A fully adaptive, high-order, fast Poisson solver for complex two-dimensional geometries

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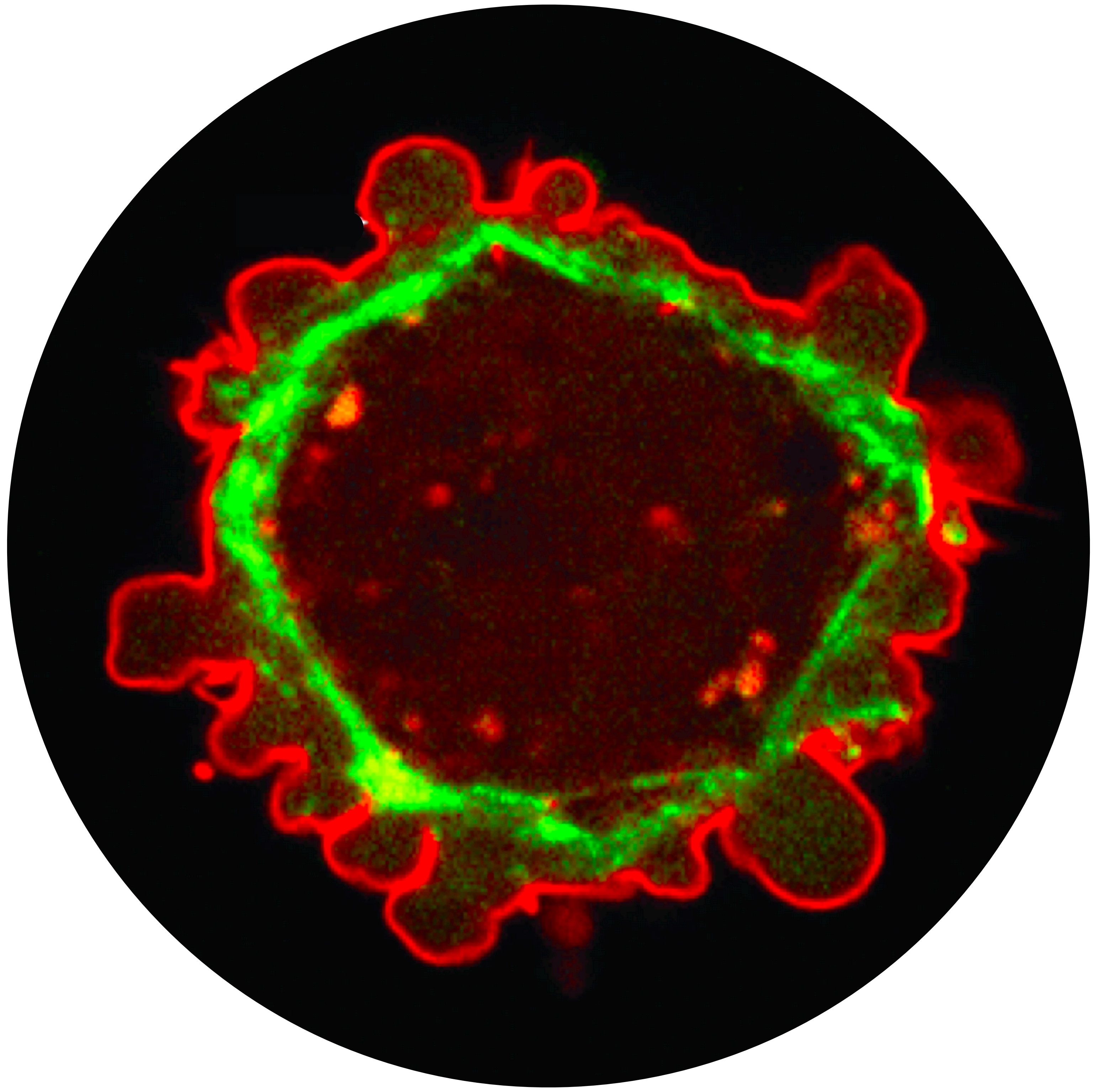

[Charras, 2008]

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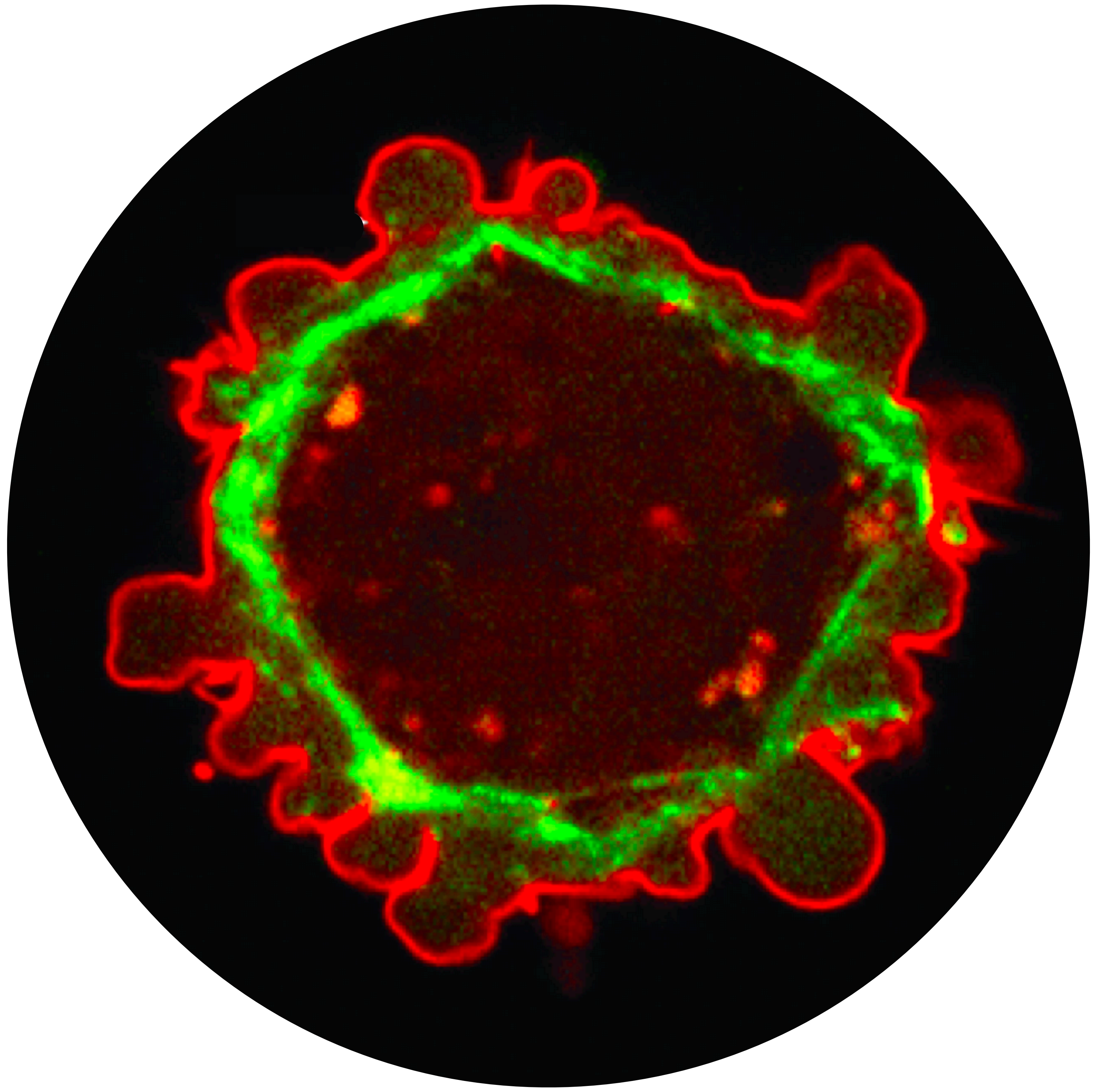
Close-to-touching regions



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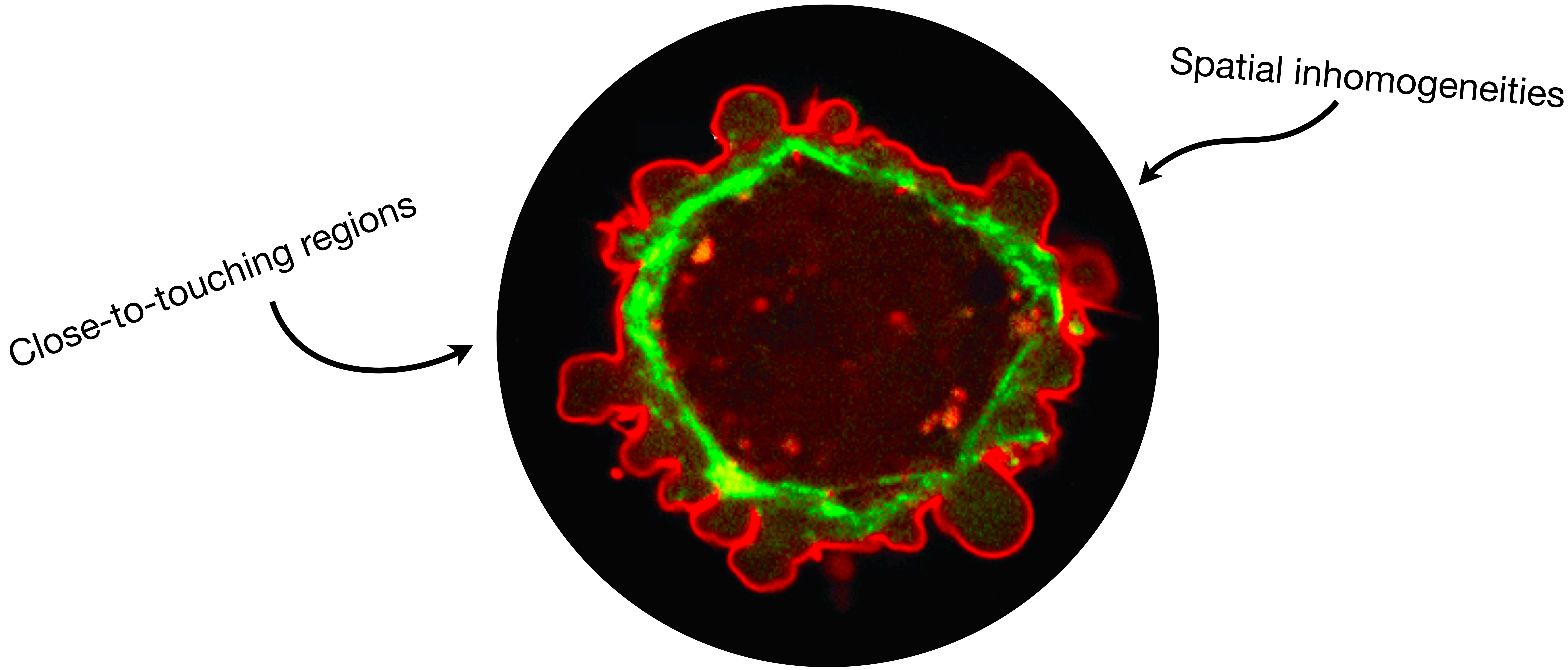

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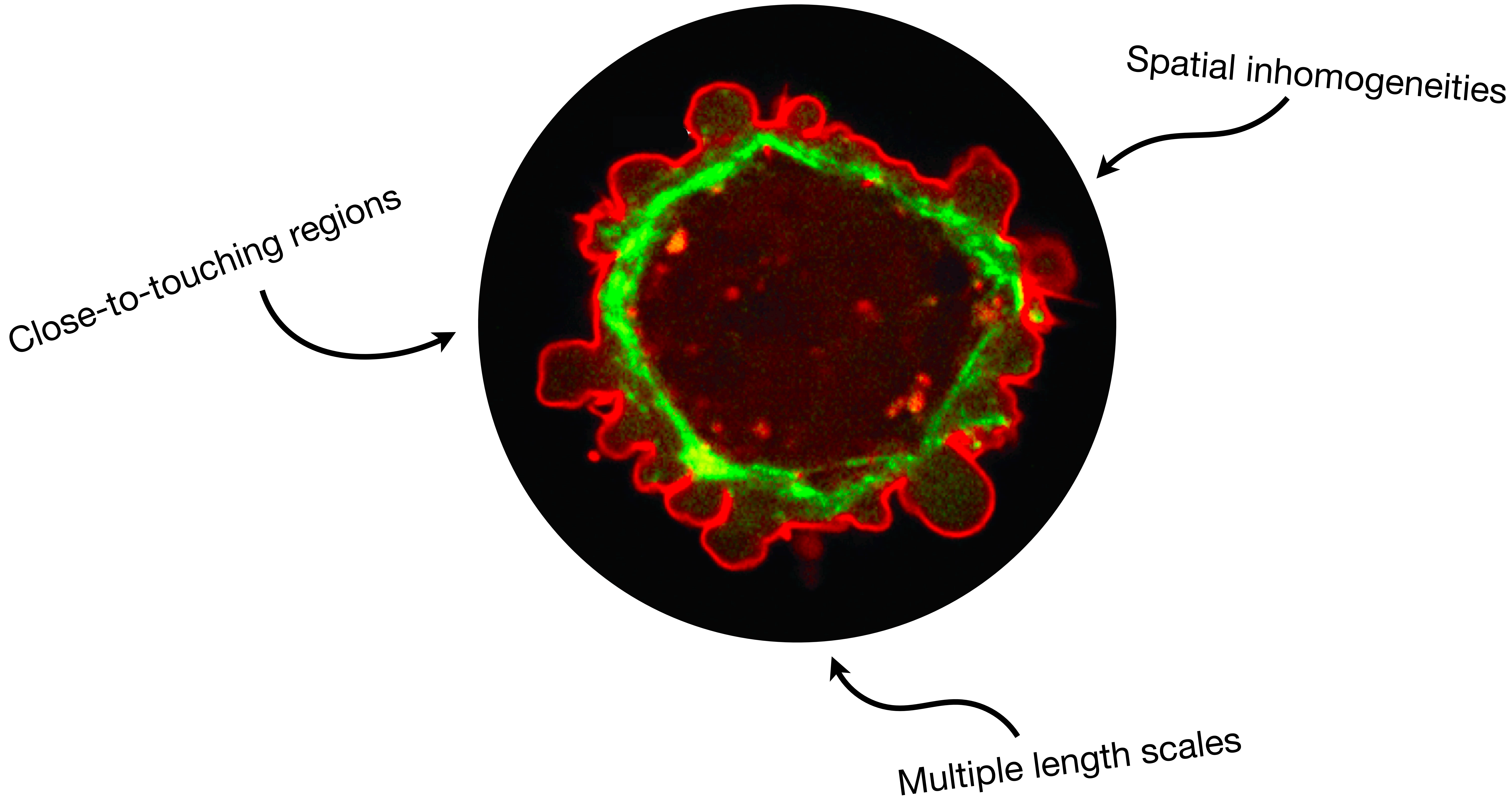

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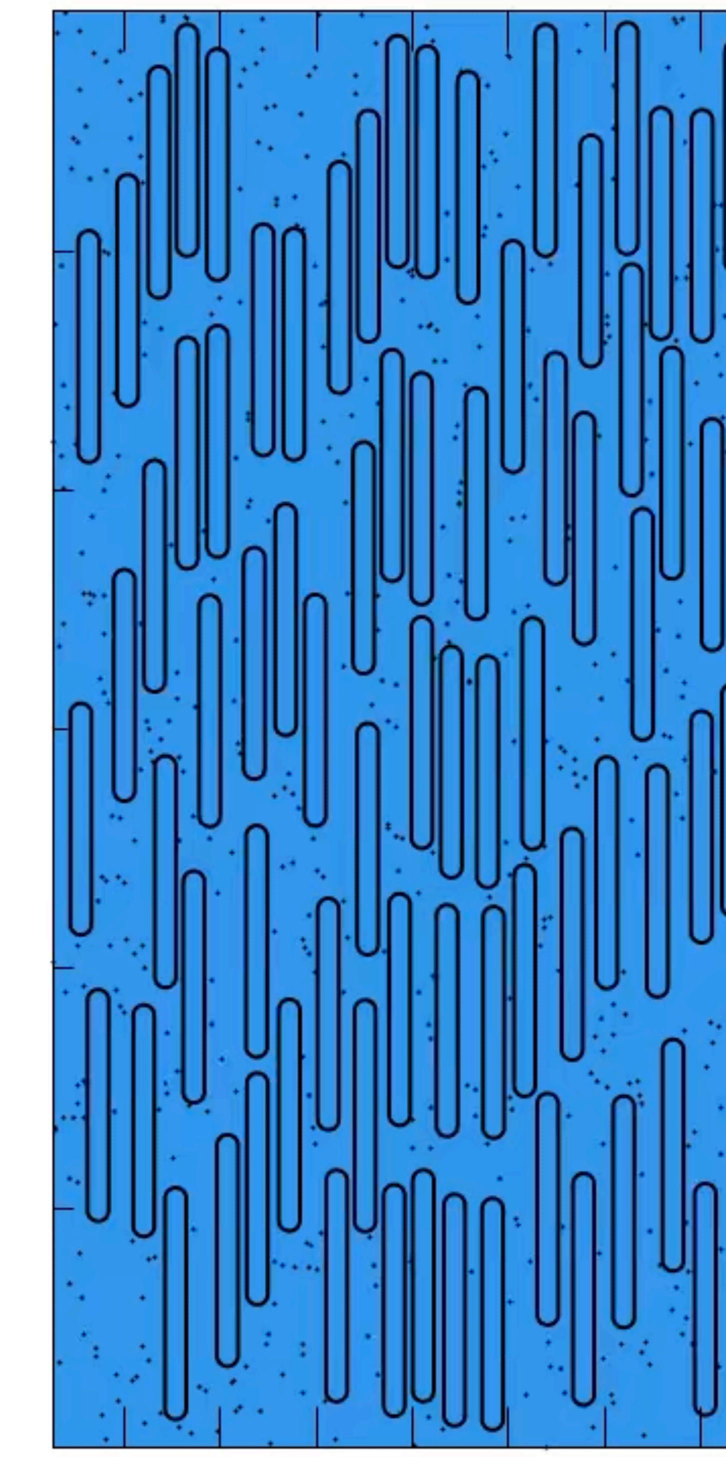
Inhomogeneous PDEs
Introduction

Many applications involve solving an inhomogeneous elliptic BVP.

$$Lu = f \quad \text{in } \Omega \qquad \text{(e.g. Poisson, Helmholtz, Stokes, ...)}$$

$$Bu = g \quad \text{on } \Gamma \qquad \text{(e.g. Dirichlet, Neumann, Robin, ...)}$$

Fluid-structure interaction
[Rycroft et al., 2020]



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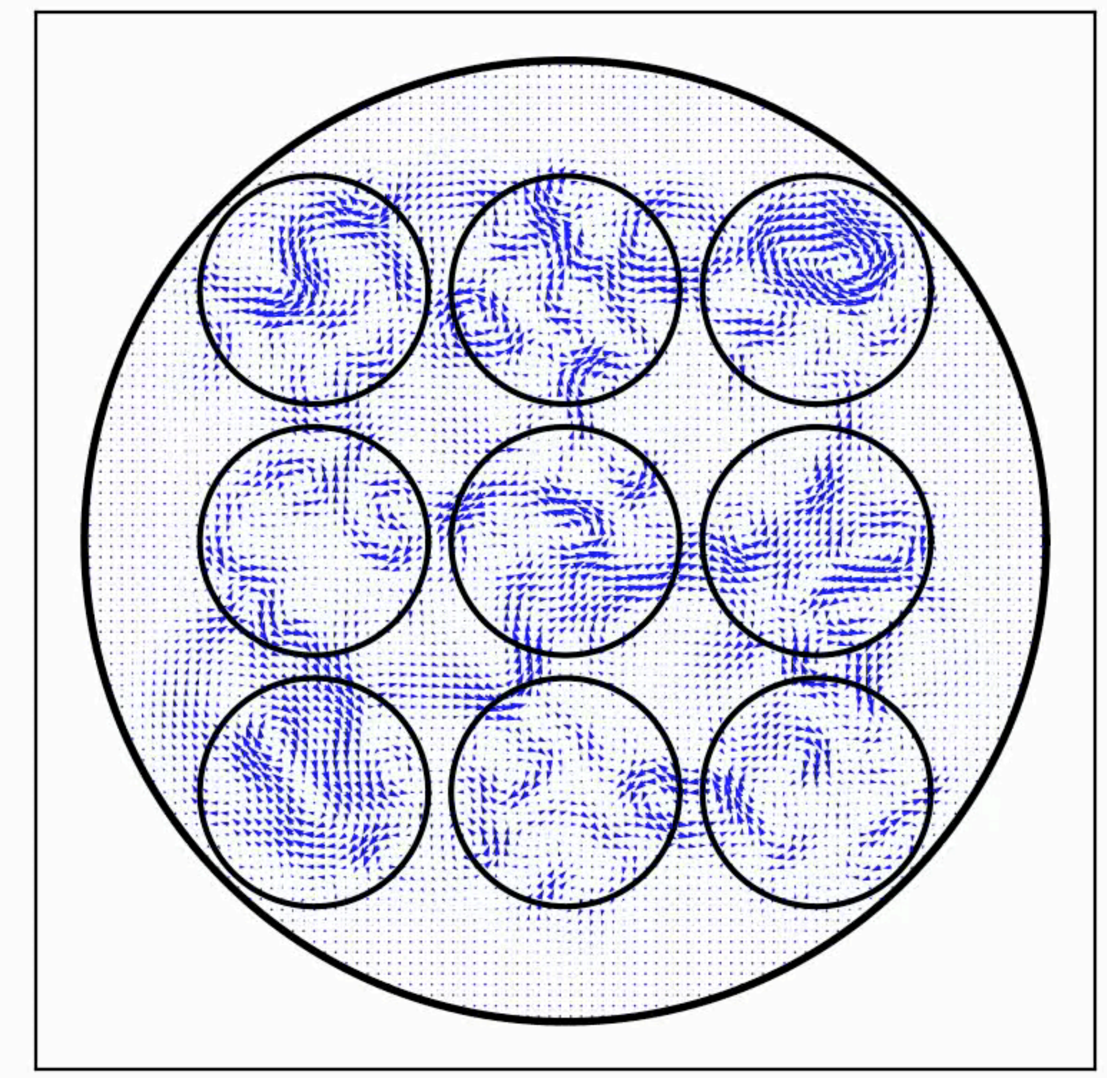

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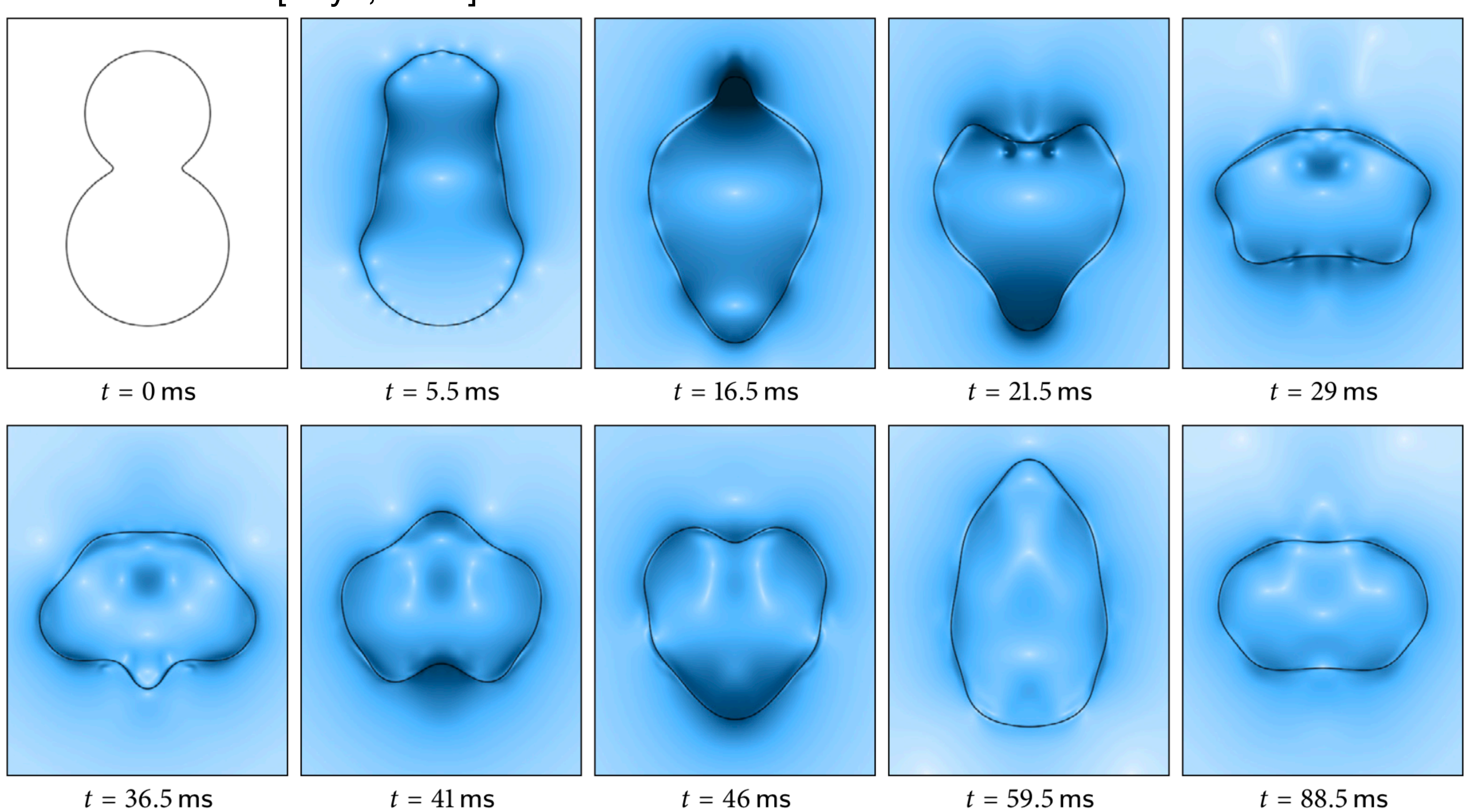

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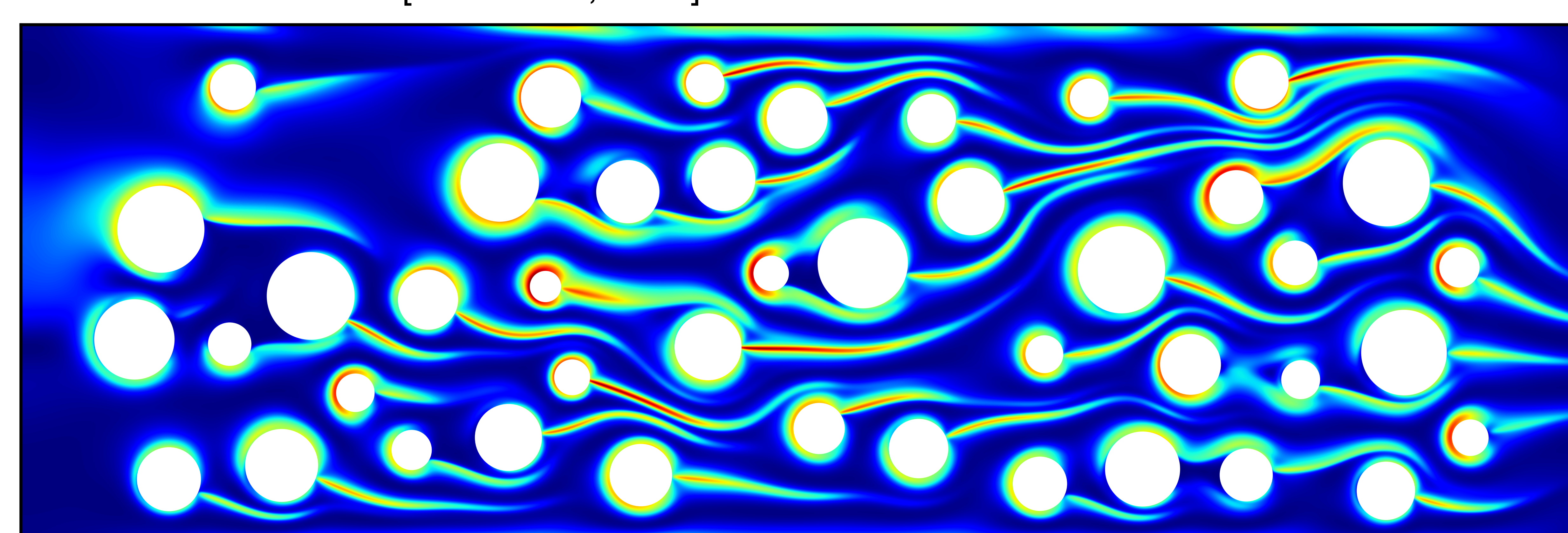

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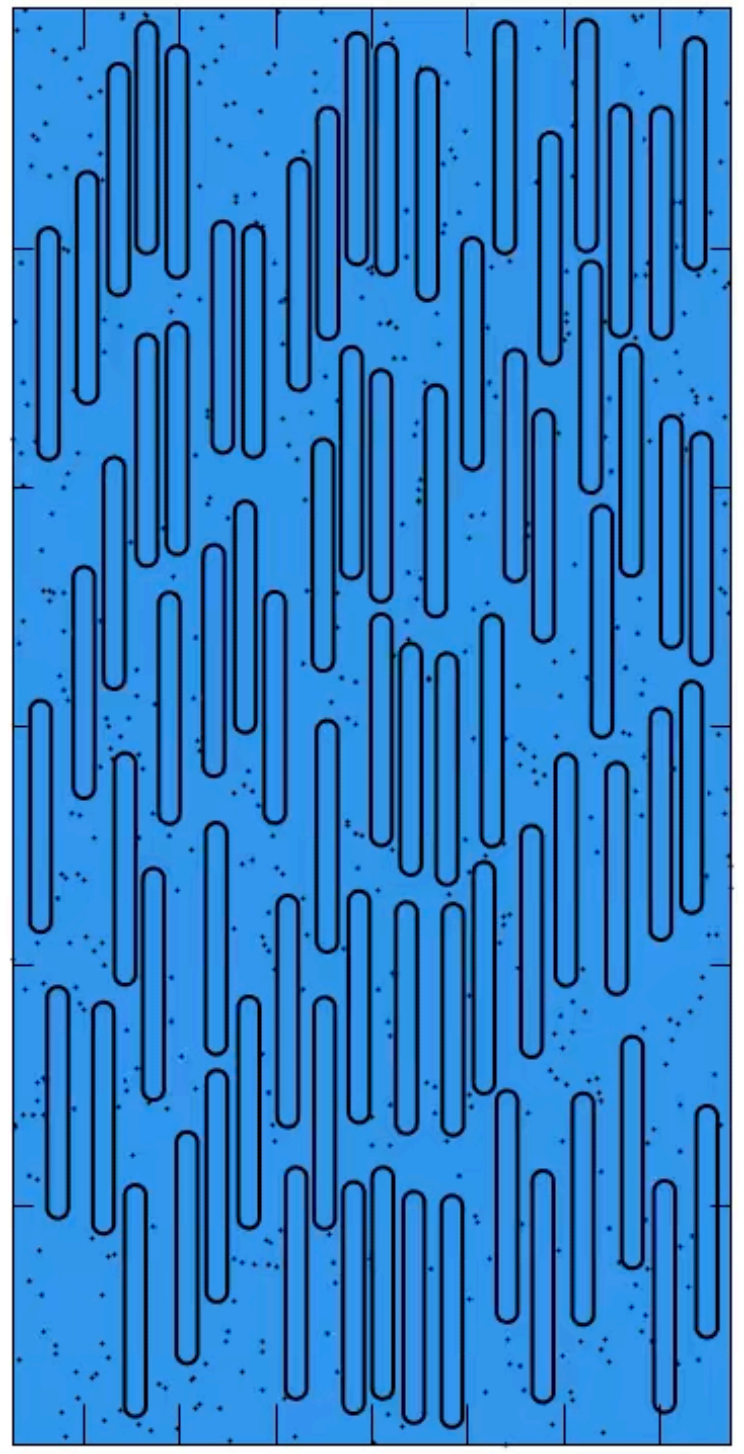

