Research Statement

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My research program concentrates on the areas of algebraic/complex geometry and symplectic geometry. I am broadly interested in the topology of algebraic and symplectic manifolds, more specifically in the study of certain homological cycles naturally attached to the underlying geometric structures. In the symplectic case, I have focused on cycles supported on Lagrangian submanifolds, while in the complex case I have worked with Pham cycles (also called vanishing cycles). By means of the monodromy representation of these cycles, I have studied several geometric questions inspired by the theory of Lie algebroids and holomorphic foliations. In the following, I will explain these questions, my contributions and part of the work I intend to develop in the near future.

1 The Monodromy problem

In complex geometry, I am widely interested in holomorphic foliations. The holomorphic foliations are a rich topic of investigation, either from the perspective of dynamics or from the perspective of algebraic geometry. For example, Hilbert's 16th problem asks about an upper bound on the number of limit cycles in planar real polynomial vector fields of a given degree. It is a problem that comes from dynamical systems, however by considering an extension of the problem to $\mathbb{C}^2$, it is possible to approach it by means of algebraic geometry techniques [19, 24]. One of these algebraic approaches is the *Center problem* [23].

Given a polynomial with two variables we consider its associated foliation. The center points of the foliation are vanishing cycles. The monodromy problem is to establish conditions on the polynomial so the orbit of the vanishing cycles generate the whole homology group. It is possible to relate the monodromy action with a diagram called the Dynkin diagram. Thus, given a polynomial, we translate questions on the subspaces generated by these orbits to combinatory aspects of the diagram.

The polynomial foliations in $\mathbb{C}^2$ are given by 1-forms $\omega = P(x,y)dy - Q(x,y)dx$, where P and Q are polynomials. The classical notation for the foliation associated to the form ω is $\mathcal{F}(\omega)$. In this context, a point $p \in \mathbb{C}^2$ is a singularity of $\mathcal{F}(\omega)$ if $P(p) = Q(p) = 0$. We say that the singularity p is a *center singularity* if there is a local chart such that p is mapped to $0 \in \mathbb{C}^2$, and a non-degenerated function $f : (\mathbb{C}^2, 0) \to \mathbb{C}$ with fibers tangent to the leaves of $\mathcal{F}(\omega)$. The degree of a foliation $\mathcal{F}(\omega)$ is the greatest degree of the polynomials P and Q, and the space of foliations of degree d is denoted by $\mathcal{F}(d)$. The closure of the set of foliations in $\mathcal{F}(d)$ with at least one center is denote by $\mathcal{M}(d)$.

It is known that $\mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$. The problem of describing its irreducible components is formulated by Lins Neto [23]. In [13], Y. Ilyashenko proves that the space of Hamiltonian foliation $\mathcal{F}(df)$, where f is a polynomial of degree $d+1$, is an irreducible component of $\mathcal{M}(d)$. In [20], H. Movasati considers the logarithmic foliations

$\mathcal{F}\left(\sum_{i=1}^{s}\lambda_i \frac{df_i}{f_i}\right)$, with $f_i \in \mathbb{C}[x,y]_{\le d_i}$ and $\lambda_i \in \mathbb{C}^*$. He proves that the set of logarithmic foliations form an irreducible component of $\mathcal{M}(d)$, where $d = \sum_{i=1}^{s} d_i - 1$. Moreover, considering pullback foliations $\mathcal{F}(P^*\omega)$, where $P = (R,S) : \mathbb{C}^2 \to \mathbb{C}^2$ is a generic morphism with $R, S \in \mathbb{C}[x,y]_{\le d_1}$, and ω is a 1-form of degree d_2, Y. Zare in [27] show that they form an irreducible component.

The main idea in the proofs of these assertions is to choose a particular polynomial f and consider deformations $df + \varepsilon\omega_1$ in $\mathcal{M}(d)$. Then, it is necessary to study the vanishing of the abelian integrals $\int_\delta \omega_1$, where δ is a homological 1-cycle in a regular fiber of f. This integral is zero on the vanishing cycle associated to the center singularity. If the monodromy action on this cycle generates the whole vector space $H_1(f^{-1}(b),\mathbb{Q})$ for a regular value b, then the deformation is relatively exact to df.

This gives rise to the next natural problem, which is summarized by C. Christopher and P. Mardešić in [4] as follows.

