

Planar Maximal Covering with Ellipses

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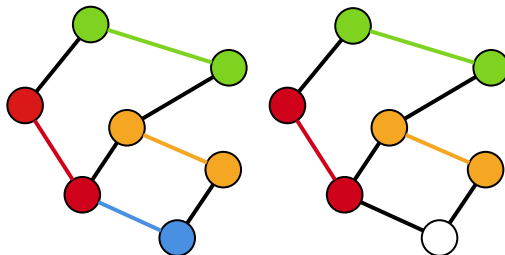
Agradecimentos à CAPES.

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- 2 Preliminaries
- 3 Maximal Covering by Disks
 - Maximum Weight Clique Problem
- 4 Maximal Covering by Ellipses
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Introduction

- Covering problems
 - Minimum Cover Problem (Karp 1972)
 - Maximal Covering Problem (Church and Velle 1974)

Figure: Minimum Vertex Cover and its maximal counterpart. The colored edges are in the cover.



Source: Elaborated by the author.

- Maximal Covering Location Problem (MCLP)
 - Introduce at first for networks. Facilities are placed on nodes, covering a radius of neighboring vertexes.
- Planar Maximal Covering Location Problem (PMCLP)
 - One disk is 3SUM-Hard (Kopelowitz, Pettie, and Porat 2014).
 - One disk: $\mathcal{O}(n^2)$ and $\mathcal{O}(n^2 \log n)$ algorithms.
 - m disks: $\mathcal{O}(n^{2m-1} \log n)$ algorithm.
- Goals
 - Develop a $\mathcal{O}(n^2 \log n)$ algorithm for the one disk case.
 - Adapt it for the m ellipses case creating a $\mathcal{O}(n^{2m})$ algorithm.

Maximal Covering by Disks

One disk

$MCD(\mathcal{P}, 1)$ is the problem of placing one disk on the plane to cover a subset of a demand set \mathcal{P} , with n points, maximizing the weights of the covered points.

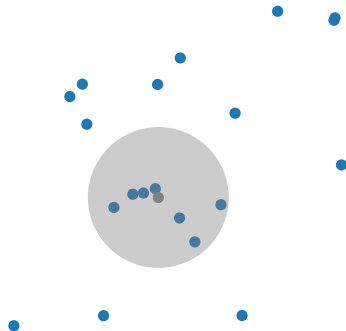
$$\max_q w(\mathcal{P} \cap D(q)),$$

- $\mathcal{P} = \{p_1, \dots, p_n\}$ is the demand set with weights $w(p_i) > 0$.
- $w(A)$, $A \subset \mathcal{P}$, is the sum of weights of the points in A .
- $D(q)$ is a unit disk with center at point q .

Maximal Covering by Disks

One disk

Figure: An instance of $MCD(\mathcal{P}, 1)$.



Source: Elaborated by the author.

Maximal Covering By Disks

One disk

Works and results found in the literature:

- *MCD* is as difficult the problem of given n numbers, find three of them that sum to 0. Proved by (Aronov and Har-Peled 2008).
- In (Drezner 1981) a $\mathcal{O}(n^2 \log n)$ algorithm was developed. The idea of our algorithm to sort the intersections by their angles comes from here.
- In (M. Chazelle and Lee 1986), a $\mathcal{O}(n^2)$ algorithm was developed. It actually solves an equivalent problem which is introduced next.

Maximum Weight Clique Problem

Let $\mathcal{D} = \{D_1, \dots, D_n\}$ be a set of n unit disks with weights $w_i > 0$. The maximum weight clique is defined as

$$\max_q \sum_{D_k \cap q \neq \emptyset} w_k,$$

- The disks are fixed with centers at $\mathcal{P} = \{p_1, \dots, p_n\}$ with $w_k = w(p_k)$.
- A clique is a non-empty intersection area of a subset of disks. We search for only a point in an optimal clique.
- An optimal solution for the maximum weight clique is an optimal solution for $MCD(\mathcal{P}, 1)$.

Maximum Weight Clique Problem

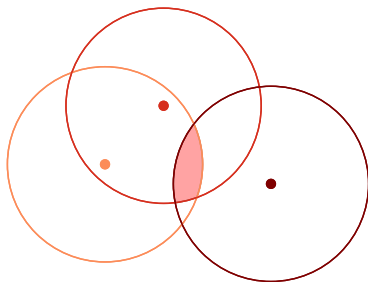
Figure: An instance of $MCD(\mathcal{P}, 1)$.



