New Exact Algorithms for Planar Maximum Covering Location by Ellipses Problems

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The Maximum Covering Location Problem (MCLP)

- Introduced in [3],
- Maximize the coverage demand vertices on a graph,
- Choose the location (vertex) of a fixed number of facilities,
- A demand vertex is considered covered if a facility is located within its coverage radius.

The Planar Maximum Covering Location Problem (PMCLP)

- Introduced in [4],
- Maximize the coverage demand vertices in $\mathbb{R}^2$,
- Choose the location (could be anywhere in $\mathbb{R}^2$) of a fixed number of facilities,
- A demand vertex is considered covered if a facility is located within its coverage radius,
- Several distance functions were studied. We are particularly interested in the Euclidean PMCLP.

Introduction

We propose algorithms for two versions of PMCLP.

Introduction

MCE

Planar Maximum Covering Location by Ellipses Problem (MCE):

- Introduced in [2],
- Mixed Non-linear optimization and a heuristic method in [2],
- Exact method, solving convex sub-problems in [1].

Our algorithm

Based on the approach used for the Euclidean PMCLP in [4].
Transform MCE into a combinatorial optimization problem.

Planar Maximum Covering Location by Ellipses with Rotation Problem (MCER):

- Introduced in [1],
- Exact method, solving many optimization sub-problems in [2],
- Heuristic method in [1].
- Much more challenging than MCE.

Our algorithm

Transforms MCER into a combinatorial optimization problem.

Introduction

Ellipse

The shape of an ellipse is given by its major-axis and minor-axis, $(a, b) \in \mathbb{R}^2_{>0}$, $a > b$.

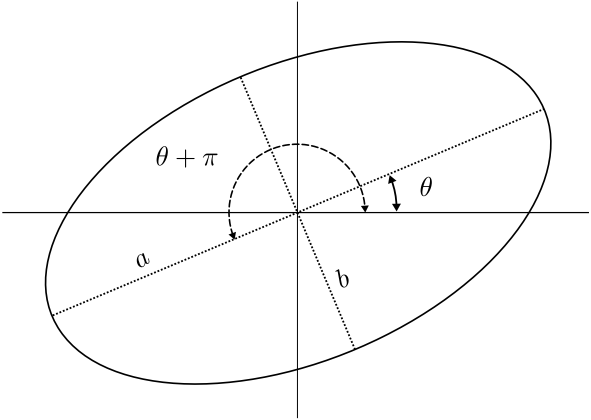


Figura: An ellipse with shape parameters a and b.

Introduction

Ellipse

An ellipse can be defined using a norm function $||\cdot||_{a,b,\theta}$ given by

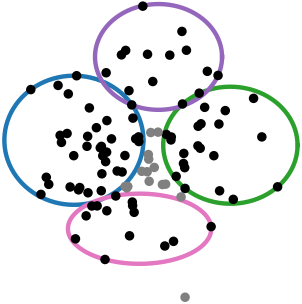
$$||x||_{a,b,\theta} = \left\|\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} x\right\|_2 .$$

Problem definition

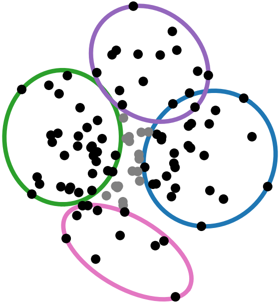
An instance of both MCE and MCER is given by

- A demand set $\mathcal{P} := \{p_1, \ldots, p_n\}$, $p_j \in \mathbb{R}^2$;
- Each point has a weight $\mathcal{W} := \{w_1, \ldots, w_n\}$, $w_j \in \mathbb{R}_{\geq 0}$;
- A list of shape parameters $\mathcal{R} := \{(a_1, b_1); \ldots; (a_m, b_m)\}$, $(a_j, b_j) \in \mathbb{R}^2_{>0}$, with $a_j > b_j$.

Problem definition



(a)



(b)

Figura: Solutions for the same instance of (a) MCE, and (b) MCER.