Monodromy problem. Under which conditions on f is the $\mathbb{Q}$-subspace of $H_1(f^{-1}(b),\mathbb{Q})$ generated by the images of a vanishing cycle of a Morse point under monodromy equal to the whole of $H_1(f^{-1}(b),\mathbb{Q})$?

Furthermore, they show a characterization of the vanishing cycles associated to a Morse point in hyperelliptic curves given by $y^2 + g(x)$, depending on whether g is decomposable. This case is closely related with the 0-dimensional monodromy problem; by using the definition of Abelian integrals of dimension zero in [12]. However, the Dynkin diagram for $y^3 + g(x)$ is a bit more complicated. Moreover, in the case $y^4 + g(x)$ there is always a pullback associated to $y \to y^2$, thus the Dynkin is expected to reflect this fact.

In my work [16, *The monodromy problem for hyperelliptic curves. D. López*], For these two cases, I prove the following two theorems.

Theorem 1.1. *Let g be a polynomial with real critical points, and degree d such that, $4 \nmid d$ and $d \le 100$. Consider the polynomial $f(x,y) = y^4 + g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

1. *The monodromy of $\delta(t)$ generates the homology $H_1(f^{-1}(t),\mathbb{Q})$.*
2. *The polynomial g is decomposable (i.e., $g = g_2 \circ g_1$), and $\pi_*\delta(t)$ is homotopic to zero in $\{y^4 + g_2(z) = t\}$, where $\pi(x,y) = (g_1(x),y) = (z,y)$. Or, the cycle $\pi_*\delta(t)$ is homotopic to zero in $\{z^2 + g(x) = t\}$, where $\pi(x,y) = (x,y^2) = (x,z)$.*

Theorem 1.2. *Let g be a polynomial with real critical points, and degree d such that, $3 \nmid d$ and $d \le 100$. Consider the polynomial $f(x,y) = y^3 + g(x)$, and let $\delta(t)$ be an associated vanishing cycle at a Morse point; then one of the following assertions holds.*

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The main idea behind the proofs of theses theorems is give an explicit description of the intersection matrices by using Lefschetz fibration theory of direct sum of polynomials. Some parts in the proof are done numerically using computer, thus we have the restriction $d \le 100$ in the degree of the polynomial g. [1]

[1]The reader can use the functions written in MATLAB, MonMatrix and VanCycleSub https://github.com/danfelmath/Intersection-matrix-for-polynomials-with-1-crit-value.git

The monodromy problem for polynomials of degree 4, on the other hand, is very interesting, because the classification of the irreducible components of $\mathcal{M}(3)$ is still an open problem. In fact, the only case which has a complete classification is $\mathcal{M}(2)$ (see [11][2, p. 601]). For polynomials $f(x,y) = h(y) + g(x)$ where $\deg(h) = \deg(g) = 4$, we determine in the Theorem 1.3, a relation between the subspaces of $H_1(f^{-1}(b), \mathbb{Q})$ generated by the monodromy action of the vanishing cycles, and the property of f being decomposable. In oder to do that, we provide an explicit description of the space of parameters of the polynomials $h(y) + g(y)$ which satisfies some conditions in the critical values.

Theorem 1.3. *Let $f(x,y) = h(x) + g(y)$, where $h \in \mathbb{R}[x]_{\leq 4}$ and $g \in \mathbb{R}[y]_{\leq 4}$ are polynomials with real critical points. There is a characterization of the equivalence class of f in terms of h, g as follows,*

1. $[f] \in \mathfrak{O}_1$ *iff* $f(x,y) = x^4 + y^4$.
2. $[f] \in \mathfrak{O}_2$ *iff* $f(x,y) = (h_2 \circ h_1)(x) + g(y)$, *where* $h_1, h_2 \in \mathbb{R}[x]_{\leq 2}$ *and g is not decomposable.*
3. $[f] \in \mathfrak{O}_3$ *iff* $f(x,y) = (h_2 \circ h_1)(x) + (g_2 \circ g_1)(y)$, *where* $h_1, h_2 \in \mathbb{R}[x]_{\leq 2}$, $g_1, g_2 \in \mathbb{R}[y]_{\leq 2}$.
4. $[f] \in \mathfrak{O}_4$ *iff* $f(x,y) = (h_2 \circ h_1)(x) + (h_2 \circ h_1)(\pm y)$, *where* $h_1, h_2 \in \mathbb{R}[x]_{\leq 2}$.
5. $[f] \in \mathfrak{O}_0$ *iff $h(x)$ and $g(y)$ are not decomposable.*

Each $\mathfrak{O}_i$ refers to an explicit subspace in $H_1(f^{-1}(b), \mathbb{Q})$ described in [16].

