Computers & Industrial Engineering 57 (2009) 201–208

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

Planar maximal covering with ellipses ☆

Mustafa S. Canbolat *, Michael von Massow

DeGroote School of Business, McMaster University, 1280 Main Street West, Hamilton, Ont., Canada L8S 4M4

ARTICLE INFO

Article history:
Received 7 March 2008
Received in revised form 6 August 2008
Accepted 14 November 2008
Available online 27 November 2008

Keywords:
Facility location
Maximal covering
Simulated Annealing
Elliptical covering
Wireless telecommunications networks
Mixed integer nonlinear programming

ABSTRACT

We consider a maximal covering location problem on the plane, where the objective is to provide maximal coverage of weighted demand points using a set of ellipses at minimum cost. The problem involves selecting k out of m ellipses. The problem occurs naturally in wireless telecommunications networks as coverage from some transmission towers takes an elliptical shape. A mixed integer nonlinear programming formulation (MINLP) for the problem is presented but MINLP solvers fail to solve it in a reasonable time. We suggest a Simulated Annealing heuristic as an alternate approach for solving the problem. Our computational results employing the heuristic show very good results with efficient processing times. We also discuss an extension to the problem where coverage occurs on a spherical surface and show that it can be solved with the same heuristic.

1. Introduction

Maximal covering location problems (MCLP) come into play when there are insufficient resources to cover all demand points or,alternately, it is also used in cases in which a firm seeks to maximize the return to the coverage selected. Any facility within the coverage distance is covered but outside of the coverage distance is not covered. The problem involves implementing different coverage categories which each have a specific cost associated with them. The objective is to cover the demand points that maximize the return. While the network version of the problem, where facilities are located on the nodes of a network, has been well studied in the literature, its planar counterpart where the facility can be placed anywhere on the plane, has been studied by several authors but is not as extensively explored. This is due to the fact that underlying distance functions make the models complicated.

The problem was first introduced by Church and ReVelle (1974) on a network. They formulate the problem and suggest a solution methodology for a given number of potential facility sites. For a detailed review on the problems studied on the networks, the user is referred to Schilling, Vaidyanathan, and Barkhi (1993) and Daskin (1995). The planar version of the problem was first considered by Mehrez and Stulman (1982) and later by Church (1984). They show that the problem can be solved over a small finite set of points which are the intersection points of circles drawn around demand points and suggested a mixed integer programming (MIP) formulation for the problem. The idea of identifying candidate points can be generalized for all distance norms. For a given distance norm, a corresponding unit ball blown up with a constant factor to the maximum coverage distance can be drawn around demand points to identify possible demand points as intersection points of these shapes. Subsequently, other researchers have considered different coverage distances such as inclined parallelograms (Younies & Wesolowsky, 2004) and block norms (Younies & Wesolowsky, 2007). Others have proposed different coverage functions (Drezner, Wesolowsky, & Drezner, 2004; Karasakal & Karasakal, 2004).

There are also a number of studies that approached to the problem in terms of its practical application. Current and O'Kelly (1992) reported on the application of the problem for locating emergency warning sirens in a case study. The authors considered two siren types with different costs and respective covering radii. Drezner and Wesolowsky (1997) considered locating a number of signal detectors with the objective of maximizing the smallest probability (to maximize the minimum protection) of the detection of an event anywhere on the plane. The authors assumed that the detection probability is a decreasing function of the distance. Recently, Jia et al. introduced a new maximal covering location model for the medical supply distribution for large-scale emergencies (2007a) and suggested heuristics based solution methodologies for the problem (2007b).

In this study we consider another practical usage of the problem. MCLP has great potential when it comes to the problem of

☆ This manuscript was processed by Area Editor Fernando Ordonez.

* Corresponding author. Tel.: +1 905 525 9140x26361; fax: +1 905 521 8995.
E-mail address: canbolat@mcmaster.ca (M.S. Canbolat).

0360-8352/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.cie.2008.11.015

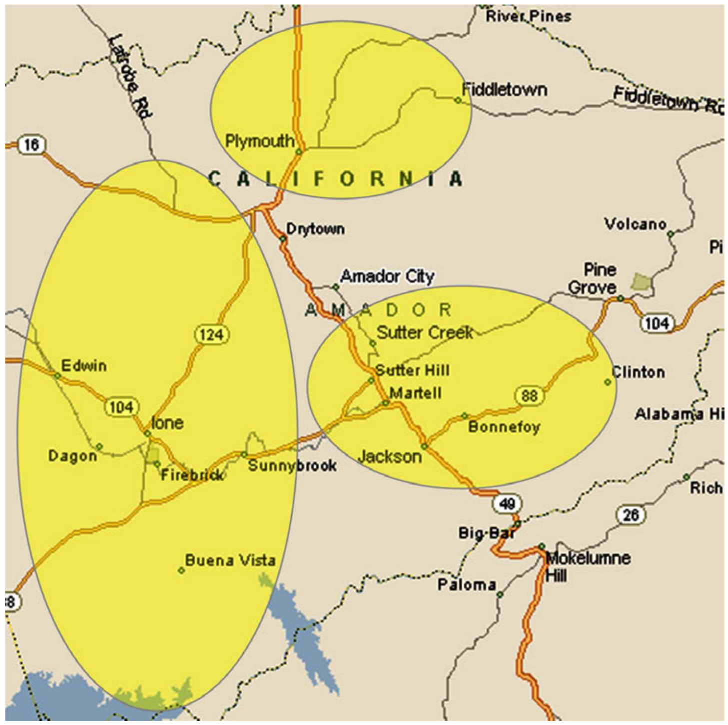

Fig. 1. Wireless high-speed internet coverage areas.

