Planar Maximal Covering with Ellipses

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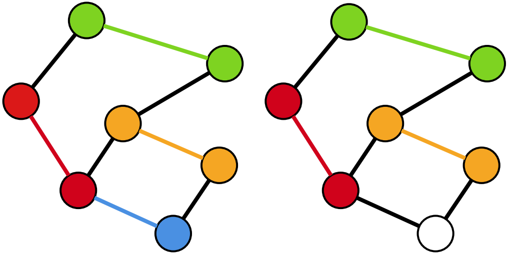
Contents

Introduction

- Covering problems
  - Minimum Cover Problem (Karp 1972)
  - Maximal Covering Problem (Richard Church and Velle 1974)

Figure: Minimum Vertex Cover and its maximal counterpart. The colored edges are in the cover.



Source: Elaborated by the author.

Introduction

- Maximal Covering Location Problem (MCLP)
  - Introduce at first for networks (Richard Church and Velle 1974). Facilities are placed on nodes, covering a radius of neighboring vertexes.
- Planar Maximal Covering Location Problem (PMCLP)
  - Introduced by (R. Church 1984).
  - One disk is 3SUM-Hard (Kopelowitz, Pettie, and Porat 2014).
  - One disk: $\mathcal{O}(n^2)$ and $\mathcal{O}(n^2 \log n)$ algorithms.
  - m disks: $\mathcal{O}(n^{2m-1} \log n)$ algorithm.
- Goals
  - Develop a $\mathcal{O}(n^2 \log n)$ algorithm for the one disk case.
  - Adapt it for the m ellipses case creating a $\mathcal{O}(n^{2m})$ algorithm.

$MCD(\mathscr{P}, 1)$ is the problem of placing one disk on the plane to cover a subset of a demand set $\mathscr{P}$, with n points, maximizing the weights of the covered points.

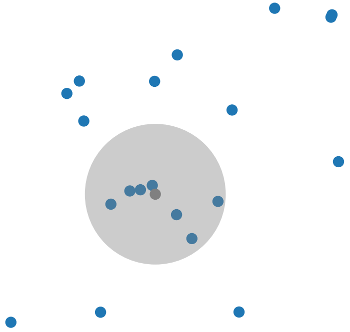
$$\max_q w(\mathscr{P} \cap D(q)),$$

- $\mathscr{P} = \{p_1, \ldots, p_n\}$ is the demand set with weights $w(p_i) > 0$.
- $w(A)$, $A \subset \mathscr{P}$, is the sum of weights of the points in A.
- $D(q)$ is a unit disk with center at point q.

Maximal Covering by Disks

One disk

Figure: An instance of $MCD(\mathscr{P}, 1)$.



Source: Elaborated by the author.

Works and results found in the literature:

- *MCD* is as difficult the problem of given n numbers, find three of them that sum to 0 (3SUM-HARD). Proved by (Aronov and Har-Peled 2008).
- In (Drezner 1981) a $\mathcal{O}(n^2 \log n)$ algorithm was developed. The idea of our algorithm to sort the intersections by their angles comes from here.
- In (M. Chazelle and Lee 1986), a $\mathcal{O}(n^2)$ algorithm was developed. It actually solves an equivalent problem which is introduced next.

Maximum Weight Clique Problem

Let $\mathscr{D} = \{D_1, \ldots, D_n\}$ be a set of n unit disks with weights $w_i > 0$. The maximum weight clique is defined as

$$\max_{q \in \mathbb{R}^2} \sum_{D_k \cap q \neq \emptyset} w_k,$$

- A clique is a non-empty intersection area of a subset of disks. We search for only a point in an optimal clique.
- The weight of a clique is the sum of the weights of the disks that intersect with it.
- In our case, we just want a point from the optimum clique.
- Given an instance $MCD(\mathscr{P}, 1)$: fix the disk centers at $\mathscr{P} = \{p_1, \ldots, p_n\}$ with weights $w_k = w(p_k)$.

Maximum Weight Clique Problem

Equivalence

Figure: An instance of $MCD(\mathscr{P}, 1)$. We will show how an instance of the Maximum Weight Clique Problem is constructed from it.

Source: Elaborated by the author.

Maximum Weight Clique Problem

Equivalence

Figure: An instance of the Maximum Weight Clique Problem obtained from an instance of $MCD(\mathscr{P}, 1)$.

