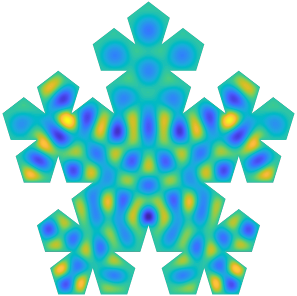
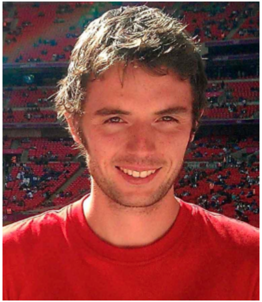
The ultraspherical spectral element method

Dan Fortunato

Alex Townsend

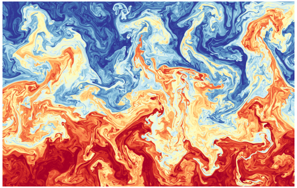




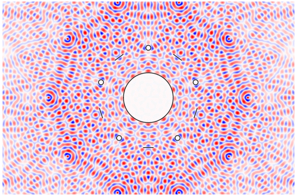
Nick Hale

- ✓ Spectrally accurate convergence to solution (e.g. exponential)
- ✓ High accuracy
- ✓ Low numerical dissipation and dispersion

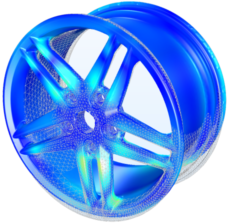
High Reynolds number flows
[Dedalus Project, 2019]

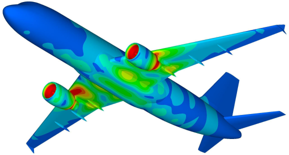


High frequency scattering
[Slevinsky & Olver, 2017]



- ✗ Lack geometric flexibility
- ✗ Globalize corner singularities

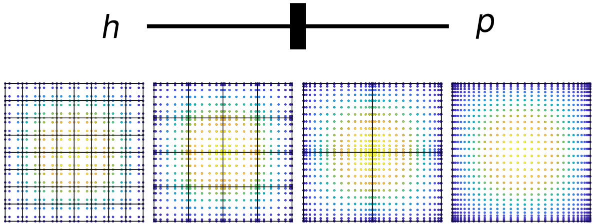




Spectral element methods and *hp*-adaptivity
Theory vs. practice

SEMs combine:

- the flexibility of finite element methods
- the convergence properties of global spectral methods



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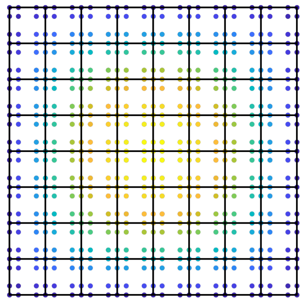

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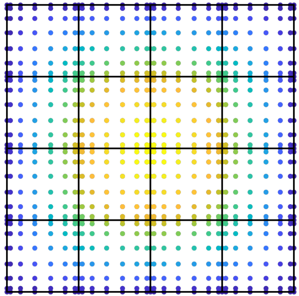

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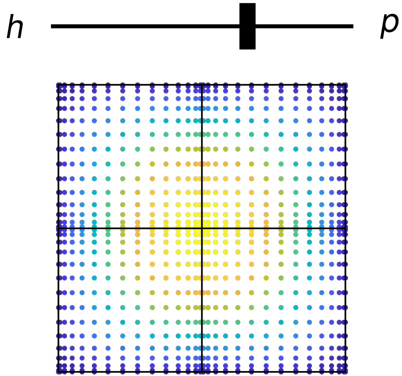

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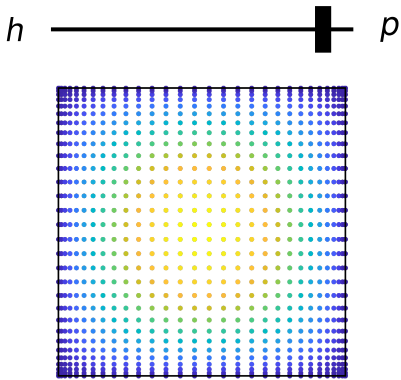

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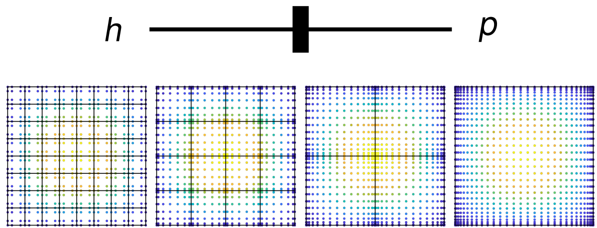


Most SEMs cost $\mathcal{O}(p^6/h^2) = \mathcal{O}(N\,p^4)$, so the slider is biased.