optimizing wireless transmitter coverage. Many satellite and antenna based transmitters have a coverage range that has an elliptical shape. Fig. 1 shows a coverage area for a California based wireless high speed Internet provider.[1]

It is clear from this diagram that companies are making decisions as described above and excluding some areas from coverage. Aguado-Agelet, Varela, Alvarez-Vazquez, Hernando, and Formella (2002) suggest that optimization techniques have not been widely used in transmitter location decisions. They introduce some solution approaches in a model that coverage requirements are predetermined and the objective is to locate transmitters such that power transmission is minimized but all of the required area is covered. It does not appear that selecting the coverage area based on minimum cost has been considered in the communications literature. To the best of our knowledge, despite its practical importance, MCLP by ellipses has not been considered in facility location literature. While most of the papers that consider circular (Euclidean) coverage, discuss antenna transmissions as an application of the problem, they do not acknowledge the fact that the circular maximal coverage is an approximation to the elliptical coverage. Our approach, therefore, provides a more precise and accurate solution than has been used before for the wireless transmission location problem. It also provides the flexibility to have ellipses with different parameters depending of the specific technical characteristics of the transmitter.

In this paper we provide a mixed integer nonlinear programming (MINLP) formulation for multi-facility MCLP with ellipses (MCLPE). We find that available solvers fail to provide a solution within a reasonable time. We then suggest a Simulated Annealing (SA) heuristic for the problem and show that it performs efficiently making the improved, more accurate approach feasible.

The paper is organized as follows. In Section 2 we present the MINLP formulation for the problem. This section includes a discussion of the geometric properties of ellipses in order to set the context for the formulation. Section 2.3 contains an example problem with computational results. In Section 3 we present the SA heuristics for the problem. We provide computational results to show that the approach performs well. In Section 4 we discuss the extension in which we evaluate spherical MCLP by ellipses. Finally in Section 5 we conclude and provide future research directions.

2. Problem definition and formulation

2.1. Problem definition

The problem involves n demand points, each with a nonnegative weight $w_j, j = 1, \cdots, n$, present in the plane. In the context of the transmitter location problem, a demand point could represent a population center and the weight the number of potential customers. We are trying to locate k elliptically shaped facilities from among a set of m facilities with a fixed elliptical coverage distance with parameters $((a_i, b_i), i = 1, \cdots, m)$. That is, we are choosing specific transmitters from among a possible technology set. The parameters (a_i, b_i) represent the semi-major and semi-minor axes of the facility i respectively, at locations defined by their foci (f_{1i}, f_{2i}) such that the facilities cover the maximum number of demand points (maximum weight) at these points. The parameters (a_i, b_i) determine the size and specific shape of the ellipse and are given. The foci (f_{1i}, f_{2i}) are the decision variables and set the location. Each facility has a cost of selection c_i.

The formulation of the problem makes use of the geometric properties of an ellipse. From these geometric properties outlined below, we derive the conditions of the coverage.

2.1.1. The geometric properties of an ellipse

The shape of an ellipse is expressed by a number called the eccentricity of the ellipse, ε. The eccentricity is a positive number

[1] http://communicationsadvantage.net/internet/coverage.html

less than 1 and greater than or equal to 0. An eccentricity of 0 implies that the two foci occupy the same point and that the ellipse is a circle. For an ellipse with semimajor axis a and semiminor axis b, the eccentricity is $\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}$. The distance from the center of an ellipse $C = (cx_i, cy_i)$ to either focus is $a\varepsilon$.

It is also the case that the sum of the distances from the foci to any point R on the edge of the ellipse is a constant value: $l_2(f_1, R) + l_2(f_2, R) = 2a$, where f_1 and f_2 are the two foci of the ellipse and l_2 is the Euclidean distance between two points. This property will help us in our formulation. Fig. 2 shows an ellipse with with a semimajor axis, a and semiminor axis, b. If we define this ellipse with its foci, we can give a coverage condition for it. Consider a demand point $X_j = (x_j, y_j)$ present in the plane. If the distance from the foci to the demand point exceeds $2a$ then we can say that the demand point is not covered by the ellipse. If this distance is less than $2a$ then the point is in the coverage area of the ellipse. This condition is an important component of our formulation as it allows us to determine coverage by a specifically located ellipse.

2.2. Notation

We use the following notation in our model formulation:

Inputs:
$w_j, j = 1, \cdots, n$: weight for demand point j.
$c_i, i = 1, \cdots, m$: cost of choosing ellipse i.
$a_i, i = 1, \cdots, m$: semimajor axis of ellipse i.
$b_i, i = 1, \cdots, m$: semiminor axis of ellipse i.
M: a big number.
m: the number of ellipses which can be located.
$k, k \leqslant m$: the number of new ellipses which are to be located.

Decision variables:
z_{ij}: binary decision variable, equal to 1 if demand point i is covered by ellipse j, and 0 otherwise.
y_i: binary decision variable, equal to 1 if ellipse i is selected and 0 otherwise.
f_1x_i: x-coordinate for focus 1 of ellipse i.
f_1y_i: y-coordinate for focus 1 of ellipse i.

