Planar Maximal Covering with Ellipses

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  - Develop a $\mathcal{O}(n^2 \log n)$ algorithm for the one disk case
  - Adapt it for the m ellipses case

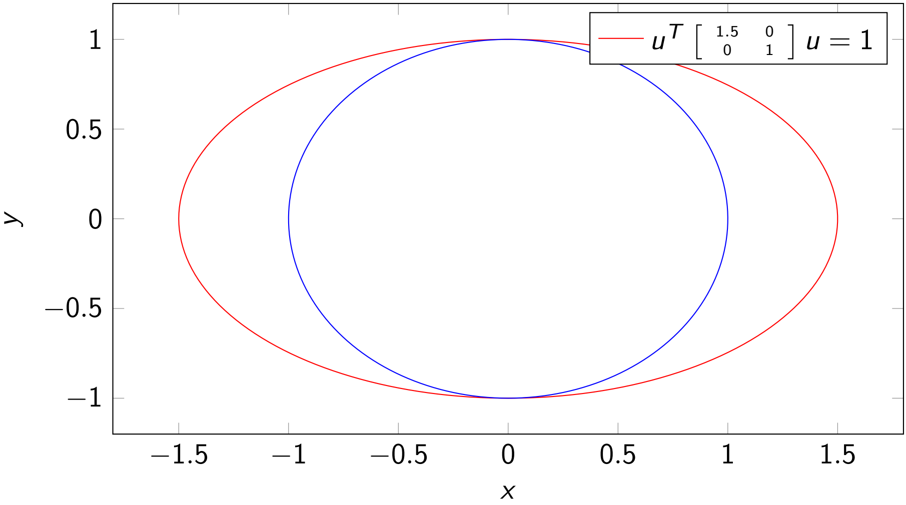
Preliminaries

Norms

Let $u \in \mathbb{R}^2$ and Q a $2x2$ positive definite matrix

- Euclidean
$$||u||_2 = \sqrt{u^T u}$$
- Elliptical
$$||u||_Q = \sqrt{u^T Q u}$$

Figure: The elliptical and euclidean norms.



Preliminaries

Ellipse

Given a center $c \in \mathbb{R}^2$ and a 2x2 p.d. matrix Q, an ellipse is the set of points that satisfy

$$||u - c||_Q = 1,$$

with $\leq$ representing the set of covered points

Axis-parallel ellipse

Any 2 by 2 diagonal d.p. matrix determines an axis-parallel ellipse, which can also be described by

$$\frac{(x - c_x)^2}{a^2} + \frac{(y - c_y)^2}{b^2} = 1,$$

where (a, b) are the shape parameters and $c = (c_x, c_y)$ is the center.

$MCD(\mathscr{P}, 1)$ is the problem of placing one disk on the plane to cover a subset of a demand set $\mathscr{P}$ maximizing the weights of the covered points.

$$\max_q w(\mathscr{P} \cap D(q)),$$

- $\mathscr{P} = \{p_1, \ldots, p_n\}$ is the demand set with weights $w(p_i) > 0$
- $w(A)$, $A \subset \mathscr{P}$, is the sum of weights of the points in A
- $D(q)$ is a unit disk with center at point q
- [MCL86] proposed a $\mathscr{O}(n^2)$ algorithm
- [Dre81] proposed a $\mathscr{O}(n^2 \log n)$ which our work is based on
- We will introduce an equivalent problem...

Maximum Weight Clique Problem

Let $\mathscr{D} = \{D_1, \ldots, D_n\}$ be a set of n unit disks with weights $w_i > 0$. The maximum weight clique is defined as

$$\max_q \sum_{D_k \cap q \neq \emptyset} w_k,$$

- The disks are fixed with centers at $\mathscr{P} = \{p_1, \ldots, p_n\}$ with $w_k = w(p_k)$
- A clique is a non-empty intersection area of a subset of disks. We search for only a point in an optimal clique.
- An optimal solution for the maximum weight clique is an optimal solution for $MCD(\mathscr{P}, 1)$.

