rangle.\end{aligned}$$

Since the component $\langle a_2, b_2, c_2\rangle$ corresponds to the join polynomial $g(y)$ which has not isolated singularities, this component is not considered. Analogously for the component $\langle a_1, b_1, c_1\rangle$. The components $\langle a_0, a_1, a_2\rangle$, $\langle b_0, b_1, b_2\rangle$ and $\langle c_0, c_1, c_2\rangle$ implies that $p_1p_2p_3 = 0$. Also $\langle b_1, b_2, a_1c_2 + a_2c_1\rangle$, $\langle a_1, a_2, b_1c_2 + b_2c_1\rangle$ and $\langle c_1, c_2, a_1b_2 + a_2b_1\rangle$ give a polynomial $h(x) + g(y) \in \mathbb{C}[x]_{\leq 2} \oplus \mathbb{C}[y]_{\leq 2}$. Therefore we are interested in the component associated to the ideal I.

If we consider the intersection of I with $a_0 = b_0 = c_0$, we reduce the numbers of equation that define I to 6 without losing the property that the degree of $p_1p_2p_3$ is 3. Actually by looking in the polynomial $p_1p_2p_3$ we note that is enough to show that $a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1 = 0$ and $a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1 = 0$ to cancel the terms x^2y and xy^2, respectively. A solution for these equation can be given by $b_1 = -\frac{b_2c_1}{c_2}(\frac{1}{2} \pm i\frac{\sqrt{3}}{2})$, $a_1 = \frac{-a_2(b_2c_1+b_1c_2)}{b_2c_2}$ and a_2, b_2, c_1, c_2 free. For example with $a_2 = 3, b_2 = 5i, c_1 = 2, c_2 = 1$, we have $b_1 = -5(i + \sqrt{3})$, $a_1 = -3(1 + i\sqrt{3})$, and $p_1p_2p_3 = 15ix^3 + 120iy^3$.

- If $\eta = \sum_{j=1}^{2} \lambda_j \frac{dp_j}{p_j}$ where $\lambda_j \in \mathbb{C}^*$ for $j = 1, 2$ a first integral is $(p_1^{\lambda_1}p_2^{\lambda_2})^n$ where $p_1 = a_5x^2 + a_4y^2 + a_3xy + a_2x + a_1y + a_0$ and $p_2 = b_2x + b_1y + b_0$. Analogously to the previous case we conclude that $\lambda_i = 1$ and $n = 1$. The ideal associated to the coefficients of the terms x^2y, xy^2, xy of the polynomial p_1p_2, is $J = \langle a_3b_2 + a_5b_1, a_3b_1 + a_4b_2, a_1b_2 + a_2b_1 + a_3b_0\rangle$.

  The prime decomposition of $\sqrt{J}$ is,

  $$\begin{aligned}\sqrt{J} =&\langle b_0, b_1, b_2\rangle \cap \langle b_1, b_2, a_3\rangle \\ &\cap \langle a_4b_2^2 - a_5b_1^2, a_3b_2 + a_5b_1, a_3b_1 + a_4b_2, a_3^2 - a_4a_5, a_1b_2 + a_2b_1 + a_3b_0, a_1a_5b_1 - a_2a_3b_1 - a_3^2b_0\rangle.\end{aligned}$$

  The component $\langle b_0, b_1, b_2\rangle$ implies that $p_1p_2 = 0$, and the component $\langle b_1, b_2, a_3\rangle$ implies that the polynomial p_1p_2 has degree 2. Hence, we focus in the component associated to the ideal

  $$\langle a_4b_2^2 - a_5b_1^2, a_3b_2 + a_5b_1, a_3b_1 + a_4b_2, a_3^2 - a_4a_5, a_1b_2 + a_2b_1 + a_3b_0, a_1a_5b_1 - a_2a_3b_1 - a_3^2b_0\rangle$$

  restricted to the conditions $a_5b_2 \neq 0$ and $a_4b_1 \neq 0$, which correspond to the coefficients of x^3 and y^3 of p_1p_2.

