ON THE MONODROMY ACTION FOR $f(x,y) = g(x) + h(y)$

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1. Example

In this text we compute an explicit example of the monodromy action in order to illustrate the codes [1] used in the proof of Lemma 2.5 in [1]. Let us consider the polynomial $f(x,y) = x^6 + y^8$. A Dynkin diagram associated to $f(x,y)$ is depicted right below, see Diagram (1).

$$\begin{tikzcd}
\delta_1 \arrow[dr] & \delta_8 \arrow[l] \arrow[r] & \delta_{15} \arrow[dl] \arrow[dr] & \delta_{22} \arrow[l] \arrow[r] & \delta_{29} \arrow[dl] \\
\delta_2 \arrow[u] \arrow[d] & \delta_9 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \delta_{16} \arrow[u] \arrow[d] & \delta_{23} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \delta_{30} \arrow[u] \arrow[d] \\
\delta_3 \arrow[ur] \arrow[dr] & \delta_{10} \arrow[l] \arrow[r] & \delta_{17} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & {\color{red}\delta_{24}} \arrow[l] \arrow[r] & \delta_{31} \arrow[ul] \arrow[dl] \\
\delta_4 \arrow[u] \arrow[d] & \delta_{11} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \delta_{18} \arrow[u] \arrow[d] & \delta_{25} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \delta_{32} \arrow[u] \arrow[d] \\
\delta_5 \arrow[ur] \arrow[dr] & \delta_{12} \arrow[l] \arrow[r] & \delta_{19} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \delta_{26} \arrow[l] \arrow[r] & \delta_{33} \arrow[ul] \arrow[dl] \\
\delta_6 \arrow[u] \arrow[d] & \delta_{13} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \delta_{20} \arrow[u] \arrow[d] & \delta_{27} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \delta_{34} \arrow[u] \arrow[d] \\
\delta_7 \arrow[ur] & \delta_{14} \arrow[l] \arrow[r] & \delta_{21} \arrow[ul] \arrow[ur] & \delta_{28} \arrow[l] \arrow[r] & \delta_{35} \arrow[ul]
\end{tikzcd} \tag{1}$$

The first code that we want to explain is *VanCycleSub*. We prove Lemma 2.5 for the vanishing cycle δ_{24} which is at the intersection of row 3 with column 4 in Diagram (1). Let us denote by $\mathcal{V}$ the subspace generated by the orbit of the monodromy action on δ_{24}. Some of the linear combinations in $\mathcal{V}$ showed in Lemma 2.5 are:

$$\delta_{24}, \delta_{10}, \delta_{17} + \delta_{31}, \delta_{16} + \delta_{18} + \delta_{30} + \delta_{32}, \delta_{22}, \delta_{23}, \delta_{25}, \delta_{26}, \delta_{27}, \delta_{28}. \tag{2}$$

Note that $\delta_8, \delta_9, \ldots, \delta_{14}$ are also in $\mathcal{V}$ since we can generate δ_{10}. Furthermore, $\delta_{k-7} + \delta_{k+7}$, with $k = 8, \ldots, 14$ or $k = 22, \ldots, 28$ are in $\mathcal{V}$ as well. Similarly, $\delta_{k-8} + \delta_{k-6} + \delta_{k+8} + \delta_{k+6}$ with $k = 9, \ldots, 13$ or $k = 23, \ldots, 27$ are in $\mathcal{V}$. However, in the proof of Lemma 2.5 it is enough to consider the vanishing cycles in (2). This is because the other missing vanishing cycles will be checked when the same procedure is carried out with the vanishing cycles in (2). For instance, if we want to check that $\delta_{19} + \delta_{21} + \delta_{33} + \delta_{35}$ belongs to $\mathcal{V}$, we only need to know that this linear combination is in the subspace generate by the monodromy action on δ_{27}.

The way to use the code *VanCycleSub* is as follows.

```
[Dim, Tlvc, U, W , CoeffNonZero]=VanCycleSub(6,8,3,4);
```

where we have that:

- **Dim** is the number of different eigenvalues of the monodromy matrix.

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[1] https://github.com/danfelmath01/Mondromy-and-tangential-problems-for-direct-sum-of-polynomials

- **Tlvc** is a matrix in which each column represents a vector of the vanishing cycles in (2). Its first column represents the vanishing cycle in the position (3, 4).
- **U** is the basis of the eigenvectors of the monodromy matrix. Each column of **U** is an eigenvector of the monodromy matrix.
- **W** satisfies **UW** = **Tlvc**. Thus, each column of **W** corresponds to the coefficients of the equation $\sum_i w_{ij}\mathbf{U}_i = \mathbf{Tlvc}_j$, where the $\mathbf{U}_i$'s are the eigenvectors of the monodromy matrix.
- **CoeffNonZero** are the coefficients different from zero. Each column contains the nonzero coefficients associated to the columns of **Tlvc**.

