New Exact Algorithms for Planar Maximum Covering Location by Ellipses Problems

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Introduction
Related problems

The Maximum Covering Location Problem (MCLP)

- Introduced in [3],
- Maximize the coverage demand vertices on a graph,
- Choose the location (vertex) of a fixed number of facilities,
- A demand vertex is considered covered if a facility is located within its coverage radius.

Introduction

Related problems

The Planar Maximum Covering Location Problem (PMCLP)

- Introduced in [4],
- Maximize the coverage demand vertices in $\mathbb{R}^2$,
- Choose the location (could be anywhere in $\mathbb{R}^2$) of a fixed number of facilities,
- A demand vertex is considered covered if a facility is located within its coverage radius,
- Several distance functions were studied. We are particularly interested in the Euclidean PMCLP.

Introduction

We propose algorithms for two versions of PMCLP.

Introduction

MCE

Planar Maximum Covering Location by Ellipses Problem (MCE):

- Introduced in [2],
- Mixed Non-linear optimization and a heuristic method in [2],
- Exact method, solving convex sub-problems in [1].

Our algorithm

Based on the approach used for the Euclidean PMCLP in [4].
Transform MCE into a combinatorial optimization problem.

Planar Maximum Covering Location by Ellipses with Rotation Problem (MCER):

- Introduced in [1],
- Exact method, solving many optimization sub-problems in [2],
- Heuristic method in [1].
- Much more challenging than MCE.

Our algorithm

Transforms MCER into a combinatorial optimization problem.

Introduction

Ellipse

The shape of an ellipse is given by its major-axis and minor-axis, $(a, b) \in \mathbb{R}^2_{>0}$, $a > b$.

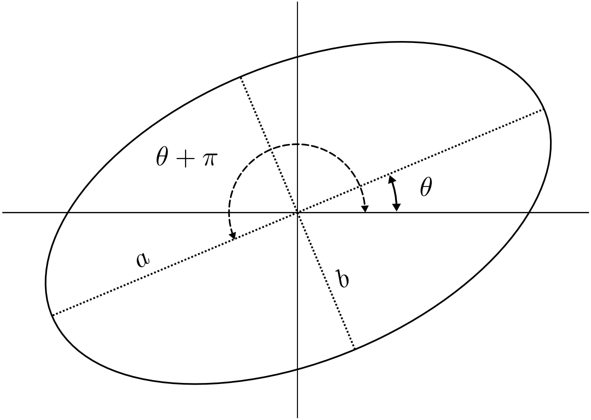


Figura: An ellipse with shape parameters a and b.

Introduction
Ellipse

An ellipse can be defined using a norm function $||\cdot||_{a,b,\theta}$ given by

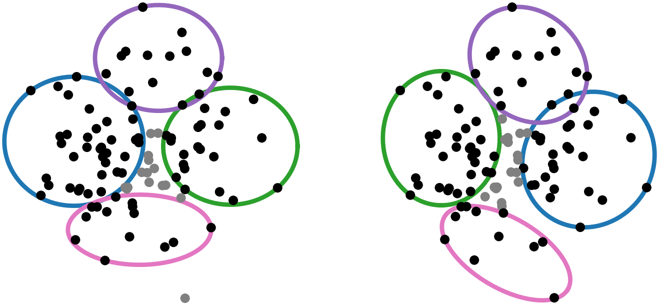
$$||x||_{a,b,\theta} = \left|\left| \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} x \right|\right|_2 .$$

Problem definition

An instance of both MCE and MCER is given by

- A demand set $\mathcal{P} := \{p_1, \ldots, p_n\}$, $p_j \in \mathbb{R}^2$;
- Each point has a weight $\mathcal{W} := \{w_1, \ldots, w_n\}$, $w_j \in \mathbb{R}_{\geq 0}$;
- A list of shape parameters $\mathcal{R} := \{(a_1, b_1); \ldots; (a_m, b_m)\}$, $(a_j, b_j) \in \mathbb{R}^2_{>0}$, with $a_j > b_j$.

Problem definition



(a) (b)

Figura: Solutions for the same instance of (a) MCE, and (b) MCER.

Problem definition

More notation

Weight function

Let $w\colon 2^{\mathcal{P}} \to \mathbb{R}$ be a function defined as

$$w(A) = \sum_{j\,:\,p_j \in A} w_j.$$

MCE's solution

$Q := (q_1, \ldots, q_m) \in \mathbb{R}^{2m}$.

MCER's solution

$Q := ((q_1, \theta_1); \ldots; (q_m, \theta_m)) \in (\mathbb{R}^2 \times [0, \pi))^m$.

