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The Condition of Polynomials in Power Form*

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Abstract. A study is made of the numerical condition of the coordinate map M_n which associates to each polynomial of degree $\leqslant n-1$ on the compact interval $[a, b]$ the n-vector of its coefficients with respect to the power basis. It is shown that the condition number $\|M_n\|_\infty \|M_n^{-1}\|_\infty$ increases at an exponential rate if the interval $[a, b]$ is symmetric or on one side of the origin, the rate of growth being at least equal to $1+\sqrt{2}$. In the more difficult case of an asymmetric interval around the origin we obtain upper bounds for the condition number which also grow exponentially.

1. Introduction. Let M_n: $\mathbf{R}^n \longrightarrow \mathbf{P}_{n-1}$ be the linear map associating to each vector $u^T = [u_1, u_2, \ldots, u_n] \in \mathbf{R}^n$ the polynomial

$$p(x) = \sum_{k=1}^{n} u_k x^{k-1} \in \mathbf{P}_{n-1}, \qquad n \geqslant 2.$$

For any $p \in \mathbf{P}_{n-1}$ we shall write $u_p = M_n^{-1} p$, where M_n^{-1} is the inverse map of M_n. We define the *condition* of the map M_n, relative to the compact interval $[a, b]$, by

$$\operatorname{cond}_\infty M_n = \|M_n\|_\infty \|M_n^{-1}\|_\infty, \tag{1.1}$$

where the norms are $\|u\|_\infty = \max_{1 \leqslant k \leqslant n} |u_k|$ (in $\mathbf{R}^n$) and $\|p\|_\infty = \max_{a \leqslant x \leqslant b} |p(x)|$ (in $\mathbf{P}_{n-1}[a, b]$). We are interested in the growth rate of $\operatorname{cond}_\infty M_n$ as $n \longrightarrow \infty$, and how this growth depends on the particular interval $[a, b]$ chosen.

The answer is relatively straightforward for symmetric intervals $[-\omega, \omega]$ and for intervals $[a, b]$ with $0 \leqslant a < b$, in which cases the condition number in (1.1) can be expressed explicitly in terms of $u_{T_{n-1}}$ (or $u_{T_{n-2}}$), where T_m denotes the Chebyshev polynomial of degree m on the appropriate interval (Theorems 3.1, 3.2). It will follow, in particular, that on $[-\omega, \omega]$ and $[0, \omega]$, $\omega > 0$, the condition grows exponentially with n, and that the minimum growth occurs precisely when $\omega = 1$, in which case $\operatorname{cond}_\infty M_n$ grows like $(1+\sqrt{2})^n$ on $[-1, 1]$ and like $(1+\sqrt{2})^{2n}$ on $[0, 1]$. This ought to be contrasted with the linear growth $\sqrt{2}\, n$ for the condition on $[-1, 1]$ of polynomials represented in terms of Chebyshev polynomials [1].

For asymmetric intervals $[a, b]$ with, say, $a < 0 < b$, $|a| < b$, the problem appears to be considerably more complex, and we are no longer able to ascertain the exact growth rate of (1.1). Instead, we obtain two upper bounds for $\operatorname{cond}_\infty M_n$, one being asymptotically sharp in the extreme case $|a| = b$, the other in the extreme case $a = 0$ (Theorem 4.1).

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2. Preliminaries on the Coefficients of Chebyshev Polynomials. In the following we need estimates for the largest coefficients in $T_n(x/\omega)$ and $T_n^*(x/\omega)$, where T_n is the Chebyshev polynomial of the first kind and T_n^* the "shifted" Chebyshev polynomial $T_n^*(x) = T_n(2x - 1)$.