2 Hausdorff integration of Lie algebroids

Poisson manifolds arise naturally from classical mechanics problems, as models of their phase spaces. They can be thought as the infinitesimal part of a global structure, which corresponds to a symplectic groupoid. In other words, the symplectic groupoids are the integration of Poisson manifolds. This idea is formalized by considering the relation between fiber-wise linear Poisson structure and Lie algebroids [10]. As usual, these integrations do not always exist. And when they do, they can present undesirable topological properties, for instance, they can be non-Hausdorff spaces.

Lie groupoids and Lie algebroids play a central role in differential geometry, serving as models for classic geometries such as actions, foliations and bundles, and with applications to symplectic geometry [18]. Every Lie groupoid yields a Lie algebroid through differentiation, setting a rich interplay between global and infinitesimal data, which is ruled by the so-called Lie Theorems. Lie 1 constructs a maximal (source-simply connected) Lie groupoid integrating the algebroid of a given groupoid, and Lie 2 shows that a Lie algebroid morphism can be integrated to a Lie groupoid morphism under a certain hypothesis [17, 18]. The hardest one, Lie 3, provides computable obstructions to the integrability of a Lie algebroid [6].

When working with Lie groupoids one usually allows the manifold of arrows to be non-Hausdorff. One reason for that is to include the monodromy and holonomy groupoids arising from foliations. Other reason is that the maximal groupoid given by Lie 1 may be non-Hausdorff even when the original groupoid is. A Poisson manifold yields a Lie algebroid on its cotangent bundle, which is integrable if and only if the Poisson manifold has a complete symplectic realization [7]. The canonical symplectic structure on the cotangent bundle integrates to a symplectic structure on the source-simply connected groupoid [5], which may be non-Hausdorff, and smaller integrations may be Hausdorff but not symplectic.

My joint work with M. del Hoyo [8, *On Hausdorff intefrations of Lie Algebroids, M. del Hoyo and D. Lopez*], is motivated by the problem of understanding Hausdorff symplectic groupoids arising from Poisson manifolds. The first main theorem is the Hausdorff version for Lie 1, Theorem 2.1, showing that every Hausdorff groupoid yields a maximal Hausdorff integration. We say that a morphism between Lie groupoids is a *Lie equivalence*, if it induces an isomorphism between the corresponding Lie algebroids.

Theorem 2.1 (Hausdorff Lie 1)**.** *Given G a Hausdorff Lie groupoid, there exists a Hausdorff Lie groupoid $\hat{G}$ and a universal Lie equivalence $\hat{\phi}:\hat{G}\to G$, in the sense that for any other Lie equivalence $\phi':G'\to G$ with G' Hausdorff, there exists a unique factorization $\hat{\phi}=\phi'\phi$.*

$$\begin{tikzcd} \hat{G} \arrow[r, "\exists!\phi"] \arrow[d, "\hat{\phi}"'] & G' \arrow[dl, "\forall\phi'"] \\ G & \end{tikzcd}$$

If $\tilde{G}$ is the source-simply connected groupoid associated to G, then it is possible to characterize every Lie equivalence $\tilde{G}\to G'$ in terms of some subgroupoids of $\tilde{G}$. Namely, these subgroupoids are given by the kernels, and they should be embedded submanifolds, with equals source and target maps, and discrete isotropy. Therefore, the proof in Theorem 2.1 is by studying which of these is the "less" subgroupoid K such that $\tilde{G}/K$ is Hausdorff.

The second main result is a Hausdorff version of Lie 2 that includes a holonomy hypothesis.

Theorem 2.2 (Hausdorff Lie 2)**.** *Let G and H be Hausdorff Lie groupoids and $\varphi:A_G\to A_H$ a Lie algebroid morphism. If $G=\hat{G}$ and the foliation F^{φ} has trivial holonomy then φ integrates to a groupoid morphism $\phi:G\to H$, which is unique.*

The idea behind of Theorem 2.2 is to integrate the Lie subalgebroid $S\subset A_G\times A_H$ associated to the graph of φ. This integration is the *holonomy groupoid* $\mathrm{Hol}(S)$ associated to the foliation defined by S. As an application of Theorem 2.2, we show that if the algebroid induced by a Poisson manifold is integrable by a Hausdorff groupoid, then the maximal Hausdorff integration is symplectic.