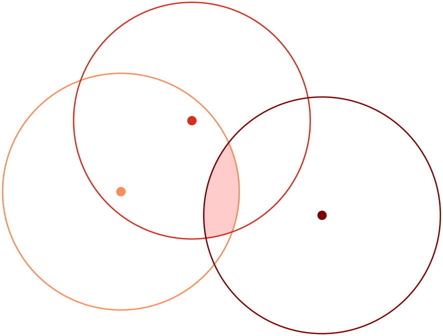


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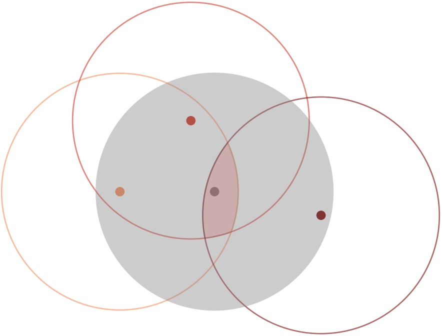
Figure: An instance of the Maximum Weight Clique Problem obtained from an instance of $MCD(\mathscr{P}, 1)$. In gray, the optimal solution.



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Defining $\Gamma_+(i,j)$ and $\Gamma_-(i,j)$:

Let D_i (at the origin) and D_j be two unit disks that have their corresponding circles intersect at two points.

- We know that the two intersection points define two arcs in D_i.
- One of the arcs bounds $D_i \cap D_j$. That is the one we want to determine.
- We can determine the polar angles of the two intersection points.
- Assuming counter-clockwise direction, we define $\Gamma_+(i,j)$ and $\Gamma_-(i,j)$ as the angles of intersection that determines the arc of D_i that bounds $D_i \cap D_j$.

Figure: $\Gamma_+(1,2)$ and $\Gamma_-(1,2)$ example.

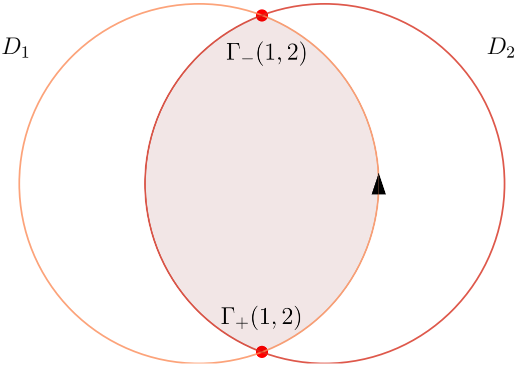

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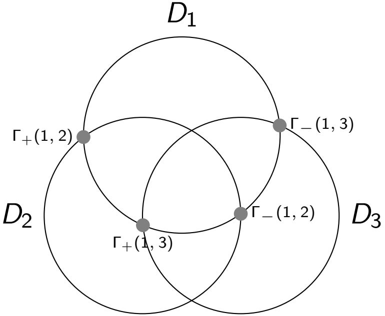
Figure: Three disks and their intersection points and angles.



Source: Elaborated by the author.

Some observations allow us to arrive at the algorithm:

- An intersection region of disks is bounded by arcs.
- The arc $A(\Gamma_+(i,j), \Gamma_-(i,j))$ (counter-clockwise) determines a region where i and j intersect.
- For every disk D_i, we want to find an angle θ, such that

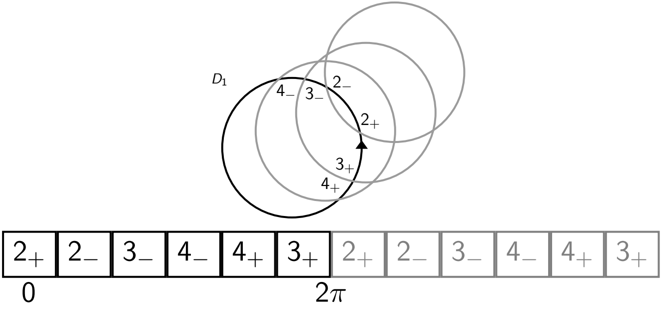
$$w(\{D_k : \theta \in A(\Gamma_+(i,k), \Gamma_-(i,k))\}),$$

  is maximized. Most overlapping intervals (circular).
- To transform it to the problem of finding the most overlapping intervals, just copy the list of intersection angles. The arcs such that $\Gamma_+(i,j) > \Gamma_-(i,j)$ will be considered.

