Planar Maximal Covering with Ellipses

Danilo F. Tedeschi
Orientadora: Dra. Marina Andretta

Instituto de Ciências Matemáticas e Computação

May 1, 2019

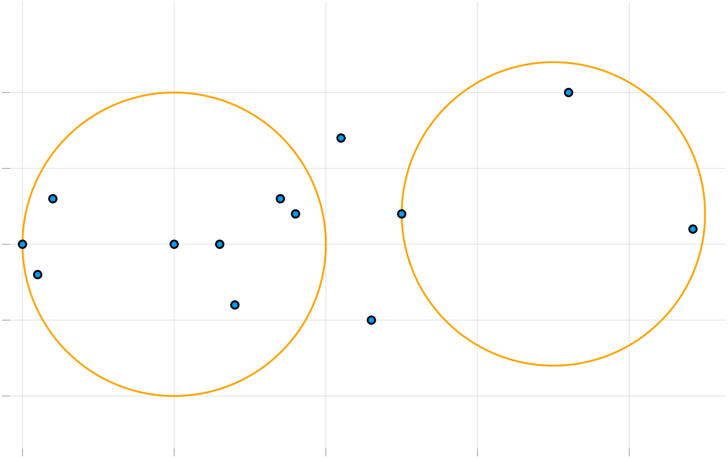
Thanks to CAPES

Contents

Introduction

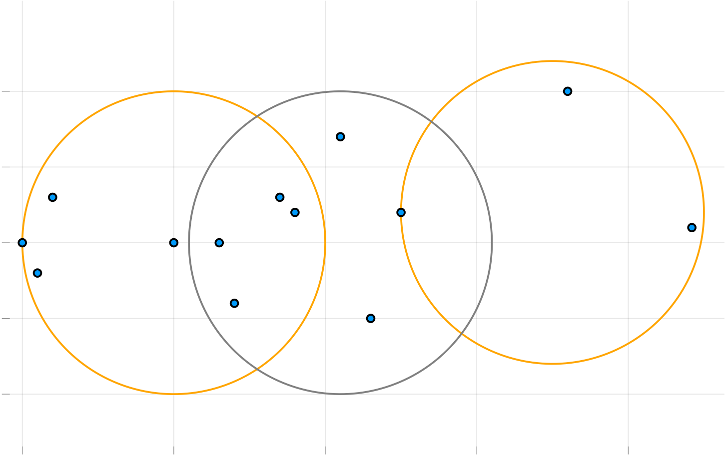
- Covering problems
  - Minimum Cover Problem
  - Maximal Covering Problem
- Maximal Covering Location Problem (MCLP)
- Planar Maximal Covering Location Problem (PMCLP)
  - One disk: $\mathcal{O}(n^2)$ and $\mathcal{O}(n^2 \log n)$ algorithms
  - m disks: $\mathcal{O}(n^{2m-1} \log n)$ algorithm
- Goals
  - Develop a $\mathcal{O}(n^2 \log n)$ algorithm for the one disk case
  - Adapt it for the m ellipses case creating a $\mathcal{O}(n^{2m})$ algorithm

Figure: Maximal cover by two disks.



Source: Elaborated by the author.

Figure: Minimum number of disks needed to cover the set of points.



Source: Elaborated by the author.