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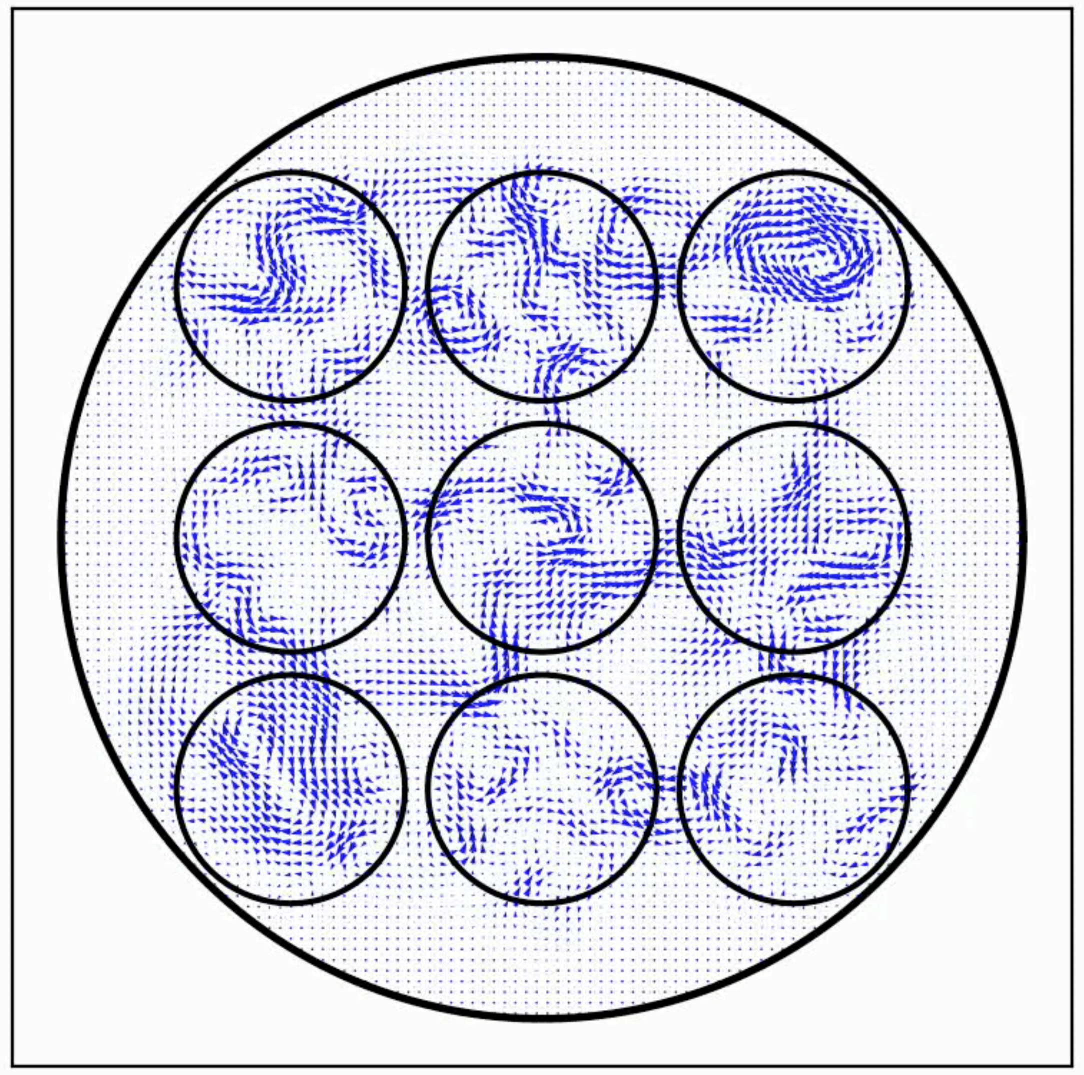

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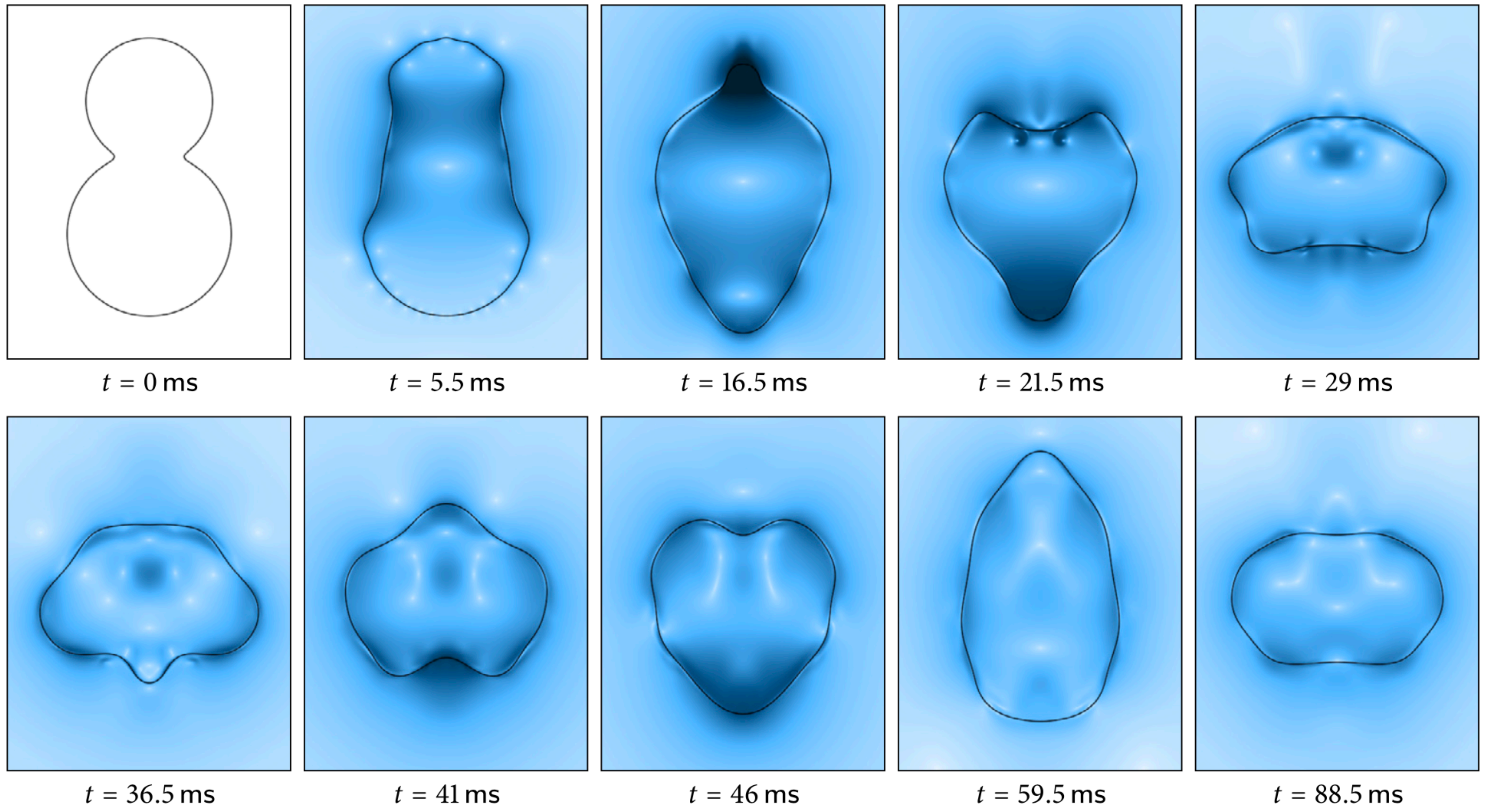

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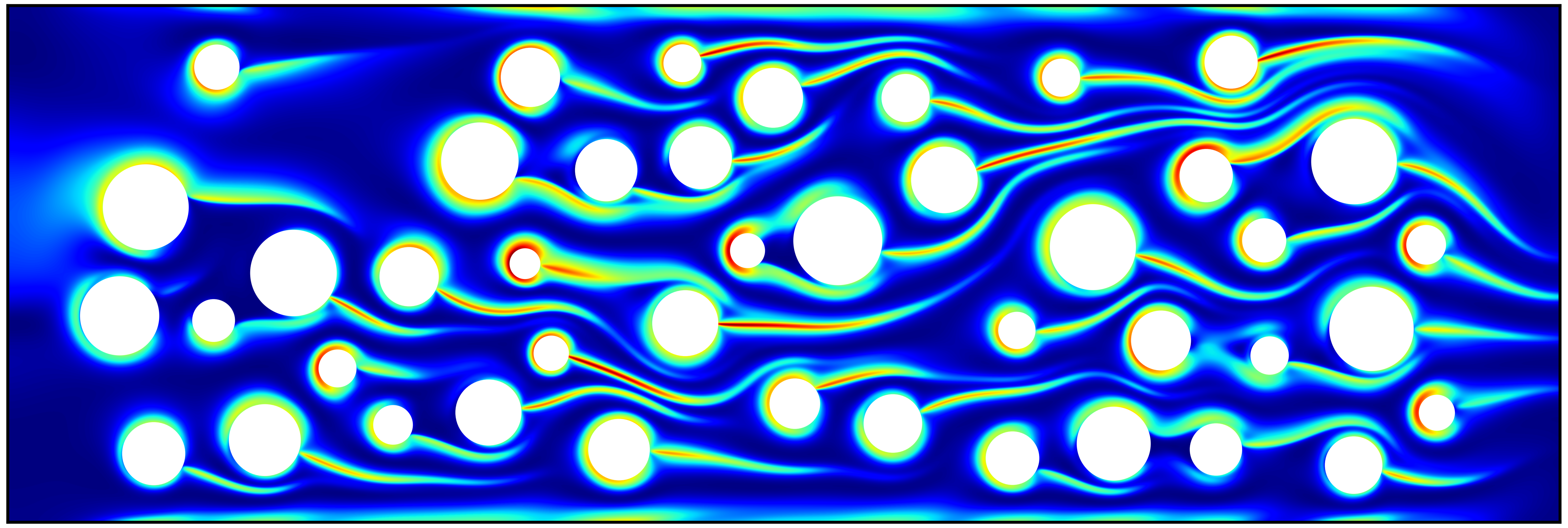
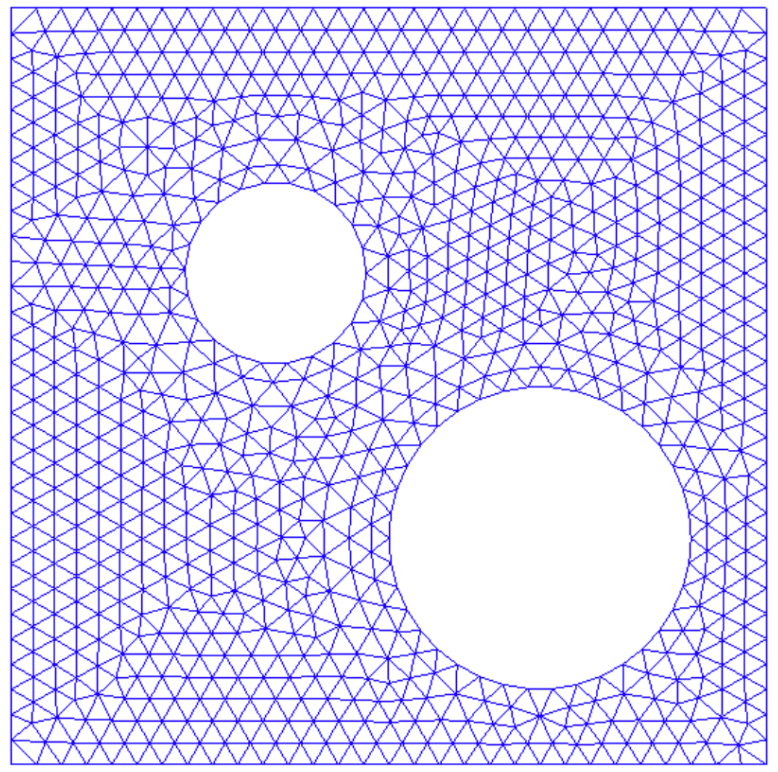


Inhomogeneous PDEs

Many approaches for inhomogeneous BVP

Traditional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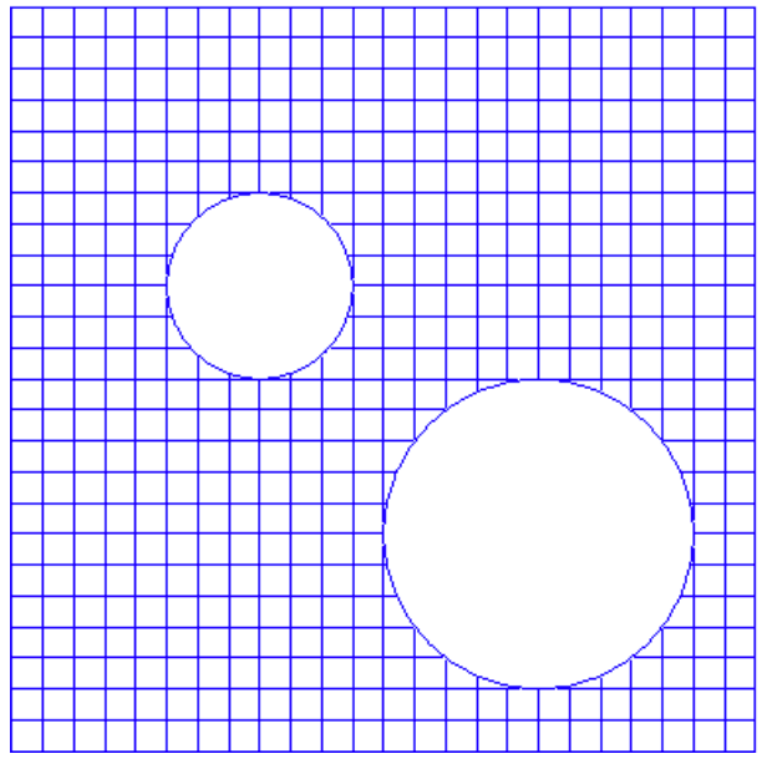
Cut-cell

- Mesh generation (or cut-cell generation)
- Directly discretize with FEM, FDM, SEM, ...

$$Ax = b$$

Discretize on a mesh.
Solve linear system for volume DoFs.

Inhomogeneous PDEs

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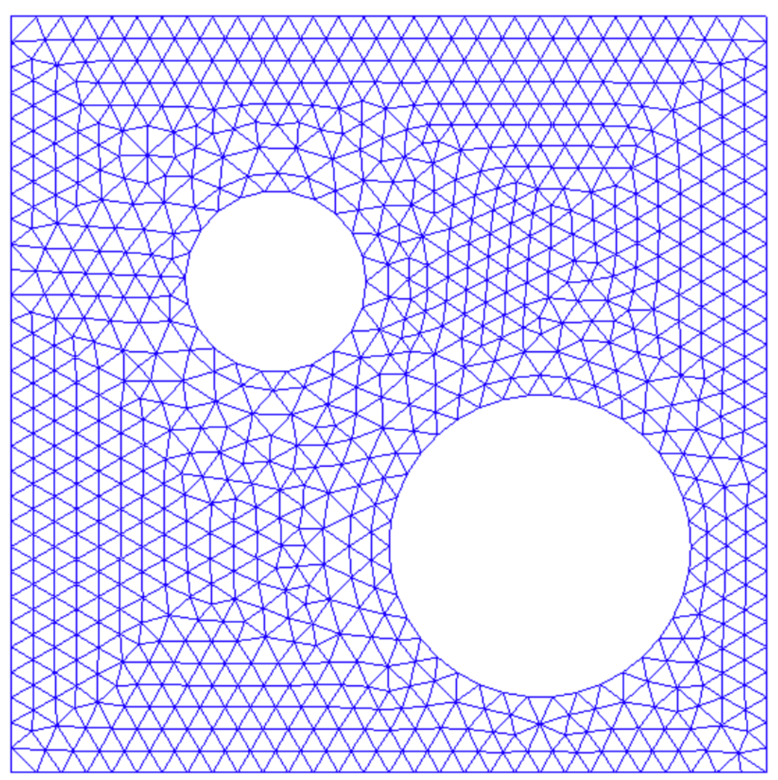

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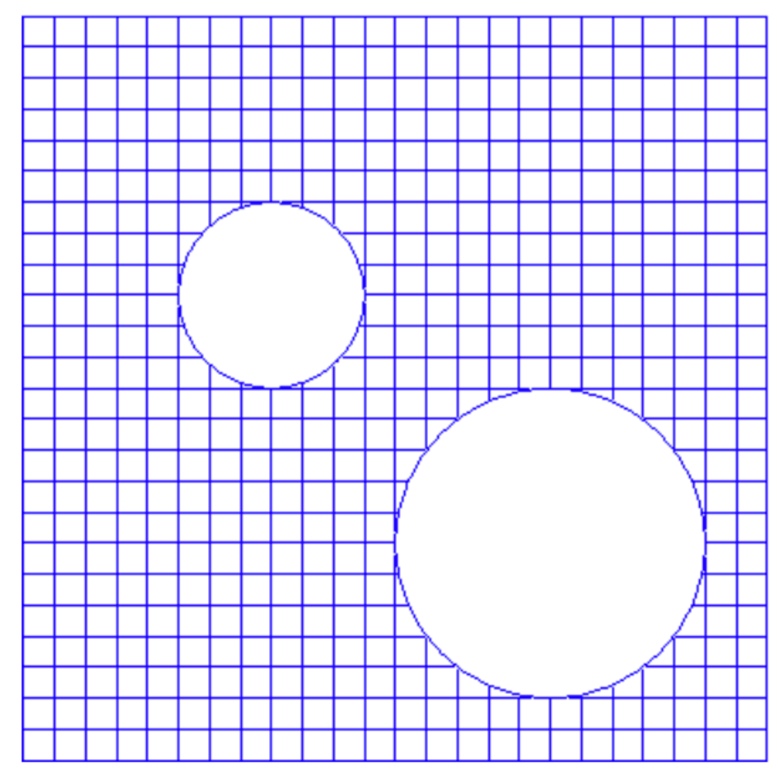

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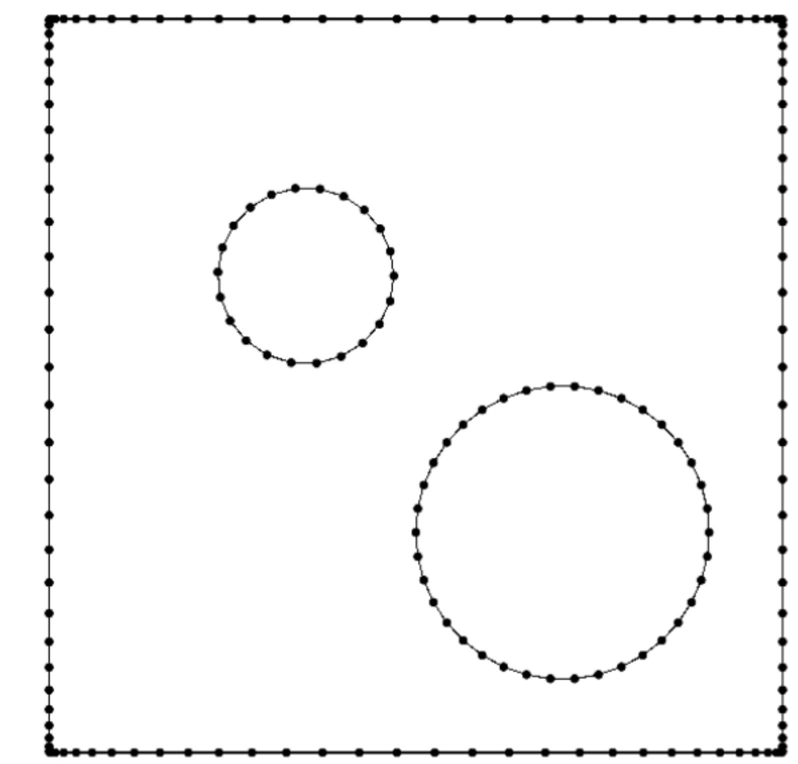
Analysis-based



- Write solution as $u = v + w$

particular solution *homogeneous solution*

- Find **some** (any!) function v such that

$$Lv = f \quad \text{in } \Omega$$

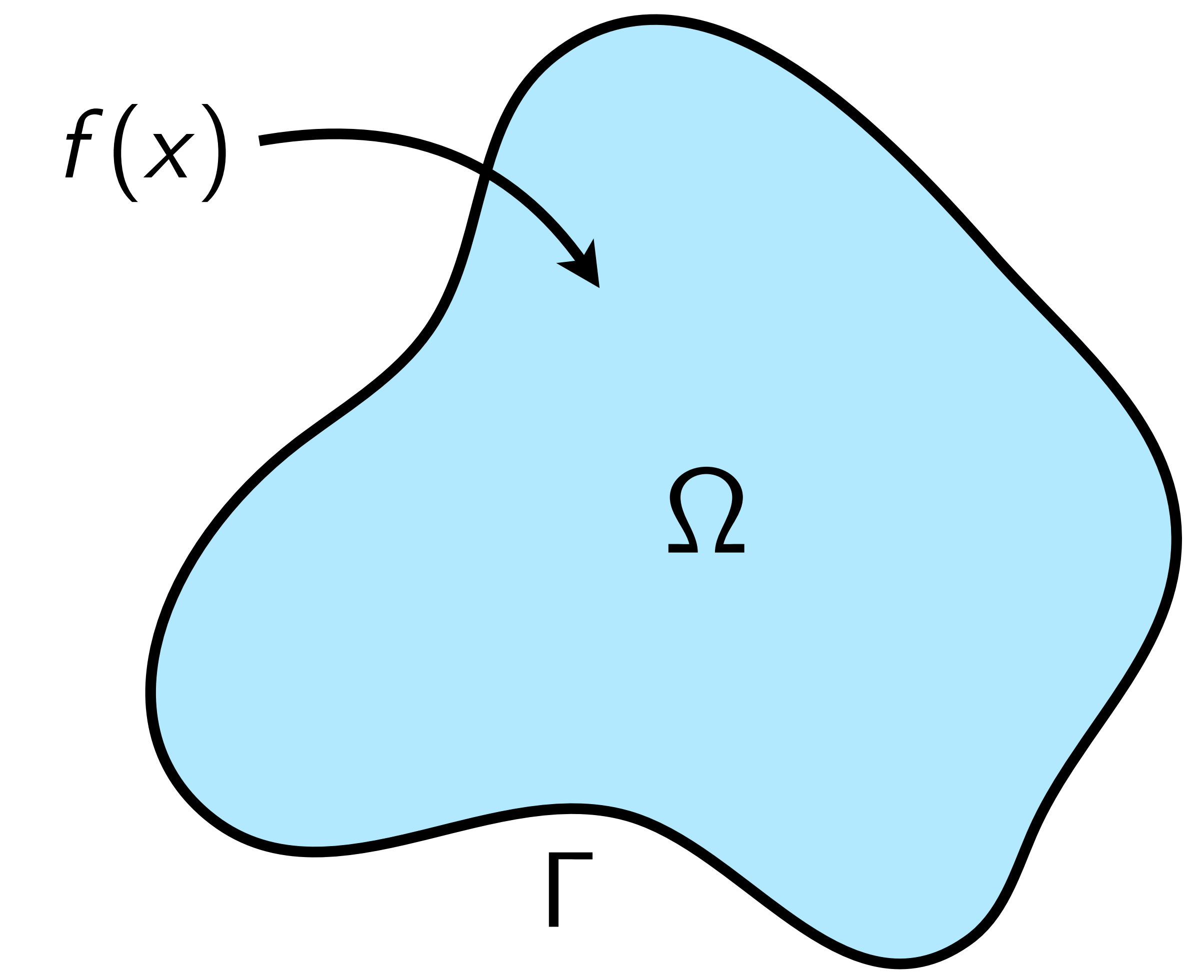
No solve required. Convolve with Green's function.

- Compute u_H to satisfy boundary conditions:

$$\begin{aligned} Lw &= 0 && \text{in } \Omega \\ Bw &= g - Bv && \text{on } \Gamma \end{aligned}$$

Solve linear system for boundary DoFs using BIE.

In many applications $f(x)$ is only given inside Ω. So we have a few options...



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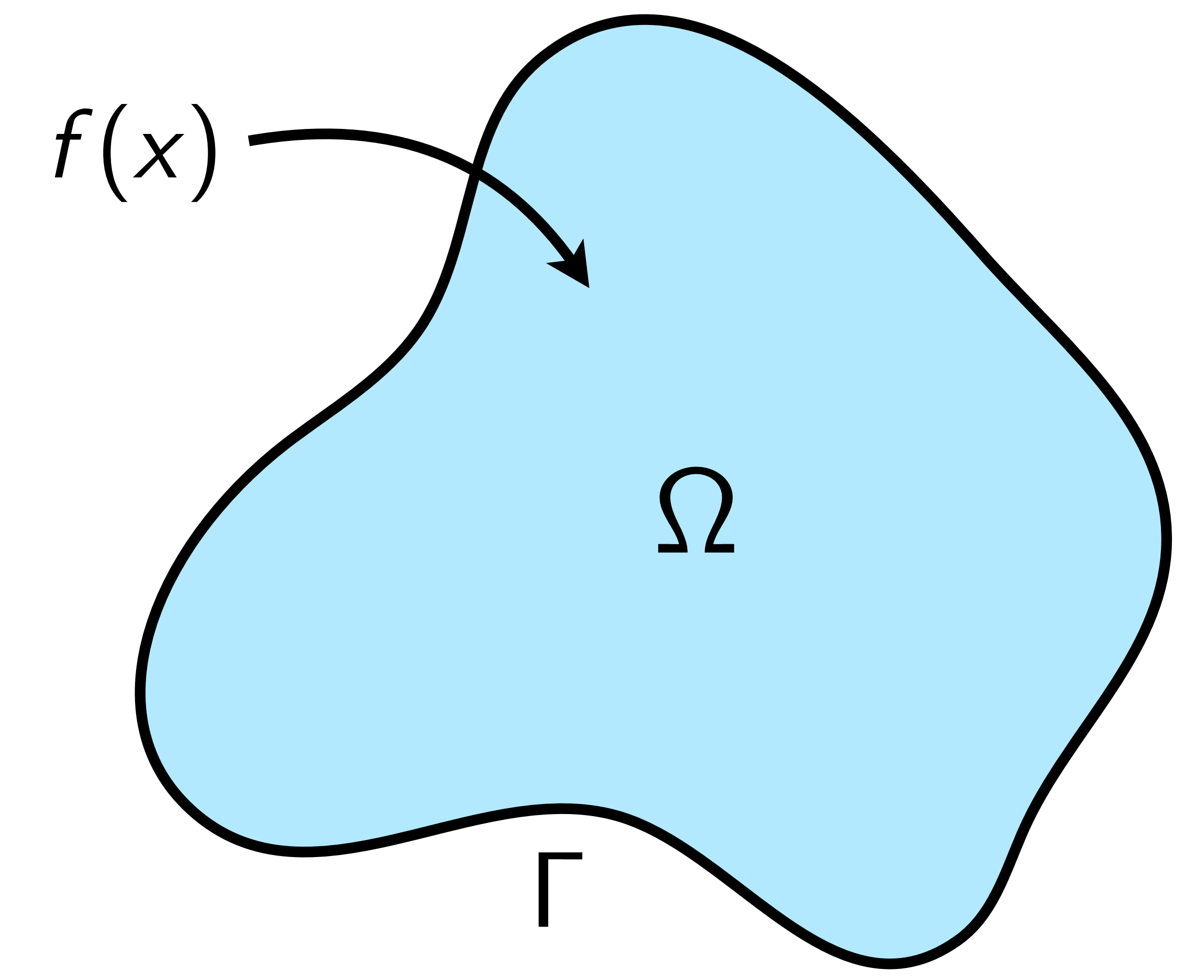
- Build accurate quadrature scheme over Ω (e.g. adaptive boxes with cut cells near boundary) and compute

$$v(x) = \int_\Omega \Phi(x, y)\, f(y)\, dy \qquad \left(\text{Poisson: } \Phi(x, y) = \frac{1}{2\pi} \log |x - y| \right)$$

