

Fixed-Shape Ellipse by Three Points

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Introduction

The shape of an ellipse is given by its major-axis and minor-axis, $(a, b) \in \mathbb{R}^2$, with $a > b > 0$.

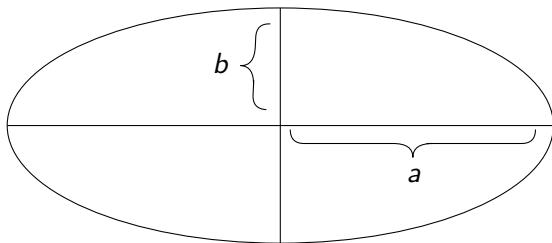


Figura: An ellipse with shape parameters a and b .

Introduction

Here, the shape will be fixed and the center and angle of rotation are free.

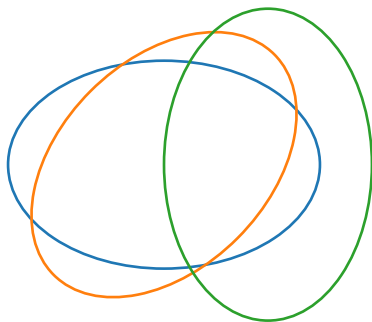


Figura: A fix-shape ellipse at different centers and with different angles of rotation.

Introduction

Problem definition

Given three points $u, v, w \in \mathbb{R}^2$, and the shape $(a, b) \in \mathbb{R}^2$ of an ellipse:

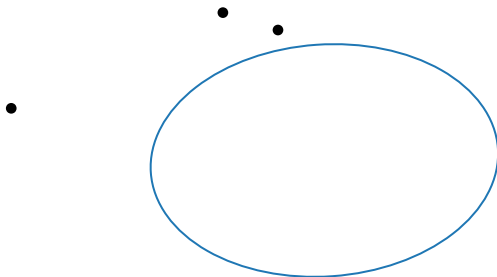


Figura: An instance of the problem.

Introduction

Problem definition

A solution is given by the ellipse's center $q \in \mathbb{R}^2$ and the angle of rotation $\theta \in [0, \pi)$, such that u, v, w lie on its border. **We want to find every solution!**

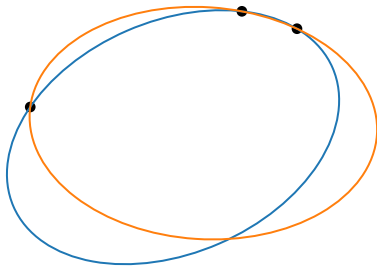


Figura: Every solution for the instance shown previously.

Introduction

The equation of an ellipse is given by:

$$\frac{\left(\begin{bmatrix} x - q_x \\ y - q_y \end{bmatrix}^T \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \right)^2}{a^2} + \frac{\left(\begin{bmatrix} x - q_x \\ q_y - y \end{bmatrix}^T \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix} \right)^2}{b^2} = 1.$$

- ▶ Fixing the points u, v, w , we get 3 equations and 3 unknowns (q_x, q_y, θ) .
- ▶ Finding every solution is difficult.

Transforming the problem

Let's make the problem simpler by transforming it into a circle problem.

An ellipse with shape (a, b) can be transformed into a circle of radius b through scaling the x -axis by $\frac{b}{a}$:

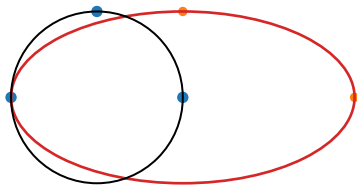


Figura: Turning an ellipse with shape (a, b) into a circle of radius b .

Transforming the problem

Let's rotate the points instead of rotating the ellipse:



Figura: Three points at their initial location.

Transforming the problem

Firstly, we rotate leaving one point fixed at $(0,0)$:



Figura: After rotation.

Transforming the problem

Then, we scale by $\frac{b}{a}$ and check the radius of the circle:

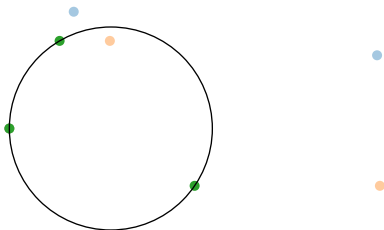


Figura: After scaling.

Transforming the problem

If the radius is b , the angle of rotation is a solution:

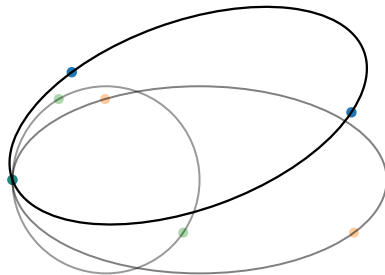


Figura: One solution for this instance.