Preliminaries

Norms

Let $u \in \mathbb{R}^2$ and Q a $2x2$ positive definite matrix

- Euclidean

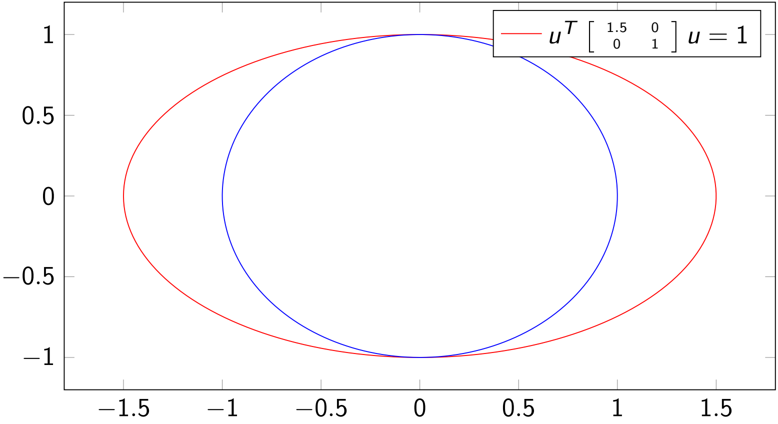
$$||u||_2 = \sqrt{u^T u}$$

- Elliptical

$$||u||_Q = \sqrt{u^T Q u}$$

Preliminaries

Figure: The elliptical and euclidean norms.



Source: Elaborated by the author.

Preliminaries

Ellipse

Given a center $c \in \mathbb{R}^2$ and a 2x2 p.d. matrix Q, an ellipse is the set of points that satisfy

$$||u - c||_Q = 1,$$

with $\leq$ representing the set of covered points

Axis-parallel ellipse

Any 2 by 2 diagonal d.p. matrix determines an axis-parallel ellipse, which can also be described by

$$\frac{(x - c_x)^2}{a^2} + \frac{(y - c_y)^2}{b^2} = 1,$$

where (a, b) are the shape parameters and $c = (c_x, c_y)$ is the center.

$MCD(\mathscr{P}, 1)$ is the problem of placing one disk on the plane to cover a subset of a demand set $\mathscr{P}$ maximizing the weights of the covered points.

$$\max_q w(\mathscr{P} \cap D(q)),$$

- $\mathscr{P} = \{p_1, \ldots, p_n\}$ is the demand set with weights $w(p_i) > 0$
- $w(A)$, $A \subset \mathscr{P}$, is the sum of weights of the points in A
- $D(q)$ is a unit disk with center at point q
- [MCL86] proposed a $\mathscr{O}(n^2)$ algorithm
- [Dre81] proposed a $\mathscr{O}(n^2 \log n)$ which our work is based on
- We will introduce an equivalent problem...

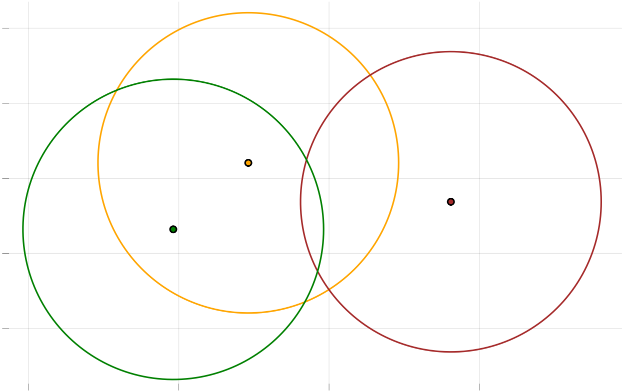
Maximum Weight Clique Problem

Let $\mathscr{D} = \{D_1, \ldots, D_n\}$ be a set of n unit disks with weights $w_i > 0$. The maximum weight clique is defined as

$$\max_q \sum_{D_k \cap q \neq \emptyset} w_k,$$

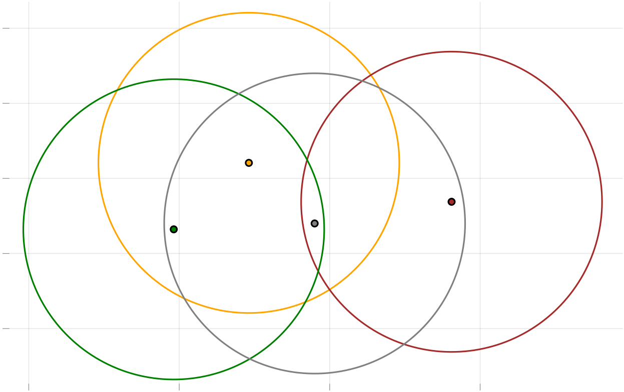
- The disks are fixed with centers at $\mathscr{P} = \{p_1, \ldots, p_n\}$ with $w_k = w(p_k)$
- A clique is a non-empty intersection area of a subset of disks. We search for only a point in an optimal clique.
- An optimal solution for the maximum weight clique is an optimal solution for $MCD(\mathscr{P}, 1)$.