  The intersection of the ideal with $a_0 = b_0 = a_1 = a_2 = 0$, reduces the numbers of equation to 4 without losing the condition of the degree of p_1p_2 is 3. A solution can be given by taking $a_3 = \sqrt{a_4a_5}$, $b_1 = \frac{-a_3b_2}{a_5}$ and a_4, a_5, b_2 free. For example $a_4 = a_5 = b_2 = 1$ implies that $a_3 = 1$, $b_1 = -1$, consequently $p_1p_2 = x^3 - y^3$.

- If $\eta = 3\frac{df}{f} - 2\frac{dg}{g}$ where the degree of f is 2 and degree of g is 3, as in theorem A.3.1. Hence the first integral is $\frac{f^3}{g^2}$. By the lemma A.2.6 follows that if there exists a polynomial $F \in \mathbb{C}[x, y]_{\leq 3}$ which is a first integral for the foliation $\mathcal{F}(\eta)$, then $\frac{f^3}{g^2}$ is a polynomial too which is a contradiction.

A.4 Numerical supplemental items

A.4.1 Orbit by monodromy action in the mirror quintic

The next code is written in MATLAB 2109b. It computes the orbit of the vector $v = \delta_2, \delta_4$, by the monodromy matrices M_0^5, M_1 of the mirror quintic Calabi-Yau threefold, modulo p (see

§2.4).

Firstly, we write the next function *pscale*, which takes any vector $[abcd]$ in a list *vect* and give us a list of p-decimal numbers, i.e. $a * p^3 + b * p^2 + c * p + d$.

```
function wn=pscale(vect,p)
for j=1:size(vect,2)
   wn(j)=vect(1,j)*p^3+vect(2,j)*p^2+vect(3,j)*p^1+vect(4,j)*p^0;
end
end
```

We use this function in order to pass from a list of vectors to a list of number. Now we describe the code for compute the obit. We start by defining the inputs,

```
v=[0;1;0;0]; %Initial vector:  [0,0,0,1] is the 3-torus delta2. [0,1,0,0] is the 3-sphere delta4.
d=5; k=5;    %(d,k) for 14 examples of monodromy for Mirror threefold.  (5,5) is the mirro quintic.
p=3;         %Prime number to consider Zp
M0=[1 1 0 0;0 1 0 0;d d 1 0;0 -k -1 1]; %Monodromy around 0
M1=[1 0 0 0;0 1 0 1;0 0 1 0;0 0 0 1];   %Monodromy around 1
```

For $p > 3$ the Lemma 2.4.1 can be modified by $mod_p(M_0^{5p}) = Id_4$, consequently, we use the next line in order to do more efficient the algorithm.

```
if p>3
   P=p;
else
   P=p^2;
end
```

The next part is the algorithm 1 in the Matlab language. At the end, we get a list of vector *Orbp* which is the orbit of the vector v by monodromy action (modulo p).

```
Orbp=mod(v,p);
Wp=pscale(Orbp,p);  %It is the list Orbp of vectors, written as list of number in p-decimal
norm=1;
while norm>0
   W=Orbp; L=size(Orbp,2);
   l=1;   c=1;
   while l <L+1
      j=0;
      while j<p+1
         i=0;
         while i<P+1
            vaux=mod(M1^j*(M0^5)^i*W(:,l),p); vp=pscale(vaux,p);
            if length(find(Wp==vp))==0
               Orbp(:,L+c)=vaux; Wp(L+c)=vp;
               c=c+1; i=i+1;
            else
               i=i+1;
            end
         end
         j=j+1;
      end
      l=l+1;
   end
   norm=size(Orbp,2)-size(W,2);
end
```

For easier viewing, we sort the list *Orbp* according to the natural order of the vectors written in p-decimal form.

```
Orbp=sortrows(transpose([Orbp; Wp]),5); % Orbit of v. In the last column there are the vectors in p-decimal form.
```

Finally, with the next lines we visualize the orbit, the size of the orbit, and the difference between the size of the orbit and the size of $H_3(\tilde{X}, \mathbb{Z}_p) = p^4$.

```
Lorbit=length(Wp); % Size of the orbit
fprintf('Orbit of v=%d\n');
Orbp
fprintf('Size of the orbit=%d\n',Lorbit);
fprintf('Size diference between the whole H1(X,Zp) and the orbit=%d\n',p^4-Lorbit);
```

A.4.2 Spanned subspace by Monodromy action of $y^e + x^d$.

