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Covering a polygonal region by rectangles

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Abstract The problem of covering a compact polygonal region, called target region, with a finite family of rectangles is considered. Tools for mathematical modeling of the problem are provided. Especially, a function, called Γ-function, is introduced which indicates whether the rectangles with respect to their configuration form a cover of the target region or not. The construction of the Γ-function is similar to that of Φ-functions which have been proved to be an efficient tool for packing problems. A mathematical model of the covering problem based on the Γ-function is proposed as well as a solution strategy. The approach is illustrated by an example and some computational results are presented.

Keywords Mathematical modeling · Optimization · Covering problem

1 Introduction

The problem of covering a compact polygonal region, called target region, with a finite family of rectangles is considered. The aim is either to find translation vectors for the rectangles so that the union of the translated rectangles form a cover for the target region, or otherwise to prove that there does not exist any covering. This problem is known to be NP-complete [2].

As opposed to, e.g., [2] where a finite number of points has to be covered, we consider the covering of a polygonal set. Therefore, the verification that a certain

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configuration of the rectangles forms a cover of the target set cannot be done by inspecting a finite set of points, other techniques (with polynomial time complexity) are necessary, e.g. those described below.

There are many relations between covering and packing or cutting problems. For an annotated survey on *Cutting and Packing* we refer to [3]. Covering problems arise naturally in a variety of applications. For a comprehensive overview we refer to [2]. As an example, query optimization in spatial databases is a source of covering problems. In this setting a query may correspond to a geometric region and be phrased in a generic form using geometric parameters. Given a set of existing geometric, parametrized, query regions and a set of points or regions, we might want to ask if there are values of the parameters that allow the query regions to cover the set of points or even regions. Another field of application (also mentioned in [2]) is shape recognition for robotics, graphics or image processing applications. In these cases it is sometimes useful to represent a shape as a collection of parts. However, given a collection of parts and a shape, it can be difficult to determine if the shape can be described by that collection of parts. If the goal is to obtain an outer approximation of the shape using the parts, then this can be posed as a covering problem.

Besides the decision problem whether a covering exists or not, related problems can be of interest. For instance, one can ask for a minimum number of (identical) rectangles needed to form a cover for the target region. Or, if there exist several covers one can look for a best cover where best means that some objective function is regarded.

In order to develop methods for solving covering problems valid mathematical models are needed. In this paper we provide tools of mathematical modeling. First of all, a cover criterion is defined applying the Φ-function technique which has been proved to be very useful when dealing with cutting and packing problems [1, 7–9]. Then a (so-called) covering function is introduced for the analytical description of relations between the target region and the family of rectangles. Based on the mathematical model, a solution strategy is given together with an example.

The paper is organized as follows. In Sect. 2 we present the problem statement and define a cover criterion applying the Φ-function technique. In Sect. 3 we discuss a constructive tool for mathematical modeling of the covering problem, called Γ-function [5], for analytical description of relations between the target region and the finite family of rectangles. The relationship between Φ-functions and Γ-function is also explained. A mathematical model of the covering problem is provided in the next section. Some basic characteristics of the model are discussed. In Sect. 5 the general solution strategy is proposed. The construction of the search tree and pruning rules are given in Sect. 6 followed by an example (Sect. 7). Results of computational experiments are given in Sect. 8 and conclusions in Sect. 9.

2 Problem statement

Let $\Omega \subset \mathbb{R}^2$ be a given canonical compact region with multi-connected components, the *target set*, which is represented as union of K convex polygonal subsets Ω_k,

$k \in I_K = \{1, \dots, K\}$. Using the corner (extreme) points ω_{kl} of Ω_k we have

$$\Omega = \bigcup_{k \in I_K} \Omega_k, \quad \text{where } \Omega_k = \operatorname{conv}\{\omega_{kl} = (\widetilde{x}_{kl}, \widetilde{y}_{kl}) : l \in I_{m_k}\}$$

and m_k is the number of corner points of Ω_k. Furthermore, a finite family

$$\Lambda = \{R_i : i \in I_n = \{1, 2, \dots, n\}\}$$

of rectangles

$$R_i = \{(x, y) \in \mathbb{R}^2 : -a_i \le x \le a_i, -b_i \le y \le b_i\}, \quad i \in I_n$$

is considered. In this representation of R_i, the origin of the corresponding eigen coordinate system coincides with the centre of R_i. The length of R_i is $2a_i$, its width $2b_i$.