free space fundamental solution



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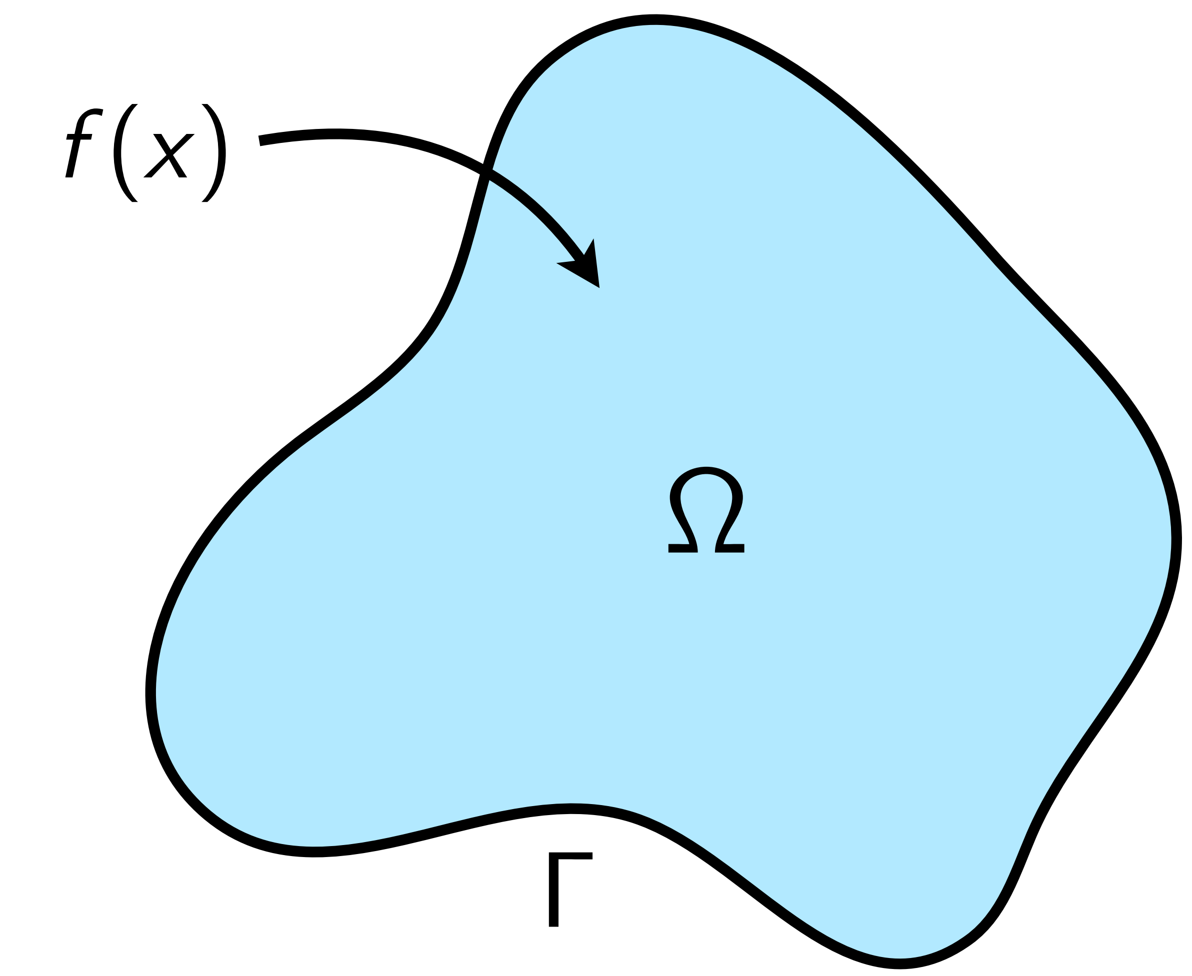
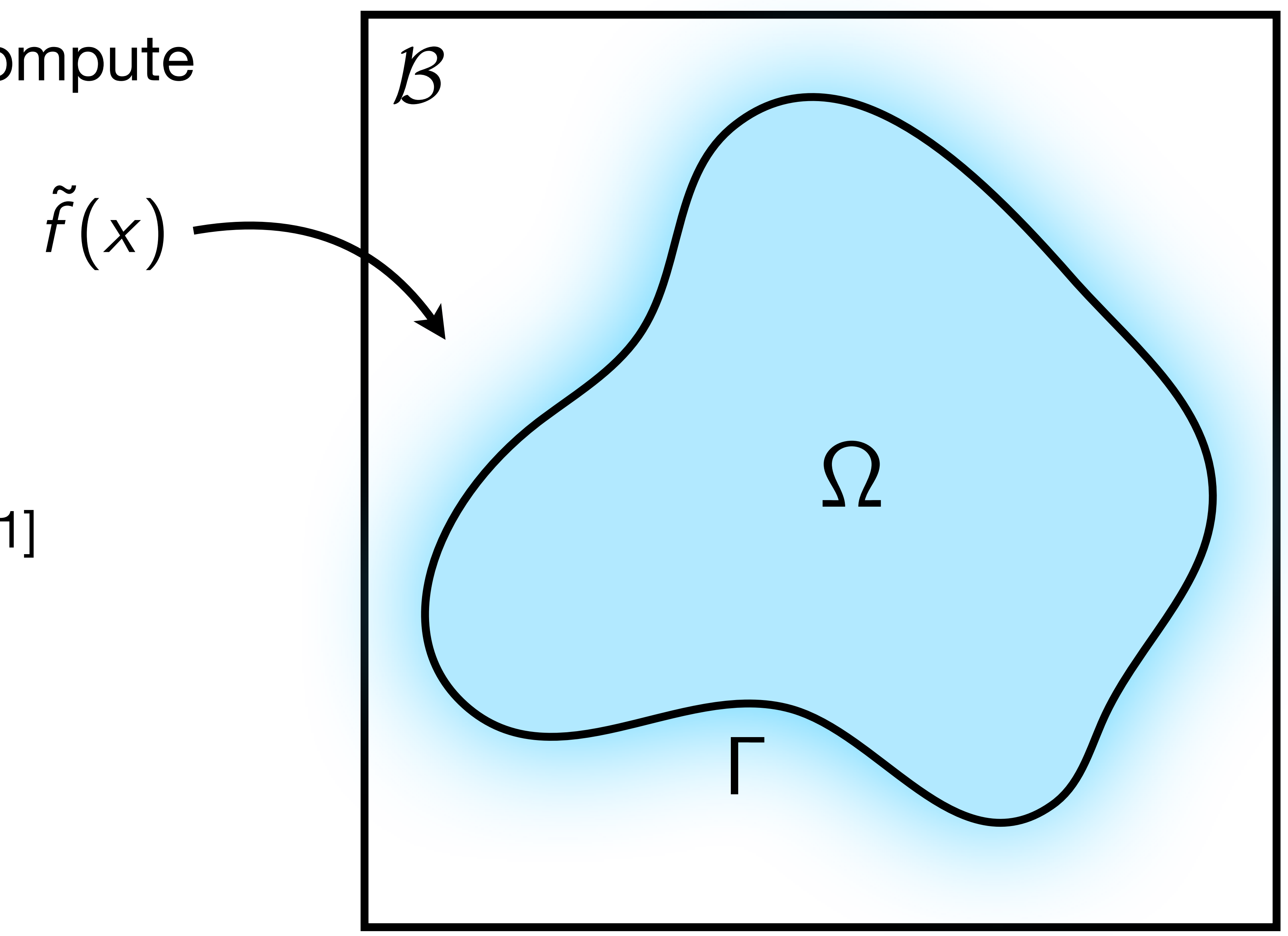


- Extend f to $\tilde{f}$ outside Ω ("function extension"). Adaptively resolve $\tilde{f}$ and compute

$$v(x) = \int_{\mathcal{B}} \Phi(x,y)\, \tilde{f}(y)\, dy$$

- Quadrature tables for boxes can be **precomputed**.
- Apply FMM to adaptive quadtree of $\tilde{f}$ ("box code"). [Ethridge & Greengard, 2001]
- Want v as smooth as u so no effort is wasted.



Function extension

Prior work

- Finite difference extension, second-order accurate [Mayo, 1984]
- Fourier continuation [Bruno & Lyon, 2010], [Bruno & Paul, 2020]
- Immersed boundary smooth extension [Stein, Guy, & Thomases, 2015]
- C^k polyharmonic extension + box code, fourth-order accurate [Askham & Cerfon, 2017]
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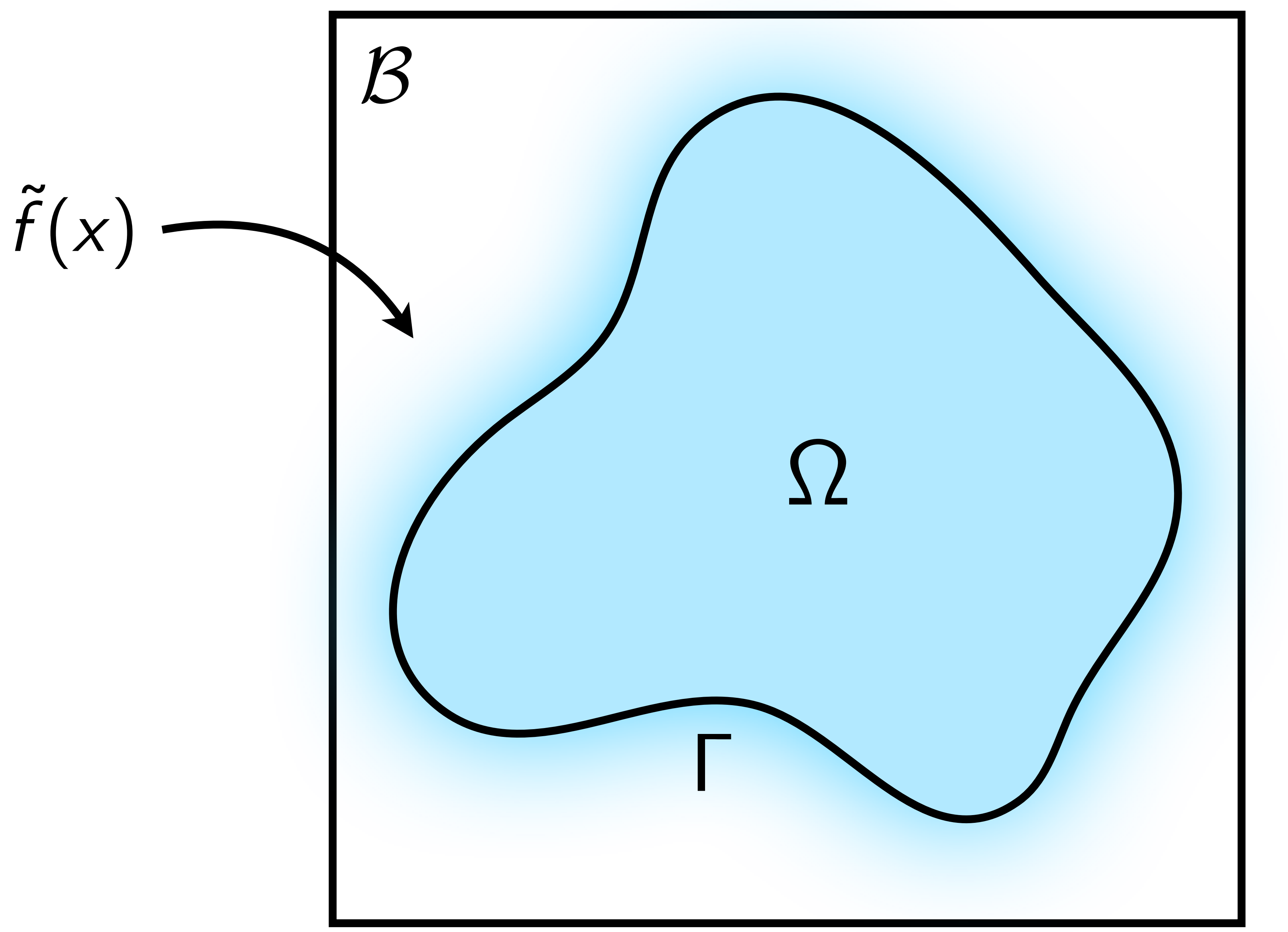

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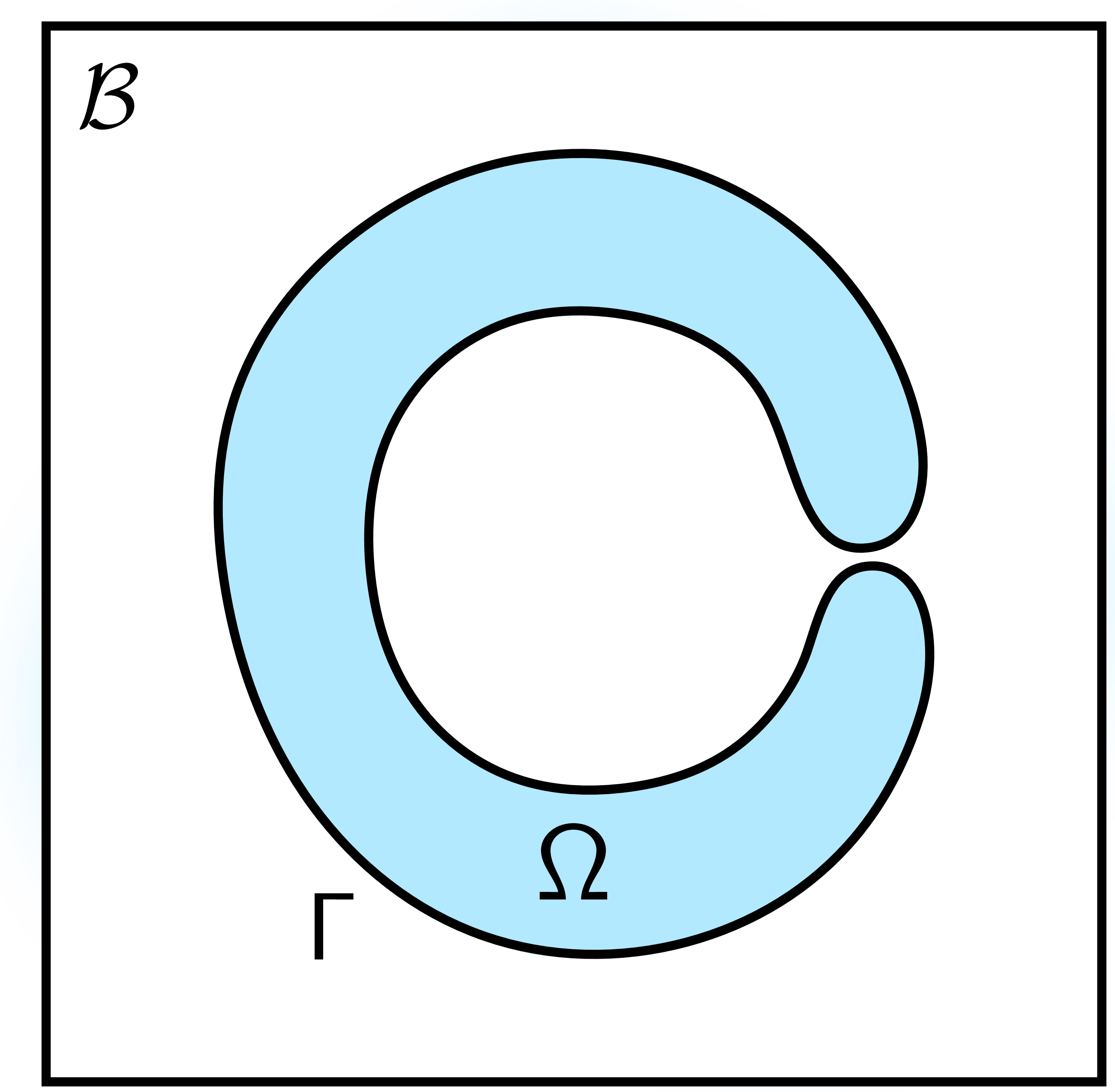

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*Smoothly extending functions in a **robust** way is **challenging**!*

- multiscale geometry
- multiscale data
- close-to-touching regions

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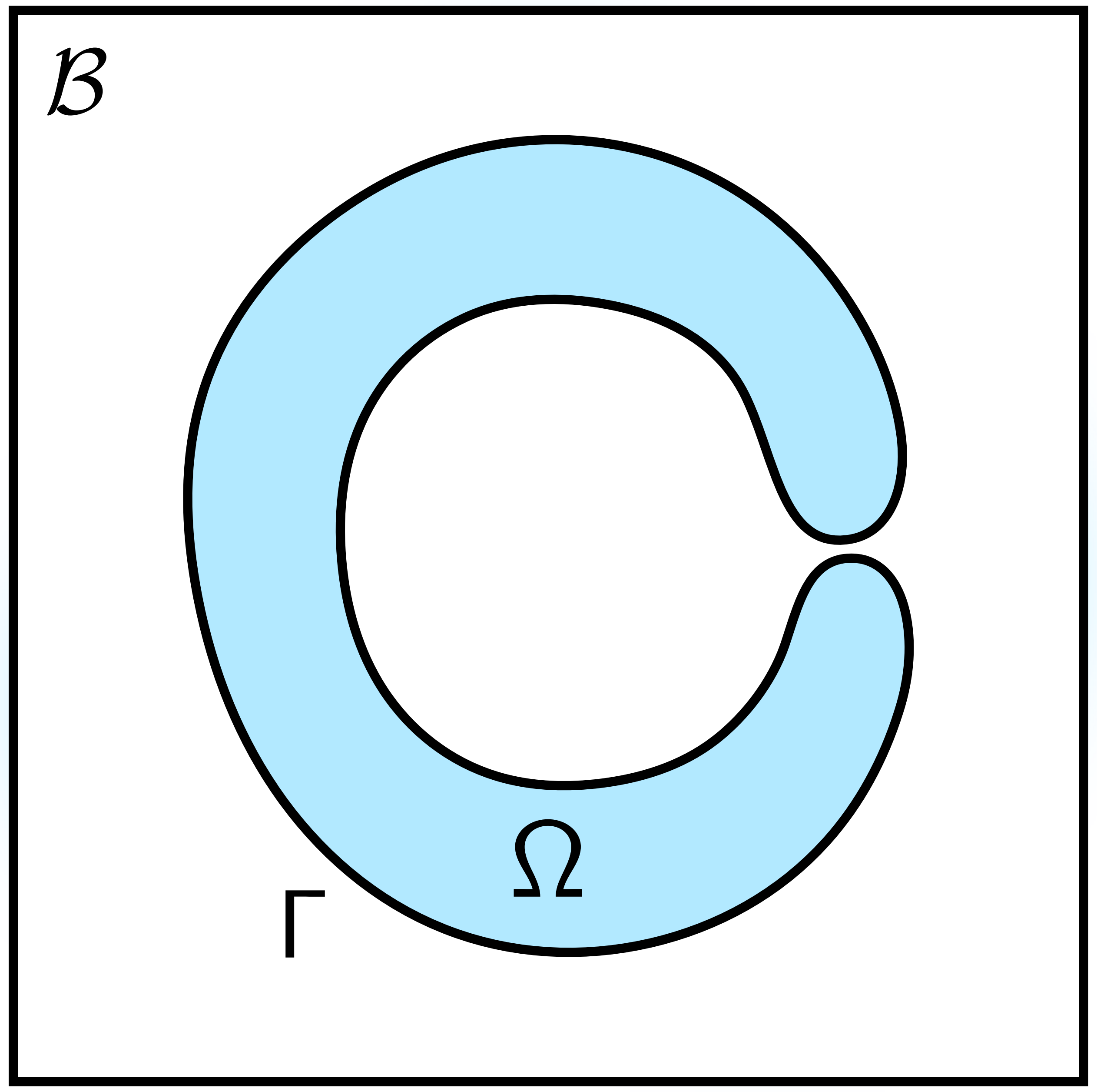

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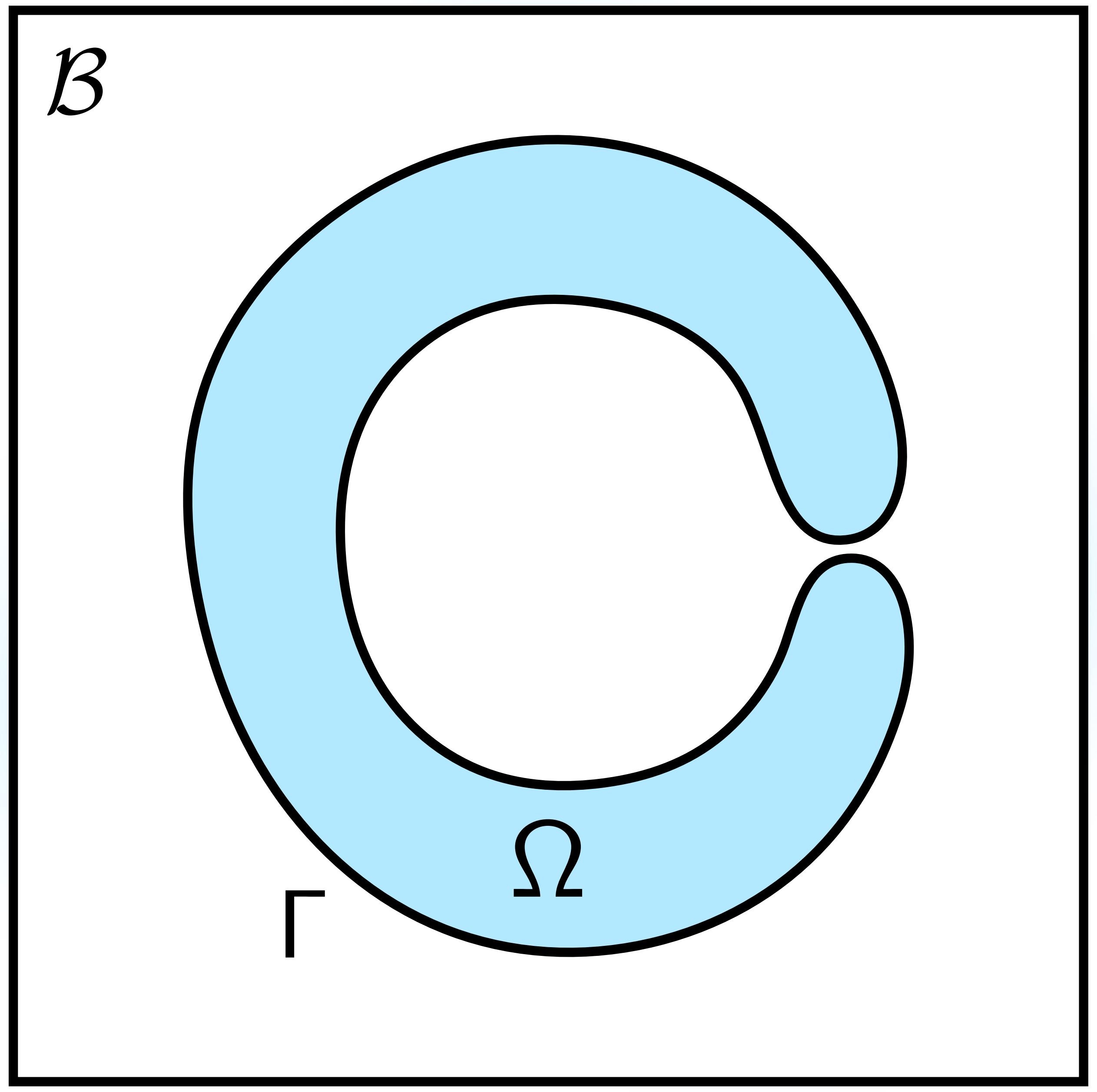

Q: What if we truncate instead of extend?

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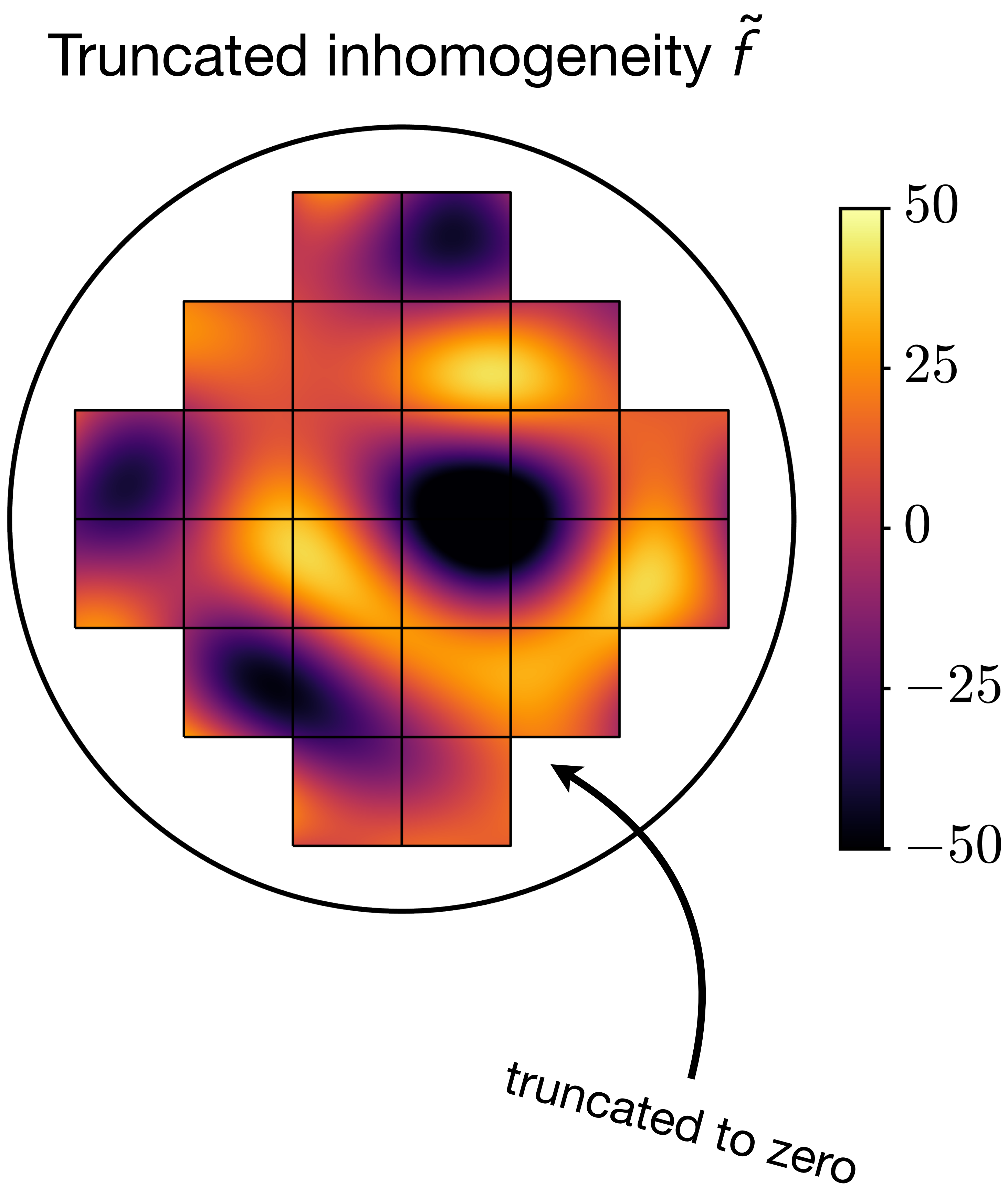
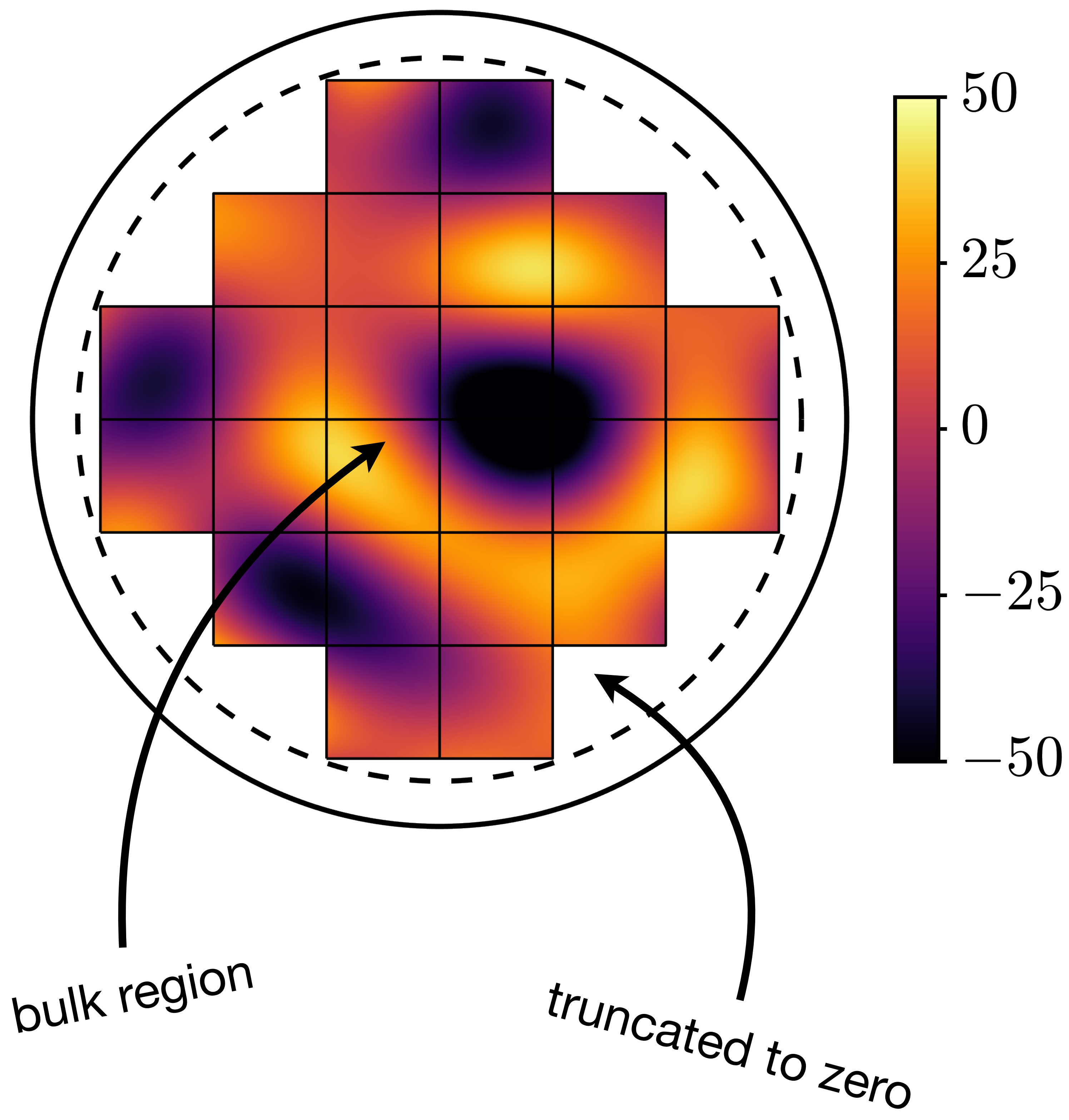


