Algorithms for Planar Maximum Covering Location by Ellipses Problems☆

Danilo Tedeschi[1], Marina Andretta[1]

Abstract

Planar Maximum Covering Location by Ellipses is an optimization problem where one wants to place fixed shape ellipses on the plane to cover demand points maximizing a function depending on the value of covered points. We propose new exact algorithms for two versions of this problem, one where the ellipses have to be parallel to the coordinate axis, and another where they can be freely rotated. Besides finding optimal solutions for previously published instances, including the ones where no optimal solution was known, both algorithms proposed by us were able to obtain optimal solutions for some new larger instances having with up to seven hundred demand points and five ellipses.

Keywords: Optimization, Covering, Combinatorial Optimization

1. Introduction

The Planar Maximum Covering Location Problem (PMCLP) was first introduced in [1], and can be seen as a category of problems where the coverage of a demand set, a collection of subsets of $\mathbb{R}^2$, is to be maximized by determining the location of facilities in $\mathbb{R}^2$, with coverage being determined by a distance function. In [1], methods for Euclidean and Rectilinear distances versions of the problem were proposed. In [2, 3], exact algorithms for the Euclidean PMCLP with only one facility are proposed; and in [4] an approximation algorithm is proposed for the version with multiple unit disks as facilities. A property of the Euclidean PMCLP, which is utilized in the algorithms developed in [2, 3, 4], and in the method proposed in [1], is that there is an optimal solution which every facility is located in the demand points, or in the intersection of two circles centered at two demand points; we will prove a similar property for ellipses in our work.

It is fair to say that PMCLP with elliptical coverage has not been vastly studied as only two articles have been found on it. In [5], a mixed-integer non-linear programming method was proposed as a first approach to the problem. For some instances, the method took too long and did not find an optimal solution. Because of that, a heuristic method was developed using a technique called Simulated Annealing, which was able to obtain solutions for the instances proposed in that study. The problem was further explored in [6], which introduced the version where the ellipses can be freely rotated, to which an exact and a heuristic method was proposed, and developed a new method for the axis-parallel version of the problem, which was able to obtain optimal solutions for instances that the method proposed by [5] could not. The exact method for the version with rotation could not obtain optimal solutions within a predefined time limit for several instances, the

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Email addresses: danilo.tedeschi@usp.br (Danilo Tedeschi), andretta@gmail.com (Marina Andretta)

heuristic method though returned solutions for every instance, and impressively enough, obtained optimal solutions for every verifiable instance.

We study two versions of PMCLP with elliptical coverage facilities in this work. For both of them, each ellipse is defined to have a fixed shape and an undefined location, which is part of the solution. In the first version, introduced in [5], all the ellipses are restricted to be axis-parallel, while in the second version, introduced in [6], this constraint is dropped, and all the ellipses can be freely rotated. The first version will be referred to as Planar Maximum Covering Location by Ellipses Problem (MCE) and the second one as Planar Maximum Covering Location by Ellipses with Rotation Problem (MCER).

2. Problem Definition

An instance of MCE and MCER is given by n distinct demand points $\mathcal{P} = \{p_1, \dots, p_n\}$, $p_j \in \mathbb{R}^2$; n weights $\mathcal{W} = \{w_1, \dots, w_n\}$, with $w_j \in \mathbb{R}_{>0}$ being the weight of the j-th point; and m shape parameters $\mathcal{R} = \{(a_1, b_1), \dots, (a_m, b_m)\}$, with (a_j, b_j) being the semi-major and semi-minor of the j-th ellipse, with $a_j > b_j > 0$. We define a list of functions $\mathcal{E} = \{E_1, \dots, E_m\}$ representing the coverage area of each facility, with $E_j \colon \mathbb{R}^2 \to \mathbb{R}^2$ for MCE, and $E_j \colon \mathbb{R}^2 \times [0, \pi) \to \mathbb{R}^2$. Let $|| \cdot ||_{a,b,\theta} \colon \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ denote the elliptical norm given by

$$||x||_{a,b,\theta} = \left|\left| \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix} x \right|\right|_2 ,$$

then, for MCE we define $E_j(q) = \{p \in \mathbb{R}^2 \colon ||p - q||_{a_j,b_j,0} \leq 1\}$; and for MCER we define $E_j(q, \theta) = \{p \in \mathbb{R}^2 \colon ||p - q||_{a_j,b_j,\theta} \leq 1\}$.

Let $w : A \subset \mathcal{P} \to \mathbb{R}$ be a function which takes a subset of the demand set and returns the sum of the weights of the points in A. Then, we define MCE as the optimization problem

$$\max_{q_1,\dots,q_m} \sum_{j=1}^{m} w(\mathcal{P} \cap E_j(q_j)),$$

and similarly MCER as

$$\max_{(q_1,\theta_1),\dots,(q_m,\theta_m)} \sum_{j=1}^{m} w(\mathcal{P} \cap E_j(q_j, \theta_j)).$$

To make the notation more clear, we denote an instance of MCE or MCER as the tuple $(\mathcal{P}, \mathcal{W}, \mathcal{R})$, and a solution of MCE as $Q := (q_1, \dots, q_m)$, and a solution of MCER as $Q := ((q_1, \theta_1); \dots; (q_m, \theta_m))$. Additionally, whenever we have an instance with only one ellipse, we omit the index referring to the facility, and define a solution of MCE as $q \in \mathbb{R}^2$, and of MCER as $(q, \theta) \in \mathbb{R}^2 \times [0, \pi)$. We also use ∂ as the boundary operator, for example, given an instance of MCE, $\partial E_1(q_1)$ denotes an ellipse with shape parameters (a_1, b_1) centered at q_1.

2.1. Facility Cost

Additionally, in [5, 6], two other parameters are present in the definition of the problem. This extension is the result of having costs associated with every facility hence, to create a decision about which ones are utilized, a new parameter $k \in \mathbb{N}$ is given limiting the number of utilized ellipses to be exactly k.

We refer to this version of the problem as MCE-k, and MCER-k. An instance of it is given by the same parameters as MCE and MCER, plus a list of costs $\mathcal{C} = \{c_1, \ldots, c_m\}$, with $c_j \in \mathbb{R}_{\geq 0}$ being the j-th ellipse's cost, and $k \in \mathbb{N}$, $k \leq m$. A solution needs another set $I := \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ to express the indexes of the utilized ellipses.

