Canad. Math. Bull. 2020, pp. 1–16
http://dx.doi.org/10.4153/S0008439520000776

Homology supported in Lagrangian submanifolds in mirror quintic threefolds

Daniel López Garcia

Abstract. In this note, we study homology classes in the mirror quintic Calabi–Yau threefold that can be realized by special Lagrangian submanifolds. We have used Picard–Lefschetz theory to establish the monodromy action and to study the orbit of Lagrangian vanishing cycles. For many prime numbers p, we can compute the orbit modulo p. We conjecture that the orbit in homology with coefficients in $\mathbb{Z}$ can be determined by these orbits with coefficients in $\mathbb{Z}_p$.

1 Introduction

Given a symplectic manifold (X,ω) of dimension $2n$, there are homology classes in $H_n(X,\mathbb{Q})$ that can be represented by Lagrangian cycles. In [17], the authors define Lagrangian cycles as cycles in a symplectic 4-manifold, whose two-simplices are given by C^1 Lagrangian maps, and a *Lagrangian homology class* is a homology class that can be represented by a Lagrangian cycle. In that article, they showed a characterization of the Lagrangian homology classes in terms of the minimizers of an area functional. Moreover, they showed for a compact Kälher 4-manifold (X,ω,J) and a homology class $\alpha \in H_2(X,\mathbb{Z})$, that α is a Lagrangian homology class if and only if $[\omega](\alpha) = 0$. If the Chern class $c_1(X)$ also annihilates α, then α can be represented by an immersed Lagrangian surface (not necessarily embedded).

The question about which part of the homology is supported in Lagrangian submanifolds can be refined a little more if we look for Lagrangian spheres. In [11], for a (X,ω,J) Kälher 4-manifold with Kodaira dimension $-\infty$, *i.e.,* for rational or ruled surfaces, it is shown that the class $\alpha \in H_2(X,\mathbb{Z})$ is represented by a Lagrangian sphere if and only if $[\omega](\alpha) = 0$, $c_1(X)(\alpha) = 0$, $\alpha^2 = -2$, and α is represented by a smooth sphere. For 4-manifolds, the dimension of the 2-cycles allows us to relate the property of being represented by Lagrangian cycles with the vanishing of the *periods* $\int_\alpha \omega$ and $\int_\alpha c_1(X)$. For higher dimension manifolds, this pairing is not well defined; hence, we do not have a natural generalization of the previous results. Despite this, it is possible to show that in any regular hypersurface of $\mathbb{P}^n$ with n even, all $(n-1)$-cycles can be written as a linear combination of cycles supported in Lagrangian spheres; see Proposition 2.4.

Received by the editors June 21, 2020; revised August 25, 2020.
Published online on Cambridge Core September 11, 2020.
AMS subject classification: **14D05, 14J30, 14J33, 53D05, 53D12.**
Keywords: Lagrangian spheres, mirror quintic threefold, monodromy action, Lefschetz fibration, vanishing cycles.

A more interesting question for $n = 4$ is to ask not only which homology classes are generated by Lagrangian spheres but which ones are supported in Lagrangian spheres. In this article, we consider a family $\widetilde{X}_\varphi$ of mirror quintic Calabi–Yau threefolds and study some classes in $H_3(\widetilde{X}_\varphi, \mathbb{Z})$ that are supported in Lagrangian 3-spheres and Lagrangian 3-tori. This family is constructed as follows. Consider the Dwork family X_φ in $\mathbb{P}^4$ given by the locus of the polynomial

$$p_\varphi := \varphi z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4 = 0,$$

with critical values in $\varphi = 0, 1, \infty$. For every $\varphi \neq 0, 1, \infty$, $\widetilde{X}_\varphi$ is obtained as a desingularization of the quotient of X_φ by the action of a finite group; see Section 3 and [5, 7, 8]. The rank of the free group $H_3(\widetilde{X}_\varphi, \mathbb{Z})$ is four, and hence it is isomorphic to $\mathbb{Z}^4$ after choosing a basis. In this basis, the homology class $\delta_2 = (0\,1\,0\,0)$ is represented by a torus associated with the singularity of X_φ when $\varphi \to 0$ and the class $\delta_4 = (0\,0\,0\,1)$ is represented by a sphere S^3 associated with the singularity of X_φ when $\varphi \to 1$. As in [5], we give an explicit description of these two cycles in Section 4, and furthermore, we show that these cycles are Lagrangian submanifolds of $\widetilde{X}_\varphi$.

The monodromy action of the family is given by symplectomorphisms at each regular fiber. It is possible to determine two matrices M_0 and M_1 such that the monodromy action over $H_3(\widetilde{X}_\varphi, \mathbb{Z})$ corresponds (with respect to the basis mentioned above) to the free subgroup of $Sp(4, \mathbb{Z})$ generated by M_0 and M_1; see Section 3. Therefore, the orbit of δ_2 and δ_4 by the action of $M_0 * M_1$ are homology classes that can be represented by Lagrangian submanifolds. Our main result is about $H_3(\widetilde{X}_\varphi, \mathbb{Z}_p)$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for some primes p, and it is summarized in the following theorem.

Theorem 1.1 *For the mirror quintic Calabi–Yau threefold $\widetilde{X} := \widetilde{X}_\varphi$ with $\varphi \neq 0, 1, \infty$, the homology classes*

$$(0\,0\,1\,1),\ (0\,1\,0\,0),\ (0\,1\,0\,1),\ (1\,0\,0\,1),\ (1\,0\,1\,1) \in H_3(\widetilde{X}, \mathbb{Z}_2) \tag{1.1}$$

$$\begin{gathered}(0\,1\,0\,0), (0\,1\,0\,1), (0\,1\,0\,2), (0\,1\,0\,3), (0\,1\,0\,4),\\ (0\,1\,1\,0), (0\,1\,1\,1), (0\,1\,1\,2), (0\,1\,1\,3), (0\,1\,1\,4),\\ (0\,1\,2\,0), (0\,1\,2\,1), (0\,1\,2\,2), (0\,1\,2\,3), (0\,1\,2\,4),\\ (0\,1\,3\,0), (0\,1\,3\,1), (0\,1\,3\,2), (0\,1\,3\,3), (0\,1\,3\,4),\\ (0\,1\,4\,0), (0\,1\,4\,1), (0\,1\,4\,2), (0\,1\,4\,3), (0\,1\,4\,4) \in H_3(\widetilde{X}, \mathbb{Z}_5)\end{gathered} \tag{1.2}$$

