Fixed-Shape Ellipse by Three Points

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Introduction

The shape of an ellipse is given by its major-axis and minor-axis, $(a, b) \in \mathbb{R}^2$, with $a > b > 0$.

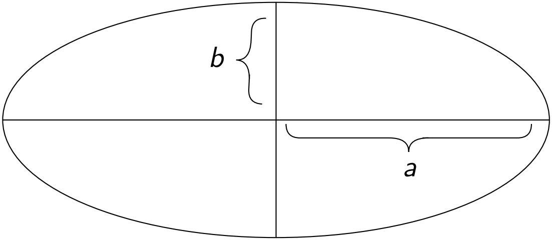

Figura: An ellipse with shape parameters a and b.

Introduction

Here, the shape will be fixed and the center and angle of rotation are free.

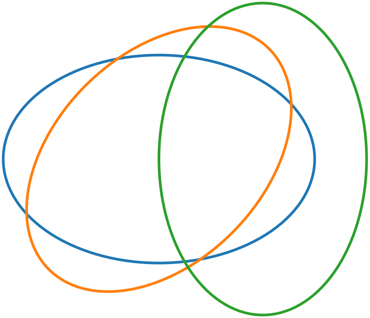

Figura: A fix-shape ellipse at different centers and with different angles of rotation.

Introduction

Problem definition

Given three points $u, v, w \in \mathbb{R}^2$, and the shape $(a, b) \in \mathbb{R}^2$ of an ellipse:

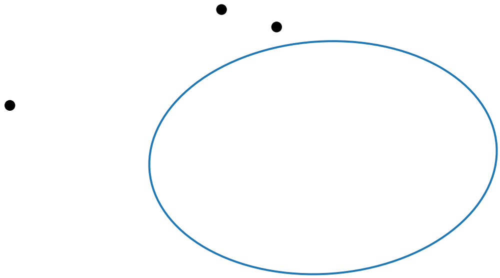

Figura: An instance of the problem.

Introduction

Problem definition

A solution is given by the ellipse's center $q \in \mathbb{R}^2$ and the angle of rotation $\theta \in [0, \pi)$, such that u, v, w lie on its border. **We want to find every solution!**

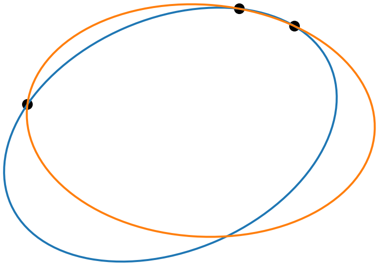

Figura: Every solution for the instance shown previously.

Introduction

The equation of an ellipse is given by:

$$\frac{\left(\begin{bmatrix} x - q_x \\ y - q_y \end{bmatrix}^T \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right)^2}{a^2} + \frac{\left(\begin{bmatrix} x - q_x \\ q_y - y \end{bmatrix}^T \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}\right)^2}{b^2} = 1.$$

- Fixing the points u, v, w, we get 3 equations and 3 unknowns (q_x, q_y, θ).
- Finding every solution is difficult.

Transforming the problem

Let's make the problem simpler by transforming it into a circle problem.
An ellipse with shape (a, b) can be transformed into a circle of radius b through scaling the x-axis by $\frac{b}{a}$:

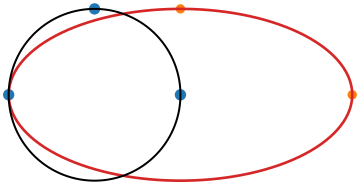

Figura: Turning an ellipse with shape (a, b) into a circle of radius b.

Transforming the problem

Let's rotate the points instead of rotating the ellipse:

Figura: Three points at their initial location.

Transforming the problem

Firstly, we rotate leaving one point fixed at $(0, 0)$:

Figura: After rotation.