Other variables:
f_2x_i: x-coordinate for focus 2 of ellipse i, which is actually equal to $f_1x_i + 2\varepsilon$, thus itself is not a decision variable.
f_2y_i: y-coordinate for focus 2 of ellipse i, this is actually as same as f_1y_i, given that an ellipse has a straight orientation, again is not a decision variable.

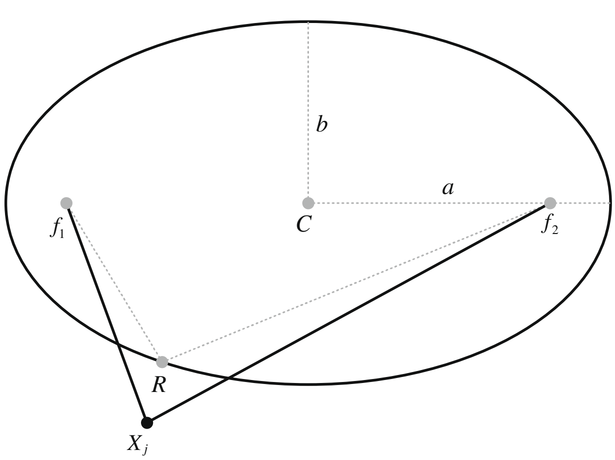

Fig. 2. Coverage conditions for an ellipse.

2.3. A MINLP formulation for MCLPE

The proposed MINLP formulation for MCLPE is as follows:

$$\text{Maximize} \sum_{i=1}^{m}\sum_{j=1}^{n} w_j z_{ij} - \sum_{i=1}^{m} c_i y_i \tag{1}$$

subject to

$$l_2(f_{1i}, X_j) + l_2(f_{2i}, X_j) - 2a_i \leqslant (1 - z_{ij})M, \quad \forall i, j \tag{2}$$

$$\sum_{i=1}^{m} y_i = k; \tag{3}$$

$$z_{ij} - y_i \leqslant 0, \quad \forall i, j \tag{4}$$

$$f_2x_i = f_1x_i + 2a_i\sqrt{1 - \frac{b_i^2}{a_i^2}}, \quad \forall i \tag{5}$$

$$f_2y_i = f_1y_i, \quad \forall i \tag{6}$$

$$\sum_{i=1}^{m} z_{ij} \leqslant 1, \quad \forall j \tag{7}$$

$$y_i, z_{ij} \in \{0, 1\} \tag{8}$$

$$i = 1, \cdots, m$$
$$j = 1, \cdots, n$$

The objective function ensures that the maximum weight of demand points will be covered by ellipse with minimum total facility selection cost. Constraint (2) ensures that z_{ij} is 1 if the sum of the distances from two foci of ellipse i is less than $2a$ which means that the demand point j is inside ellipse i. Constraint (3) is a user defined constraint which determines the number of ellipses to be selected. Constraint (4) makes sure that no point is assigned to an unselected ellipse. Constraints (5) and (6) determine the values of f_2x_i and f_2y_i in the objective function, in terms of f_1x_i and f_1y_i respectively. Constraint (7) requires that demand point i is covered by at most one ellipse. Finally Constraint (8) imposes a binary restriction on the decision variables y_i and z_{ij}.

We now define two properties that are helpful in the solution of of the model as formulated. The first helps reduce the feasible region.

Property 1. *There exists an optimal solution to the problem within the convex hull of the demand points.*

Proof. Let P be a point located outside the convex hull of the demand points. Assume an optimal solution includes an ellipse with a semimajor axis a and foci coordinates f_{1P} and f_{2P} centred at point P. Demand point X_j is covered by this ellipse if $l_2(f_{1P}, X_j) + l_2(f_{2P}, X_j) \leqslant 2a$.

Now consider a point Z which is the nearest point to P on the edge of the convex hull of the demand points. By the definition of a convex hull, point Z will be closer to every point within the convex hull than point P. If follows directly then that for the same ellipse now centred at points Z,that $l_2(f_{1Z}, X_j) + l_2(f_{2Z}, X_j) \leqslant l_2(f_{1P}, X_j) + l_2(f_{2P}, X_j)$ for all j. Therefore point P will never cover more demand points than point Z and an optimal solution is within the convex hull of the demand points. □

The second property establishes a reasonable value of the large number multiplier M used in constraint (2). This value has to be made as small as possible; using unnecessarily large values typically leads to very bad branch-and-bound trees in the solution process hence affects the solution quality (Bosch & Trick, 2005).

Property 2. *Let* $s_1 = \max_{1\leqslant j\leqslant n}\{x_j\} - \min_{1\leqslant j\leqslant n}\{x_j\}$ *and* $s_2 = \max_{1\leqslant j\leqslant n}\{y_j\} - \min_{1\leqslant j\leqslant n}\{y_j\}$. *Then M can be chosen as,*

$$M = s_1 + s_2 + a$$

Table 1
Ellipse parameters.

i	a_i	b_i	c_i
1	6	4	2
2	8	5	3.2
3	10	6	4.8

Proof. We know that M must be $\geqslant \sqrt{(s_1 - a\varepsilon)^2 + s_2^2} + \sqrt{(s_1 + a\varepsilon)^2 + s_2^2} - 2a$ to be an effective constraint.

