Your Paper

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Abstract

Your abstract.

1 Introduction

1.1 Pre-req

Let f be a homogeneous polynomial in $\mathbb{C}[x, y, z]$ of degree $d + 1$ defining a curve $C = V(f)$ and $\delta = P\partial_x + Q\partial_y + R\partial_z$ be a derivation with P, Q, R of the same degree a. Assume that $\delta(f) \in Der(f) = \{\delta|\delta(f) \in (f)\}$, without loss of generality we can take $\delta(f) = 0$, that is $\delta \in Der_0(f) = \{\delta|\delta(f) = 0\}$. Let $\mathcal{T}_f$ be the sheaf of logarithmic derivations of f defined by the exact sequence

$$0 \to \mathcal{T}_f \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{I}_J(d) \to 0 \tag{1}$$

where $\mathcal{I}_J$ is the ideal sheaf defined by the three partial derivatives of f. If δ is a minimal degree derivation then we have an induced section $\tilde{\delta} \in H^0(\mathcal{T}_f(a))$ and a scheme of points Z satisfying the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^2}(-a) \arrow[r, equal] \arrow[d, hook, "\tilde{\delta}"] & \mathcal{O}_{\mathbb{P}^2}(-a) \arrow[d, hook, "\delta"] & & \\
\mathcal{T}_f \arrow[r, hook] \arrow[d, two heads] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \arrow[r, two heads, "\alpha"] \arrow[d, two heads] \arrow[rr, bend left, "\nabla f"] & \mathcal{I}_J(d) \arrow[r, hook] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^2}(d) \\
\mathcal{I}_Z(a-d) \arrow[r, hook] & F \arrow[r, two heads] & \mathcal{I}_J(d) &
\end{tikzcd}$$

The length of Z can be computed from the second Chern class $c_2(\mathcal{T}_f(a))$ with the additional knowledge that $c_2(\mathcal{T}_f) = d^2 - \sum_{p \in Sing(C)} \tau_p(C)$ where $\tau_p(C)$ is the Tjurina number of the singularity p (the sum for every singular point is called the total Tjurina number).

Lemma 1.1. *If C is smooth then $a = d$ and $n = d^2$*

Proof. By dualizing the sequence

$$0 \to \mathcal{T}_f \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(d) \to 0$$

we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{T}_f(d) \to 0 \tag{2}$$

which implies that $H^0(\mathcal{T}_f(d-1)) = 0$ and $H^0(\mathcal{T}_f(d)) \neq 0$ so $a = d$ and $n = d^2$. $\square$

Lemma 1.2. *Let $\mathcal{O}_{Z(\delta)}$ be the structure sheaf of the zeroes of the three homogeneous forms defining δ (we will denote this scheme as $Z(\delta)$). Then:*

$$\mathcal{E}xt^1(F, \mathcal{O}_{\mathbb{P}^2}) \cong \mathcal{O}_{Z(\delta)} \tag{3}$$

Proof. Dualizing the second column gives us the exact sequence:

$$0 \to F^* \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(a) \to \mathcal{E}xt^1(F, \mathcal{O}_{\mathbb{P}^2}) \to 0 \tag{4}$$

which implies that $\mathcal{E}xt^1(F, \mathcal{O}_{\mathbb{P}^2})$ is a quotient of $\mathcal{O}_{\mathbb{P}^2}(a)$ by the image Φ of $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(a)$. Analyzing the support of the exact sequence

$$0 \to F^* \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \Phi \to 0$$

shows us that Φ is zero exactly at $Z(\delta)$, hence it corresponds to $\mathcal{I}_{Z(\delta)}(a)$. □

Lemma 1.3. *Z is contained in $Z(\delta)$.*

Proof. By dualizing the third row we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to F^* \to \mathcal{O}_{\mathbb{P}^2}(d-a) \to \mathcal{E}xt^1(\mathcal{I}_J(d), \mathcal{O}_{\mathbb{P}^2}) \to \mathcal{E}xt^1(F, \mathcal{O}_{\mathbb{P}^2}) \to \mathcal{E}xt^1(\mathcal{I}_Z(a-d), \mathcal{O}_{\mathbb{P}^2}) \to 0 \tag{5}$$

which simplifies into

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to F^* \to \mathcal{O}_{\mathbb{P}^2}(d-a) \to \omega_J \to \mathcal{O}_{Z(\delta)} \to \omega_Z \to 0 \tag{6}$$

and hence the support of ω_Z is contained in $Z(\delta)$. □

1.2 Addition of a line

Now let $l = l_1 * x + l_2 * y + l_3 * z$ be the equation of a line L with $l_i \in \mathbb{C}$. We can construct a derivation of degree $a+1$ in $Der_0(f * l)$ from $Der_0(f)$ resulting in a morphism:

$$\delta \mapsto l\delta := l * \delta - \frac{\delta(l) * \delta_E}{d+2} \tag{7}$$

where $\delta_E = x * \partial_x + y * \partial_y + z * \partial_z$ is the Euler derivation. There is then the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^2}(-a-1) \arrow[r, equal] \arrow[d, hook, "\tilde{\delta}"] & \mathcal{O}_{\mathbb{P}^2}(-a-1) \arrow[d, hook, "l\delta"] & \\
\mathcal{T}_f(-1) \arrow[r, hook] \arrow[d, two heads] & \mathcal{T}_{f*l} \arrow[r, two heads] \arrow[d, two heads] & \mathcal{O}_L(a-d-|Z' \cap L|) \arrow[d, equal] \\
\mathcal{I}_Z(a-d-1) \arrow[r, hook] & \mathcal{I}_{Z'}(a-d) \arrow[r, two heads] & \mathcal{O}_L(a-d-|Z' \cap L|)
\end{tikzcd}$$

which implies that Z' can be divided by points in the line L and points outside the line, given by Z. An interesting fact is that the points in $Z' \cap L$ must be inside $Z(\delta(l))$, while $Z = Z' \setminus L$ are necessarily in the eigenscheme of δ:

Lemma 1.4. *$Z' \setminus Z$ is contained in $Z(\delta(l))$ and Z is in the eigenscheme of δ. Furthermore, $Z(\delta(l)) \subset Z(\delta)$.*

Proof. We start by remembering that any point in Z' must be in the vanishing of the three homogeneous polynomials constituting $l\delta$, so let $p = (p_1; p_2; p_3) \in Z' \subset V(l\delta)$. We have that $0 = (l * \delta - \frac{\delta(l) * \delta_E}{d+2})|_p$, suppose first that p is not in Z and thus $p \in L$ so that $l(p) = 0$. Then $0 = \frac{\delta(l) * \delta_E}{d+2}|_p = \frac{\delta(l)(p)}{d+2}(p_1, p_2, p_3)$ which implies that $\delta(l)(p) = 0$ so p is in $V(\delta(l))$. Now if $p \in Z$ then $p \in Z' \setminus L$, hence $l(p) = \alpha \neq 0$ and $\alpha * (P(p), Q(p), R(p)) = \frac{\delta(l)(p)}{d+1} * (p_1, p_2, p_3)$. We have two cases to consider: If $\delta(l)(p) = 0$ then $P(p) = Q(p) = R(p) = 0$ and if $\delta(l)(p) \neq 0$ we would have that p is a fixed point of the rational map induced by the derivation δ. Either case implies that p is in the eigenscheme of δ.

To prove that $Z(\delta(l)) \subset Z(\delta)$ we consider the following diagram:

$$\begin{tikzcd}
 & \mathcal{O}_{\mathbb{P}^2}(-a) \arrow[r, equal] \arrow[d, hook, "\delta"] & \mathcal{O}_{\mathbb{P}^2}(-a) \arrow[d, hook, "\delta(l)"] \\
\mathcal{T}_f \arrow[r, hook] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \arrow[r, two heads, "\nabla(l)"] \arrow[d, two heads] & \mathcal{O}_{\mathbb{P}^2} \arrow[d, two heads] \\
\mathcal{T}_f \arrow[r, hook] & F \arrow[r, two heads] & \mathcal{O}_{Z(\delta(l))}
\end{tikzcd}$$

By comparing the supports in the third row we conclude that $Z(\delta(l))$ is in $supp(F)$, which is $Z(\delta)$. □

Corollary 1.4.1. *If $\mathcal{T}_f$ is free then $Z' \subset Z(\delta(l)) \cap L$*

Proof. Since $\mathcal{T}_f$ is free we have that Z is empty and $Z' \subset L$. Any point $p \in Z'$ must be in the zero set of the section $l\delta = l * \delta - \frac{\delta(l)}{d+2}\delta_E$, but since $l(p) = 0$ we have that $\frac{\delta(l)(p)}{d+2}\delta_E(p) = \frac{\delta(l)(p)}{d+2}(p_1, p_2, p_3) = 0$ so $\delta(l)(p) = 0$. □

Corollary 1.4.2. *Let L be a generic line so that it intersects the curve X given by f only in simple points. Then:*

- *$\mathcal{T}_f$ free implies that Z' has length a*
- *X smooth implies that Z' has length $h^0(\mathcal{O}_Z) + d = d^2 + d$*

Proof. The computation of $c_2\mathcal{T}_f(a)$ and $c_2\mathcal{T}_{f*l}(a+1)$ can be simplified in the following way: Let $n = h^0(\mathcal{O}_Z)$, $n' = h^0(\mathcal{O}_Z)$, $j = h^0(\mathcal{O}_J)$ and $j' = h^0(\mathcal{O}_{J'})$. Then $n + j = d^2 - a * d + d^2$ and $n' + j' = (d+1)^2 - (a+1) * (d+1) + (a+1)^2$.