Problem definition

More notation

Weight function

Let $w \colon 2^{\mathcal{P}} \to \mathbb{R}$ be a function defined as

$$w(A) = \sum_{j \colon p_j \in A} w_j.$$

MCE's solution

$Q := (q_1, \ldots, q_m) \in \mathbb{R}^{2m}$.

MCER's solution

$Q := ((q_1, \theta_1); \ldots; (q_m, \theta_m)) \in (\mathbb{R}^2 \times [0, \pi))^m$.

Problem definition
MCE

Let $E_j\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the coverage region of the j-th ellipse defined as

$$E_j(q) = \{x \in \mathbb{R}^2\colon ||x - q||_{a,b,0} \leq 1\}.$$

Then, MCE is defined as the optimization problem:

$$\max_Q w\left(\bigcup_{j=1}^{m} \mathcal{P} \cap E_j(q_j)\right).$$

Problem definition

MCER

Let $E_j : \mathbb{R}^2 \times [0, \pi) \to \mathbb{R}^2$ be the coverage region of the j-th ellipse defined as

$$E_j(q, \theta) = \{x \in \mathbb{R}^2 \colon ||x - q||_{a,b,\theta} \leq 1\}.$$

Then, MCER is defined as the optimization problem:

$$\max_{Q} w \left(\bigcup_{j=1}^{m} \mathcal{P} \cap E_j(q_j, \theta_j) \right).$$

Remark

For the one-facility MCE and MCER, we omit the index referring to the ellipse.

MCE

$\{p_1, p_2, p_3\} \subset E(q)$

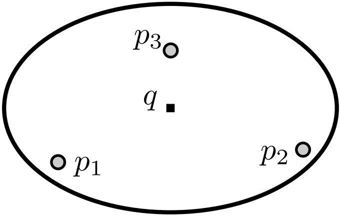


Figura: A solution for the one-facility MCE.

MCE

$$\{p_1, p_2, p_3\} \subset E(q) \implies q \in E(p_1) \cap E(p_2) \cap E(p_3).$$

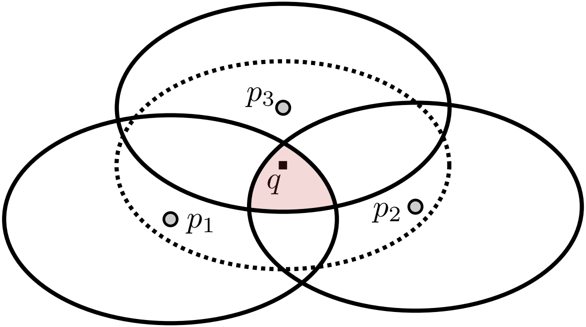


Figura: A solution for the one-facility MCE.

MCE

In general we have

$$A = \mathcal{P} \cap E(q) \implies q \in \cap_{p \in A} E(p),$$

and

$$q' \in \cap_{p \in A} E(p) \implies A \subset E(q').$$

Intersection region of ellipses

By [6], we have that if $|A| > 1$, there is at least one intersection between two ellipses in the border of $\cap_{p \in A} E(p)$.

MCE

$\{q_1, q_2, q_3\} \subset \bigcup_{1 \le i < j \le 3} \partial E(p_i) \cap \partial E(p_j)$.

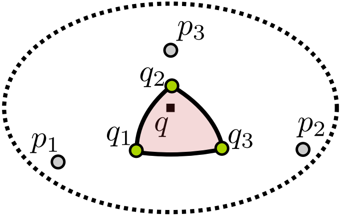


Figura: A solution for the one-facility MCE.

MCE

In general, we have

$$|\partial E(u) \cap \partial E(v)| \leq 2,$$

and that $\partial E(u) \cap \partial E(v)$ can be determined analytically.

Based on [4], we define a Candidate List Set (CLS) for each facility as follows.