Potential theory

Truncating a volume potential

$$v(x) = \int_{\mathcal{B}} \Phi(x, y)\, \tilde{f}(y)\, dy$$

Truncated inhomogeneity $\tilde{f}$



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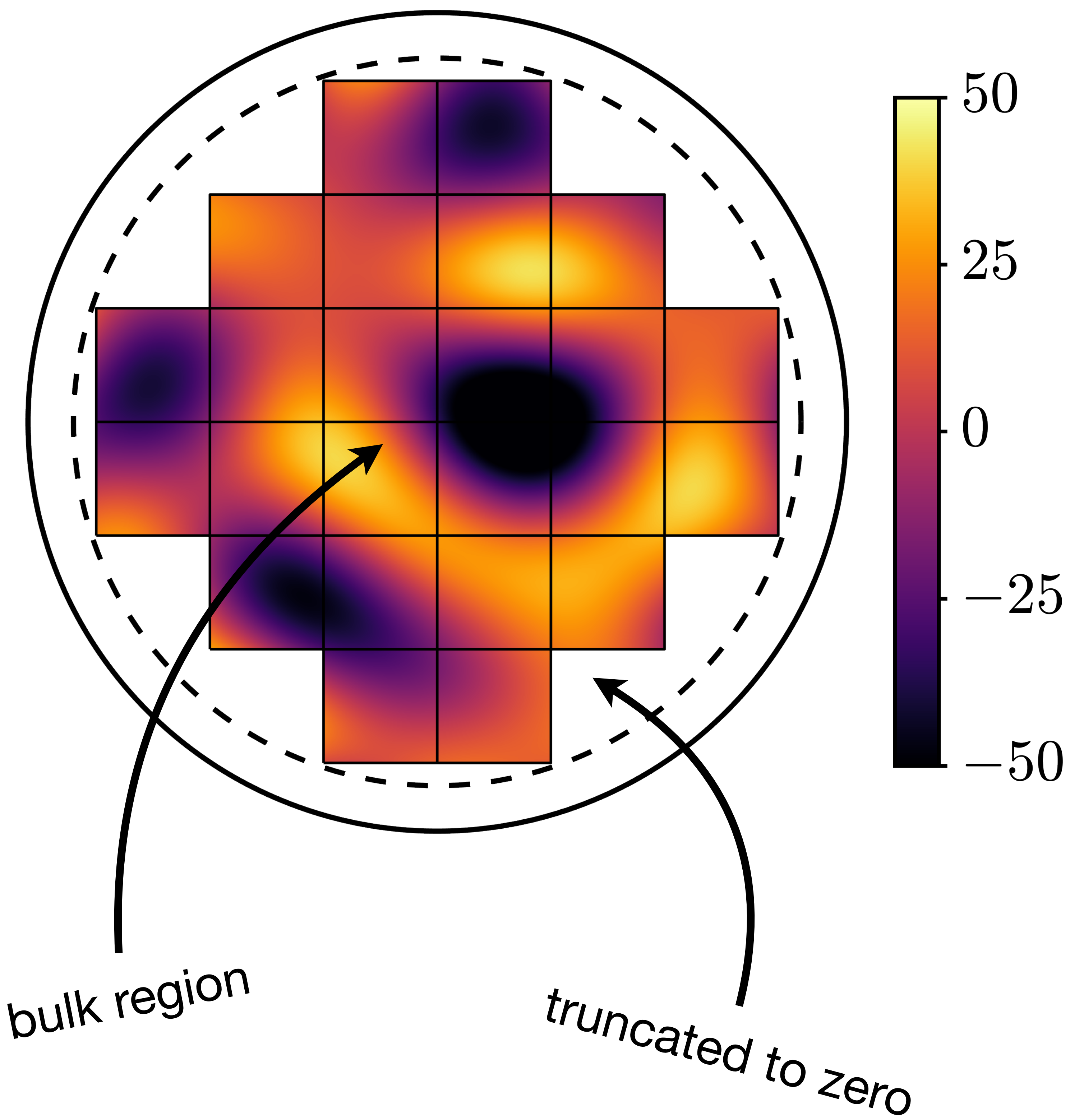

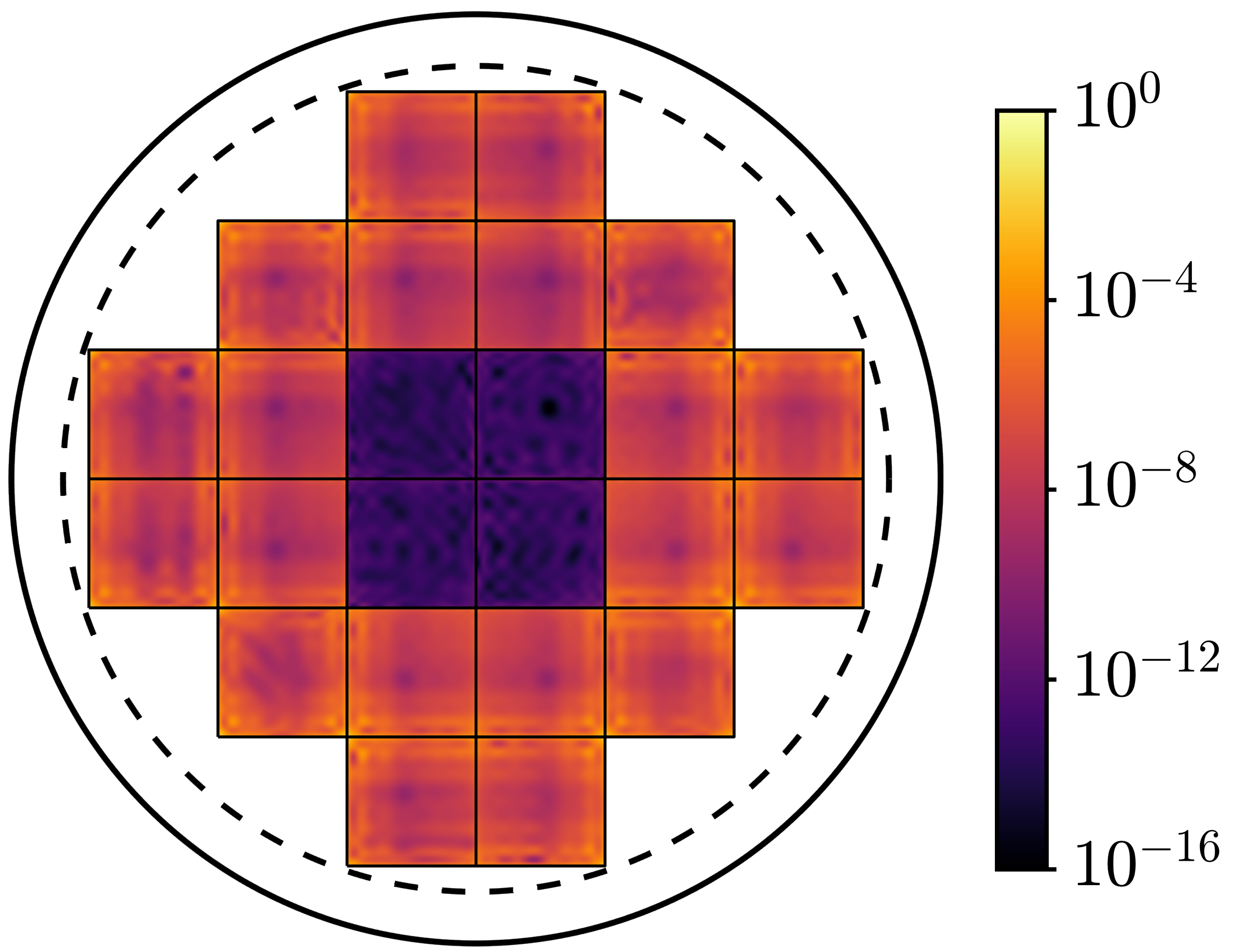

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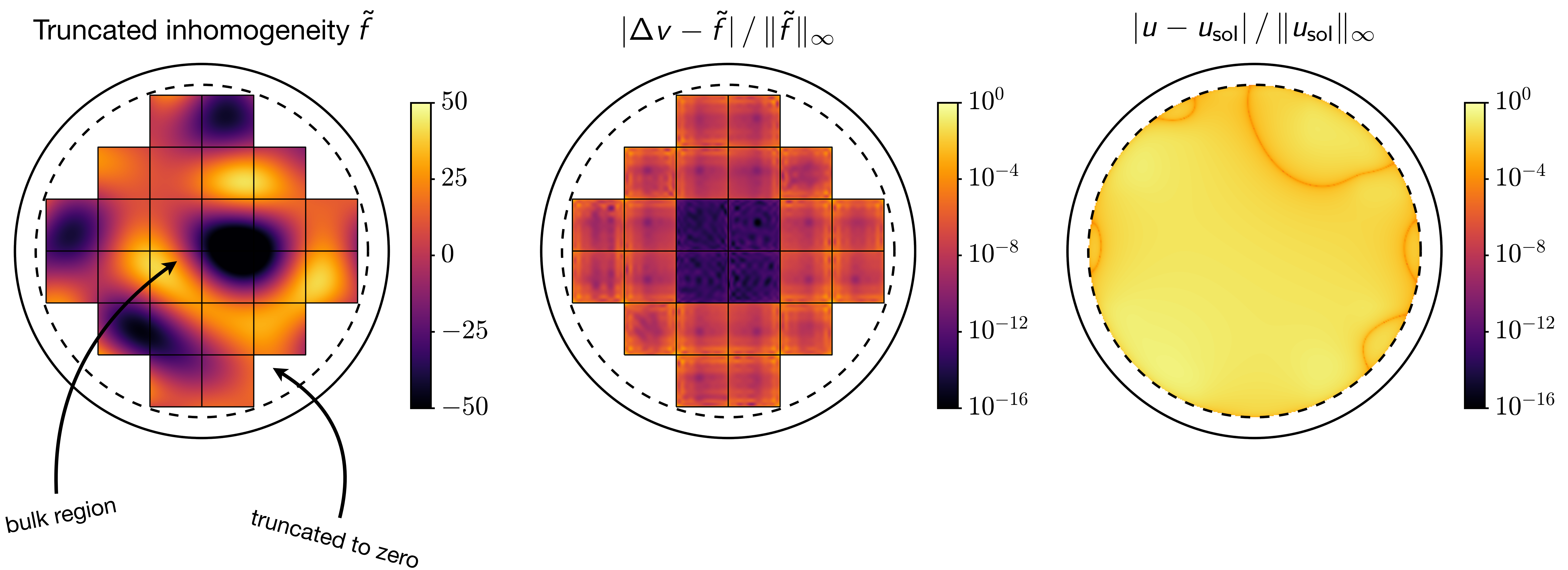

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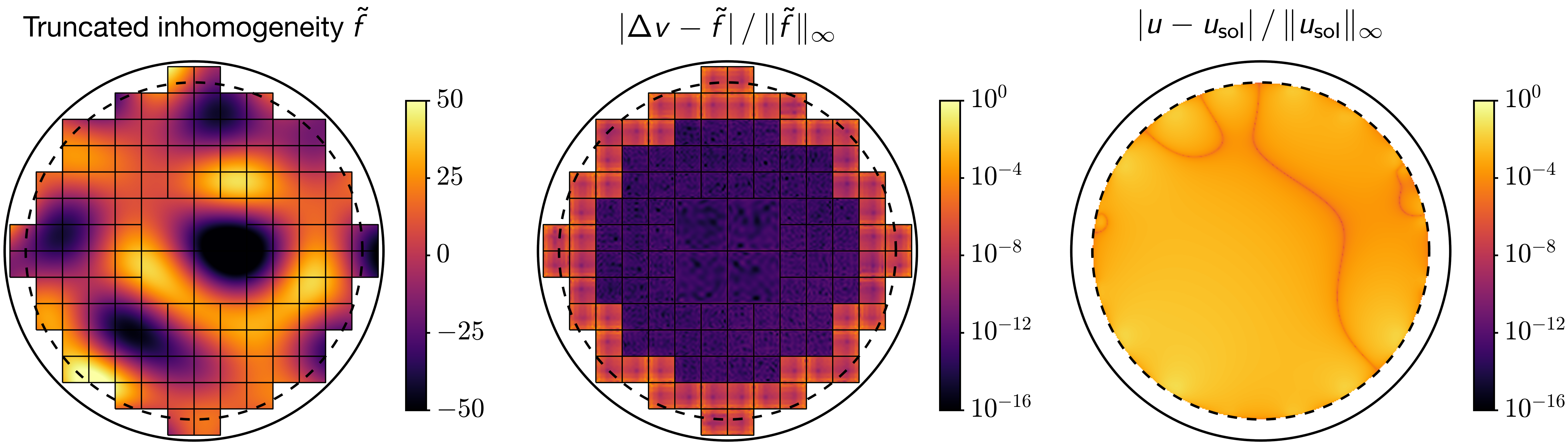

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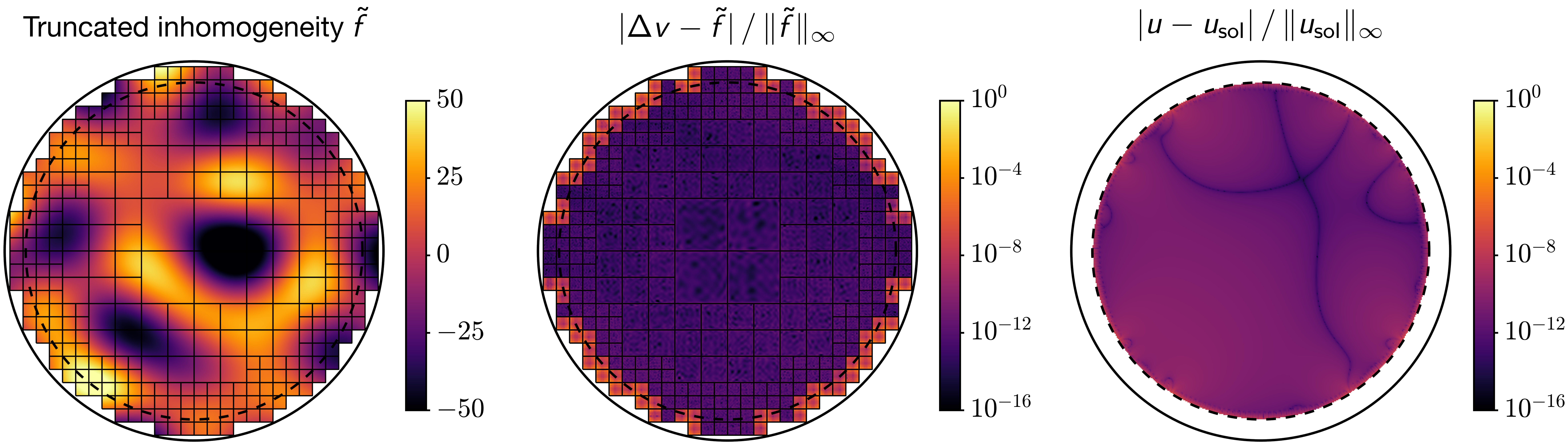

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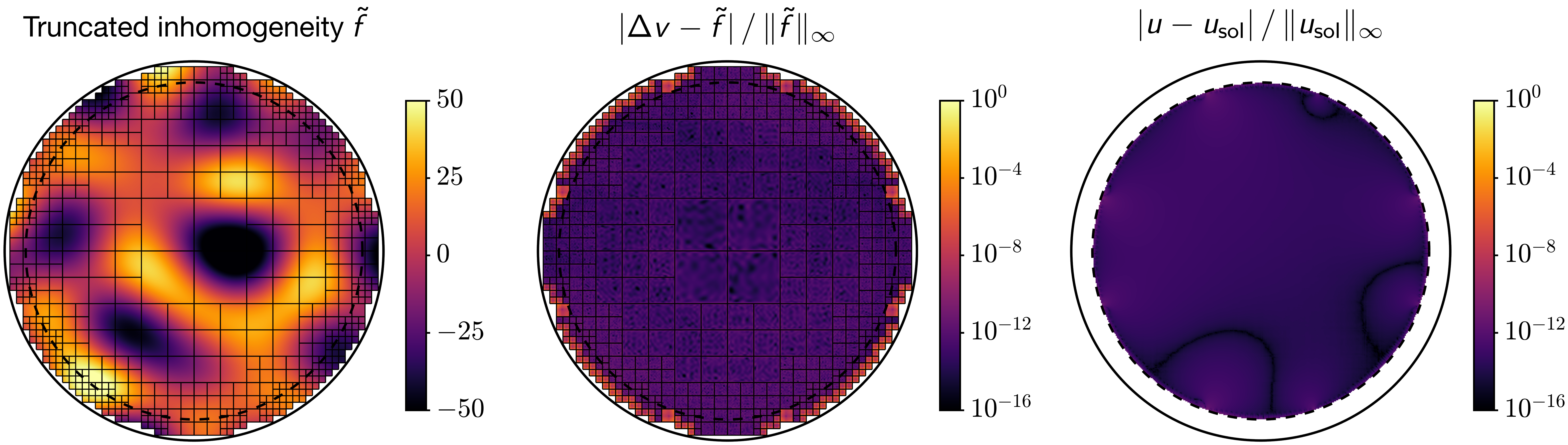

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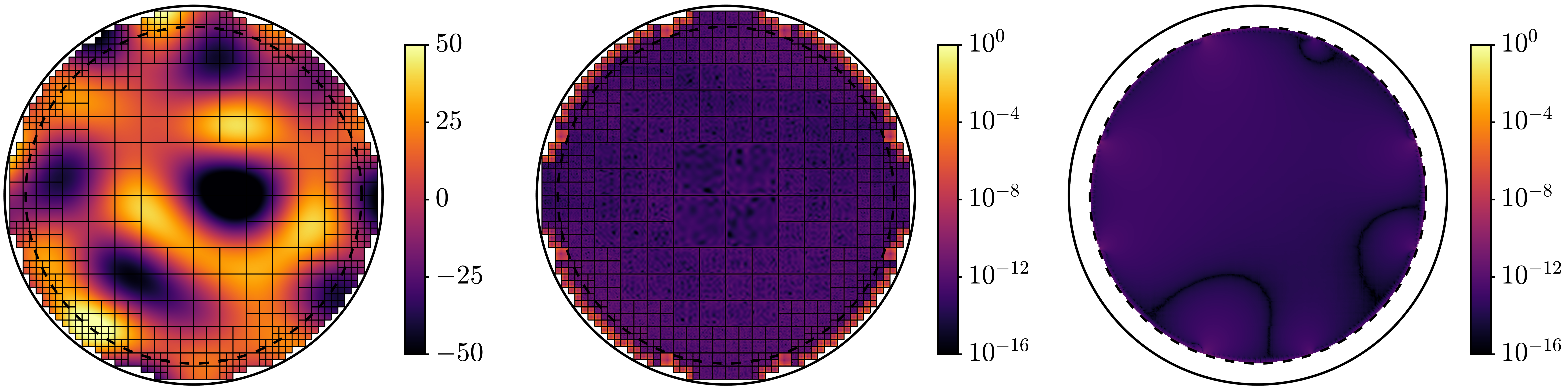
Truncated inhomogeneity $\tilde{f}$ | $|\Delta v - \tilde{f}| \,/\, \|\tilde{f}\|_\infty$ | $|u - u_{\mathrm{sol}}| \,/\, \|u_{\mathrm{sol}}\|_\infty$

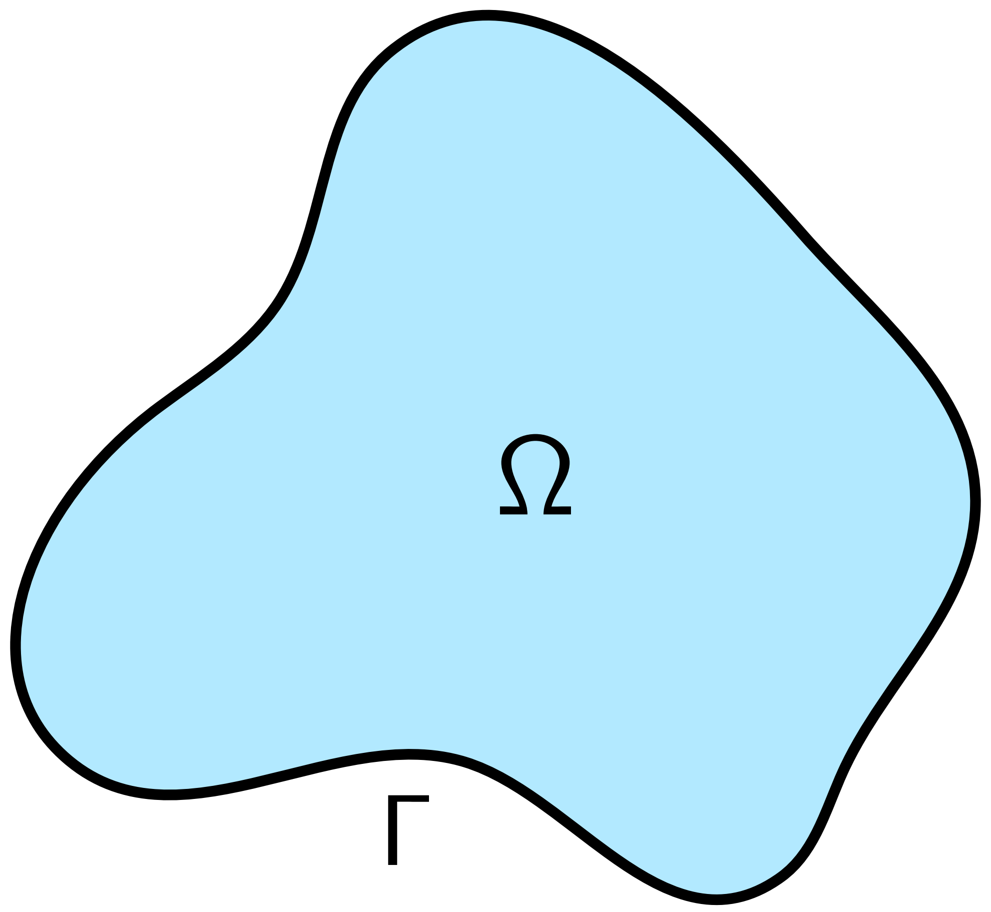


Theorem (F., Stein, & Barnett)

Let the truncated region $\operatorname{supp}(f - \tilde{f})$ *be well separated from each bulk box. Then the volume potential v induced by $\tilde{f}$ is as smooth on each bulk box as that induced by the original f.*

Inhomogeneous PDEs

Our approach



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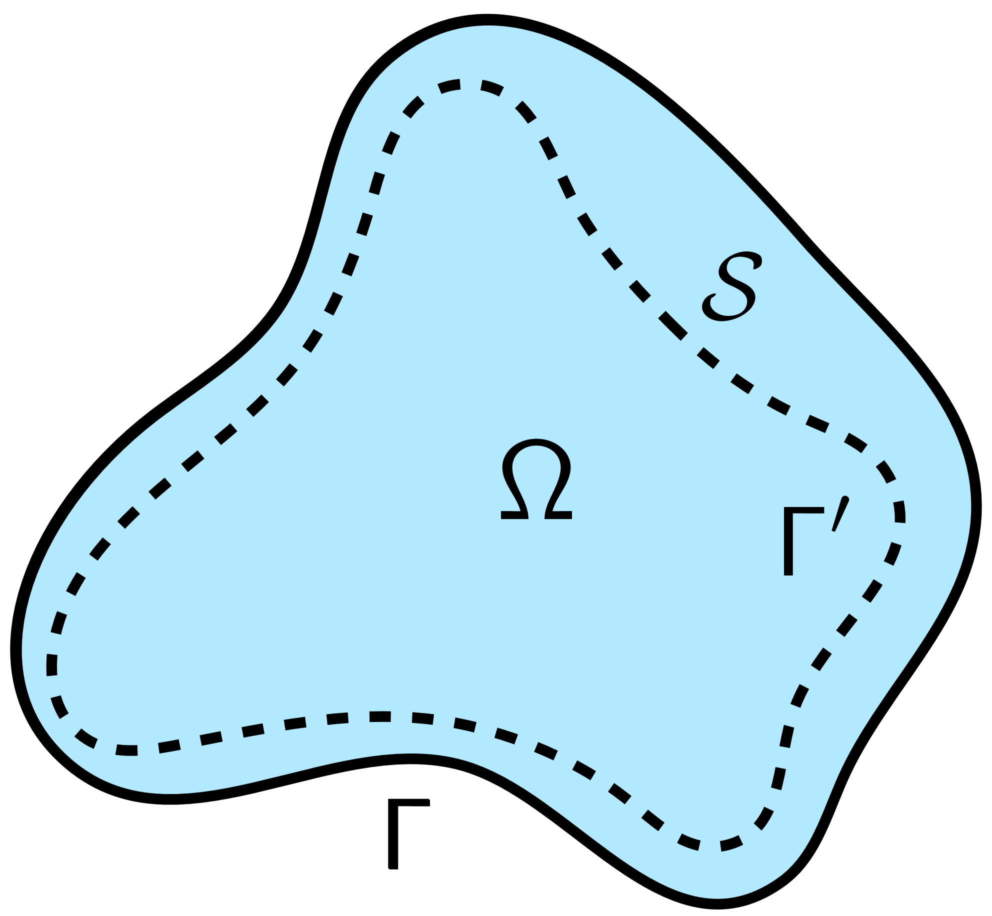

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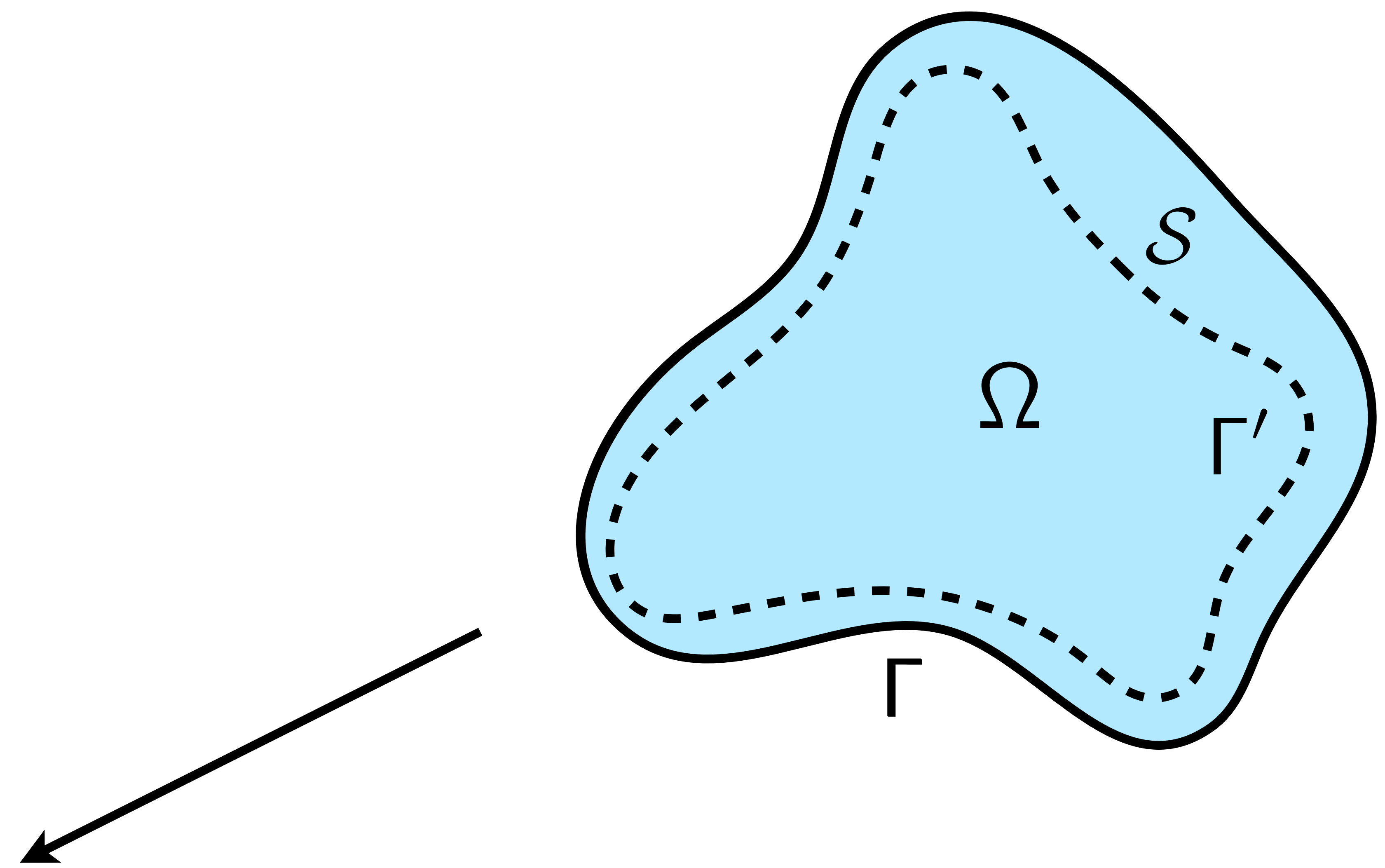