Maximum Weight Clique Problem
Algorithm

Transforming it to the most overlapping intervals.

Figure: The intersections list of a disk with three other disks.



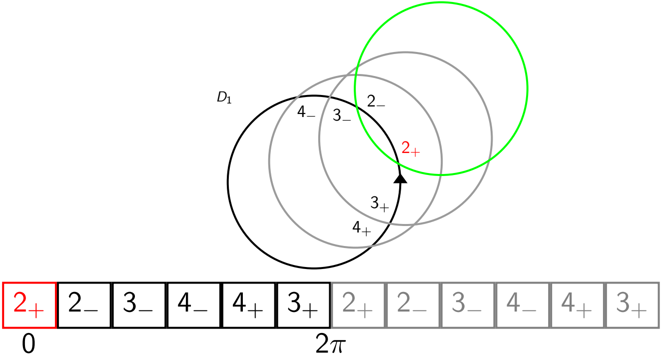
Source: Elaborated by the author.

Our algorithm for the Maximum Weight Clique Problem:

For every disk D_i, do:

- Get the sorted list of intersection angles with D_i $A = \cap_j \Gamma_+(i,j) \cup \Gamma_-(i,j)$.
- Traverse it twice starting at the angle with smallest value.
  - Keep a set of active disks. When an opening angle is visited, make the disk active, otherwise remove it from the set.
  - Update the optimal solution. Use the closing angle.

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



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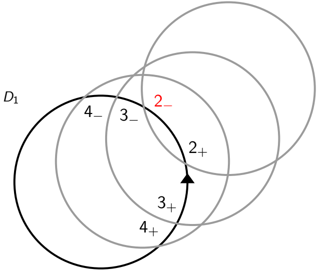

2_+	2_-	3_-	4_-	4_+	3_+	2_+	2_-	3_-	4_-	4_+	3_+

0 2π

Source: Elaborated by the author.

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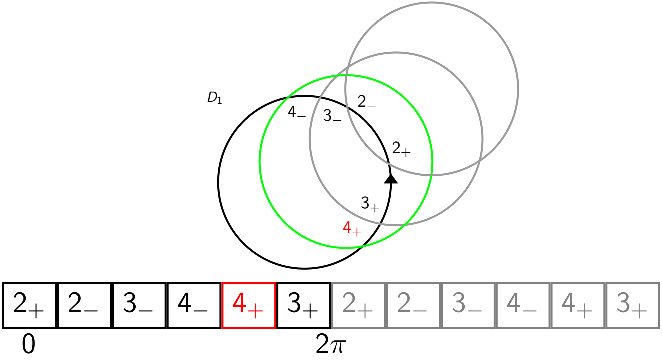

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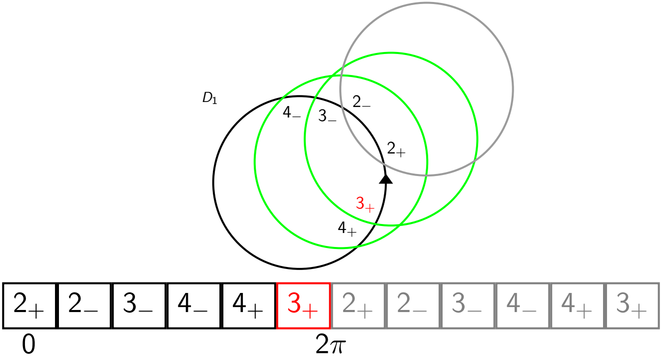

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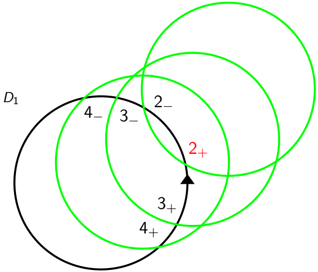

2_+	2_-	3_-	4_-	4_+	3_+	2_+	2_-	3_-	4_-	4_+	3_+
0					2π						

Source: Elaborated by the author.