It is well known that

$$T_n\left(\frac{x}{\omega}\right) = \sum_{k=0}^{[n/2]} c_k x^{n-2k}, \tag{2.1}$$

where

$$c_k = (-1)^k \frac{n}{2} \frac{(n-k-1)!}{k!(n-2k)!} \left(\frac{2}{\omega}\right)^{n-2k}, \qquad 0 \leqslant k \leqslant [n/2].$$

For fixed t, with $0 < t < ½$, we put $k = tn$, and let $n \longrightarrow \infty$. Using Stirling's formula, we find

$$|c_{tn}| \sim \frac{n^{-1/2}}{2\sqrt{2\pi}} \frac{1}{\sqrt{t(1-t)(1-2t)}} \left(\frac{2}{\omega}\right)^n e^{ng(t)}, \qquad n \longrightarrow \infty,$$

where

$$g(t) = (1-t)\ln(1-t) - t\ln t - (1-2t)\ln(1-2t) - 2t\ln(2/\omega), \qquad 0 < t < ½.$$

From $g(0) = 0$, $g(½) = -\ln(2/\omega)$, $g'(t) = \ln[(1-2t)^2\omega^2/4t(1-t)]$, it is seen that $g(t)$ has a unique maximum on $[0, ½]$, assumed at

$$t = t_0 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\omega^2}}\right).$$

Since

$$g(t_0) = \ln\frac{1-t_0}{1-2t_0} = \ln[½(1+\sqrt{1+\omega^2})], \sqrt{t_0(1-t_0)(1-2t_0)} = ½\omega(1+\omega^2)^{-3/4},$$

we thus find for the maximum coefficient of $T_n(x/\omega)$ the asymptotic approximation

$$\|u_{T_n(x/\omega)}\|_\infty \sim \frac{1}{\sqrt{2\pi}} \frac{(1+\omega^2)^{3/4}}{\omega} n^{-1/2} \left(\frac{1+\sqrt{1+\omega^2}}{\omega}\right)^n, \qquad n \longrightarrow \infty. \tag{2.2}$$

For $\omega = 1$, this gives

$$\|u_{T_n}\|_\infty \sim \frac{2^{1/4}}{\sqrt{\pi}} n^{-1/2}(1+\sqrt{2})^n, \qquad n \longrightarrow \infty \quad (\omega = 1), \tag{2.2'}$$

which agrees with a result attributed to an (anonymous) referee in J. R. Rice [3, p. 304].

Since $T_n^*(x^2) = T_{2n}(x)$, the analogous result for $T_n^*(x/\omega)$ is readily obtained from (2.2) by replacing n by $2n$ and ω by $\sqrt{\omega}$,

$$\|u_{T_n^*(x/\omega)}\|_\infty \sim \frac{1}{2\sqrt{\pi}} \frac{(1+\omega)^{3/4}}{\sqrt{\omega}} n^{-1/2} \left(\frac{2+\omega+2\sqrt{1+\omega}}{\omega}\right)^n, \qquad n \longrightarrow \infty. \tag{2.3}$$

For $\omega = 1$, this gives

$$\|u_{T_n^*}\|_\infty \sim \frac{2^{-1/4}}{\sqrt{\pi}} n^{-1/2}(3 + 2\sqrt{2})^n, \qquad n \longrightarrow \infty \quad (\omega = 1). \tag{2.3'}$$

In Table 2.1 we compare the true values of $\|u_{T_n(x/\omega)}\|_\infty$ with their asymptotic approximations in (2.2) for selected values of n and ω.

ω	$n = 5$		$n = 10$		$n = 20$		$n = 40$	
	true	(2.2)	true	(2.2)	true	(2.2)	true	(2.2)
10	5.00(−1)	9.36(−1)	1.00	1.09	2.00	2.09	1.06(1)	1.09(1)
5	1.00	1.11	2.00	2.12	1.06(1)	1.09(1)	4.02(2)	4.11(2)
1	2.00(1)	2.46(1)	1.28(3)	1.43(3)	6.55(6)	6.79(6)	2.12(14)	2.17(14)
.2	5.00(4)	9.65(4)	5.00(9)	7.17(9)	5.00(19)	5.59(19)	5.00(39)	4.82(39)
.1	1.60(6)	5.82(6)	5.12(12)	1.33(13)	5.24(25)	9.91(25)	5.50(51)	7.72(51)

