ON THE MONODROMY ACTION FOR $f(x,y) = g(x) + h(y)$

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1. Example

In this text we compute an explicit example in order to illustrate the codes [1] using in the proof of the Lemma 2.5 in [1]. Consider the polynomial $f(x,y) = x^6 + y^8$, thus a Dynkin diagram associated to $f(x,y)$ is presented in (1).

$$\begin{array}{ccccccccc}
\delta_1 & \longleftarrow & \delta_8 & \longrightarrow & \delta_{15} & \longleftarrow & \delta_{22} & \longrightarrow & \delta_{29} \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\
\delta_2 & \longleftarrow & \delta_9 & \longrightarrow & \delta_{16} & \longleftarrow & \delta_{23} & \longrightarrow & \delta_{30} \\
\downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\
\delta_3 & \longleftarrow & \delta_{10} & \longrightarrow & \delta_{17} & \longleftarrow & \color{red}{\delta_{24}} & \longrightarrow & \delta_{31} \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\
\delta_4 & \longleftarrow & \delta_{11} & \longrightarrow & \delta_{18} & \longleftarrow & \delta_{25} & \longrightarrow & \delta_{32} \\
\downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\
\delta_5 & \longleftarrow & \delta_{12} & \longrightarrow & \delta_{19} & \longleftarrow & \delta_{26} & \longrightarrow & \delta_{33} \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\
\delta_6 & \longleftarrow & \delta_{13} & \longrightarrow & \delta_{20} & \longleftarrow & \delta_{27} & \longrightarrow & \delta_{34} \\
\downarrow & \nearrow & \downarrow & \nwarrow & \downarrow & \nearrow & \downarrow & \nwarrow & \downarrow \\
\delta_7 & \longleftarrow & \delta_{14} & \longrightarrow & \delta_{21} & \longleftarrow & \delta_{28} & \longrightarrow & \delta_{35}
\end{array} \tag{1}$$

The first code that we explain *VanCycleSub*. We prove Lemma 2.5, for the vanishing cycle δ_{24} which is in the row 3, and the column 4 in (1). Let us denote by $\mathcal{V}$ the subspace generated by the orbit of the monodromy action on δ_{24}.

Some of the linear combination in $\mathcal{V}$ described in Lemma 2.5 are:

$$\delta_{24}, \delta_{10}, \delta_{17} + \delta_{31}, \delta_{16} + \delta_{18} + \delta_{30} + \delta_{32}, \delta_{22}, \delta_{23}, \delta_{25}, \delta_{26}, \delta_{27}, \delta_{28}. \tag{2}$$

Note that, because we can generate δ_{10}, then $\delta_8, \delta_9, \dots, \delta_{14}$ are also in $\mathcal{V}$. Moreover, $\delta_{k-7} + \delta_{k+7}$, with $k = 8, \dots, 14$ or $k = 22, \dots, 28$ are in $\mathcal{V}$. Also, $\delta_{k-8} + \delta_{k-6} + \delta_{k+8} + \delta_{k+6}$ with $k = 9, \dots, 13$ or $k = 23, \dots, 27$ are in $\mathcal{V}$

However, for the proof of Lemma 2.5 it is enough to consider the vanishing cycles in (2), since the others that are missing will be checked when the same procedure is carried out with the vanishing cycles in (2). For example, if we can to check that $\delta_{19} + \delta_{21} + \delta_{33} + \delta_{35}$ is in $\mathcal{V}$, we just need to know that those cycles are in the subspace generate by the monodromy action on δ_{27}.

The use of *VanCycleSub* is as follows

```
[Dim, Tlvc, U, W , CoeffNonZero]=VanCycleSub(6,8,3,4);
```

and we get:

- **Dim** is the number of different eigenvalues of the monodromy matrix.

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[1]https://github.com/danfelmath01/Mondromy-and-tangential-problems-for-direct-sum-of-polynomials

- **Tlvc** is a matrix in which each column represents a vector of (2). Its first column represent the vanishing in the position (3, 4).
- **U** is the basis of the eigenvectors of the monodromy matrix. Each column of **U** is an eigenvector of the monodromy matrix.
- **W** satisfy **UW** = **Tlvc**. Thus, each column of **W** are the coefficients of the equation $\sum_i w_{ij}\mathbf{U}_i = \mathbf{Tlvc}_j$, where U_i are the eigenvectors of the monodromy matrix.
- **CoeffNonZero** are the coefficients different from zero. In each column there are the non zero coefficients associated to the columns of **Tlvc**.