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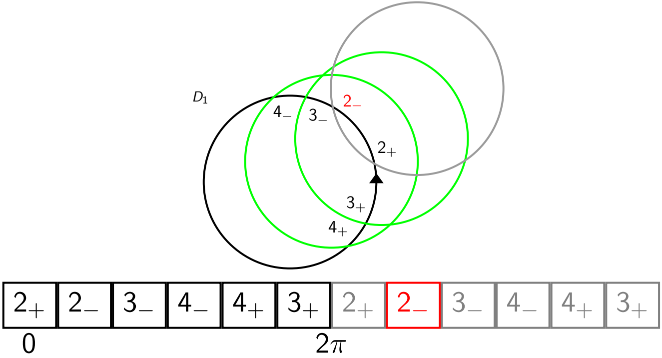

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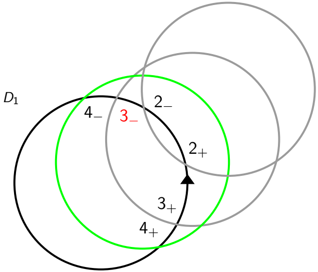


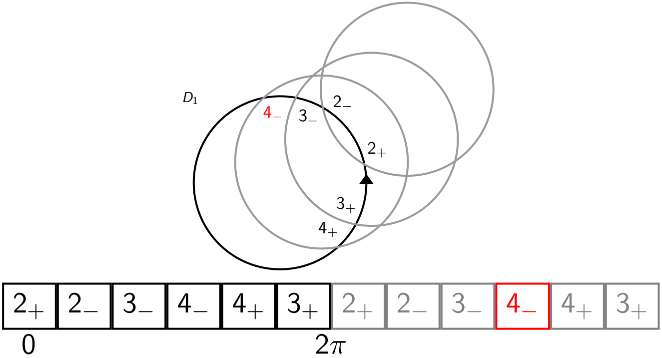
2_+	2_-	3_-	4_-	4_+	3_+	2_+	2_-	3_-	4_-	4_+	3_+

0 2π

Source: Elaborated by the author.

Maximum Weight Clique Problem
Algorithm

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



Source: Elaborated by the author.

The run-time complexity of the algorithm is $\mathcal{O}(n^2 \log n)$.

- There are $\mathcal{O}(n^2)$ intersections among n disks.
- Sorting takes $\mathcal{O}(n^2 \log n)$.
- The traversal takes $\mathcal{O}(n)$ for every disk.
- It can be implemented in $\mathcal{O}(K \log n)$ where K is the number of intersections (L. Bentley and A. Ottmann 1979).
- The algorithm is basically finding the most number of overlapping intervals n times.

As it was mentioned, the solution found by this algorithm is a solution for the Maximal Covering by One Disk.

Works found in the literature:

- In (Berg, Cabello, and Har-Peled 2006) a $\mathcal{O}(n^{2m-1})$ algorithm was proposed. Also a $(1-\epsilon)-$approximation that runs in $\mathcal{O}(n \log n)$ was introduced.
- In (He et al. 2015) a heuristic method using an algorithm called mean-shift was developed. The mean-shift algorithm converges to a local density maxima of any probability distribution and it is used to find a smaller candidate list of centers for the disks.

Because of the similarities, we will discuss only the multiple ellipses algorithm later.

Ellipses

Ellipse

Given a center $c \in \mathbb{R}^2$ and $Q \in \mathbb{R}^{2x2}$ a positive definite matrix, an ellipse is the set of points that satisfy

$$||u - c||_Q^2 = (u - c)^T Q(u - c) = 1,$$

with $\leq$ representing the set of covered points.

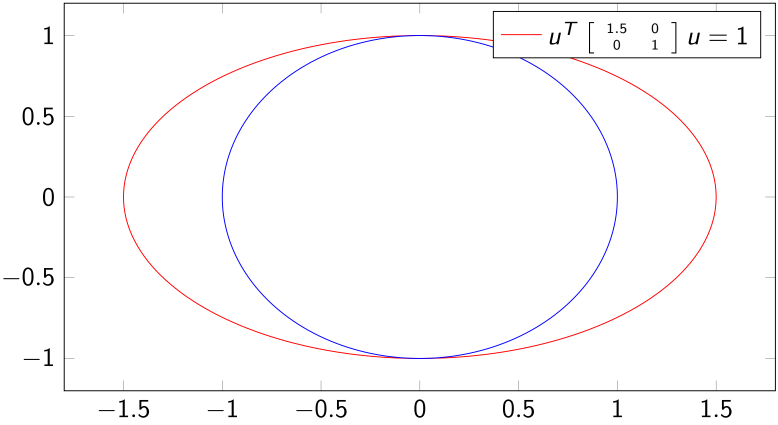
Axis-parallel ellipse

Any 2 by 2 diagonal p.d. matrix determines an axis-parallel ellipse, which can also be described by

$$\frac{(x - c_x)^2}{a^2} + \frac{(y - c_y)^2}{b^2} = 1,$$

where (a, b) are the shape parameters and $c = (c_x, c_y)$ is the center.