Transforming the problem

Then, we scale by $\frac{b}{a}$ and check the radius of the circle:

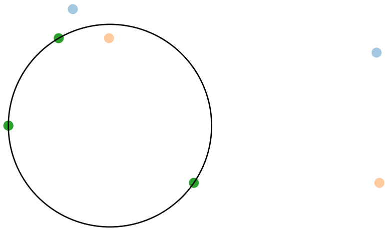

Figura: After scaling.

Transforming the problem

If the radius is b, the angle of rotation is a solution:

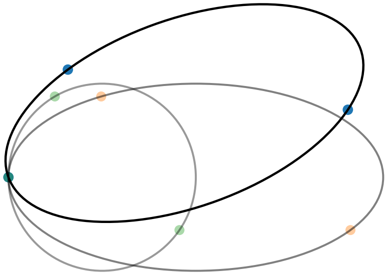

Figura: One solution for this instance.

Transforming the problem

Formally, we can transform the problem by:

- Translate the points so $u = (0, 0)$.
- Rotate by θ and scale the x-axis by $\frac{b}{a}$.
- Find the θ's which produce a circle with radius b.

This transformation is expressed by:

$$\varphi(p, \theta) = \begin{bmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix},$$

for $p = u, v, w$.

This is an one variable problem on a closed interval!

Fixed-shape ellipse by three points

There is a known formula for the radius of a circumscribed circle [JY60, p. 189]:

$$R = \frac{\|\varphi(v,\theta)\|_2 \, \|\varphi(w,\theta)\|_2 \, \|\varphi(v,\theta) - \varphi(w,\theta)\|_2}{4A(\theta)}$$

- R is the radius.
- $A(\theta)$ is the area of the triangle defined by the points $\varphi(u,\theta), \varphi(v,\theta), \varphi(w,\theta)$.

Fixed-shape ellipse by three points

We define the function $\xi : [0, \pi) \mapsto \mathbb{R}$:

$$\xi(\theta) = 16b^2 A(\theta)^2 - \|\varphi(v, \theta)\|_2^2 \, \|\varphi(w, \theta)\|_2^2 \, \|\varphi(v, \theta) - \varphi(w, \theta)\|_2^2$$

The roots of ξ are solutions of our problem.

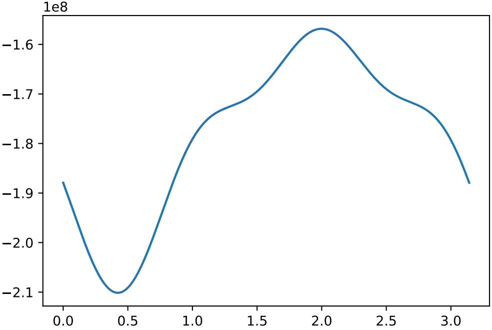

Figura: An example of ξ.

Fixed-shape ellipse by three points

There is no clear pattern in ξ.

- It can be written as $\{\cos^j x \sin^k x : j, k \in \mathbb{N}\}$.
- Degree 6, at most 12 roots in $[0, 2\pi)$ [Pow81, p. 150]

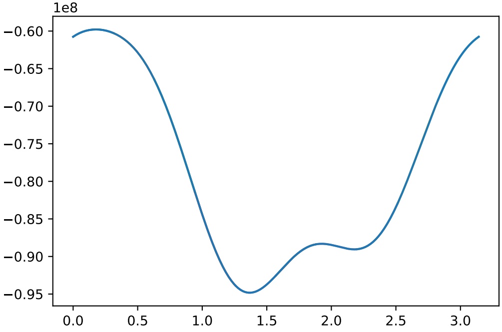

Figura: Another example of ξ.

Fixed-shape ellipse by three points

An example with two roots.