Then, **Dim** = 35. The matrix **Tlvc** is in (3).

$$
\begin{array}{c}
1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24\\25\\26\\27\\28\\29\\30\\31\\32\\33\\34\\35
\end{array}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{3}
$$

The norm of the coefficients appearing in the matrix **W** are presented in the matrix (4).

$$
\begin{array}{c}
1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24\\25\\26\\27\\28\\29\\30\\31\\32\\33\\34\\35
\end{array}
\begin{pmatrix}
0.1722 & 0.1722 & 0.4808 & 0.5511 & 0.0713 & 0.2124 & 0.3004 & 0.1722 & 0.2124 & 0.0713\\
0.1722 & 0.1722 & 0.4808 & 0.5511 & 0.0713 & 0.2124 & 0.3004 & 0.1722 & 0.2124 & 0.0713\\
0.1394 & 0.1394 & 0.3594 & 0.3415 & 0.1394 & 0.2935 & 0.0000 & 0.1394 & 0.2935 & 0.1394\\
0.1394 & 0.1394 & 0.3594 & 0.3415 & 0.1394 & 0.2935 & 0.0000 & 0.1394 & 0.2935 & 0.1394\\
0.1928 & 0.1928 & 0.2637 & 0.3562 & 0.0798 & 0.2018 & 0.2854 & 0.1928 & 0.2018 & 0.0798\\
0.1928 & 0.1928 & 0.2637 & 0.3562 & 0.0798 & 0.2018 & 0.2854 & 0.1928 & 0.2018 & 0.0798\\
0.0837 & 0.0837 & 0.1841 & 0.1110 & 0.2022 & 0.1964 & 0.2777 & 0.0837 & 0.1964 & 0.2022\\
0.0837 & 0.0837 & 0.1841 & 0.1110 & 0.2022 & 0.1964 & 0.2777 & 0.0837 & 0.1964 & 0.2022\\
0.1523 & 0.1523 & 0.1983 & 0.2153 & 0.1523 & 0.2804 & 0.0000 & 0.1523 & 0.2804 & 0.1523\\
0.1523 & 0.1523 & 0.1983 & 0.2153 & 0.1523 & 0.2804 & 0.0000 & 0.1523 & 0.2804 & 0.1523\\
0.0877 & 0.0877 & 0.1030 & 0.0671 & 0.2117 & 0.1904 & 0.2692 & 0.0877 & 0.1904 & 0.2117\\
0.0877 & 0.0877 & 0.1030 & 0.0671 & 0.2117 & 0.1904 & 0.2692 & 0.0877 & 0.1904 & 0.2117\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.2500 & 0.2500 & 0.4330 & 0.0000 & 0.2500 & 0.0000 & 0.0000 & 0.2500 & 0.0000 & 0.2500\\
0.2500 & 0.2500 & 0.4330 & 0.0000 & 0.2500 & 0.0000 & 0.0000 & 0.2500 & 0.0000 & 0.2500\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.2500 & 0.2500 & 0.2500 & 0.0000 & 0.2500 & 0.0000 & 0.0000 & 0.2500 & 0.0000 & 0.2500\\
0.2500 & 0.2500 & 0.2500 & 0.0000 & 0.2500 & 0.0000 & 0.0000 & 0.2500 & 0.0000 & 0.2500\\
0.1063 & 0.1063 & 0.1450 & 0.1409 & 0.2566 & 0.1547 & 0.2188 & 0.1063 & 0.1547 & 0.2566\\
0.1063 & 0.1063 & 0.1450 & 0.1409 & 0.2566 & 0.1547 & 0.2188 & 0.1063 & 0.1547 & 0.2566\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.2637 & 0.2637 & 0.1928 & 0.4873 & 0.1092 & 0.1475 & 0.2086 & 0.2637 & 0.1475 & 0.1092\\
0.2637 & 0.2637 & 0.1928 & 0.4873 & 0.1092 & 0.1475 & 0.2086 & 0.2637 & 0.1475 & 0.1092\\
0.1983 & 0.1983 & 0.1523 & 0.2804 & 0.1983 & 0.2153 & 0.0000 & 0.1983 & 0.2153 & 0.1983\\
0.1983 & 0.1983 & 0.1523 & 0.2804 & 0.1983 & 0.2153 & 0.0000 & 0.1983 & 0.2153 & 0.1983\\
0.2075 & 0.2075 & 0.2415 & 0.5083 & 0.2075 & 0.1972 & 0.0000 & 0.2075 & 0.1972 & 0.2075\\
0.2075 & 0.2075 & 0.2415 & 0.5083 & 0.2075 & 0.1972 & 0.0000 & 0.2075 & 0.1972 & 0.2075\\
0.1030 & 0.1030 & 0.0877 & 0.0789 & 0.2487 & 0.1620 & 0.2292 & 0.1030 & 0.1620 & 0.2487\\
0.1030 & 0.1030 & 0.0877 & 0.0789 & 0.2487 & 0.1620 & 0.2292 & 0.1030 & 0.1620 & 0.2487\\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000\\
0.2776 & 0.2776 & 0.2982 & 0.8883 & 0.1150 & 0.1318 & 0.1864 & 0.2776 & 0.1318 & 0.1150\\
0.2776 & 0.2776 & 0.2982 & 0.8883 & 0.1150 & 0.1318 & 0.1864 & 0.2776 & 0.1318 & 0.1150
\end{pmatrix}. \tag{4}
$$

