Algorithms for Planar Maximum Covering Location by Ellipses Problems*

Danilo Tedeschi[1], Marina Andretta[1]

Abstract

Planar Maximum Covering Location by Ellipses is an optimization problem where one wants to place fixed shape ellipses on the plane to cover demand points maximizing a function depending on the value of covered points. We propose new exact algorithms for two versions of this problem, one where the ellipses have to be parallel to the coordinate axis, and another where they can be freely rotated. Besides finding optimal solutions for previously published instances, including the ones where no optimal solution was known, both algorithms proposed by us were able to obtain optimal solutions for some new larger instances having with up to seven hundred demand points and five ellipses.

Keywords: Optimization, Covering, Combinatorial Optimization

1. Introduction

The Planar Maximum Covering Location Problem (PMCLP) was first introduced in [1], and can be seen as a category of problems where the coverage of a demand set, a collection of subsets of $\mathbb{R}^2$, is to be maximized by determining the location of facilities in $\mathbb{R}^2$, with coverage being determined by a distance function. In [1], methods for Euclidean and Rectilinear distances versions of the problem were proposed. In [2, 3], exact algorithms for the Euclidean PMCLP with only one facility are proposed; and in [4] an approximation algorithm is proposed for the version with multiple unit disks as facilities. A property of the Euclidean PMCLP, which is utilized in the algorithms developed in [2, 3, 4], and in the method proposed in [1], is that there is an optimal solution which every facility is located in the demand points, or in the intersection of two circles centered at two demand points; we will prove a similar property for ellipses in our work.

It is fair to say that PMCLP with elliptical coverage has not been vastly studied as only two articles have been found on it. In [5], a mixed-integer non-linear programming method was proposed as a first approach to the problem. For some instances, the method took too long and did not find an optimal solution. Because of that, a heuristic method was developed using a technique called Simulated Annealing, which was able to obtain solutions for the instances proposed in that study. The problem was further explored in [6], which introduced the version where the ellipses can be freely rotated, to which an exact and a heuristic method was proposed, and developed a new method for the axis-parallel version of the problem, which was able to obtain optimal solutions for instances that the method proposed by [5] could not. The exact method for the version with rotation could not obtain optimal solutions within a predefined time limit for several instances, the

*This paper is the results of the research project funded by CAPES.
Email addresses: danilo.tedeschi@usp.br (Danilo Tedeschi), andretta@gmail.com (Marina Andretta)

heuristic method though returned solutions for every instance, and impressively enough, obtained optimal solutions for every verifiable instance.

We study two versions of PMCLP with elliptical coverage facilities in this work. For both of them, each ellipse is defined to have a fixed shape and an undefined location, which is part of the solution. In the first version, introduced in [5], all the ellipses are restricted to be axis-parallel, while in the second version, introduced in [6], this constraint is dropped, and all the ellipses can be freely rotated. The first version will be referred to as Planar Maximum Covering Location by Ellipses Problem (MCE) and the second one as Planar Maximum Covering Location by Ellipses with Rotation Problem (MCER).

2. Problem Definition

An instance of MCE and MCER is given by n demand points $\mathcal{P} = \{p_1, \ldots, p_n\}$, $p_j \in \mathbb{R}^2$; n weights $\mathcal{W} = \{w_1, \ldots, w_n\}$, with $w_j \in \mathbb{R}$, $w_j > 0$ being the weight of the j-th point; and m shape parameters $\mathcal{R} = \{(a_1, b_1), \ldots, (a_m, b_m)\}$, with (a_j, b_j) being the semi-major and semi-minor of the j-th ellipse, with $a_j > b_j$. We define a list of functions $\mathcal{E} = \{E_1, \ldots, E_m\}$ representing the coverage area of each facility. For MCE $E_j \colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$E_j(q) = \{p \in \mathbb{R}^2 \colon (p_x - q_x)^2/a_j^2 + (p_y - q_y)^2/b_j^2 \leq 1\}. \tag{1}$$