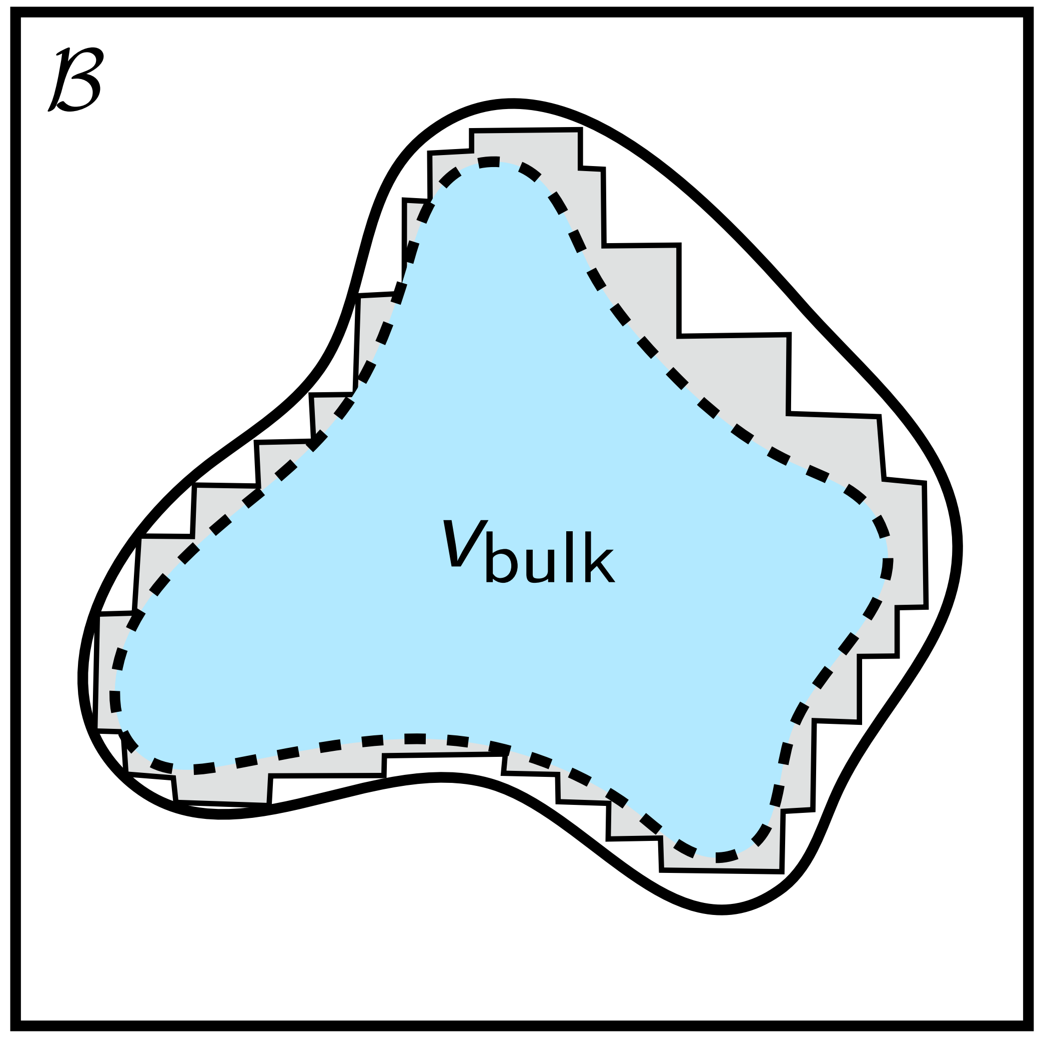

Truncated volume potential

Inhomogeneous PDEs

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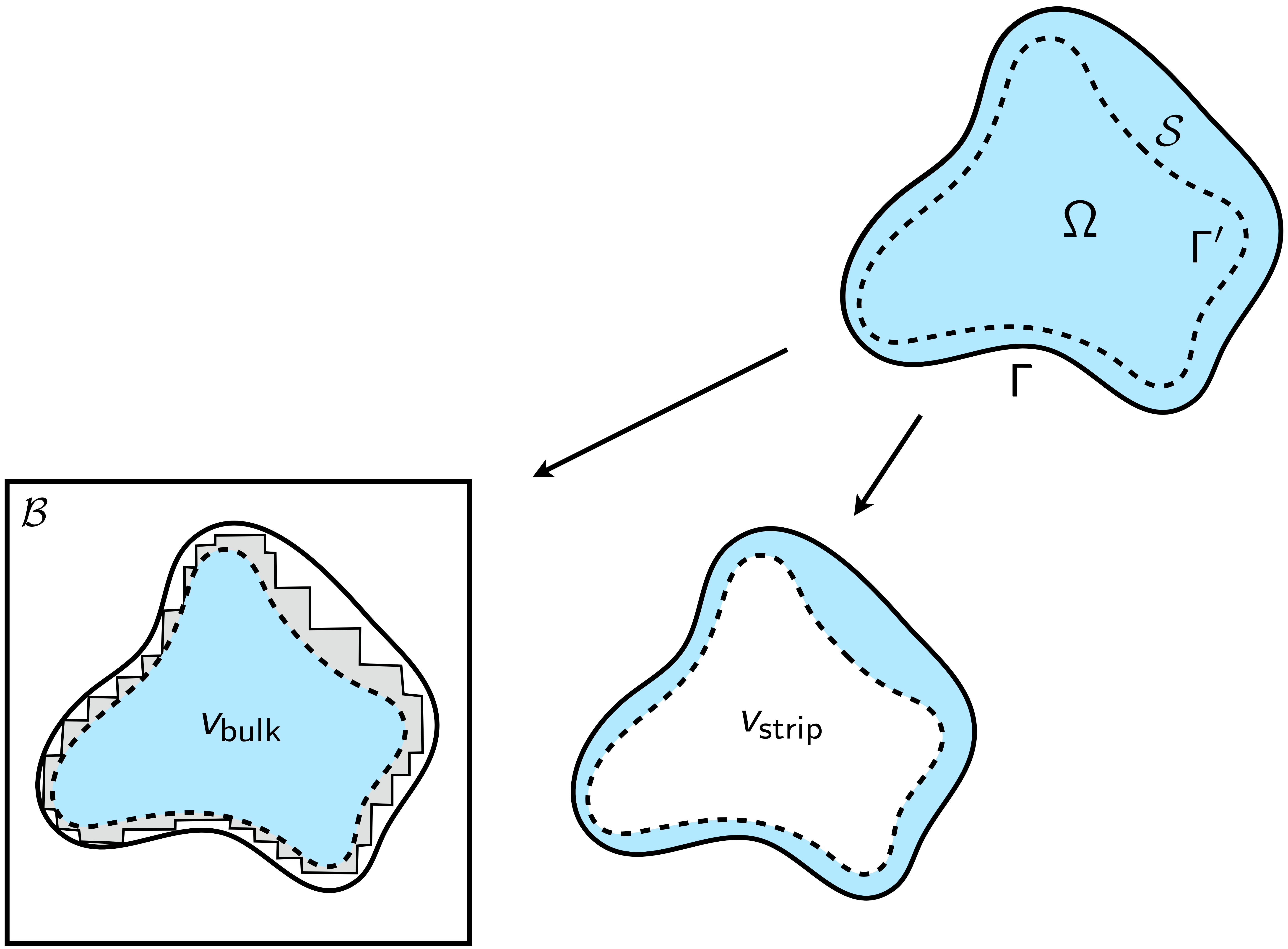

Truncated volume potential

Strip solve

Inhomogeneous PDEs

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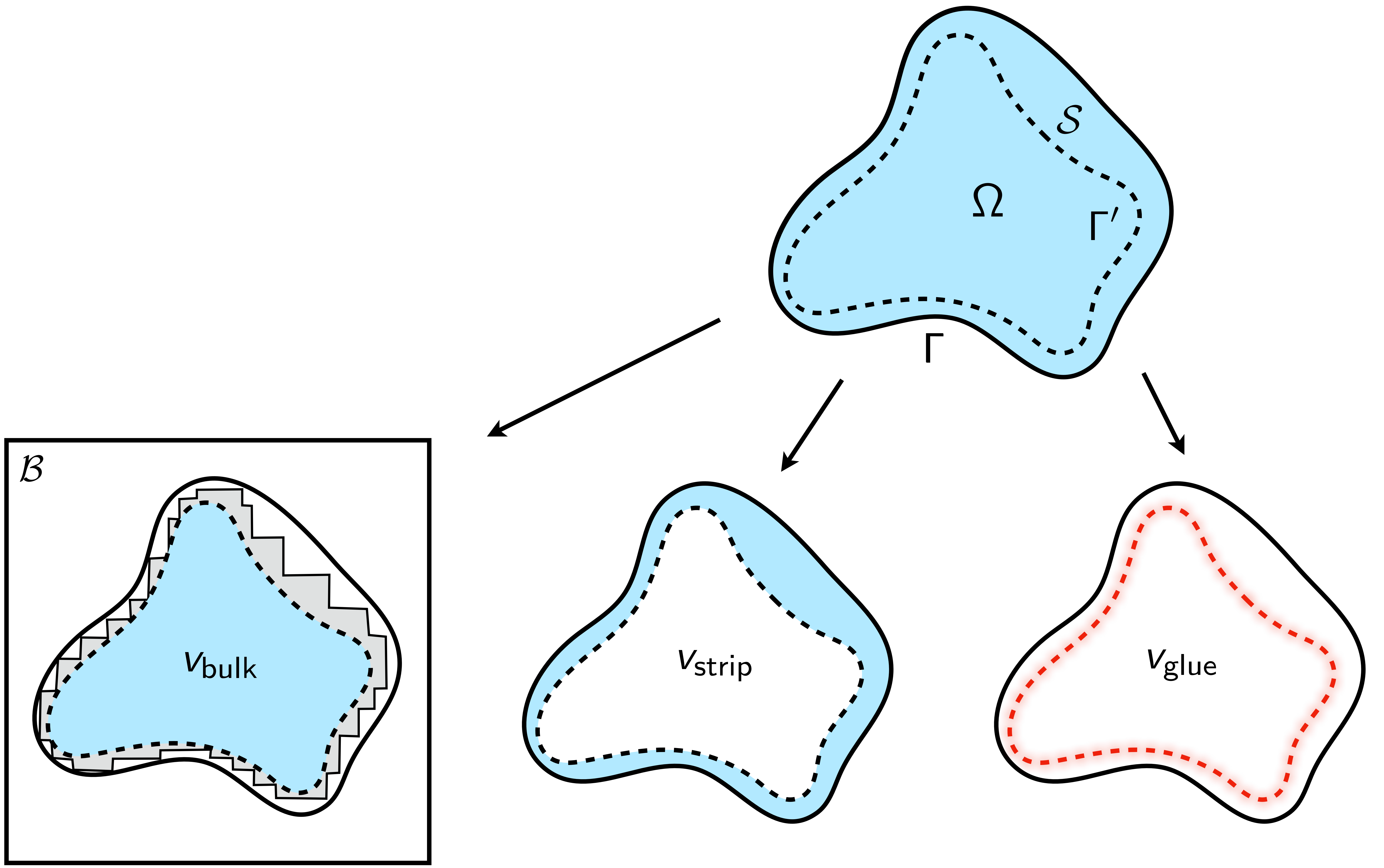

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Layer potentials

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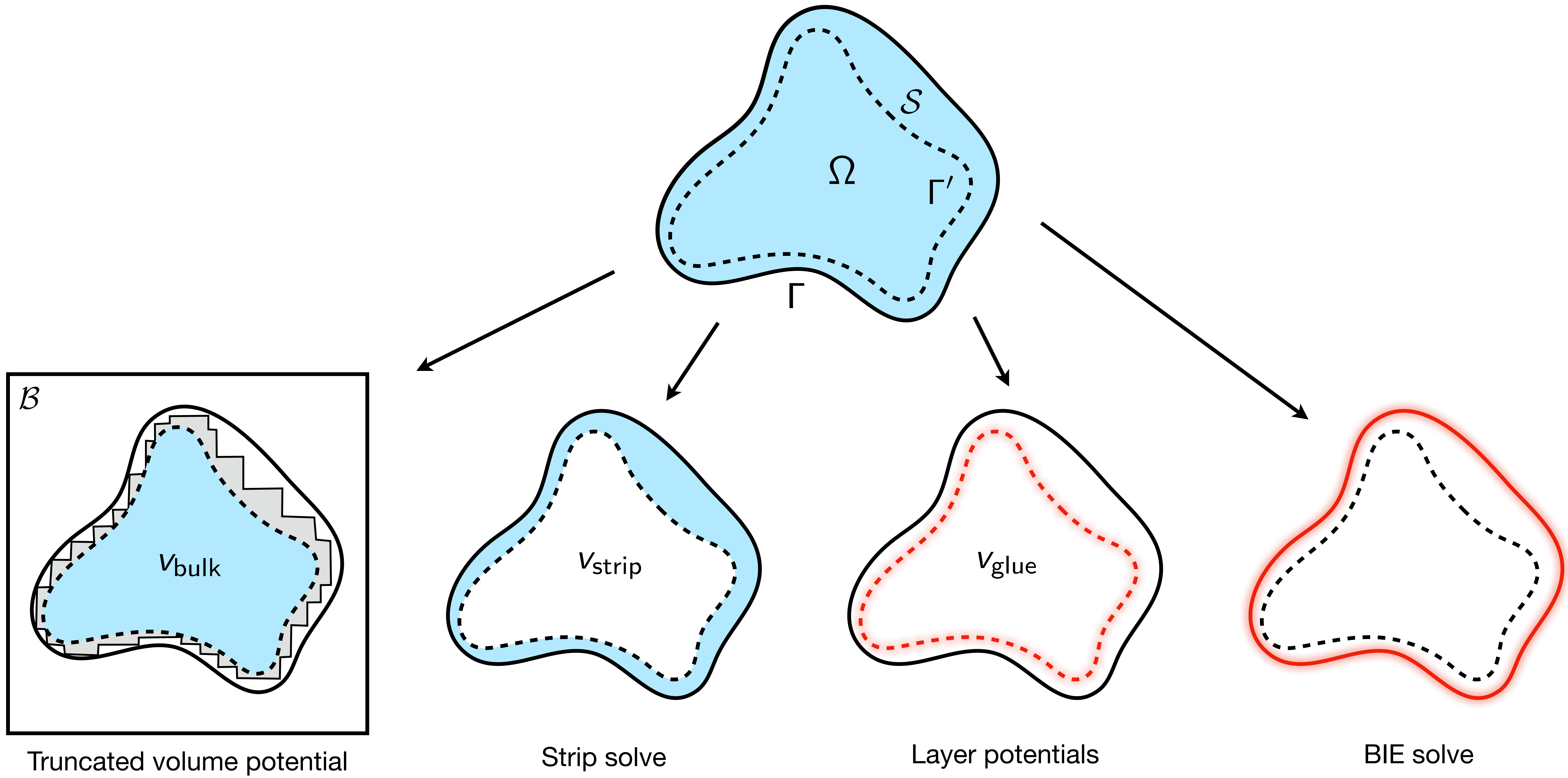

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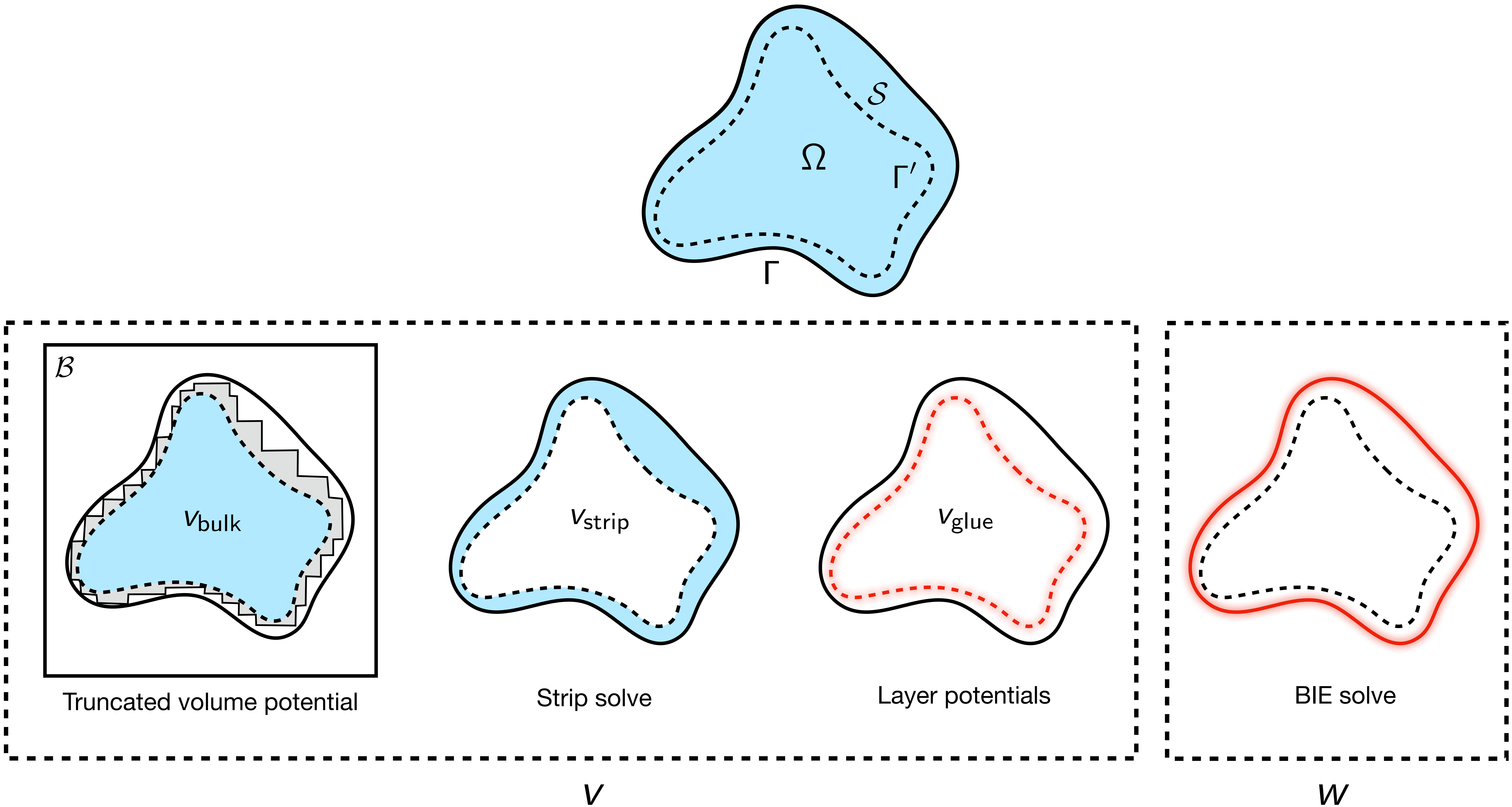
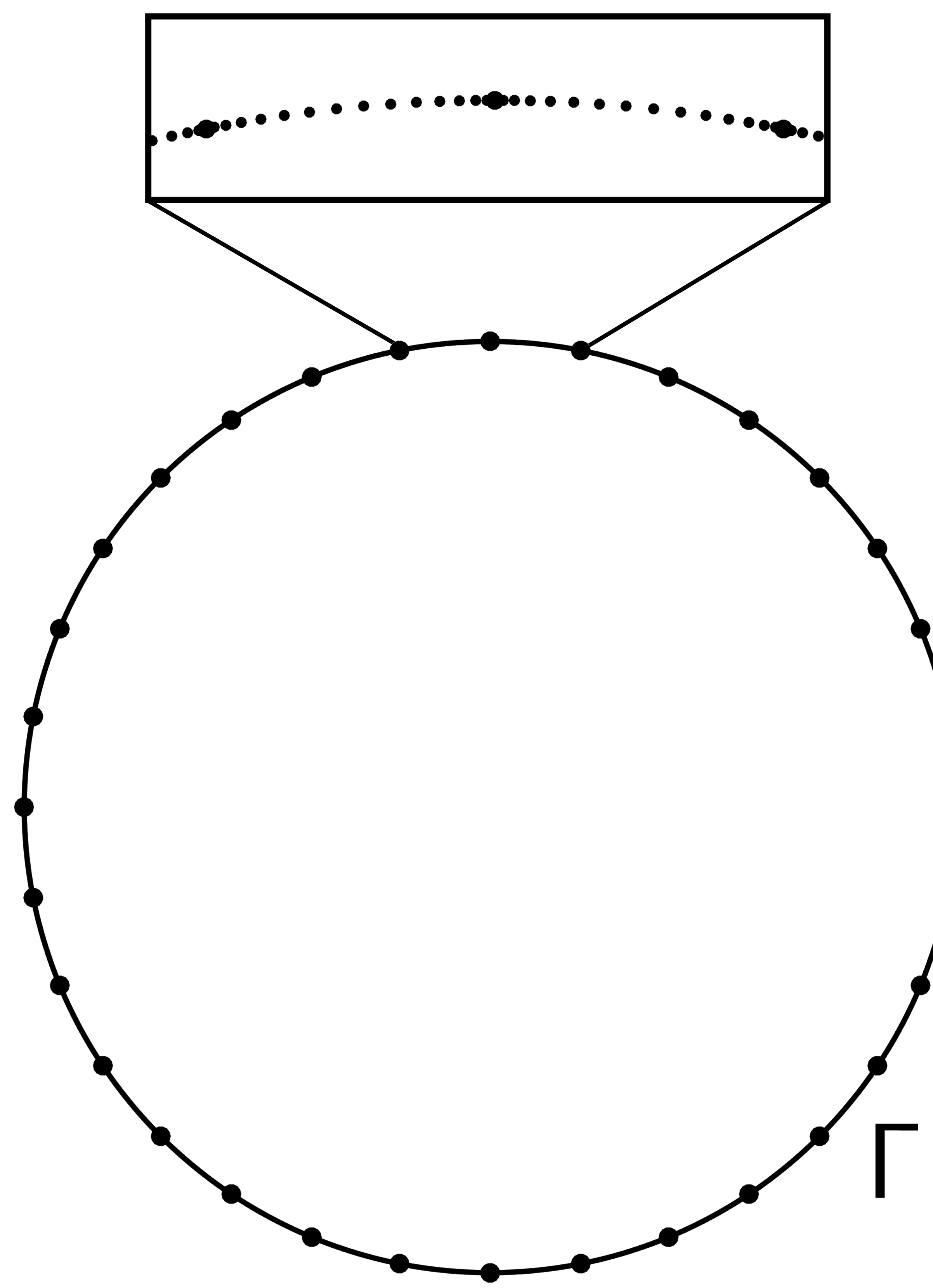


Defining the truncation region

What happens on an adaptive geometry?



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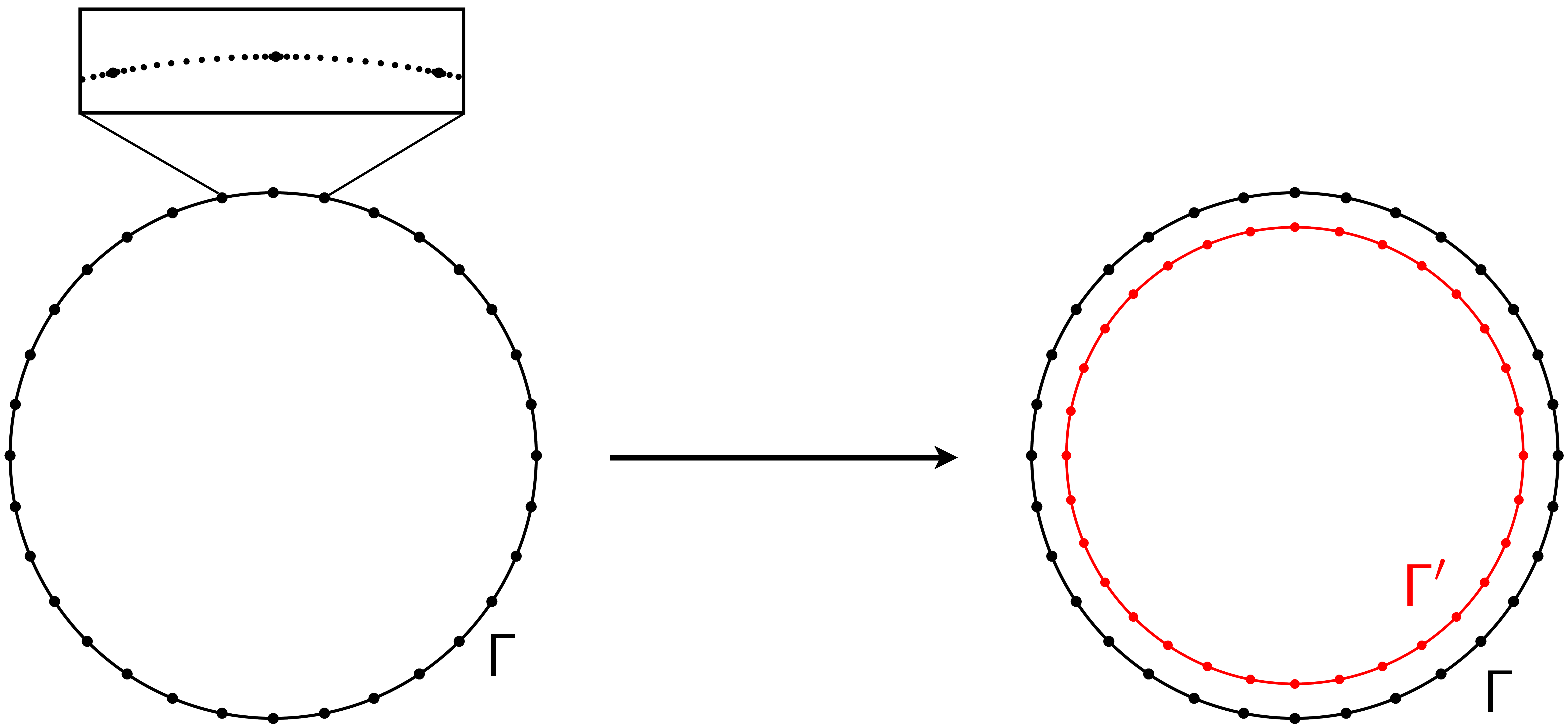

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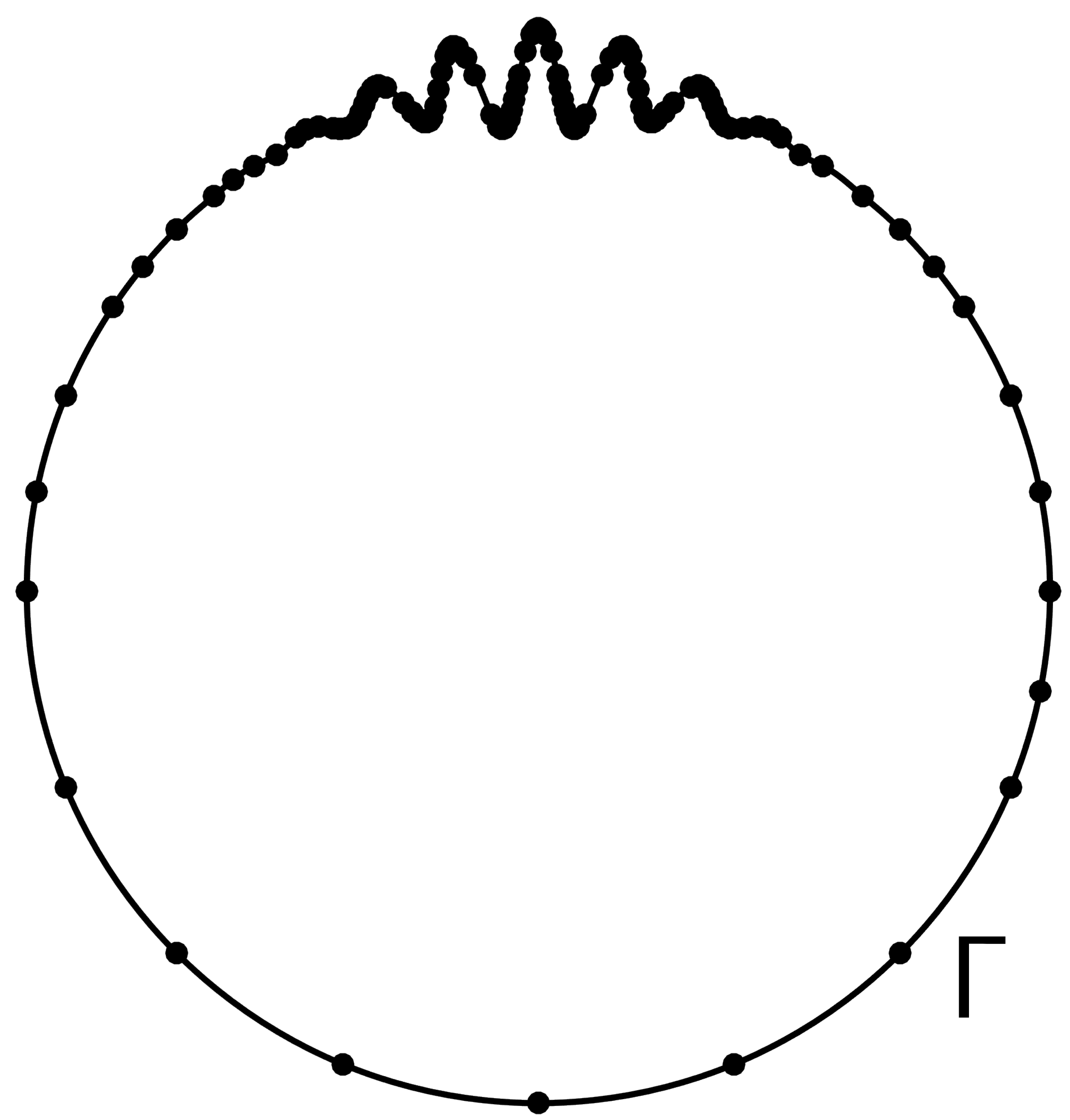

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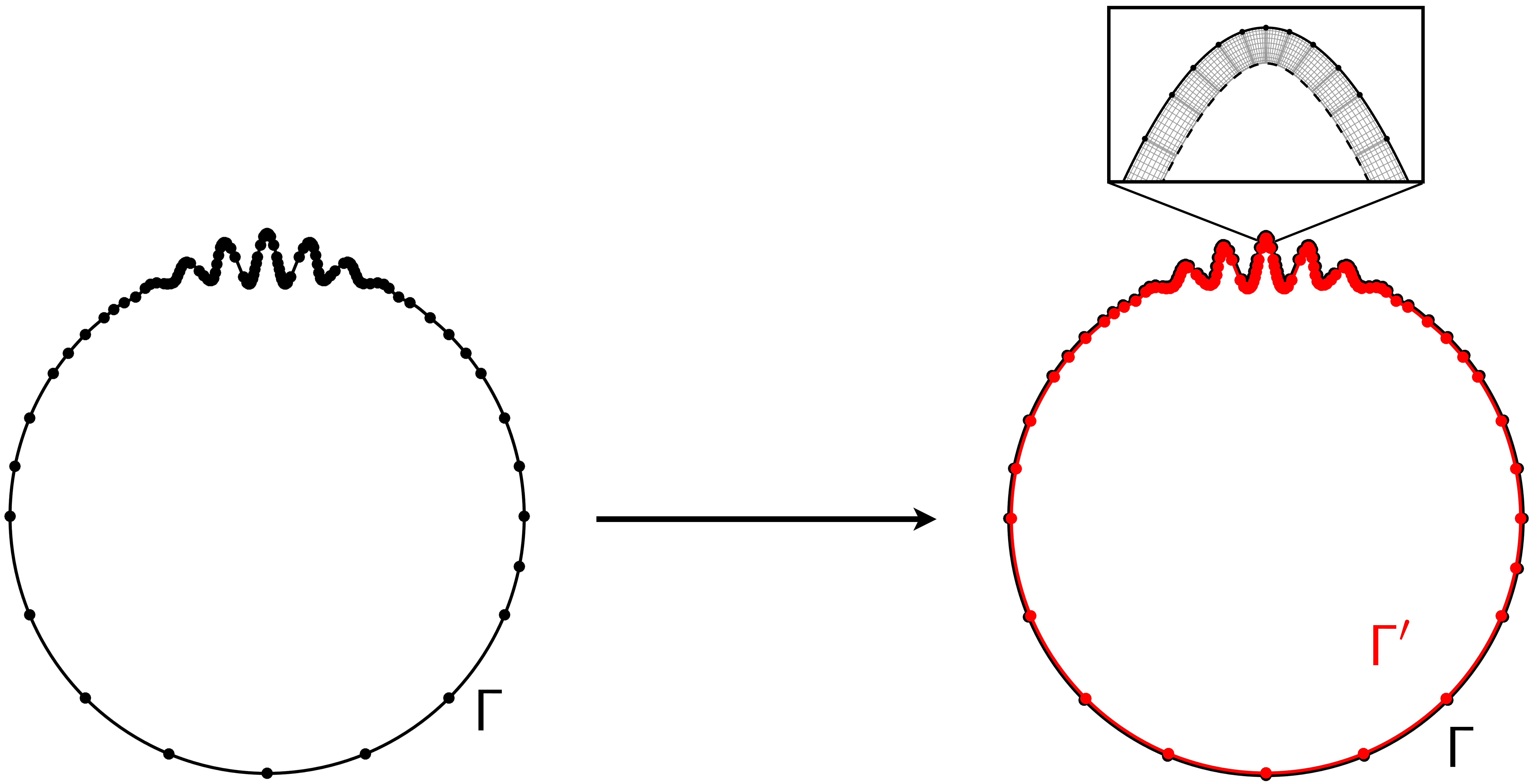