Source: Elaborated by the author.

Maximum Weight Clique Problem

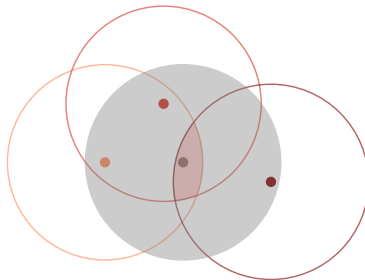
Figure: An instance of the Maximum Weight Clique Problem obtained from an instance of $MCD(\mathcal{P}, 1)$.



Source: Elaborated by the author.

Maximum Weight Clique Problem

Figure: An instance of the Maximum Weight Clique Problem obtained from an instance of $MCD(\mathcal{P}, 1)$. In gray, the optimal solution.



Source: Elaborated by the author.

Maximum Weight Clique Problem

Algorithm

Defining $\Gamma_+(i,j)$ and $\Gamma_-(i,j)$:

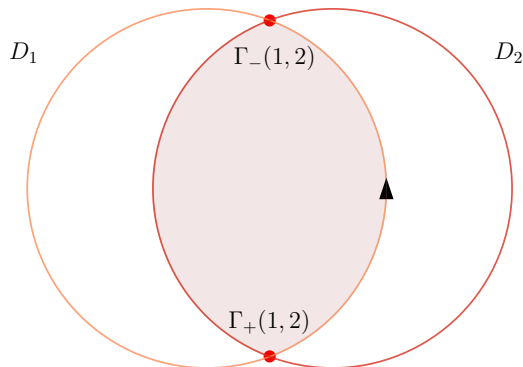
Let D_i (at the origin) and D_j be two unit disks that intersect at two points.

- We know that the two intersection points define two arcs in D_i .
- One of the arcs bounds $D_i \cap D_j$. That is the one we want to determine.
- We can determine the polar angles of the two intersection points.
- Assuming counter-clockwise direction, we define $\Gamma_+(i,j)$ and $\Gamma_-(i,j)$ as the angles of intersection that determines the arc of D_i that bounds $D_i \cap D_j$.

Maximum Weight Clique Problem

Algorithm

Figure: $\Gamma_+(1, 2)$ and $\Gamma_-(1, 2)$ example.

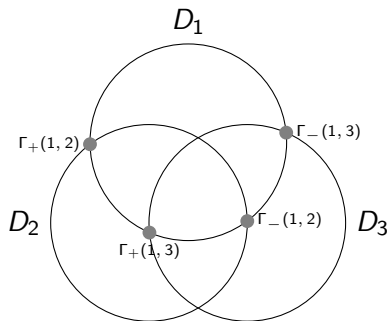


Source: Elaborated by the author.

Maximum Weight Clique Problem

Algorithm

Figure: Three disks and their intersection points and angles.



Source: Elaborated by the author.

Maximum Weight Clique Problem

Algorithm

For a disk D_i , a counter-clockwise traversal visits every $\Gamma_+(i, j)$ and $\Gamma_-(i, j)$ in counter-clockwise order.

- An intersection region of disks is bounded by arcs.
- The arc $\Gamma_+(i, j), \Gamma_-(i, j)$ (counter-clockwise) determines a region where i and j intersect.
- In a counter-clockwise traversal, the arcs where $\Gamma_+(i, j) > \Gamma_-(i, j)$ can be a problem for the implementation. The traversal is on a circle, where to start?
- If the traversal reaches a opening angle, we mark the intersecting disk as active. If it reaches a closing angle we unmark it.
- We want to find a point in the traversal with the most active disks.

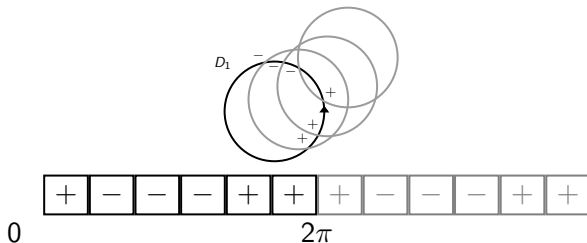
Maximum Weight Clique Problem

Algorithm

For every disk, traverse the sorted list of intersection angles twice, keeping a set of active disks, and the current best solution.

Doing the traversal twice solves the problem of deciding where to start.

Figure: The intersections list of a disk with three other disks.

