1 Intro

Points:

$$\begin{aligned} p_i &= (x_i, y_i) \\ p_1 &= (0,0) \end{aligned} \tag{1}$$

If p_i are not co-linear and the ellipse with form:

$$x^2 + qy^2 = a^2b^2 \tag{2}$$

can be determined by three points:

$$\frac{(x-x_1)(x-x_2)+q(y-y_1)(y-y_2)}{(y-y_1)(x-x_2)-(y-y_2)(x-x_1)} = \frac{(x_3-x_1)(x_3-x_2)+q(y_3-y_1)(y_3-y_2)}{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)} \tag{3}$$

q is fixed, we want to find θ, let

$$M(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

Now, lets define

$$u_i = M(\theta)(p_i - c) \tag{5}$$

$$c = (h,k) \tag{6}$$

as $u_1 = 0$, we have

$$\frac{X^t(X-u_2)}{det(X, X-u_2)} = \frac{u_3^t(u_3-u_2)}{det(u_3, u_3-u_2)} \tag{7}$$

1.1 Right hand side of the equation

The dot product after, with the rotation applied is:

$$\begin{aligned} u^t w = \cos^2\theta(u_x w_x + q u_y w_y) + \\ \sin\theta\cos\theta((q-1)(u_y w_x + u_x w_y)) + \\ \sin^2\theta(u_y w_y + q u_x w_x) \end{aligned}$$

The determinant, also with the rotation is:

$$\begin{aligned} det(M(\theta)v, M(\theta)w) = \\ \begin{vmatrix} M(\theta)_1 v & M(\theta)_2 v \\ M(\theta)_1 w & M(\theta)_2 w \end{vmatrix} = \\ |M(\theta)| \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} = \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} = v_x w_y - v_y w_x \end{aligned}$$

Now, grouping the constants:

$$\Gamma_1 = \frac{p_{3x}(p_{3x} - p_{2x}) + qp_{3y}(p_{3y} - p_{2y})}{det(p_3, p_3 - p_2)}$$
$$\Gamma_2 = \frac{(q-1)(p_{3y}(p_{3x} - p_{2x}) + p_{3x}(p_{3y} - p_{2y}))}{det(p_3, p_3 - p_2)}$$
$$\Gamma_3 = \frac{p_{3y}(p_{3y} - p_{2y}) + qp_{3x}(p_{3x} - p_{2x})}{det(p_3, p_3 - p_2)}$$

Getting everything together we get

$$\frac{u_3^t(u_3 - u_2)}{det(p_3, p_3 - p_2)} = \Gamma_1 \cos^2\theta + \Gamma_2 \cos\theta \sin\theta + \Gamma_3 \sin^2\theta \tag{8}$$

1.2 The left rand side of the equation

Looking at the first part:

$$(X - u_1)^t(X - u_2) = X^t(X - u_2) = X^tX - X^tM(\theta)p_2 =$$
$$x^2 + qy^2 - x((p_{2x} - h)\cos\theta - (p_{2y} - k)\sin\theta) - qy((p_{2x} - h)\sin\theta + (p_{2y} - k)\cos\theta)$$

and

$$det(X, X - u_2) = \begin{vmatrix} x & y \\ x - M(\theta)_1 p_2 & y - M(\theta)_2 p_2 \end{vmatrix} =$$
$$\begin{vmatrix} x & y \\ x - (p_{2x}\cos\theta - p_{2y}\sin\theta) & y - (p_{2x}\sin\theta + p_{2y}\cos\theta) \end{vmatrix} =$$
$$y(p_{2x}\cos\theta - p_{2y}\sin\theta) - x(p_{2x}\sin\theta + p_{2y}\cos\theta)$$