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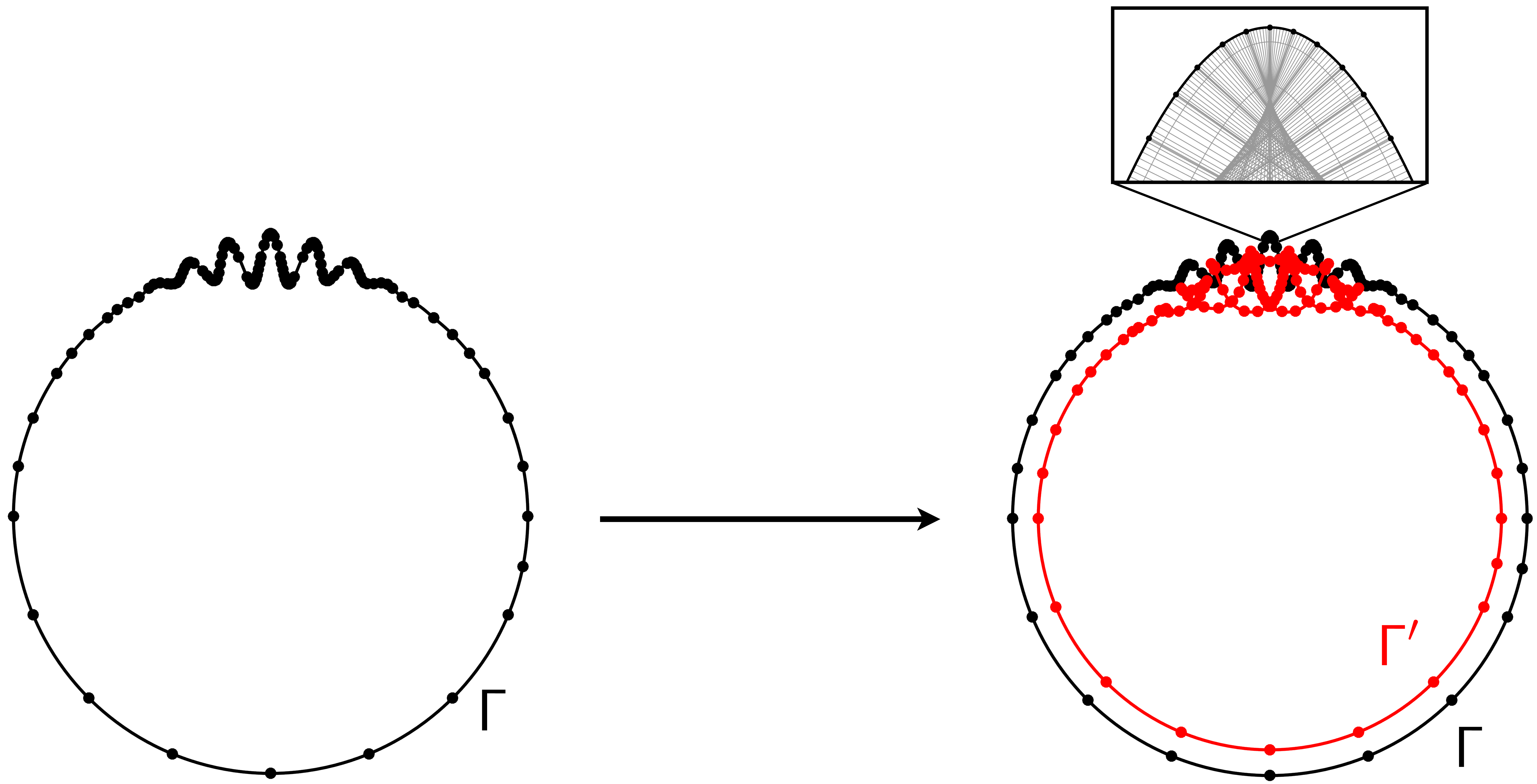

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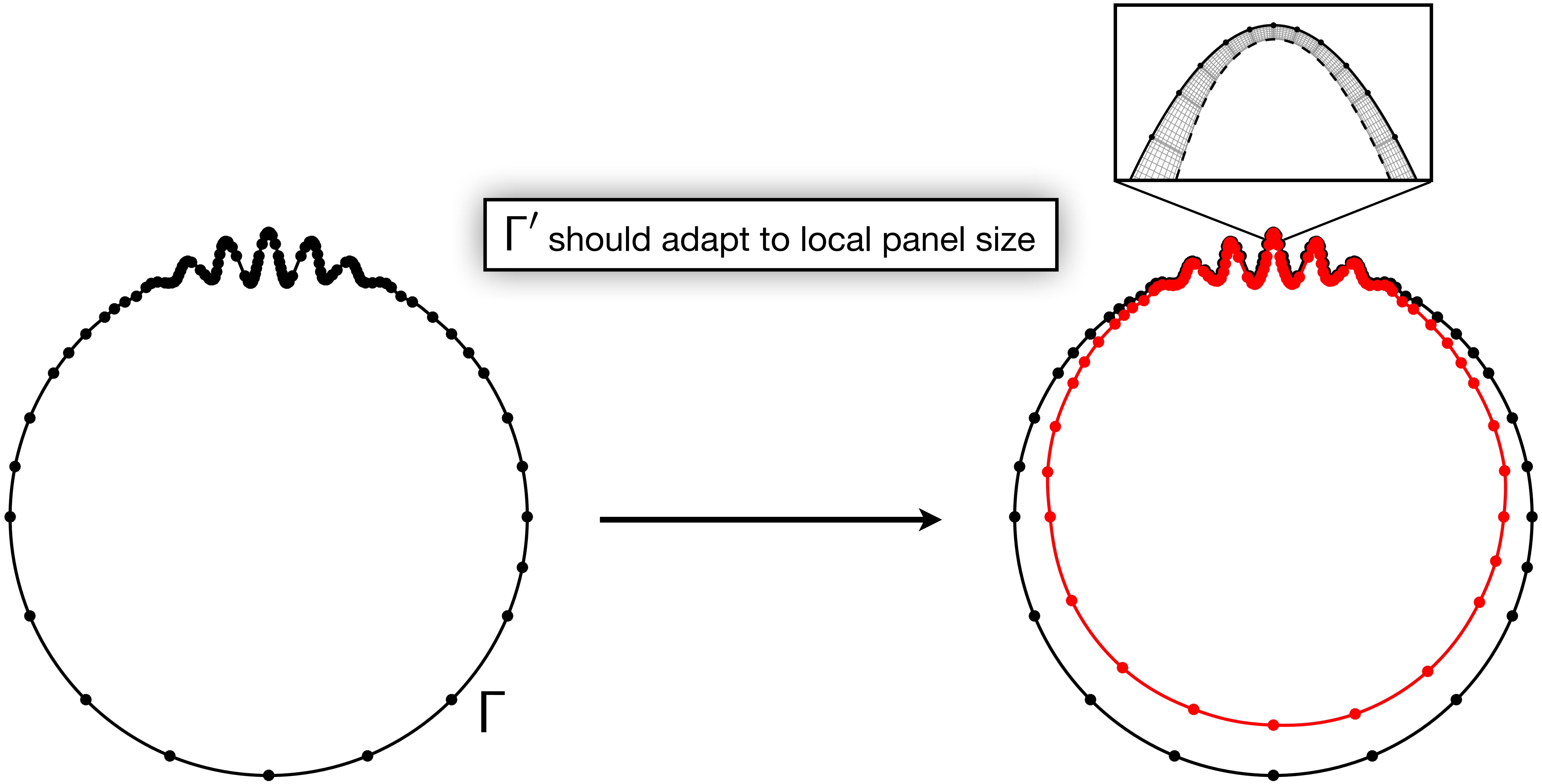

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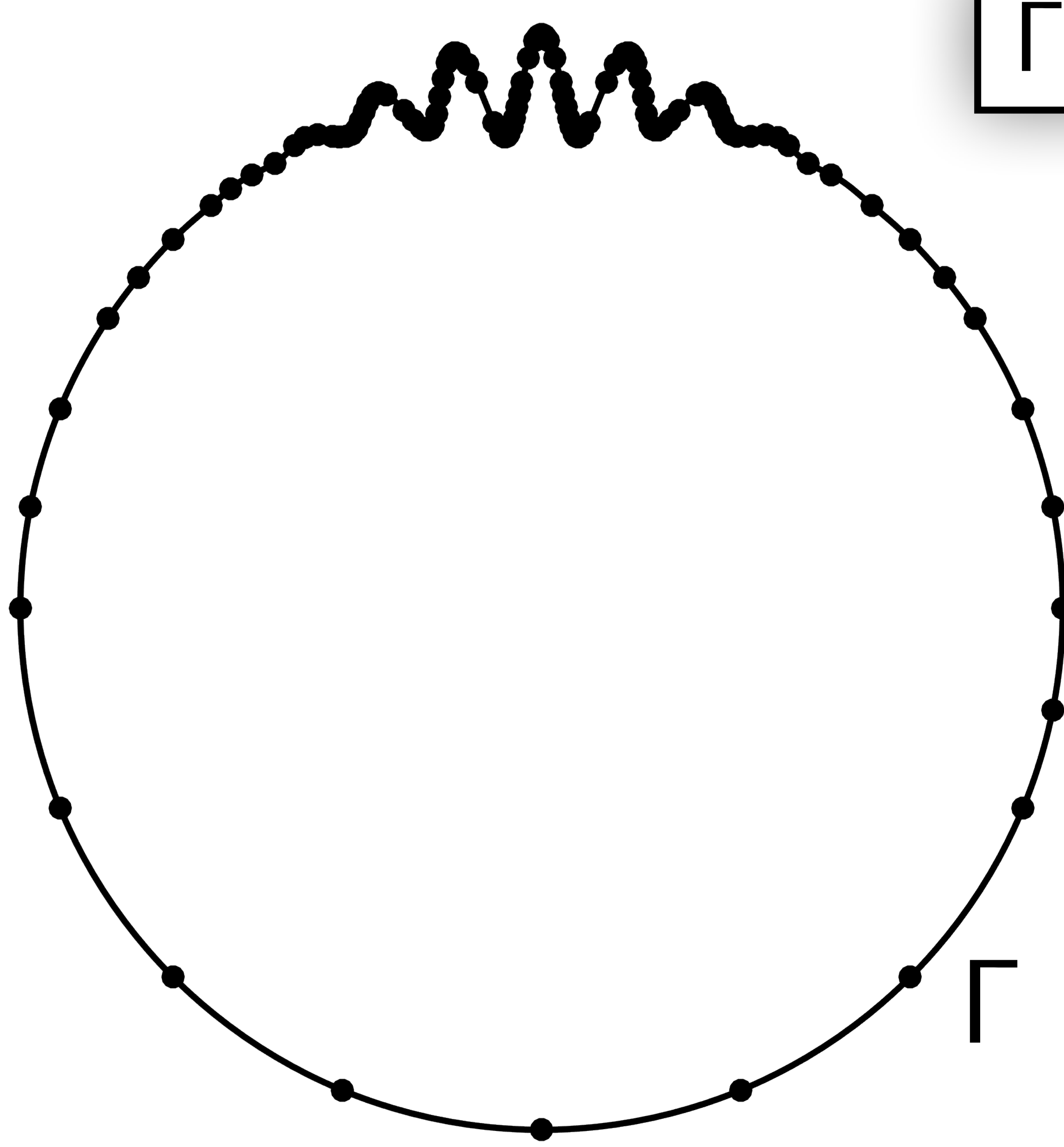
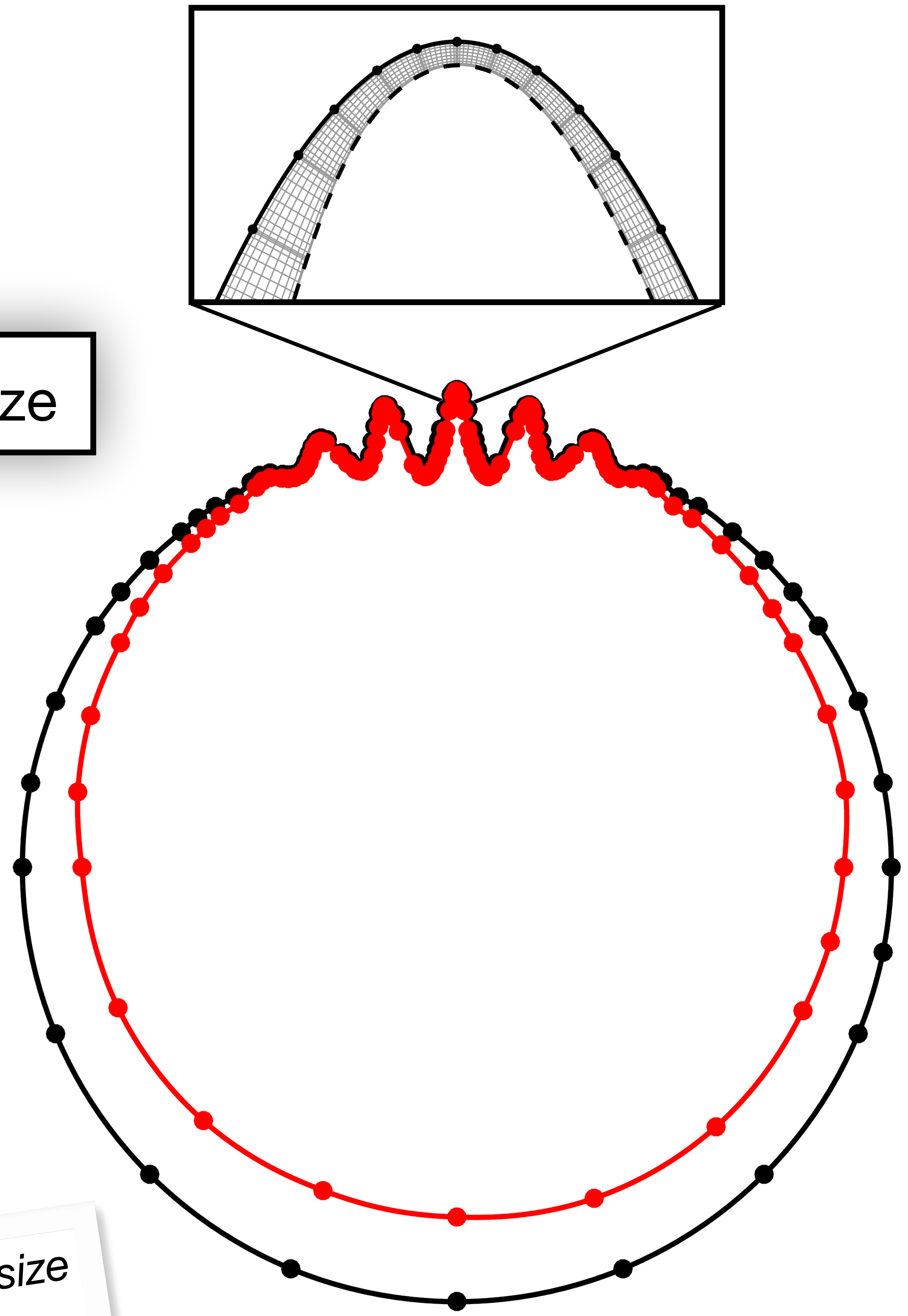




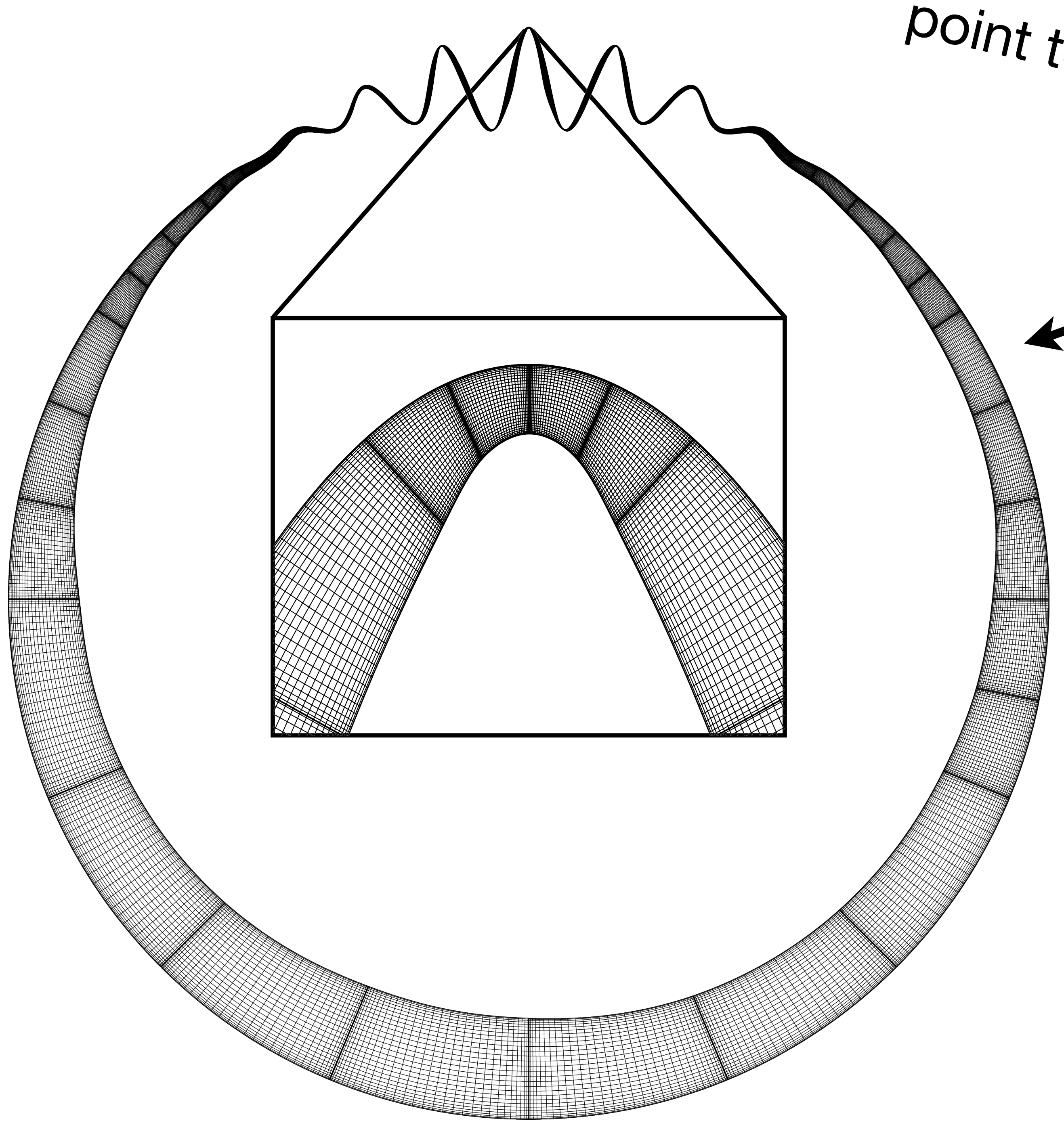
Γ' should adapt to local panel size

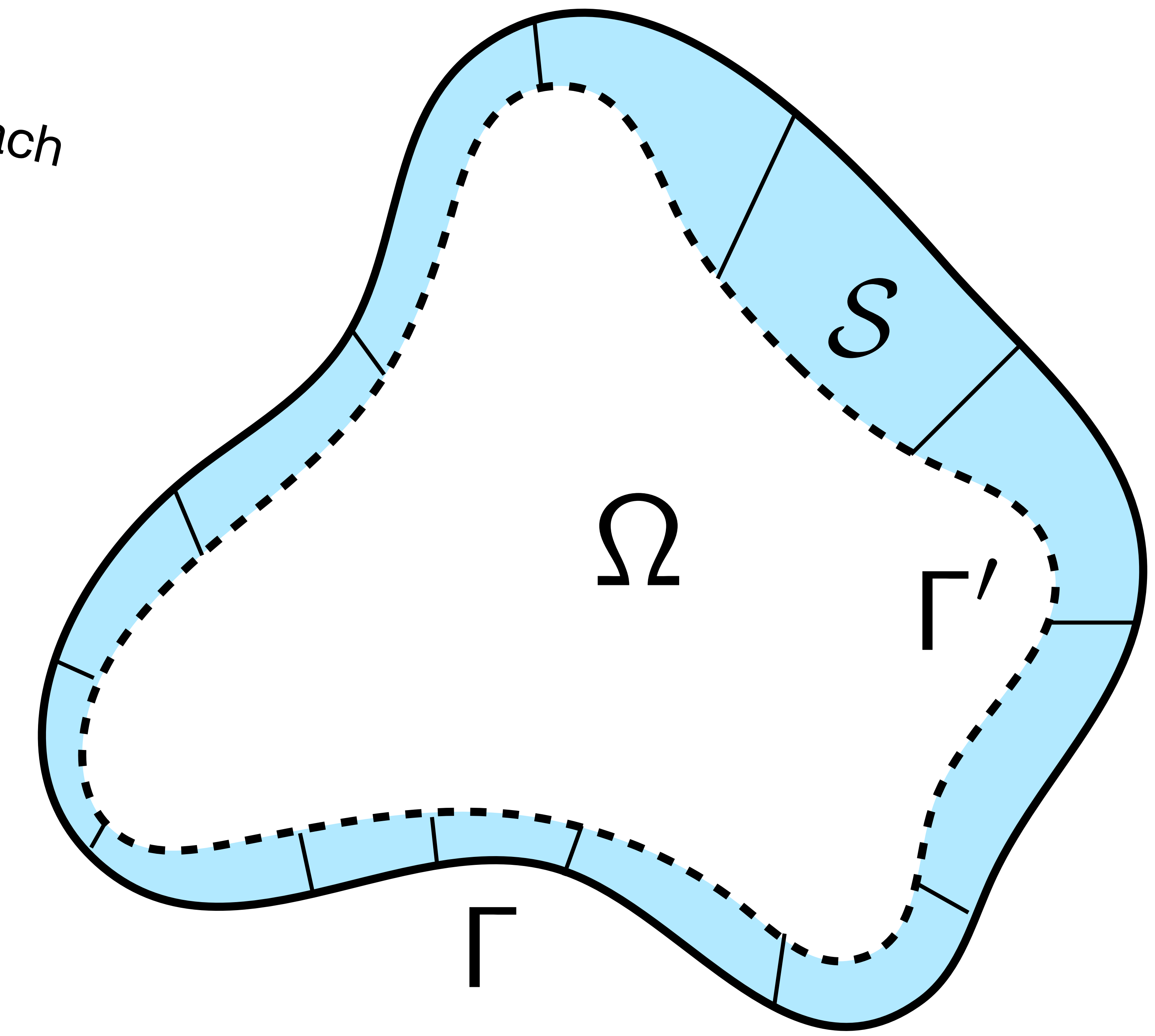
1. Define piecewise width from local panel size
2. Smooth each junction
3. Blend together using matched asymptotics
4. Perturb in the normal direction

$$\begin{aligned} \Delta v_{\text{strip}}(\boldsymbol{x}) &= f(\boldsymbol{x}), & \boldsymbol{x} &\in \mathcal{S}, \\ v_{\text{strip}}(\boldsymbol{x}) &= 0, & \boldsymbol{x} &\in \Gamma \cup \Gamma'. \end{aligned}$$

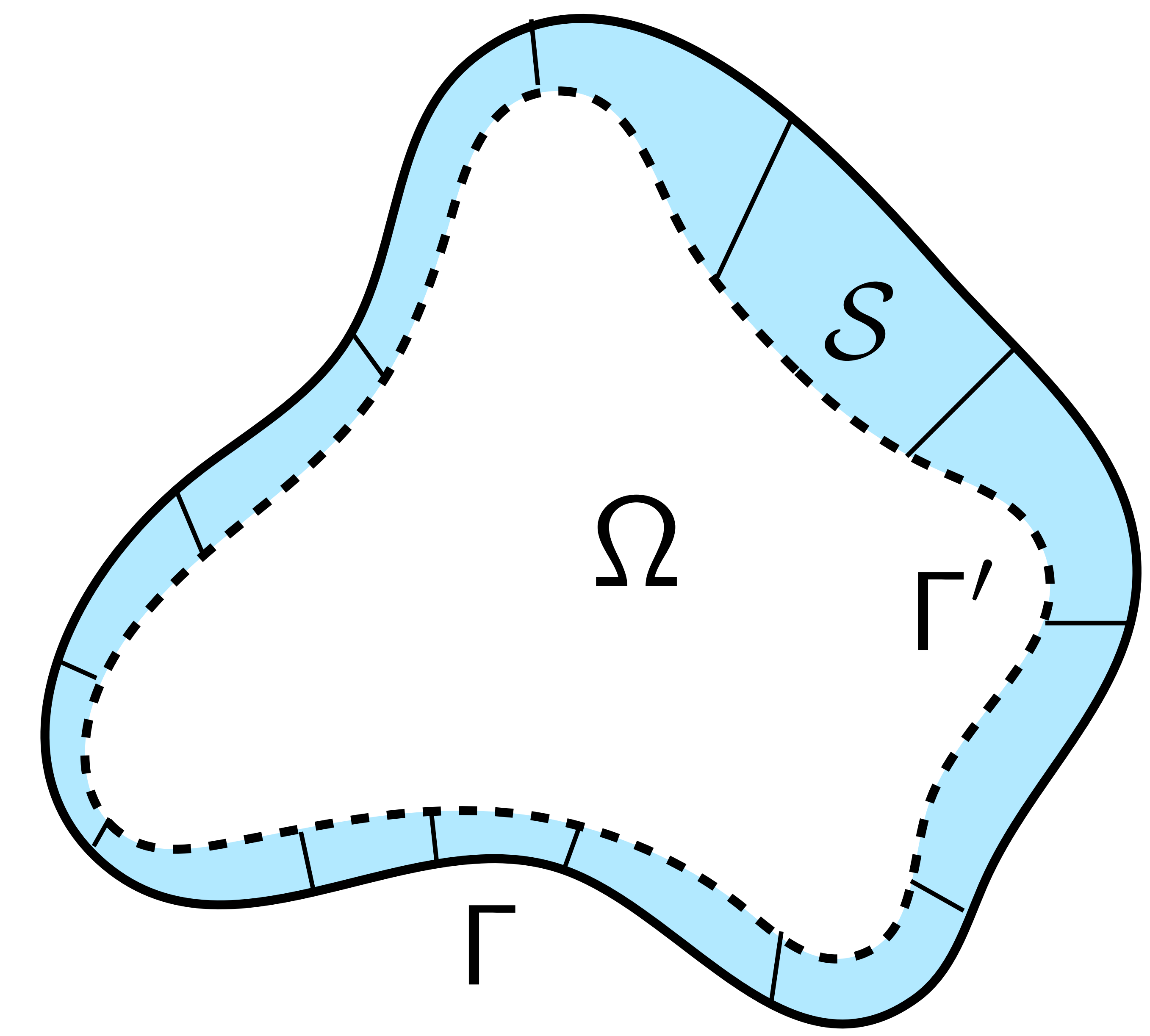
We use a spectral collocation method in $\mathcal{S}$ and solve using the “1D” HPS method.

Jacobian is numerically computed at each point to form differentiation matrices.





We use the hierarchical Poincaré–Steklov scheme to build a **fast direct solver** in $\mathcal{S}$.



[Martinsson, 2013]
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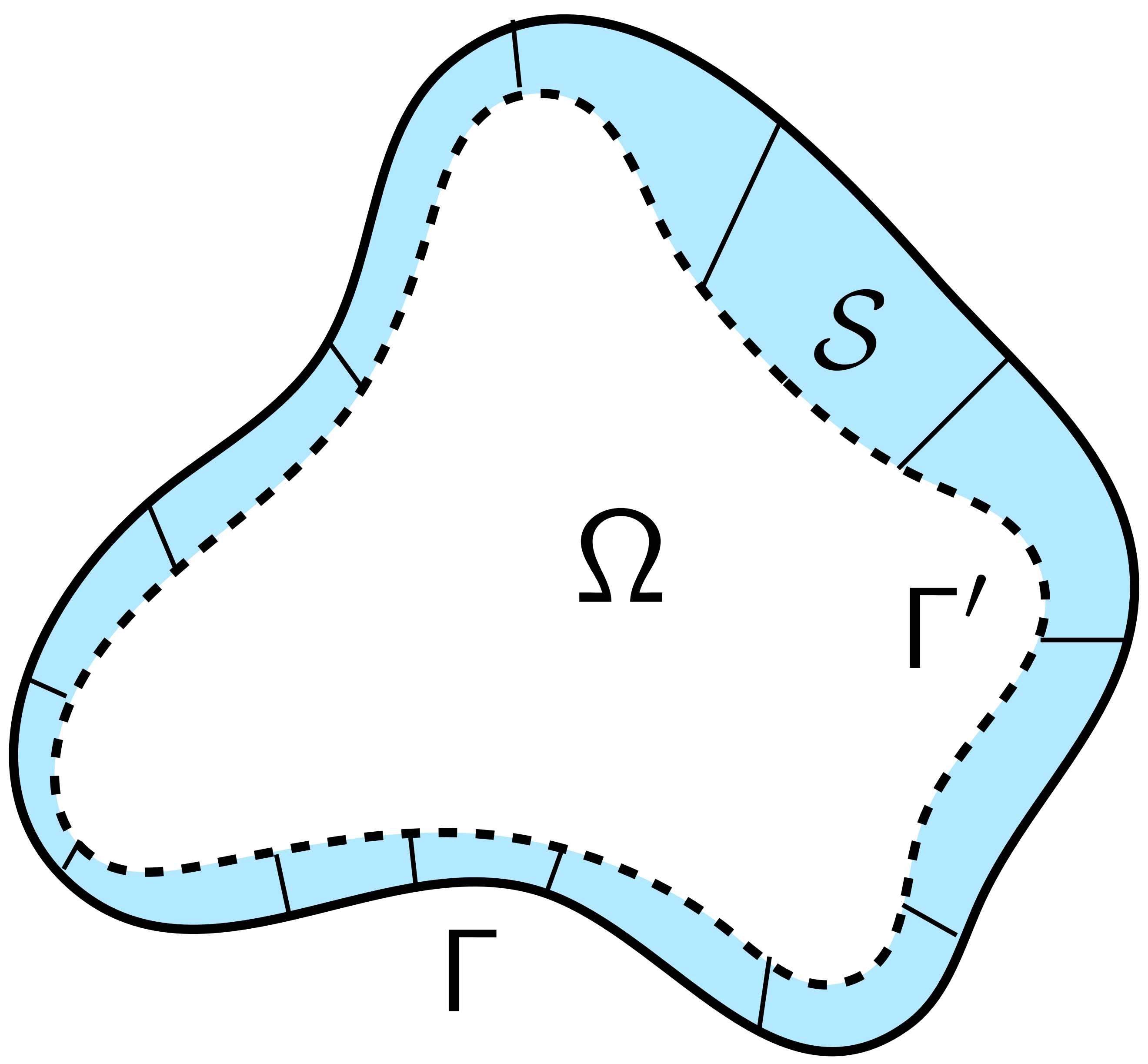

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Solving the strip problem
A fast direct solver for the strip