TABLE 2.1. The quality of the asymptotic formula (2.2)

We also note that

$$\|u_{T_n(x/\omega)}\|_\infty \geqslant \|u_{T_{n-1}(x/\omega)}\|_\infty, \qquad n = 1, 2, 3, \ldots, \omega \leqslant 1, \tag{2.4}$$

where equality holds only for $n = 1$, $\omega = 1$. This follows easily from the three-term recurrence relation for Chebyshev polynomials and from the alternating character of the coefficients c_k in (2.1). The inequality in (2.4) holds for all $\omega \leqslant 2$, if n is restricted to $n \geqslant 2$, and it indeed holds for any fixed ω, if n is sufficiently large, as is seen from (2.2).

3. The Condition of M_n for Symmetric Intervals and for Intervals on One Side of the Origin.

We shall always assume (without loss of generality) that our basic interval $[a, b]$ is centered to the right of the origin, so that $0 \leqslant |a| \leqslant b$. The Chebyshev polynomial T_m, adjusted to the interval $[a, b]$, will be denoted by $T_m[a, b]$,

$$T_m[a, b](x) = T_m\left(\frac{2x - a - b}{b - a}\right), \qquad a \leqslant x \leqslant b.$$

Relative to any such interval $[a, b]$, the norm of the map M_n is easily seen to be

$$\|M_n\|_\infty = \sum_{k=1}^{n} b^{k-1} = \begin{cases} \dfrac{b^n - 1}{b - 1}, & b \neq 1, \\ n, & b = 1. \end{cases} \tag{3.1}$$

More delicate is the determination of $\|M_n^{-1}\|_\infty$, as this amounts to finding the norms of the linear functionals λ_k: $p \longmapsto p^{(k-1)}(0)/(k-1)!$, $p \in \mathbf{P}_{n-1}[a, b]$, $k = 1, 2, \ldots, n$. Indeed,

$$\|M_n^{-1}\|_\infty = \max_{1 \leqslant k \leqslant n} \|\lambda_k\|_\infty. \tag{3.2}$$

While it is known [5, Satz 6.11] that, for $2 \leqslant k \leqslant n$, the extremal in $\mathbf{P}_{n-1}[a, b]$ for the functional λ_k is a Zolotarev polynomial of degree $n-1$, it appears difficult, in the case of a general interval $[a, b]$, to pinpoint the parameter involved in the Zolotarev polynomial, and there may correspond different Zolotarev polynomials to different values of k. For these reasons the case of an arbitrary interval will be dealt with by other (less sophisticated and cruder) methods in Section 4.

For symmetric intervals $[-\omega, \omega]$, $\omega > 0$, on the other hand, the appropriate Zolotarev polynomials are known to be the Chebyshev polynomials T_{n-1} or T_{n-2}; indeed, $\|\lambda_k\|_\infty = |T_{n-1}^{(k-1)}[-\omega, \omega](0) + T_{n-2}^{(k-1)}[-\omega, \omega](0)|/(k-1)!$, $k = 1, 2, \ldots, n$, $n \geqslant 2$ [5, p. 167], and therefore,

$$\max_{1 \leqslant k \leqslant n} \|\lambda_k\|_\infty = \|u_{T_{n-1}[-\omega,\omega] + T_{n-2}[-\omega,\omega]}\|_\infty.$$

Since $T_n[-\omega, \omega](x) = T_n(x/\omega)$, and T_m is an even or odd polynomial, depending on the parity of m, we thus have, in view of (3.1), (3.2):

THEOREM 3.1. *The condition number* (1.1) *on* $[-\omega, \omega]$ *is given by*

$$\operatorname{cond}_\infty M_n = \frac{\omega^n - 1}{\omega - 1} \max\{\|u_{T_{n-1}(x/\omega)}\|_\infty, \|u_{T_{n-2}(x/\omega)}\|_\infty\}, \tag{3.3}$$

where $(\omega^n - 1)/(\omega - 1)$ (*here and in the sequel*) *is to be interpreted as having the value* n *if* $\omega = 1$.