Since L contributes with $d+1$ simple points in any case, we know how the total Tjurina number behaves: we have that $j' = j + d + 1$. So by subtracting one equation from the other we have that $n' - n + d + 1 = -d^2 + a*d - a^2 + d^2 + 2d + 1 - ad - a - d - 1 + a^2 + 2a + 1 = -a^2 + 2d + 1 - a - d - 1 + a^2 + 2a + 1 = d + a + 1$ so that $n' - n = a$. The first case is when $n = 0$ so that $n' = a$, the second case happens when $n = d^2$ and $a = d$ so that $n' = d^5 + d$. □

Possible question: Are all a points the entire $L \cap Z(\delta(l))$????

Proposition 1.5. *If $\mathcal{T}_f$ or better, the additional points are given by: is free then $Z' = L \cap Z(\delta(l))$*

Proof. TODO!!!!! □

Example 1.1. *Consider the union of two conics sharing a unique tangent given by $x = 0$, we can write explicitly $f = (z^2 - x * y) * (z^2 - x * y + x * z) * (z^2 - x * y - x * z)$. Denote $X = V(f)$ and let $L = V(a * x + b * y + c * z)$ be a general line.*

A minimal derivation of f has degree 2 and can be found by computing the canonical derivation of the pencil of two conics sharing one tangent, say $\delta = P * \partial_x + Q * \partial_y + R * \partial_z$ with P, Q, R of degree 2. Explicitly we have that $P = -x^2, Q = 2 * z^2 + x * y, R = x * z$.

We can also compute the length of Z and Z' with the additional knowledge of the Tjurina numbers for the singularities of $X \cup L$ and conclude that X is free (hence Z is empty) and Z' has length 2. So we have the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^2}(-3) \arrow[r, equal] \arrow[d, hook, "\tilde{\delta}"] & \mathcal{O}_{\mathbb{P}^2}(-3) \arrow[d, hook, "l\delta"] & \\
\mathcal{T}_f(-1) \arrow[r, hook] \arrow[d, two heads] & \mathcal{T}_{f*l} \arrow[r, two heads] \arrow[d, two heads] & \mathcal{O}_L(-3 - |Z' \cap L|) \arrow[d, equal] \\
\mathcal{O}_{\mathbb{P}^2}(-4) \arrow[r, hook] & \mathcal{I}_{Z'}(-3) \arrow[r, two heads] & \mathcal{O}_L(-3 - |Z' \cap L|)
\end{tikzcd}$$

The third row implies that Z' is contained in L so that $|Z' \cap L| = 2$. By the lemma above we also conclude that $Z' \subset V(\delta(l)) = V(a * P + b * Q + c * R)$, so it is possible to provide a form for both points in Z' in terms of a, b, c by considering the intersection of $V(\delta(l))$ and L:

Im working in the affine patch $z = 1$ because in my tests with code, Z' never touched the line $z = 0$. But I'm not sure how to argue that here.

We work in the affine patch $z = 1$, $a*P+b*Q+c*R = 0$ implies that $-a*x^2+x*(c+b*y)+2*b = 0$. Using the equation of the line $b*y = -a*x-c$ we find $-2*a*x^2+2*b = 0$ and hence (for a general line) $Z' = \{(\sqrt{\frac{b}{a}} : \frac{-\sqrt{a*b}-c}{b} : 1), (-\sqrt{\frac{b}{a}} : \frac{\sqrt{a*b}-c}{b} : 1)\}$. Denote the first point by C and the second point by D.

There are two distinguished lines for this example: $x = 0$ and $z = 0$. The first is the unique tangent for the singular point $(0 : 1 : 0)$ and the second is the unique line passing through both singularities $(0 : 1 : 0), (1 : 0 : 0)$. The intersection of each of those two lines with L gives then two points $A = (0 : -c : b)$ and $B = (-b : a : 0)$ in L. By projecting A, B, C, D to the line $z = 0$ we can compute the cross-ratio $CR(A, B; C, D) = \frac{AC}{BC} * \frac{BD}{AD}$:

$$CR(A,B;C,D) = \frac{\sqrt{\frac{b}{a}}}{\sqrt{a*b}+c-\sqrt{a}*\sqrt{b}} * \frac{-\sqrt{a*b}+c+\sqrt{a*b}}{-\sqrt{\frac{b}{a}}} = -1 \tag{8}$$

So the cross-ratio is constant for any a, b, c, provided that $a \neq 0$ and $b \neq 0$. (I think there is a better argument here instead of requiring the non-vanishing conditions of a,b...).