are represented by Lagrangian 3-tori. The homology classes

$$\begin{gathered}(0\,0\,0\,1),\ (0\,0\,1\,0), (0\,1\,1\,0), (0\,1\,1\,1), (1\,0\,0\,0),\\ (1\,0\,1\,0), (1\,1\,0\,0), (1\,1\,0\,1), (1\,1\,1\,0), (1\,1\,1\,1) \in H_3(\widetilde{X}, \mathbb{Z}_2),\end{gathered} \tag{1.3}$$

$$(0\,0\,0\,1),\ (0\,0\,1\,1),\ (0\,0\,2\,1),\ (0\,0\,3\,1),\ (0\,0\,4\,1) \in H_3(\widetilde{X}, \mathbb{Z}_5) \tag{1.4}$$

are represented by Lagrangian 3-spheres. For $p = 3, 7, 11, 13, 17, 19, 23$, any homology class in $H_3(\widetilde{X}, \mathbb{Z}_p)$ different from $(0\,0\,0\,0)$ can be represented by Lagrangian 3-tori and by Lagrangian 3-spheres.

Table 1: Fourteen values for equation (3.3) with the corresponding Calabi–Yau threefold.

(d,k)	A	B	A-model of equation (3.3)
$(5,5)$	$1/5$	$2/5$	$X(5) \subset \mathbb{P}^4$
$(2,4)$	$1/8$	$3/8$	$X(8) \subset \mathbb{P}^4(1,1,1,1,4)$
$(1,4)$	$1/12$	$5/12$	$X(2,12) \subset \mathbb{P}^5(1,1,1,1,4,6)$
$(16,8)$	$1/2$	$1/2$	$X(2,2,2,2) \subset \mathbb{P}^7$
$(12,7)$	$1/3$	$1/2$	$X(2,2,3) \subset \mathbb{P}^6$
$(8,6)$	$1/4$	$1/2$	$X(2,4) \subset \mathbb{P}^5$
$(4,5)$	$1/6$	$1/2$	$X(2,6) \subset \mathbb{P}^5(1,1,1,1,1,3)$
$(2,3)$	$1/4$	$1/3$	$X(4,6) \subset \mathbb{P}^5(1,1,1,2,2,3)$
$(1,2)$	$1/6$	$1/6$	$X(6,6) \subset \mathbb{P}^5(1,1,2,2,3,3)$
$(6,5)$	$1/6$	$1/4$	$X(3,4) \subset \mathbb{P}^5(1,1,1,1,1,2)$
$(3,4)$	$1/6$	$1/3$	$X(6) \subset \mathbb{P}^4(1,1,1,1,2)$
$(1,3)$	$1/10$	$3/10$	$X(5) \subset \mathbb{P}^4(1,1,1,2,5)$
$(4,4)$	$1/4$	$1/4$	$X(4,4) \subset \mathbb{P}^5(1,1,1,1,2,2)$
$(9,6)$	$1/3$	$1/3$	$X(3,3) \subset \mathbb{P}^5$

In general, for a manifold M, a class $\delta \in H_k(M,\mathbb{Z})$ is called *primitive* if there is no $m \in \mathbb{Z}$ and $\delta' \in H_k(M,\mathbb{Z})$ such that $\delta = m\delta'$. We believe that for any prime different to 2 and 5, all classes in $H_3(\widetilde{X},\mathbb{Z}_p)$ different to $(0\ 0\ 0\ 0)$ can be represented by Lagrangian 3-tori and by a Lagrangian 3-spheres. This is a consequence of the following conjecture.

Conjecture 1.2 *Let δ be a primitive class in $H_3(\widetilde{X},\mathbb{Z})$. If $\mathrm{mod}_2(\delta)$ is a homology class in the list (1.1) and $\mathrm{mod}_5(\delta)$ is a homology class in the list (1.2), then δ is represented by a Lagrangian 3-torus. If $\mathrm{mod}_2(\delta)$ is a homology class in the list (1.3) and $\mathrm{mod}_5(\delta)$ is a homology class in the list (1.4), then δ is represented by a Lagrangian 3-sphere.*

We have analogous results for 14 other examples of Calabi–Yau threefolds that appear in Table 1. However, in these cases, we do not know if the vectors δ_2 and δ_4 are associated with Lagrangian submanifolds as in the Dwork family case.

2 Basics on Picard–Lefschetz Theory

We recall some facts about Lefschetz fibration in symplectic geometry. These results are in the literature; see, for example, [1, 2, 18, 19]. We collect them here to set notation and for quick reference throughout the article.

Let Y be a complex manifold. A *Lefschetz fibration* is a surjective analytic map $f : Y \to \mathbb{P}^1$ with a finite number of critical points, such that for any critical point p, there is a chart with Morse coordinates. This means that there is a coordinate system around p such that $f(z) = f(p) + z_1^2 + \cdots + z_n^2$ for z in a neighborhood of p.

Every projective manifold $Y \hookrightarrow \mathbb{P}^N$ has a natural symplectic form ω given by the pullback of the Fubini study form in $\mathbb{P}^N$. Since the fibers of f over regular values are complex submanifolds of Y, the restriction of ω to each regular fiber remains symplectic. Furthermore, the regular fibers of the Lefschetz fibration $f : Y \to \mathbb{P}^1$ are symplectomorphic. This follows from the following symplectic version of the Ehresmann lemma.

Proposition 2.1 *Let (E, ω) be a symplectic manifold and let B be a connected manifold. Consider $f : E \to B$ a proper surjective map with a finite set of critical values C, such that ω is symplectic at every regular fiber of f. Then the regular fibers are symplectomorphic.*

Proof Using ω, we can decompose the tangent bundle TE, over the set of regular values, as a direct sum of a vertical bundle VE and a horizontal bundle $HE := (VE)^{\omega}$. Here, the vertical space V_eE is the space of vectors tangent to the fibers of f, and the horizontal space H_eE is its symplectic complement. This is well defined, since the restriction of ω to the fibers is symplectic.