"In practice, hp-adaptivity means $p \lesssim 6$." [Sherwin, 2014]

Spectral element methods and *hp*-adaptivity
Theory vs. practice

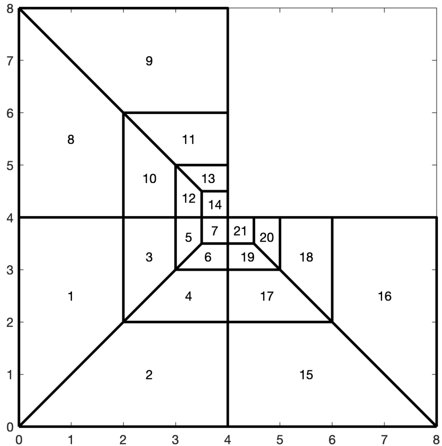
"As expected, the numerical results indicate that in the case of smooth solutions, one should fix the mesh and vary the polynomial order according to the desired accuracy (p-convergence)."
[Sherwin, 2014]

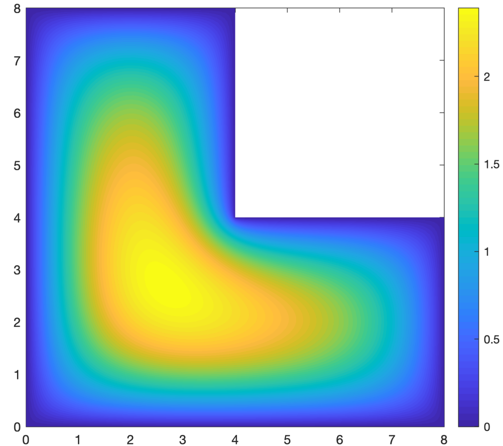
"While flow discontinuities are understandably better resolved with h-refinement, it is found that in regions of smooth flow, p-refinement offers a higher accuracy with the same number of degrees of freedom." [Li & Jameson, 2010]

"Within each of these elements the solution is represented by Nth-order polynomials, where $N = 5$–15 is most common but $N = 1$–100 or beyond is feasible." [Fischer, 2016]

Spectral element methods and *hp*-adaptivity

Want to choose *hp* based on physical considerations, not computational ones.





Spectral methods can be sparse and well-conditioned

Fourier spectral method

Periodic problems

Coefficient-based

Trigonometric polynomials $\{e^{ikx}\}$

Differentiation $\{e^{ikx}\} \to \{e^{ikx}\}$ is **sparse**:

$$\frac{d}{dx} e^{ikx} = ike^{ikx}$$

Well-conditioned

Spectral collocation

Non-periodic problems

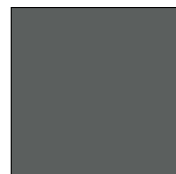
Value-based

Chebyshev points $x_k = \{\cos \frac{k\pi}{N}\}$

Differentiation $\{x_k\} \to \{x_k\}$ is **dense**:

$$\frac{d}{dx} u(x_k) = \mathbb{I}'_{\{u(x_j)\}}(x_k)$$

Ill-conditioned



Spectral methods can be sparse and well-conditioned

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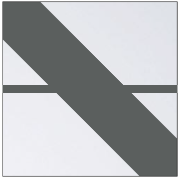


Chebyshev spectral method

Non-periodic problems

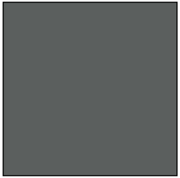
Coefficient-based

Chebyshev polynomials $\{T_k(x)\}$

Differentiation $\{T_k(x)\} \to \{T_k(x)\}$ is **dense**:

$$\frac{d}{dx} T_k(x) = \sum_j a_j T_j(x)$$

Ill-conditioned



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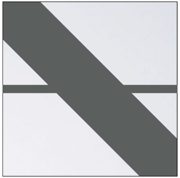