Solving MCE-k (MCER-k) can be done by considering the $\binom{m}{k}$ instances of MCE (MCER), and then taking the best one as the optimal solution, taking into account the cost of each utilized ellipse. As this step can be seen as trivial, we propose algorithms for MCE and MCER, and then in section 8, as we use the instances from [5, 6] in the numerical experiments, we analyze the results obtained by our implementations of the algorithms proposed in this work for MCE-k and MCER-k.

3. An algorithm for MCE

Similarly to the method developed in [1] for the Euclidean PMCLP, we will describe a Candidate List Set (CLS) of possible locations for each ellipse and then propose an algorithm that constructs solutions combining the possible locations in each ellipse's CLS. Based on the approach of [3, 4], we will construct the CLS for each ellipse by working with a problem equivalent to MCE for only one facility.

Let $(\mathcal{P}, \mathcal{W}, \{(a, b)\})$ be an instance of MCE, then consider the problem where n ellipses $E(p_1), \ldots, E(p_n)$ are given, and a solution is defined as a point $q \in \mathbb{R}^2$, with an optimal one maximizing $w(\{p_i \in \mathcal{P} \colon q \in E(p_i)\})$. These two problems are equivalent as, let $q \in \mathbb{R}^2$, and $A \subset \mathcal{P}$; then $q \in \cap_{p \in A} E(p)$ if, and only if $\mathcal{P} \cap E(q) = A$.

To construct the CLS for each ellipse, let us consider the intersection region $\cap_{p \in A} E(p)$, for some $A \subset \mathcal{P}$, $|A| > 1$. In [1], this region is said to have vertices that are in the set of pairwise intersections of the circles with centers in A. Using the results of [7], which develops an algorithm to determine the intersection region of n fixed-radii disks in an strictly convex normed space, it is possible to prove that this is also true for ellipses. Based on that, given an instance of MCE, we define the CLS for each ellipse considering also the case where an ellipse could cover only one point in an optimal solution.

Lemma 1. *Let E be the coverage region of an axis-parallel ellipse with shape parameters (a, b); and $v \in \mathbb{R}^2$, $v \neq 0$, then $|\partial E(0) \cap \partial E(v)| \leq 2$, and $\partial E(0) \cap \partial E(v)$ can be determined analytically.*

Proof. By [7], we have that the number of intersections between two strictly convex circles of fixed radii is at most two. To determine the intersection points, consider the equality between the equations of $\partial E(0)$ and $\partial E(v)$: $x^2/a^2 + y^2/b^2 = (x - v_x)^2/a^2 + (y - v_y)^2/b^2$. This expression can be reduced to $y = \alpha x + \beta$, for some α, β. Using $\partial E(0)$'s equation, we obtain at most two values for x, which consequently, by $y = \alpha x + \beta$, determine the intersection points between the two ellipses. □

Definition 1. Given an instance of MCE, for any $k \in \{1, \ldots, m\}$, we define the CLS for the k-th ellipse as

$$S_k = \bigcup_{1 \leq i < j \leq n} \partial E_k(p_i) \cap \partial E_k(p_j) \bigcup \mathcal{P}. \tag{1}$$

By Lemma 1, the CLS for each ellipse can be computed in $\mathcal{O}(n^2)$. Next, we establish a lemma stating that the set of solutions obtained by combining the possible locations in each ellipse's CLS contains at least one optimal solution.

Theorem 1. *Given an instance of MCE, and $S_1, \dots, S_m$ as defined by Definition 1, then the set $\Omega = \{(q_1, \dots, q_m)\colon \text{ for all } q_k \in S_k\}$ contains an optimal solution of MCE and $|\Omega| \le n^{2m}$.*

Proof. Let Q^* be an optimal solution of MCE for the given instance. Then, we are going to prove that there exists $Q' \in \Omega$, which is also optimal.

For each $k = 1, \dots, m$, let $X_k = \{p_i \in \mathcal{P}\colon p_i \in E_k(q_k^*)\}$.

if $|X_k| \le 1$, then there is at least one element in S_k that makes $X_k \subset \mathcal{P} \cap E_k(q_k)$.

if $|X_k| > 1$, then let $Y_k = \cap_{p \in X_k} E_k(p)$, by the results of [7], we have that the boundary of Y_k has vertices in the pairwise intersections of $\{\partial E_k(p)\colon p \in X_k\}$. Therefore, at least one vertex of Y_k is in S_k, and any of those vertices produce a solution that covers at least the same points covered by Q^*.

Lastly, we have that $|S_k| \le 2\binom{n}{2} + n = n(n+1)/2 \le n^2$ hence $|\Omega| \le n^{2m}$. □

With all this in hand, we define Algorithm 1, which goes through every possible combination in the CLS of each ellipse. As evaluating each solution can be done in $\mathcal{O}(nm)$, we have that Algorithm 1 has a $\mathcal{O}(mn^{2m+1})$ runtime complexity. In section 8, we give more details about the implementation of Algorithm 1 and analyze some numerical experiments for instances proposed in [5, 6], and for some new ones.

Algorithm 1 Algorithm for MCE

Input: A set of points $\mathcal{P} = \{p_1, \dots, p_n\}$, a list of weights $\mathcal{W} = \{w_1, \dots, w_n\}$, and a list of shape parameters $\mathcal{R} = \{(a_1, b_1), \dots, (a_m, b_m)\}$.