For MCER, we define $E_j \colon \mathbb{R}^2 \to \mathbb{R}^2 \times [0, \pi)$ as

$$E_j(q, \theta) = \left\{ p \in \mathbb{R}^2 \colon \left\| \begin{pmatrix} \cos\theta/a_j & \sin\theta/a_j \\ \sin\theta/b_j & -\cos\theta/b_j \end{pmatrix} \begin{pmatrix} p_x - q_x \\ p_y - q_y \end{pmatrix} \right\|_2 \leq 1 \right\}. \tag{2}$$

Let $w : A \subset \mathcal{P} \to \mathbb{R}$ be a function which takes a subset of the demand set and returns the sum of the weights of the points in A. Then, we define MCE as the optimization problem

$$\max_{q_1, \ldots, q_m} \sum_{j=1}^{m} w(\mathcal{P} \cap E_j(q_j)), \tag{3}$$

and similarly MCER as

$$\max_{(q_1,\theta_1), \ldots, (q_m,\theta_m)} \sum_{j=1}^{m} w(\mathcal{P} \cap E_j(q_j, \theta_j)). \tag{4}$$

To make the notation more clear, we denote an instance of MCE or MCER as the tuple $(\mathcal{P}, \mathcal{W}, \mathcal{R})$, and a solution of MCE as $Q := (q_1, \ldots, q_m)$, and a solution of MCER as $Q := ((q_1, \theta_1); \ldots; (q_m, \theta_m))$. We also use ∂ as the boundary operator, for example, $\partial E_1(q_1)$ denotes an ellipse with shape parameters (a_1, b_1) centered at q_1 given an instance of MCE.

3. An algorithm for MCE

We will develop a method which is similar to the one developed in [2] for only one euclidean disk, and the exact algorithm developed for multiple Euclidean disks in [4]. We first describe a Candidate List set (CLS) for each facility, which is finite set of possible locations for each ellipse, which we use to converting MCE into a discrete optimization problem. We, then prove that using the possible solutions obtained from the combination of every ellipse's CLS an optimal solution can be obtained.

Let $(\mathcal{P}, \mathcal{W}, \mathcal{R})$ be an instance of MCE, then for each $j = 1 \dots m$, consider n ellipses with shape parameters (a_j, b_j) centered at each one of the points in $\mathcal{P}$. If we have $q_j \in \mathbb{R}^2$ in a subset of the coverage areas of those ellipses, then q_j is a solution of MCE covering the centers of those ellipses. In other words, if $q_j \in \mathbb{R}^2$, and $X \subset \{1, \dots, n\}$, $X \neq \emptyset$, such that $q_j \in \cap_{i \in X} E_j(p_i)$, then $\mathcal{P} \cap E_j(q_j) = \{p_i \colon i \in X\}$. From that observation, we can constrain each q_j to be in $\cap_{i \in X} E_j(p_i)$, for some $X \subset \{1, \dots, n\}$, $X \neq \emptyset$.

In [7], an algorithm is proposed for the problem of determining the intersection of disks of fixed radii from a strictly convex normed plane. We say that $(\mathbb{R}^2, ||\cdot||)$ is a strictly convex normed plane if the unit disk given by the norm $||\cdot||$ is strictly convex. Note that for any ellipse, there is a strictly convex normed plane whose unit circle is that ellipse. For that reason, we state some results from [7] here, which we use on the development of an algorithm for MCE.

Let $(\mathbb{R}^2, ||\cdot||)$ be a strictly convex normed plane, and $\mathcal{D} = \{D_1, \dots, D_n\}$ be a set of n unit disks in that space centered at different points, with the condition that $\cap_{i=1}^n D_i \neq \emptyset$. In [7], an algorithm was developed to construct this intersection in $\mathcal{O}(n \lg n)$, some of its preliminary results are:

- $\partial \cap_{i=1}^n D_i$ is formed by arcs of $\partial D_1, \dots, \partial D_n$.
- The vertices of $\partial \cap_{i=1}^n D_i$ is contained in the set of pairwise-intersection of the circles $\partial D_1, \dots, \partial D_n$.
- $|\partial D_i \cap \partial D_j| \leq 2$.