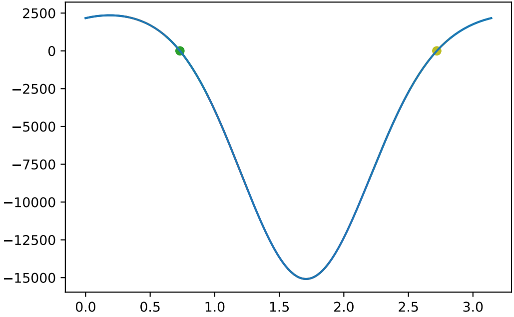

Figura: Yet another example of ξ.

Polynomial Interpolation

It is a way to approximate a function by a simpler one (a polynomial).

- A degree n of the interpolation is determined.
- $n + 1$ points are chosen, such that the polynomial has to pass through.
- Can be calculated using Lagrange's formula.
- We can find every root of a polynomial by determining the eigenvalues of a matrix called The Companion Matrix [HJ86, p. 195].
- Depending on the points, the interpolation can be a bad approximation. It can get worse even if n is increased (Runge's Phenomenon) [Pow81, p. 37].

Chebyshev Interpolation

Chebyshev Polynomial

$T_n : [-1, 1] \mapsto [-1, 1]$ is the n-degree Chebyshev polynomial [MH03]:

$$T_n(\cos t) = \cos(nt)$$

Also, it can be defined recursively:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x) \end{aligned}$$

Chebyshev Interpolation

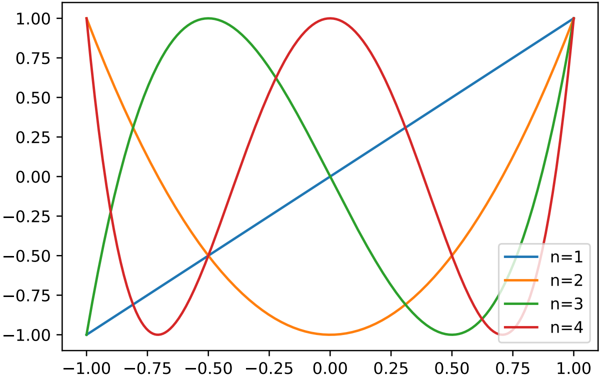

Figura: Chebyshev Polynomials of degree $1, 2, 3, 4$.

Chebyshev Interpolation

Interpolation on the roots of T_n, also known as Chebyshev Nodes:

$$x_k = \cos\left(\pi \frac{2k-1}{2n}\right)$$

The interpolation of a function $f : [-1, 1] \mapsto \mathbb{R}$ can be written directly using Chebyshev polynomial as basis:

$$f(x) \approx \sum_{k=0}^{n} a_k T_k(x)$$

- A simple change of coordinates lets the interpolation to be done on any closed interval!
- This can be done in $\mathcal{O}(n^3)$ [Boy13].

Chebyshev Interpolation

Why is it good?

- Numerically stable! Way better then polynomials in the power format [Gau79].
- No Runge's Phenomenon, the interpolation converges to f.
  - $\mathcal{O}(n^{-m})$ if f is m times differentiable [GO77, p. 28].
  - $\mathcal{O}(C^n)$, for $C < 1$, if f is analytical in a neighborhood of $[-1, 1]$ [BT04].
- Very used in practice: present in external libraries like NumPy for Python.

Chebyshev Interpolation

Choosing the degree of the interpolation:

- There is no guaranteed way to choose it.
- A good rule is to examine the last coefficient (the last coefficient rule-of-thumb [Boy01, p .50]).
- For a predefined ϵ, choose n, such that:

$$|a_n| \leq \epsilon$$

- There are other ways like checking the error on a Lobatto grid [BG07].

Chebyshev Interpolation

For $n = 32$, a precision of 10^{-10} is expected.