In this section we provide the explanation of the codes used in the proof of Proposition 3.3.1. Furthermore, this codes are useful in more general contexts, as we show below.

To start, it is necessary to get the MATLAB's functions MonMatrix and VanCycleSub. These are available in https://github.com/danfelmath/Intersection-matrix-for-polynomials-with-1-crit-value.git.

The function MonMatrix computes the monodromy matrix for the polynomial

$$f := x_1^{m_1} + x_2^{m_2} + \ldots + x_{n+1}^{m_{n+1}} \tag{A.1}$$

in a basis similar to that described in §3.3. That is, for each variable x_i, consider a perturbation of $x_i^{m_i}$, such that the critical values are reals and different. Thus, we have a real curve similar to the Figure 3.3. Moreover, we can suppose that the critical points induce the 0-Dynkin diagram

$$\gamma^i_{L+1} \text{ -- } \gamma^i_1 \text{ -- } \gamma^1_{L+2} \text{ -- } \gamma^i_2 \text{ -- } \gamma^i_{L+3} \text{ -- } \gamma^i_3 \quad \cdots,$$

where $L = \lfloor \frac{m_i-1}{2} \rfloor$, and γ^i_j with $j = 1, \ldots, m_i - 1$ are the 0-dimensional vanishing cycles associated to $x_i^{m_i}$. The last vanishing cycle on the right in this Dynkin diagram is γ^i_L or $\gamma^i_{m_i-1}$, depending on whether m_i is odd or even, respectively.

Then, we consider the basis given by the join cycles of the vanishing cycles

$$\gamma^1_{j_1} * \gamma^2_{j_2} * \cdots * \gamma^n_{j_n} * \gamma^{n+1}_{j_{n+1}},$$

where $j_i = 1, \ldots, m_i - 1$. Furthermore, we consider the order $\gamma^i_{L+1} > \gamma^i_1 > \gamma^i_{L+2} > \gamma^i_2 > \gamma^i_{L+3} > \gamma^i_3 > \cdots$, for any i, and the order for the join cycles given by

$$\gamma^1_{j_1} * \gamma^2_{j_2} * \cdots * \gamma^n_{j_n} * \gamma^{n+1}_{j_{n+1}} > \gamma^1_{j'_1} * \gamma^2_{j'_2} * \cdots * \gamma^n_{j'_n} * \gamma^{n+1}_{j'_{n+1}}$$

if and only if $\gamma^i_{j_i} = \gamma^i_{j'_i}$ for $i = 1, \ldots, I-1$ and $\gamma^I_{j_I} > \gamma^I_{j'_I}$. We use the ordered basis $(\gamma^1_{j_1} * \gamma^2_{j_2} * \cdots * \gamma^n_{j_n} * \gamma^{n+1}_{j_{n+1}}, >)$ where $j_i = 1, \ldots, m_i - 1$, in order to write the intersection and monodromy matrices associated to the fibration given by A.1.

For example, in §3.3 we consider $f(x,y) = x^6 + y^4$, and we use the notation δ^j_i to denote the vanishing cycle in the row i and column j of the Dynkin diagram 3.4. Since we consider $x_1 = x$, $x_2 = y$, we have the basis

$$\gamma^1_3 * \gamma^2_2, \gamma^1_3 * \gamma^2_1, \gamma^1_3 * \gamma^2_3, \gamma^1_1 * \gamma^2_2, \gamma^1_1 * \gamma^2_1, \ldots, \gamma^1_5 * \gamma^2_3.$$

Thus, $\delta^j_1 = \gamma^1_{\sigma(j)} * \gamma^2_2$, $\quad \delta^j_2 = \gamma^1_{\sigma(j)} * \gamma^2_1$, $\quad \delta^j_3 = \gamma^1_{\sigma(j)} * \gamma^2_3$, where σ is the permutation $(1,2,3,4,5) \xrightarrow{\sigma} (3,1,4,2,5)$.