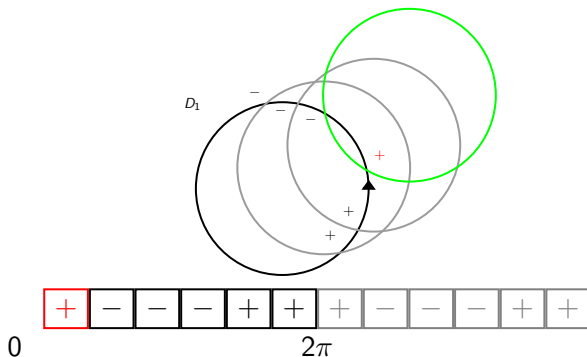


Source: Elaborated by the author.

Maximum Weight Clique Problem

Algorithm

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).

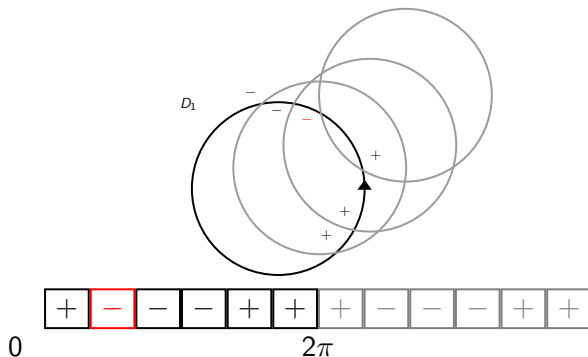


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Maximum Weight Clique Problem

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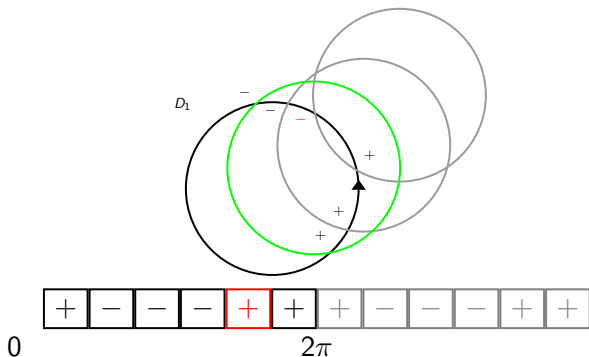


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Maximum Weight Clique Problem

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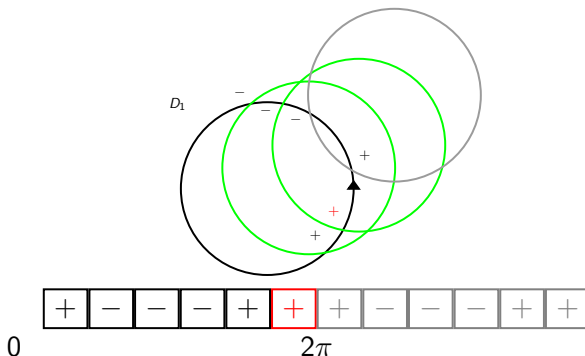


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Maximum Weight Clique Problem

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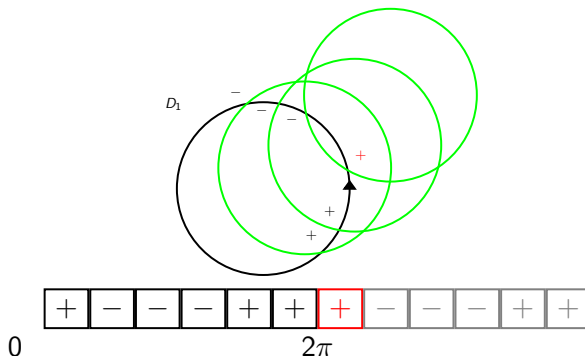


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Maximum Weight Clique Problem

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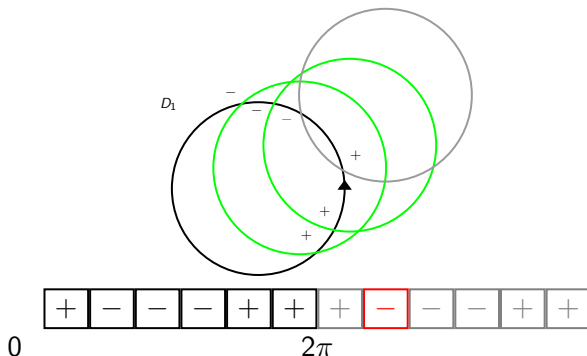


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Maximum Weight Clique Problem