Theorem 2.3. *Given (M,π) a Poisson manifold, if the induced Lie algebroid $A\Rightarrow M$ is integrable by a Hausdorff groupoid $G\rightrightarrows M$, then $\hat{G}\rightrightarrows M$ is a Hausdorff symplectic groupoid.*

The Lie algebroid associated to (M,π) is the contagent bundle $A=T^*M$, thus the canonical symplectic form defines an isomorphism between the Lie algebroids $TA\xrightarrow{\omega_{can}}T^*A$. After showing that the conditions of the holonomy holds, Theorem 2.2 integrates ω_{can} to a isomorphism of the Lie groupoids $T\hat{G}\to T^*\hat{G}$, which defines a symplectic structure in $\hat{G}$.

3 Homology supported in Lagrangian submanifolds

Given a symplectic manifold, the study of its Lagrangian submanifolds it is an interesting problem. For example, the Lagrangian submanifolds endowed with the intersection between them give rise to the *Floer homology.* In a symplectic manifold (X,ω) of dimension

$2n$, a Lagrangian submanifold has dimension n, therefore, a natural question is which part of the homology group $H_n(X,\mathbb{Q})$ can be represented by Lagrangian submanifolds.

Considering a symplectic structure in a fibration such that the fibers are symplectic manifolds, it is possible to show that the vanishing cycles are Lagrangian submanifolds and the monodromy action is given by symplectomorphisms. In the case of mirror quintic, there is an explicit form for the monodromy matrices and for two Lagrangian cycles which are supported in a 3-sphere and 3-torus. We study the orbit of these cycles by monodromy action.

In [25], the authors define Lagrangian cycles as cycles in a symplectic 4-manifold, whose two-simplices are given by C^1 Lagrangian maps and a *Lagrangian homology class* is a homology class which can be represented by a Lagrangian cycle. In that article, they show a characterization of the Lagrangian homology classes in terms of the minimizers of an area functional. Moreover, they show for a compact Kälher 4-manifold (X,ω,J) and a homology class $\alpha\in H_2(X,\mathbb{Z})$, that α is a Lagrangian homology class if and only if $[\omega](\alpha)=0$. If the Chern class $c_1(X)$ also annihilates α, then α can be represented by an immersed Lagrangian surface (not necessarily embedded).

The question about which part of the homology is supported in Lagrangian submanifolds, can be refined a little more if we look for Lagrangian spheres. In [14], for a ruled surfaces (X,ω,J), it is shown that the class $\alpha\in H_2(X,\mathbb{Z})$ is represented by a Lagrangian sphere if and only if $[\omega](\alpha)=0$, $c_1(X)(\alpha)=0$, $\alpha^2=-2$ and α is represented by a smooth sphere. For 4-manifolds, the dimension of the 2-cycles allows us to relate the property of being represented by Lagrangian cycles with the vanishing of the *periods* $\int_\alpha\omega$ and $\int_\alpha c_1(X)$. For higher dimension manifolds this pairing is not well-defined, hence we do not have a natural generalization of the previous results. Despite of this, it is possible to show that in any regular hypersurface of $\mathbb{P}^n$ with n even, all $(n-1)$-cycles can be written as a linear combination of cycles supported in Lagrangian spheres. It is a consequence of Lefschetz fibration theory.

A more interesting question for $n=4$, is to ask not only which homology classes are generated by Lagrangian spheres but which ones are supported in Lagrangian spheres. In my work [15, *Homology supported in Lagrangian submanifolds in mirror quintic threefolds. D. López. Can. Math. Bull.*], a family $\tilde{X}_\varphi$ of mirror quintic Calabi-Yau threefolds is considered and some classes in $H_3(\tilde{X}_\varphi,\mathbb{Z})$ which are supported in Lagrangian 3-spheres and Lagrangian 3-tori are studied. This family is constructed as follows. Consider the Dwork family X_φ in $\mathbb{P}^4$ given by the locus of the polynomial

$$p_\varphi := \varphi z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4 = 0,$$

with critical values in $\varphi=0,1,\infty$. For every $\varphi\neq 0,1,\infty$, $\tilde{X}_\varphi$ is obtained as a desingularization of the quotient of X_φ by the action of a finite group (see [1, 3, 9]). The rank of the free group $H_3(\tilde{X}_\varphi,\mathbb{Z})$ is 4, hence it is isomorphic to $\mathbb{Z}^4$ after choosing a basis. In this basis the homology class $\delta_2=(0\ 1\ 0\ 0)$ is represented by a torus associated to the singularity of X_φ when $\varphi\to 0$ and the class $\delta_4=(0\ 0\ 0\ 1)$ is represented by a sphere S^3 associated to the singularity of X_φ when $\varphi\to 1$.

