Points:

$$\begin{aligned} p_i &= (x_i, y_i) \\ p_1 &= (0, 0) \end{aligned} \tag{1}$$

If p_i are not co-linear and the ellipse with form:

$$x^2 + qy^2 = a^2b^2 \tag{2}$$

can be determined by three points:

$$\frac{(x-x_1)(x-x_2)+q(y-y_1)(y-y_2)}{(y-y_1)(x-x_2)-(y-y_2)(x-x_1)} = \frac{(x_3-x_1)(x_3-x_2)+q(y_3-y_1)(y_3-y_2)}{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)} \tag{3}$$

q is fixed, we want to find θ, let

$$M(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

Now, lets define

$$u_i = M(\theta)(p_i - c) \tag{5}$$

$$c = (h, k) \tag{6}$$

we can rearrange 3

$$\frac{(X-u_1)^t(X-u_2)}{det(X-u_1, X-u_2)} = \frac{(u_3-u_1)^t(u_3-u_2)}{det(u_3-u_1, u_3-u_2)} \tag{7}$$

Analyzing the second part:

$$\begin{aligned} u^t v = \cos^2\theta(u_x v_x + q u_y v_y) + \\ \sin\theta\cos\theta((q-1)(u_y v_x + u_x v_y)) + \\ \sin^2\theta(u_y v_y + q u_x v_x) \end{aligned}$$

Now looking at $det(u_3, u_3 - u_2)$

$$\begin{aligned} det(u_3, u_3-u_2) = det(M(\theta)v, M(\theta)w) = \\ \begin{vmatrix} M(\theta)_1 v & M(\theta)_2 v \\ M(\theta)_1 w & M(\theta)_2 w \end{vmatrix} = \end{aligned} \tag{8}$$

$$\begin{aligned} (v_x\cos\theta - v_y\sin\theta)(w_x\sin\theta + w_y\cos\theta) - (\sin\theta v_x + \cos\theta v_y)(w_x\cos\theta - w_y\sin\theta) = \\ \cos^2\theta(v_x w_y - v_y w_x) + \sin^2\theta(-v_y w_x + v_x w_y) + \sin\theta\cos\theta(v_x w_x - v_y w_y - v_x w_x + v_y w_y) = \\ v_x w_y(\cos^2\theta + \sin^2\theta) - v_y w_x(\cos^2\theta + \sin^2\theta) = \\ v_x w_y - v_y w_x = det(v, w) = det(u_3, u_3 - u_2) = det(p_3, p_3 - p_2) \end{aligned}$$

Then, the second part is:

$$\frac{u_3^t(u_3 - u_2)}{det(p_3, p_3 - p_2)} \tag{9}$$

The main equation is:

$$\frac{(X - u_1)^t(X - u_2)}{det(X - u_1, X - u_2)} = \frac{u_3^t(u_3 - u_2)}{det(p_3, p_3 - p_2)} \tag{10}$$

Looking at the first part:

$$(X - u_1)^t(X - u_2) = X^t(X - u_2) = X^tX - X^tM(\theta)p_2 =$$
$$x^2 + qy^2 - x((p_{2x} - h)\cos\theta - (p_{2y} - k)\sin\theta) - qy((p_{2x} - h)\sin\theta + (p_{2y} - k)\cos\theta)$$

and

$$det(X, X - u_2) = \begin{vmatrix} x & y \\ x - M(\theta)_1 p_2 & y - M(\theta)_2 p_2 \end{vmatrix} =$$
$$\begin{vmatrix} x & y \\ x - (p_{2x}\cos\theta - p_{2y}\sin\theta) & y - (p_{2x}\sin\theta + p_{2y}\cos\theta) \end{vmatrix} =$$
$$y(p_{2x}\cos\theta - p_{2y}\sin\theta) - x(p_{2x}\sin\theta + p_{2y}\cos\theta)$$

Making:

$$\gamma = p_{2x}\cos\theta - p_{2y}\sin\theta \tag{11}$$
$$\gamma = p_{2x}\frac{1 - t^2}{1 + t^2} - p_{2y}\frac{2t}{1 + t^2} \tag{12}$$
$$\delta = p_{2x}\sin\theta + p_{2y}\cos\theta \tag{13}$$
$$\gamma = p_{2x}\frac{2t}{1 + t^2} + p_{2y}\frac{1 - t^2}{1 + t^2} \tag{14}$$
$$\Phi = \frac{u_3^t(u_3 - u_2)}{det(p_3, p_3 - p_2)} = \Phi_a\cos^2\theta + \Phi_b\sin(2\theta) + \Phi_c\sin^2\theta = \tag{15}$$
$$\Phi = \Phi_a(\frac{1 - t^2}{1 + t^2})^2 + \Phi_b(\frac{4t(1 - t^2)}{(1 + t^2)^2}) + \Phi_c(\frac{2t}{1 + t^2})^2 \tag{16}$$
$$\frac{1}{4}((\Phi_a - \Phi_c)\cos 2\theta + 2\Phi_b\sin 2\theta + \Phi_a + \Phi_c) = \tag{17}$$
$$\Phi_a\cos 2\theta + \Phi_b\sin 2\theta + \Phi_c \tag{18}$$

Then, we get:

$$\frac{(X - u_1)^t(X - u_2)}{det(X - u_1, X - u_2)} = \frac{x^2 + qy^2 - x\gamma - qy\delta}{y\gamma - x\delta} = \Phi \tag{19}$$