Output: An optimal solution for MCE.

procedure $MCE(\mathcal{P}, \mathcal{W}, \mathcal{R})$
    return $MCE_{bt}(\mathcal{P}, \mathcal{W}, \mathcal{R}, 1)$
end procedure

procedure $MCE_{bt}(Z, \mathcal{W}, \mathcal{R}, j)$
    if $j = m + 1$ **then**
        return 0
    end if
    $(q_j^*, \dots, q_m^*) \leftarrow (0, \dots, 0)$
    Let S_j be the CLS for the j-th ellipse as defined by Definition 1.
    for $q_j \in S_j$ **do**
        $Cov \leftarrow \mathcal{P} \cap E_j(q_j)$
        $(q_{j+1}, \dots, q_m) \leftarrow MCE_{bt}(Z \setminus Cov, \mathcal{W}, \mathcal{R}, j+1)\}$
        if $w(\cup_{k=j}^m Z \cap E_k(q_k)) > w(\cup_{k=j}^m Z \cap E_k(q_k^*))$ **then**
            $(q_j^*, \dots, q_m^*) \leftarrow (q_j, \dots, q_m)$
        end if
    end for
    return $(q_j^*, \dots, q_m^*)$
end procedure

4. Determining every location of an ellipse given its shape and three points

In this section, we introduce the problem of determining every location, here defined as the center and angle of rotation, of an ellipse with fixed shape parameters, such that it contains three given points. This problem comes up in the development of an algorithm for MCER in the next section. It is important to point out that no prior studies were found on it, or even on related problems. We propose an algorithm for it that involves determining the eigenvalues of a 6×6 complex matrix. We also analyze its efficiency in terms of numerical accuracy and display some solutions that it was able to obtain.

4.1. Problem definition

Given the shape parameters of an ellipse (a, b), $a > b > 0$, and three points $u, v, w \in \mathbb{R}^2$, let $E\colon \mathbb{R}^2 \times [0, \pi) \to \mathbb{R}^2$ be the coverage region of an ellipse with shape parameters (a, b), we refer to the problem of obtaining $(q, \theta) \in \mathbb{R}^2 \times [0, \pi)$, such that $\{u, v, w\} \subset \partial E(q, \theta)$ as Ellipse by Three Points Problem (E3P). Because of its application here in our work, we are only interested in a method that can obtain every solution of E3P.

4.2. Transforming E3P into a circle problem

Initially, E3P is a problem of determining the values of three unknown continuous variables (q_x, q_y), and θ. However, as it will be shown, we can reduce this number to only one, as it is possible to obtain q uniquely from θ. Let us assume that point u is at the origin. If it is not, a simple translation by $-u$ applied to the three points can be made to put u at the origin. Assume as well that (q, θ) is a solution of E3P.

Applying a rotation of $-\theta$ to the coordinate system makes the ellipse in the original solution become axis-parallel. Then, that ellipse can be transformed into a circle of radius b by squeezing the x-axis by b/a. This two-step transformation can be written as a function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$\varphi(p, \theta) = \begin{bmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}.$$

An example of this transformation can be seen in Figure 1. As φ^{-1} is well-defined, instead of solving E3P, we can work with the univariate problem of determining an angle of rotation $\theta \in [0, \pi)$ that makes the triangle with vertices $\varphi(u, \theta), \varphi(v, \theta), \varphi(w, \theta)$ be circumscribed in a circle of radius b. To make the notation less cluttered, we denote by $\Lambda(\theta)$ the triangle with vertices $\varphi(u, \theta), \varphi(v, \theta), \varphi(w, \theta)$.

As circles are uniquely defined by three non-collinear points, the circumcircle of $\Lambda(\theta)$ is unique, and its radius and center can be determined analytically [8]. Let $|\Lambda(\theta)|$ denote the area of $\Lambda(\theta)$, using the formula from [9, p. 189] for the radius of a circumcircle of a triangle, and imposing that radius to be equal b, we define a function $\xi\colon [0, \pi) \to \mathbb{R}$ as

$$\xi(\theta) = 16b^2|\Lambda(\theta)|^2 - \|\varphi(v, \theta)\|_2^2 \, \|\varphi(w, \theta)\|_2^2 \, \|\varphi(v, \theta) - \varphi(w, \theta)\|_2^2 \,,$$

whose roots are angles of rotation which determine solutions of E3P through the inverse transformation φ^{-1}. From a root $\hat{\theta}$ of ξ, let $\hat{q}$ be the center of the circumcircle of $\Lambda(\hat{\theta})$, the solution $(\varphi^{-1}(\hat{q}), \hat{\theta})$ of E3P is obtained.

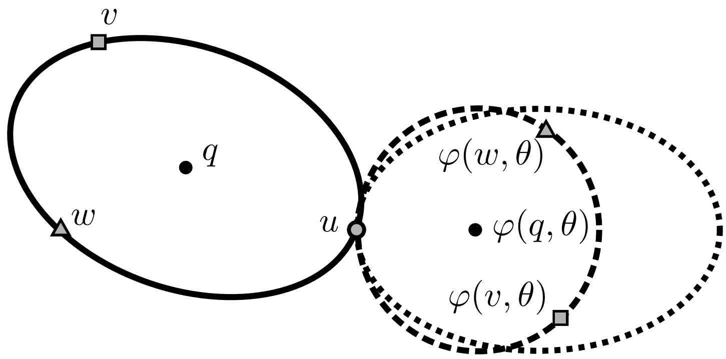


Figure 1: Transforming a solution of E3P into a solution of the circumcircle problem.

The algorithm for MCER described in the next section, of which E3P is a subproblem, goes through every solution of several instances of E3P. This is only possible if the number of solutions of E3P is finite, and it is only viable if that number is small. Next we introduce a lemma regarding that matter.

Lemma 2. *E3P has at most six solutions.*

Proof. The first thing to notice is that ξ is a real trigonometric polynomial of degree 6. Its term of highest degree is the multiplication of the norms $\|\varphi(v,\theta)\|_2^2 \, \|\varphi(w,\theta)\|_2^2 \, \|\varphi(v,\theta) - \varphi(w,\theta)\|_2^2$. In [10, p. 150], where a definition of real trigonometric polynomial is also given, it is stated that a n-degree real trigonometric polynomial can have up to $2n$ roots in $[0, 2\pi)$. Therefore, E3P has at most 12 solutions in $[0, 2\pi]$. Half of these solutions, though, are duplicated as ellipses are symmetric to their axis. $\square$

4.3. Converting ξ into a polynomial

In [11, p. 195], a theorem is presented stating that for every univariate polynomial of degree n, there exists a companion matrix, which is a $n \times n$ matrix, such that its eigenvalues are the zeros of that polynomial. Finding every eigenvalue of a matrix can be done using the QR algorithm, which runs in $\mathcal{O}(n^3)$ and uses $\mathcal{O}(n^2)$ memory (a very complete introduction to it can be found in [12]). For example, for a degree-4 polynomial $\sum_{k=0}^{4} a_k x^4$, a possible companion matrix is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{a_0}{a_4} & -\dfrac{a_1}{a_4} & -\dfrac{a_2}{a_4} & -\dfrac{a_3}{a_4} \end{bmatrix}.$$

In practice, we can use the very well-known LAPACK software library to obtain the eigenvalues of a matrix; for more information about LAPACK see [13]. This approach works for both real or complex polynomials, because of that, based on [14], we describe a way of converting ξ into a complex polynomial.