The allocation (or the placement) of the target set Ω and of the rectangles R_i within $\mathbb{R}^2$ is characterized by translation vectors $u_0 = (x_0, y_0)$ and $u_i = (x_i, y_i)$, $i \in I_n$, respectively. We denote rectangle R_i translated by vector u_i by $R_i(u_i)$, and region Ω translated by u_0 by $\Omega(u_0)$. Furthermore, the family of translated rectangles $R_i(u_i)$, $i \in I_n$, is denoted by $\Lambda(u)$ where $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^{2n}$. In case, the translation vector $u^0 = (u_1^0, u_2^0, \dots, u_n^0) \in \mathbb{R}^{2n}$ is fixed, then we define

$$P(u^0, w) = \bigcup_{i=1}^{n} R_i(u_i^0 + w) = \bigcup_{i=1}^{n} R_i(u_i^0) + w, \quad w \in \mathbb{R}^2,$$

which describes the translation of all rectangles without changing their relative position to each other. Polygonal set $P(u^0, w)$ represents the region covered by the union of the translated rectangles $R_i(u_i^0 + w)$, $i \in I_n$.

Region $P(u^0, w)$ in general consists of several components (connected subsets). Therefore we use a description of $P(u^0, w)$ as union of components $P_s(u^0, w)$, $s \in I_\tau = \{1, \dots, \tau\}$ where τ is the number of components of $P(u^0, w)$. Since, in general, P_s is non-convex we use a description of P_s as union of convex polygons P_{st}, $t \in I_{n_s} = \{1, \dots, n_s\}$, i.e.

$$P(u^0, w) = \bigcup_{s \in I_\tau} P_s(u^0, w) = \bigcup_{s \in I_\tau} \bigcup_{t \in I_{n_s}} P_{st}(u^0, w).$$

In general, a set P_{st} can be identified with one of the given rectangles. We say, $\Lambda(u^0)$ is a *cover* of $\Omega(u_0)$ if there exists a vector $w_0 \in \mathbb{R}^2$ such that

$$\Omega(u_0) \subset P(u^0, w_0) = \bigcup_{i \in I_n} R_i(u_i^0 + w_0). \tag{1}$$

It is obvious, in order to get a cover for $\Omega(0)$, i.e. $u_0 = (0, 0)$, one has to replace w_0 in (1) by $w_0 - u_0$. Then, $\Omega(0) \subset \bigcup_{i=1}^{n} R_i(u_i^0 + w_0 - u_0)$.

The search for a suitable translation vector $u^0 \in \mathbb{R}^{2n}$ and of $w_0 \in \mathbb{R}^2$ to obtain a cover can be unified so that $u_0 = (0, 0)$ and $w_0 = (0, 0)$ can be used. Summarizing, in this paper the following decision problem is considered:

Decide whether there exists a vector $u^0 \in \mathbb{R}^{2n}$ such that $\Lambda(u^0)$ is a cover of $\Omega(0)$ subject to $w_0 = (0, 0)$.
Moreover, in case of existence, compute at least one such vector u^0.

To formulate a cover criterion we use the concept of Φ-functions (cf. [8]), which have been proved to be a useful tool for mathematical modeling of packing problems. A Φ-function $\Phi_{AB} : \mathbb{R}^4 \to \mathbb{R}$ of a pair of translated objects $A(u) \subset \mathbb{R}^2$ and $B(v) \subset \mathbb{R}^2$ ($u, v \in \mathbb{R}^2$) is an everywhere defined continuous function which has a negative value if the interior of the (translated) objects intersect each other, i.e. if $\operatorname{int} A(u) \cap B(v) \neq \emptyset$ ($\operatorname{int} S$ denotes the interior of set S), $\Phi_{AB}(u, v) = 0$ if the two objects are in contact, i.e. if $\operatorname{int} A(u) \cap B(v) = \emptyset$ but $A(u) \cap B(v) \neq \emptyset$, and it is strictly positive if the two objects have no common points, i.e. if $A(u) \cap B(v) = \emptyset$.

In order to formulate a cover criterion by means of Φ-functions, we consider the complementary sets

$$h_s(u^0, w) = \operatorname{cl}(\mathbb{R}^2 \setminus P_s(u^0, w)), \quad s = 1, \dots, \tau,$$

where $\operatorname{cl}(S)$ denotes the closure of set S. Since $P(u^0, w) = \bigcup_{s \in I_\tau} P_s(u^0, w)$, the complement $h(u^0, w) = \operatorname{cl}(\mathbb{R}^2 \setminus P(u^0, w))$ of $P(u^0, w)$ has the form

$$h(u^0, w) = \bigcap_{s \in I_\tau} h_s(u^0, w). \tag{2}$$

Furthermore, since set $h_s(u^0, w)$ is the complement of the union of rectangles, it can be represented as union of convex polygonal sets C_{st}, $t \in I_{\lambda_s} = \{1, \dots, \lambda_s\}$, i.e.

$$h_s(u^0, w) = \bigcup_{t \in I_{\lambda_s}} C_{st}(u^0, w). \tag{3}$$

Each set C_{st} is a non-empty intersection of not more than four half-planes, e.g. it can either be a half-plane, a cone, a half-strip, or a rectangle.