The monodromy action of the family is given by symplectomorphisms at each regular fiber. It is possible to determine two matrices M_0 and M_1 such that the monodromy action over $H_3(\tilde{X}_\varphi,\mathbb{Z})$ corresponds to the free subgroup of $Sp(4,\mathbb{Z})$ generated by M_0 and M_1. Therefore, the orbit of δ_2 and δ_4 by the action of $M_0 * M_1$ are homology classes which can be represented by Lagrangian submanifolds. The main result in my article is about

$H_3(\tilde{X}_\varphi, \mathbb{Z}_p)$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for some primes p, and it is summarized in the following theorem.

Theorem 3.1. *For the mirror quintic Calabi-Yau threefold $\tilde{X} := \tilde{X}_\varphi$ with $\varphi \neq 0, 1, \infty$, the homology classes*

$$(0\ 0\ 1\ 1),\ (0\ 1\ 0\ 0),\ (0\ 1\ 0\ 1),\ (1\ 0\ 0\ 1),\ (1\ 0\ 1\ 1) \in H_3(\tilde{X}, \mathbb{Z}_2) \tag{3.1}$$

$$\begin{gathered}(0\ 1\ 0\ 0), (0\ 1\ 0\ 1), (0\ 1\ 0\ 2), (0\ 1\ 0\ 3), (0\ 1\ 0\ 4),\\ (0\ 1\ 1\ 0), (0\ 1\ 1\ 1), (0\ 1\ 1\ 2), (0\ 1\ 1\ 3), (0\ 1\ 1\ 4),\\ (0\ 1\ 2\ 0), (0\ 1\ 2\ 1), (0\ 1\ 2\ 2), (0\ 1\ 2\ 3), (0\ 1\ 2\ 4),\\ (0\ 1\ 3\ 0), (0\ 1\ 3\ 1), (0\ 1\ 3\ 2), (0\ 1\ 3\ 3), (0\ 1\ 3\ 4),\\ (0\ 1\ 4\ 0), (0\ 1\ 4\ 1), (0\ 1\ 4\ 2), (0\ 1\ 4\ 3), (0\ 1\ 4\ 4) \in H_3(\tilde{X}, \mathbb{Z}_5)\end{gathered} \tag{3.2}$$

are represented by Lagrangian 3-tori. The homology classes

$$\begin{gathered}(0\ 0\ 0\ 1),\ (0\ 0\ 1\ 0), (0\ 1\ 1\ 0), (0\ 1\ 1\ 1), (1\ 0\ 0\ 0),\\ (1\ 0\ 1\ 0), (1\ 1\ 0\ 0), (1\ 1\ 0\ 1), (1\ 1\ 1\ 0), (1\ 1\ 1\ 1) \in H_3(\tilde{X}, \mathbb{Z}_2)\end{gathered} \tag{3.3}$$

$$(0\ 0\ 0\ 1),\ (0\ 0\ 1\ 1),\ (0\ 0\ 2\ 1),\ (0\ 0\ 3\ 1),\ (0\ 0\ 4\ 1)\ \in H_3(\tilde{X}, \mathbb{Z}_5) \tag{3.4}$$

are represented by Lagrangian 3-spheres. For $p = 3, 7, 11, 13, 17, 19, 23$, any homology class in $H_3(\tilde{X}, \mathbb{Z}_p)$ different from $(0\ 0\ 0\ 0)$ can be represented by Lagrangian 3-tori and by Lagrangian 3-spheres.