By using the two identities $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$, which relate trigonometric functions with complex numbers in the unit circle $\mathbb{S} = \{z \in \mathbb{C} \colon |z| = 1\}$, we can rewrite ξ as a function of the variable $z = e^{i\theta} \in \mathbb{S}$. In [14], it is stated that this substitution when utilized for the task of determining the roots of a real trigonometric polynomial does not yield loss of accuracy.

As ξ is a real trigonometric polynomial of degree 6, z appears with exponents from -6 up to 6. Multiplying ξ by z^6 and extending the domain of z to $\mathbb{C}$, we obtain a complex polynomial $g(z) = \sum_{k=0}^{12} c_k z^k$, for some $c_0, \ldots, c_{12} \in \mathbb{C}$. In practice, we utilize symbolic computation to obtain the actual coefficients of g in terms of an instance of E3P.

Let $angle\colon \mathbb{C}\colon [0, 2\pi)$ be a function that takes a complex number and returns its angle, then given a root $\hat{z}$ of g, if $|\hat{z}| = 1$ and $angle(\hat{z}) \in [0, \pi)$, then $\hat{\theta} = angle(\hat{z})$ is a root of ξ.

Observing that for any $z \in \mathbb{C}$, $angle(-z) = \pi + angle(z)$, and that for any ellipse the angles of rotation θ and $\theta + \pi$ are equivalent, we conclude that $g(-z) = g(z)$. This implies that all the odd-degree coefficients of g are zero. Therefore, we can use the substitution $y = z^2$ to obtain a degree-6 polynomial $f(y) = \sum_{k=1}^{6} c_{2k} y^k$ whose roots can be used to determine the roots of ξ: from a root $\hat{y}$ of f, $\hat{y} \in \mathbb{S}$, we have that $\hat{\theta} = angle(\hat{y})/2$ is a root of ξ.

Therefore, using the QR algorithm to obtain the eigenvalues of a companion matrix of the polynomial f, we can conclude that an algorithm to obtain every solution of E3P can be implemented, and that such algorithm takes a constant number of operations to do so.

4.4. Choosing a precision constant

In this section, we describe an experiment we made to choose a precision constant for comparing if a root of f returned as an eigenvalue of its companion matrix is in the unit circle. The implementation was coded in C++, and LAPACK's ZGEEV was utilized to obtain the eigenvalues of the companion matrix of f, more information about the implementation is given in **??**. For the experiment, we defined $K \in \mathbb{R}$, $K > 0$, and considered instances with the ellipse's shape parameters $(K, \frac{K}{2})$, for $K \in \{10^j : j = 0, \ldots, 10\}$.

The experiment considered instances of E3P where the three points are the vertices of an ellipse rotated by $\theta \in [0, \pi)$. Such instances only have one solution, and therefore, roots with multiplicity greater than one are expected. For each value of K, we ran the algorithm for 100 instances generated randomly by sampling θ according to a uniform distribution. For each instance, we took the root $\hat{z}$ which produced the closest solution to the known one. Then, for each K, as it can be seen in Figure 2a, we considered the maximum and the average distance to the unit circle $|1 - |\hat{z}||$; and, as it presented in Figure 2b, the maximum and average error $|f(\hat{z})|$.

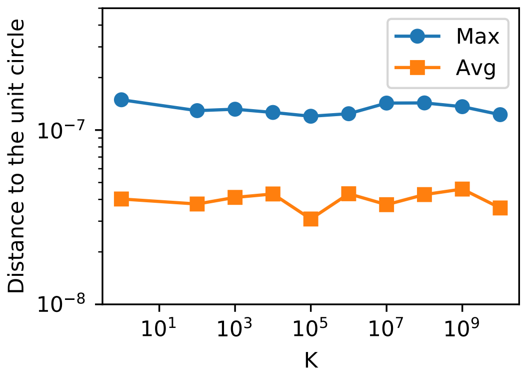
From this experiment, we decided to adopt a precision constant of 10^{-6} to consider a root of f to be in the unit circle, and as an additional check, we adopted a precision constant of 10^{-9} to consider a root to be a solution of E3P.

4.5. Instances with four and six solutions

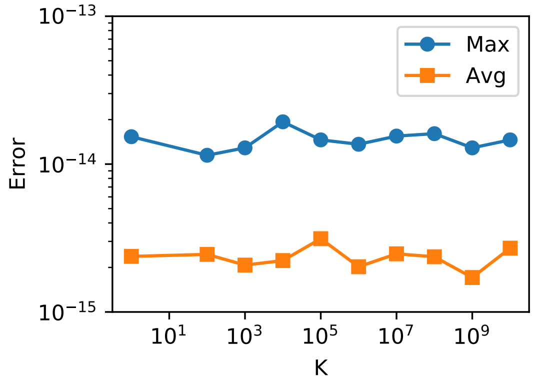
Any instance of E3P, as stated by Lemma 2 can have up to six solutions. At first, though, this bound seemed to be loose as for randomly generated instances, we were not able to obtain instances with more than two solutions.

After some investigation, we were able to construct some four-solution instances (an example is displayed in Figure 3a). An interesting property of those solutions is their three points form an isosceles triangle.

Six-solution instances were found by taking a particular case of the four-solution instances, we took the three points as the vertices of a equilateral triangle. An example of that is shown in Figure 3b.

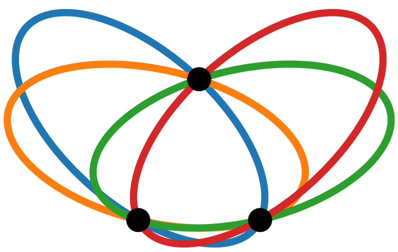


(a)

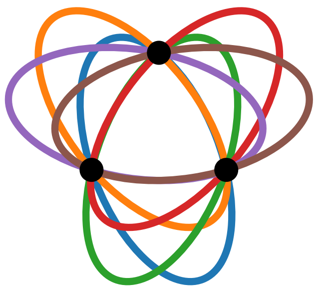


(b)

Figure 2: (a) shows the maximum and average distance to the unit circle $|1 - |\hat{z}||$. (b) shows the maximum and average error $|f(\hat{z})|$.