Example 1.2. *Consider a pencil of two conics sharing two tangents given by $C_1 = x^2 - z^2$ and $C_2 = y^2$ and pick the union of two smooth conics belonging to the arrangement. Let $L = V(a*x+b*y+c*z)$ be a general line. Then the canonical derivation is given by $\delta' = y*\delta$ and hence the minimal derivation is $\delta = P*\partial_x + Q*\partial_y + R*\partial_z$ with P, Q, R of degree 1 which implies that $\delta(l) = a*P+b*Q+c*R$ is also of degree 1. Using the fact that the Tjurina number for each singularity is 3 we conclude that the length of Z is $c_2(\mathcal{T}_C(1)) = 1$, the same can be done for Z' since we just need to add each intersection with L in the total Tjurina number. If L is generic we conclude that Z' has length $c_2(\mathcal{T}_{C\cup L}(2)) = 2$. To understand the geometric configuration of Z and Z' we consider the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r, equal] \arrow[d, hook, "\tilde{\delta}"] & \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[d, hook, "l\delta"] & \\
\mathcal{T}_f(-1) \arrow[r, hook] \arrow[d, two heads] & \mathcal{T}_{f*l} \arrow[r, two heads] \arrow[d, two heads] & \mathcal{O}_L(-2-|Z'\cap L|) \arrow[d, equal] \\
\mathcal{I}_Z(-3) \arrow[r, hook] & \mathcal{I}_{Z'}(-2) \arrow[r, two heads] & \mathcal{O}_L(-2-|Z'\cap L|)
\end{tikzcd}$$

Using the third row of the diagram and the previous lemma we can also conclude that the point Z lies outside of L and hence $Z \cap L = 1$ for a generic line L. Let L_δ be the line defined by $\delta(l)$, the intersection $L_\delta \cap L$ is precisely the unique point in $Z' \setminus Z$. We can compute this point using the explicit form for the minimal derivation:

$$P = z, Q = 0, R = x \implies \delta(l) = a*z + c*x$$

$$a*x+b*y+c*z = 0 \implies L \cap L_\delta = \{(\frac{-a}{c} : \frac{-(c^2-a^2)}{bc} : 1)\}$$

Now the splitting, I'm not sure where to put it *Now, $|Z' \cap L| = 1$ implies that there is a surjection:*

$$\mathcal{T}_{f*l}|_L \to \mathcal{O}_L(-3) \to 0 \tag{9}$$

Since $c_1(\mathcal{T}_{f*l}) = -4$ we need only to consider the possible splittings $(-1, -3)$ and $(0, -4)$. If we had $\mathcal{T}_{f*l}|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(-4) \to \mathcal{O}_L(-3)$ there would be a cokernel corresponding to a point in L. But since this morphism is a surjection we conclude this case does not occur, so we have $\mathcal{T}_{f*l}|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-3)$.

Example 1.3. *Consider our last example with L a sufficiently generic line passing through the unique point Z we still have that the length of Z' is 2. I'm a bit confused here because of the third exact row*

$$0 \to \mathcal{I}_Z(-3) \to \mathcal{I}_{Z'}(-2) \to \mathcal{O}_L(-2-|Z'\cap L|) \to 0 \tag{10}$$

Since Z is in L by construction the exact sequence above seems to be in contradiction. Because it is saying that L passes through Z' but does not intersect with Z. I think the contradiction is I assumed *that $l\delta$ is a minimal derivation so we conclude that δ itself is a minimal derivation for $C \cup L$ and thus it is free?*

The original purpose of the example 1.3 was to look at $L_\delta = \delta(l)$ to conclude that $\mathcal{T}_{f*l}|_{L_\delta} = \mathcal{O}_{L_\delta}(-4) \oplus \mathcal{O}_{L_\delta}$ as we spoke in our last meeting.

Proposition 1.6. *Let $X = V(f)$ be a free curve with $\mathcal{T}_f \cong \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b)$ where $a \leq b$ and suppose that l is a linear form such that $H^0(\mathcal{T}_{l*f}(a)) \neq 0$. Then $\mathcal{T}_{l*f}$ is free.*

Proof. Let ξ be a derivation of degree a such that $\xi(l * f) = 0$. The Leibniz rule implies that $f * \xi(l) = -l * \xi(f)$ so $\xi(f) \in (f)$ and hence ξ is in $Der(f)$ (and $\xi(f) = -\frac{\xi(l)}{l} * f$). There is a canonical morphism given by $\mathcal{T}_{l*f} \to \mathcal{T}_f$ sending ξ to $\xi - \frac{\xi(f)}{f * deg(f)} * \delta_E$ $\square$