In order to compute the intersection matrix by using the function MonMatrix, it is enough to write *Im=MonMatrix(m,p)*, where m is a vector whose coordinate $m(i)$ corresponds with the degree m_i. The parameter p should be a integer number such that: If $p = 0$, then Im is the intersection matrix, else Im is the monodromy matrix. For the previous example, we get the matrix 3.7, by writing the lines

```
m=[6,4];
Im=MonMatrix(m, 0)
```

The function VanCycleSub computes the subspace spanned by the monodromy action of the fibration given by A.1, acting on each vanishing cycle. In other words, this function compute the Krylov space of the monodromy matrix and each one of the the vectors of the basis previously described. That is by computing the eigenvalues and eigenvector of the monodromy matrix as in the proof of Proposition 3.3.1. The usage of this function is as follows: *[Dim, Wout, Vout]=VanCycleSub(m)*, where the vector m represented again the degrees m_i.

The output *Dim* is the number of different eigenvalues of the monodromy matrix. Note that the dimension of the homology group $H_n(f^{-1}(b))$, is $N = \prod_{i=1}^{n+1}(m_i - 1)$. If $Dim = N$, then the array *Wout* of size $N \times N \times N$, represents the Krylov space of each vanishing cycle. Thus, the columns of the matrix *Wout(:,:,j)* are a basis of the subspace generated by the monodromy action on the vector $e_j = (0 \cdots 0\ 1\ 0 \cdots 0)$. The vector e_j corresponds with the j-th joint cycle according to the previously defined order.

Finally, *Vout* is a matrix where the column j is a list of the vanishing cycles in the Krylov subspace of the vector e_j with the monodromy matrix. Actually, this list is the position given by the order, associated to these vanishing cycles. Note that these vanishing cycles correspond to the rows of *Wout(:,:,j)* with a single 1 and zeros in the others. Continuing the example,

```
m=[6,4];
[Dim, Wout,Vout]=VanCycleSub(m)
```

In this case the second column of $Vout$ is the list $(2, 5, 8, 11, 14)$, which are the positions associated to the vanishing cycles δ_2^k, with $k = 1, \ldots, 5$ (see Dynkin diagram 3.4). The fifth column is the list $(5, 11)$; it is because the vanishing cycle associated to the position 5th and 11th are δ_2^2 and δ_2^4, respectively, and $gcd(d, 2) = 2$ (see Proposition 3.3.1).

A.4.3 Dynkin diagrams for 4th degree polynomials $h(y) + g(x)$

Here, we provide a MATLAB's code, which determine the *non simple* Dynkin diagrams associated to the polynomials $h(y) + g(x) \in \mathbb{R}[x, y]_{\le 4}$ (see §3.5). By non simple Dynkin diagram, we mean Dynkin diagram where any vanishing cycles is simple. The Milnor number in this case is 9, hence we consider partition $(d_1, d_2, \ldots, d_M)$ of 9.

We use two MATLAB's functions, DynkinComb and DynkinNonSimple. These are available in https://github.com/danfelmath/Intersection-matrix-for-polynomials-with-1-crit-value.git.

The function DynkinComb computes the possibles Dynkin diagrams associated to the partition $(d_1, d_2, \ldots, d_M)$ of 9, according to the rules described in §3.2. The usage is as follows

```
[Vout,IoutR]=DynkinComb(Par,Ds)
```

where *Par* is a vector expression which storages a partition of 9, and *Ds* is the value 1 or 2. If Ds=1, then the Dynkin diagram is the associated to n 3.10. If Ds=1, then the Dynkin diagram is the associated to 3.11. The array *Vout* storages matrices which represent the Dynkin diagrams with this partition *Par* in the equalities of the critical values and which satisfy the rules in §3.5. Each matrix in this array is a 3×3 matrix, with integer numbers. The repetition of any of these integer values means that the critical values are equals. For example for the partition $(4, 2, 2, 1)$, we compute

```
>> [Vout,IoutR]=DynkinComb([4,4,1],1);
>> Vout

Vout(:,:,1) =

1     4     4
```

```
1       4       4
3       1       1


Vout(:,:,2) =

2       1       2
1       7       1
2       1       2
```