Problem definition

MCE

Let $E_j : \mathbb{R}^2 \to \mathbb{R}^2$ be the coverage region of the j-th ellipse defined as

$$E_j(q) = \{x \in \mathbb{R}^2 \colon ||x - q||_{a,b,0} \leq 1\}.$$

Then, MCE is defined as the optimization problem:

$$\max_{Q} w \left(\bigcup_{j=1}^{m} \mathcal{P} \cap E_j(q_j) \right).$$

Problem definition
MCER

Let $E_j : \mathbb{R}^2 \times [0, \pi) \to \mathbb{R}^2$ be the coverage region of the j-th ellipse defined as

$$E_j(q, \theta) = \{x \in \mathbb{R}^2 \colon ||x - q||_{a,b,\theta} \leq 1\}.$$

Then, MCER is defined as the optimization problem:

$$\max_{Q} w \left(\bigcup_{j=1}^{m} \mathcal{P} \cap E_j(q_j, \theta_j) \right).$$

Remark

For the one-facility MCE and MCER, we omit the index referring to the ellipse.

MCE

$\{p_1, p_2, p_3\} \subset E(q)$

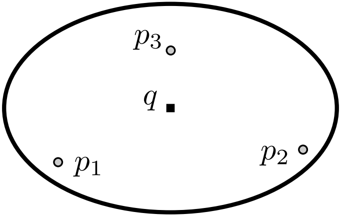


Figura: A solution for the one-facility MCE.

MCE

$$\{p_1, p_2, p_3\} \subset E(q) \implies q \in E(p_1) \cap E(p_2) \cap E(p_3).$$

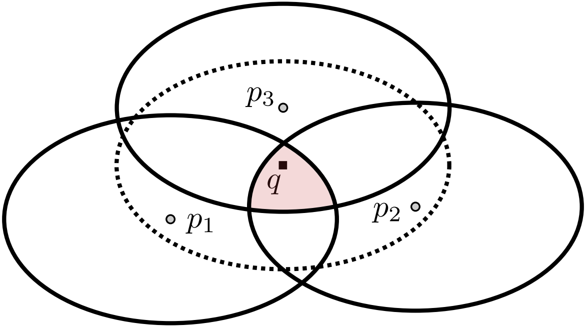


Figura: A solution for the one-facility MCE.

MCE

In general we have

$$A = \mathcal{P} \cap E(q) \implies q \in \cap_{p \in A} E(p),$$

and

$$q' \in \cap_{p \in A} E(p) \implies A \subset E(q').$$

Intersection region of ellipses

By [5], we have that if $|A| > 1$, there is at least one intersection between two ellipses in the border of $\cap_{p \in A} E(p)$.

MCE

$\{q_1, q_2, q_3\} \subset \bigcup_{1 \leq i < j \leq 3} \partial E(p_i) \cap \partial E(p_j)$.

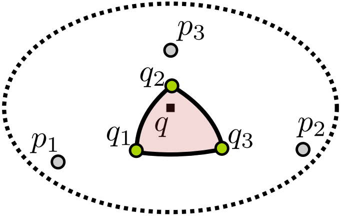


Figura: A solution for the one-facility MCE.

MCE

In general, we have

$$|\partial E(u) \cap \partial E(v)| \leq 2,$$

and that $\partial E(u) \cap \partial E(v)$ can be determined analytically.

Based on [4], we define a Candidate List Set (CLS) for each facility as follows.

Definition

Given an instance of MCE, for all $k \in \{1, \ldots, m\}$, we define the CLS for the k-th ellipse as

$$S_k = \mathcal{P} \cup \left(\bigcup_{1 \leq i < j \leq n} \partial E_k(p_i) \cap \partial E_k(p_j) \right).$$

MCE

Main result

Theorem

Given an instance of MCE, and $S_1, \ldots, S_m$ as defined previously, then the set

$$\Omega = \{(q_1, \ldots, q_m)\colon \text{ for all } q_k \in S_k\}$$

contains an optimal solution of MCE and $|\Omega| \leq n^{2m}$.

- Notice that $|S_k| \leq n(n+1)/2 \leq n^2$.
- An algorithm with $\mathcal{O}(mn^{2m+1})$ runtime complexity can be implemented.

Determining Every Center and Angle of Rotation of An Ellipse Given Its Shape and Three Points that It Must Contain

Given

- The coverage region function of an ellipse $E\colon \mathbb{R}^2 \times [0, \pi) \to \mathbb{R}^2$.
- Three points $u, v, w \in \mathbb{R}^2$.