Figure: An instance of the Maximum Weight Clique Problem obtained from an instance of $MCD(\mathscr{P}, 1)$.



Source: Elaborated by the author.

Figure: An instance of the Maximum Weight Clique Problem obtained from an instance of $MCD(\mathscr{P}, 1)$. In gray, the optimal solution.

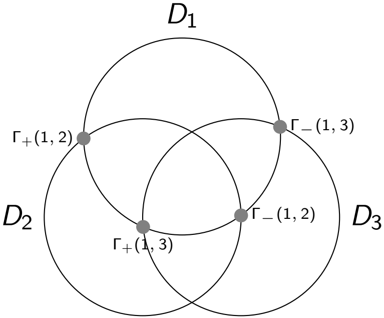


Source: Elaborated by the author.

Defining $\Gamma_+(i,j)$ and $\Gamma_-(i,j)$:
Let D_i and D_j be two unit disks that intersect at two points.

- Assume D_i is at the origin
- We can find the polar angle of the intersection points. Let's call them $\Gamma_+(i,j)$ and $\Gamma_-(i,j)$ with the following condition:
  - The arc defined by $\Gamma_+(i,j)$ and $\Gamma_-(i,j)$ bounds the region $D_i \cap D_j$
- Later it will be shown how to find which one is which algorithmically.

Figure: Three disks and their intersection points and angles.



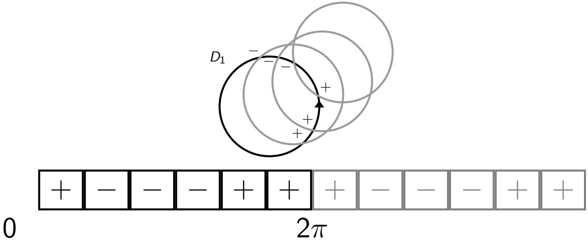
Source: Elaborated by the author.

For a disk D_i, a counter-clockwise traversal visits every $\Gamma_+(i,j)$ and $\Gamma_-(i,j)$ in counter-clockwise order.

- An intersection region of disks is bounded by arcs.
- The arc $\Gamma_+(i,j), \Gamma_-(i,j)$ (counter-clockwise) determines a region where i and j intersect.
- In a counter-clockwise traversal, the arcs where $\Gamma_+(i,j) > \Gamma_-(i,j)$ can be a problem for the implementation. The traversal is on a circle, where to start?
- If the traversal reaches a opening angle, we mark the intersecting disk as active. If it reaches a closing angle we unmark it.
- We want to find a point in the traversal with the most active disks.

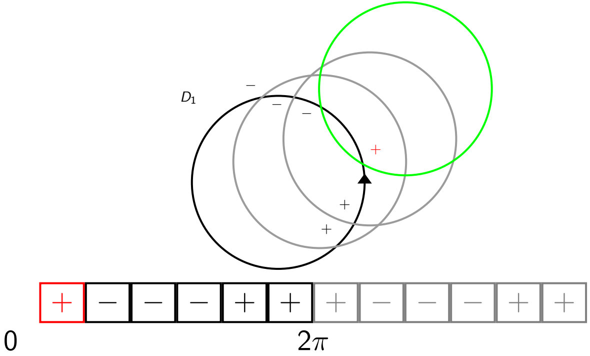
For every disk, traverse the sorted list of intersection angles twice, keeping a set of active disks, and the current best solution.
Doing the traversal twice solves the problem of deciding where to start.