Based on those, we introduce the next definition for the k-th ellipse's CLS, which we refer to as S_k.

Definition 1. Given an instance of MCE, for any $k \in \{1, \dots, m\}$, we define the CLS for the k-th ellipse as

$$S_k = \bigcup_{1 \leq i < j \leq n} \partial E_k(p_i) \cap \partial E_k(p_j) \bigcup \mathcal{P}. \tag{5}$$

The set of solutions S_k, can be computed in $\mathcal{O}(n^2)$ as determining the intersections between two axis-parallel ellipses can be done analytically.

Lemma 1. *Given an instance of MCE, and $S_1, \dots, S_m$ as defined by Definition 1, then the set $\Omega = \{(q_1, \dots, q_m) \colon \text{ for all } q_k \in S_k\}$ contains an optimal solution of MCE and $|\Omega| \leq n^{2m}$.*

Proof. Let Q^* be an optimal solution of MCE for the given instance. Then, we are going to prove that there exists $Q' \in \Omega$, which is also optimal.

For each $k = 1, \dots, m$, let $X_k = \{p_i \in \mathcal{P} \colon p_i \in E_k(q_k^*)\}$.

If $|X_k| = 0$, then any $q_k \in S_k$ makes $X_k \subset \mathcal{P} \cap E_k(q_k)$.

if $|X_k| = 1$, then there is at least one element in S_k that makes $X_k \subset \mathcal{P} \cap E_k(q_k)$.

if $|X_k| > 1$, then let $Y_k = \cap_{p \in X_k} E_k(p)$, by the results of [7], we have that the boundary of Y_k has vertices in the pairwise intersections of $\{\partial E_k(p) \colon p \in X_k\}$. Therefore, at least one vertex of Y_k is in S_k, and any of those vertices produce a solution that covers at least the same points covered by Q^*.

Lastly, we have that $|S_k| \leq 2\binom{n}{2} + n = n(n+1)/2 \leq n^2$ hence $|\Omega| \leq n^{2m}$.

With all this in hand, we can go ahead and define an algorithm for MCE.

Algorithm 1 can be proved to have a runtime complexity of $\mathcal{O}(mn^{2m+1})$.

Algorithm 1 Algorithm for MCE

Input: A set of points $\mathcal{P} = \{p_1, \dots, p_n\}$, a list of weights $\mathcal{W} = \{w_1, \dots, w_n\}$, and a list of shape parameters $\mathcal{R} = \{(a_1, b_1), \dots, (a_m, b_m)\}$.
Output: An optimal solution for MCE.

procedure $MCE(\mathcal{P}, \mathcal{W}, \mathcal{R})$
  return $MCE_{bt}(\mathcal{P}, \mathcal{W}, \mathcal{R}, 1)$
end procedure

procedure $MCE_{bt}(Z, \mathcal{W}, \mathcal{R}, j)$
  if $j = m + 1$ **then**
    return 0
  end if
  $(q_j^*, \dots, q_m^*) \leftarrow (0, \dots, 0)$
  Let S_j be the CLS for the j-th ellipse as defined by Definition 1.
  for $q_j \in S_j$ **do**
    $Cov \leftarrow \mathcal{P} \cap E_j(q_j)$
    $(q_{j+1}, \dots, q_m) \leftarrow MCE_{bt}(Z \setminus Cov, \mathcal{W}, \mathcal{R}, j + 1)\}$
    if $w(\cup_{k=j}^{m} Z \cap E_k(q_k)) > w(\cup_{k=j}^{m} Z \cap E_k(q_k^*))$ **then**
      $(q_j^*, \dots, q_m^*) \leftarrow (q_j, \dots, q_m)$
    end if
  end for
  return $(q_j^*, \dots, q_m^*)$
end procedure