Ultraspherical spectral method

Non-periodic problems

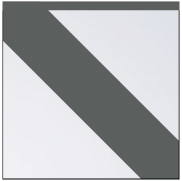
Coefficient-based

Chebyshev polynomials $\{T_k(x)\}$

Differentiation $\{T_k(x)\} \to \{C_k^{(\lambda)}(x)\}$ is **sparse**:

$$\frac{d}{dx} T_k(x) = kC_{k-1}^{(1)}(x)$$
$$\frac{d^2}{dx^2} T_k(x) = 2kC_{k-2}^{(2)}(x)$$
$$\vdots$$

Well-conditioned



The ultraspherical spectral method in 2D
Solving PDEs on rectangles

Solve the elliptic PDE

$$\begin{aligned}\mathcal{L}u(x,y) &= f(x,y) \text{ in } [-1,1]^2\\ u(x,y) &= g(x,y) \text{ on boundary}\end{aligned}$$

where

$$\mathcal{L} = \sum_{i=0}^{2}\sum_{j=0}^{2-i} a_{ij}(x,y)\frac{\partial^{i+j}}{\partial x^i \partial y^j}$$

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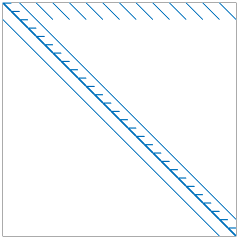
where

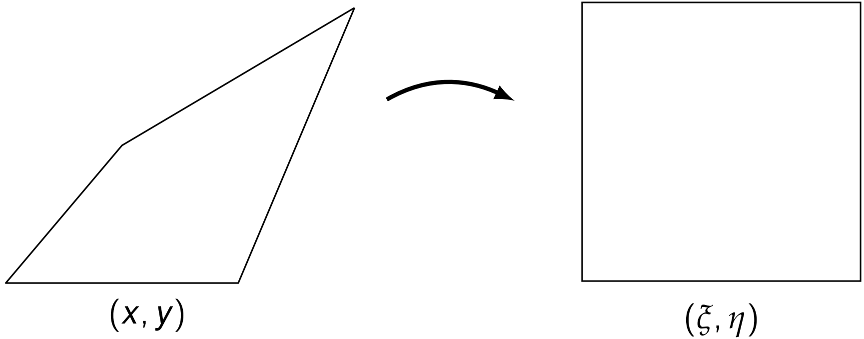
$$\mathcal{L} = \sum_{i=0}^{2}\sum_{j=0}^{2-i} a_{ij}(x,y)\frac{\partial^{i+j}}{\partial x^i \partial y^j} \approx \sum_{k=1}^{K}\left(\mathcal{L}_k^y \otimes \mathcal{L}_k^x\right)$$

1-D ultraspherical method



Almost banded-block-banded
Woodbury solve: $\mathcal{O}(p^4)$
Conditioning: $\mathcal{O}(p^3)$

The ultraspherical spectral method in 2D
Solving PDEs on kites



The ultraspherical spectral method in 2D

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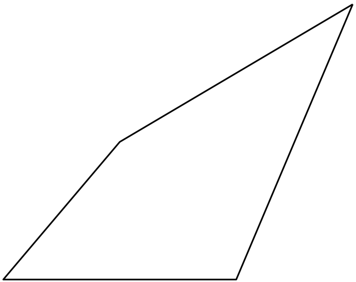
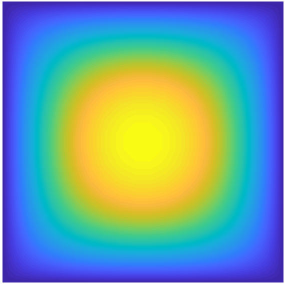
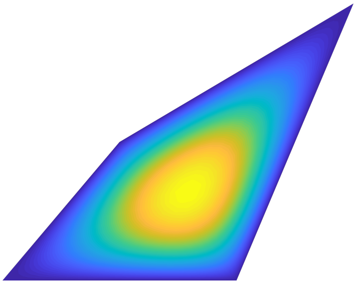


(x, y)



(ξ, η)



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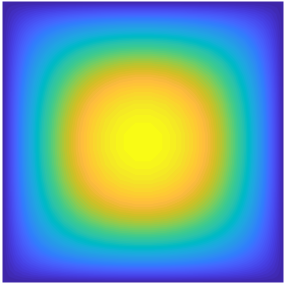