Let us call E3P the problem whose solution is given by $(q, \theta) \in \mathbb{R}^2 \times [0, \pi)$, such that

$$\{u, v, w\} \subset \partial E(q, \theta).$$

We want to compute every solution of E3P.
We did not find any work on E3P in the literature.

E3P

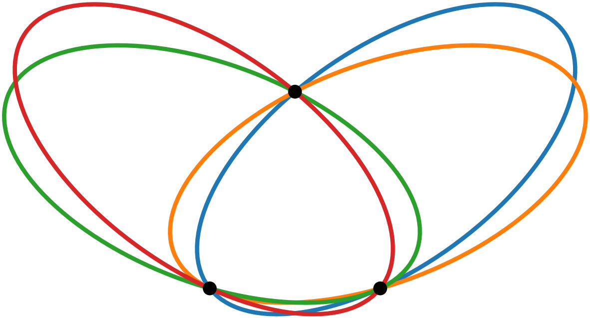


Figura: Example of every solution for an instance of E3P.

E3P

Transforming the problem

Let us define a function $\varphi\colon \mathbb{R}^2 \times [0, \pi) \to \mathbb{R}^2$ that transforms the problem as follows.

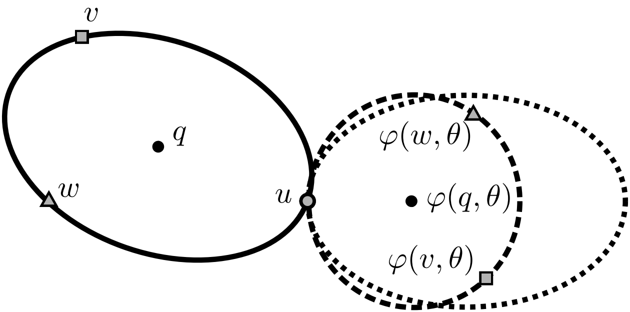


Figura: Transforming a solution of E3P into a solution of the circumcircle problem.

E3P

Transforming the problem

If u is at the origin, this function can be described as

$$\varphi(p,\theta) = \begin{bmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}.$$

- For a fixed angle, φ is bijective, we refer to φ^{-1} as its inverse.
- Let us denote by $\Lambda(\theta)$ as the triangle with vertices $\varphi(u), \varphi(v), \varphi(w)$.
- E3P is equivalent to determining θ, such that the circumscribed circle of $\Lambda(\theta)$ has radius b.

A circle is uniquely defined by $\Lambda(\theta)$, and its radius and center can be determined analytically [7].
Let $|\Lambda(\theta)|$ be the area of $\Lambda(\theta)$, and imposing that the radius of that circle is equal to b, we define a function $\xi\colon [0,\pi) \to \mathbb{R}$ whose roots determine solutions of E3P.

$$\xi(\theta) = 16b^2|\Lambda(\theta)|^2 - \|\varphi(v,\theta)\|_2^2 \, \|\varphi(w,\theta)\|_2^2 \, \|\varphi(v,\theta) - \varphi(w,\theta)\|_2^2 \, .$$

E3P

Lemma

E3P has at most six solutions.

- ξ can be written as $\sum_{0 \leq j+k \leq 6} c_{j,k} \cos^j \theta \sin^k \theta$,
- It is a real trigonometric polynomial,
- By [6, p. 150], it has at most 12 roots in $[0, 2\pi)$,
- As ellipses are symmetrical, we can dismiss half of the roots.

References I

ANDRETTA, M., AND BIRGIN, E.
Deterministic and stochastic global optimization techniques for planar covering with ellipses problems.
European Journal of Operational Research 224, 1 (Jan. 2013), 23–40.

CANBOLAT, M. S., AND MASSOW, M. V.
Planar maximal covering with ellipses.
Comput. Ind. Eng. 57, 1 (Aug. 2009), 201–208.

CHURCH, R., AND VELLE, C. R.
The maximal covering location problem.
Papers in Regional Science 32, 1 (1974), 101–118.

References II

Church, R. L.
The planar maximal covering location problem. (symposium on location problems: in memory of leon cooper).
Journal of Regional Science 24, 2 (May 1984), 185–201.

Martín, P., and Martini, H.
Algorithms for ball hulls and ball intersections in normed planes.
Journal of Computational Geometry 6, 1 (2015), 99–107.

Powell, M. J. D. M. J. D.
Approximation theory and methods.
Cambridge [England] ; New York : Cambridge University Press, 1981.
Includes index.

Weisstein, E. W.
Circumcircle From MathWorld—A Wolfram Web Resource.
Last visited on 9/4/2020.

References III