4. Determining every location of an ellipse given its shape and three points

In this section, we introduce the problem of determining every location, here defined as the center and angle of rotation, of an ellipse with fixed shape parameters, such that it contains three given points. This problem comes up in the development of an algorithm for MCER in the next section. It is important to point out that no prior studies were found on it, or even on related problems. We propose an algorithm for it that involves determining the eigenvalues of a 6×6 complex matrix. We also analyze its efficiency in terms of numerical accuracy and display some solutions that it was able to obtain.

4.1. Problem definition

Given the shape parameters of an ellipse (a, b), $a > b > 0$, and three points $u, v, w \in \mathbb{R}^2$, let $E\colon \mathbb{R}^2 \times [0, \pi) \to \mathbb{R}^2$ be the coverage region of an ellipse with shape parameters (a, b), we refer to the problem of obtaining $(q, \theta) \in \mathbb{R}^2 \times [0, \pi)$, such that $\{u, v, w\} \subset \partial E(q, \theta)$ as Ellipse by Three Points Problem (E3P). Because of its application here in our work, we are only interested in a method that can obtain every solution of E3P.

4.2. Transforming E3P into a circle problem

Initially, E3P is a problem of determining the values of three unknown continuous variables (q_x, q_y), and θ. However, as it will be shown, we can reduce this number to only one, as it is possible to obtain q uniquely from θ. Let us assume that point u is at the origin. If it is not, a simple translation by $-u$ applied to the three points can be made to put u at the origin. Assume as well that (q, θ) is a solution of E3P.

Applying a rotation of $-\theta$ to the coordinate system makes the ellipse in the original solution become axis-parallel. Then, that ellipse can be transformed into a circle of radius b by squeezing the x-axis by b/a. This two-step transformation can be written as a function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$\varphi(p, \theta) = \begin{bmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}. \tag{6}$$

An example of this transformation can be seen in Figure 1. As φ^{-1} is well-defined, instead of solving E3P, we can work with the univariate problem of determining an angle of rotation $\theta \in [0, \pi)$ that makes the triangle with vertices $\varphi(u, \theta), \varphi(v, \theta), \varphi(w, \theta)$ be circumscribed in a circle of radius b. To make the notation less cluttered, we denote by $\Lambda(\theta)$ the triangle with vertices $\varphi(u, \theta), \varphi(v, \theta), \varphi(w, \theta)$.

Let $A(\theta)$ denote the area of the triangle $\Lambda(\theta)$. Using the formula given in [8, p. 189] for the radius of the circumscribed circle of a triangle, we can obtain a function $\xi\colon [0, \pi) \to \mathbb{R}$ given by

$$\xi(\theta) = 16b^2 A(\theta)^2 - \|\varphi(v, \theta)\|_2^2 \, \|\varphi(w, \theta)\|_2^2 \, \|\varphi(v, \theta) - \varphi(w, \theta)\|_2^2 \,, \tag{7}$$

whose roots determine solutions of E3P.

Lemma 2. *E3P has at most six solutions.*

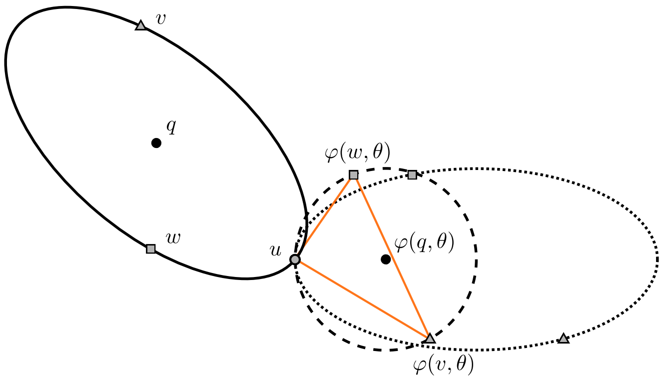


Figure 1: Transforming a solution of E3P into a solution of the circumradius problem.