Making:

$$\gamma = p_{2x}\cos\theta - p_{2y}\sin\theta \tag{9}$$
$$\delta = p_{2x}\sin\theta + p_{2y}\cos\theta \tag{10}$$
$$\Phi = \Gamma_1 \cos^2\theta + \Gamma_2 \cos\theta \sin\theta + \Gamma_3 \sin^2\theta \tag{11}$$

Then, we get:

$$\frac{(X - u_1)^t(X - u_2)}{det(X - u_1, X - u_2)} = \frac{x^2 + qy^2 - x\gamma - qy\delta}{y\gamma - x\delta} = \Phi \tag{12}$$

1.3 Doing all the maths

Now, we need to rearrange everything so it would look like an ellipse equation.

$$x^2 + qy^2 - x(\gamma - \Phi\delta) - qy(\delta + \frac{\Phi\gamma}{q}) = 0$$
$$(x - \frac{\gamma - \Phi\delta}{2})^2 + q(y - \frac{q\delta + \Phi\gamma}{2q})^2 = R$$
$$R = \frac{(\gamma - \Phi\delta)^2}{4} + \frac{(q\delta + \Phi\gamma)^2}{4q^2}$$

As q is a fixed parameter and we are rotating the points, we need to impose that $R = a^2$, so it satisfies the equation we started for an ellipse.

2 Having $R = a^2$, it looks way easier than it is...

Let us do it:

$$\frac{(\gamma - \Phi\delta)^2}{4} + \frac{(q\delta + \Phi\gamma)^2}{4q^2} = a^2$$
$$q^2(\gamma^2 - 2\Phi\gamma\delta + \Phi^2\delta^2) + q^2\delta^2 + 2q\delta\Phi\gamma + \Phi^2\gamma^2 = 4q^2a^2 =$$

$$q^2\gamma^2 + q^2\Phi^2\delta^2 + q^2\delta^2 + \Phi^2\gamma^2 + 2\Phi\gamma\delta(q - q^2) = 4q^2a^2$$

$$\gamma^2(q^2 + \Phi^2) + \delta^2[q^2(1 + \Phi^2)] + \gamma\delta[2\Phi(q - q^2)] = 4q^2a^2$$

Then,

$$\gamma^2 = p_{2x}^2 \cos^2\theta - p_{2x}p_{2y}\sin 2\theta + p_{2y}^2\sin^2\theta \quad (13)$$
$$\delta^2 = p_{2x}^2 \sin^2\theta + p_{2x}p_{2y}\sin 2\theta + p_{2y}^2\cos^2\theta \quad (14)$$
$$\delta\gamma = \cos^2\theta(p_{2x}p_{2y}) + \frac{\sin(2\theta)}{2}(p_{2x}^2 - p_{2y}^2) - \sin^2\theta(p_{2x}p_{2y}) \quad (15)$$
$$\Phi^2 = \Gamma_1^2\cos^4\theta + 2\Gamma_1\Gamma_2\cos^3\theta\sin\theta + (2\Gamma_1\Gamma_3 + \Gamma_2^2)\cos^2\theta\sin^2\theta + 2\Gamma_2\Gamma_3\cos\theta\sin^3\theta + \Gamma_3^2\sin^4\theta \quad (16)$$

We have

$$\cos^2\theta[(q^2 + \Phi^2)p_{2x}^2 + q^2(\Phi^2 + 1)p_{2y}^2 + 2\Phi(q - q^2)p_{2x}p_{2y}] +$$
$$\sin^2\theta[(q^2 + \Phi^2)p_{2y}^2 + q^2(\Phi^2 + 1)p_{2x}^2 - 2\Phi(q - q^2)p_{2x}p_{2y}] +$$
$$\sin(2\theta)[\Phi(q - q^2)(p_{2x}^2 - p_{2y}^2) - (q^2 + \Phi^2)p_{2x}p_{2y} + q^2(\Phi^2 + 1)p_{2x}p_{2y}] = 4q^2a^2$$