Hence, for this partition and the Dynkin diagram 3.10, there are two possibilities. For example, the second means that the critical values satisfy $a_1 = a_4 = a_8 = a_9$, $a_2 = a_3 = a_5 = a_6$ and a_7 is different from the others. The list *IoutR* storages each Dynkin diagram for the partition *Par* and an indication of whether or not they satisfy the rules 1 and 2. For the previous example, we have that the matrix in the position 201 of the list *IoutR* is

```
IoutR(:,:,201) =

2       1       2       0       0
1       7       1       1       1
2       1       2       0       0
```

The 3×3 sub-matrix in the right indicates the Dynkin diagram. The position $(2,4)$ and $(2,5)$ indicate that the rules 1 and 2 hold, respectively. In other cases of Dynkin diagrams the position $(2,4)$ or $(2,5)$ are 0.

The function DynkinNonSimple determines which of these Dynkin diagram have at least one non simple vanishing cycle. The usages is as follows

```
[Vout,Ra]=DynkinComb(Par,Ds)
```

Where *Par* and *Ds* are as in the previous function. The array *Vout* contains all the Dynkin diagrams (3.10 or 3.11 depending on *Ds*) with non simple cycles for the partition *Par* of 9. For the previous example, we have

```
>> [Vout,Ra]=DynkinNonSimple([4,4,1],1);
>> [Vout]

Vout =

2       1       2
1       7       1
2       1       2
```

The array *Ra* contains the rank of the vanishing cycles for the Dynkin diagrams in Vout. In the previous example,

```
>> Ra

Ra =

7       6       7
6       4       6
7       6       7
```

This example corresponds to $\mathfrak{O}_4$ in Table 3.1.

A.4.4 Irreducible components of polynomials space

The next codes for Singular computes the irreducible components of the polynomials in $\mathbb{C}[x]_{\le d} \oplus \mathbb{C}[y]_{\le d}$ with a critical values degree fixed. The code uses the library "foliation.lib" written by H. Movasati.

The first one, computes the irreducible components for a polynomial in 1 variable $F(x) = x^d + x^{d-2}r_{d-2} + ...xr_1$:, with parameters $r_1, r_2, \dots, r_{d-2}$. By linear operations, we can suppose that $r_0 = r_{d-1} = 0$ and $r_d = 1$. Initially, we use the next libraries,

```
LIB "foliation.lib";
LIB "primdec.lib";
LIB "rinvar.lib";
```

Then, the algorithm has four steps:

1. Compute the *discriminant* of F. The zeros of the discriminant, Δ, of $F - c$ are the points c such that the variety given by the polynomial F is not smooth, i.e. the critical values.

```
int d=4;  //degree of polynomial
ring R1=(0,r(1..d-1),c),(x),dp;
// Definition of the polynomial F(x,y)=x^d+x^(d-2)r(d-2)+...x r(1):
poly F=x^d;
for (int i=1; i<=d-2; i=i+1)
{
 F=F+r(i)*x^i;
}
number disc=discriminant(F-c);
poly disc2=substpar(disc,c,x); // We write the polynomial disc in the variable x.
int S=int(coef(disc2,x)[2,1]); // Sign of the first, for the Vieta's formula.
// We change the ring.  R1 is used  because in Foliation.lib,  F has to be tame polynomial.
ring R2 = 0, (t(1..d-1)),lp;
```

2. Computing the equation of the image of the map given by Vieta's formula. Consider the map

$$\mathbb{C}^{d-1} \xrightarrow{\varphi} \mathbb{C}^{d-1}$$
$$(t_1, t_2, \dots, t_{d-1}) \xrightarrow{\varphi} (\varepsilon(1)\sum_i t_i, \varepsilon(2)\sum_{i\neq j} t_i t_j, \dots, \varepsilon(d-1)t_1 t_2 \cdots t_{d-1}),$$