It follows from (2.2) that for $\omega > 1$, $\omega = 1$, $0 < \omega < 1$, the condition of M_n for large n grows, respectively, like $(1 + \sqrt{1 + \omega^2})^n$, $(1 + \sqrt{2})^n$, $[(1 + \sqrt{1 + \omega^2})/\omega]^n$ (disregarding a factor $n^{\pm 1/2}$ and constant factors), so that the growth is smallest, asymptotically, when $\omega = 1$. Selected numerical values of cond M_n are shown in Table 3.1.

ω	$n = 5$	$n = 10$	$n = 20$	$n = 40$
10	1.11(4)	1.11(9)	2.11(19)	1.10(40)
5	7.81(2)	4.39(6)	2.17(14)	7.74(29)
1	4.00(1)	5.76(3)	5.45(7)	3.51(15)
.2	6.25(3)	6.25(8)	6.25(18)	6.25(38)
.1	8.89(4)	2.84(11)	2.91(24)	3.05(50)

TABLE 3.1. The condition of M_n on $[-\omega, \omega]$

Another special case which can be disposed of similarly is the case of an interval $[a, b]$ with $0 \leqslant a < b$. Here (see, e.g., [4, p. 93]) $\|\lambda_k\|_\infty = |T_{n-1}^{(k-1)}[a, b](0)|/(k-1)!$, and we can state

THEOREM 3.2. *The condition number* (1.1) *on* $[a, b]$, *where* $0 \leqslant a < b$, *is given by*

$$\operatorname{cond}_\infty M_n = \frac{b^n - 1}{b - 1} \|u_{T_{n-1}[a,b]}\|_\infty. \tag{3.4}$$

We note that the expression on the right of (3.4), even for an arbitrary interval $[a, b]$, is always a lower bound for $\operatorname{cond}_\infty M_n$, since

$$\|M_n^{-1}\|_\infty = \sup_{p\in \mathbf{P}_{n-1}[a,b]} \frac{\|M_n^{-1}p\|_\infty}{\|p\|_\infty} \geqslant \|u_{T_{n-1}[a,b]}\|_\infty. \tag{3.5}$$

To illustrate Theorem 3.2, we consider the interval $[0, \omega]$, $\omega > 0$. Here, $T_{n-1}[0, \omega](x) = T_{n-1}^*(x/\omega)$, and depending on whether $\omega > 1$, $\omega = 1$, or $0 < \omega < 1$, Eq. (2.3) shows that the condition grows, respectively, like $(2 + \omega + 2\sqrt{1 + \omega})^n$, $(3 + 2\sqrt{2})^n$ and $[(2 + \omega + 2\sqrt{1 + \omega})/\omega]^n$, thus again slowest, asymptotically, when $\omega = 1$. Selected numerical values are shown in Table 3.2.

ω	$n = 5$	$n = 10$	$n = 20$	$n = 40$
10	3.56(4)	4.93(10)	1.80(23)	3.27(48)
5	5.00(3)	8.91(8)	3.67(19)	8.47(40)
1	1.28(3)	1.12(7)	7.34(14)	2.16(30)
.2	1.00(5)	3.20(11)	6.23(24)	3.02(51)
.1	1.42(6)	1.46(14)	1.53(30)	3.27(62)

TABLE 3.2. The condition of M_n on $[0, \omega]$

4. The Condition of M_n on an Arbitrary Interval. We now wish to make some progress towards the more difficult problem of estimating $\operatorname{cond}_\infty M_n$ for an arbitrary right-centered interval $[a, b]$, $0 \leqslant |a| \leqslant b$. We content ourselves with establishing upper bounds for $\operatorname{cond}_\infty M_n$. (A trivial, but not very useful, lower bound can be had from (3.1) and (3.5).)