Definition

Given an instance of MCE, for all $k \in \{1, \ldots, m\}$, we define the CLS for the k-th ellipse as

$$S_k = \mathcal{P} \cup \left(\bigcup_{1 \le i < j \le n} \partial E_k(p_i) \cap \partial E_k(p_j) \right).$$

MCE

Main result

Theorem

Given an instance of MCE, and $S_1, \ldots, S_m$ as defined previously, then the set

$$\Omega = \{(q_1, \ldots, q_m) \colon \text{ for all } q_k \in S_k\}$$

contains an optimal solution of MCE and $|\Omega| \leq n^{2m}$.

- Notice that $|S_k| \leq n(n+1)/2 \leq n^2$.
- An algorithm with $\mathcal{O}(mn^{2m+1})$ runtime complexity can be implemented.

Determining Every Center and Angle of Rotation of An Ellipse Given Its Shape and Three Points that It Must Contain

Given

- The coverage region function of an ellipse $E\colon \mathbb{R}^2 \times [0, \pi) \to \mathbb{R}^2$.
- Three points $u, v, w \in \mathbb{R}^2$.

Let us call E3P the problem whose solution is given by $(q, \theta) \in \mathbb{R}^2 \times [0, \pi)$, such that

$$\{u, v, w\} \subset \partial E(q, \theta).$$

We want to compute every solution of E3P.
We did not find any work on E3P in the literature.

E3P

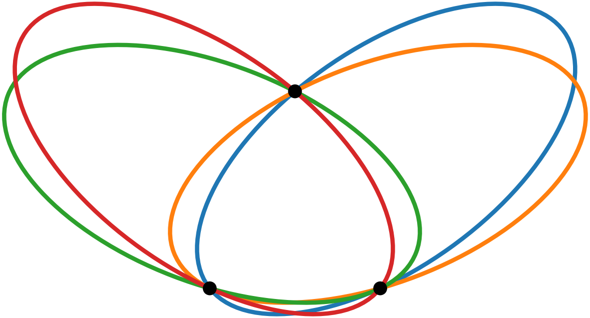


Figura: Example of every solution for an instance of E3P.

Let us define a function $\varphi\colon \mathbb{R}^2 \times [0, \pi) \to \mathbb{R}^2$ that transforms the problem as follows.

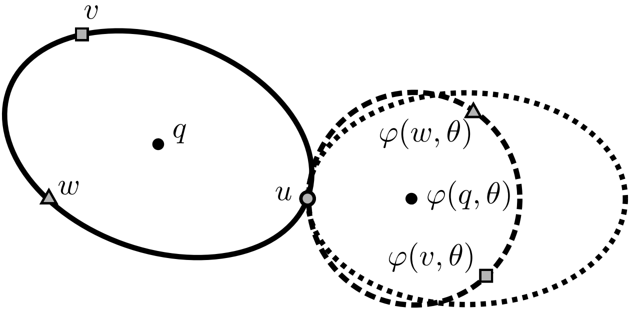


Figura: Transforming a solution of E3P into a solution of the circumcircle problem.

If u is at the origin, this function can be described as

$$\varphi(p,\theta) = \begin{bmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}.$$

- For a fixed angle, φ is bijective, we refer to φ^{-1} as its inverse.
- Let us denote by $\Lambda(\theta)$ as the triangle with vertices $\varphi(u), \varphi(v), \varphi(w)$.
- E3P is equivalent to determining θ, such that the circumscribed circle of $\Lambda(\theta)$ has radius b.