Figure: The ellipse seen as just a linear transformation of a circle.



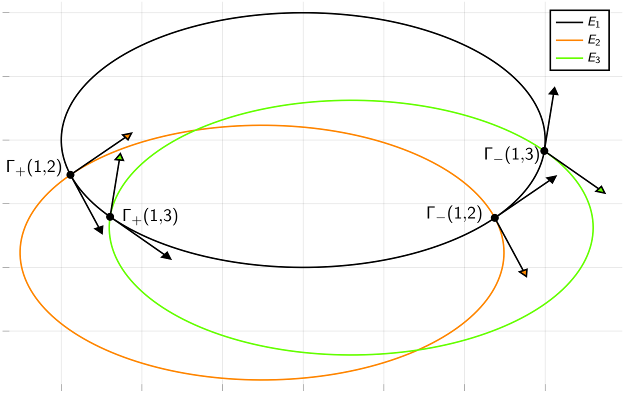
Source: Elaborated by the author.

Let $MCE(\mathscr{P}, a, b)$ be an instance of the maximal covering by one ellipse, with E being an ellipse with shape parameters $(a, b) \in \mathbb{R}^2_{>0}$, and $\mathscr{P} = \{p_1, \ldots, p_n\}$ is a set of n points with each point having a positive weight w_i, an optimal solution of $MCE(\mathscr{P}, a, b)$ is given by

$$\max_q w(\mathscr{P} \cap E(q)),$$

- $E(q)$ is an axis-parallel ellipse with center point q.
- $w(A)$, $A \subset \mathscr{P}$, is the sum of the weights of every point in $\mathscr{P}$.
- Same algorithm for one disk.

Figure: Intersection points of E_1 with E_2 and E_3 along with opening and closing angles indicators.



Source: Elaborated by the author.

Let $MCE(\mathscr{P}, \mathscr{E})$ be an instance of the maximal covering by ellipses, an optimal solution is given by

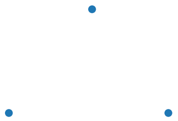
$$\max_{q_1,\ldots,q_m} w(\bigcup_{i=1}^{m} \mathscr{P} \cap E_i(q_i)),$$

- $\mathscr{P}$ is a set of n points, $\mathscr{E}$ is a set of m ellipses.
- (Canbolat and Massow 2009) is the very first study on the problem. Slow exact method, a heuristic one was proposed.
- (Andretta and Birgin 2013) proposed a method that breaks the problem into smaller optimization ones. Also, they developed a method for the non-axis-parallel case.

Maximal Covering by Ellipses

m ellipses

Figure: *m* ellipses adaptation.



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Maximal Covering by Ellipses

m ellipses

Figure: m ellipses adaptation. Max. Weight Clique.

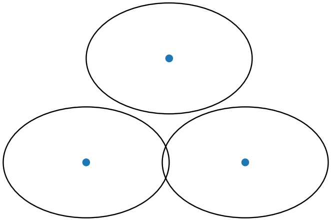

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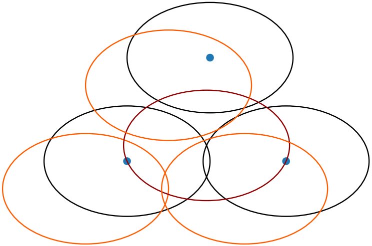
Figure: m ellipses adaptation. Max. Weight Clique (every solution).



Source: Elaborated by the author.

Figure: m ellipses adaptation. Every solution using E_1.

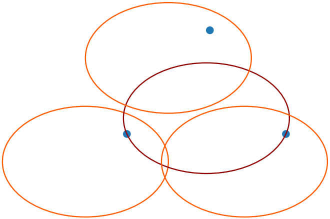


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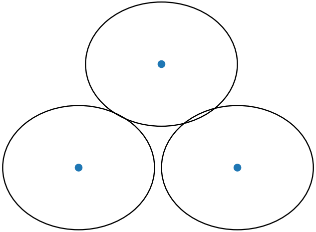
Figure: Max. Weight Clique for E_2.



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Maximal Covering by Ellipses

m ellipses

Figure: Every solution using E_2.