Let $b \in B$ be a regular value and let $U \subset B \backslash C$ be a neighborhood of b. We take a vector field W defined on U, without singularities. Since f is a submersion on U, the map f_* is an isomorphism between H_eE and $T_{f(e)}B$ for all $f(e) \in U$; thus, we can take the vector field $V := f^* W$ on E_U. Because the fibers are compact, the flow θ of V is defined in a neighborhood of E_b for all t in some interval I. Therefore $\varphi_t := \theta(-, t)$ is a diffeomorphism between E_b and some other fiber in a neighborhood.

In order to show that φ_t preserves the symplectic form at the fibers, it is enough to show that $\frac{d}{d\tau}\big|_{\tau=t} \varphi_\tau^* \omega_b = 0$ for $t \in I$, where ω_b is the form ω restricted to the fiber E_b. This follows, noting that

$$\frac{d}{d\tau}\bigg|_{\tau=t} \varphi_\tau^* \omega_b = \varphi_t^*(\mathcal{L}_V \omega_b) = \varphi_t^*(d\imath_V \omega_b + \imath_V d\omega_b) = \varphi_t^*(d\imath_V \omega_b),$$

and that $\imath_V \omega_b = 0$ since V is in HE. ∎

Let $\gamma : [0,1] \to B \backslash C$ be a simple path. We denote by $P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ the symplectomorphism given by the lifting of γ as in the previous proposition.

Corollary 2.2 *Let Y be a projective manifold and let $f : Y \to \mathbb{P}^1$ be a Lefschetz fibration with critical values C. For a simple path $\gamma : [0,1] \to \mathbb{P}^1 \backslash C$, the map P_γ is a symplectomorphism.*

From now on, we will denote by $X := Y_b \hookrightarrow Y$ a regular fiber of f. Since these fibers are symplectomorphic, we simply denote any symplectic fiber by (X, ω_X). Thus, we have a map $\pi_1(\mathbb{P}^1 \backslash C) \to \mathrm{Symp}(X, \omega_X)$ that descends to homology, inducing the so-called *monodromy action* $\pi_1(\mathbb{P}^1 \backslash C) \curvearrowright H_*(X, \mathbb{Z})$ given by $(\gamma, \delta) \to (P_\gamma)_* \delta$.

Let $\gamma : [0,1] \to \mathbb{P}^1$ be a simple path such that $\gamma(1) \in C$ and $\gamma(t) \in \mathbb{P}^1 \backslash C$ for $t \in [0,1)$. Let p be a critical point in $f^{-1}(\gamma(1))$. The set of points

$$V_\gamma = \left\{ z \in f^{-1}(Im(\gamma)) \mid \lim_{t \to 1} P_\gamma(t)(z) = p \right\}$$

is called a *Lefschetz thimble,* and the intersection of V_γ with the fiber $f^{-1}(\gamma(0))$ is called a *vanishing cycle* δ_γ.

Proposition 2.3 *The Lefschetz thimble V_γ is a Lagrangian submanifold of (Y,ω), and the vanishing cycle δ_γ is a Lagrangian sphere of (X,ω_X).*

Proof In a compact neighborhood U of p, we can suppose that $f(z) = f(p) + z_1^2 + \cdots + z_n^2$ and that γ is a real curve in $\mathbb{C}$ with $\gamma(1) = 0$, and $\gamma(0) > 0$. Let $H : U \to \mathbb{R}$ be the map given by $H(z) = \mathrm{Re}(f(z))$. The Hamiltonian vector field X_H is horizontal, because H is constant in the fibers of f and $\omega(X_H, V) = dH(V) = 0$ for any vertical vector field V. Since JV is also vertical, ∇H is horizontal. On the other hand, $-\nabla H$ projects to $\frac{\partial}{\partial x}$ and so V_γ is the unstable set of p .

By a direct computation, H is a Morse function with index n. Using the unstable manifold theorem [3, Thm. 4.2], we conclude that V_γ is an n-ball inside Y. To see that V_γ is isotropic, consider $u, v \in T_z V_\gamma$ for any $z \in V_\gamma$. Since the horizontal component of V_γ is one-dimensional, we have $\omega_z(u,v) = \omega_X(z)(u_v, v_v)$, where u_v and v_v are the vertical components of u and v. As the fibers over $\gamma(t)$ with $t \in [0,1)$ are symplectomorphic via φ_t, we have that

$$\omega_X(z)(u_v, v_v) = \omega_X\big(z(t)\big)\big(u_v(t), v_v(t)\big),$$

where $z(t) = \varphi_t(z)$, $u_v(t) = (\varphi_t(z))_* u_v$, and $v_v(t) = (\varphi_t(z))_* v_v$. In the limit, the tangent space is a point; then by continuity, we can conclude the result. ∎

Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ and let $b \in \mathbb{C}$ be some regular value of f. Suppose that the origin is an isolated critical point of the highest-grade homogeneous piece of f. The $(n-1)$-homology group of the fiber over b is generated by the vanishing cycles (see [10], [14, §7.4]). As a consequence, we can prove the next proposition.

Proposition 2.4 *Let $F \in \mathbb{C}[z_0, \ldots, z_n]$ be a homogeneous polynomial with n even. Suppose that F defines a smooth variety X in $\mathbb{P}^n$. Then any homology class $\delta \in H_{n-1}(X,\mathbb{Z})$ can be written as a finite sum $\delta = \sum_j a_j \delta_j$, where $a_j \in \mathbb{Z}$ and δ_j is supported in a Lagrangian $(n-1)$-sphere.*

Proof Consider a hyperplane that intersects transversally X, and let Z be its intersection. We can suppose that the hyperplane section is $Z = X \cap \{z_0 = 0\}$. Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be the polynomial $F(1, z_1, \ldots, z_n)$, and we define the affine variety $U := X \backslash Z = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid f(z_1, \ldots, z_n) = 0\}$. The pair (X, U) induces the exact sequence in homology

$$\cdots \longrightarrow H_n(X,U) \longrightarrow H_{n-1}(U) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(X,U) \longrightarrow \cdots,$$

where the map $H_k(U) \to H_k(X)$ comes from the inclusion $U \subset X$. By the Leray–Thom–Gysin isomorphism, we have $H_k(X,U) \simeq H_{k-2}(Z)$. By the Lefschetz hyperplane section theorem, we know that $H_k(Z) \simeq H_k(\mathbb{P}^{n-2})$ if $k \neq n-2$ (see [14, §5.4]). Since n is even, we have that $H_{n-3}(Z) = 0$; then the map

$$H_{n-1}(U) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-3}(Z) = 0$$

is surjective. The vanishing cycles associated with the fibration $f : \mathbb{C}^n \to \mathbb{C}$ generate the homology group $H_{n-1}(U)$, and they are supported in Lagrangian spheres of U. ■

3 Monodromy Action on Mirror Quintic Threefolds

In this section, we recall the definition of a mirror quintic Calabi–Yau threefold and its monodromy action coming from the Picard–Fuchs equations. We also list the monodromy action of 14 other examples of Calabi–Yau threefolds. For a more detailed description, the reader is referred to [5, 8, 15, 16].