A circle is uniquely defined by $\Lambda(\theta)$, and its radius and center can be determined analytically [10].
Let $|\Lambda(\theta)|$ be the area of $\Lambda(\theta)$, and imposing that the radius of that circle is equal to b, we define a function $\xi\colon [0,\pi) \to \mathbb{R}$ whose roots determine solutions of E3P.

$$\xi(\theta) = 16b^2|\Lambda(\theta)|^2 - \|\varphi(v,\theta)\|_2^2 \, \|\varphi(w,\theta)\|_2^2 \, \|\varphi(v,\theta) - \varphi(w,\theta)\|_2^2 \, .$$

E3P

Transforming the problem

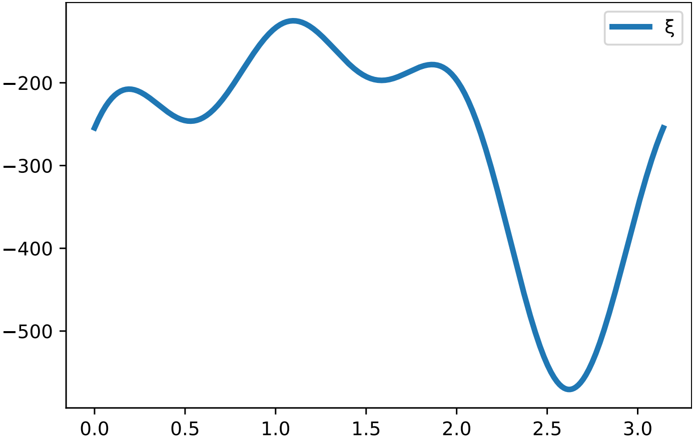


Figura: A plot of ξ in $[0, \pi)$ for an instance of E3P.

E3P

Lemma

E3P has at most six solutions.

- ξ can be written as $\sum_{0 \leq j+k \leq 6} c_{j,k} \cos^j \theta \sin^k \theta$,
- It is a real trigonometric polynomial of degree 6, can be written as

$$\sum_{k=0}^{6} a_k \cos k\theta + \sum_{k=1}^{6} b_k \sin k\theta$$

- A n-degree real trig. poly. can have up to $2n$ roots in $[0, 2\pi)$ [7, p. 150],
- As ellipses are symmetrical, we can dismiss half of the roots.

We will convert ξ into a complex polynomial on $z = e^{i\theta}$ using the identities

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We define a function $g\colon \mathbb{S} \to \mathbb{C}$ given by

$$g(z = e^{i\theta}) = \xi(\theta).$$

Extending the domain to $\mathbb{C}$ we define a polynomial

$$f(z) = z^6 g(z).$$

- The exponents of g go from -6 to 6,
- The roots of f that are in $\mathbb{S}$ are roots of g.
- In practice, we used symbolic computation for this task,
- No loss of accuracy [9].

E3P

Finding the roots of ξ

We can cut in half the degree of f by observing that

$$angle(-z) = \pi + angle(z).$$

As an ellipse rotated by θ is identical to an ellipse rotated by $\pi + \theta$, we get that

$$f(-z) = f(z).$$

Therefore, every odd coefficient of f is zero, and we can use another substitution $y = z^2$ to obtain a 6-degree polynomial.

E3P

Determining the roots of a polynomial

For every univariate polynomial of degree n, there exists a companion matrix, which is a $n \times n$ matrix, such that its eigenvalues are the zeros of that polynomial [5, p. 195].

E3P

Determining the roots of a polynomial

For a degree-4 polynomial $\sum_{k=0}^{4} a_k x^k$, a companion matrix is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{a_0}{a_4} & -\frac{a_1}{a_4} & -\frac{a_2}{a_4} & -\frac{a_3}{a_4} \end{bmatrix}.$$

- QR algorithm can determine the eigenvalues in $\mathcal{O}(n^3)$ [8].
- In practice, we use LAPACK's ZGEEV routine.

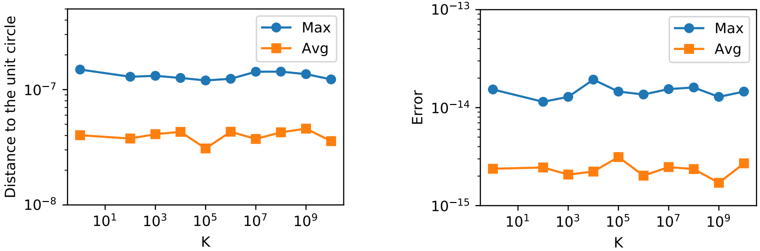
We need to choose a precision constant to check if a root is in $\mathbb{S}$. In practice, every eigenvalue $\hat{z}$ returned by the QR algorithm, we consider it to be in $\mathbb{S}$ if

$$|1 - \hat{z}| < \epsilon.$$

- Experiments with three points $(a, 0); (-a, 0); (a, b)$ rotated by a random angle.