(a)



(b)

Figure 3: (a) The four solutions for an instance of E3P where the three points form an isosceles triangle. (b) The six solutions for an instance of E3P where the three points form an equilateral triangle.

It should be pointed out that neither non-isosceles instances with four solutions nor non-equilateral instances with six solutions could be found. Further investigating these possible properties of E3P is left as future work.

5. An Algorithm for MCER

The version of PMCLP where the facilities are ellipses that can be freely rotated was first introduced in [6] where an exact and a heuristic method were developed for it. In comparison with MCE, this problem introduces a new variable that is responsible for determining the angle of rotation of every ellipse, making MCER a more challenging problem. We propose an algorithm for MCER which is able to obtain optimal solutions for every instance proposed in [6] including the ones its exact method could not, and its heuristic obtained non-optimal ones.

Definition 2. Two solutions Q and Q' of MCER, are said to be equivalent to each other if

$\cup_{j=1}^{m} \mathcal{P} \cap E_j(q_j, \theta_j) = \cup_{j=1}^{m} \mathcal{P} \cap E_j(q'_j, \theta'_j)$. Also, if $\cup_{j=1}^{m} \mathcal{P} \cap E_j(q_j, \theta_j) \subset \cup_{j=1}^{m} \mathcal{P} \cap E_j(q'_j, \theta'_j)$, then we say that $Q' \succ Q$.

Next, we introduce a lemma which states that any optimal solution of MCER has an equivalent solution where every ellipse that covers at least two points has two points on its border.

Lemma 3. *Let Q^* be a solution of MCER for an instance $(\mathcal{P}, \mathcal{W}, \{(a, b)\})$. If $|\mathcal{P} \cap E(q^*, \theta^*)| \geq 2$, then there exists a solution Q for the same instance, such that $Q \succ Q^*$, and $|\mathcal{P} \cap \partial E(q, \theta)| \geq 2$.*

Proof. First, let $\theta = \theta^*$ and ignore the angle of rotation as it does not change, and assume that we are dealing with an axis-parallel ellipse.

Let $A = \mathcal{P} \cap E(q^*, \theta^*)$ and $X = \cap_{p \in A} E(p, \theta^*)$ be the region of intersection of ellipses centered at each point in A. By [7], the vertices of ∂X are in the set of pairwise intersections of $\{\partial E(p, \theta^*) : p \in A\}$. Setting q as any of these vertices makes $E(q, \theta)$ have two points on its border. □

Next, we introduce a notation that helps us characterize angles which given an ellipse rotated by it and two points, it is possible to find a center for the ellipse, such that it contains both points.

Definition 3. Let E be the coverage region of an ellipse and $u, v \in \mathbb{R}^2$. An angle $\theta \in [0, \pi)$ is said to be (E, u, v)-feasible if there is $q \in \mathbb{R}^2$ such that $\{u, v\} \subset \partial E(q, \theta)$. In addition to that, the set of (E, u, v)-feasible angles is referred to as

$$\Phi(u, v) := \{\theta \in [0, \pi) : \theta \text{ is a } (E, u, v)\text{-feasible angle}\}.$$

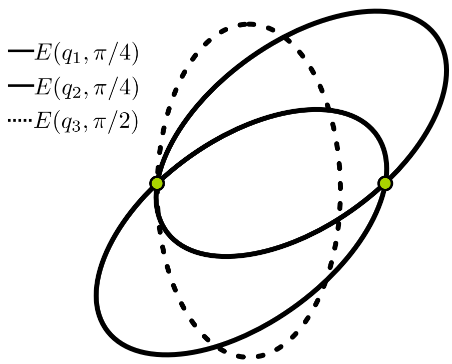


Figure 4: A (E, u, v)-feasible angle and a not (E, u, v)-feasible angle.

Let $x \in \mathbb{R}^2$, and $\angle x$ be the angle between x and the vector $(1, 0)$. Then we have that if $\Phi(u, v) \neq \emptyset$, then $\angle(u - v) \in \Phi(u, v)$ as it is the angle that makes the ellipse's major-axis be parallel to the line that passes through u and v.

Next we open a parenthesis to discuss the problem of deciding for what angles of rotation it is possible to find a center for an ellipse, so it contains two given points. We give the result for two points that have the same y-coordinate, but this can be generalized.

Lemma 4. *Given an instance $(\mathcal{P}, \mathcal{W}, \{(a, b)\})$ of MCER, if $u, v \in \mathcal{P}$ have the same y-coordinate and $||u - v||_2 \leq 2a$, then $\Phi(u, v) = [0, \alpha] \cup [\pi - \alpha, \pi)$, for some $\alpha \in [0, \pi/2]$.*

Proof. Consider an axis-parallel ellipse with shape parameters $(a, b) \in \mathbb{R}^2_{>0}$ centered at the origin, and a line represented by the equation $y = mx + c$, with $m, c \in \mathbb{R}$. Suppose that this line intersects the ellipse at least at one point. By plugging the line's equation into $x^2/a^2 + y^2/b^2 = 1$, it is possible to obtain the distance between the intersection points. The final expression is given by

$$D(m, c) = \frac{\sqrt{(a^2m^2 + b^2 - c^2)(4a^2b^2(1 + m^2))}}{(a^2m^2 + b^2)},$$

with $D : \mathbb{R}^2 \mapsto \mathbb{R}_{\geq 0}$ being a function of the line parameters (m, c). If $D(m, c) = ||u - v||_2$, then there exists $q_1, q_2 \in \mathbb{R}^2$, such that $\{u, v\} \subset \partial E(q_1, \tan m)$ and $\{u, v\} \subset \partial E(q_2, \pi - \tan m)$. It is also possible to see that, when m is fixed, $D(m, c)^2$ is a parabola, and that $D(m, c)$ is maximized at $c = 0$. Following that, we define a function $L : \mathbb{R} \mapsto \mathbb{R}$ as

$$L(m) := D(m, 0)^2 = \frac{(a^2m^2 + b^2)(4a^2b^2(1 + m^2))}{(a^2m^2 + b^2)^2},$$

which describes the maximum distance between points of an ellipse-line intersection considering all lines with m angular coefficient. From that, if $L(m) \geq ||v - u||_2^2$, then there exists $q_1, q_2 \in \mathbb{R}^2$, such that $\{u, v\} \subset \partial E(q_1, \tan m)$, and $\{u, v\} \subset \partial E(q_2, \pi - \tan m)$.