Transforming the problem

Formally, we can transform the problem by:

- ▶ Translate the points so $u = (0, 0)$.
- ▶ Rotate by θ and scale the x -axis by $\frac{b}{a}$.
- ▶ Find the θ 's which produce a circle with radius b .

This transformation is expressed by:

$$\varphi(p, \theta) = \begin{bmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix},$$

for $p = u, v, w$.

This is an one variable problem on a closed interval!

Fixed-shape ellipse by three points

There is a known formula for the radius of a circumscribed circle [JY60, p. 189]:

$$R = \frac{\|\varphi(v, \theta)\|_2 \|\varphi(w, \theta)\|_2 \|\varphi(v, \theta) - \varphi(w, \theta)\|_2}{4A(\theta)}$$

- ▶ R is the radius.
- ▶ $A(\theta)$ is the area of the triangle defined by the points $\varphi(u, \theta), \varphi(v, \theta), \varphi(w, \theta)$.

Fixed-shape ellipse by three points

We define the function $\xi : [0, \pi) \mapsto \mathbb{R}$:

$$\xi(\theta) = 16b^2 A(\theta)^2 - \|\varphi(v, \theta)\|_2^2 \|\varphi(w, \theta)\|_2^2 \|\varphi(v, \theta) - \varphi(w, \theta)\|_2^2$$

The roots of ξ are solutions of our problem.

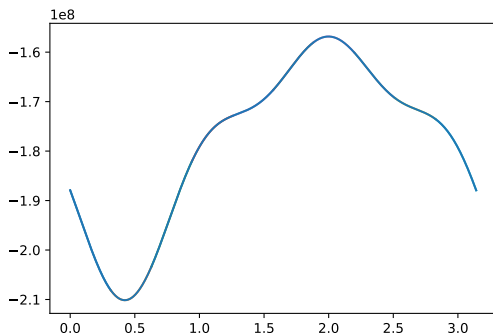


Figura: An example of ξ .

Fixed-shape ellipse by three points

There is no clear pattern in ξ .

- It can be written as

$$\xi(\theta) = \sum_{0 \leq j, k \leq \text{Deg}(\xi)} \alpha_{j,k} \cos^j \theta \sin^k \theta.$$

- Degree 6, at most 12 roots in $[0, 2\pi)$ [Pow81, p. 150]

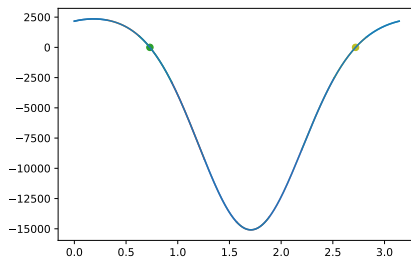


Figura: Another example of ξ .

Fixed-shape ellipse by three points

An example with two roots.

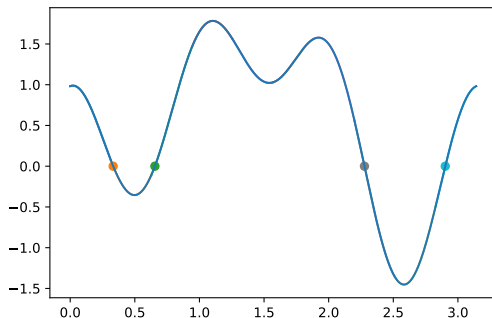


Figura: An example of ξ with 4 roots.

Polynomial Interpolation

It is a way to approximate a function by a simpler one (a polynomial).

- ▶ A degree n of the interpolation is determined.
- ▶ $n + 1$ points are chosen, such that the polynomial has to pass through.
- ▶ Can be calculated using Lagrange's formula.
- ▶ We can find every root of a polynomial by determining the eigenvalues of a matrix called The Companion Matrix [HJ86, p. 195].
- ▶ Depending on the points, the interpolation can be a bad approximation. It can get worse even if n is increased (Runge's Phenomenon) [Pow81, p. 37].