where $\varepsilon(j) = (-1)^j \alpha_{d-1}$ and α_{d-1} is the lead coefficient of $\Delta(F-c)$. By taking some equalities $t_i = t_j$, we define a variety V in the domain which corresponds with a partition $(d-1) = d_1 + d_2 + d_3 + d_4 + ... + d_{d-1}$ (in this notation some d_i could be 0).

```
ideal Phi0=elemSymmId(d-1); //Polynomials of the image Phi (Vieta's formula) Phi.
ideal Phi;
for (int i=1; i<=d-1; i=i+1)
{
 Phi[d-i]=(Phi0[i]*(-1)^i)/S;
}
///////HERE WE SPECIFY THE CONDITIONS ON CRITICAL VALUE DEGREE////////
//ideal I=t(1)-t(2), t(1)-t(3);  // I_(3,0)
ideal I=t(1)-t(2);  // I_(2,1)
//ideal I=0; //I_(111)
/////////////////////////
def R3 = ImageVariety(I, Phi);
setring R3;
ideal Wa=imageid;
```

3. Pullback of the variety $\varphi(V)$ via the function $disc(F-c) := \Delta(F-c)$. We have the map $\Delta(F-c) = (c-t_1)^{d_1}(c-t_2)^{d_2}(c-t_3)^{d_3}(c-t_4)^{d_4}\dots(c-t_{d-1})^{d_{d-1}}$. This map defines a point in $\mathbb{C}^{d-1}$ for any polynomial in $\mathbb{C}[x]_{\le d}$. In $\mathbb{C}^{d-1}$ we have the variety $\varphi(V)$. Hence, by taking pullback we get equations in $\mathbb{C}^{d-1} = (r_{d-1},...r_1)$, which correspond to the ideal $I_{(d_1,d_2,\dots,d_{d-1})}$ associated to the partition $(d_1, d_2, \dots, d_{d-1})$.

```
// Ring where there are all variables and parameters:
ring R4 =(0,r(1..d-1),c), (x,Y(1..d-1),R(1..d-1)),lp;
// We recall the discriminant from the ring R1
// Then we replace the parameters r(i) with the variables R(i).
// We storage the coefficients of the polynomial in the variable c (or x) in 2th row of  M.

poly Disc1=imap(R1,disc2);
for (int i=1; i<=d; i=i+1)
{
 Disc1=subst(Disc1,r(i),R(i));
}
matrix M=coeffs(Disc1, x);
// Image variety:
ideal W=imap(R3,Wa);
//Doing pullback:
ideal Iddd=W;
for (int i=1; i<=d-1; i=i+1)
{
 Iddd=subst(Iddd,Y(i),M[i,1]);
}
```

4. The irreducible components $I_{(d_1,d_2,\dots,d_{d-1})}$. We compute a primary decomposition of the ideal $I_(d_1, d_2, \dots, d_{d-1})$. Actually, we can compute the radicals of the irreducible components, i.e. the prime decomposition of $I_(d_1, d_2, \dots, d_{d-1})$.

```
list C_1=primdecGTZ(Iddd);
```

For polynomials $h(y) + g(x)$ such that the critical values of h coincide with the critical values of g, we have a similar code. Again, this is divided in four steps:

1. Compute the discriminant of h and g. We compute the zeros of $\Delta(h-c)$ and $\Delta(g-c)$ in function of c. They are the critical values of h and g.

```
int d=4;
ring R1=(0,r(1..d-1),s(1..d-1),c),(x),dp;
poly h=x^d; poly g=x^d;
for (int i=1; i<=d-2; i=i+1)
{
   h=h+r(i)*x^i; g=g+s(i)*x^i;
}
number disch=discriminant(h-c);
 poly disch2=substpar(disch,c,x);   // We write the polynomial disc in the variable x.
int Sh=int(coef(disch2,x)[2,1]); // Sign of the first coefficient in la Vieta's formula.
number discg=discriminant(g-c);
poly discg2=substpar(discg,c,x);  // We write the polynomial disc in the variable x.
int Sg=int(coef(discg2,x)[2,1]); // Sign of the first coefficient in la Vieta's formula.
// We change the ring. R1 is used  because in Foliation.lib, h and g has to be tame.
ring R2 = 0, (t(1..d-1),l(1..d-1)),lp;
```