The family of hypersurfaces in $\mathbb{P}^4$ given by a generic polynomial of degree 5 is denoted $\mathbb{P}^4[5]$. The elements of $\mathbb{P}^4[5]$ are quintic Calabi–Yau threefolds, with Hodge numbers $h^{1,1} = 1$ and $h^{2,1} = 101$. Let $\{X_\varphi\}_\varphi$ be the one-parameter family of hypersurfaces in $\mathbb{P}^4$ given by

$$p_\varphi = \varphi z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4, \quad \varphi \neq 0, 1. \tag{3.1}$$

Consider the finite group

$$G = \left\{(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{C}^5 \mid \xi_i^5 = 1, \xi_0\xi_1\xi_2\xi_3\xi_4 = 1\right\}$$

acting on $\mathbb{P}^4$, as $(\xi_0, \xi_1, \dots, \xi_4) \cdot [z_0 : z_1 : \cdots : z_4] = [\xi_0 z_0 : \cdots : \xi_4 z_4]$. It is known that the action of G is free away from the curves $C_{ijk} := \{z_i^5 + z_j^5 + z_k^5 = 0,\ z_l = 0 \text{ for all } l \neq i, j, k\}$ for $0 \le i < j < k \le 4$ (see [13]). The *mirror quintic Calabi-Yau threefold*, mirror quintic for short, is the variety $\widetilde{X}_\varphi$ obtained after resolving the orbifolds singularities of the quotient X_φ/G. The manifold $\widetilde{X}_\varphi$, has Hodge numbers $h^{1,1} = 101$ and $h^{2,1} = 1$ and Betti number $b_3 = 4$. In terms of the mirror symmetry, $\mathbb{P}^4[5]$ is called the *A-model* and $\{\widetilde{X}_\varphi\}_\varphi$ the *B-model* (see, for example, [9]).

The variety $\widetilde{X}_\varphi$ has a holomorphic 3 form η that vanishes nowhere. Moreover, $H^{3,0}$ is spanned by η. The *periods* of η are functions $\int_\Delta \eta$, where the homology class $\delta = [\Delta] \in H_3(\widetilde{X}_\varphi, \mathbb{Z})$ is supported in the submanifold Δ. The fourth-order linear differential equation

$$\left(\theta^4 - \varphi\left(\theta + \frac{1}{5}\right)\left(\theta + \frac{2}{5}\right)\left(\theta + \frac{3}{5}\right)\left(\theta + \frac{4}{5}\right)\right) y = 0, \quad \theta = \varphi\frac{\partial}{\partial\varphi}$$

is called a Picard–Fuchs equation, and its solutions are the periods of η.

The Picard–Fuchs ODE has 3 regular singular points $\varphi = 0, 1, \infty$. The analytic continuation of this ODE gives us the monodromy operators M_0, M_1, M_∞. Since the monodromy is a representation $\rho : \pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty\}) \to Sp(4)$, we have the relation $M_0 M_1 M_\infty = \mathrm{Id}$. There exists a basis such that the monodromy operators in this basis are written as

$$M_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

see, for example, [4, 7, 8].

The matrices M_0 and M_1 are conjugated to the matrices

$$T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

appearing in [7, 8], via the matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 5 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus, $P^{-1}T_iP = M_i$, $i = 0, 1$. In [5], the matrices for the monodromy are

$$S_\infty = \begin{pmatrix} 51 & 90 & -25 & 0 \\ 0 & 1 & 0 & 0 \\ 100 & 175 & -49 & 0 \\ -75 & -125 & 35 & 1 \end{pmatrix} \quad \text{and} \quad S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

these matrices are associated with the equation

$$p_\psi = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4,$$

with singularities at $\psi^5 = 1, \infty$. The change of variable $\psi = \varphi^{\frac{-1}{5}}$ gives us the family defined by equation (3.1). Moreover, the matrix M_0^5 is conjugated to S_∞. In fact, with the matrix

$$M = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.2}$$

we obtain the equations $M^{-1}S_1M = M_1$ and $M^{-1}S_\infty M = M_0^5$.

More generally, it is known that the differential equation

$$\big(\theta^4 - \varphi(\theta + A)(\theta + 1 - A)(\theta + B)(\theta + 1 - B)\big)y = 0, \quad \theta = \varphi\frac{\partial}{\partial\varphi} \tag{3.3}$$

corresponds to the Picard–Fuchs equation of a mirror Calabi–Yau threefold for 14 values of (A, B), and the singularities are in $\varphi = 0, 1, \infty$. We have listed the *A-model* of these 14 examples in Table 1.

The notation $X(d_1, d_2, \ldots, d_l) \subset \mathbb{P}^n(w_1, w_2, \ldots, w_n)$ denotes a complete intersection of l hypersurfaces of degrees $d_1, d_2, \ldots, d_l$ in the weighted projective space with weight $(w_1, w_2, \ldots, w_n)$; see, for example, [7]. For these cases, the monodromy matrices correspond to the same M_1 as before and

$$M_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix}.$$

4 Lagrangian Sphere and Lagrangian Torus in Mirror Quintic Threefold

In the basis of homology used in [5], there are two homology classes that are supported on Lagrangian submanifolds. We observe that one class is realized by a Lagrangian 3-sphere and the other by a Lagrangian 3-torus.

Consider the mirror quintic Calabi–Yau threefold $\widetilde{X}_\psi$ associated with the equation

$$p_\psi = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4,$$

with singularities in $\psi = 1, \infty$. Let η be the holomorphic form on $\widetilde{X}_\psi$. The basis of the matrices S_∞ and S_0 are the periods $\int_{\Delta_k} \eta$ with $k = 1, 2, 3, 4$. The cycle Δ_2 is a torus associated with the degeneration of the manifold as ψ goes to ∞; see [5, §3]. In coordinates it can be described as

$$\begin{aligned} \Delta_2 = \Big\{ [1 : z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^4 \mid |z_1| = |z_2| = |z_3| = r \\ \text{and } z_4 \text{ given by } p_\psi = 0 \text{ when } \psi \to \infty \Big\} \end{aligned} \tag{4.1}$$

for $r > 0$ small enough, and z_4 is defined as the branch of the solution $p_\psi(z) = 0$ that tends to zero as $\psi \to \infty$. The cycle Δ_2 does not intersect the curves C_{ijk}, and so its quotient by the group G is again a torus.

