Fixed-Shape Ellipse by Three Points

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Introduction

The shape of an ellipse is given by its major-axis and minor-axis, $(a, b) \in \mathbb{R}^2$, with $a > b > 0$.

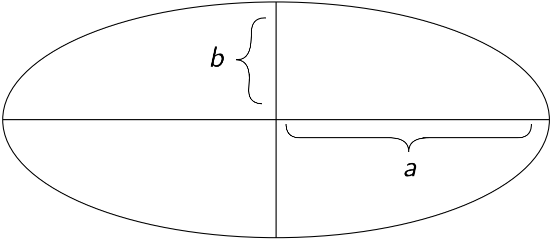


Figura: An ellipse with shape parameters a and b.

Introduction

Here, the shape will be fixed and the center and angle of rotation are free.

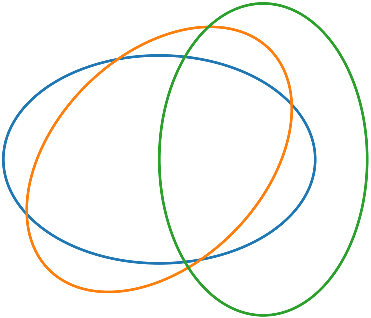


Figura: A fix-shape ellipse at different centers and with different angles of rotation.

Given three points $u, v, w \in \mathbb{R}^2$, and the shape $(a, b) \in \mathbb{R}^2$ of an ellipse:

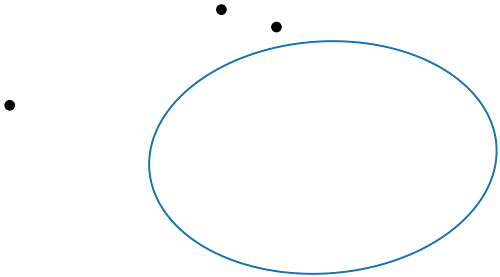


Figura: An instance of the problem.

Introduction

Problem definition

A solution is given by the ellipse's center $q \in \mathbb{R}^2$ and the angle of rotation $\theta \in [0, \pi)$, such that u, v, w lie on its border. **We want to find every solution!**

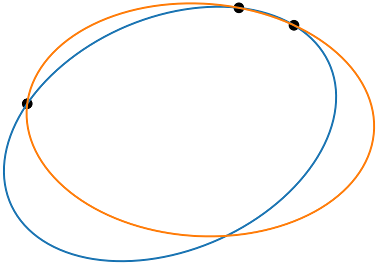


Figura: Every solution for that instance.

Introduction

The equation of an ellipse is given by:

$$\frac{\left(\begin{bmatrix} x - q_x \\ y - q_y \end{bmatrix}^T \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right)^2}{a^2} + \frac{\left(\begin{bmatrix} x - q_x \\ q_y - y \end{bmatrix}^T \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}\right)^2}{b^2} = 1.$$

- Fixing the points u, v, w, we get 3 equations and 3 unknowns (q_x, q_y, θ).
- Finding every solution is difficult.

Transforming the problem

Let's make the problem simpler by transforming it into a circle problem.
Given any non-colinear points, there is an unique circumscribed circle. Also, we can turn an axis-parallel ellipse into a circle:

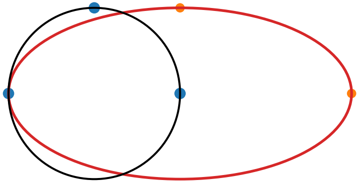
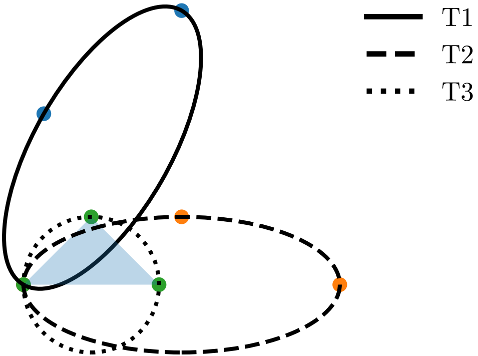


Figura: Turning an ellipse with shape (a, b) into a circle of radius b.

Transforming the problem

Let's rotate the points instead of rotating the ellipse. It is equivalent:



Transforming the problem

Formally, we can transform the problem by:

- Translate the points so $u = (0, 0)$.
- Scale the x-axis by $\frac{b}{a}$.

Then, we rotate the points. The rotated points are $\varphi(u, \theta), \varphi(v, \theta), \varphi(w, \theta)$:

$$\varphi(p, \theta) = \begin{bmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}.$$

After that, we need to look at the circumscribed circle formed by $\varphi(u, \theta), \varphi(v, \theta), \varphi(w, \theta)$.

Transforming the problem

An angle is a solution if, and only if the circumscribed circle of the points $\varphi(u,\theta), \varphi(v,\theta), \varphi(w,\theta)$ has radius b.

- This is a problem of one variable!

Fixed-shape ellipse by three points

There is a known formula for the radius of a circumscribed circle:

$$R = \frac{\|\varphi(v,\theta)\|_2 \, \|\varphi(w,\theta)\|_2 \, \|\varphi(v,\theta) - \varphi(w,\theta)\|_2}{4A(\theta)}$$

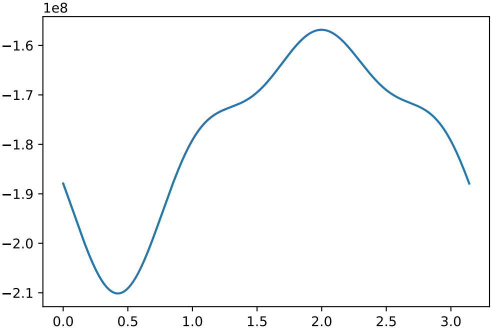
- R is the radius.
- $A(\theta)$ is the area of the triangle defined by the points $\varphi(u,\theta), \varphi(v,\theta), \varphi(w,\theta)$.

Fixed-shape ellipse by three points

We define the function $\xi : [0, \pi) \mapsto \mathbb{R}$:

$$\xi(\theta) = 16b^2 A(\theta)^2 - \|\varphi(v,\theta)\|_2^2 \, \|\varphi(w,\theta)\|_2^2 \, \|\varphi(v,\theta) - \varphi(w,\theta)\|_2^2$$

The roots of ξ are solutions of our problem.



Fixed-shape ellipse by three points

There is no clear pattern in ξ.

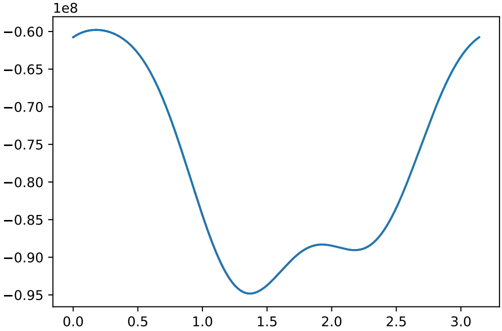


Figura: Another example of ξ.

Fixed-shape ellipse by three points

An example with two roots.

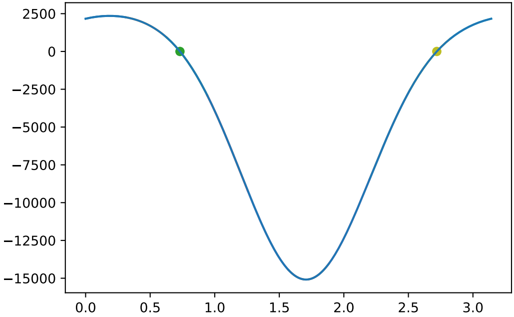


Figura: Another example of ξ.