Figure: An instance of $MCD(\mathscr{P}, 1)$.

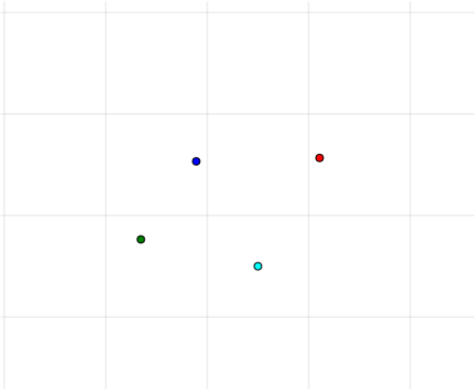
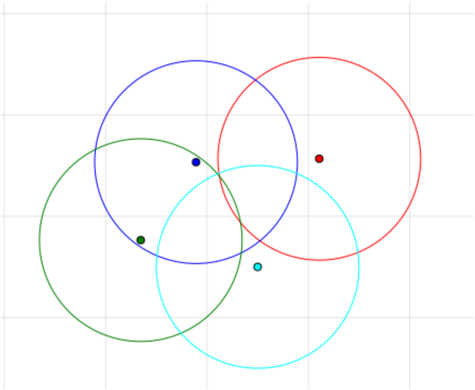
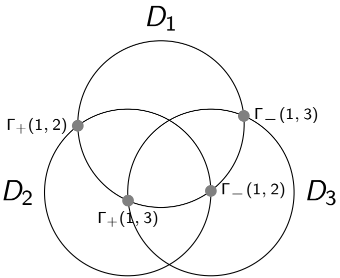


Figure: An instance of Maximum Weight Clique Problem.



Let $\Gamma_+(i,j)$ and $\Gamma_-(i,j)$ be the opening and closing angles (with respect to D_i) of intersections of disks i and j. Also, $\Gamma_+(i,j), \Gamma_-(i,j) \in [0, 2\pi]$.

Figure: Three disks and their intersection points.



For a disk D_i, a counter-clockwise traversal visits every $\Gamma_+(i,j)$ and $\Gamma_-(i,j)$ in counter-clockwise order.

- An intersection region of disks is bounded by arcs.
- The arc $\Gamma_+(i,j), \Gamma_-(i,j)$ (counter-clockwise) determines a region where i and j intersect.
- In a counter-clockwise traversal, the arcs where $\Gamma_+(i,j) > \Gamma_-(i,j)$ can be a problem for the implementation. Work-around: repeat it.

The algorithm is described simply as:
For every disk, traverse the sorted list of intersection angles twice, keeping a set of active disks, and the current best solution.

Figure: The intersections list of a disk with three other disks.

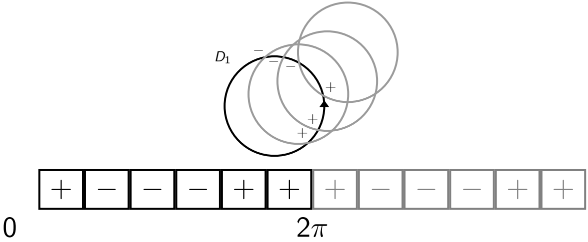
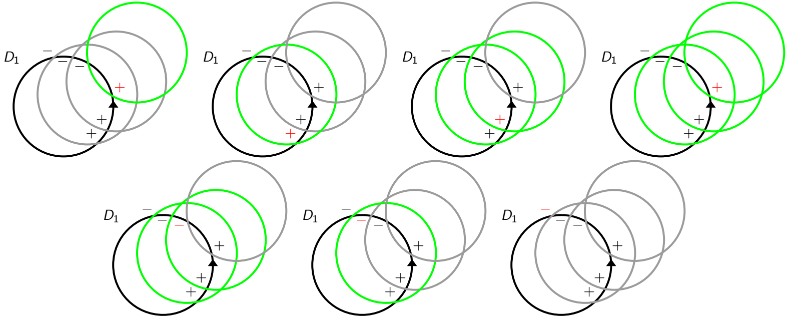


Figure: A traversal for D_1 with green disks representing the active set and red signs representing the current angle being visited (some are omitted).