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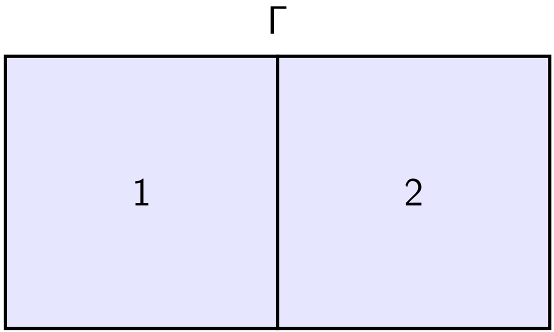
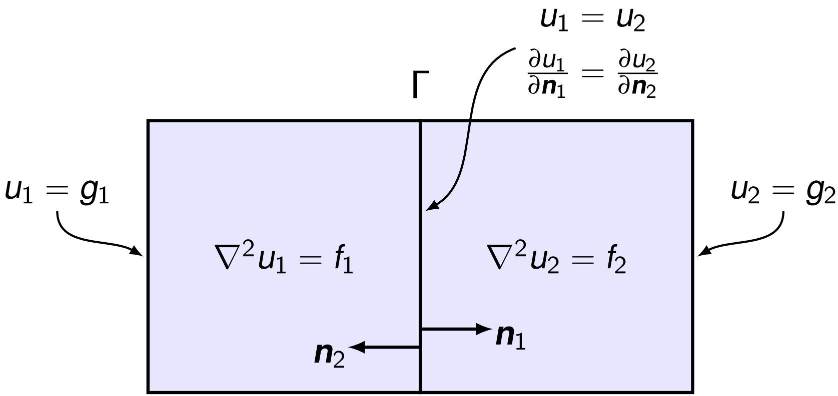
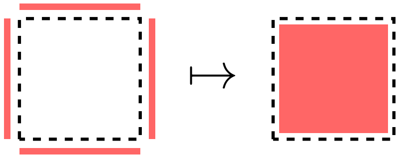

Γ
1
2

The ultraspherical spectral element method
Two glued squares



1. Solution operator: $S \in \mathbb{R}^{n^2 \times 4n}$
   - Maps n coefficients of Dirichlet data on each side to $n \times n$ coefficients of the solution.



The ultraspherical spectral element method
Building blocks

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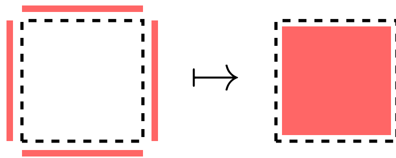
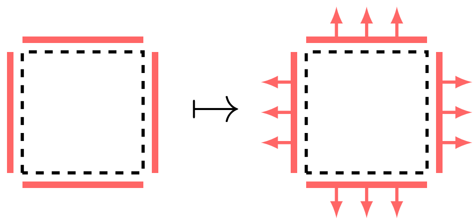


2. Dirichlet-to-Neumann map: $DtN \in \mathbb{R}^{4n \times 4n}$
   - Maps n coefficients of Dirichlet data on each side to n coefficients of the normal derivative of the solution on each side.



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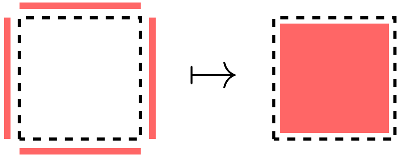

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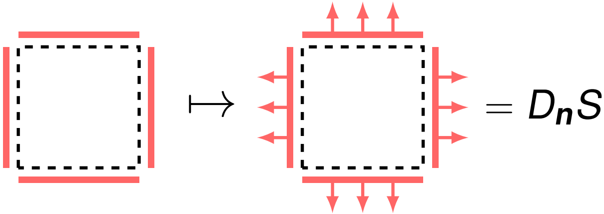

The ultraspherical spectral element method
Constructing the solution operator

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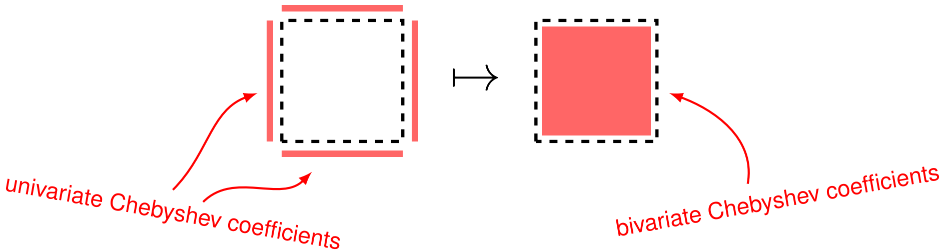