Proposition 4.1 *The cycle Δ_2 is a Lagrangian submanifold of (X_ψ, ω), where ω is the symplectic form given by the pullback of the Fubini–Study form.*

Proof Consider the Hamiltonian S^1-space $(\mathbb{C}^5 \backslash \{0\}, \omega_{can}, S^1, \mu)$, where $\mu(z) = \frac{-\|z\|^2+1}{2}$. By the Marsden–Weinstein–Meyer theorem, there exists a symplectic form in the reduction $\mu^{-1}(0)/S^1 = \mathbb{P}^4$, and in this case, it corresponds to the Fubini–Study form ω_{FS}; see, for example, [12, §5] or [6, §23]. Furthermore, if we denote the reduction by

$$\begin{tikzcd} \mu^{-1}(0) = S^9 \arrow[r, "\imath"] \arrow[d, "\pi"'] & \mathbb{C}^5 \backslash \{0\} \arrow[dl, "pr"] \\ \mathbb{P}^4 & \end{tikzcd}$$

the reduced form satisfies $\pi^* \omega_{FS} = \imath^* \omega_{can}$. The canonical form can be written as $\omega_{can} = \frac{1}{2} \sum_j d|z_j|^2 \wedge d\theta_j$. Therefore, for $\varepsilon > 0$ small enough, the set

$$T := \left\{ (z_0, z_1, z_2, z_3, z_4) \in \mathbb{C}^5 \mid |z_0| = \varepsilon, |z_1| = |z_2| = |z_3| = r, |z_4|^2 = 1 - \varepsilon^2 - 3r^2 \right\} \subset S^9,$$

is a Lagrangian submanifold of $(\mathbb{C}^5, \omega_{can})$. Besides, Δ_2 is the intersection of X_ψ with the projection of T to $\mathbb{P}^4$. Consequently, the tangent space of Δ_2 is contained in the tangent space of $\pi(T)$. Since $0 = (\pi^* \omega_{FS})|_T = (\omega_{FS})|_{\pi(T)}$, we conclude that $(\omega_{FS})|_{\Delta_2} = 0$. ∎

The cycle Δ_4 is associated with the degeneration of the manifold when ψ goes to 1 [5, §3]. In coordinates can be described as

$$\begin{aligned} \Delta_4 = \Big\{[1 : z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^4 \mid z_1, z_2, z_3 \text{ reals and} \\ z_4 \text{ given by } p_\psi = 0 \text{ when } \psi \to 1\Big\}, \end{aligned} \tag{4.2}$$

where z_4 is defined as the branch of of the solution of $p_\psi(z) = 0$ which is an S^3 when $\psi \to 1$. It follows from the next proposition that Δ_4 is an Lagrangian sphere S^3.

Proposition 4.2 *The cycle Δ_4 is a vanishing cycle.*

Proof In the chart $z_0 = 1$, consider the function $f : \mathbb{C}^4 \to \mathbb{C}$ given by $f(z_1, \ldots, z_4) = p_\psi(1, z_1, \ldots, z_4)$. The critical points of f, $(\xi^{k_1}\psi, \xi^{k_2}\psi, \xi^{k_3}\psi, \xi^{k_4}\psi)$ where $\xi = e^{\frac{2\pi i}{5}}$, $k_i = 1, \ldots, 5$ and $5| \sum_{j=1}^4 k_j$ are non degenerated. After calculating the quotient by the finite group G, these critical points are identified with $(\psi, \ldots, \psi)$.

For real $\psi > 1$ close enough to 1 and by taking $z_j = x_j + i y_j$, we have that the map can be locally defined as

$$f(z_1, \ldots, z_4) = (1 - \psi^5) + \sum x_j^2 - \sum y_j^2 + 2i \prod x_j y_j,$$

and so the vanishing cycle δ_γ in Proposition 2.3 is the sphere $(\psi^5 - 1) = \sum x_j^2$. ∎

Let $\delta_1, \delta_2, \delta_3, \delta_4$ be the basis on which the matrices S_1 and S_∞ are written. Consider the isomorphism between $span\{\delta_i\}_{i=1}^4$ and $\mathbb{R}^4$ with the canonical basis, given by $\sum_{i=1}^4 n_i \delta_i \to (n_1, n_2, n_3, n_4)$. Thus, the monodromy acting on a vector $\delta = \sum_{i=1}^4 n_i \delta_i$ corresponds to

$$S_j(\delta) = (n_1 \ n_2 \ n_3 \ n_4) S_j \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} \quad \text{with } j = 1, \infty.$$

From [5] and the Picard–Lefschetz formula, we know that the monodromy matrices satisfy $S_\infty \Delta_2 = \Delta_2$ and $S_1 \Delta_2 = \Delta_2 + \Delta_4$. Therefore, $\Delta_2 = \delta_2 \equiv [0\,1\,0\,0]$ and $\Delta_4 = \delta_4 \equiv [0\,0\,0\,1]$.

5 Orbits for δ_2 and δ_4

Let H be the subgroup of $Sp(4)$, generated by M_0 and M_1. Moreover, the vectors $\delta_2 = (0\,1\,0\,0)$ and $\delta_4 = (0\,0\,0\,1)$ are invariants by the change of basis M defined in (3.2). In this section, we compute the orbit of δ_2 and δ_4 by the action of H in $\mathbb{Z}_p$, for some prime numbers p.

For the mirror quintic $\widetilde{X}$, any element in $H_3(\widetilde{X}, \mathbb{Z})$ that is in the orbit $H \cdot \delta_4$ is a homology class supported in a Lagrangian 3-sphere, and any element in the orbit $H \cdot \delta_2$ is a homology class supported in a Lagrangian 3-torus. So far, we have not computed the orbits in $\mathbb{Z}$. However, considering $H_3(\widetilde{X}, \mathbb{Z}_p)$ for some primes p, it is possible to compute the orbits. The next lemma helps us to reduce the possible words appearing in $H \bmod p\mathbb{Z}$.