Proof. The first thing to notice is that ξ is a real trigonometric polynomial of degree 6. Its term of highest degree is the multiplication of the norms $\|\varphi(v,\theta)\|_2^2 \, \|\varphi(w,\theta)\|_2^2 \, \|\varphi(v,\theta) - \varphi(w,\theta)\|_2^2$. In [9, p. 150], where a definition of real trigonometric polynomial is also given, it is stated that a n-degree real trigonometric polynomial can have up to $2n$ roots in $[0, 2\pi)$. Therefore, E3P has at most 12 solutions in $[0, 2\pi]$. Half of these solutions, though, are duplicated as ellipses are symmetric to their axis.

4.3. Converting ξ into a polynomial

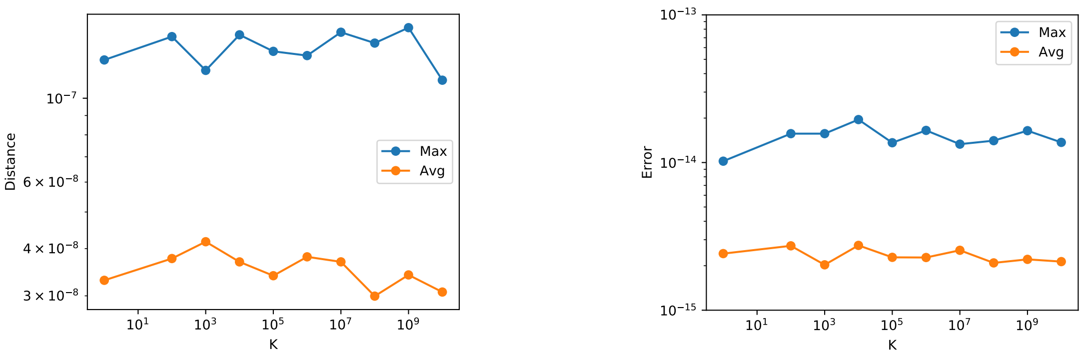
In [10, p. 195], a theorem is presented stating that for every univariate polynomial of degree n, there exists a companion matrix, which is a $n \times n$ matrix, such that its eigenvalues are the zeros of that polynomial. Finding every eigenvalue of a matrix can be done using the QR algorithm, which runs in $\mathcal{O}(n^3)$ and uses $\mathcal{O}(n^2)$ memory (a very complete introduction to it can be found in [11]). For example, for a degree-4 polynomial $\sum_{k=0}^{4} a_k x^4$, a possible companion matrix is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{a_0}{a_4} & -\dfrac{a_1}{a_4} & -\dfrac{a_2}{a_4} & -\dfrac{a_3}{a_4} \end{bmatrix}.$$

In practice, we can use the very well-known LAPACK software library to obtain the eigenvalues of a matrix; for more information about LAPACK see [12]. This approach works for both real or complex polynomials, because of that, based on [13], we describe a way of converting ξ into a complex polynomial.

By using the two identities $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$, which relate trigonometric functions with complex numbers in the unit circle $\mathbb{S} = \{z \in \mathbb{C} \colon |z| = 1\}$, we can rewrite ξ as a function of the new variable $z = e^{i\theta} \in \mathbb{S}$. In [13], it is stated that this substitution when utilized for the task of determining the roots of a real trigonometric polynomial does not yield loss of accuracy.

As ξ is a real trigonometric polynomial of degree 6, z appears with exponents from -6 up to 6. Multiplying ξ by z^6 and extending the domain of z to $\mathbb{C}$, we obtain a complex polynomial $g(z) = \sum_{k=0}^{12} c_k z^k$, for some $c_0, \ldots, c_{12} \in \mathbb{C}$. In practice, symbolic computation can be used to obtain the actual coefficients of g in terms of an instance of E3P.