In our specific case **Dim** = 35. The matrix **Tlvc** is given by Matrix (3).

$$\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24\\25\\26\\27\\28\\29\\30\\31\\32\\33\\34\\35 \end{array}
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{pmatrix}. \tag{3}$$

The norm of the coefficients appearing in the matrix **W** are presented in Matrix (4).

$$\begin{array}{c}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24\\25\\26\\27\\28\\29\\30\\31\\32\\33\\34\\35\end{array}\begin{pmatrix}
0.1722 & 0.1722 & 0.4808 & 0.5511 & 0.0713 & 0.2124 & 0.3004 & 0.1722 & 0.2124 & 0.0713\\
0.1722 & 0.1722 & 0.4808 & 0.5511 & 0.0713 & 0.2124 & 0.3004 & 0.1722 & 0.2124 & 0.0713\\
0.1394 & 0.1394 & 0.3594 & 0.3415 & 0.1394 & 0.2935 & 0.0000 & 0.1394 & 0.2935 & 0.1394\\
0.1394 & 0.1394 & 0.3594 & 0.3415 & 0.1394 & 0.2935 & 0.0000 & 0.1394 & 0.2935 & 0.1394\\
0.1928 & 0.1928 & 0.2637 & 0.3562 & 0.0798 & 0.2018 & 0.2854 & 0.1928 & 0.2018 & 0.0798\\
0.1928 & 0.1928 & 0.2637 & 0.3562 & 0.0798 & 0.2018 & 0.2854 & 0.1928 & 0.2018 & 0.0798\\
0.0837 & 0.0837 & 0.1841 & 0.1110 & 0.2022 & 0.1964 & 0.2777 & 0.0837 & 0.1964 & 0.2022\\
0.0837 & 0.0837 & 0.1841 & 0.1110 & 0.2022 & 0.1964 & 0.2777 & 0.0837 & 0.1964 & 0.2022\\
0.1523 & 0.1523 & 0.1983 & 0.2153 & 0.1523 & 0.2804 & 0.0000 & 0.1523 & 0.2804 & 0.1523\\
0.1523 & 0.1523 & 0.1983 & 0.2153 & 0.1523 & 0.2804 & 0.0000 & 0.1523 & 0.2804 & 0.1523\\
0.0877 & 0.0877 & 0.1030 & 0.0671 & 0.2117 & 0.1904 & 0.2692 & 0.0877 & 0.1904 & 0.2117\\
0.0877 & 0.0877 & 0.1030 & 0.0671 & 0.2117 & 0.1904 & 0.2692 & 0.0877 & 0.1904 & 0.2117\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.2500 & 0.2500 & 0.4330 & 0.0000 & 0.2500 & 0.0000 & 0.0000 & 0.2500 & 0.0000 & 0.2500\\
0.2500 & 0.2500 & 0.4330 & 0.0000 & 0.2500 & 0.0000 & 0.0000 & 0.2500 & 0.0000 & 0.2500\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.2500 & 0.2500 & 0.2500 & 0.0000 & 0.2500 & 0.0000 & 0.0000 & 0.2500 & 0.0000 & 0.2500\\
0.2500 & 0.2500 & 0.2500 & 0.0000 & 0.2500 & 0.0000 & 0.0000 & 0.2500 & 0.0000 & 0.2500\\
0.1063 & 0.1063 & 0.1450 & 0.1409 & 0.2566 & 0.1547 & 0.2188 & 0.1063 & 0.1547 & 0.2566\\
0.1063 & 0.1063 & 0.1450 & 0.1409 & 0.2566 & 0.1547 & 0.2188 & 0.1063 & 0.1547 & 0.2566\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.2637 & 0.2637 & 0.1928 & 0.4873 & 0.1092 & 0.1475 & 0.2086 & 0.2637 & 0.1475 & 0.1092\\
0.2637 & 0.2637 & 0.1928 & 0.4873 & 0.1092 & 0.1475 & 0.2086 & 0.2637 & 0.1475 & 0.1092\\
0.1983 & 0.1983 & 0.1523 & 0.2804 & 0.1983 & 0.2153 & 0.0000 & 0.1983 & 0.2153 & 0.1983\\
0.1983 & 0.1983 & 0.1523 & 0.2804 & 0.1983 & 0.2153 & 0.0000 & 0.1983 & 0.2153 & 0.1983\\
0.2075 & 0.2075 & 0.2415 & 0.5083 & 0.2075 & 0.1972 & 0.0000 & 0.2075 & 0.1972 & 0.2075\\
0.2075 & 0.2075 & 0.2415 & 0.5083 & 0.2075 & 0.1972 & 0.0000 & 0.2075 & 0.1972 & 0.2075\\
0.1030 & 0.1030 & 0.0877 & 0.0789 & 0.2487 & 0.1620 & 0.2292 & 0.1030 & 0.1620 & 0.2487\\
0.1030 & 0.1030 & 0.0877 & 0.0789 & 0.2487 & 0.1620 & 0.2292 & 0.1030 & 0.1620 & 0.2487\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.2776 & 0.2776 & 0.2982 & 0.8883 & 0.1150 & 0.1318 & 0.1864 & 0.2776 & 0.1318 & 0.1150\\
0.2776 & 0.2776 & 0.2982 & 0.8883 & 0.1150 & 0.1318 & 0.1864 & 0.2776 & 0.1318 & 0.1150
\end{pmatrix}. \tag{4}$$

The non-zero coefficients appearing in $\mathbf{W}$ are presented in Matrix (5).