Lemma 5.1 *We have that*

$$\operatorname{mod}_p(M_0^p) = \operatorname{Id}_4, \quad p \neq 2,3, \quad \operatorname{mod}_2(M_0^4) = \operatorname{Id}_4, \quad \operatorname{mod}_3(M_0^9) = \operatorname{Id}_4,$$
$$\operatorname{mod}_p(M_1^p) = \operatorname{Id}_4 \quad \forall \textit{ prime } p.$$

Proof Computing the power of theses matrix, we have

$$M_0^m = \begin{pmatrix} 1 & m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ dm & a_m & 1 & 0 \\ b_m & c_m & -m & 1 \end{pmatrix} \quad \text{and} \quad M_1^m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $a_m = \frac{d}{2}m(m+1)$, $b_m = \frac{d}{2}m(1-m)$, and $c_m = \frac{d}{6}m(1-m^2) - km$. Thus, it is enough to show that $p|a_p$, $p|b_p$, and $p|c_p$. However, this is immediate, because $2|(p+1)$, $2|(1-p)$, and $6|(1-p^2)$, for $p \neq 2,3$ prime. ∎

Let v be the vector δ_2 (or δ_4), and we denote by orb_p the list of the vector in the orbit of v modulo p. First, we start with $orb_p = \{\operatorname{mod}_p(v)\}$, and we compute the vectors $\operatorname{mod}_p(wM_1^jM_0^i)$ for $j = 0, \dots, p$, $i = 0, \dots p$, and $w \in orb_p$. If these vectors are not in orb_p, we add them to orb_p. This step is repeated with the new orb_p; if there are no new vectors, then the orbit is complete. This process is finite, because we have at most p^4 different vectors in $(\mathbb{Z}/p\mathbb{Z})^4$. We summarize the algorithm[1] to compute the orbit of v modulo p as follows:

Proof of Theorem 1.1 Consider the free group H when $d = k = 5$. Given p, we denote the orbit of δ_2 and δ_4 modulo p as $orb_p(\delta_2)$ and $orb_p(\delta_4)$, respectively. By using the previous algorithm, we have

$$\begin{aligned} orb_2(\delta_2) &= \{(0\,0\,1\,1),\ (0\,1\,0\,0),\ (0\,1\,0\,1),\ (1\,0\,0\,1),\ (1\,0\,1\,1)\}, \\ orb_2(\delta_4) &= \{(0\,0\,0\,1),\ (0\,0\,1\,0), (0\,1\,1\,0), (0\,1\,1\,1), (1\,0\,0\,0), \\ &\qquad (1\,0\,1\,0), (1\,1\,0\,0), (1\,1\,0\,1), (1\,1\,1\,0), (1\,1\,1\,1)\}, \\ orb_5(\delta_2) &= \{(0\,1\,0\,0), (0\,1\,0\,1), (0\,1\,0\,2), (0\,1\,0\,3), (0\,1\,0\,4), \\ &\qquad (0\,1\,1\,0), (0\,1\,1\,1), (0\,1\,1\,2), (0\,1\,1\,3), (0\,1\,1\,4), \\ &\qquad (0\,1\,2\,0), (0\,1\,2\,1), (0\,1\,2\,2), (0\,1\,2\,3), (0\,1\,2\,4), \\ &\qquad (0\,1\,3\,0), (0\,1\,3\,1), (0\,1\,3\,2), (0\,1\,3\,3), (0\,1\,3\,4), \end{aligned}$$

[1]We have written a MATLAB code for the computation. It is available in www.github.com/danfelmath/mirrorquintic.git .

```
Input: v, M_0, M_1, p
Output: Orb_p
Orb_p := mod_p(v)
norm := 1
while norm > 0 do
    W := Orb_p; L := length(Orb_p)
    l := 1; c := 1
    while l ≤ L do
        j:=0
        while j ≤ p do
            i := 0
            while i ≤ p do
                v_aux := mod_p(W(l) * (M_1^j M_0^i))
                if v_aux ∉ Orb_p then
                    Orb_p(L + c) := v_aux
                    c := c + 1; i := i + 1
                else
                    i := i + 1
                end
            end
            j : j + 1
        end
        l := l + 1
    end
    norm := length(Orb_p) − length(W)
end
```

$$(0\,1\,4\,0), (0\,1\,4\,1), (0\,1\,4\,2), (0\,1\,4\,3), (0\,1\,4\,4)\}.$$

$$orb_5(\delta_4) = \{(0\,0\,0\,1),\ (0\,0\,1\,1),\ (0\,0\,2\,1),\ (0\,0\,3\,1),\ (0\,0\,4\,1)\},$$

$$orb_p(\delta_2) = orb_p(\delta_4) = (\mathbb{Z}/p\mathbb{Z})^4 \backslash (0\,0\,0\,0),\ \text{for } p = 3, 7, 11, 13, 17, 19, 23.$$

■

From the map $H_3(\widetilde{X}, \mathbb{Z}) \xrightarrow{\text{mod}_p} H_3(\widetilde{X}, \mathbb{Z}_p)$, we have that if $\delta \in H_3(\widetilde{X}, \mathbb{Z})$ is a primitive class and it is in the orbit of δ_2 (or δ_4), then $\text{mod}_p(\delta) \in orb_p(\delta_2)$ (or $\text{mod}_p(\delta_4) \in orb_p(\delta_4)$) for all p. We think that the converse should be true, that is, Conjecture 1.2.

For the other examples of quintic threefolds appearing in Table 1, we have analogous results. However, in this case, we do not know if the vectors $\delta_2 = (0\,1\,0\,0)$ and $\delta_4 = (0\,0\,0\,1)$ are really supported in a Lagrangian submanifold. In Table 2, we present the orbits of the vectors δ_2 and δ_4 modulo p for the fourteen cases of (d, k). If the orbit is $(\mathbb{Z}/p\mathbb{Z})^4 \backslash (0\,0\,0\,0)$, we call it *complete*. The orbits for the vector δ_2 are presented in Table 3, and the orbits for the vector δ_4 are presented in Table 4.

Table 2: Orbit of vectors δ_2 and δ_4 by the monodromy action for the fourteen mirror Calabi-Yau threefolds.