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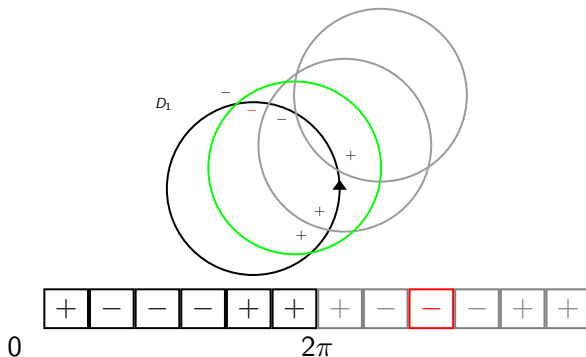


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Maximum Weight Clique Problem

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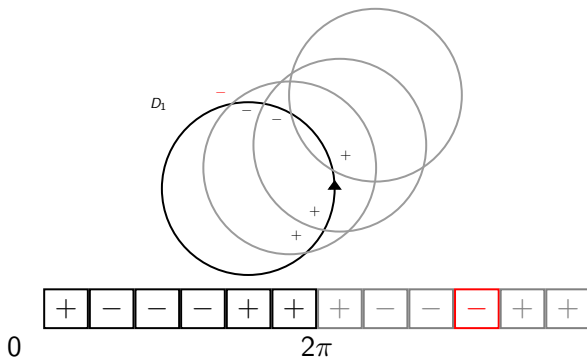


Source: Elaborated by the author.

Maximum Weight Clique Problem

Algorithm

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



Source: Elaborated by the author.

Maximum Weight Clique Problem

Algorithm

The run-time complexity of the algorithm is $\mathcal{O}(n^2 \log n)$.

- There are $\mathcal{O}(n^2)$ intersections among n disks.
- Sorting takes $\mathcal{O}(n^2 \log n)$.
- The traversal takes $\mathcal{O}(n)$ for every disk.
- It can be implemented in $K \log n$ where K is the number of intersections (L. Bentley and A. Ottmann 1979).

Maximum Covering by Disks

Multiple disks

Works found in the literature:

- In (Berg, Cabello, and Har-Peled 2006) a $\mathcal{O}(n^{2m-1})$ algorithm was proposed. Also a $(1 - \epsilon)$ -approximation that runs in $\mathcal{O}(n \log n)$ was introduced.
- In (HE et al. 2015) a heuristic method using an algorithm called mean-shift was developed. The mean-shift algorithm converges to a local density maxima of any probability distribution and it is used to find a smaller candidate list of centers for the disks.

Because of the similarities, we will discuss only the multiple ellipses algorithm later.

Ellipse

Given a center $c \in \mathbb{R}^2$ and $Q \in \mathbb{R}^{2 \times 2}$, an ellipse is the set of points that satisfy

$$\|u - c\|_Q^2 = (u - c)^T Q (u - c) = 1,$$

with \leq representing the set of covered points.

Axis-parallel ellipse

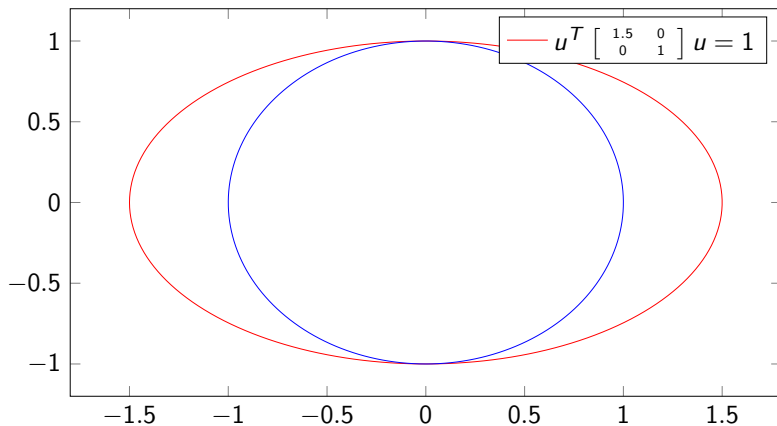
Any 2 by 2 diagonal d.p. matrix determines an axis-parallel ellipse, which can also be described by

$$\frac{(x - c_x)^2}{a^2} + \frac{(y - c_y)^2}{b^2} = 1,$$

where (a, b) are the shape parameters and $c = (c_x, c_y)$ is the center.

Ellipses

Figure: The ellipse seen as just a linear transformation of a circle.



Source: Elaborated by the author.