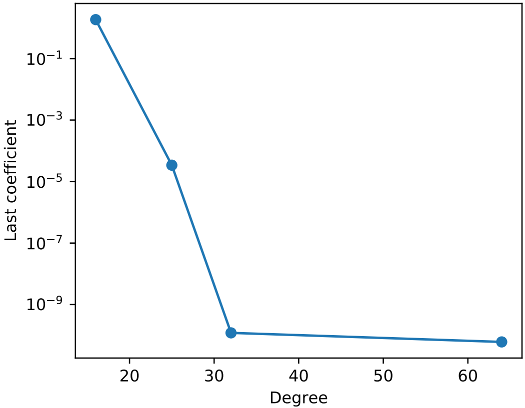

Figura: $|a_n|$ for the interpolation of ξ for an instance.

Chebyshev Interpolation

Roots

The roots of a Chebyshev polynomial can be found though determining the eigenvalues of a Chebyshev companion matrix [Boy13]. This is it for $n = 5$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{a_0}{2a_5} & -\frac{a_1}{2a_5} & -\frac{a_2}{2a_5} & -\frac{a_3}{2a_5} & -\frac{a_4}{2a_5} \end{bmatrix}$$

- This matrix is a Hessenberg matrix.
- Its eigenvalues can be found by a QR decomposition in $\mathcal{O}(n^3)$.

Chebyshev Interpolation

Roots

The largest error on roots that were found for $n = 32$ is around 10^{-14} for an instance:

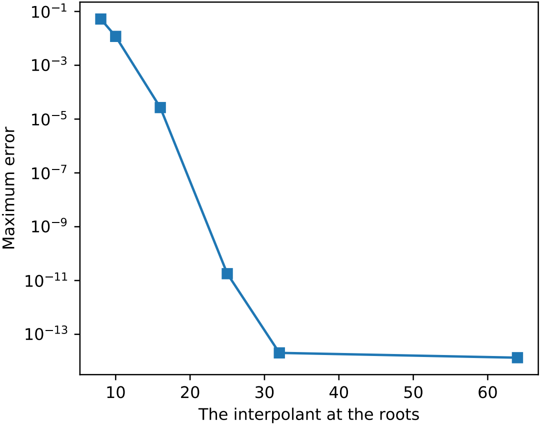

Figura: $|\xi(\hat{\theta})|$, where $\hat{\theta}$ is a root of f_n.

Chebyshev Interpolation

Roots

The experiments were made using Python with the NumPy library.
The running time is really low, even for $n = 64$.

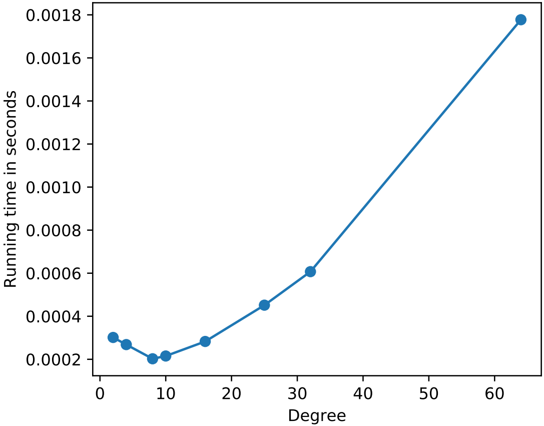

Figura: The running time to find the roots of f_n.

Example

An example with 4 solutions.

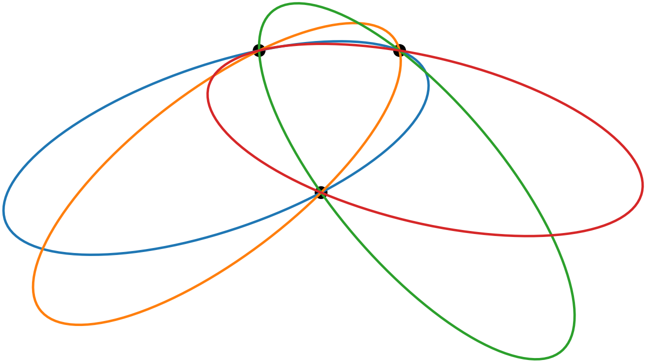

Figura: An example with 4 solutions.

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