Our main tool is the following simple observation.

LEMMA 4.1. *Let $s^T = [s_1, s_2, \ldots, s_n]$ be any vector of n distinct nodes in $[a, b]$ and $V_n(s)$ the corresponding Vandermonde matrix*

$$V_n(s) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_n \\ \cdot & \cdot & \cdots & \cdot \\ s_1^{n-1} & s_2^{n-1} & \cdots & s_n^{n-1} \end{bmatrix} \quad (a \leqslant s_\nu \leqslant b,\ \nu = 1, 2, \ldots, n). \tag{4.1}$$

Then

$$\|M_n^{-1}\|_\infty \leqslant n \|V_n^{-1}(s)\|_\infty. \tag{4.2}$$

Proof. Let

$$p(x) = \sum_{k=1}^{n} u_k x^{k-1}, \qquad a \leqslant x \leqslant b,$$

be an arbitrary polynomial of degree $\leqslant n - 1$. From

$$\sum_{k=1}^{n} s_\nu^{k-1} u_k = p(s_\nu), \qquad \nu = 1, 2, \dots, n,$$

or, equivalently,

$$V_n^T(s)u = \pi, \quad u^T = [u_1, u_2, \dots, u_n], \quad \pi^T = [p(s_1), p(s_2), \dots, p(s_n)],$$

one gets immediately

$$\|u\|_\infty \leqslant \|u\|_1 \leqslant \|[V_n^{-1}(s)]^T\|_1 \|\pi\|_1 \leqslant n\|V_n^{-1}(s)\|_\infty \|\pi\|_\infty \leqslant n\|V_n^{-1}(s)\|_\infty \|p\|_\infty,$$

hence (4.2). □

It is tempting to optimize the bound in (4.2) by minimizing $\|V_n^{-1}(s)\|_\infty$ over all admissible node vectors s. Unfortunately, the corresponding optimal nodes are not known explicitly. We expect, however, the Chebyshev points on $[a, b]$ to provide a reasonably good alternative. In order to carry out the necessary computations, we need the following properties of Vandermonde matrices.

LEMMA 4.2 (SHIFT PROPERTY). *Let* $t = [t_1, t_2, \dots, t_n]^T$ *and* $t - \mu = [t_1 - \mu, t_2 - \mu, \dots, t_n - \mu]^T$. *Then*

$$V_n^{-1}(t - \mu) = V_n^{-1}(t)(D_n^{-1}P_nD_n)^T, \tag{4.3}$$

where $D_n = \mathrm{diag}(1, \mu, \mu^2, \dots, \mu^{n-1})$ *and* P_n *is the initial* $(n \times n)$*-segment of the Pascal triangle, that is*

$$D_n^{-1}P_nD_n = \begin{bmatrix} 1 & \mu & \mu^2 & \mu^3 & \cdots \\ 0 & 1 & \binom{2}{1}\mu & \binom{3}{2}\mu^2 & \cdots \\ 0 & 0 & 1 & \binom{3}{1}\mu & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}_{(n\times n)}. \tag{4.4}$$

Proof. It is well known (see, e.g., [2]) that $V_n^{-1}(t) = [u_{\kappa\lambda}]$, where

$$\prod_{\substack{\nu=1\\ \nu\neq\kappa}}^{n} \frac{x - t_\nu}{t_\kappa - t_\nu} \equiv \sum_{\lambda=1}^{n} u_{\kappa\lambda}x^{\lambda-1}.$$