Figure: The intersections list of a disk with three other disks.



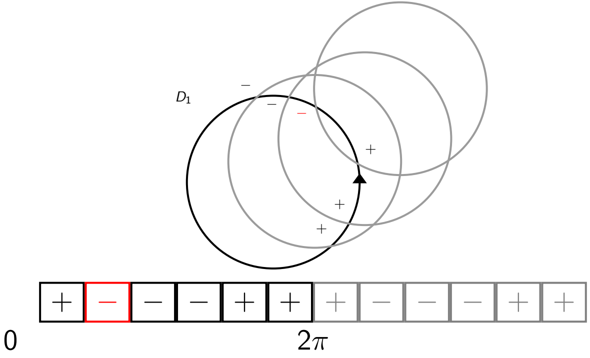
Source: Elaborated by the author.

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



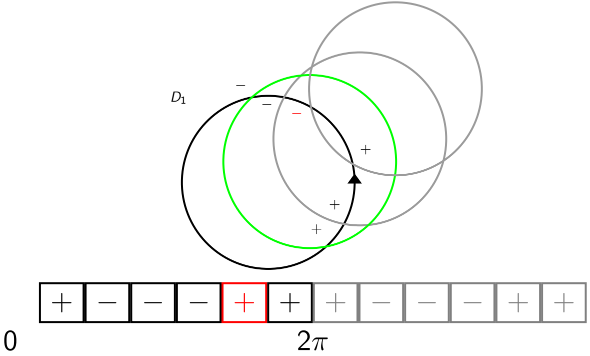
Source: Elaborated by the author.

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



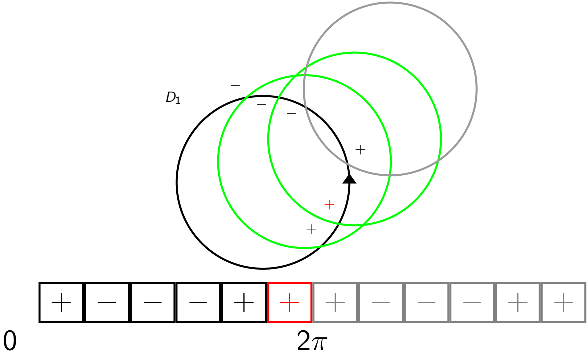
Source: Elaborated by the author.

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



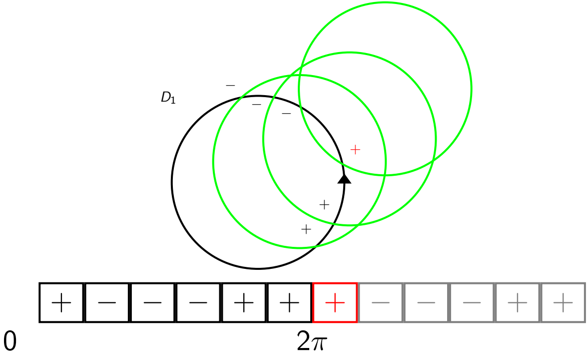
Source: Elaborated by the author.

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



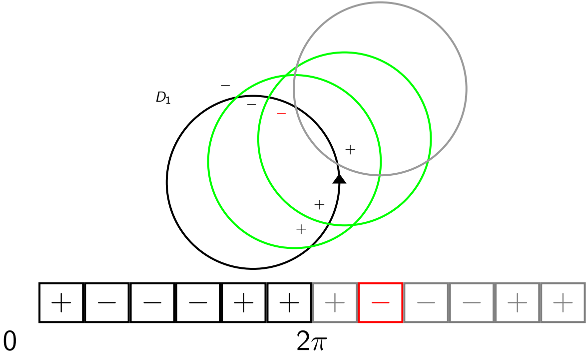
Source: Elaborated by the author.