(a)

(b)

Figure 2: (a) shows the maximum and average distance to the unit circle $|1 - |\hat{z}||$. (b) shows the maximum and average error $|f(\hat{z})|$, where $\hat{z}$ is a root of f returned by our algorithm.

Let $angle\colon \mathbb{C}\colon [0, 2\pi)$ be a function that returns the angle of a complex number, then given a root $\hat{z}$ of g, if $|\hat{z}| = 1$ and $angle(\hat{z}) \in [0, \pi)$, then $\hat{\theta} = angle(\hat{z})$ is a root of ξ. Observing that for any $z \in \mathbb{C}$, $angle(-z) = \pi + angle(z)$, which is the same symmetry ellipses have, we conclude that $g(-z) = g(z)$, implying that all the odd-degree coefficients of g are zero. Using the substitution $y = z^2$, we can obtain a degree-6 polynomial $f(y) = \sum_{k=1}^{6} c_{2k} y^k$. Then, from a root $\hat{y}$ of f, we obtain a root $\hat{\theta} = angle(\hat{y})/2$ of ξ. Lastly, let $C(\theta) \in \mathbb{R}^2$ be the center of the circumcircle of $\Lambda(\theta)$, then from a root $\hat{\theta}$ of ξ, we obtain the solution $(\varphi^{-1}(C(\hat{\theta})) + u, \hat{\theta})$ of E3P.

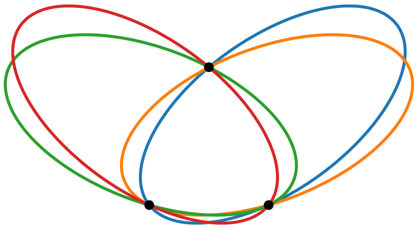
It worth mentioning that a pattern on the coefficients of f was identified, and maybe it can be used for further improvements. Analyzing the polynomials produced for several instances, we observed that $c_k = \overline{c_{6-k}}$, for all $k = 0 \ldots 6$.

4.4. Choosing a precision constant

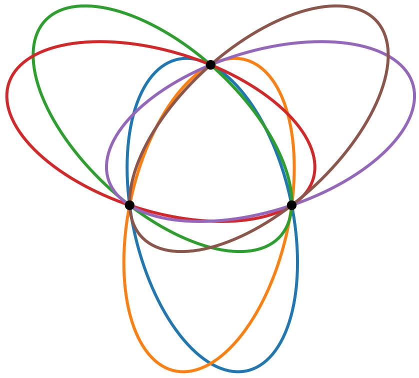
We developed an experiment to choose a precision constant for comparing if a root of f is in the unit circle. The implementation was coded in C++, and LAPACK's ZGEEV was utilized to obtain the eigenvalues of the companion matrix of the polynomial f, more information about the implementation is given in section 6. For the experiment, we defined $K \in \mathbb{R}$, $K > 0$, and consider instances with ellipse's shape parameters $(K, \frac{K}{2})$, for $K \in \{10^j : j = 0, \ldots, 10\}$.

The experiment considers instances of E3P where the three points are the vertices of an ellipse rotated by $\theta \in [0, \pi)$. Such instances only have one solution, and therefore, roots with multiplicity greater than one are expected, which can be seen as a special case. For each value of K, we ran the algorithm for 100 instances generated randomly by sampling θ according to a uniform distribution. For each instance, we took the root $\hat{z}$ which produced the closest solution to the priorly known one. Then, for each K, as it can be seen in Figure 2a, we considered the maximum and the average distance to the unit circle $|1 - |\hat{z}||$; and, as it can be seen in Figure 2b, the maximum and average error $|f(\hat{z})|$.