E3P

Choosing a precision constant



(a) (b)

Figura: (a) $|1 - \hat{z}|$. (b) $|f(\hat{z})|$.

- We define $\epsilon = 10^{-6}$.
- We further check if $|f(\hat{z})| < 10^{-9}$ to consider $\hat{z}$ to be a root of f.

E3P

These possible properties were observed in practice, and could be used in future work:

- Instances with 6 solutions seem to always come from an equilateral triangle's vertices.
- Instances with 4 solutions seem to always come from an isosceles triangle's vertices.
- The coefficients of f seem to have the following symmetry property:

$$c_k = \overline{c_{6-k}}.$$

MCER

The algorithm for MCE is based on the fact that there is at most 2 centers for an ellipse to contain two points.

For MCER, we will use a similar idea based on the results for E3P.

- We will prove that any solution (q, θ) for the one-facility MCER can always be classified as one out of three possible types.

MCER

It covers at most one point.

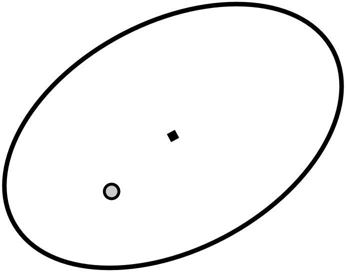


Figura: Example of solution for the one-facility MCER.

MCER

It covers at least three points, and there exists $\{u, v, w\} \subset E(q, \theta)$, such that their E3P's instance has at least one solution.

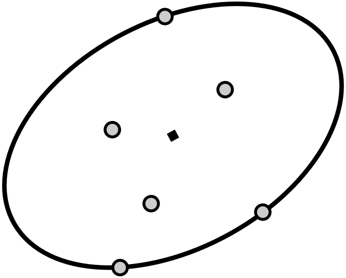


Figura: Example of solution for the one-facility MCER.

MCER

It covers at least two points, and there is no $\{u, v, w\} \subset E(q, \theta)$, such that their E3P's instance has at least one solution.

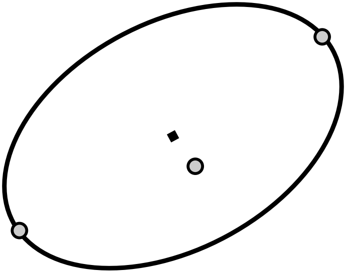


Figura: Example of solution for the one-facility MCER.

MCER

Let Q and Q' be two solutions of MCER, then we say that $Q \succ Q'$ if, and only if

$$\bigcup_{j=1}^{m} \mathcal{P} \cap E_j(q'_j, \theta'_j) \subset \bigcup_{j=1}^{m} \mathcal{P} \cap E_j(q_j, \theta_j).$$

Lemma

Let Q be a solution for the one-facility MCER, such that $|\mathcal{P} \cap E(q, \theta)| \geq 2$. There exists a solution Q', such that $Q' \succ Q$ and $|\mathcal{P} \cap \partial E(q', \theta')| \geq 2$.

MCER

Feasible angle

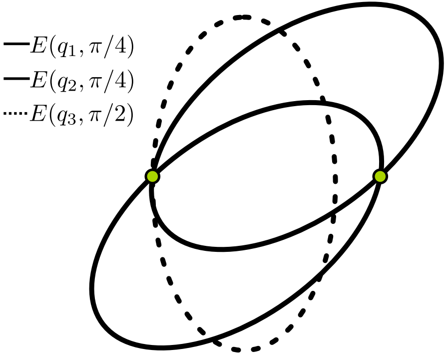


Figura: The solid border ellipses are rotated by a (E, u, v)-feasible angle, while the ellipse with a dashed border is rotated by a not (E, u, v)-feasible angle.