The run-time complexity of the algorithm is $\mathcal{O}(n^2 \log n)$.

- There are $\mathcal{O}(n^2)$ intersections among n disks
- Sorting takes $\mathcal{O}(n^2 \log n)$
- The traversal takes $\mathcal{O}(n)$ for every disk.
- It can be implemented in $K \log n$ where K is the number of intersections.

Multiple disks

For every disk, try every possible assignment that the traversal goes through. [dBCHP06] proposed a $\mathcal{O}(n^{2m-1})$ algorithm.

Maximal Covering by Ellipses

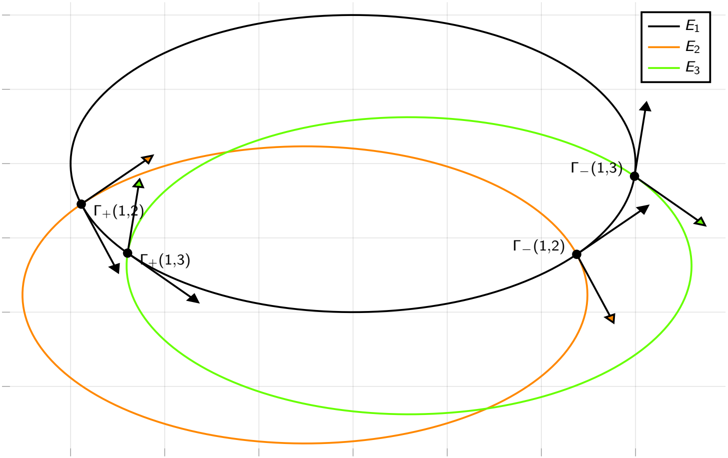
Let $MCE(\mathscr{P}, a, b)$ be an instance of the maximal covering by one ellipse, with E being an ellipse with shape parameters $(a, b) \in \mathbb{R}^2_{>0}$, an optimal solution of $MCE(\mathscr{P}, a, b)$ is given by

$$\max_{q} |\mathscr{P} \cap (q)|,$$

- $E(q)$ is an axis-parallel ellipse with center point q
- Assuming unit weights for now
- Same algorithm for one disk

Maximal Covering by Ellipses

Figure: Intersection points of E_1 with E_2 and E_3 along with opening and closing angles indicators.



Let $MCE(\mathscr{P}, \mathscr{E})$ be an instance of the maximal covering by ellipses, an optimal solution is given by

$$\max_{q_1,\ldots,q_m} \left| \bigcup_{i=1}^{m} \mathscr{P} \cap E_i(q_i) \right|,$$

- $\mathscr{E}$ is a set of m ellipses
- [ABR13, CvM09] are the works found on the problem which the ellipses have costs
- [ABR13] also developed a method for the non-axis-parallel case

A set of points $\mathscr{P} = \{p_1, \ldots, p_n\}$, and the shape parameters (a, b) of an ellipse.

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References I

M. Andretta, E. G. Birgin, and M. Raydan, *Deterministic and stochastic global optimization techniques for planar covering with elipses problems*, European Journal of Operational Research **22** (2013), 23–40.

M. S. Canbolat and M. von Massow, *Planar maximal covering with ellipses*, Computers and Industrial Engineering **57** (2009), 201–208.

Mark de Berg, Sergio Cabello, and Sariel Har-Peled, *Covering many or few points with unit disks*, vol. 45, 09 2006, pp. 55–68.

Zvi Drezner, *Note—on a modified one-center model*, Management Science **27** (1981), 848–851.

B M. Chazelle and D Lee, *On a circle placement problem*, Computing **36** (1986), 1–16.