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



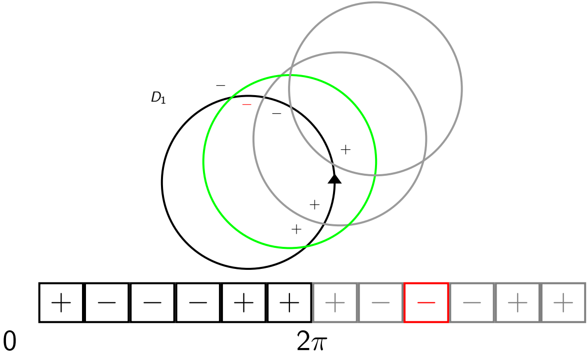
Source: Elaborated by the author.

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



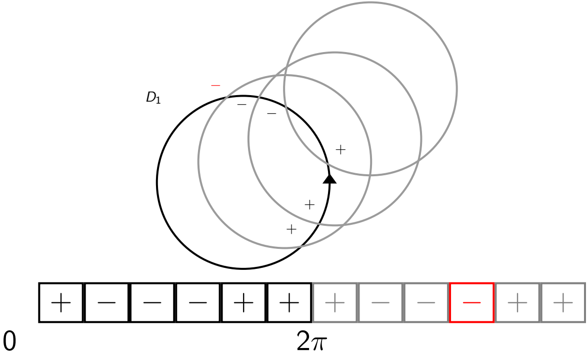
Source: Elaborated by the author.

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



Source: Elaborated by the author.

Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



Source: Elaborated by the author.

The run-time complexity of the algorithm is $\mathcal{O}(n^2 \log n)$.

- There are $\mathcal{O}(n^2)$ intersections among n disks
- Sorting takes $\mathcal{O}(n^2 \log n)$
- The traversal takes $\mathcal{O}(n)$ for every disk.
- It can be implemented in $K \log n$ where K is the number of intersections.

Works found in the literature:

- In [dBCHP06] a $\mathcal{O}(n^{2m-1})$ algorithm was proposed. Also a $(1-\epsilon)-$approximation that runs in $\mathcal{O}(n \log n)$ was introduced.
- In [HFC⁺15] a heuristic method using an algorithm called mean-shift was developed. The mean-shift algorithm converges to a local density maxima of any probability distribution and it is used to find a smaller candidate list of centers for the disks.

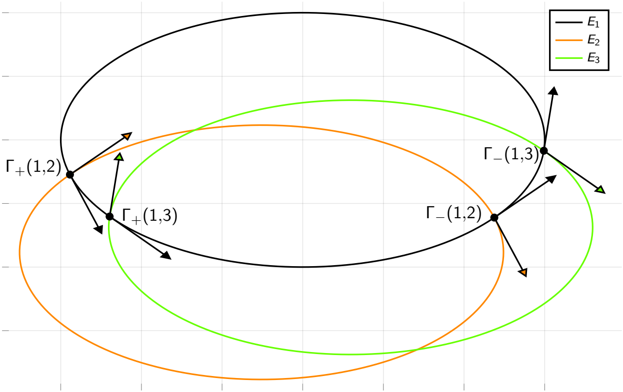
Because of the similarities, we will discuss only the multiple ellipses algorithm later.

Let $MCE(\mathscr{P}, a, b)$ be an instance of the maximal covering by one ellipse, with E being an ellipse with shape parameters $(a, b) \in \mathbb{R}^2_{>0}$, an optimal solution of $MCE(\mathscr{P}, a, b)$ is given by

$$\max_{q} |\mathscr{P} \cap (q)|,$$

- $E(q)$ is an axis-parallel ellipse with center point q
- Assuming unit weights for now
- Same algorithm for one disk

Figure: Intersection points of E_1 with E_2 and E_3 along with opening and closing angles indicators.



Source: Elaborated by the author.