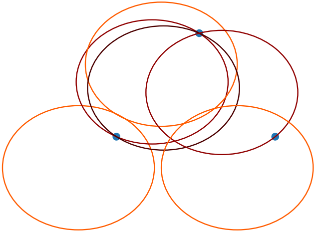

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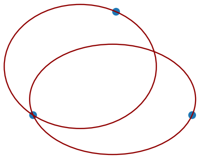
Figure: Optimal solution.



Source: Elaborated by the author.

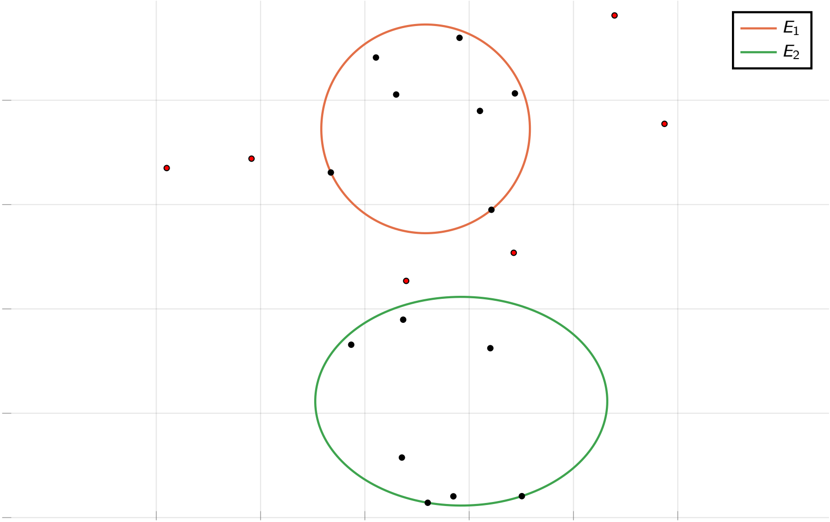
Maximal Covering by Ellipses

The algorithm for m ellipses

for $E_i \in \mathscr{E}$ **do**
    Fix an ellipse with the same shape as E_i at each point.
    $Z_i \leftarrow$ every solution of Maximum Weight Clique for it.
end for

function $f(\mathscr{P}, j)$
if $j = m$ **then**
    return 0
end if
$ans \leftarrow 0$
for $z \in Z_j$ **do**
    Fix E_j at z
    $Q \leftarrow$ points covered by $E_j(z)$
    $ans \leftarrow \max\{ans, f(\mathscr{P} \setminus Q, j+1) + w(Q)\}$
end for
return ans

Figure: Optimal solution with two ellipses for a random instance.



Source: Elaborated by the author.

Maximal Covering by Ellipses

- The algorithm for m ellipses tries every possible assignment of coverage for every one of the ellipses.
- Run-time complexity of $\mathcal{O}(n^{2m})$.
- Simpler than the m disks algorithm proposed by (Berg, Cabello, and Har-Peled 2006). Achieves a similar complexity ($\mathcal{O}(n^{2m-1})$).
- Small improvements can be made in the pre-processing exhibited earlier in oder to reduce the size of the search space:
  - Non-maximal coverage sets.
  - Ellipses that are too distant do not need to be checked.
- The unit-weight assumption can be easily dropped.

Future Work

Primary goals:

- Study the $(1-\epsilon)$-approximation method for the planar covering with disks in (Berg, Cabello, and Har-Peled 2006) and develop an adapted version of the algorithm for ellipses with the same time complexity of $\mathcal{O}(n \log n)$.
- Develop an exact method for the version of the problem introduced in (Andretta and Birgin 2013) where the ellipses can be freely rotated.

Secondary goals:

- Develop a probabilistic approximation algorithm based on (Aronov and Har-Peled 2008) which proposed a Monte Carlo approximation for the problem of finding the deepest point in a arrangement of regions. The method runs in $\mathcal{O}(n\epsilon^2 \log n)$ and can be applied to solve the case with one ellipse. The case with more than one ellipse is left as a challenge for us for the next steps of our research.
- In (He et al. 2015), the task of finding every center candidate, after eliminating all the non-essential ones, is done in $\mathcal{O}(n^5)$ run-time complexity. We want to generalize this for the elliptical distance function and achieve a better run-time complexity. We also intend to use the mean-shift algorithm to try to develop a greedy version for the ellipses version.

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