(d,k)	Prime	Orbit
$(5,5)$	$p=5$	(0001), (0011), (0021), (0031), (0041), (0100), (0101), (0102), (0103), (0104), (0110), (0111), (0112), (0113), (0114), (0120), (0121), (0122), (0123), (0124), (0130), (0131), (0132), (0133), (0134), (0140), (0141), (0142), (0143), (0144)
	$p=2,3,7,11,13,17,19,23$	Complete
$(2,4)$	$p=2$	(0001), (0011), (0100), (0101), (0110), (0111)
	$p=3,5,7,11,13,17,19,23$	Complete
$(1,4)$	$p=2,3,5,7,11,13,17,19,23$	Complete
$(16,8)$	$p=2$	(0001), (0011), (0100), (0101), (0110), (0111)
	$p=3,5,7,11,13,17,19,23$	Complete
$(12,7)$	$p=2$	(0001), (0011), (0100), (0101), (0110), (0111)
	$p=3$	(0001), (0002), (0021), (0022), (0100), (0101), (0102), (0110), (0111), (0112), (0210), (0211), (0212), (0220), (0221), (0222)
	$p=5,7,11,13,17,19,23$	Complete
$(8,6)$	$p=2$	(0001), (0011), (0100), (0101), (0110), (0111)
	$p=3,5,7,11,13,17,19,23$	Complete
$(4,5)$	$p=2$	(0001), (0011), (0100), (0101), (0110), (0111)
	$p=3,5,7,11,13,17,19,23$	Complete
$(2,3)$	$p=2$	(0001), (0011), (0100), (0101), (0110), (0111)
	$p=3,5,7,11,13,17,19,23$	Complete
$(1,2)$	$p=2,3,5,7,11,13,17,19,23$	Complete
$(6,5)$	$p=2$	(0001), (0011), (0100), (0101), (0110), (0111)
	$p=3$	(0001), (0002), (0011), (0012), (0100), (0101), (0102), (0120), (0121), (0122), (0210), (0211), (0212), (0220), (0221), (0222)
	$p=5,7,11,13,17,19,23$	Complete
$(3,4)$	$p=3$	(0001), (0002), (0021), (0022), (0100), (0101), (0102), (0110), (0111), (0112), (0210), (0211), (0212), (0220), (0221), (0222)
	$p=2,5,7,11,13,17,19,23$	Complete

(continued)

Table 2: Continued

$(1,3)$	$p = 2, 3, 5, 7, 11, 13, 17, 19, 23$	Complete
$(4,4)$	$p = 2$	$(0001), (0011), (0100), (0101), (0110), (0111)$
	$p = 3, 5, 7, 11, 13, 17, 19, 23$	Complete
$(9,6)$	$p = 3$	$(0001), (0011), (0021), (0100), (0101), (0102), (0110), (0111), (0112), (0120), (0121), (0122)$
	$p = 2, 5, 7, 11, 13, 17, 19, 23$	Complete

Table 3: Orbit of vector δ_2 by the monodromy action for the fourteen mirror Calabi–Yau threefolds.

(d,k)	Prime	Orbit
$(5,5)$	$p = 2$	$(0011), (0100), (0101), (1001), (1011)$
	$p = 5$	$(0100), (0101), (0102), (0103), (0104), (0110), (0111), (0112), (0113), (0114), (0120), (0121), (0122), (0123), (0124), (0130), (0131), (0132), (0133), (0134), (0140), (0141), (0142), (0143), (0144)$
	$p = 3, 7, 11, 13, 17, 19, 23$	Complete
$(2,4)$	$p = 2$	$(0100), (0101)(0110), (0111)$
	$p = 3, 5, 7, 11, 13, 17, 19, 23$	Complete
$(1,4)$	$p = 2$	$(0010), (0100), (0101), (0110), (0111), (1001), (1010), (1110), (1111)$
	$p = 3, 5, 7, 11, 13, 17, 19, 23$	Complete
$(16,8)$	$p = 2$	$(0100), (0101), (0110), (0111)$
	$p = 3, 5, 7, 11, 13, 17, 19, 23$	Complete
$(12,7)$	$p = 2$	$(0011), (0100), (0101)$
	$p = 3$	$(0021), (0022), (0100), (0101), (0102), (0210), (0211), (0212)$
	$p = 5, 7, 11, 13, 17, 19, 23$	Complete
$(8,6)$	$p = 2$	$(0100), (0101), (0110), (0111)$
	$p = 3, 5, 7, 11, 13, 17, 19, 23$	Complete
$(4,5)$	$p = 2$	$(0011), (0100), (0101)$
	$p = 3, 5, 7, 11, 13, 17, 19, 23$	Complete
$(2,3)$	$p = 2$	$(0011), (0100), (0101)$
	$p = 3, 5, 7, 11, 13, 17, 19, 23$	Complete
$(1,2)$	$p = 2$	$(0010), (0100), (0101), (0110), (0111), (1001), (1010), (1110), (1111)$
	$p = 3, 5, 7, 11, 13, 17, 19, 23$	Complete

(continued)

Table 3: Continued

$(6,5)$	$p=2$ $p=3$	$(0011),(0100),(0101)$ $(0011),(0012),(0100),(0101),(0102),(0220),(0221),(0222)$
	$p=5,7,11,13,17,19,23$	Complete
$(3,4)$	$p=2$	$(0010),(0100),(0101),(0110),(0111),(1001),(1010),(1110),(1111)$
	$p=3$	$(0021),(0022),(0100),(0101),(0102),(0210),(0211),(0212)$
	$p=5,7,11,13,17,19,23$	Complete
$(1,3)$	$p=2$	$(0011),(0100),(0101),(1001),(1011)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(4,4)$	$p=2$	$(0100),(0101),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(9,6)$	$p=2$	$(0010),(0100),(0101),(0110),(0111),(1001),(1010),(1110),(1111)$
	$p=3$	$(0100),(0101),(0102),(0110),(0111),(0112),(0120),(0121),(0122)$
	$p=5,7,11,13,17,19,23$	Complete

Table 4: Orbit of vector δ_4 by the monodromy action for the fourteen mirror Calabi–Yau threefolds.