Then using simple algebra,

$$\begin{aligned}&\sqrt{(s_1 - a\varepsilon)^2 + s_2^2} + \sqrt{(s_1 + a\varepsilon)^2 + s_2^2} - 2a \\ &\leqslant \sqrt{(s_1 + a\varepsilon)^2 + s_2^2} + \sqrt{(s_1 + a\varepsilon)^2 + s_2^2} \leqslant \sqrt{(s_1 + a\varepsilon + s_2)^2} \\ &\leqslant s_1 + s_2 + a\end{aligned}$$

we can see that we have a sufficiently large value of M without being unnecessarily large. □

A final consideration before moving on to the computational examples is the determination of an upper bound for the problem. Most MINLP solvers use branching and bounding in their process and, therefore, providing a good upper bound for the problem can improve the solution time. We propose using a rectangle with sides $2a$ and $2b$ as an approximation to the corresponding elliptical shape with semimajor axis a and semiminor axis b. This will provide a good upper bound for the problem. MCLP with rectangular shapes can be modeled as a mixed integer programming problem (See Younies & Wesolowsky (2004)) and getting an optimal solution for these type of problems will be relatively easier.

2.3.1. Computational examples

We specify a problem locating a ellipse on the plane to serve a number of demand points. We generate uniformly distributed random points with equal weights on the plane for $n = 25$, $n = 50$ and $n = 100$. Both the x and y coordinates are specified over the range of 0–50. The randomly generated coordinates of the demand points used in the computations are given in Appendix A. There are three ellipses available with the parameters listed in Table 1. These are given and the selection will depend on the number of ellipses allowed, k and the number and relative location of the demand points. We assign the cost parameters c_i proportional to the area of the ellipse given. That means coverage for a larger area, requires more cost. This idea gives a decision maker a flexibility to consider different coverage areas and observe the changes in the maximal coverage.

The problems are modeled in GAMS and solved using MINLP solvers available on the NEOS server[2]. The NEOS Server is a freely available Internet based client–server application that provides access to a library of optimization software. Among the optimization solvers hosted on the NEOS server, there are a number of solvers that are capable of solving MINLP problems and that support GAMS inputs. We evaluated two different solvers (SBB and BARON) for different values of n and k. We incorporate the upper bounds discussed in the formulation phase and limit the search area to the convex hull of the demand points. A time limit of 3600 s is defined for each case. If the solver does not find an optimal solution within the time limit we terminate the process and report the best integer solution, if there is one. Performance of the solvers and solution results are reported in Table 2.

The overall performance of SBB for this problem was considerably better than that of BARON. In fact, BARON was unable to provide proven optimal results for any of the test problems. However, some of the best integer solutions found by BARON are as same as the ones that are found and proven to be optimal by SBB. Therefore, we report these results as optimal. SBB found proven optimal solutions for the problem of selecting a single facility ($k = 1$) among three facilities for all test problems. Similarly, SBB was only successful in selecting two facilities among three ($k = 2$) and locating them optimally for up to $n = 25$. For the others, SBB found integer solutions, but could not prove their optimality. Neither solver was able to provide an integer solution to locate two facilities for $n = 100$. Fig. 3 is a graphical representation of the optimal solution found by SBB for $n = 25$ and $k = 2$. The dots represent the randomly generated demand points with equal weights, some of which are covered by the optimal selection of two ellipses.

3. A Simulated Annealing heuristic for the MCLP by ellipses

We can see from the results that this bruteforce approach of using general off-the-shelf MINLP solvers to solve MCLPE does not seem efficient at all. While some of the solvers might provide better results than the others as in this procedure, it is clear that for smaller size problems, it becomes problematic as the size of the problem instances increases. Therefore for larger sizes of real life problems there is need for an efficient heuristic procedure. We propose a Simulated Annealing (SA) heuristic for the problem. The SA heuristic was initially proposed by Kirkpatrick, Gelat, and Vecchi (1983) as an adaptation of the Metropolis-Hastings algorithm (Metropolis, Rosenbluth, Rosenbluth, Teller, & Teller, 1953). The name originates from physical process of annealing in metallurgy. SA is a technique requiring heating and slow cooling of materials in order to achieve a minimum energy crystalline structure. The SA algorithm mimics this process with the aim of finding a good solution while providing the opportunity to escape from local optima. The opportunities to jump from local optima are greater early in the process when the temperature is higher. As the process 'cools' the focus is on finding an optimal solution and the probability of a jump to a new neighborhood is reduced.

Previous work has shown that SA has great potential for problems with mixed discrete and continuous variables such as ours (Bennage & Dhingra, 1995). More recently, Arostegui, Kadipasaoglu, and Khumawala (2007) empirically evaluated several different heuristics in different location problems. While tabu search was better for some types of location problems, SA was slightly better statistically for an unconstrained (relative to solutions or time) facility location problem like ours. We also note that Taheri and Zomaya (2007) and Paik and Soni (2007) use SA approaches in communications technology and find that it works effectively, providing good solutions. Given this context, we choose to explore the possibility of efficiently finding good solutions to the MCLPE using an SA approach.

The SA algorithm for the MCLPE is as follows:

input : $T = T_0,\ \gamma,\ N = N_{max},\ F_0$
output: Best heuristic solution
while $N \leq N_{max}$ **do**
  randomly choose $\bar{Z} \in N(Z)$;
  if $F(\bar{Z}) \leq F(Z)$ **then**
    $\bar{Z} \leftarrow Z$;
  else
    $\Delta = F(\bar{Z}) - F(Z)$;
    $p = \exp(-\Delta/T)$;
    $\bar{Z} \leftarrow Z$ with probability p;
  end
  $T \leftarrow T\gamma$;
end

[2] http://www-neos.mcs.anl.gov/

Table 2
MINLP solution reports.