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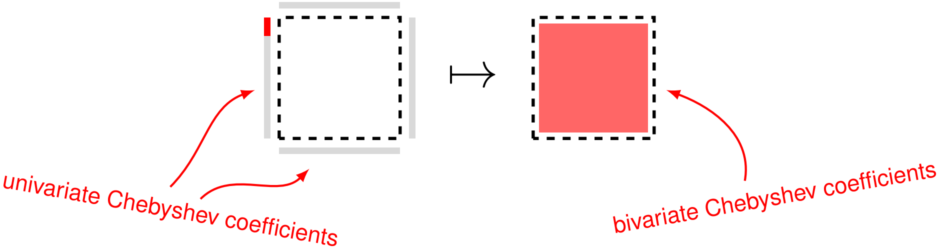

How does the n^{th} Dirichlet coefficient affect the solution?

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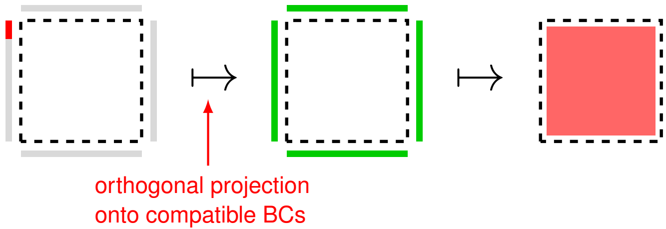

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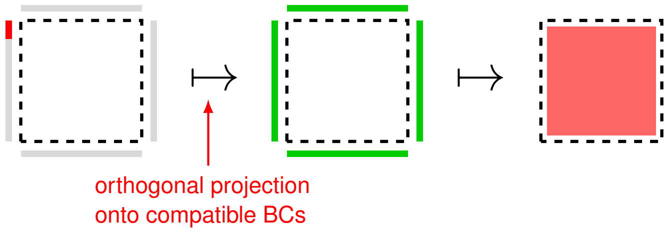

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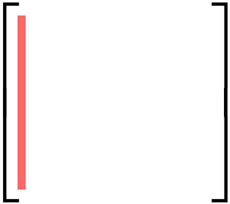
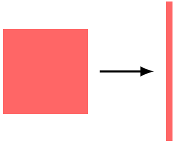
$S =$ 



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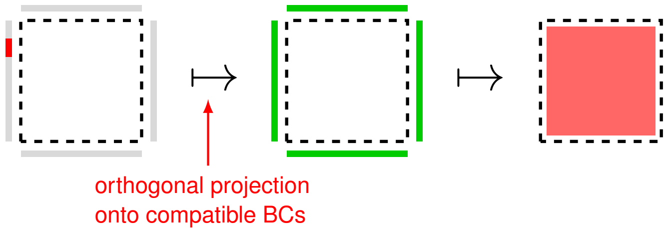

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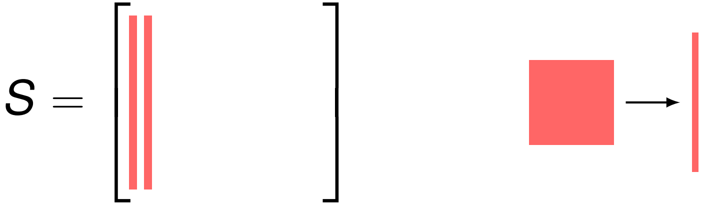

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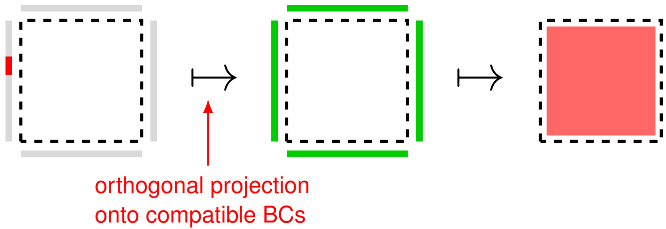

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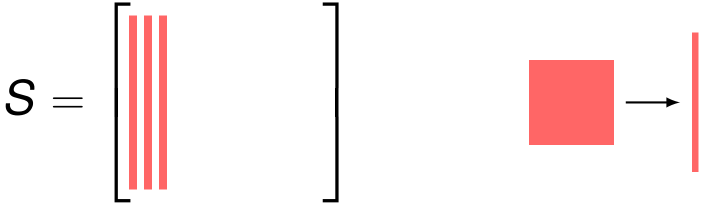