The non-zero coefficients appearing $\mathbf{W}$ are in (5)

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 5 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 6 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 7 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 8 & 6 & 6 & 6 \\
7 & 7 & 7 & 7 & 7 & 7 & 11 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 8 & 8 & 12 & 8 & 8 & 8 \\
9 & 9 & 9 & 9 & 9 & 9 & 21 & 9 & 9 & 9 \\
10 & 10 & 10 & 10 & 10 & 10 & 22 & 10 & 10 & 10 \\
11 & 11 & 11 & 11 & 11 & 11 & 25 & 11 & 11 & 11 \\
12 & 12 & 12 & 12 & 12 & 12 & 26 & 12 & 12 & 12 \\
15 & 15 & 15 & 21 & 15 & 21 & 31 & 15 & 21 & 15 \\
16 & 16 & 16 & 22 & 16 & 22 & 32 & 16 & 22 & 16 \\
19 & 19 & 19 & 25 & 19 & 25 & 34 & 19 & 25 & 19 \\
20 & 20 & 20 & 26 & 20 & 26 & 35 & 20 & 26 & 20 \\
21 & 21 & 21 & 27 & 21 & 27 & 0 & 21 & 27 & 21 \\
22 & 22 & 22 & 28 & 22 & 28 & 0 & 22 & 28 & 22 \\
25 & 25 & 25 & 29 & 25 & 29 & 0 & 25 & 29 & 25 \\
26 & 26 & 26 & 30 & 26 & 30 & 0 & 26 & 30 & 26 \\
27 & 27 & 27 & 31 & 27 & 31 & 0 & 27 & 31 & 27 \\
28 & 28 & 28 & 32 & 28 & 32 & 0 & 28 & 32 & 28 \\
29 & 29 & 29 & 34 & 29 & 34 & 0 & 29 & 34 & 29 \\
30 & 30 & 30 & 35 & 30 & 35 & 0 & 30 & 35 & 30 \\
31 & 31 & 31 & 0 & 31 & 0 & 0 & 31 & 0 & 31 \\
32 & 32 & 32 & 0 & 32 & 0 & 0 & 32 & 0 & 32 \\
34 & 34 & 34 & 0 & 34 & 0 & 0 & 34 & 0 & 34 \\
35 & 35 & 35 & 0 & 35 & 0 & 0 & 35 & 0 & 35
\end{pmatrix}. \tag{5}$$

In order to prove that the vectors in (2) are in $\mathcal{V}$, we check that the position of the coefficients different from zero in each column of $\mathbf{W}$ are contained in the position of the coefficients different from zero in the first column of $\mathbf{W}$. In other words, we check that the columns in (5) are contained in the first column of (5).

References

[1] D. Lopez Garcia and F. Valencia: *On the monodromy action for* $f(x,y) = g(x) + h(y)$, https://arxiv.org/abs/2311.05563

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