		Selected facility(ies)		# of points covered		Running time (s)		Optimality?	
k	n	SBB	BARON	SBB	BARON	SBB	BARON	SBB	BARON
1	10	1	1	3	3	1.81	> 3600	POS	Optimal
1	15	1	1	3	3	6.61	> 3600	POS	Optimal
1	20	1	1	3	3	15.22	> 3600	POS	Optimal
1	25	1	1	3	3	23.18	> 3600	POS	Optimal
1	30	3	3	4	4	62.67	> 3600	POS	Optimal
1	35	3	1	4	3	95.40	> 3600	POS	NOS
1	40	3	2	8	2	98.20	> 3600	POS	NOS
1	45	3	2	8	5	143.13	> 3600	POS	NOS
1	50	3	1	9	2	166.98	> 3600	POS	NOS
1	100	3	n/a	17	n/a	1284.90	> 3600	POS	n/a
2	10	1,2	1,2	5	5	15.76	> 3600	POS	Optimal
2	15	1,2	n/a	7	n/a	79.05	> 3600	POS	n/a
2	20	1,2	n/a	7	n/a	285.17	> 3600	POS	n/a
2	25	1,2	n/a	9	n/a	992.43	> 3600	POS	n/a
2	30	1,2	n/a	8	n/a	1000.12	> 3600	BIS	n/a
2	35	1,2	n/a	8	n/a	1030.24	> 3600	BIS	n/a
2	40	1,2	n/a	8	n/a	1075.68	> 3600	BIS	n/a
2	45	1,2	n/a	8	n/a	1148.75	> 3600	BIS	n/a
2	50	1,2	n/a	8	n/a	1586.16	> 3600	BIS	n/a
2	100	n/a	n/a	n/a	n/a	> 3600	> 3600	n/a	n/a

POS, proven optimal solution; NOS, non-optimal solution; BIS, best integer solution; n/a, no solution.

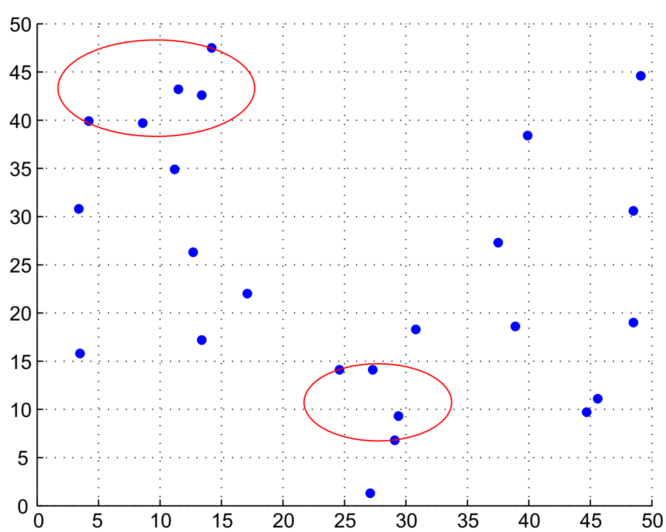

Fig. 3. MINLP solution for $n = 25$ $k = 2$.

The process starts with a parameter T_0 that represents a high temperature, which is reduced after each iteration. After an initial solution is generated, a random search is conducted to move from the current solution to a neighborhood solution. The selection of a neighborhood range is at the discretion of the user. Specifying the neighborhood range effectively is a key to the successful implementation of an SA algorithm. A new solution with a better objective value will always be accepted. There is also an opportunity to accept an inferior solution based on a probability p which is given by $p = \exp(-\frac{\Delta}{T})$, where Δ is the difference between the new solution and the current solution, and T is the current temperature.

When the temperature is high, the probability of accepting a worse solution is higher, so that in the initial steps, it is easier to escape from a local optimum. When the temperature decreases gradually, the probability will also decrease, so that it will be harder to move from the current solution. The process is summarized below.

We used the following parameters in our SA process which are similar to those used in previous SA literature:

T_0 (Initial temperature) = 500;
N (The number of iterations remaining) = 1000;
γ (The proportion by which the temperature reduced after each iteration) = $1 - (5/N)$.

Neighbourhood size and function is highly problem specific. Initial trials have been made with a number of different neighbourhood sizes and functions and the best one is kept for each test problem. We run the SA heuristic for 30 times for each of the test problems specified in the previous section. The results are reported in Table 3. Fig. 4 is the illustration of solution for $n = 100$ and $k = 2$. Note that for this test instance we were not be able to receive any solution from the MINLP solvers that we used.

The SA heuristic performs well. It finds solutions for all of the problems within reasonable computation times. The solutions generated are confirmed to be optimal for those problems that the MINLP solvers found proven optimal solutions. These computational results suggest that SA can be an effective tool in solving the MCLPE. However, its efficiency needs to be compared with other meta heuristics approaches such as Tabu Search or Genetic Algorithm in order to claim that it is the best heuristic approach for this problem. We propose to work on this issue in the future. Our focus here was to show that a more precise approach to problems such as the telecommunications transmitter problem is possible.