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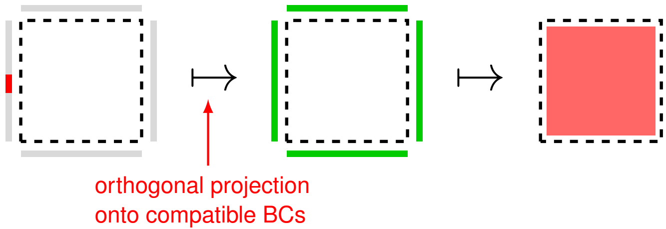

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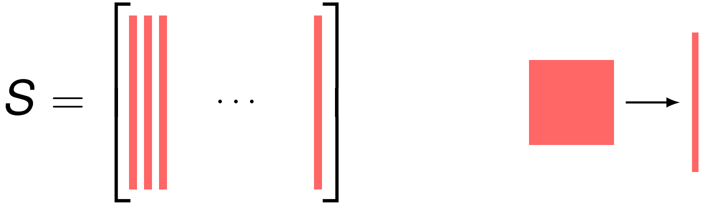
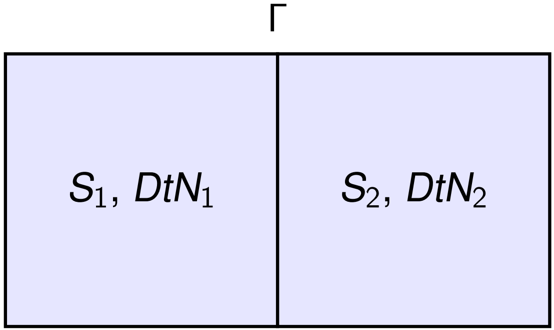
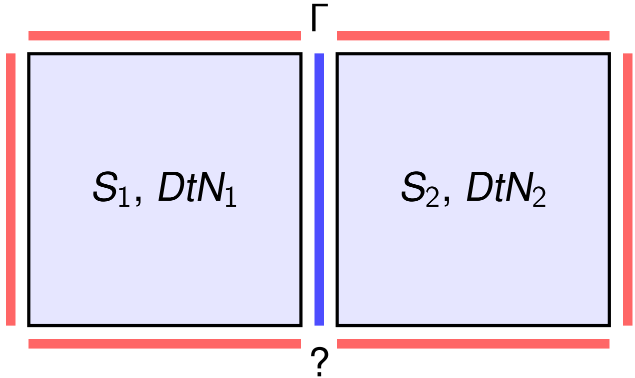


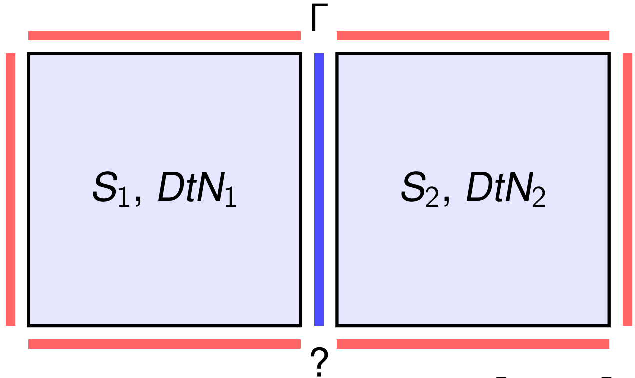
The ultraspherical spectral element method
Merging operators




Γ
$S_1,\ DtN_1$
$S_2,\ DtN_2$
?

The ultraspherical spectral element method

Merging operators

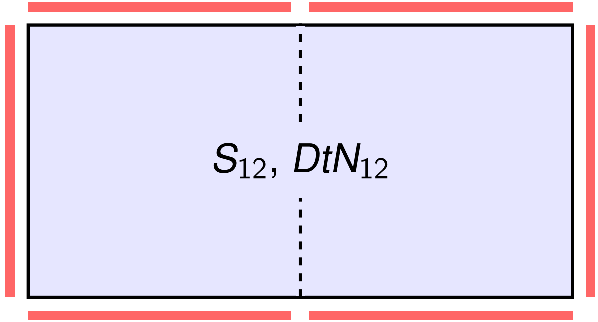


$$S_{12} = -\left(DtN_1^{\Gamma,\Gamma} + DtN_2^{\Gamma,\Gamma} \right)^{-1} \begin{bmatrix} DtN_1^{\Gamma,1} \\ DtN_2^{\Gamma,2} \end{bmatrix}$$

$$DtN_{12} = \begin{bmatrix} DtN_1^{\Gamma,1} & 0 \\ 0 & DtN_2^{\Gamma,2} \end{bmatrix} + \begin{bmatrix} DtN_1^{1,\Gamma} \\ DtN_2^{2,\Gamma} \end{bmatrix} S_{12}$$