It is possible, by calculating the derivatives, to conclude that L has its maximum at $m = 0$, is increasing in $[0, \infty)$, is decreasing in $(-\infty, 0]$, and attains every value in the interval $(4b^2, 4a^2]$. Notice that L never hits $4b^2$ because that is the distance between the intersection of the ellipse with a vertical line.

If $\inf L \geq ||u - v||_2^2$, then $\Phi(u, v) = [0, \pi)$. Otherwise, let $\beta \in \mathbb{R}$, $\beta \geq 0$, such that $L(\beta) = ||u - v||_2^2$, then as $m > \beta$, we have $L(m) < ||u - v||_2^2$, which means that it is impossible to make the ellipse contain u, and v. As L is an even function, the same can be said for $m < \beta$. Therefore, we conclude that $\Phi(u, v) = [0, \tan(\beta)] \cup [\pi - \tan(\beta), \pi)$. □

Following that, we introduce a lemma that is responsible for connecting the developments of this chapter with the results of section 4. This lemma makes it possible to describe a type of solution which, for sure, is part of the equivalence class of any optimal solution. It states that, for any ellipse that covers more than two points in a given optimal solution, an equivalent solution exists with at least one of the two properties:

- The ellipse contains at least three points.
- The ellipse contains two points for any feasible angle.

Lemma 5. *Let Q^* be a solution of MCER for the instance $(\mathcal{P}, \mathcal{W}, \{(a, b)\})$, such that $|\mathcal{P} \cap E(q^*, \theta^*)| \geq 2$. If for all $\bar{Q} \succ Q^*$, $|\mathcal{P} \cap \partial E(\bar{q}, \bar{\theta})| < 3$, then there exists $\{u, v\} \subset \mathcal{P} \cap E(q^*, \theta^*)$, such that for all $\theta \in \Phi(u, v)$ there exists $q \in \mathbb{R}^2$, such that (q, θ) is equivalent to Q^*.*

Proof. According to Lemma 3, there exists $\{u, v\} \subset \mathcal{P} \cap E(q^*, \theta^*)$, such that $Q' \succ Q^*$ exists, and $\{u, v\} \subset \partial E(q', \theta^*)$. Therefore, $\theta^* \in \Phi(u, v)$.

Suppose that u and v have the same y-coordinate, if they do not, a rotation can be applied to make them do. Then, by Lemma 4, $\Phi(u, v) = [0, \alpha] \cup [\pi - \alpha, \pi)$, for some $\alpha \in [0, \pi/2]$. Then, if we rotate the coordinate system by $\pi - \alpha$, we obtain $\Phi(u, v) = [0, 2\alpha]$.

With this result in hand, we can use a continuity argument to complete our proof as follows. Let $\delta : \Phi(u, v) \mapsto \mathbb{R}^2$ be a continuous function which takes an angle $\theta \in \Phi(u, v)$ and returns a center, such that $\{u, v\} \subset \partial E(\delta(\theta), \theta)$, and, from solution Q', $\delta(\theta') = q'$. Notice that, in general, for any angle in $\Phi(u, v)$, there are two possible centers that make $\{u, v\} \subset \partial E(\delta(\theta), \theta)$ (see Figure 4 for an example), however, imposing $\delta(\theta') = q'$ makes δ be a well-defined continuous function. This is shown in Figure 5 where δ is plotted for the whole interval $\Phi(u, v)$.

Let $w \in \mathcal{P} \setminus \{u, v\}$, then we define $f_w : [0, \pi) \mapsto \mathbb{R}_{\geq 0}$ to be a function that takes an angle of rotation θ and returns the elliptical distance $|| \cdot ||_{a,b,\theta}$ to the center $\delta(\theta)$; that is $f_w(\theta) = ||w - \delta(\theta)||_{a,b,\theta}$. We have that if $w \in \mathcal{P} \cap E(q^*, \theta^*)$, then $f_w(\theta^*) \leq 1$; and if $w \notin \mathcal{P} \cap E(q^*, \theta^*)$, then $f_w(\theta^*) > 1$.

Therefore, if there exists $\theta \in \Phi(u, v)$, such that for all $q \in \mathbb{R}^2$, (q, θ) is not equivalent to Q^*, then there exists either $w \in \mathcal{P} \cap E(q^*, \theta^*)$, with $f_w(\theta) > 1$, or $w \notin \mathcal{P} \cap E(q^*, \theta^*)$, with $f_w(\theta) \leq 1$. Because f_w is continuous, there exists $\bar{\theta} \in \Phi(u, v)$, such that $f_w(\theta) = 1$, implying that $|\mathcal{P} \cap \partial E(\delta(\bar{\theta}), \bar{\theta})| \geq 3$. □

In Figure 5, a visualization of Lemma 5 is presented. An initial solution is given by the dashed-border ellipse and its center, represented by a star point. From it, the continuous function δ is defined by moving the ellipse through the rotation angles in $\Phi(u, v)$ while maintaining u, v on it. Six angles were chosen from $\Phi(u, v)$ to be shown in Figure 5, among those were 0 and $\max\{\Phi(u, v)\}$; their corresponding ellipses are displayed with solid-line borders. Consistently with Lemma 5, the points in $\mathcal{P} \setminus \{u, v, w\}$ stay within the ellipse's cover for any angle of rotation, and, for point w, there exists an angle, such that it is on the ellipse, which is a solution of E3P.