In the interest of comparison, Fig. 5 shows the elliptical coverage model versus the rectangular approximation. This involves the same sample problem as presented in Fig. 4. The MCLPE solution is the dotted line. An ellipse with a shaded line is superimposed on the rectangle to highlight the demand points that are in the optimal solution with the rectangle but would actually fall outside of the transmitter's coverage area. If we consider the larger ellipse at the top of Fig. 4, we can see that it covers demand points that are not covered by the rectangle. The rectangle covers demand points that are not covered by the MCLPE. However, if we consider the actual coverage area represented by the shaded ellipse there are more that fall outside the area under the rectangle. It is clear, therefore, that the MCLPE approach provides a better solution because it more accurately represents the coverage area of the transmitter and provides a better location for that transmitter.

Table 3
SA solution reports.

k	n	Selected facility(ies)	# of points covered			Average running times (s)	Optimality of best solution
			Worst	Average	Best		
1	10	1	1	1.8	3	1.10	Optimal
1	15	1	2	2.7	3	1.23	Optimal
1	20	1	2	2.9	3	1.32	Optimal
1	25	1	2	2.5	3	1.60	Optimal
1	30	3	2	3.8	4	1.74	Optimal
1	35	3	3	3.5	4	1.72	Optimal
1	40	3	5	7.4	8	1.88	Optimal
1	45	3	5	6.7	8	1.83	Optimal
1	50	3	6	7.9	9	1.92	Optimal
1	100	3	12	15.8	17	2.76	Optimal
2	10	1,2	3	4.3	5	1.94	Optimal
2	15	1,2	4	6.2	7	2.12	Optimal
2	20	1,2	4	6.4	7	2.28	Optimal
2	25	1,2	6	7.3	9	2.41	Optimal
2	30	1,2	6	7.5	10	2.72	BIS
2	35	1,2	7	9.3	10	3.42	BIS
2	40	1,3	8	10.8	12	3.60	BIS
2	45	1,3	10	13.1	14	3.67	BIS
2	50	1,3	12	14.7	15	4.12	BIS
2	100	2,3	23	26.2	28	7.28	BIS

BIS, best integer solution.

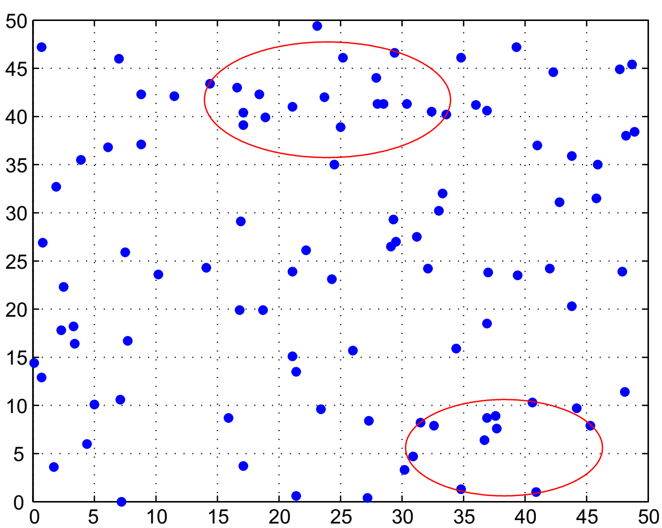

Fig. 4. SA solution for $n = 100$ $k = 2$.

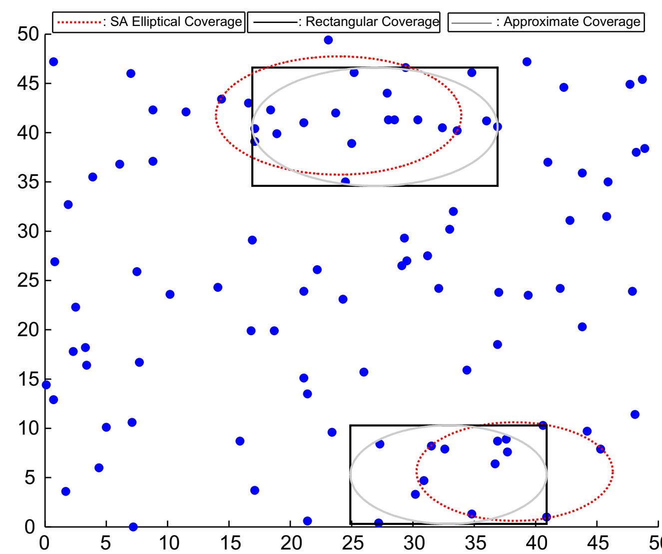

Fig. 5. Elliptical coverage vs. approximate coverage.

4. Covering spheres with ellipses

Planar distance functions are good approximations of the spherical distances on problems that actually occur on the world spheroid provided that the distance considered is small enough. When the distance gets larger, the approximation would be less accurate. For the maximal covering problems considered in this study, which have potential real world application, it would likely be more appropriate to use the spherical distance norm in place of the Euclidean distance.

The shortest distance between any two given points on a sphere is on the great circle passing through these two points and will be the shorter arc that joins these two points. It is clear that the maximum possible distance between any two points on a unit sphere will be π. Love, Morris, and Wesolowsky (1988) provides a formula for the shortest distance between two points on a unit sphere as,

$$A(X_1, X_2) = \cos^{-1}[\cos x_1 \cos x_2 \cos(y_1 - y_2) + \sin x_1 \sin x_2] \qquad (9)$$

where (x_1, y_1) and (x_2, y_2) are the latitude and longitude of point one and point two respectively. In an approach analogous to that used for the ellipse on a cartesian plane, we can define the following property for an ellipse given on a sphere.