MCER

Feasible angle

Definition

Let E be the coverage region of an ellipse and $u, v \in \mathbb{R}^2$. An angle $\theta \in [0, \pi)$ is said to be (E, u, v)-feasible if there is $q \in \mathbb{R}^2$ such that $\{u, v\} \subset \partial E(q, \theta)$. In addition to that, the set of (E, u, v)-feasible angles is referred to as

$$\Phi(u, v) := \{\theta \in [0, \pi) : \theta \text{ is a } (E, u, v)\text{-feasible angle}\}.$$

MCER

Feasible angle

Lemma

Given an instance of the one-facility MCER, if $u, v \in \mathcal{P}$ *have the same y-coordinate and* $||u - v||_2 \leq 2a$*, then* $\Phi(u, v) = [0, \alpha] \cup [\pi - \alpha, \pi)$*, for some* $\alpha \in [0, \pi/2]$.

MCER

Feasible angle

$L(m)$ is the maximum distance between the intersection of a line from the set $\{y = mx + c \colon c \in \mathbb{R}\}$ and an ellipse $\partial E(0,0)$.

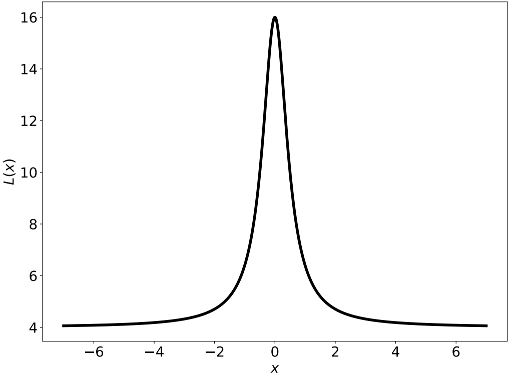


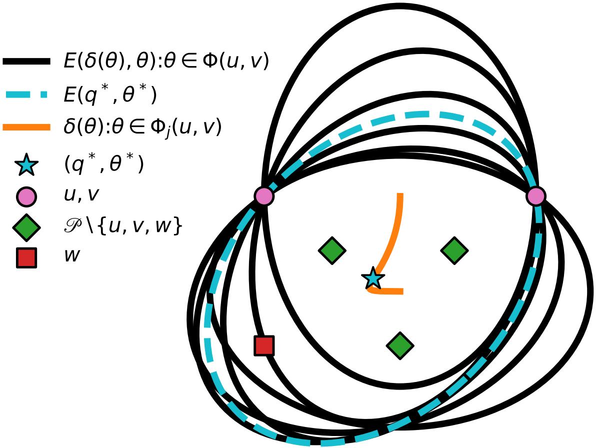
Figura: Plot of function L in the interval $[-7, 7]$.

MCER

The only problem now is with respect to solutions which we can find another solution covering the same points with three points on the ellipse.

Next, we prove that in this case, for any feasible angle of rotation, a solution exists covering the same set of points.

MCER



$E(\delta(\theta), \theta) : \theta \in \Phi(u, v)$
$E(q^*, \theta^*)$
$\delta(\theta) : \theta \in \Phi_j(u, v)$
(q^*, θ^*)
u, v
$\mathscr{P} \setminus \{u, v, w\}$
w

MCER

Let Q and Q' be two solutions of MCER, we say that they are equivalent if, and only if

$$\bigcup_{j=1}^{m} \mathcal{P} \cap E_j(q_j, \theta_j) = \bigcup_{j=1}^{m} \mathcal{P} \cap E_j(q'_j, \theta'_j).$$

Lemma

Let Q^* be a solution of the one-facility MCER, such that $|\mathcal{P} \cap E(q^*, \theta^*)| \geq 2$. If for all $\bar{Q} \succ Q^*$, $|\mathcal{P} \cap \partial E(\bar{q}, \bar{\theta})| < 3$, then there exists $\{u, v\} \subset \mathcal{P} \cap E(q^*, \theta^*)$, such that for all $\theta \in \Phi(u, v)$ there exists $q \in \mathbb{R}^2$, such that (q, θ) is equivalent to Q^*.