The elements $u'_{\kappa\lambda}$ of $V_n^{-1}(t - \mu)$, therefore, are the coefficients of the polynomial

$$\prod_{\nu\neq\kappa} \frac{x + \mu - t_\nu}{t_\kappa - t_\nu} = \sum_{\rho=1}^{n} u_{\kappa\rho}(x + \mu)^{\rho-1} = \sum_{\rho=1}^{n} u_{\kappa\rho} \sum_{\lambda=1}^{\rho} \binom{\rho - 1}{\lambda - 1} x^{\lambda-1}\mu^{\rho-\lambda}$$

$$= \sum_{\lambda=1}^{n} x^{\lambda-1} \sum_{\rho=\lambda}^{n} u_{\kappa\rho}\binom{\rho - 1}{\lambda - 1}\mu^{\rho-\lambda},$$

that is,

$$u'_{\kappa\lambda} = \sum_{\rho=\lambda}^{n} u_{\kappa\rho}\binom{\rho - 1}{\lambda - 1}\mu^{\rho-\lambda}.$$

This, written in matrix form, is precisely (4.3). □

In the following two lemmas,

$$\cos\theta_\nu,\quad \theta_\nu = \frac{2\nu-1}{2n}\pi, \qquad \nu = 1, 2, \ldots, n,$$

denote the Chebyshev points on $[-1, 1]$.

LEMMA 4.3. *If* $t_\nu = \tau\cos\theta_\nu$, $\nu = 1, 2, \ldots, n$, $\tau > 0$, *then*

$$n\|V_n^{-1}(t)\|_\infty \leqslant \frac{3^{3/4}}{4(\sqrt{2}-1)}(\tau+1)\left|T_n\left(\frac{i}{\tau}\right)\right| \qquad (i = \sqrt{-1}). \tag{4.5}$$

Proof. From [2, Theorem 5.2]** one has

$$n\|V_n^{-1}(t)\|_\infty \leqslant \frac{(\tau+1)n}{2(\sqrt{2}-1)}\left|\frac{T_n(i/\tau)}{T_n(i)}\right|\left\|V_n^{-1}\left(\frac{1}{\tau}t\right)\right\|_\infty,$$

and from [2, Example 6.2]

$$n\left\|V_n^{-1}\left(\frac{1}{\tau}t\right)\right\|_\infty \leqslant \frac{3^{3/4}}{2}\,|T_n(i)|.$$

LEMMA 4.4. *If* $t_\nu = \tau(1 + \cos\theta_\nu)$, $\nu = 1, 2, \ldots, n$, $\tau > 0$, *then*

$$n\|V_n^{-1}(t)\|_\infty \leqslant \frac{\tau}{\sqrt{1+2\tau}}\,T_n\left(\frac{1}{\tau}+1\right). \tag{4.6}$$

Proof. From [2, Eq. (4.1′)] one obtains

$$n\|V_n^{-1}(t)\|_\infty \leqslant \frac{T_n(1/\tau+1)}{\min\limits_{1\leqslant\nu\leqslant n}\left\{\dfrac{1/\tau+1+\cos\theta_\nu}{\sin\theta_\nu}\right\}}, \tag{4.7}$$

having used $|T_n'(\cos\theta_\nu)| = n/\sin\theta_\nu$. An elementary calculation will show that

$$f(\theta) = \frac{1/\tau+1+\cos\theta}{\sin\theta}$$

has a unique minimum on $0 < \theta < \pi$ at $\theta = \theta_0$, where $\cos\theta_0 = -\tau/(\tau+1)$. Thus

$$\min_{0<\theta<\pi} f(\theta) = \frac{1/\tau+1-\tau/(\tau+1)}{\sqrt{1-\tau^2/(\tau+1)^2}} = \frac{1}{\tau}\sqrt{1+2\tau},$$

from which (4.6) follows by virtue of (4.7). □

Now the Chebyshev points on $[a, b]$ are given by

$$s_\nu = \frac{a+b}{2} + \frac{b-a}{2}\cos\theta_\nu = a + \frac{b-a}{2}(1+\cos\theta_\nu), \qquad \nu = 1, 2, \ldots, n. \tag{4.8}$$