Let's call sines and cosines s and r, with the condition that

$$s^2 + r^2 - 1 = 0$$

Renaming the constants on Φ^2:

$$\Phi^2 = \Phi_1 r^4 + \Phi_2 r^3 s + \Phi_3 r^2 s^2 + \Phi_4 r s^3 + \Phi_5 s^4$$

$$\Phi = \Gamma_1 r^2 + \Gamma_2 rs + \Gamma_3 s^2$$

Now let's work on each of the terms separately:

2.1 First one $\cos^2(\theta)$

$$\begin{aligned}r^2(\Phi^2(p_{2x}^2 + q^2 p_{2y}^2) + 2\Phi(q - q^2)p_{2x}p_{2y} + q^2(p_{2x}^2 + p_{2y}^2)) = \\ (p_{2x}^2 + q^2 p_{2y}^2)(\Phi_1 r^6 + \Phi_2 r^5 s + \Phi_3 r^4 s^2 + \Phi_4 r^3 s^3 + \Phi_5 r^2 s^4) + \\ 2(q - q^2)p_{2x}p_{2y}(\Gamma_1 r^4 + \Gamma_2 r^3 s + \Gamma_3 r^2 s^2) + \\ q^2(p_{2x}^2 + p_{2y}^2))r^2\end{aligned}$$

Grouping them

$$\begin{aligned}r^6[(p_{2x}^2 + q^2 p_{2y}^2)\Phi_1] + \\ r^5 s[\Phi_2(p_{2x}^2 + q^2 p_{2y}^2)] + \\ r^4 s^2[\Phi_3(p_{2x}^2 + q^2 p_{2y}^2)] + \\ r^4[2(q - q^2)p_{2x}p_{2y}\Gamma_1] + \\ r^3 s^3[\Phi_4(p_{2x}^2 + q^2 p_{2y}^2)] + \\ r^3 s[2(q - q^2)p_{2x}p_{2y}\Gamma_2] + \\ r^2 s^4[\Phi_5(p_{2x}^2 + q^2 p_{2y}^2)] + \\ r^2 s^2[2(q - q^2)p_{2x}p_{2y}\Gamma_3] + \\ r^2[q^2(p_{2x}^2 + p_{2y}^2)]\end{aligned}$$

2.2 Second one $\sin^2\theta$

$$\begin{aligned}s^2\theta[(q^2 + \Phi^2)p_{2y}^2 + q^2(\Phi^2 + 1)p_{2x}^2 - 2\Phi(q - q^2)p_{2x}p_{2y}] = \\ s^2(\Phi^2(p2y^2 + q^2 p_{2x}^2) - 2\Phi(q - q^2)p_{2x}p_{2y} + q^2(p_{2x}^2 + p_{2y}^2))\end{aligned}$$

Almost the same as the first one:

$$
\begin{aligned}
r^6[\Phi_1(p_{2y}^2 + q^2 p_{2x}^2)]+ \\
r^5 s[\Phi_2(p_{2y}^2 + q^2 p_{2x}^2)]+ \\
r^4 s^2[\Phi_3(p_{2y}^2 + q^2 p_{2x}^2)]+ \\
r^4[-2(q - q^2)p_{2x}p_{2y}\Gamma_1]+ \\
r^3 s^3[\Phi_4(p_{2y}^2 + q^2 p_{2x}^2)]+ \\
r^3 s[-2(q - q^2)p_{2x}p_{2y}\Gamma_2]+ \\
r^2 s^4[\Phi_5(p_{2y}^2 + q^2 p_{2x}^2)]+ \\
r^2 s^2[-2(q - q^2)p_{2x}p_{2y}\Gamma_3]+ \\
r^2[q^2(p_{2x}^2 + p_{2y}^2)]
\end{aligned}
$$