From this experiment, we decided to adopt a precision constant of 10^{-6} to consider a root of f to be in the unit circle, and as an additional check, we adopted a precision constant of 10^{-9} to consider a root to be a solution of E3P.



(a) An instance of E3P with four solutions.



(b) An instance of E3P with six solutions.

4.5. Instances with four and six solutions

Any instance of E3P, as stated by Theorem 2 can have up to six solutions. At first, though, this bound seemed to be loose as for randomly generated instances, we were not able to obtain instances with more than two solutions.

After some investigation, though, we were able to construct some four-solution instances (an example is displayed in Figure 3a). An interesting property of those solutions is that every one of them has their three points form an isosceles triangle.

Obtaining six-solution instances, on the other hand, was done by taking a particular case of the isosceles-triangle approach. As it can be seen in Figure 3b, the three points on every one of the six ellipses' border form an equilateral triangle.

It should be pointed out that neither non-isosceles instances with four solutions nor non-equilateral instances with six solutions could be found. Further investigating these possible properties of E3P is left as future work.

5. An Algorithm for MCER

6. Implementation Details

7. Numerical Experiments

References

References

[1] R. L. Church, The planar maximal covering location problem. (symposium on location problems: in memory of leon cooper), Journal of Regional Science 24 (2) (1984) 185–201. `doi:10.1111/j.1467-9787.1984.tb01031.x`. URL `https://doi.org/10.1111/j.1467-9787.1984.tb01031.x`

[2] Z. Drezner, Note—on a modified one-center model, Management Science 27 (1981) 848–851. `doi:10.1287/mnsc.27.7.848`.

[3] B. M. Chazelle, D. Lee, On a circle placement problem, Computing 36 (1986) 1–16. `doi:10.1007/BF02238188`.

[4] M. de Berg, S. Cabello, S. Har-Peled, Covering many or few points with unit disks, Theory of Computing Systems 45 (3) (2008) 446–469. `doi:10.1007/s00224-008-9135-9`. URL `https://doi.org/10.1007/s00224-008-9135-9`

[5] M. S. Canbolat, M. von Massow, Planar maximal covering with ellipses, Computers and Industrial Engineering 57 (2009) 201–208.

[6] M. Andretta, E. Birgin, Deterministic and stochastic global optimization techniques for planar covering with ellipses problems, European Journal of Operational Research 224 (1) (2013) 23–40. doi:10.1016/j.ejor.2012.07.020.
URL https://doi.org/10.1016/j.ejor.2012.07.020

[7] P. Martín, H. Martini, Algorithms for ball hulls and ball intersections in normed planes, Journal of Computational Geometry Vol 6 (2015) No 1 (2015)–. doi:10.20382/JOCG.V6I1A4.
URL https://journals.carleton.ca/jocg/index.php/jocg/article/view/187

[8] R. Johnson, Y. Young, Advance Euclidean Geometry (modern Geometry): An Elementary Treatise on the Geometry of the Triangle and the Circle, Dover books on advanced mathematics, Dover, 1960.
URL https://books.google.com.br/books?id=HdCjnQEACAAJ

[9] M. J. D. M. J. D. Powell, Approximation theory and methods, Cambridge [England] ; New York : Cambridge University Press, 1981, includes index.

[10] R. A. Horn, C. R. Johnson (Eds.), Matrix Analysis, Cambridge University Press, New York, NY, USA, 1986.

[11] D. S. Watkins, The qr algorithm revisited, SIAM Rev. 50 (1) (2008) 133–145. doi:10.1137/060659454.
URL http://dx.doi.org/10.1137/060659454

[12] E. Anderson, Z. Bai, C. Bischof, S. Blackford, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, D. Sorensen, LAPACK Users' Guide, 3rd Edition, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1999.

[13] P. Weidner, The durand-kerner method for trigonometric and exponential polynomials, Computing 40 (2) (1988) 175–179. doi:10.1007/BF02247945.
URL https://doi.org/10.1007/BF02247945