The proof is based on the observation that

$$\text{mod}_{\,p}(M_0^p) = Id \quad \text{and} \quad \text{mod}_{\,p}(M_1^p) = Id, \text{ for } p \neq 2, 3 \text{ prime number},$$

Hence, it is possible to set an algorithm to compute the orbit of the cycles δ_2 and δ_4 by the monodromy action with coefficients in $\mathbb{Z}_p$. The algorithm used to prove Theorem 3.1 is described below,

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Input: v, M_0, M_1, p
Output: Orb_p
$W_{old} := \emptyset$
$W_{new} := \{mod_p(v)\}$
while $W_{old} \neq W_{new}$ **do**
  $W_{old} := W_{new}$
  $W_{new} := W_{new} \cup \{M_1^j (M_0)^i u\} \quad u \in W_{old},\ i = 0, \ldots, p^2 - 1,\ j = 0, \ldots, p-1$
end
$Orb_p := W_{new}$

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where v is the δ_2 or δ_4. [2]

In general for a manifold M, a class $\delta \in H_k(M, \mathbb{Z})$ is called *primitive* if there is no $m \in \mathbb{Z}$ and $\delta' \in H_k(M, \mathbb{Z})$ such that $\delta = m\delta'$. I believe that for any prime different to 2 and 5, all classes in $H_3(\tilde{X}, \mathbb{Z}_p)$ different to $(0\ 0\ 0\ 0)$ can be represented by Lagrangian 3-tori and by a Lagrangian 3-spheres. This is a consequence of the following conjecture.

Conjecture 3.2. *Let δ be a primitive class in $H_3(\tilde{X}, \mathbb{Z})$. If $mod_2(\delta)$ is a homology class in the list (3.1) and $mod_5(\delta)$ is a homology class in the list (3.2), then δ is represented by a Lagrangian 3-torus. If $mod_2(\delta)$ is a homology class in the list (3.3) and $mod_5(\delta)$ is a homology class in the list (3.4), then δ is represented by a Lagrangian 3-sphere.*

[2] I have written a MATLAB code for the computation. It is available in https://github.com/danfelmath/mirrorquintic.git

4 Future Projects

I expect that the Theorems 1.1 and 1.2 in [16, *The monodromy problem for hyperelliptic curves. D. López*] holds without the assumption in the degree. That means, to find a closed formula for the eigenvectors of the intersection matrices in order to generalize the numerical approach. Moreover, I believe that it is possible to generalize these theorems for any direct sum of polynomials $h(y) + g(x)$.

As an application of these results, I pretend to study the non-generic cases in the logarithmic and pullback foliations described in [20] and [27], respectively. For example, in [20] it is considered dF, where F is the product of some lines in general position. Without the condition of general position, the monodromy action in the fibration given by F generates smaller subspaces in $H_1(F^{-1}(b), \mathbb{Q})$, therefore perturbation of dF are not necessarily relatively exacts. On the other hand, I am working in a generalization of the results in [20] to foliations with center in $\mathbb{P}^2$.

From Theorem 2.3, follows that a Poisson manifold with a Hausdorff integration admits a Hausdorff complete symplectic realization (Corollary 7.7 in [8, *On Hausdorff intefrations of Lie Algebroids, M. del Hoyo and D. Lopez*]). An interesting question which I would like to work on is the converse of this Corollary, more precisely: does a Hausdorff symplectic complete realization give rise to a Hausdorff symplectic groupoid? It would be a Hausdorff version of [7, Thm 8], namely, a Poisson manifold admits a complete symplectic realization if and only if it is integrable.

The strategy used in [15, *Homology supported in Lagrangian submanifolds in mirror quintic threefolds. D. López. Can. Math. Bull.*], could be interesting in order to obtain information about homology cycles supported in Lagrangian submanifolds in hypersurfaces of $\mathbb{P}^{n+1}$. For some hypersurfaces in $\mathbb{P}^{n+1}$, for example the Fermat variety, it is possible to give an explicit basis for the n-homology in terms o vanishing cycles (see [21]). These vanishing cycles are embedded Lagrangian spheres in the hypersurface. Moreover, the monodromy matrices are explicit. Therefore, it is possible to ask about the conditions of the cycles to decide whether they are in the orbit by monodromy action.

The interest of an explicit description of a basis for homology in [21] is due to calculating examples for which Hodge's conjecture is true. Hence, the question about the monodromy orbits of Lagrangian spheres could be related to some algebraic conditions in the homology cycles.