MCER

Proof

- Take $u, v \in E(q^*, \theta^*)$, such that there exists $Q' \succ Q$ with $\{u, v\} \subset \partial E(q', \theta')$.
- We can rotate the coordinate system to make $\Phi(u, v) = [0, 2\alpha]$.
- Define $\delta \colon \Phi(u, v) \to \mathbb{R}^2$, such that

$$\begin{aligned} \text{for all } \theta \in \Phi(u, v) \implies \{u, v\} \subset \partial E(\delta(\theta), \theta) \\ \delta(\theta') = q' \end{aligned}$$

- That is, $\delta(\theta)$ returns the center to make the ellipse contain the two points u and v.

MCER

Proof

- For any w covered, by continuity, we have that

$$\exists\theta \in \Phi(u,v)\colon ||w - \delta(\theta)||_{a,b,\theta} > 1$$
$$\Leftrightarrow$$
$$\exists\bar{\theta} \in \Phi(u,v)\colon |w - \delta(\bar{\theta})||_{a,b,\bar{\theta}} = 1.$$

- That is, if for any feasible angle w becomes uncovered, there must be another angle that puts it on the ellipse.
- The same argument can be used for a point that is initially uncovered.

MCE

With this lemma, we can choose any feasible angle for a pair of points or three points to put on the ellipse.

Let $u \in \mathbb{R}^2$, we define $\angle u$ denotes the minimal angle between the vector u and the x-axis.

- If $\Phi(u, v) \neq \emptyset$, then $\angle(u - v) \in \Phi(u, v)$.
- The longest segment that intersects an ellipse is its major-axis.

Following this, we define the CLS for each ellipse.

Definition

Given an instance of MCER. Then, for all $j \in \{1, \ldots, m\}$, we define the CLS of the j-th ellipse as $S_j = S_j^{(1)} \cup S_j^{(2)} \cup S_j^{(3)}$ with

$$S_j^{(1)} = \bigcup_{u \in \mathcal{P}} \{(u, 0)\}$$

$$S_j^{(2)} = \bigcup_{\{u,v\} \subset \mathcal{P}} \{(q, \angle(u - v)) \in \mathbb{R}^2 \times [0, \pi) : \{u, v\} \subset \partial E_j(q, \angle(u - v))\}$$

$$S_j^{(3)} = \bigcup_{\{u,v,w\} \subset \mathcal{P}} \{(q, \theta) \in \mathbb{R}^2 \times [0, \pi) : \{u, v, w\} \subset \partial E_j(q, \theta)\}.$$

MCER

Finally, we can construct a finite set which contains an optimal solution.

Theorem

Given an instance of MCER, let Ω be a set of solutions defined as

$$\Omega = \{Q \in (\mathbb{R}^2 \times [0, \pi))^m \colon (q_j, \theta_j) \in S_j \text{ for all } j \in \{1, \ldots, m\}\},$$

Then there exists an optimal solution $Q^* \in \Omega$, and $|\Omega| \leq n^{3m}$.

MCER

proof

$$|S_j^{(2)}| \leq 2\binom{n}{2} = \frac{n(n+1)}{2} \leq n^2,$$

$$|S_j^{(3)}| \leq 6\binom{n}{3} = n((n-1)^2+1) \leq n^3.$$

- For an ellipse, if an equivalent solution is not in $S_j^{(2)}$, we can apply the previous lemma, which implies that an equivalent solution is in $S_j^{(3)}$.
- An $\mathcal{O}(mn^{3m+1})$ runtime algorithm can be implemented.

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