Property 3. *Let f_1 and f_2 be the foci of an ellipse given on a sphere and let a be the semimajor axis of this ellipse given as a spherical distance. Then any points that satisfy the following inequality will be covered by this ellipse.*

$$A(X_i, f_1) + A(X_i, f_2) \leqslant 2a \qquad (10)$$

Proof. This is similar to the case for an ellipse on the plane. An ellipse can be drawn on a sphere by placing two pins at the foci and connecting the ends of a piece of string with length 2a to the pins. If a pencil is placed at an inferior point of the string and moved along, an ellipse can be drawn on the sphere. Any point remains inside the elliptical shape can be said covered by the ellipse. □

Although the distance function and the coordinate metrics used in the process are different from those in the Euclidean distance case, this problem can be solved by the heuristic suggested in this study.

5. Conclusion and future research directions

We presented a maximum coverage location problem using ellipses and an efficient solution algorithm. This problem has real world application but cannot be solved using readily available MINLP software. We find, however, that it can be efficiently solved using a Simulated Annealing heuristic. We present good computational results found in reasonable processing times. We also presented a potential extension to the problem of placing ellipses on a sphere rather than on the plane.

This formulation and solution methodology set the foundation for a number of interesting lines of future research. These build on the premise of locating wireless transmitters. The first potential extension is the inclusion of inclined ellipses to allow for different positions of transmission towers. Similarly one could include flexibility in the specific shape of the ellipse by specifying a fixed coverage area but allowing the semi-major and semi-minor axes to be decision variables.

One could also consider the inclusion of a budget constraint as an alternative to choosing a specified number of ellipses. This would allow for the choice of any number of ellipses up to and including m subject to a budget constraint. If the weights are interpreted as revenues from each demand point, the problem would not necessarily require a budget constraint and would become a purely profit maximizing MCPLE problem.

A further extension would be the incorporation of a continuous decline in the coverage. It is clear that the signal strength decreases as distance from the transmitter increases. The inclusion of a decay function would incorporate signal quality into the objective function through the maximization of the weighted and signal weighted coverage at minimum cost.

Finally, given issues in natural geography and potential limits on broadcast licenses, it would be interesting to consider the problem of elliptical coverage in presence of barriers. One might consider the big triangle small triangle (BTST) approach suggested by Drezner and Suzuki (2004) which can be used for maximal covering by a single ellipse, in order to find a global optimal point in a reasonable time.

Acknowledgements

We would like to acknowledge the contribution of two anonymous referees to making the manuscript. Their comments have ensured that the manuscript is more complete and clearer and easier to read.

Appendix A

Data sets used in the examples.

$n = 25$			$n = 50$			$n = 100$					
j	x_j	y_j	j	x_j	y_j	j	x_j	y_j	j	x_j	y_j
1	11.5	43.2	1	19.7	5.9	1	18.7	19.9	51	32.6	7.9
2	13.4	42.6	2	30.2	45.1	2	30.2	3.3	52	36.9	8.7
3	4.2	39.9	3	44.3	48	3	17.1	3.7	53	40.6	10.3
4	39.9	38.4	4	1.7	21	4	42.8	31.1	54	23.1	49.4
5	14.2	47.5	5	43.3	7.8	5	29.4	46.6	55	37.7	7.6
6	17.1	22.0	6	13	3.2	6	33	30.2	56	5	10.1
7	24.6	14.1	7	2.6	9	7	42.3	44.6	57	3.9	35.5
8	30.8	18.3	8	11.8	1.8	8	0.1	14.4	58	28	41.3
9	37.5	27.3	9	15	17.8	9	7.7	16.7	59	48.1	11.4
10	12.7	26.3	10	28.1	18.5	10	0.7	47.2	60	21.4	13.5
11	49.1	44.6	11	19.2	18.4	11	21.4	0.6	61	43.8	20.3
12	27.3	14.1	12	45.6	23.8	12	44.2	9.7	62	45.9	35
13	45.6	11.1	13	21.9	15.9	13	43.8	35.9	63	17.1	39.1
14	3.4	30.8	14	48.8	40.5	14	48.2	38	64	39.3	47.2
15	48.5	30.6	15	49.6	13.6	15	28.5	41.3	65	36	41.2
16	48.5	19.0	16	47.6	3.6	16	24.3	23.1	66	18.9	39.9
17	8.6	39.7	17	35.5	41	17	2.3	17.8	67	36.9	18.5
18	11.2	34.9	18	48.7	23.8	18	21.1	41	68	1.9	32.7
19	27.1	1.3	19	15.7	37.8	19	33.6	40.2	69	16.6	43
20	29.4	9.3	20	18.2	39	20	32.4	40.5	70	16.8	19.9
21	3.5	15.8	21	4.6	10.7	21	23.4	9.6	71	36.9	40.6
22	29.1	6.8	22	4.1	18.6	22	22.2	26.1	72	29.1	26.5
23	38.9	18.6	23	24.9	26	23	31.2	27.5	73	27.9	44
24	13.4	17.2	24	19.3	49.3	24	0.8	26.9	74	33.3	32
25	44.7	9.7	25	3	12.3	25	39.4	23.5	75	2.5	22.3
			26	1.2	46.4	26	45.3	7.9	76	41	37
			27	5.9	13.6	27	45.8	31.5	77	3.4	16.4
			28	39	34.3	28	15.9	8.7	78	8.8	42.3

(*continued on next page*)

Appendix A (*continued*)