Chebyshev Interpolation

Chebyshev Polynomial

$T_n : [-1, 1] \mapsto [-1, 1]$ is the n -degree Chebyshev polynomial [MH03]:

$$T_n(\cos t) = \cos(nt)$$

Also, it can be defined recursively:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

Chebyshev Interpolation

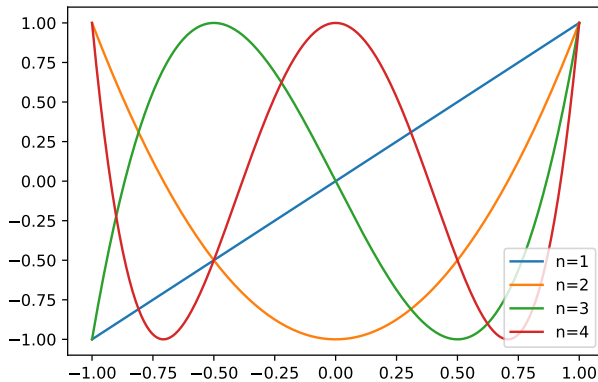


Figura: Chebyshev Polynomials of degree 1, 2, 3, 4.

Chebyshev Interpolation

Interpolation on the roots of T_n , also known as Chebyshev Nodes:

$$x_k = \cos\left(\pi \frac{2k-1}{2n}\right)$$

The interpolation of a function $f : [-1, 1] \mapsto \mathbb{R}$ can be written directly using Chebyshev polynomial as basis:

$$f(x) \approx \sum_{k=0}^n a_k T_k(x)$$

- ▶ This can be done in $\mathcal{O}(n^3)$ [Boy13].
- ▶ A simple change of coordinates lets the interpolation to be done on any closed interval!

Chebyshev Interpolation

Why is it good?

- ▶ Numerically stable! Way better than polynomials in the power format [Gau79].
- ▶ No Runge's Phenomenon, the interpolation converges to f .
 - ▶ $\mathcal{O}(n^{-m})$ if f is m times differentiable [GO77, p. 28].
 - ▶ $\mathcal{O}(C^n)$, for $C < 1$, if f is analytical in a neighborhood of $[-1, 1]$ [BT04].
- ▶ Very used in practice:
 - ▶ Present in external libraries like NumPy for Python.
 - ▶ Matlab tool Chebfun: allows functions to be treated as vectors.

Example

ξ and its approximation of degree 8.

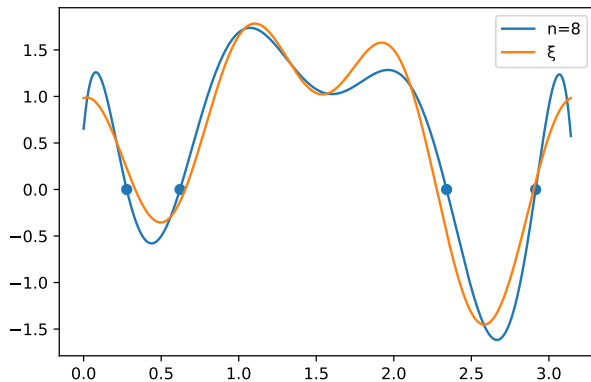


Figura: An example of degree 8 approximation.

Example

ξ and its approximation of degree 10.

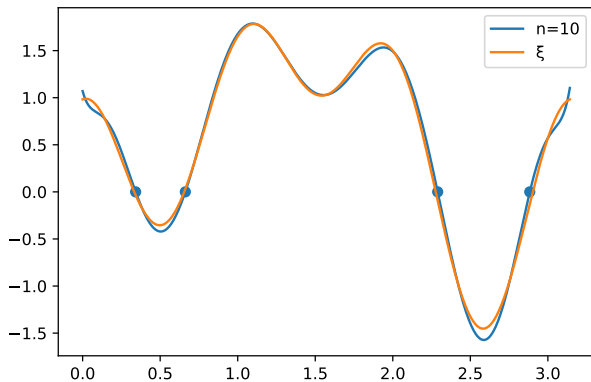


Figura: An example of degree 10 approximation.

Example

ξ and its approximation of degree 12.

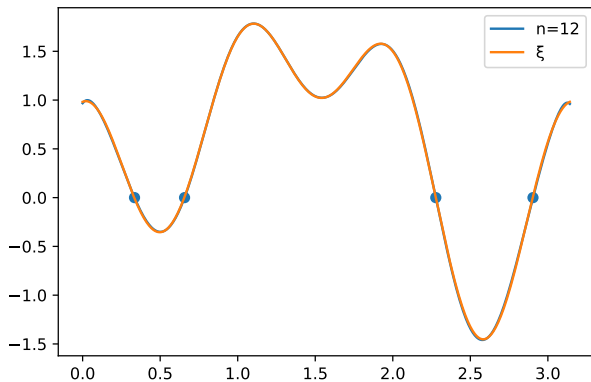


Figura: An example of degree 12 approximation.