2.3 The last one $\sin\theta\cos\theta$

$$
\begin{aligned}
2rs[\Phi(q - q^2)(p_{2x}^2 - p_{2y}^2) - (q^2 + \Phi^2)p_{2x}p_{2y} + q^2(\Phi^2 + 1)p_{2x}p_{2y}] = \\
2rs[\Phi^2(p_{2x}p_{2y}(q^2 - 1)) + \Phi(q - q^2)(p_{2x}^2 - p_{2y}^2)] = \\
(2p_{2x}p_{2y}(q^2 - 1))(\Phi_1 r^5 s + \Phi_2 r^4 s^2 + \Phi_3 r^3 s^3 + \Phi_4 r^2 s^4 + \Phi_5 r s^5)+ \\
2(q - q^2)(p_{2x}^2 - p_{2y}^2)(\Gamma_1 r^3 s + \Gamma_2 r^2 s^2 + \Gamma_3 r s^3)
\end{aligned}
$$

Grouping the monomiums:

$$
\begin{aligned}
r^5 s[2p_{2x}p_{2y}(q^2 - 1)\Phi_1]+ \\
r^4 s[2p_{2x}p_{2y}(q^2 - 1)\Phi_2]+ \\
r^3 s^3[2p_{2x}p_{2y}(q^2 - 1)\Phi_3]+ \\
r^3 s[2(q - q^2)(p_{2x}^2 - p_{2y}^2)\Gamma_1]+ \\
r^2 s^4[2p_{2x}p_{2y}(q^2 - 1)\Phi_4]+ \\
r^2 s^2[2(q - q^2)(p_{2x}^2 - p_{2y}^2)\Gamma_2]+ \\
r s^5[2p_{2x}p_{2y}(q^2 - 1)\Phi_5]+ \\
r s^3[2(q - q^2)(p_{2x}^2 - p_{2y}^2)\Gamma_3]
\end{aligned}
$$

2.4 Putting all this mess together once again

Let's do this

$$
\begin{aligned}
r^6[(p_{2x}^2 + q^2 p_{2y}^2 + p_{2y}^2 + q^2 p_{2x}^2)\Phi_1]+ \\
r^5 s[(p_{2x}^2 + q^2 p_{2y}^2 + p_{2y}^2 + q^2 p_{2x}^2)\Phi_2 + 2p_{2x}p_{2y}(q^2-1)\Phi_1]+ \\
r^4 s^2[(p_{2x}^2 + q^2 p_{2y}^2 + p_{2y}^2 + q^2 p_{2x}^2)\Phi_3]+ \\
r^3 s^3[2p_{2x}p_{2y}(q^2-1)\Phi_3 + (p_{2x}^2 + q^2 p_{2y}^2 + p_{2y}^2 + q^2 p_{2x}^2)\Phi_4]+ \\
r^3 s[2(q-q^2)(p_{2x}^2 - p_{2y}^2)\Gamma_1]+ \\
r^2 s^4[(p_{2x}^2 + q^2 p_{2y}^2 + p_{2y}^2 + q^2 p_{2x}^2)\Phi_3 + 2p_{2x}p_{2y}(q^2-1)\Phi_4]+ \\
r^2 s^2[2(q-q^2)(p_{2x}^2 - p_{2y}^2)\Gamma_2]+ \\
r^2[2q^2(p_{2x}^2 + p_{2y}^2)]+ \\
r s^5[2p_{2x}p_{2y}(q^2-1)\Phi_5]+ \\
r s^3[2(q-q^2)(p_{2x}^2 - p_{2y}^2)\Gamma_3] = 4a^2q^2
\end{aligned}
$$

Renaming the constants, we get:

$$
\begin{aligned}
A_1 r^6 + A_2 r^5 s + A_3 r^4 s^2 + A_4 r^3 s^3 + A_5 r^3 s + A_6 r^2 s^4 + A_6 r^2 s^2 + A_7 r^2 + A_8 r s^5 + A_9 r s^3 - 4a^2 q^2 = 0 \\
r^2 + s^2 - 1 = 0
\end{aligned}
$$