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 5 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 6 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 7 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 8 & 6 & 6 & 6 \\
7 & 7 & 7 & 7 & 7 & 7 & 11 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 8 & 8 & 12 & 8 & 8 & 8 \\
9 & 9 & 9 & 9 & 9 & 9 & 21 & 9 & 9 & 9 \\
10 & 10 & 10 & 10 & 10 & 10 & 22 & 10 & 10 & 10 \\
11 & 11 & 11 & 11 & 11 & 11 & 25 & 11 & 11 & 11 \\
12 & 12 & 12 & 12 & 12 & 12 & 26 & 12 & 12 & 12 \\
15 & 15 & 15 & 21 & 15 & 21 & 31 & 15 & 21 & 15 \\
16 & 16 & 16 & 22 & 16 & 22 & 32 & 16 & 22 & 16 \\
19 & 19 & 19 & 25 & 19 & 25 & 34 & 19 & 25 & 19 \\
20 & 20 & 20 & 26 & 20 & 26 & 35 & 20 & 26 & 20 \\
21 & 21 & 21 & 27 & 21 & 27 & 0 & 21 & 27 & 21 \\
22 & 22 & 22 & 28 & 22 & 28 & 0 & 22 & 28 & 22 \\
25 & 25 & 25 & 29 & 25 & 29 & 0 & 25 & 29 & 25 \\
26 & 26 & 26 & 30 & 26 & 30 & 0 & 26 & 30 & 26 \\
27 & 27 & 27 & 31 & 27 & 31 & 0 & 27 & 31 & 27 \\
28 & 28 & 28 & 32 & 28 & 32 & 0 & 28 & 32 & 28 \\
29 & 29 & 29 & 34 & 29 & 34 & 0 & 29 & 34 & 29 \\
30 & 30 & 30 & 35 & 30 & 35 & 0 & 30 & 35 & 30 \\
31 & 31 & 31 & 0 & 31 & 0 & 0 & 31 & 0 & 31 \\
32 & 32 & 32 & 0 & 32 & 0 & 0 & 32 & 0 & 32 \\
34 & 34 & 34 & 0 & 34 & 0 & 0 & 34 & 0 & 34 \\
35 & 35 & 35 & 0 & 35 & 0 & 0 & 35 & 0 & 35
\end{pmatrix}. \tag{5}$$

In order to show that the vectors corresponding to the vanishing cycles in (2) are in $\mathcal{V}$, we have to check that the position of the coefficients different from zero in each column of $\mathbf{W}$ are contained in the position of the coefficients different from zero in the first column of $\mathbf{W}$. In other words, we check that the list of indexes defined by the columns of Matrix (5) is contained in the the first column of Matrix (5).

The second code that we want to explain is *Proof_lemma_2_5*. This code works faster than the previous one with each given evanescent cycle for (d, e) since it computes the monodromy matrix, the eigenvalues and the eigenvectors only once.

The way to use of *Proof_lemma_2_5* is as follows.

```
[Proof, FailVanCycle]=Proof_lemma_2_5(6,8);
```

where we have that:

- **Proof** is a logical value. The value 1 means that Lemma 2.5 is true for the given $(d, e) = (6, 8)$. The value 0 means that Lemma 2.5 fails for the given (d, e) for some vanishing cycles.
- **FailVanCycle** is a list of the vanishing cycles for which Lemma 2.5 is not true when $Proof = 0$.

In our particular example we get that **Proof** $= 1$ and **FailVanCycle** $= 0$. Nevertheless, if we consider for instance $(d, e) = (6, 3)$:

```
[Proof, FailVanCycle]=Proof_lemma_2_5(6,3);
```

then we obtain **Proof** $= 0$ and **FailVanCycle** $= 0$.

Remark: Pick $d, e > 1$ such that $\gcd(d, e) \leq 2$. If **Proof** $= 0$ for some values (d, e) then the rounding function used to determine whether a coefficient is zero must be checked. To do this the list **FailVanCycle** would be useful. However, for the values with $\gcd(d, e) \leq 2$ studied in the paper [1] **Proof** $= 1$.

References

[1] D. Lopez Garcia and F. Valencia: *On the monodromy action for* $f(x,y) = g(x) + h(y)$, https://arxiv.org/abs/2311.05563

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