Chebyshev Interpolation

Choosing the degree of the interpolation:

- ▶ There is no guaranteed way to choose it.
- ▶ A good rule is to examine the last coefficient (the last coefficient rule-of-thumb [Boy01, p .50]).
- ▶ For a predefined ϵ , choose n , such that:

$$|a_n| \leq \epsilon$$

- ▶ There are other ways like checking the error on a Lobatto grid [BG07].

Chebyshev Interpolation

For $n = 32$, a precision of 10^{-10} is expected.

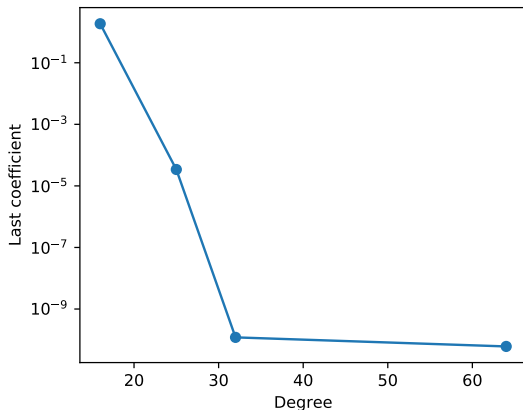


Figura: $|a_n|$ for the interpolation of ξ for an instance.

Chebyshev Interpolation

Roots

The roots of a Chebyshev polynomial can be found though determining the eigenvalues of a Chebyshev companion matrix [Boy13]. For $n = 5$ we have:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{a_0}{2a_5} & -\frac{a_1}{2a_5} & -\frac{a_2}{2a_5} & -\frac{a_3}{2a_5} & -\frac{a_4}{2a_5} \end{bmatrix}$$

- ▶ This matrix is a Hessenberg matrix.
- ▶ Its eigenvalues can be found by a QR decomposition in $\mathcal{O}(n^3)$.

Chebyshev Interpolation

Roots

The largest error on roots that were found for $n = 32$ is around 10^{-14} for an instance:

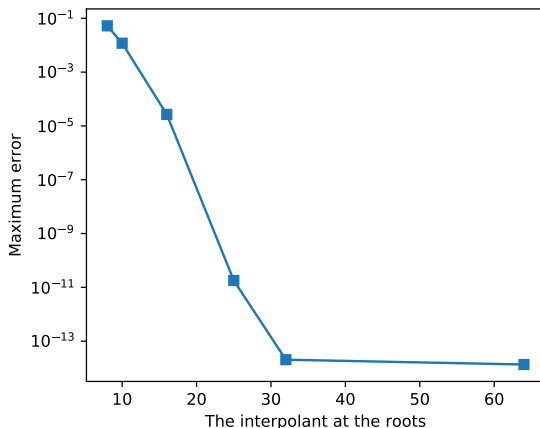


Figura: $|\xi(\hat{\theta})|$, where $\hat{\theta}$ is a root of f_n .

Chebyshev Interpolation

Roots

The experiments were made using Python with the NumPy library.
The running time is really low, even for $n = 64$.

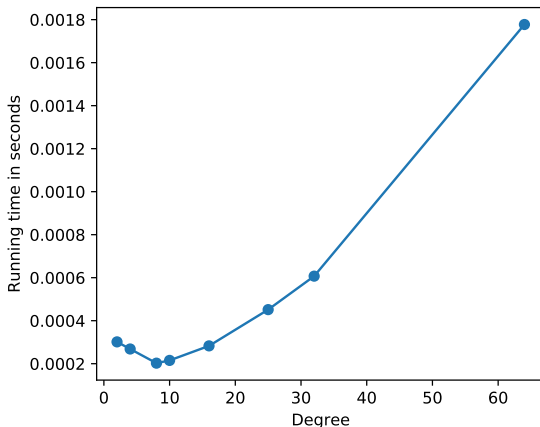


Figura: The running time to find the roots of f_n .

Example

An example with 4 solutions.

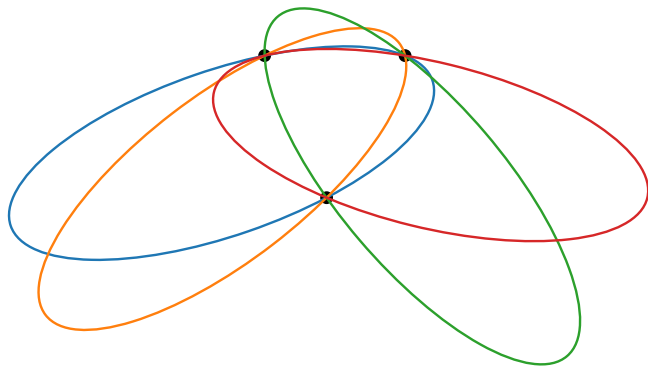











Figura: An example with 4 solutions.

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