Rearranging

$$
\begin{aligned}
x^2 + qy^2 - x(\gamma - \Phi(\delta)) - qy(\delta + \frac{\Phi\gamma}{q}) = 0 \\
(x - \frac{\gamma - \Phi\delta}{2})^2 + q(y - \frac{q\delta + \Phi\gamma}{2q})^2 = R \\
R = \frac{(\gamma - \Phi\delta)^2}{4} + \frac{(q\delta + \Phi\gamma)^2}{4q^2}
\end{aligned}
$$

1 Wrong one

Then, as q is fixed, we need to impose:

$$
\frac{(\gamma - \Phi\delta)^2}{4} + \frac{(q\delta + \Phi\gamma)^2}{4q^2} = a^2
$$

$$
q^2(\gamma^2 - 2\Phi\gamma\delta + \Phi^2\delta^2) + q^2\delta^2 + 2q\delta\Phi\gamma + \Phi^2\gamma^2 = 4q^2a^2 =
$$

$$
q^2\gamma^2 + q^2\Phi^2\delta^2 + q^2\delta^2 + \Phi^2\gamma^2 + 2\Phi\gamma\delta(q - q^2) = 4q^2a^2
$$

$$
\gamma^2(q^2 + \Phi^2) + \delta^2[q^2(1 + \Phi^2)] + \gamma\delta[2\Phi(q - q^2)] = 4q^2a^2
$$

Then,

$$
\gamma^2 = p_{2x}^2 \cos^2\theta - p_{2x}p_{2y} \sin 2\theta + p_{2y}^2 \sin^2\theta \tag{20}
$$

$$
\delta^2 = p_{2x}^2 \sin^2\theta + p_{2x}p_{2y} \sin 2\theta + p_{2y}^2 \cos^2\theta \tag{21}
$$

$$
\delta\gamma = \cos^2\theta(p_{2x}p_{2y}) + \frac{\sin(2\theta)}{2}(p_{2x}^2 - p_{2y}^2) - \sin^2\theta(p_{2x}p_{2y}) \tag{22}
$$

$$
\Phi^2 = \tag{23}
$$

We have

$$
\begin{aligned}
\cos^2\theta[(q^2 + \Phi^2)p_{2x}^2 + q^2(\Phi^2 + 1)p_{2y}^2 + 2\Phi(q - q^2)p_{2x}p_{2y}] + \\
\sin^2\theta[(q^2 + \Phi^2)p_{2y}^2 + q^2(\Phi^2 + 1)p_{2x}^2 - 2\Phi(q - q^2)p_{2x}p_{2y}] + \\
\sin(2\theta)[\Phi(q - q^2)(p_{2x}^2 - p_{2y}^2) - (q^2 + \Phi^2)p_{2x}p_{2y} + q^2(\Phi^2 + 1)p_{2x}p_{2y}] = 4q^2a^2
\end{aligned}
$$

—

$$
\begin{aligned}
(\frac{1 - t^2}{1 + t^2})^2[(q^2 + \Phi^2)p_{2x}^2 + q^2(\Phi^2 + 1)p_{2y}^2 + 2\Phi(q - q^2)p_{2x}p_{2y}] + \\
(\frac{2t}{1 + t^2})^2[(q^2 + \Phi^2)p_{2y}^2 + q^2(\Phi^2 + 1)p_{2x}^2 - 2\Phi(q - q^2)p_{2x}p_{2y}] + \\
(\frac{4t(1 - t^2)}{1 + t^2})[\Phi(q - q^2)(p_{2x}^2 - p_{2y}^2) - (q^2 + \Phi^2)p_{2x}p_{2y} + q^2(\Phi^2 + 1)p_{2x}p_{2y}] = 4q^2a^2
\end{aligned}
$$

Which can be written as

$$A\cos^2\theta + B\sin(2\theta) + C\sin^2\theta = D \tag{24}$$
$$A, B, C, D > 0 \tag{25}$$

Which can be rearranged as:

$$\frac{A-C}{2}\cos(2\theta) + B\sin(2\theta) + \frac{A+C-2D}{2} = 0$$

Rewritting everything (keep up!):

$$\begin{aligned}\hat{A} &= \frac{A-C}{2}\\ \hat{B} &= B\\ \hat{C} &= \frac{A+C-2D}{2}\\ \hat{A}\cos(2\theta) + \hat{B}\sin(2\theta) + \hat{C} &= 0\end{aligned}$$

Then,

$$\hat{A}\cos(t) + \hat{B}\sin(t) + \hat{C} = \hat{C} + \sqrt{\hat{A}^2 + \hat{B}^2}\sin(t+\alpha) = 0 \tag{26}$$
$$t + \alpha = \arcsin\frac{-\hat{C}}{\sqrt{\hat{A}^2 + \hat{B}^2}} \tag{27}$$

where

$$\tan\alpha = \frac{\hat{A}}{\hat{B}} \Rightarrow \alpha = \arctan\frac{\hat{A}}{\hat{B}}$$

Then, finally

$$\theta = \arcsin\frac{-\hat{C}}{2\sqrt{\hat{A}^2 + \hat{B}^2}} - \frac{1}{2}\arctan\frac{\hat{A}}{\hat{B}} \tag{28}$$

Similirarly we can achieve the other result:

$$\theta = \arccos\frac{-\hat{C}}{2\sqrt{\hat{A}^2 + \hat{B}^2}} + \frac{1}{2}\arctan\frac{\hat{A}}{\hat{B}} \tag{29}$$