Each of these two representations suggests an application of the shift property in Lemma 4.2, the first with $t_\nu = \tau\cos\theta_\nu$, $\mu = -(a+b)/2$, the second with $t_\nu = \tau(1+\cos\theta_\nu)$, $\mu = -a$, where $\tau = (b-a)/2$ in both. Observing also that

$$\|V_n^{-1}(t-\mu)\|_\infty \leqslant \|V_n^{-1}(t)\|_\infty\|D_n^{-1}P_nD_n\|_1 = (1+|\mu|)^{n-1}\|V_n^{-1}(t)\|_\infty,$$

**Theorem 5.2 in [2] is stated for n even; the same theorem, however, also holds if n is odd.

and using Lemmas 4.3 and 4.4 to estimate $\|V_n^{-1}(t)\|_\infty$, we can easily estimate $\|V_n^{-1}(s)\|_\infty$ for the nodes in (4.8), hence $\|M_n^{-1}\|_\infty$ by Lemma 4.1, and finally $\operatorname{cond}_\infty M_n$, using (3.1). The result is stated as

THEOREM 4.1. *The condition number* (1.1) *on* $[a, b]$, *where* $0 \leqslant |a| \leqslant b$, *satisfies the inequality*

$$(4.9)\quad \operatorname{cond}_\infty M_n \leqslant \frac{3^{3/4}}{4(\sqrt{2}-1)}\, \frac{2+b-a}{2+b+a}\, \frac{b^n-1}{b-1} \left(1+\frac{b+a}{2}\right)^n \left|T_n\left(\frac{2i}{b-a}\right)\right|,$$

as well as the inequality

$$(4.10)\quad \operatorname{cond}_\infty M_n \leqslant \frac{b-a}{2(1+|a|)\sqrt{1+b-a}}\, \frac{b^n-1}{b-1}\, (1+|a|)^n T_n\left(\frac{2}{b-a}+1\right).$$

Theorem 4.1 holds for arbitrary intervals $[a, b]$, subject to $|a| \leqslant b$, but is of interest only in the case $a \leqslant 0 < b$ of an interval containing the origin. It will be useful to characterize such an interval by its "degree of asymmetry"

$$\alpha = (b+a)/(b-a), \qquad 0 \leqslant \alpha \leqslant 1,$$

and its half-width

$$\tau = (b-a)/2,$$

in terms of which $b = (1+\alpha)\tau$, $a = -(1-\alpha)\tau$.

We first examine the extreme cases $\alpha = 0$ (perfect symmetry) and $\alpha = 1$ (perfect asymmetry), typified by the intervals $[-\omega, \omega]$ and $[0, \omega]$, $\omega > 0$. In the first case, by virtue of

$$2\left|T_n\left(\frac{i}{\omega}\right)\right| = \left(\frac{1+\sqrt{1+\omega^2}}{\omega}\right)^n + \left(\frac{1-\sqrt{1+\omega^2}}{\omega}\right)^n \sim \left(\frac{1+\sqrt{1+\omega^2}}{\omega}\right)^n, \quad n \longrightarrow \infty,$$

we find that the bound in (4.9) has the correct exponential growth rate as $n \longrightarrow \infty$, which can be obtained from (3.3) and (2.2), while the bound in (4.10) grows at a larger exponential rate. (We say here that a sequence $\{c_n\}$ has exponential growth rate γ if $|c_{n+1}/c_n| \sim \gamma$ as $n \longrightarrow \infty$.) The reverse is true in the second case, as can be seen from

$$\begin{aligned} 2T_n\left(\frac{2}{\omega}+1\right) &= \left(\frac{2+\omega+2\sqrt{1+\omega}}{\omega}\right)^n + \left(\frac{2+\omega-2\sqrt{1+\omega}}{\omega}\right)^n \\ &\sim \left(\frac{2+\omega+2\sqrt{1+\omega}}{\omega}\right)^n, \quad n \longrightarrow \infty, \end{aligned}$$

and comparison with (3.4), (2.3). We, therefore, expect (4.9) to be sharper than (4.10) if the interval $[a, b]$ is more nearly symmetric (i.e., α small), and (4.10) better than (4.9) for more asymmetric intervals (α close to 1). That this is indeed the case can be seen by forming the ratio ρ of the exponential growth rates in (4.9) and (4.10), and expressing the result in terms of α and τ,

$$\rho = \frac{1+\alpha\tau}{1+(1-\alpha)\tau}\, \lambda(\tau), \qquad \lambda(\tau) = \frac{1+\sqrt{1+\tau^2}}{1+\tau+\sqrt{1+2\tau}}.$$