The ultraspherical spectral element method

Merging operators



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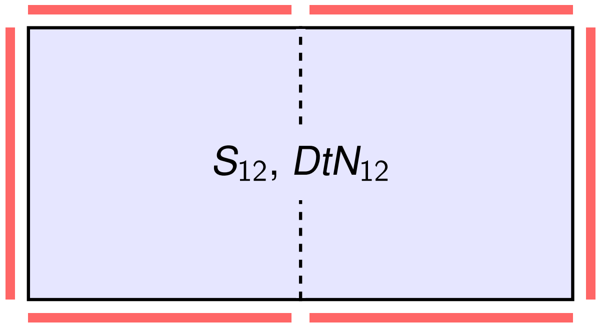

Recurse!

$$S_{12} = -\left(DtN_1^{\Gamma,\Gamma} + DtN_2^{\Gamma,\Gamma}\right)^{-1} \begin{bmatrix} DtN_1^{\Gamma,1} \\ DtN_2^{\Gamma,2} \end{bmatrix}$$

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The ultraspherical spectral element method
Hierarchical Poincaré–Steklov scheme

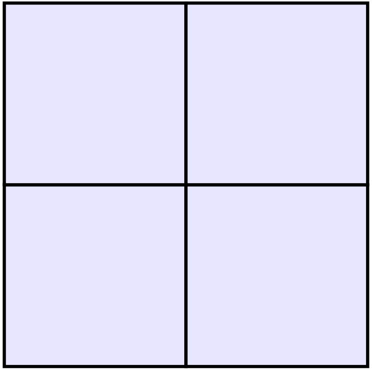
A hierarchical variant of the Schur complement method.

Gunnar Martinsson

Adrianna Gillman

[Martinsson, 2013]

[Gillman & Martinsson, 2014]



A hierarchical variant of the Schur complement method.

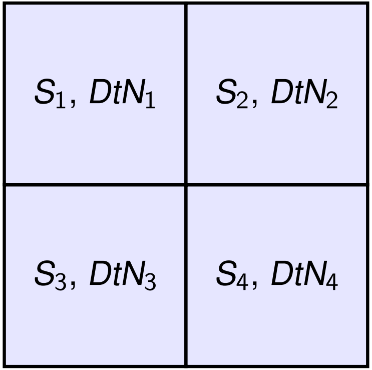

Gunnar Martinsson


Adrianna Gillman

[Martinsson, 2013]

[Gillman & Martinsson, 2014]

Build element operators



The ultraspherical spectral element method
Hierarchical Poincaré–Steklov scheme

A hierarchical variant of the Schur complement method.

Gunnar Martinsson

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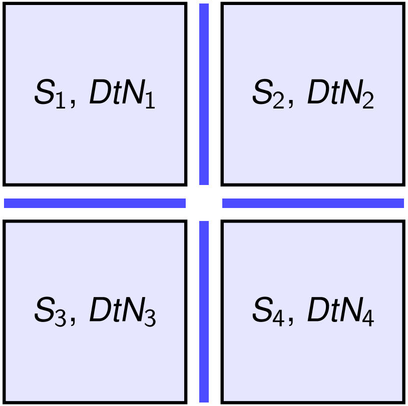
[Martinsson, 2013]

[Gillman & Martinsson, 2014]

Merge operators



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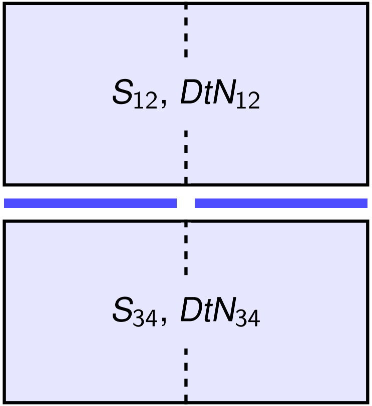
Merge operators



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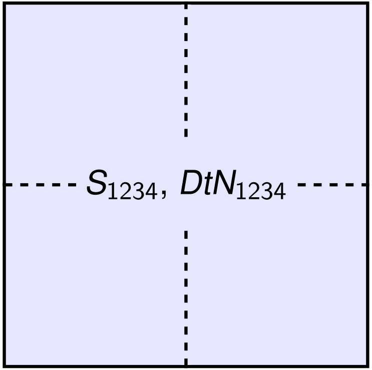
Merge operators