Maximal Covering by Ellipses

One ellipse

Let $MCE(\mathcal{P}, a, b)$ be an instance of the maximal covering by one ellipse, with E being an ellipse with shape parameters $(a, b) \in \mathbb{R}_{>0}^2$, an optimal solution of $MCE(\mathcal{P}, a, b)$ is given by

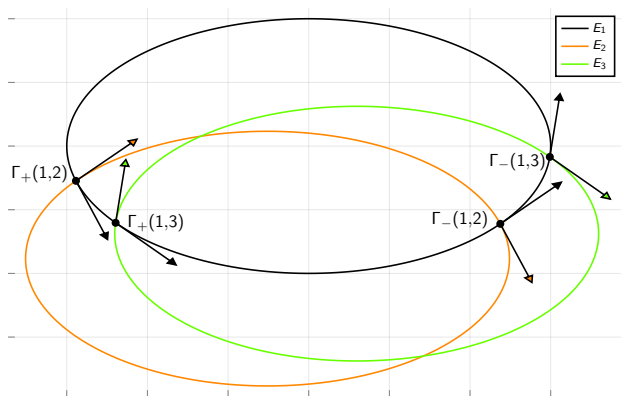
$$\max_q |\mathcal{P} \cap E(q)|,$$

- $E(q)$ is an axis-parallel ellipse with center point q .
- Assuming unit weights for now.
- Same algorithm for one disk.

Maximal Covering by Ellipses

One ellipse

Figure: Intersection points of E_1 with E_2 and E_3 along with opening and closing angles indicators.



Source: Elaborated by the author.

Maximal Covering by Ellipses

m ellipses

Let $MCE(\mathcal{P}, \mathcal{E})$ be an instance of the maximal covering by ellipses, an optimal solution is given by

$$\max_{q_1, \dots, q_m} \left| \bigcup_{i=1}^m \mathcal{P} \cap E_i(q_i) \right|,$$

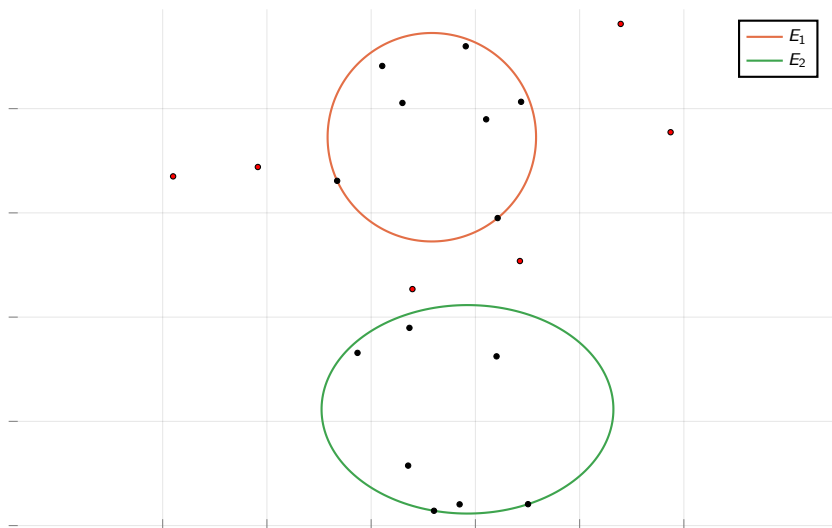
- \mathcal{E} is a set of m ellipses.
- (Canbolat and Massow 2009) is the very first study on the problem. Slow exact method, a heuristic one was proposed.
- (Andretta and Birgin 2013) proposed a method that breaks the problem into smaller optimization ones. Also, they developed a method for the non-axis-parallel case.

Maximal Covering by Ellipses

Pre-processing that finds every possible coverage for ellipse E_i

```
1:  $A \leftarrow \bigcup_{j \in I_i} \{\Gamma_+(i, j) \cup \Gamma_-(i, j)\}$ 
2:  $Z \leftarrow \{\}$ 
3:  $Cov \leftarrow \{p_i\}$ 
4: for  $cnt = 1..2$  do
5:   for  $a \in A$  do
6:     Let  $p_a$  be the point that intersects  $E_i$  at angle  $a$ .
7:     if  $a$  is a starting angle then
8:        $Cov \leftarrow Cov \cup \{p_a\}$ 
9:     else
10:       $Cov \leftarrow Cov \setminus \{p_a\}$ 
11:    end if
12:     $Z \leftarrow Z \cup \{Cov\}$ 
13:  end for
14: end for
```


Figure: Optimal solution with two ellipses for a random instance.