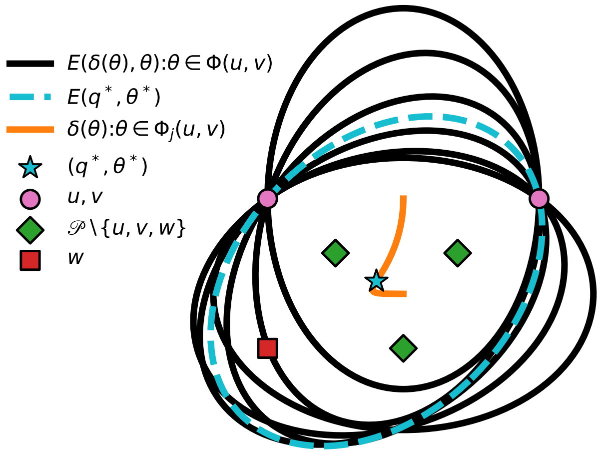


Figure 5: A visualization of Lemma 5.

Definition 4. Let $(\mathcal{P}, \mathcal{W}, \mathcal{R})$ be an instance of MCER. Then, for all $j \in \{1, \ldots, m\}$, we define the

CLS of the j-th ellipse as $S_j = S_j^{(1)} \cup S_j^{(2)} \cup S_j^{(3)}$ with

$$\begin{aligned}
S_j^{(1)} &= \bigcup_{u \in \mathcal{P}} \{(u, 0)\} \\
S_j^{(2)} &= \bigcup_{\{u,v\} \subset \mathcal{P}} \{(q, \angle(u - v)) \in \mathbb{R}^2 \times [0, \pi) : \{u, v\} \subset \partial E_j(q, \angle(u - v))\} \\
S_j^{(3)} &= \bigcup_{\{u,v,w\} \subset \mathcal{P}} \{(q, \theta) \in \mathbb{R}^2 \times [0, \pi) : \{u, v, w\} \subset \partial E_j(q, \theta)\}.
\end{aligned}$$

This definition breaks the construction of the CLS into three separated cases. The first one, $S_j^{(1)}$, represents solutions where the j-th ellipse covers only one point. The second one, $S_j^{(2)}$, takes into account solutions where the j-th ellipse covers at least two points, and no equivalent solution with three points on the ellipse exists. The last case, $S_j^{(3)}$, considers solutions where there exists an equivalent one with three points on the j-th ellipse.

To compute $S_j^{(2)}$, we can observe that, given two points u, v, determining every $q \in \mathbb{R}^2$, such that $\{u, v\} \subset \partial E_j(q, \angle(u-v))$ can be transformed into the problem of determining the set $\partial E_j(u, \angle(u-v)) \cap \partial E_j(v, \angle(u - v))$, which, by Lemma 1, is composed of at most two points, which can be determined analytically. Therefore, we have that $S_j^{(2)}$ can be computed in $\mathcal{O}(n^2)$ operations.

To compute $S_j^{(3)}$, we have to call the algorithm described in section 4 to determine every solution of E3P for every triplet of points in $\mathcal{P}$. Even though that algorithm is $\mathcal{O}(1)$, it has a high constant factor, thus skipping it, in practice, is a good suggestion. Given three points and an ellipse with shape parameters (a, b), the following two conditions are sufficient for E3P to have no solutions, and therefore, if any of them is true, we can skip calling the algorithm to determine every solution of E3P for that instance:

- The maximum distance between any of the points is greater than $2a$;
- The triangle's area with vertices on these three points have area greater than $\frac{3\sqrt{3}}{4}\pi ab$, which can be proved to be the greatest area of an inscribed triangle in an ellipse with shape parameters (a, b).

Overall, constructing every ellipse's CLS can be implemented to have a $\mathcal{O}(n^3)$ runtime complexity. Following this, we introduce a theorem, which connects the results for MCER so far, to prove that the set of solutions constructed using the CLSs described by Definition 4 contains an optimal solution.

Theorem 2. *Let $(\mathcal{P}, \mathcal{W}, \mathcal{R})$ be an instance of MCER, and Ω be a set of solutions defined as*

$$\Omega = \{Q \in (\mathbb{R}^2 \times [0, \pi))^m \colon (q_j, \theta_j) \in S_j \text{ for all } j \in \{1, \ldots, m\}\},$$

Then there exists an optimal solution $Q^* \in \Omega$, and $|\Omega| \leq n^{3m}$.

Proof. The first thing to notice is that Ω is defined as the combination of every possible solution from each CLS. To prove that it contains an optimal solution Q^*, we only need to prove that for all $j \in \{1, \ldots, m\}$, there exists $(q_j, \theta_j) \in S_j$, such that $\mathcal{P} \cap E_j(q_j^*, \theta_j^*) \subset \mathcal{P} \cap E_j(q_j, \theta_j)$. To do that, we use Lemma 5 and break the possible optimal solutions into three cases.

In the first case, we consider solutions where the j-th ellipse covers less than one point, that is, $|\mathcal{P} \cap E_j(q_j^*, \theta_j^*)| \leq 1$. It is possible to see that $S_j^{(1)}$ takes this possibility into account as it includes in Ω every solution that has an ellipse centered at a point from $\mathcal{P}$. From that, we can also conclude that $|S_j^{(1)}| \leq n$.

In the second case, we consider solutions where the j-th ellipse covers at least two points, and there is no $Q' \succ Q^*$, such that $|\mathcal{P} \cap \partial E_j(q'_j, \theta'_j)| \geq 3$. This case is addressed by Lemma 5, which states that there are equivalent solutions to Q^* with two points $u, v \in \mathcal{P} \cap E_j(q_j^*, \theta_j^*)$ on the j-th ellipse for every (E_j, u, v)-feasible angle. As $\angle(u - v)$ is a (E_j, u, v)-feasible angle, we have that there exists $(q_j, \theta_j) \in S_j^{(2)}$, such that $\mathcal{P} \cap E_j(q_j, \theta_j) = \mathcal{P} \cap E_j(q_j^*, \theta_j^*)$. Also, by Lemma 1, we have that $|S_j^{(2)}| \leq 2\binom{n}{2}$.

For the last case, we are left with solutions where the j-th ellipse covers more than two points, and there exists an equivalent solution with three points on it. As $S_j^{(3)}$ contains every center and angle of rotation that puts three points on the j-th ellipse, an equivalent solution for this case is present in the set of solutions Ω. Also, by Lemma 2 we can conclude that $|S_j^{(3)}| \leq 6\binom{n}{3}$.

