

Points:

$$\begin{aligned} p_i &= (x_i, y_i) \\ p_1 &= (0, 0) \end{aligned} \tag{1}$$

If p_i are not co-linear and the ellipse with form:

$$x^2 + qy^2 = a^2b^2 \tag{2}$$

can be determined by three points:

$$\frac{(x - x_1)(x - x_2) + q(y - y_1)(y - y_2)}{(y - y_1)(x - x_2) - (y - y_2)(x - x_1)} = \frac{(x_3 - x_1)(x_3 - x_2) + q(y_3 - y_1)(y_3 - y_2)}{(y_3 - y_1)(x_3 - x_2) - (y_3 - y_2)(x_3 - x_1)} \tag{3}$$

q is fixed, we want to find θ , let

$$M(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \tag{4}$$

Now, lets define

$$u_i = M(\theta)p_i \tag{5}$$

we can rearrange 3

$$\frac{(X - u_1)^t(X - u_2)}{\det(X - u_1, X - u_2)} = \frac{(u_3 - u_1)^t(u_3 - u_2)}{\det(u_3 - u_1, u_3 - u_2)} \tag{6}$$

Analyzing the second part:

$$\begin{aligned} \frac{(u_3 - u_1)^t(u_3 - u_2)}{\det(u_3 - u_1, u_3 - u_2)} &= \frac{u_3^t(u_3 - u_2)}{\det(u_3, u_3 - u_2)} = \\ \frac{p_3^t M(\theta)^t (M(\theta)(p_3 - p_2))}{\det(u_3, u_3 - u_2)} &= \frac{p_3^t (p_3 - p_2)}{\det(u_3, u_3 - u_2)} \end{aligned} \tag{7}$$

Now looking at $\det(u_3, u_3 - u_2)$

$$\begin{aligned} \det(u_3, u_3 - u_2) &= \det(M(\theta)v, M(\theta)w) = \\ &= \begin{vmatrix} M(\theta)_1 v & M(\theta)_2 v \\ M(\theta)_1 w & M(\theta)_2 w \end{vmatrix} = \end{aligned} \tag{8}$$

$$\begin{aligned} &(v_x \cos \theta - v_y \sin \theta)(w_x \sin \theta + w_y \cos \theta) - (\sin \theta v_x + \cos \theta v_y)(w_x \cos \theta - w_y \sin \theta) = \\ &\cos^2 \theta (v_x w_y - v_y w_x) + \sin^2 \theta (-v_y w_x + v_x w_y) + \sin \theta \cos \theta (v_x w_x - v_y w_y - v_x w_x + v_y w_y) = \\ &v_x w_y (\cos^2 \theta + \sin^2 \theta) - v_y w_x (\cos^2 \theta + \sin^2 \theta) = \\ &v_x w_y - v_y w_x = \det(v, w) = \det(u_3, u_3 - u_2) = \det(p_3, p_3 - p_2) \end{aligned}$$

Then, the second part is:

$$\frac{p_3^t(p_3 - p_2)}{\det(p_3, p_3 - p_2)} \quad (9)$$

The main equation is:

$$\frac{(X - u_1)^t(X - u_2)}{\det(X - u_1, X - u_2)} = \frac{p_3^t(p_3 - p_2)}{\det(p_3, p_3 - p_2)} \quad (10)$$

Looking at the first part:

$$(X - u_1)^t(X - u_2) = X^t(X - u_2) = X^tX - X^tM(\theta)p_2 = x^2 + qy^2 - x(p_{2x} \cos \theta - p_{2y} \sin \theta) - qy(p_{2x} \sin \theta + p_{2y} \cos \theta)$$

and

$$\begin{aligned} \det(X, X - u_2) &= \begin{vmatrix} x & y \\ x - M(\theta)_1 p_2 & y - M(\theta)_2 p_2 \end{vmatrix} = \\ &= \begin{vmatrix} x & y \\ x - (p_{2x} \cos \theta - p_{2y} \sin \theta) & y - (p_{2x} \sin \theta + p_{2y} \cos \theta) \end{vmatrix} = \\ &= y(p_{2x} \cos \theta - p_{2y} \sin \theta) - x(p_{2x} \sin \theta + p_{2y} \cos \theta) \end{aligned}$$

Making:

$$\gamma = p_{2x} \cos \theta - p_{2y} \sin \theta \quad (11)$$

$$\delta = p_{2x} \sin \theta + p_{2y} \cos \theta \quad (12)$$

$$D = \frac{p_3^t(p_3 - p_2)}{\det(p_3, p_3 - p_2)} \quad (13)$$

Then, we get:

$$\frac{(X - u_1)^t(X - u_2)}{\det(X - u_1, X - u_2)} = \frac{x^2 + qy^2 - x\gamma - qy\delta}{y\gamma - x\delta} = D \quad (14)$$

Rearranging

$$\begin{aligned} x^2 + qy^2 - x(\gamma - D\delta) - qy(\delta + \frac{D\gamma}{q}) &= 0 \\ (x - \frac{\gamma - D\delta}{2})^2 + q(y - \frac{q\delta + D\gamma}{2q})^2 - \frac{(\gamma - D\delta)^2}{4} - \frac{(q\delta + D\gamma)^2}{4q} &= 0 \end{aligned}$$

Then, as q is fixed, we need to impose:

$$\frac{(\gamma - D\delta)^2}{4} + \frac{(q\delta + D\gamma)^2}{4q} = 1$$

$$q(\gamma^2 - 2D\gamma\delta + D^2\delta^2) + q^2\delta^2 + 2q\delta D\gamma + D^2\gamma^2 = 4q =$$

$$q\gamma^2 + qD^2\delta^2 + q^2\delta^2 + D^2\gamma^2 = 4q$$

$$\gamma^2(q + D^2) + \delta^2(qD^2 + q^2) = 4q$$

Then,

$$\gamma^2 = p_{2x}^2 \cos^2 \theta - p_{2x}p_{2y} \sin 2\theta + p_{2y}^2 \sin^2 \theta \quad (15)$$

$$\delta^2 = p_{2x}^2 \sin^2 \theta + p_{2x}p_{2y} \sin 2\theta + p_{2y}^2 \cos^2 \theta \quad (16)$$

We have

$$\begin{aligned} & \cos^2 \theta ((q + D^2)p_{2x}^2 + (qD^2 + q^2)p_{2y}^2) + \\ & \sin^2 \theta ((q + D^2)p_{2y}^2 + (qD^2 + q^2)p_{2x}^2) + \\ & \sin 2\theta ((qD^2 + q^2)p_{2x}p_{2y} - (q + D^2)p_{2x}p_{2y}) = 4q \end{aligned}$$

Which can be written as

$$A \cos^2 \theta + 2B \sin \theta \cos \theta + C \sin^2 \theta = D \quad (17)$$

$$A, B, C, D > 0 \quad (18)$$

Which can be rearranged as:

$$\frac{A - C}{2} \cos(2\theta) + B \sin(2\theta) + \frac{A + C - 2D}{2} = 0$$

Rewritting everything (keep up!):

$$\hat{A} \cos(2\theta) + \hat{B} \sin(2\theta) + \hat{C} = 0$$