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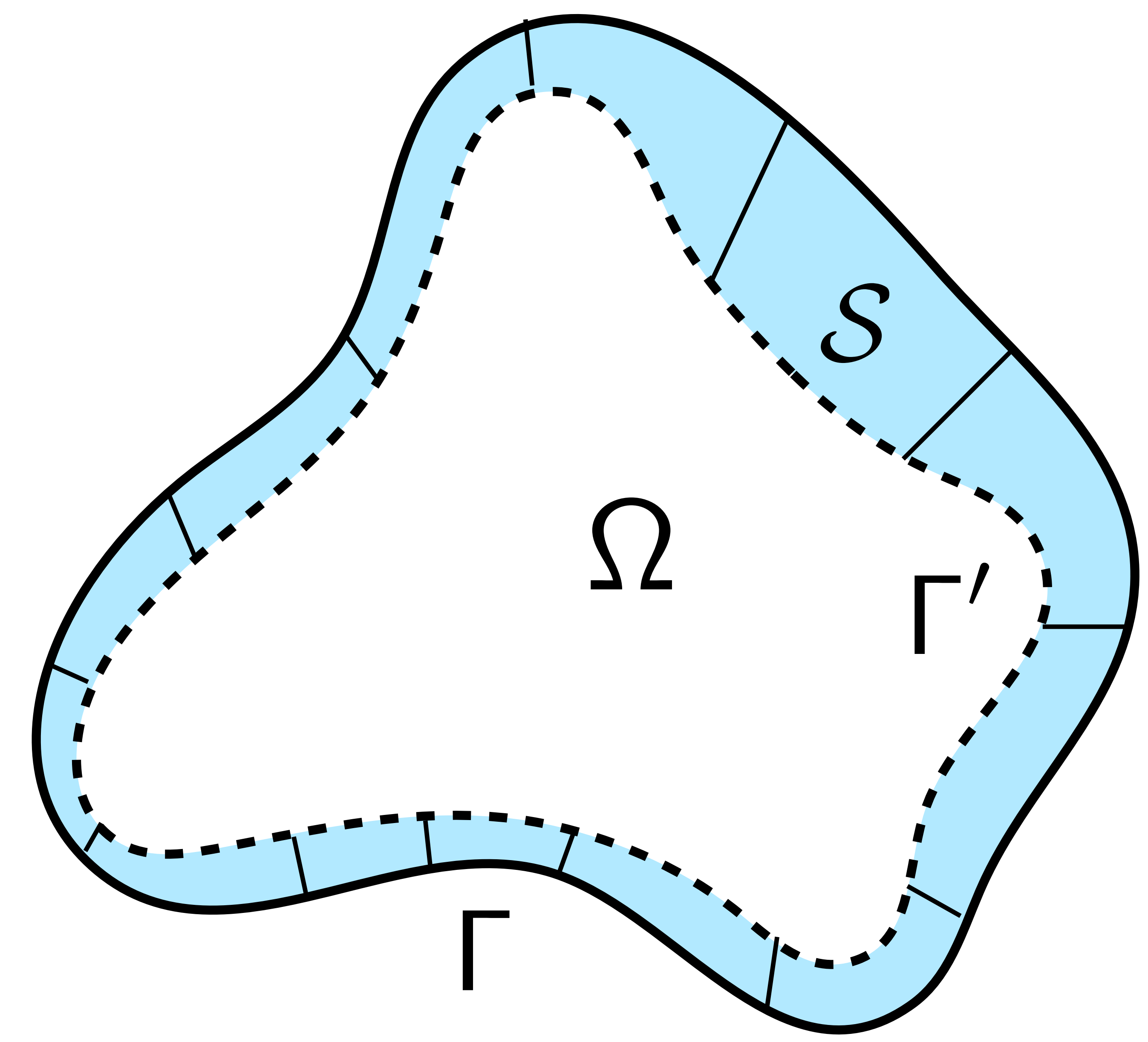
- Compute S and DtN on parent via Schur complement



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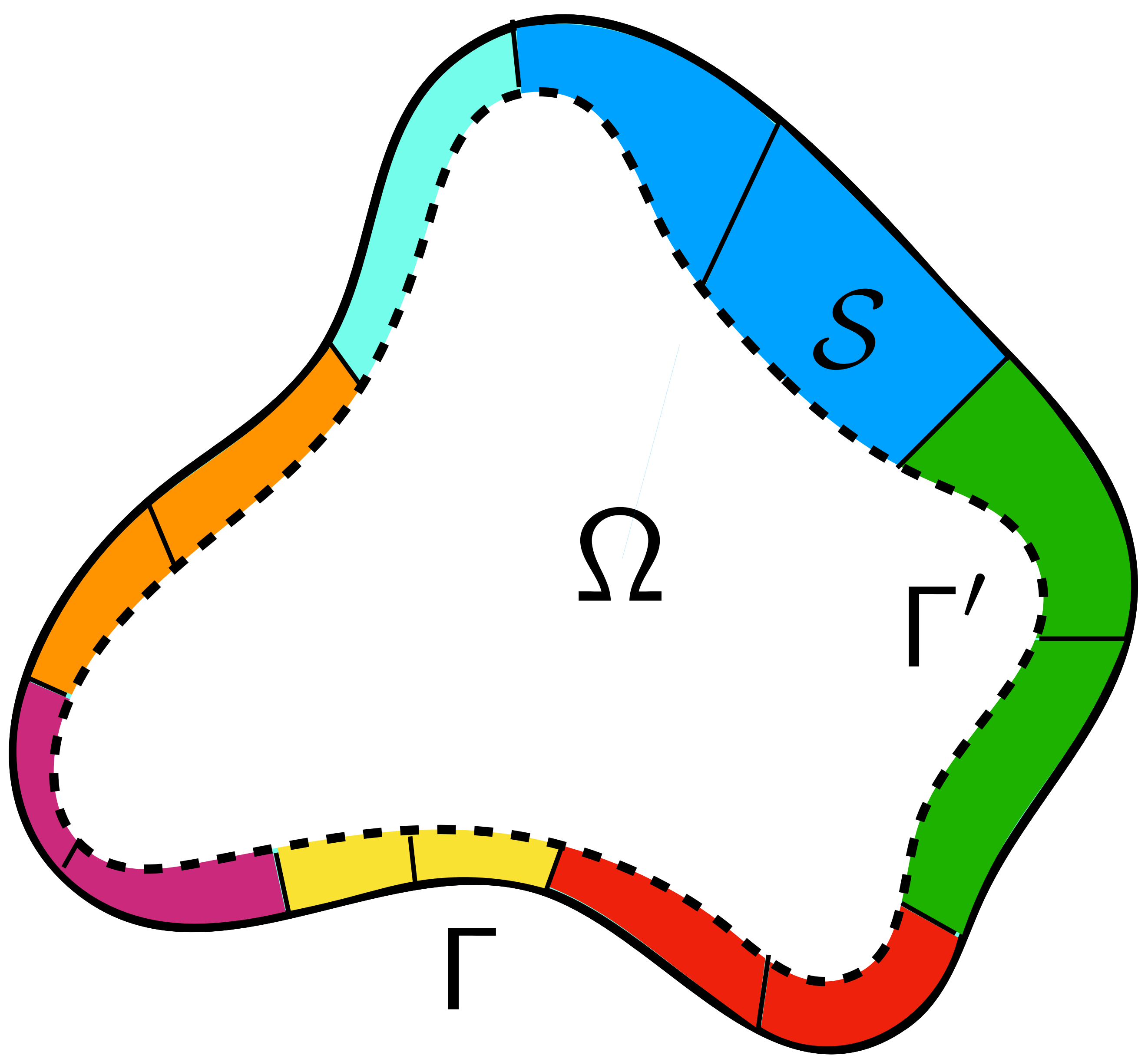

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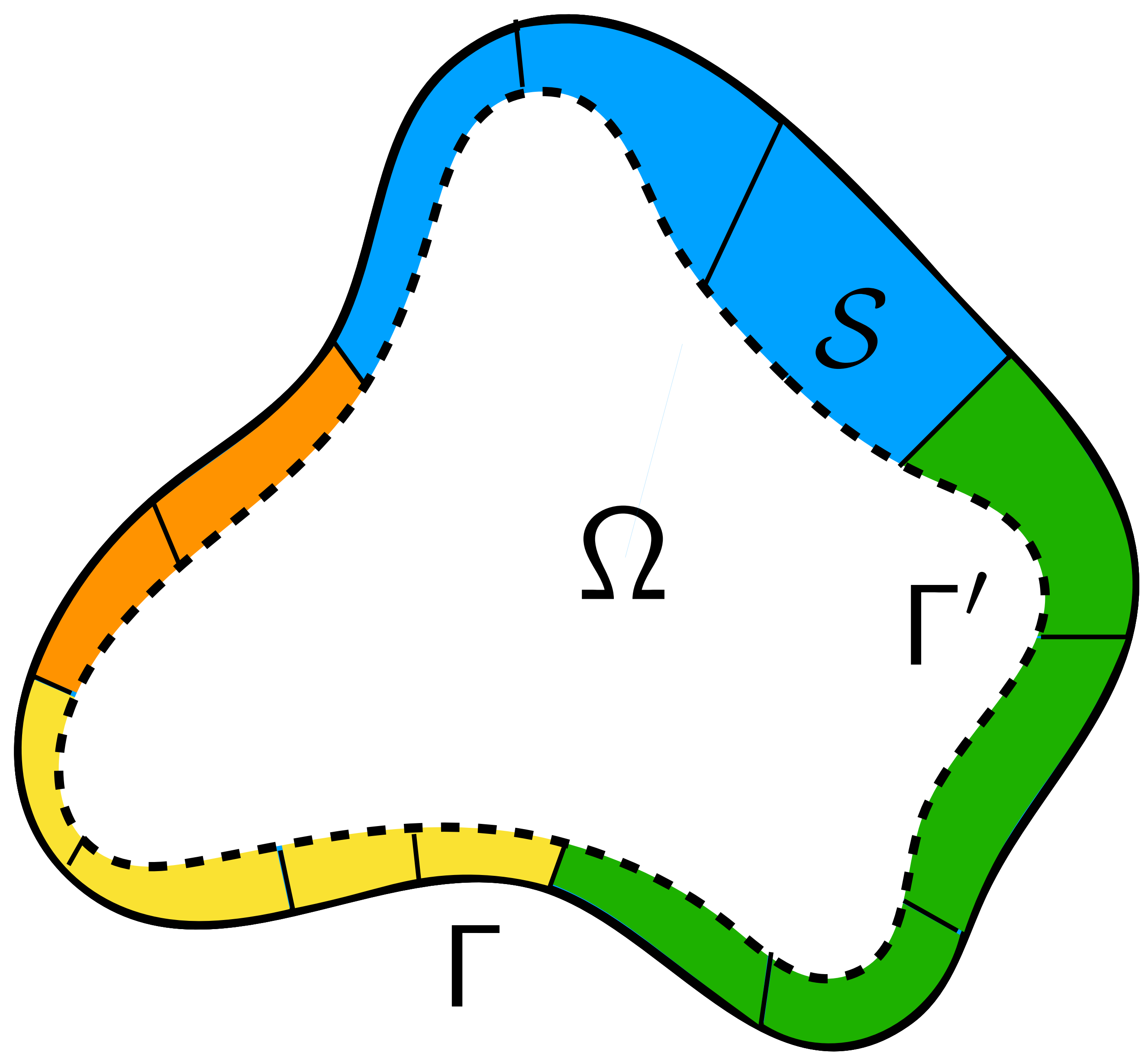

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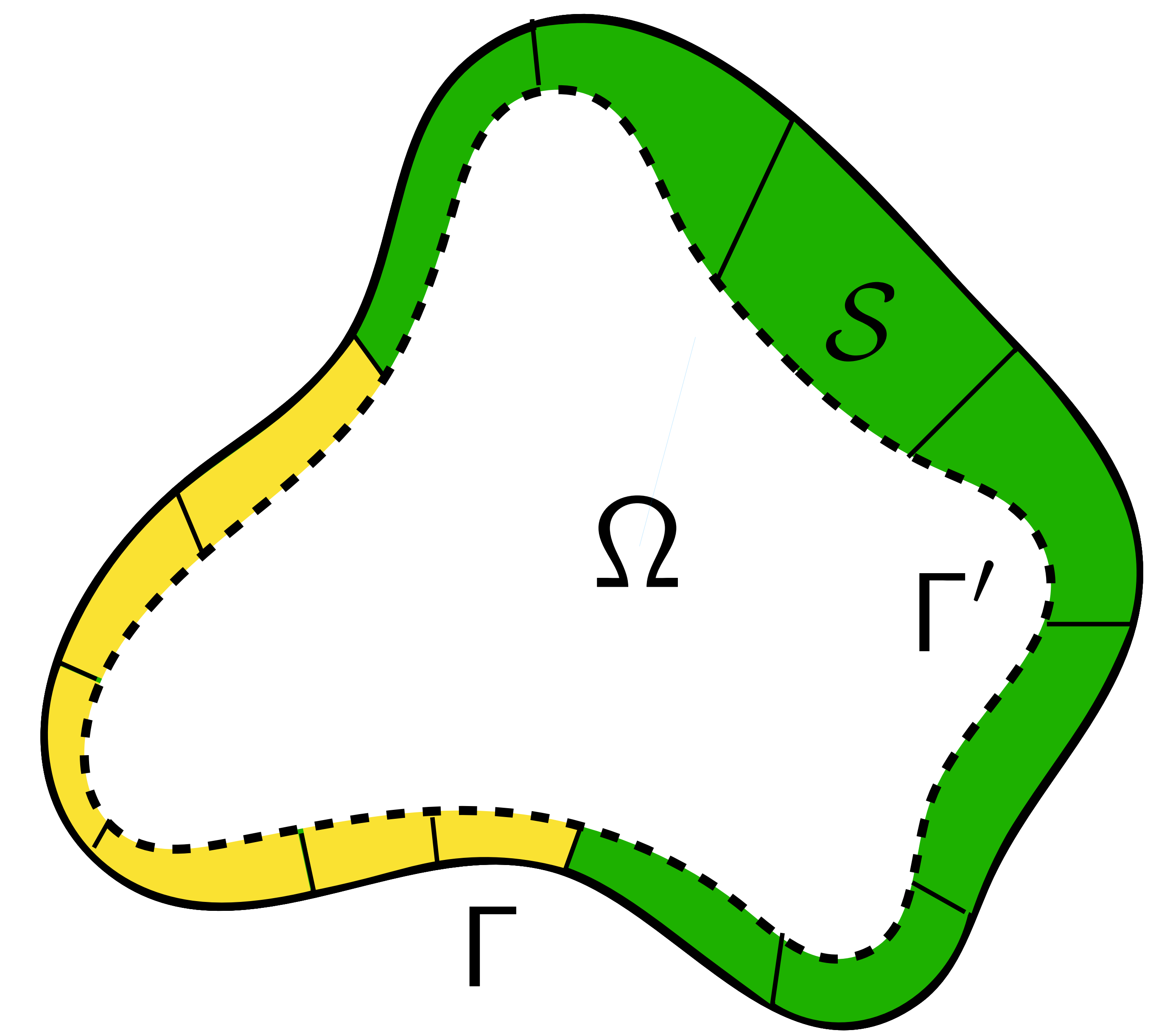

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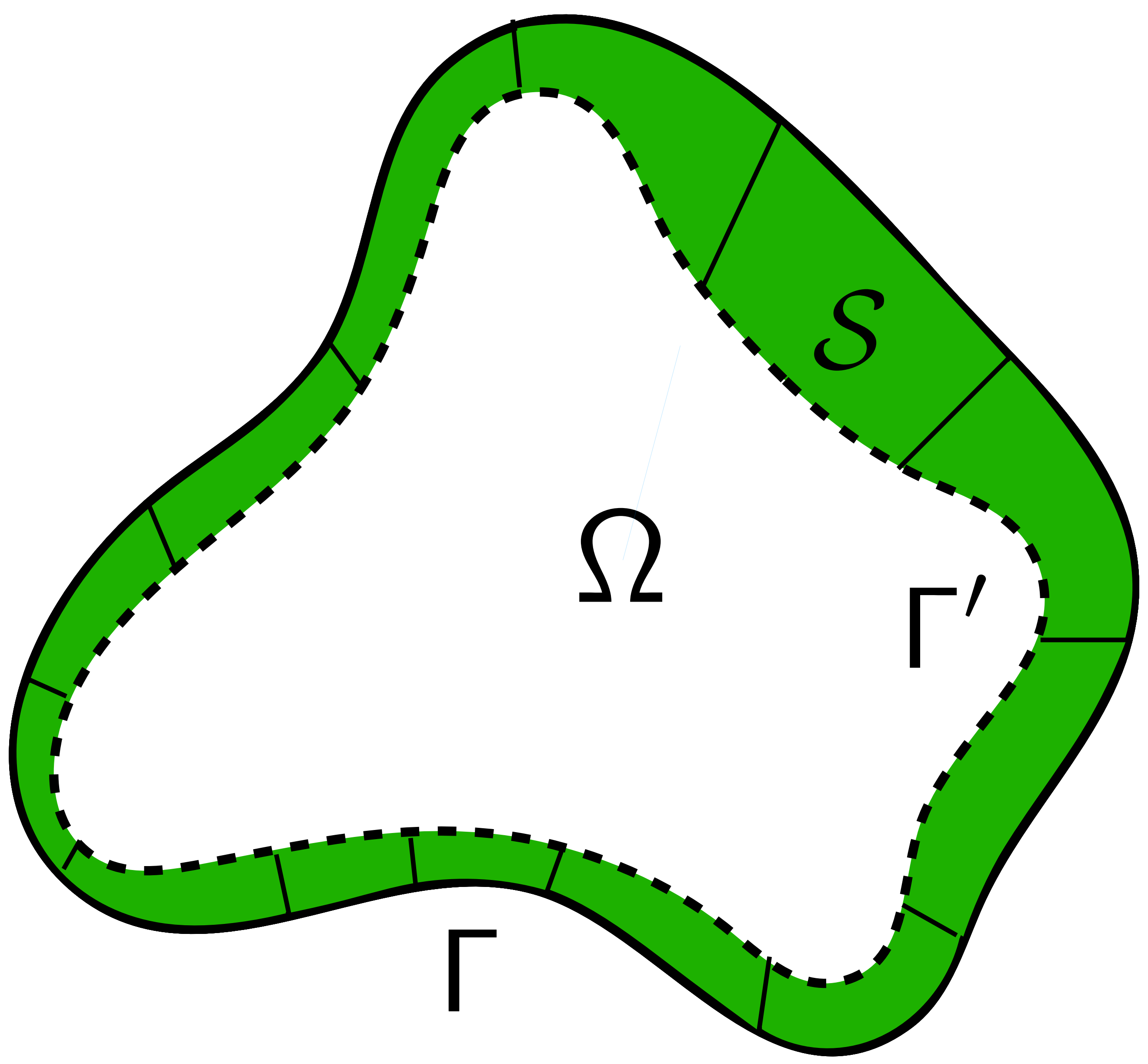

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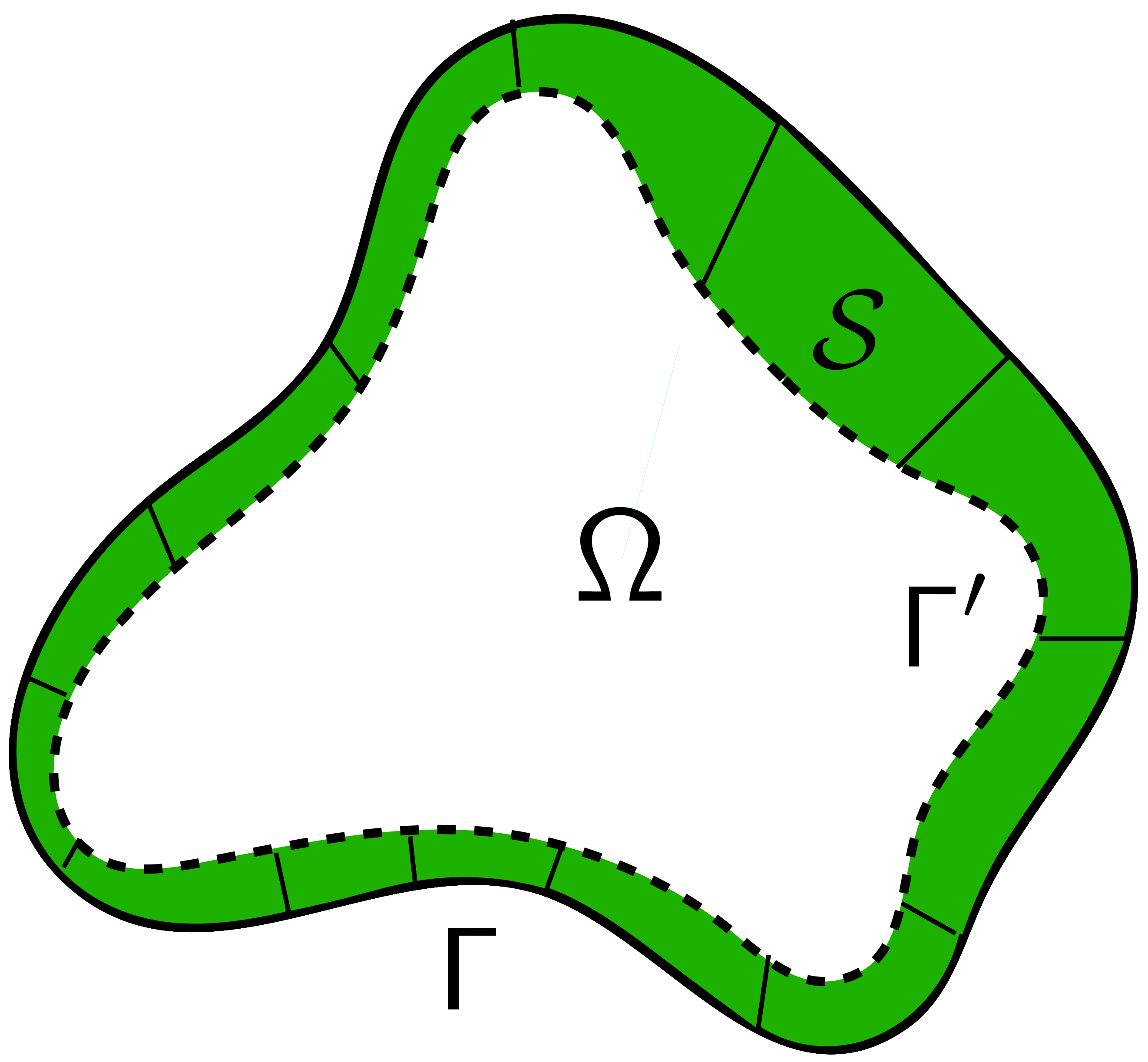

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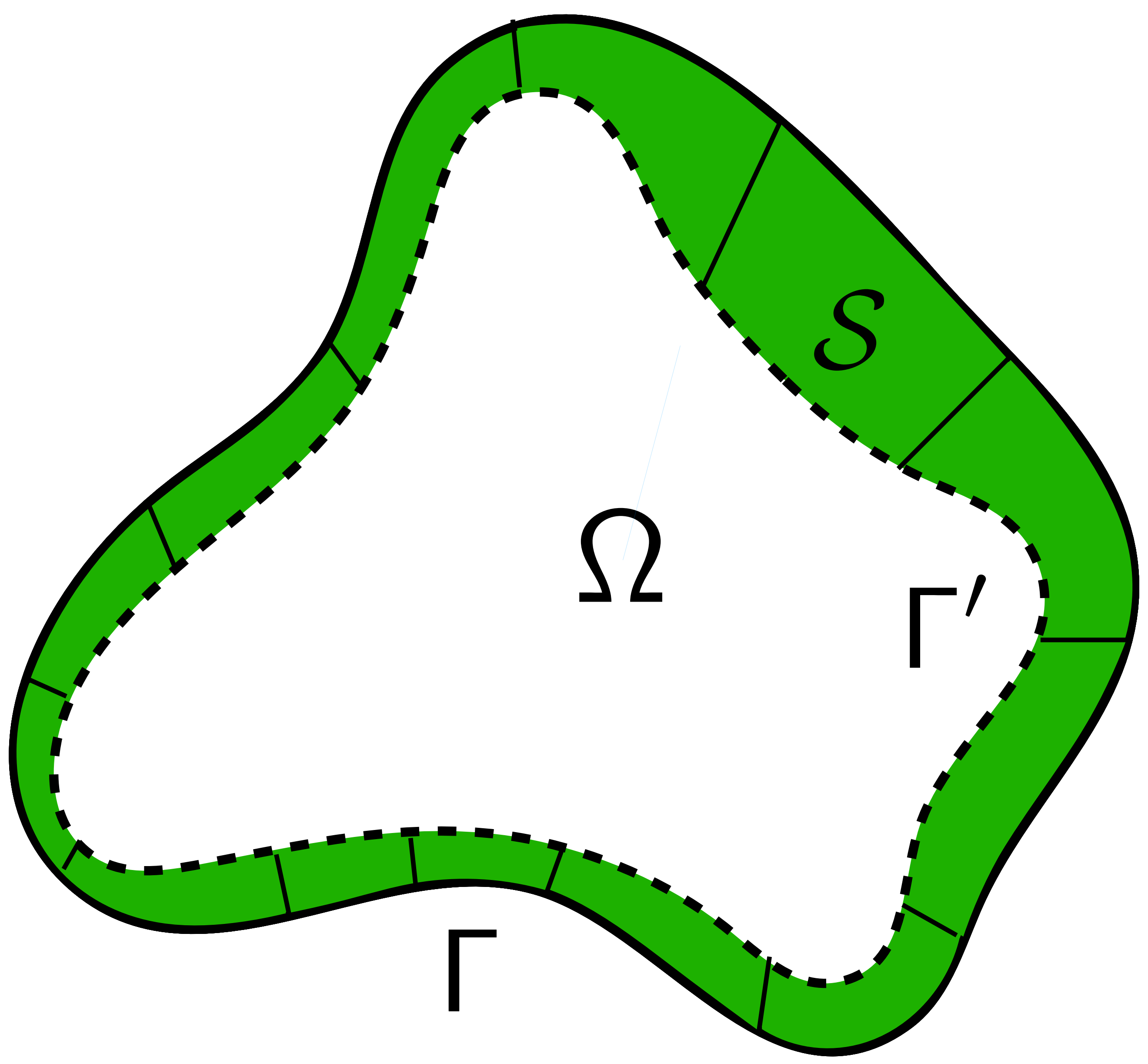

Cost: $\underset{①}{\mathcal{O}(p^6 n_{\text{panel}})} + \underset{②}{\mathcal{O}(p^3 n_{\text{panel}})} + \underset{③}{\mathcal{O}(p^2 n_{\text{panel}})} = \mathcal{O}(n_{\text{panel}})$

We typically use p = 16 on each panel and upsample the SEM grid to 2p = 32.

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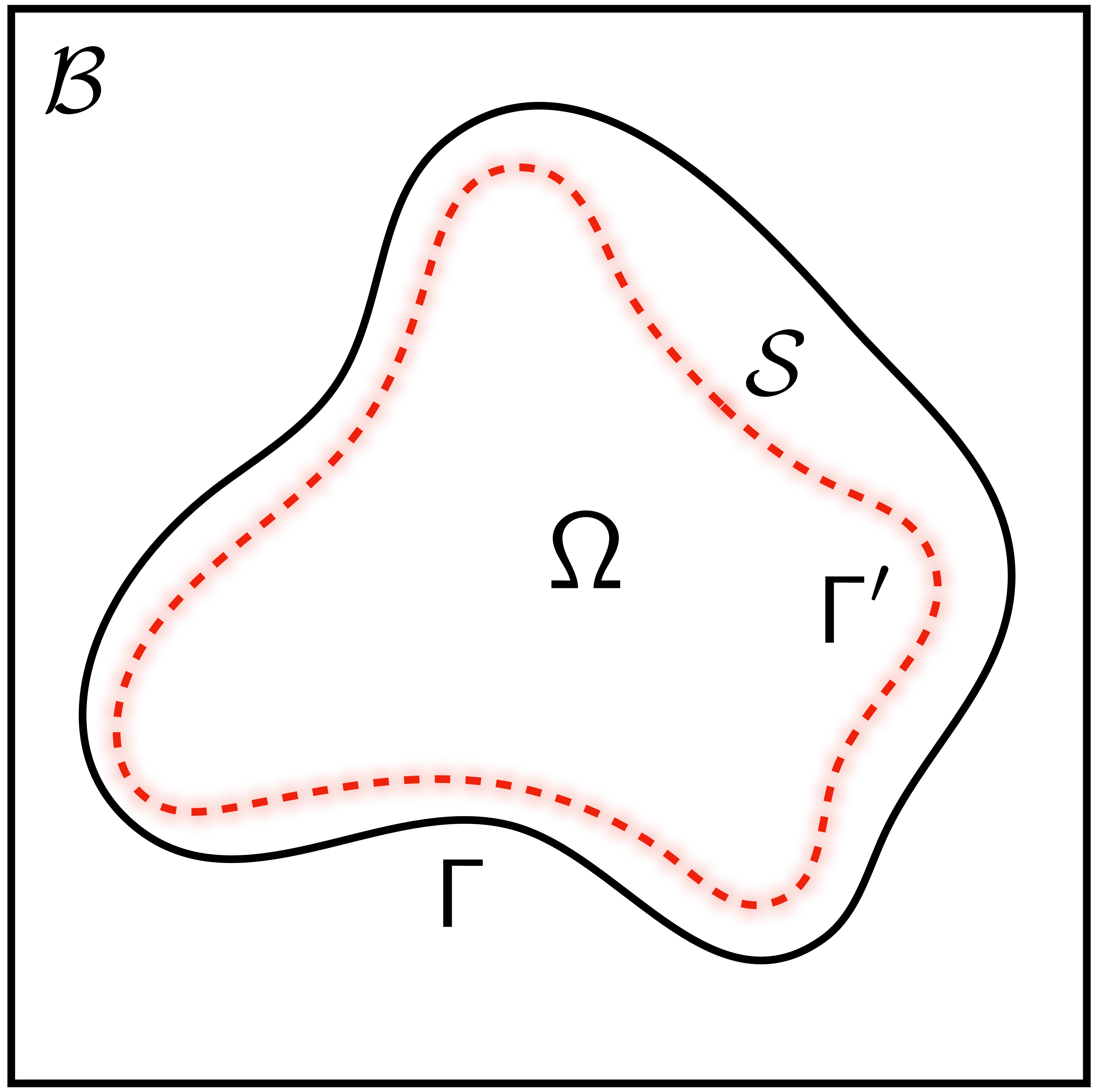
Patching the solutions
Layer potentials

The solutions v_{bulk} and v_{strip} may differ in value and normal derivative along Γ'.



Patching the solutions

Layer potentials

The solutions v_{bulk} and v_{strip} may differ in value and normal derivative along Γ'.