2. Computing the equation of the image of the map given by Vieta's formula. Consider the map

$$\mathbb{C}^{2(d-1)} \xrightarrow{\varphi} \mathbb{C}^{2(d-1)}$$

$$(t_1, \dots, t_{d-1}, l_1, \dots l_{d-1}) \xrightarrow{\varphi} (\varphi_h, \varphi_g) :=$$
$$(\varepsilon_h(1)\sum_i t_i, \dots, \varepsilon_h(d-1)t_1t_2\cdots t_{d-1}, \varepsilon_g(1)\sum_i l_i, \dots, \varepsilon_g(d-1)l_1l_2\cdots l_{d-1}),$$

where $\varepsilon_h(j) = (-1)^j \alpha^h_{d-1}$ and α^h_{d-1} is the lead coefficient of $\Delta(h-c)$, analogously $\varepsilon_g(j)$. The equation in the domain $t_i = l_i$ for $i = 1, \dots d-1$, correspond with the condition that the critical values of h are equal to the critical values of g. These equations define the variety V in the domain.

```
// Phi_h is consider elemSymmd in the first (d-1) variable.
// The definition for Phi_g is  little trickly, because we only can define for  the firts variables.
 ideal Ph=elemSymmId(d-1);
 ideal Pg=Ph;
 for (int i=1; i<=d-1; i=i+1)
 {
  Pg=subst(Pg,t(i),l(i));
 }
 for (int i=1; i<=d-1; i=i+1)
 {
  Ph[i]=(Ph[i]*(-1)^i div Sh); Pg[i]=(Pg[i]*(-1)^i div Sg);
 }
 ideal Phi=Ph,Pg;
 ///////HERE WE SPECIFY THE CONDITIONS ON CRITICAL VALUE DEGREE////////
 ideal I=t(1)-l(1), t(2)-l(2),t(3)-l(3); // The critical values of g and h are equal.
 //////////////////////////
 def R3 = ImageVariety(I, Phi);
 setring R3;
 ideal Wa=imageid;
```

3. Pullback of the variety $\varphi(V) = (\varphi_h, \varphi_g)(V)$ via the function $(\Delta(h-c), \Delta(g-c))$. We have the maps $\Delta(h-c) = (c-r_1)(c-r_2)\cdots(c-r_{d-1})$ and $\Delta(g-c) = (c-s_1)(c-s_2)\cdots(c-s_{d-1})$. This map defines a point in $\mathbb{C}^{2(d-1)}$ for any polynomial in $\mathbb{C}[x,y]_{\le d}$. In $\mathbb{C}^{2(d-1)}$ we have the variety $\varphi(V)$. Hence, by taking pullback we get equations in $\mathbb{C}^{2(d-1)} = (r_{d-1}, \dots, r_1, s_{d-1}, \dots, s_1)$, which correspond of the ideal of the polynomials $h(y) + g(x)$ such that the critical values of h and g are equals.

```
 // Ring where there are all variables and parameters:
 ring R4 =(0,r(1..d-1),s(1..d-1),c), (x,Y(1..2*(d-1)),R(1..d-1),S(1..d-1)),lp;
 // We recall the discrimant from the ring R1
 // Then we replace the parameters r(i), s(j) with the variables R(i), S(j).
 // We storage the coeff. of the polynomial in the variable c (or x) in 2th row Mh and Mg.

 poly Disch1=imap(R1,disch2); poly Discg1=imap(R1,discg2);
 for (int i=1; i<=d; i=i+1)
 {
  Disch1=subst(Disch1,r(i),R(i)); Discg1=subst(Discg1,s(i),S(i));
 }
 matrix Mh=coeffs(Disch1, x); matrix Mg=coeffs(Discg1, x);
 // Image variety:
 ideal W=imap(R3,Wa);
 //Doing pullback
 ideal T=W;
 for (int i=1; i<=d-1; i=i+1)
 {
  T=subst(T,Y(i),Mh[i,1]); T=subst(T,Y(i+(d-1)),Mg[i,1]);
 }
 // Counting components
```

4. The irreducible components. We compute the prime decomposition of the ideal of the polynomials $h(y) + g(y)$, such that the critical values of h and g are equals.

```
  list C_1=primdecGTZ(T);
```

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