Let $MCE(\mathscr{P}, \mathscr{E})$ be an instance of the maximal covering by ellipses, an optimal solution is given by

$$\max_{q_1,\ldots,q_m} \left| \bigcup_{i=1}^{m} \mathscr{P} \cap E_i(q_i) \right|,$$

- $\mathscr{E}$ is a set of m ellipses
- [CvM09] is the very first study on the problem. Slow exact method, a heuristic one was proposed.
- [ABR13] proposed a method that breaks the problem into smaller optimization ones. Also, they developed a method for the non-axis-parallel case.

Maximal Covering by Ellipses

Pre-processing that finds every possible coverage for ellipse E_i

$A \leftarrow \bigcup_{j \in I_i} \{\Gamma_+(i,j) \cup \Gamma_-(i,j)\}$
$Z \leftarrow \{\}$
for $cnt = 1..2$ **do**
  for $a \in A$ **do**
    Let p_a be the point that intersects E_i at angle a.
    if a is a starting angle **then**
      $Cov \leftarrow Cov \cup \{p_a\}$
    else
      $Cov \leftarrow Cov \setminus \{p_a\}$
    end if
    $Z \leftarrow Z \cup \{Cov\}$
  end for
end for

Maximal Covering by Ellipses

- The algorithm for m ellipses tries every possible assignment of coverage for every one of the ellipses
- Run-time complexity of $\mathcal{O}(n^2)^m = \mathcal{O}(n^{2m})$
- Simpler than the m disks algorithm proposed by [dBCHP06]. Achieves a similar complexity ($\mathcal{O}(n^{2m-1})$).
- Small improvements can be made in the pre-processing exhibited earlier in oder to reduce the size of the search space:
  - Non-maximal coverage sets.
  - Ellipses that are too distant do not need to be checked.
- The unit-weight assumption can be easily dropped

Future Work

Primary goals:

- Study the $(1-\epsilon)$-approximation method for the planar covering with disks in [dBCHP06] and develop an adapted version of the algorithm for ellipses with the same time complexity of $\mathcal{O}(n \log n)$.
- Develop an exact method for the version of the problem introduced in [ABR13] where the ellipses can be freely rotated.

Future Work

Secondary goals:

- Develop a probabilistic approximation algorithm based on [AH08] which proposed a Monte Carlo approximation for the problem of finding the deepest point in a arrangement of regions. The method runs in $\mathcal{O}(n\epsilon^2 \log n)$ and can be applied to solve the case with one ellipse. The case with more than one ellipse is left as a challenge for us for the next steps of our research.
- In [HFC$^+$15], the task of finding every center candidate, after eliminating all the non-essential ones, is done in $\mathcal{O}(n^5)$ run-time complexity. We want to generalize this for the elliptical distance function and achieve a better run-time complexity. We also intend to use the mean-shift algorithm to try to develop a greedy version for the ellipses version.

M. Andretta, E. G. Birgin, and M. Raydan, *Deterministic and stochastic global optimization techniques for planar covering with elipses problems*, European Journal of Operational Research **22** (2013), 23–40.

Boris Aronov and Sariel Har-Peled, *On approximating the depth and related problems*, SIAM J. Comput. **38** (2008), no. 3, 899–921.

M. S. Canbolat and M. von Massow, *Planar maximal covering with ellipses*, Computers and Industrial Engineering **57** (2009), 201–208.

Mark de Berg, Sergio Cabello, and Sariel Har-Peled, *Covering many or few points with unit disks*, vol. 45, 09 2006, pp. 55–68.

Zvi Drezner, *Note—on a modified one-center model*, Management Science **27** (1981), 848–851.

References II

Zhou HE, Bo Fan, T. C. E. Cheng, Shou-Yang Wang, and Chin-Hon Tan, *A mean-shift algorithm for large-scale planar maximal covering location problems*, European Journal of Operational Research **250** (2015).

B M. Chazelle and D Lee, *On a circle placement problem*, Computing **36** (1986), 1–16.