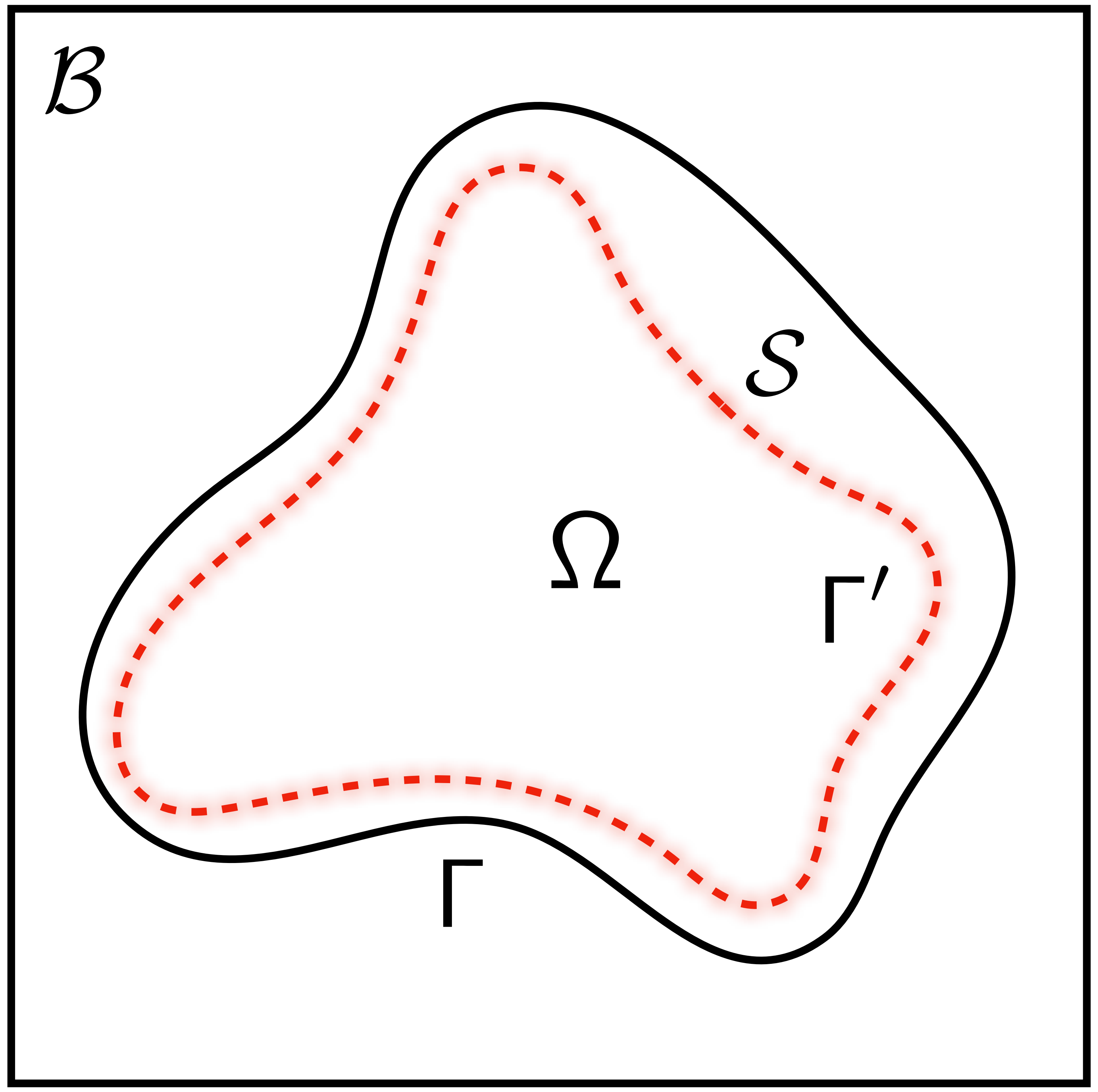
The single- and double-layer potentials

$$S[\sigma](\boldsymbol{x}) = \int_{\Gamma'} \Phi(\boldsymbol{x},\boldsymbol{y})\,\sigma(\boldsymbol{y})\,d\boldsymbol{y} \qquad D[\tau](\boldsymbol{x}) = \int_{\Gamma'} \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n_y}}\,\tau(\boldsymbol{y})\,d\boldsymbol{y}$$

are harmonic and satisfy the jump relations

$$S[\sigma]\big|_{\Gamma'}^{+} - S[\sigma]\big|_{\Gamma'}^{-} = 0 \qquad D[\tau]\big|_{\Gamma'}^{+} - D[\tau]\big|_{\Gamma'}^{-} = \tau$$

$$\partial_{\boldsymbol{n}} S[\sigma]\big|_{\Gamma'}^{+} - \partial_{\boldsymbol{n}} S[\sigma]\big|_{\Gamma'}^{-} = -\sigma \qquad \partial_{\boldsymbol{n}} D[\tau]\big|_{\Gamma'}^{+} - \partial_{\boldsymbol{n}} D[\tau]\big|_{\Gamma'}^{-} = 0$$



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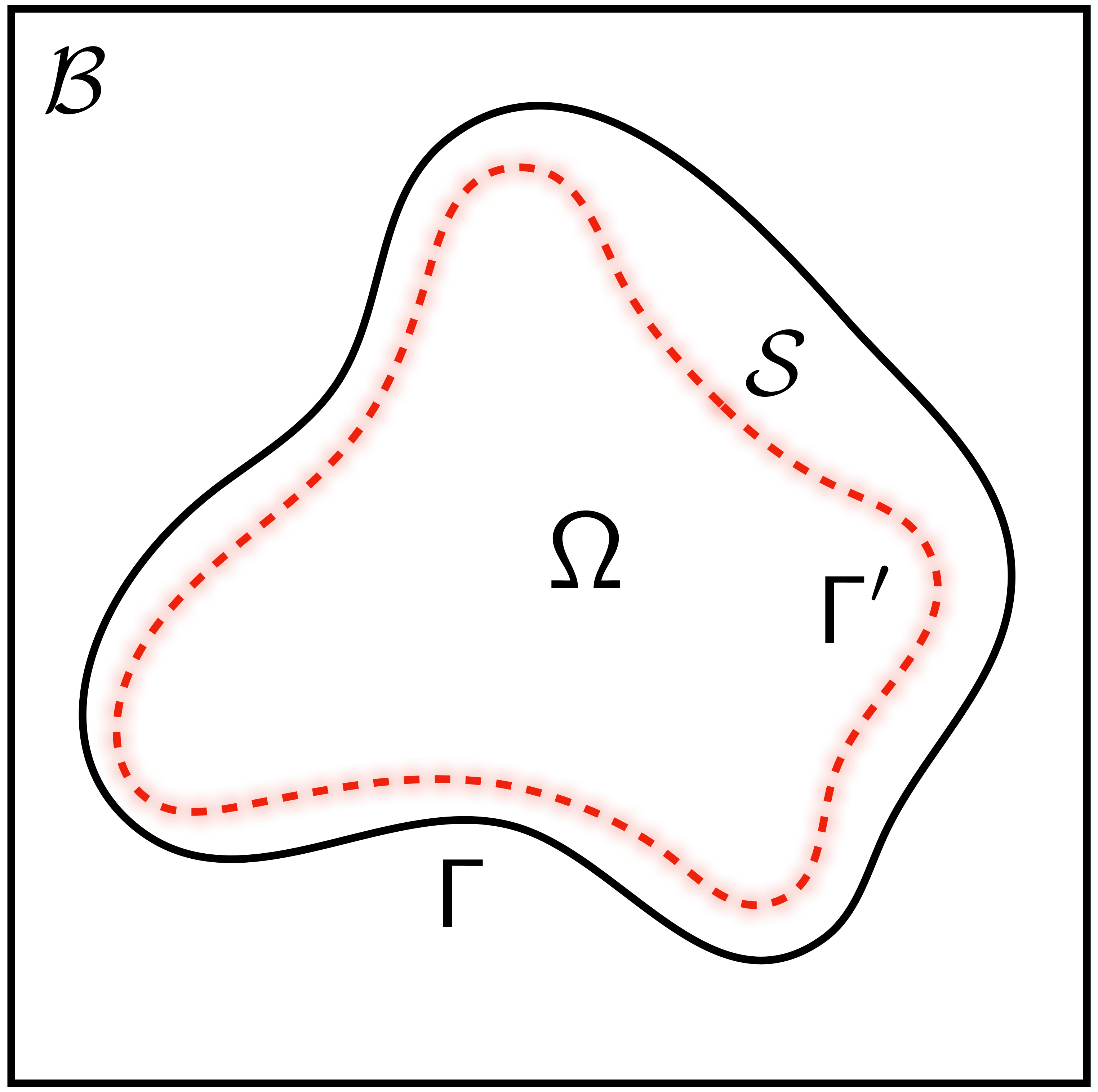
So define $v_{\text{glue}} = S[\sigma] - D[\tau]$ with

$$\sigma = \partial_{\boldsymbol{n}} v_{\text{bulk}}|_{\Gamma'} - \partial_{\boldsymbol{n}} v_{\text{strip}}|_{\Gamma'} \qquad \tau = v_{\text{bulk}}|_{\Gamma'} - v_{\text{strip}}|_{\Gamma'}$$



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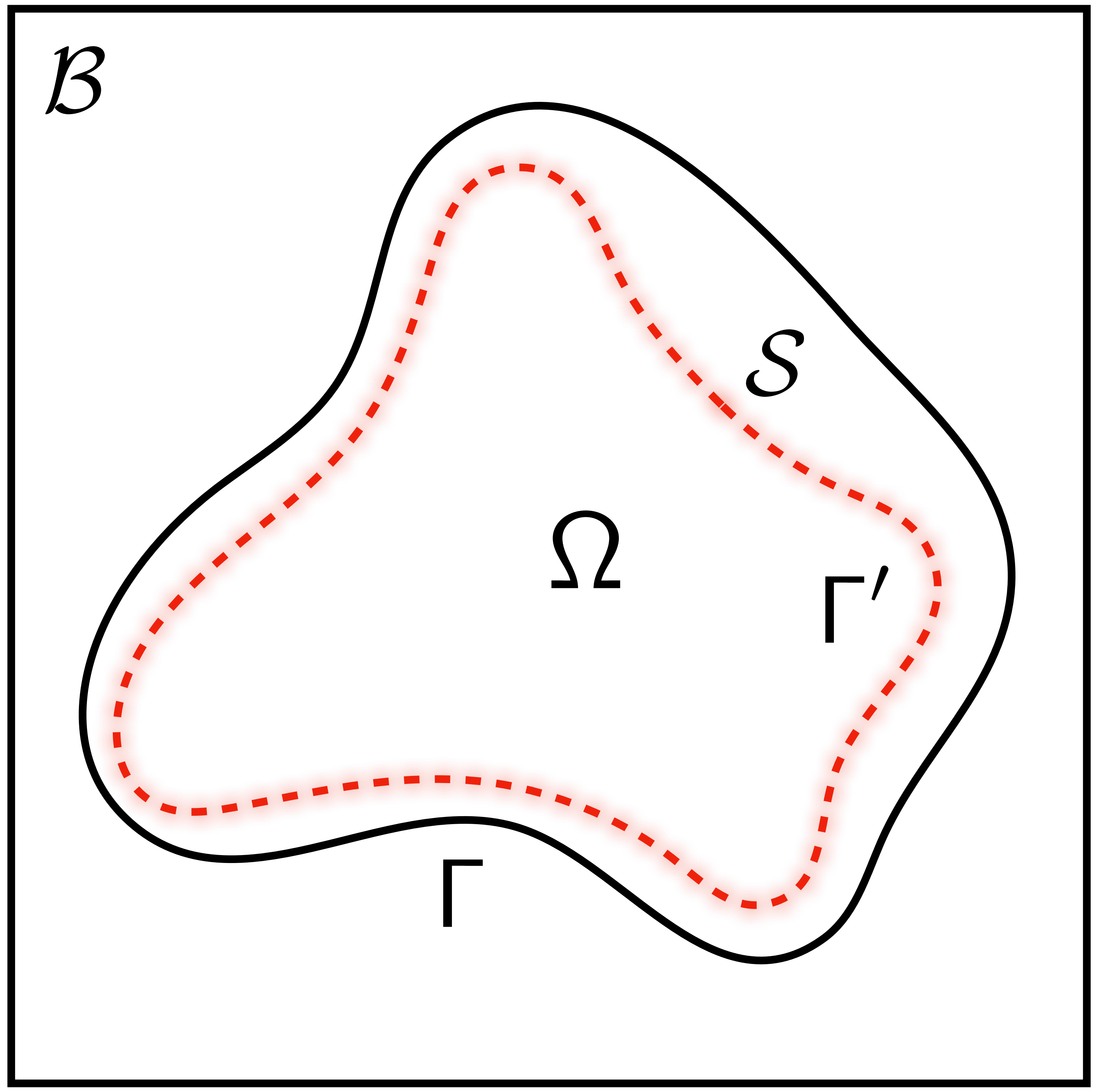
Then

$$v = \begin{cases} v_{\text{bulk}} + v_{\text{glue}} & \text{in } \Omega \setminus \mathcal{S} \\ v_{\text{strip}} + v_{\text{glue}} & \text{in } \mathcal{S} \end{cases}$$

satisfies $\Delta v = f$ in Ω and is smooth across Γ'.



Correcting for boundary data
Homogeneous solution

Finally, the boundary conditions may not be satisfied.

A homogeneous correction may computed using standard BIE techniques:

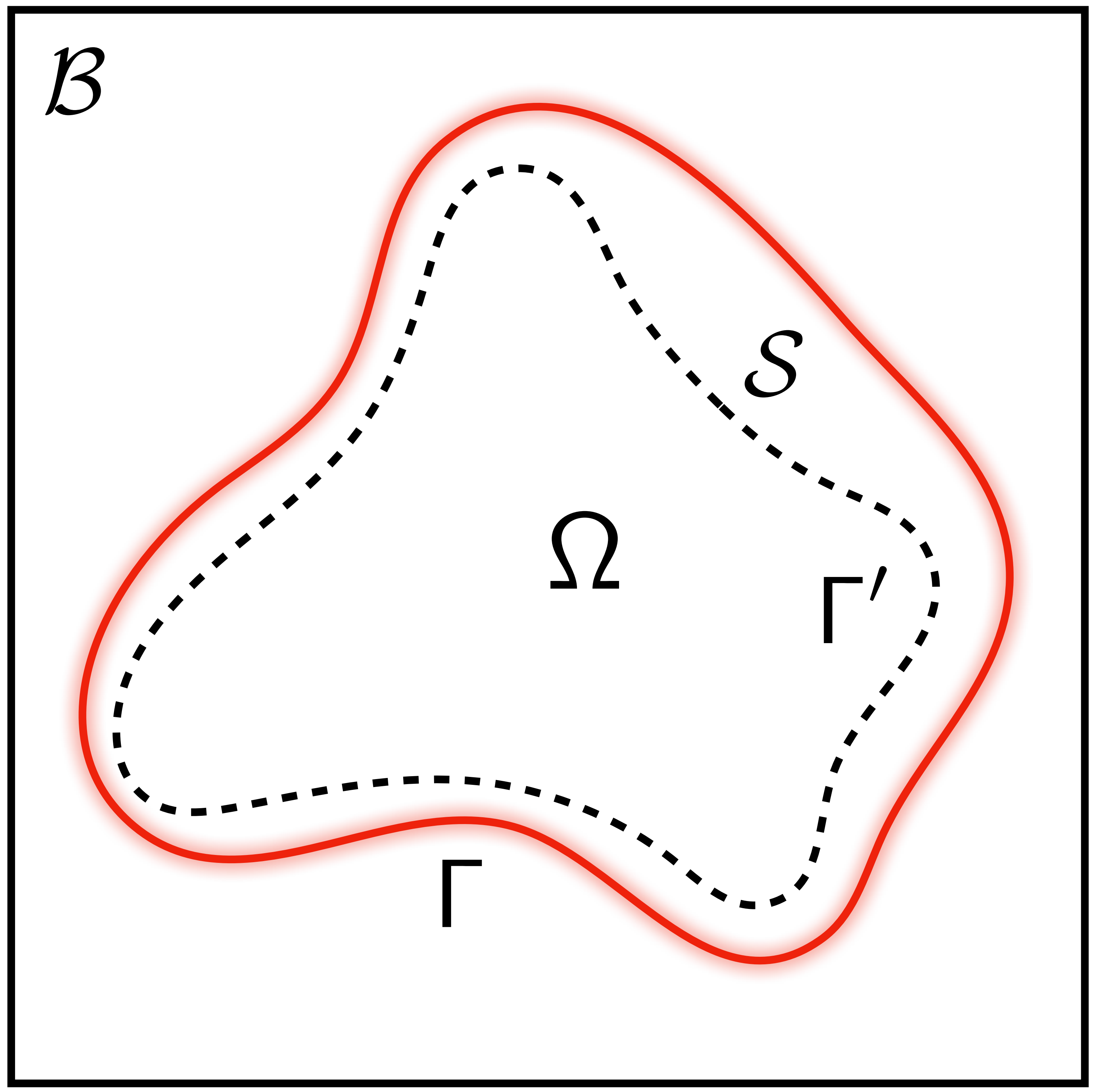
$$\begin{aligned} \Delta w &= 0 && \text{in } \Omega \\ w &= g - v_{\text{glue}}\big|_{\Gamma} && \text{on } \Gamma \end{aligned}$$

Then,

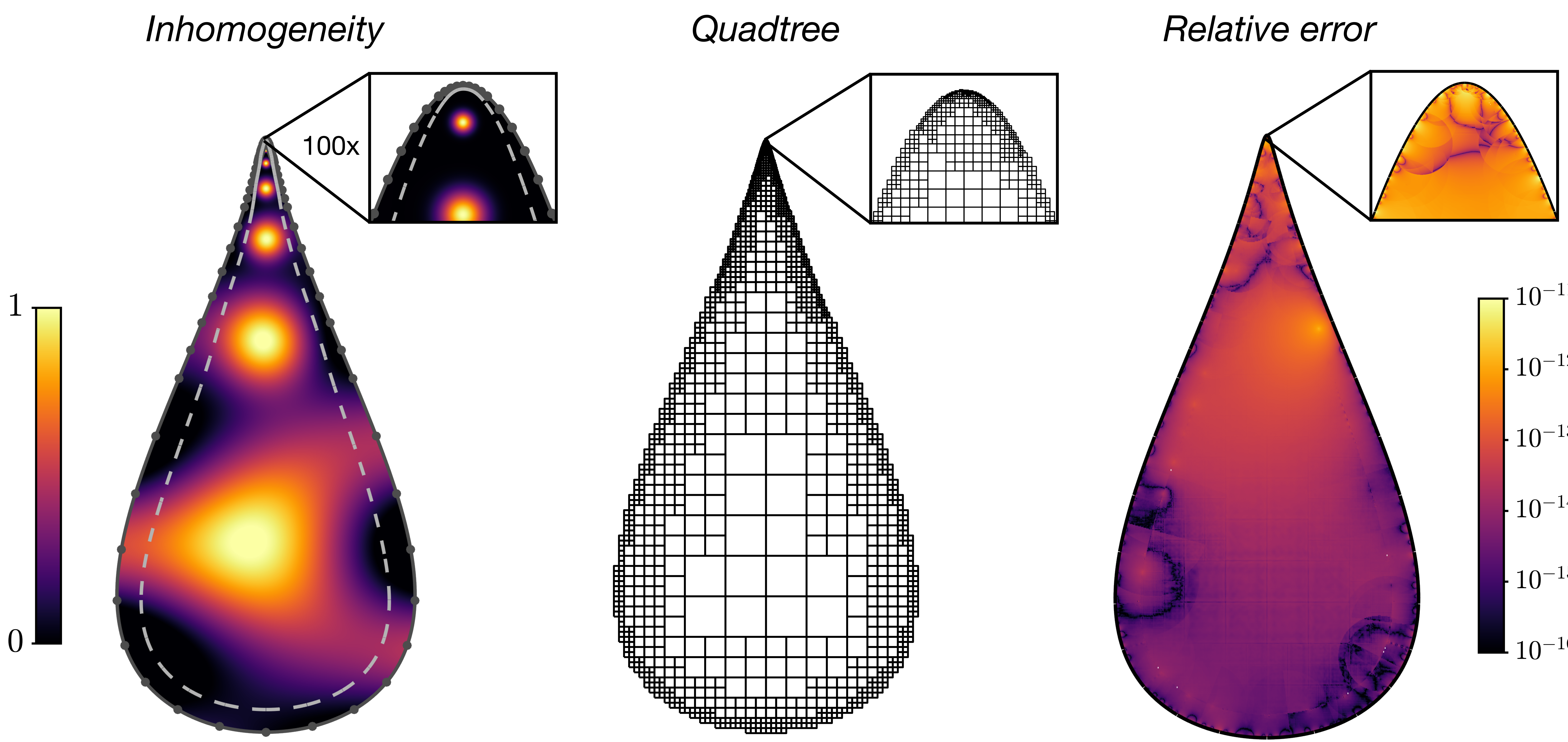
$$u = \begin{cases} v_{\text{bulk}} + v_{\text{glue}} + w & \text{in } \Omega \setminus \mathcal{S} \\ v_{\text{strip}} + v_{\text{glue}} + w & \text{in } \mathcal{S} \end{cases}$$

satisfies

$$\begin{aligned} \Delta u &= f && \text{in } \Omega \\ u &= g && \text{on } \Gamma \end{aligned}$$



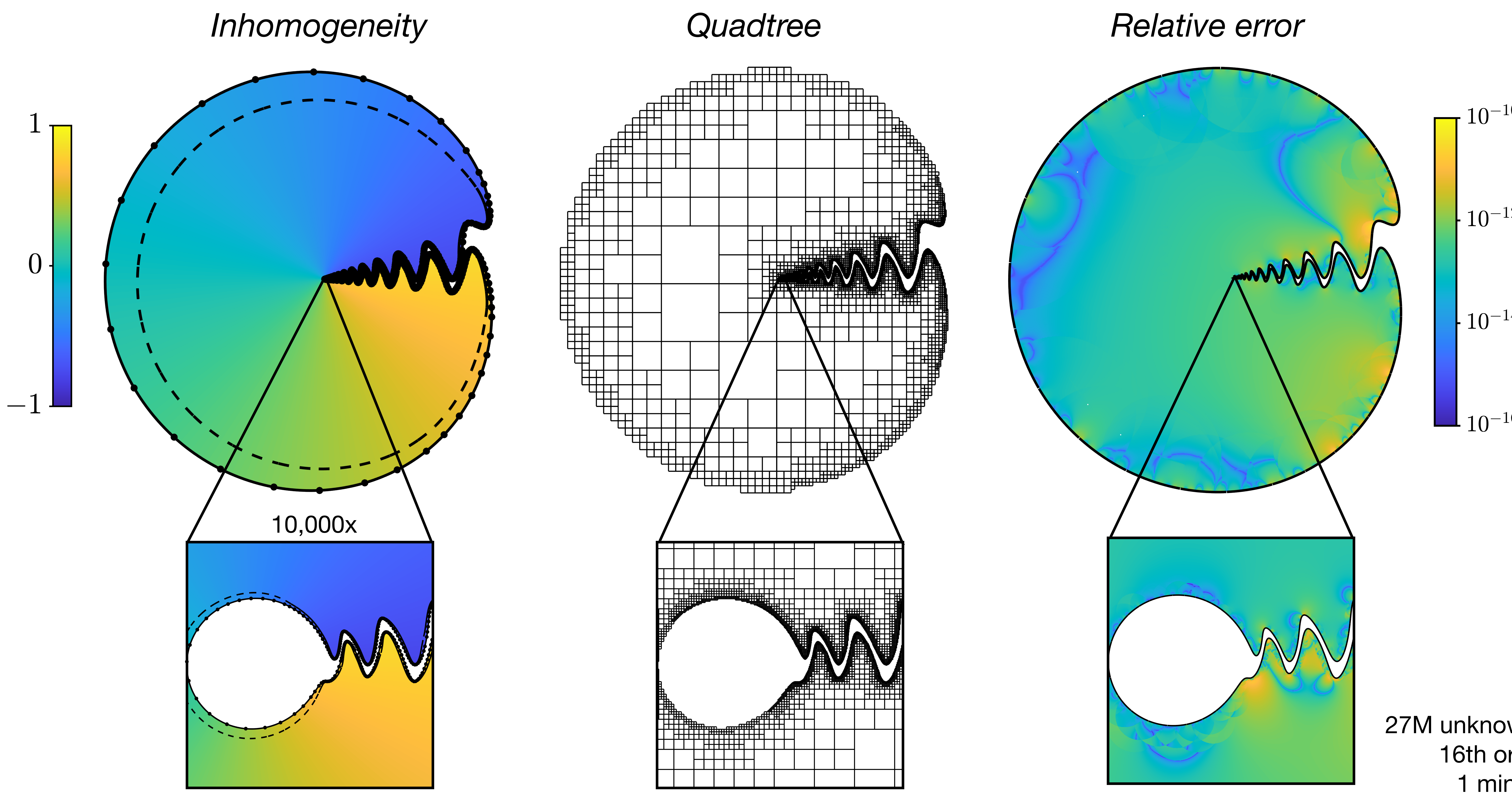
Example
A rounded cusp



750k unknowns
16th order
2 seconds

Example

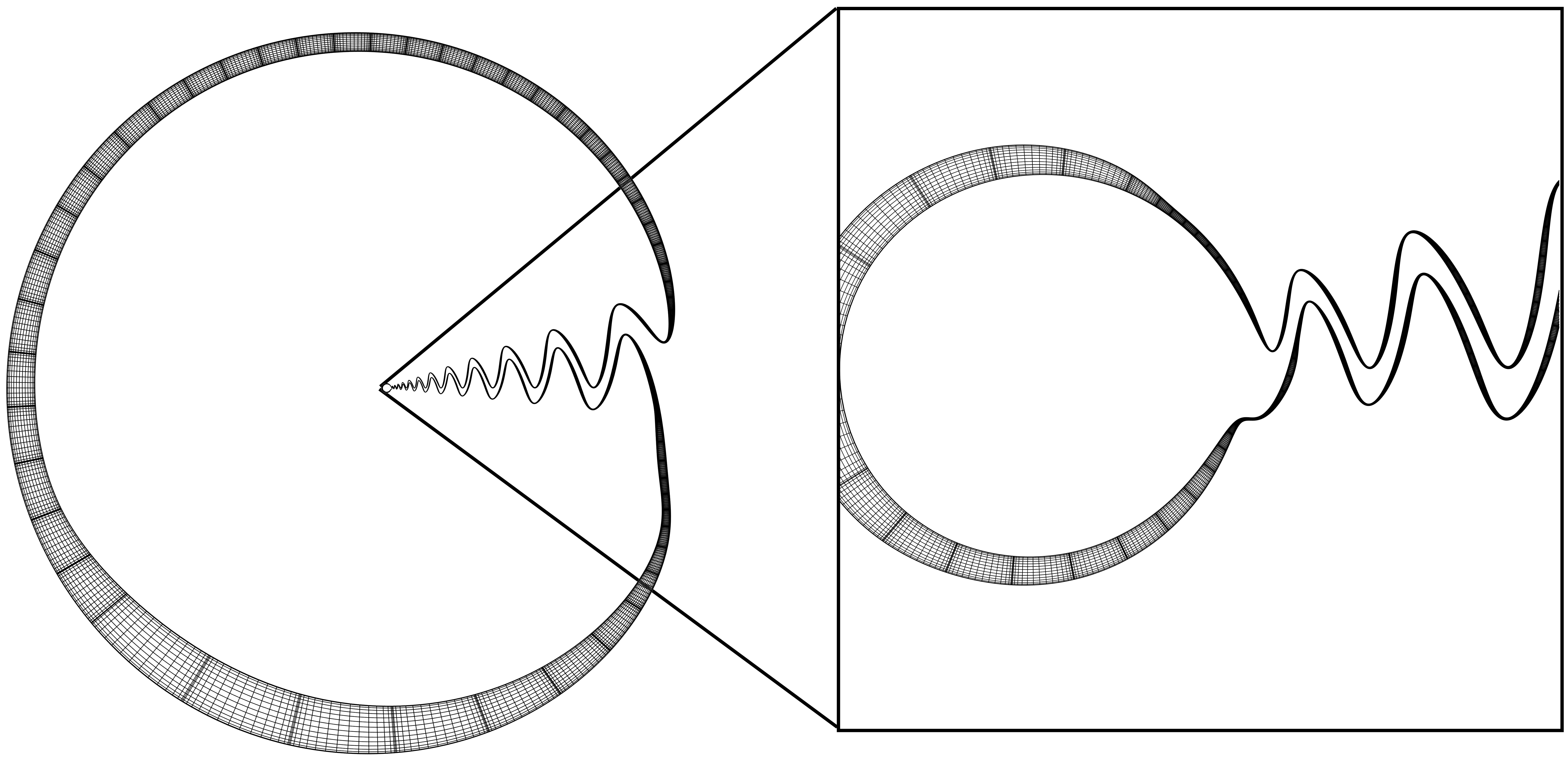
A multiscale inhomogeneity and close-to-touching geometry



27M unknowns
16th order
1 minute

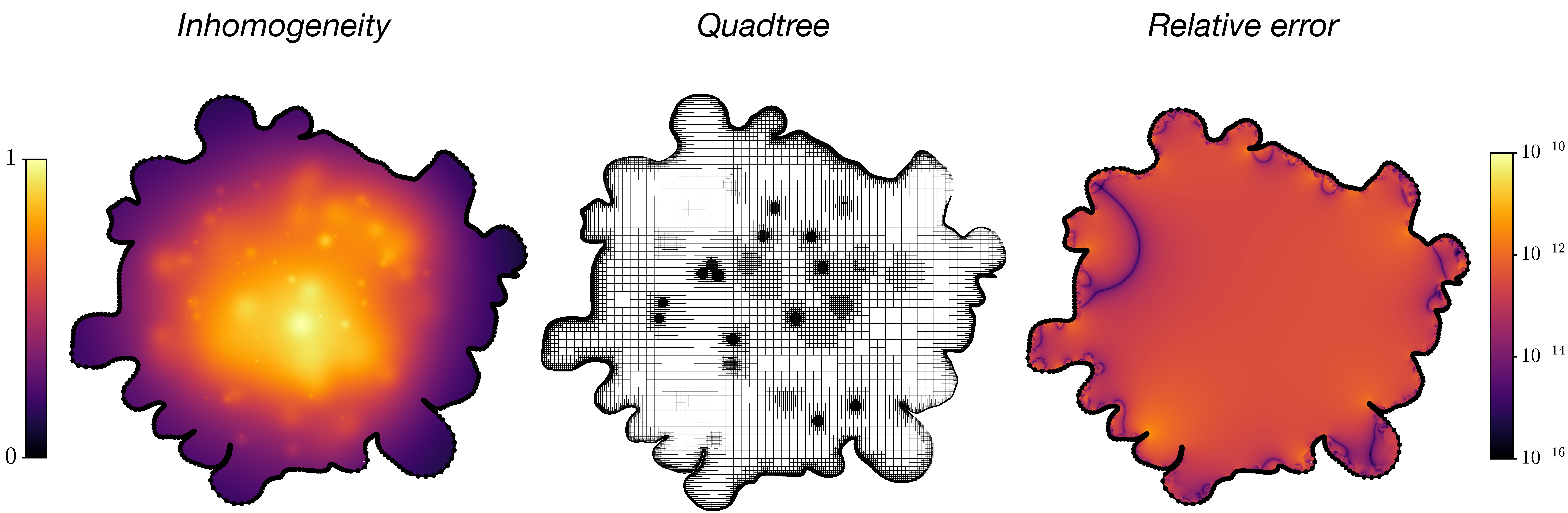
Example

A multiscale inhomogeneity and close-to-touching geometry



Example

Cell blebbing geometry extract from microscopy



12M unknowns
16th order
30 seconds

Conclusion

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Conclusion

- Function extension can be challenging in multiscale geometries. Our scheme avoids extension.
- Particular solutions can be computed adaptively and efficiently using truncated volume potentials.
- Surface-conforming truncation idea designed to extend to 3D.

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