One verifies that $\lambda(\tau) < 1$ for all τ, with $\lambda(0) = \lambda(\infty) = 1$, so that $\rho < 1$ certainly if $1 + \alpha\tau < 1 + (1 - \alpha)\tau$, i.e., $\alpha < \frac{1}{2}$. Thus, (4.9) *is asymptotically sharper than* (4.10) *whenever* $\alpha < \frac{1}{2}$. The condition on α is best possible for $\tau \longrightarrow \infty$, but too stringent for specific finite values of τ. If $\tau = 1$, e.g., one finds (4.9) better than (4.10) whenever $\alpha <$ .8216 ... , and as $\tau \longrightarrow 0$, (4.9) is always better.

We illustrate Theorem 4.1 in Figure 4.1, where we plot the exponential growth rates of the bounds in (4.9) and (4.10) for intervals of fixed half-width $\tau = 1$, and asymmetries α varying from 0 to 1. (The growth rates are $(1 + \alpha)^2(1 + \sqrt{2})$ and $(1 + \alpha)(2 - \alpha)(2 + \sqrt{3})$, respectively.) The true asymptotic growth rate presumably interpolates somehow between the boundary values $1 + \sqrt{2}$ and $2(2 + \sqrt{3})$ (cf. the dashed line in Figure 4.1).

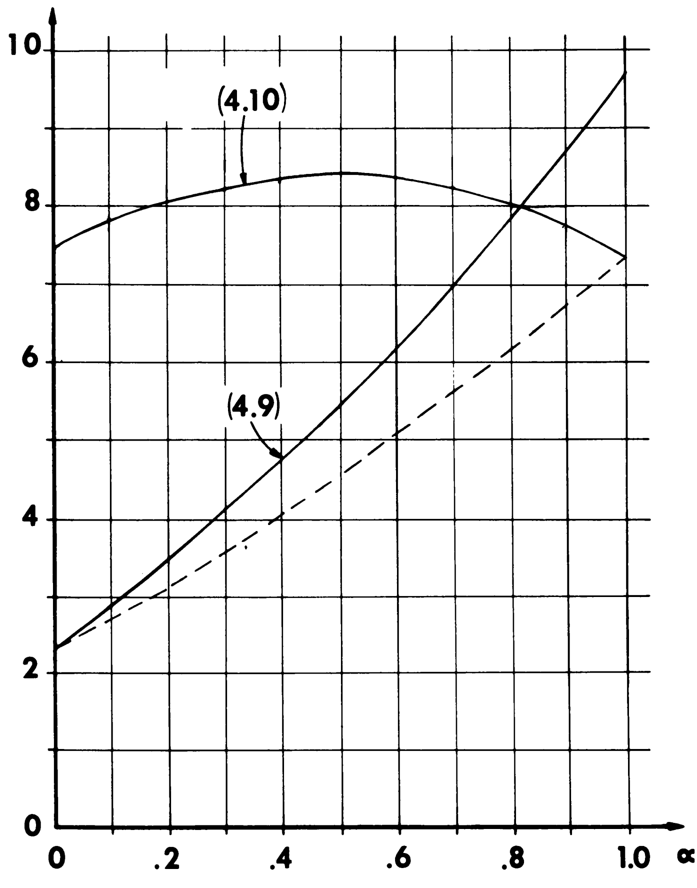

FIGURE 4.1. The asymptotic growth rates of the bounds in (4.9) and (4.10) for $a = -1 + \alpha$, $b = 1 + \alpha$, $0 \leqslant \alpha \leqslant 1$.

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