Source: Elaborated by the author.

Maximal Covering by Ellipses

- The algorithm for m ellipses tries every possible assignment of coverage for every one of the ellipses.
- Run-time complexity of $\mathcal{O}(n^{2m})$.
- Simpler than the m disks algorithm proposed by (Berg, Cabello, and Har-Peled 2006). Achieves a similar complexity ($\mathcal{O}(n^{2m-1})$).
- Small improvements can be made in the pre-processing exhibited earlier in order to reduce the size of the search space:
 - Non-maximal coverage sets.
 - Ellipses that are too distant do not need to be checked.
- The unit-weight assumption can be easily dropped.

Primary goals:

- Study the $(1 - \epsilon)$ -approximation method for the planar covering with disks in (Berg, Cabello, and Har-Peled 2006) and develop an adapted version of the algorithm for ellipses with the same time complexity of $\mathcal{O}(n \log n)$.
- Develop an exact method for the version of the problem introduced in (Andretta and Birgin 2013) where the ellipses can be freely rotated.

Secondary goals:

- Develop a probabilistic approximation algorithm based on (Aronov and Har-Peled 2008) which proposed a Monte Carlo approximation for the problem of finding the deepest point in a arrangement of regions. The method runs in $\mathcal{O}(n\epsilon^2 \log n)$ and can be applied to solve the case with one ellipse. The case with more than one ellipse is left as a challenge for us for the next steps of our research.
- In (HE et al. 2015), the task of finding every center candidate, after eliminating all the non-essential ones, is done in $\mathcal{O}(n^5)$ run-time complexity. We want to generalize this for the elliptical distance function and achieve a better run-time complexity. We also intend to use the mean-shift algorithm to try to develop a greedy version for the ellipses version.

- Andretta, M. and E.G. Birgin (2013). “Deterministic and stochastic global optimization techniques for planar covering with ellipses problems”. In: *European Journal of Operational Research* 224.1, pp. 23–40. ISSN: 0377-2217. DOI: <https://doi.org/10.1016/j.ejor.2012.07.020>. URL: <http://www.sciencedirect.com/science/article/pii/S0377221712005619>.
- Aronov, Boris and Sarel Har-Peled (2008). “On Approximating the Depth and Related Problems”. In: *SIAM J. Comput.* 38.3, pp. 899–921. DOI: [10.1137/060669474](https://doi.org/10.1137/060669474). URL: <https://doi.org/10.1137/060669474>.
- Berg, Mark de, Sergio Cabello, and Sarel Har-Peled (2006). “Covering Many or Few Points with Unit Disks”. In: vol. 45, pp. 55–68. DOI: [10.1007/11970125_5](https://doi.org/10.1007/11970125_5).
- Canbolat, M. S. and M. von Massow (2009). “Planar maximal covering with ellipses”. In: *Computers and Industrial Engineering* 57, pp. 201–208.

- Church, Richard and Charles R. Velle (1974). "THE MAXIMAL COVERING LOCATION PROBLEM". In: *Papers in Regional Science* 32.1, pp. 101–118. DOI: 10.1111/j.1435-5597.1974.tb00902.x. eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1111/j.1435-5597.1974.tb00902.x>. URL: <https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1435-5597.1974.tb00902.x>.
- Drezner, Zvi (1981). "Note—On a Modified One-Center Model". In: *Management Science* 27, pp. 848–851. DOI: 10.1287/mnsc.27.7.848.
- HE, Zhou et al. (2015). "A Mean-Shift Algorithm for Large-Scale Planar Maximal Covering Location Problems". In: *European Journal of Operational Research* 250. DOI: 10.1016/j.ejor.2015.09.006.
- Karp, R. (1972). "Reducibility among combinatorial problems". In: *Complexity of Computer Computations*. Ed. by R. Miller and J. Thatcher. Plenum Press, pp. 85–103.
- Kopelowitz, Tsvi, Seth Pettie, and Ely Porat (2014). *Higher Lower Bounds from the 3SUM Conjecture*. arXiv: 1407.6756 [cs.DS].

- L. Bentley, Jon and Thomas A. Ottmann (1979). "Algorithms for Reporting and Counting Geometric Intersections". In: *Computers, IEEE Transactions on C-28*, pp. 643–647. DOI: 10.1109/TC.1979.1675432.
- M. Chazelle, B and D Lee (1986). "On a circle placement problem". In: *Computing* 36, pp. 1–16. DOI: 10.1007/BF02238188.