(d,k)	Prime	Orbit
$(5,5)$	$p=2$	$(0001),(0010),(0110),(0111),(1000),(1010),(1100),(1101),(1110),(1111)$
	$p=5$	$(0001),(0011),(0021),(0031),(0041)$
	$p=3,7,11,13,17,19,23$	Complete
$(2,4)$	$p=2$	$(0001),(0011)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(1,4)$	$p=2$	$(0001),(0011),(1000),(1011),(1100),(1101)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(16,8)$	$p=2$	$(0001),(0011)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(12,7)$	$p=2$	$(0001),(0110),(0111)$
	$p=3$	$(0001),(0002),(0110),(0111),(0112),(0220),(0221),(0222)$
	$p=5,7,11,13,17,19,23$	Complete
$(8,6)$	$p=2$	$(0001),(0011)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(4,5)$	$p=2$	$(0001),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete

(continued)

Table 4: Continued

$(2,3)$	$p=2$	$(0001),(0110),(0111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(1,2)$	$p=2$	$(0001),(0011),(1000),(1011),(1100),(1101)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(6,5)$	$p=2$	$(0001),(0110),(0111)$
	$p=3$	$(0001),(0002),(0120),(0121),(0122),(0210),(0211),(0212)$
	$p=5,7,11,13,17,19,23$	Complete
$(3,4)$	$p=2$	$(0001),(0011),(1000),(1011),(1100),(1101)$
	$p=3$	$(0001),(0002),(0110),(0111),(0112),(0220),(0221),(0222)$
	$p=5,7,11,13,17,19,23$	Complete
$(1,3)$	$p=2$	$(0001),(0010),(0110),(0111),(1000),(1010),(1100),(1101),(1110),(1111)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(4,4)$	$p=2$	$(0001),(0011)$
	$p=3,5,7,11,13,17,19,23$	Complete
$(9,6)$	$p=2$	$(0001),(0011),(1000),(1011),(1100),(1101),$
	$p=3$	$(0001),(0011),(0021)$
	$p=5,7,11,13,17,19,23$	Complete

Acknowledgment I thank Hossein Movasati for suggesting the study of monodromy action of quintic mirror and for helpful discussions. A discussion with Emanuel Scheidegger about the Lagrangian torus was also very useful. Furthermore, I would like to thank Roberto Villaflor for his comments and suggestions on a first version of the paper.

References

[1] D. Auroux, V. Muñoz, and F. Presas, *Lagrangian submanifolds and Lefschetz pencils*. J. Symplectic Geom. 3(2005), 171–219.

[2] D. Auroux and I. Smith, *Lefschetz pencils, branched covers and symplectic invariants*. In: Symplectic 4-manifolds and algebraic surfaces, Lecture Notes in Math., 1938, Springer, Berlin, 2008, pp. 1–53. http://dx.doi.org/10.1007/978-3-540-78279-7_1

[3] A. Banyaga and D. Hurtubise, *Lectures on Morse homology*, Kluwer Texts in Mathematical Sciences, 29, Kluwer Academic Publishers, Dordrecht, 2004. http://dx.doi.org/10.1007/978-1-4020-2696-6

[4] C. Brav and H. Thomas, *Thin monodromy in Sp(4)*. Compositio Math. **150**(2014), 333–343. http://dx.doi.org/10.1112/S0001043X13007550

[5] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory*. Nucl. Phys. B **359**(1991), 21–74. http://dx.doi.org/10.1016/0550-3213(91)90292-6

[6] A. Cannas da Silva, *Symplectic geometry*. In: Handbook of differential geometry. Vol. II, Elsevier/North-Holland, Amsterdam, 2006. http://dx.doi.org/10.1016/S1874-5741(06)80006-3
[7] Y.-H. Chen and Y. Yang, *Monodromy of Picard-Fuchs differential equations for calabi-yau threefolds. With an appendix by Cord Erdenberger*. J. Reine Angew. Math. **616**(2008), 167–203. http://dx.doi.org/10.1515/CRELLE.2008.021
[8] C. Doran and J. Morgan, *Mirror symmetry and integral variations of Hodge structure underlying one parameter families of Calabi-Yau threefolds.* In: Mirror symmetry, V, AMS/IP Stud. Adv. Math., 38, American Mathematical Society, Providence, RI, 2006, pp. 517–537.
[9] M. Kontsevich, *Homological algebra of mirror symmetry.* In: Proceedings of the International Congress of Mathematicians, Vols. 1–2 (Zurich, 1994), Birkhäuser, Basel, 1995, pp. 120–139.
[10] K. Lamotke, *The topology of complex projective varieties after S. Lefschetz*. Topology **20**(1981), 15–51. http://dx.doi.org/10.1016/0040-9383(81)90013-6
[11] T.-J. Li and W. Wu, *Lagrangian spheres, symplectic surfaces and the symplectic mapping class group*. Geom. Topol. **16**(2012), 1121–1169. http://dx.doi.org/10.2140/gt.2012.16.1121
[12] D. McDuff and S. Dietmar, *Introduction to symplectic topology*. 3rd ed., Oxford Graduate Texts in Mathematics, Oxford University Press, Oxford, 2017. http://dx.doi.org/10.1093/oso/9780198794899.001.0001
[13] D. R. Morrison, *Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians.* J. Amer. Math. Soc. **6**(1993), 223–247. http://dx.doi.org/10.2307/21252798
[14] H Movasati. *A Course in Hodge theory, with emphasis on multiple integrals*. 2017. http://w3.impa.br/~hossein/myarticles/hodgetheory.pdf.
[15] H. Movasati, *Gauss-Manin connection in disguise: Calabi-Yau modular forms*. Surveys of Modern Mathematics, 13 (International Press, Somerville, MA), Higher Education Press, Beijing, 2017.
[16] Y. Nohara and K. Ueda, *Homological mirror symmetry for the quintic 3-fold.* Geom. Topol. **16**(2012), 1967–2001. http://dx.doi.org/10.2140/gt.2012.16.1967
[17] R. Schoen and J. Wolfson, *Minimizing area among Lagrangian surfaces: the mapping problem.* J. Differ. Geom. **58**(2001), 1–86.
[18] P. Seidel, *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics, European Mathematical Society, Zürich, 2008. http://dx.doi.org/10.4171/063
[19] R. P. Thomas, *An exercise in mirror symmetry*. In: Proceedings of the International Congress of Mathematicians, Vol. II, Hindustan Book Agency, New Delhi, 2010, pp. 624–651.

Instituto de Matematica Pura e Aplicada (IMPA), Estrada Dona Castorina 110, Rio de Janeiro 22460-320, Brazil
e-mail: daflopez@impa.br