$n = 25$			$n = 50$			$n = 100$					
j	x_j	y_j	j	x_j	y_j	j	x_j	y_j	j	x_j	y_j
			29	40.6	36.5	29	37	23.8	79	11.5	42.1
			30	5.2	7.5	30	17.1	40.4	80	32.1	24.2
			31	38.1	31.7	31	10.2	23.6	81	3.3	18.2
			32	9.5	20.8	32	14.4	43.4	82	8.8	37.1
			33	28.1	35.9	33	25.2	46.1	83	6.1	36.8
			34	28.2	9.9	34	26	15.7	84	34.4	15.9
			35	48.5	34.7	35	36.7	6.4	85	29.5	27
			36	26.9	40	36	0.7	12.9	86	7.5	25.9
			37	40.5	13.8	37	40.9	1	87	47.9	23.9
			38	9.7	43.5	38	37.6	8.9	88	34.8	46.1
			39	6.6	3.9	39	27.3	8.4	89	30.9	4.7
			40	38.3	37.2	40	48.7	45.4	90	4.4	6
			41	49.3	46.4	41	48.9	38.4	91	25	38.9
			42	45.3	27.7	42	23.7	42	92	34.8	1.3
			43	25.5	34.1	43	42	24.2	93	30.4	41.3
			44	25	8.1	44	27.2	0.4	94	24.5	35
			45	2.9	40	45	21.1	23.9	95	7.2	0
			46	33.9	36.9	46	7.1	10.6	96	16.9	29.1
			47	29.6	8.5	47	21.1	15.1	97	31.5	8.2
			48	44.7	19.5	48	14.1	24.3	98	29.3	29.3
			49	10.8	11.1	49	1.7	3.6	99	18.4	42.3
			50	17.4	16.9	50	47.7	44.9	100	7	46

References

Aguado-Agelet, F., Varela, A. M., Alvarez-Vazquez, L. J., Hernando, J. M., & Formella, A. (2002). Optimization methods for optimal transmitter locations in a mobile wireless system. *IEEE Transactions on Vehicular Technology, 51*(6), 1316–1321.

Arostegui, M., Kadipasaoglu, S., & Khumawala, B. (2007). An empirical comparison of tabu search simulated annealing and genetic algorithms for facilities location problems. *International Journal of Production Economics, 103*(2), 742–754.

Bennage, W. A., & Dhingra, A. K. (1995). Single and multiobjective structural optimization in discrete-continuous variables using simulated annealing. *International Journal for Numerical Methods in Engineering, 38*(16), 2753–2773.

Bosch, R., & Trick, M. (2005). *Chapter 3: Integer programming in search methodologies*. US: Springer.

Church, R. (1984). The planar maximal covering location problem. *Journal of Regional Science, 24*, 185–201.

Church, R., & ReVelle, C. (1974). The maximal covering location problem. *Papers of the Regional Science Association, 6*(32), 101–118.

Current, J. R., & O'Kelly, M. E. (1992). Locating emergency warning sirens. *Decision Sciences, 23*, 221–234.

Daskin, M. S. (1995). *Network and discrete location: Models, algorithms, and applications*. New York: John Wiley and Sons.

Drezner, Z., & Suzuki, A. (2004). The big triangle small triangle method for the solution of nonconvex facility location problems. *Operations Research, 52*(1), 128–135.

Drezner, Z., & Wesolowsky, G. O. (1997). On the best location of signal detectors. *IIE Transactions, 29*, 1007–1015.

Drezner, Z., Wesolowsky, G. O., & Drezner, T. (2004). The gradual covering problem. *Naval Research Logistics, 51*, 841–855.

Jia, H., Ordó ñez, F., & Dessouky, M. M. (2007a). A modeling framework for facility location of medical services for large-scale emergencies. *IIE Transactions, 39*(1), 41–55.

Jia, H., Ordó ñez, F., & Dessouky, M. M. (2007b). Solution approaches for facility location of medical supplies for large-scale emergencies. *Computers and Industrial Engineering, 52:*, 257–276.

Karasakal, O., & Karasakal, E. K. (2004). Maximal covering location model in the presence of partial coverage. *Computers and Operations Research, 31*(9), 1515–1526.

Kirkpatrick, S., Gelat, C. D., & Vecchi, M. P. (1983). Optimization by simulated annealing. *Science, 220*, 671–680.

Love, R. F., Morris, J. G., & Wesolowsky, G. O. (1988). *Facilities location: Models and methods*. Amsterdam: North Holland.

Mehrez, A., & Stulman, A. (1982). The maximal covering location problem with facility placement on the entire plane. *Journal of Regional Science, 22*, 361–365.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., & Teller, E. (1953). Equation of state calculation by fast computing machines. *Journal of Chemical Physics, 21*(6), 1087–1092.

Paik, C., & Soni, S. (2007). A simulated annealing based solution approach for the two-layered location registration and paging areas partitioning problem in cellular mobile networks. *European Journal of Operations Research, 178*(2), 579–594.

Schilling, D. A., Vaidyanathan, J., & Barkhi, R. (1993). A review of covering problems in facility location. *Location Science, 1*, 25–55.

Taheri, J., & Zomaya, A. (2007). A simulated annealing approach for mobile location management. *Computer Communications, 30*(4), 714–730.

Younies, H., & Wesolowsky, G. O. (2004). A mixed integer formulation for maximal covering by inclined parallelograms. *European Journal of Operational Research, 159*, 83–94, 200.

Younies, H., & Wesolowsky, G. O. (2007). A mixed integer formulation for maximal covering by inclined parallelograms. *Journal of the Operational Research Society, 58*, 740–750.