Combining the three cases, as $S_j = S_j^{(1)} \cup S_j^{(2)} \cup S_j^{(3)}$, we get the following bound for $|S_j|$:

$$\begin{aligned}
|S_j| &\leq 6\tbinom{n}{3} + 2\tbinom{n}{2} + n &&= n(n-1)(n-2) + n(n-1) + n \\
|S_j| &\leq 6\tbinom{n}{3} + 2\tbinom{n}{2} + n &&= n((n-1)^2 + 1) \leq n^3.
\end{aligned}$$

Therefore, we conclude that $|\Omega| \leq |S_1| \times \cdots \times |S_m| \leq n^{3m}$. □

Finally, we define Algorithm 2, which backtracks every possible combination of solutions considering the CLS of every ellipse. As evaluating each solution can be implemented to take $\mathcal{O}(nm)$ operations, we have that Algorithm 2 has a $\mathcal{O}(mn^{3m+1})$ runtime complexity. In the next section, we describe some implementation details and improvements that, in practice, can lower the size of S_j significantly, and also can make the backtracking process described in Algorithm 2 skip many non-optimal solutions.

6. Reducing the CLS size

As for the algorithms for both MCE and MCER, the number of solutions they go through is directly proportional to the size of each ellipse's CLS, reducing their size can significantly improve the performance of both algorithms.

For MCE (the MCER's case is analogous), let $q, q' \in S_j$ be two possible locations in the CLS for the j-th ellipse. If $\mathcal{P} \cap E_j(q') \subset \mathcal{P} \cap E_j(q)$, then q' is redundant and we can remove it from S_j, as it produces a solution which is either non-optimal or equivalent to an optimal one.

To remove redundant elements from a CLS, we use the same tree-like data structure described in [6], which keeps every maximal subset of covered points by an ellipse, and supports a query operation to verify if a subset is maximal or not. First, we sort the elements in S_j by the number of covered demand points, non-decreasingly. Then, we iterate over it, removing elements which make the ellipse cover non-maximal subsets of demand points, when compared to the elements of S_j that have already been processed.

Algorithm 2 Algorithm for MCER

Input: A set of points $\mathcal{P} = \{p_1, \dots, p_n\}$, a list of weights $\mathcal{W} = \{w_1, \dots, w_n\}$, and a list of shape parameters $\mathcal{R} = \{(a_1, b_1), \dots, (a_m, b_m)\}$.
Output: An optimal solution for the given instance of MCER.

procedure $MCER(\mathcal{P}, \mathcal{W}, \mathcal{R})$
  return $MCER_{bt}(\mathcal{P}, \mathcal{W}, \mathcal{R}, 1)$
end procedure

procedure $MCER_{bt}(Z, \mathcal{W}, \mathcal{R}, j)$
  $(q_j^*, \theta_j*); \dots; (q_m^*, \theta_m^*) \leftarrow (0, 0); \dots; (0, 0)$ ▷ Setting to 0 as a default value.
  Let S_j be the CLS for the j-th ellipse as defined in Definition 4
  for all $(q_j, \theta_j) \in S_j$ **do**
    if $j < m$ **then**
      $(q_{j+1}, \theta_{j+1}); \dots; (q_m, \theta_m) \leftarrow MCER_{bt}(Z \setminus Cov, \mathcal{W}, \mathcal{R}, j+1)$
    end if
    if $w(\bigcup_{k=j}^{m} \mathcal{P} \cap E_k(q_k, \theta_k)) > w(\bigcup_{k=j}^{m} \mathcal{P} \cap E_k(q_k^*, \theta_k^*))$ **then**
      $(q_j^*, \theta_j*); \dots; (q_m^*, \theta_m^*) \leftarrow (q_j, \theta_j); \dots; (q_m, \theta_m)$
    end if
  end for
  return $(q_j^*, \theta_j*); \dots; (q_m^*, \theta_m^*)$
end procedure

7. Prunning the Backtracking Tree

Without any improvement, backtracking through every possible combination of every ellipse's CLS can take a very long time, possibly going through a lot of non-optimal solutions. For this reason, we introduce a sufficient condition for the MCER's case (the MCE's case is analogous), based on MCER for one ellipse, which can be used to skip solutions that for sure are non-optimal.

Definition 5. Given an instance $(\mathcal{P}, \mathcal{W}, \mathcal{R})$ of MCER. We define OPT_j as the value of the best solution with the first j ellipses fixed at locations $(q_1, \theta_1); \dots; (q_j, \theta_j)$, and $Z_j = \mathcal{P} \setminus \cup_{k=1}^{j} E_k(q_k, \theta_k)$.

Then, we can obtain an upper-bound for OPT_j by using, for each $k \in \{j+1, \dots, m\}$, the solution (q'_k, θ'_k) of MCER for instance $(Z_k, \{w_i \colon p_i \in Z_k\}, \{(a_k, b_k)\})$. As these solutions only consider the best cover individually for each ellipse, we have the following inequality

$$OPT_j \leq w\left(\bigcup_{k=1}^{j} \mathcal{P} \cap E_k(q_k, \theta_k)\right) + w\left(\bigcup_{k=j+1}^{m} \mathcal{P} \cap E_k(q'_k, \theta'_k)\right).$$

This upper-bound for OPT_j can then be used in the backtracking process to skip solutions that are not better than any optimal solution. Let OPT_{lo} be a lower bound for the optimal solution, we have that if

$$w\left(\bigcup_{k=1}^{j} \mathcal{P} \cap E_k(q_k, \theta_k)\right) + w\left(\bigcup_{k=j+1}^{m} \mathcal{P} \cap E_k(q'_k, \theta'_k)\right) \leq OPT_{lo}, \tag{2}$$

then $OPT_j \leq OPT_{lo}$, which implies that OPT_j is less than or equal the value of any optimal solution. This defines a sufficient condition for us to dismiss every solution which have the location of the first j ellipses fixed at $(q_1, \theta_1); \ldots; (q_j, \theta_j)$. In practice, we can use the value of the best solution found so far as the lower-bound for the optimal solution.

It is worth pointing out that these improvement suggestions do not have an effect in a possible worst case scenario. We are adopting them in our implementation because they showed good results in practice. For example, without taking the suggestion given by Equation 2, MCER-k's algorithm takes nine seconds to obtain an optimal solution for instance AB060, going through 336,494,451 solutions, while the implementation using Equation 2 to prune the backtracking tree for the same instance takes less than one second, and evaluates only 1809 solutions in total.

8. Numerical Experiments

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