Recently, I am working on a project jointly with R. Villaflor, in applications of the results in [26]. The main goal is to do a library in SINGULAR to compute the periods and intersection matrices of a family of algebraic cycles in a variety. So far, we can determine these matrices for linear cycles in a Fermat variety (see [22]). Moreover, in the case of the Fermat quartic sixfold, we can prove the rational Hodge conjecture. The library in SINGULAR could be found at https://github.com/danfelmath/Hodge-Project.git

References

[1] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Physics B*, 359(1):21–74, 1991.

[2] D. Cerveau and A. L. Neto. Irreducible components of the space of holomorphic foliations of degree two in $\mathbb{CP}(n), n \geq 3$. *Annals of mathematics*, pages 577–612, 1996.

[3] Y.-H. Chen, Y. Yang, N. Yui, and C. Erdenberger. Monodromy of picard-fuchs differential equations for calabi-yau threefolds. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2008(616):167–203, 2008.

[4] C. Christopher and P. Mardešić. The monodromy problem and the tangential center problem. *Functional analysis and its applications*, 44(1):22–35, 2010.

[5] A. Coste, P. Dazord, and A. Weinstein. Groupoides symplectiques. *Publications du Departement de mathematiques (Lyon)*, pages 1–62, 1987.

[6] M. Crainic and R. Fernandes. Integrability of lie brackets. *Annals of Math*, 157:575–620, 2003.

[7] M. Crainic and R. Fernandes. Integrability of poisson brackets. *J. Differ. Geom.*, 66:71–137, 2004.

[8] M. del Hoyo and D. López G. On hausdorff integrations of lie algebroids. *Monatshefte für Mathematik*, (194):811–833, 2021.

[9] C. Doran and J. Morgan. Mirror symmetry and integral variations of Hodge structure underlying one parameter families of Calabi-Yau threefolds. *V, AMS/IP Studies in Advanced Mathematics*, 38, 2006.

[10] J. Dufour and N. Zung. Poisson structures and their normal forms. *Birkhuser, Verlag*, 2000.

[11] H. Dulac. *Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un centre*, volume 32. Gauthier-Villars, 1908.

[12] L. Gavrilov and H. Movasati. The infinitesimal 16th Hilbert problem in dimension zero. *Bulletin des sciences mathematiques*, 131, 2007.

[13] Y. Ilyashenko. The origin of limit cycles under perturbation of the equation $dw/dz = -r_z/r_w$, where $r(z,w)$ is a polynomial. *Matematicheskii Sbornik*, 120(3):360–373, 1969.

[14] T. Li and W. Wu. Lagrangian spheres, symplectic surfaces and the symplectic mapping class group. *Geometry & Topology*, 16(2):1121–1169, 2012.

[15] D. López G. Homology supported in Lagrangian submanifolds in mirror quintic threefolds. *Canadian Mathematical Bulletin*, pages 1–16, 2020.

[16] D. López G. The monodromy problem for hyperelliptic curves. *Bulletin des Sciences Mathématiques*, 170, 2021.

[17] K. Mackenzie and P. Xu. Integration of lie bialgebroids. *Topology*, 39:445–467, 2000.

[18] I. Moerdijk and J. Mrcun. *Introduction to foliations and Lie groupoids*, volume 91. Cambridge University Press, 2003.

[19] H. Movasati. Abelian integrals in holomorphic foliations. *Revista Matemática Iberoamericana*, 20(1):183–204, 2004.

[20] H. Movasati. Center conditions: rigidity of logarithmic differential equations. *Journal of Differential Equations*, 197(1):197–217, 2004.

[21] H. Movasati. *A Course in Hodge theory, with emphasis on multiple integrals.* http://w3.impa.br/~hossein/myarticles/hodgetheory.pdf. To be published by IP, Boston, 2017.

[22] H. Movasati and R. Villaflor. Periods of linear algebraic cycles. *Pure and Applied Mathematics Quarterly*, 2018.

[23] A. L. Neto. Foliations with a morse center. *J. Singul*, 9:82–100, 2014.

[24] R. Roussarie. *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, volume 164. Birkhauser, 1998.

[25] R. Schoen and J. Wolfson. Minimizing area among Lagrangian surfaces: the mapping problem. *Journal of Differential Geometry*, 58(1):1–86, 2001.

[26] R. Villaflor. Periods of complete intersection algebraic cycles. *Manuscripta mathematica*, 2019.

[27] Y. Zare. Center conditions: pull back of differential equations. *Transactions of the American Mathematical Society*, 2017.