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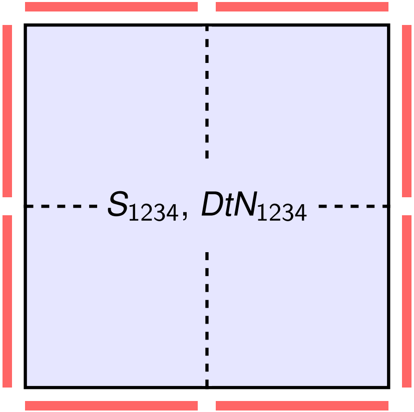
Inject Dirichlet data



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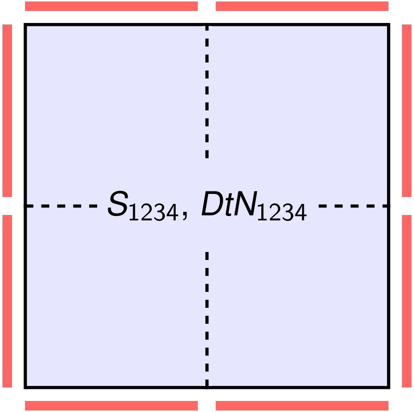
Apply merged operators



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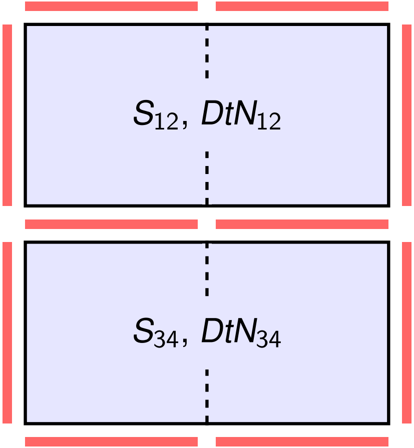
Apply merged operators



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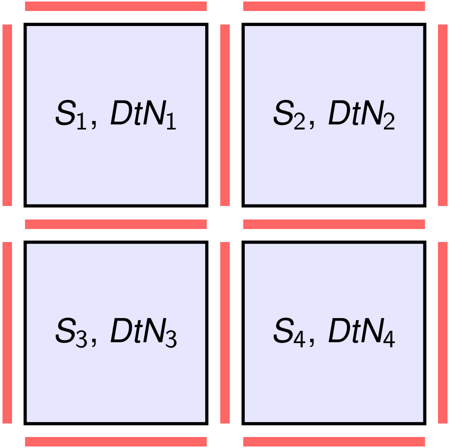

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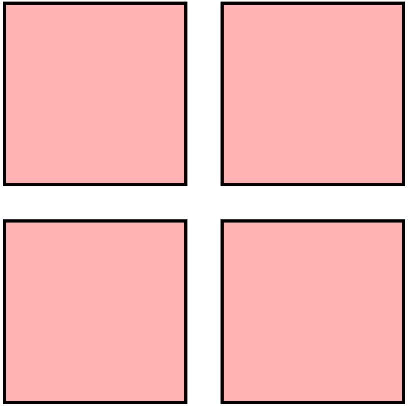
Apply merged operators



The ultraspherical spectral element method

The sparsity of the ultraspherical spectral method allows us to build solution operators on each leaf in $\mathcal{O}(p^4)$ instead of $\mathcal{O}(p^6)$. For $N = p^2/h^2$ degrees of freedom:

$$\underbrace{\frac{p^4}{h^2}}_{\text{leaf computation}} + \underbrace{\frac{p^3}{h^3}}_{\text{merge cost}} + \underbrace{\frac{p^2}{h} + \frac{p}{h}\log\frac{p}{h}}_{\text{solve cost}} \approx \frac{p^4}{h^2} + \frac{p^3}{h^3} \approx Np^2 + N^{3/2}$$

The storage complexity scales as $\mathcal{O}(p^3/h^2)$.

Demo

Ongoing work

- **Benchmarking**: rigorous timing tests to determine practicality.
- **Adaptivity**: automatically detect where to refine h and p.
- **Timestepping**: solution operator can be reused for fast implicit solves.
- **Skinny elements**: high accuracy on elements with small aspect ratio.
- **Parallelizablity**: leaf computations decouple.

Thank you

(Open-source code coming soon.)