According to relation (1), family $\Lambda(u^0)$ is a cover of $\Omega(u_0)$ if

$$\operatorname{int}(\Omega(u_0)) \cap h(u^0, w_0) = \emptyset. \tag{4}$$

For a moment we consider the translation vector u^0 to be fixed, i.e. a fixed configuration of the rectangles. In order to highlight such a situation we omit the argument u^0, i.e. we use $P(w)$ instead of $P(u^0, w)$ etc.

For fixed u^0, let $\Phi_{kst}(w, v)$ be a Φ-function for the pair of convex sets $C_{st}(w)$ and $\Omega_k(v)$ with $w, v \in \mathbb{R}^2$, $k \in I_K$, $t \in \{1, \dots, \lambda_s\}$, $s \in \{1, \dots, \tau\}$, then

$$\Phi(w, v) = \min_{k \in I_K} \max_{s \in I_\tau} \min_{t \in I_{\lambda_s}} \Phi_{kst}(w, v), \tag{5}$$

is a Φ-function for $h(w) = \operatorname{cl}(\mathbb{R}^2 \setminus P(w))$ and $\Omega(v)$ since for every subset Ω_k, $k \in I_K$, there must be at least one connected component $P_s(w)$ so that its complement $h_s(w)$ does not intersect the interior of Ω_k. Thus, a cover criterion for $\Omega(v)$ and $\Lambda(u_1^0 + w, u_2^0 + w, \dots, u_n^0 + w)$ has the form

$$\Phi(w, v) \geq 0 \quad \Longleftrightarrow \quad \Omega(v) \subset P(u^0, w). \tag{6}$$

This means, for a fixed configuration of the rectangles given by u^0, every $w \in \mathbb{R}^2$ with $\Phi(w, v) \geq 0$ yields a cover for the target set.

In the rest of the paper we assume $u_0 = (0, 0)$ in general.

3 Tools of mathematical modeling

As became apparent in the previous section, a modeling tool for the covering problem which is based on the usage of Φ-functions, is strongly dependent on the relative position of the rectangles to each other which is involved within u^0. On the other hand, modifications in u^0 do not change the principal configuration in many cases, e.g. if rectangle R_j is left to rectangle R_i for some $u = (x_1, y_1, \ldots, x_n, y_n)$, i.e. $x_j + a_j < x_i - a_i$, it remains left for small changes. Only if x_j is sufficiently increased (or x_i is decreased) then the relative position of R_i and R_j varies.

In order to construct a tool for the analytical description of relations between $\Omega(0)$ and $\Lambda(u)$ we consider a partition of $\mathbb{R}^{2n}$ into (convex) subsets $\mathbb{R}^{2n}_q$, $q \in I_\eta = \{1, \ldots, \eta\}$ which is determined by different relative positions of the rectangles. For every translation vector $u \in \mathrm{int}(\mathbb{R}^{2n}_q)$ the configuration $\Lambda(u)$ of the rectangles leads to the same structure of $h(u, 0)$, i.e. its representation according to (2) and (3) does not change for all $u \in \mathrm{int}(\mathbb{R}^{2n}_q)$.

Considering $u^0 \in \mathbb{R}^{2n}_q$ to be fixed we can apply, as proposed in [5], the cover criterion (6). But in our case, the translation vector $u \in \mathbb{R}^{2n}$ has to be found.

The definition of the partition of $\mathbb{R}^{2n}$ into subsets $\mathbb{R}^{2n}_q$, $q \in I_\eta$ where different subsets are determined by different configurations of the rectangles is as follows. In order to explain what we mean with *configuration* we consider the interaction between two rectangles, say $R_i(u_i)$ and $R_j(u_j)$ with $i \neq j$, $u_i = (x_i, y_i)$ and $u_j = (x_j, y_j)$, i.e.

$$R_i(u_i) = \{(x, y) \in \mathbb{R}^2 : -a_i \leq x - x_i \leq a_i, -b_i \leq y - y_i \leq b_i\},$$
$$R_j(u_j) = \{(x, y) \in \mathbb{R}^2 : -a_j \leq x - x_j \leq a_j, -b_j \leq y - y_j \leq b_j\}.$$

There arise several cases, mainly determined by

$$R_i(u_i) \cap R_j(u_j) = \emptyset \quad \text{or} \quad R_i(u_i) \cap R_j(u_j) \neq \emptyset.$$

The first relation leads to four (convex) subcases:

$$\begin{array}{lll}
(0.\mathrm{a}) & x_j + a_j < x_i - a_i, & R_j(u_j) \text{ is left to } R_i(u_i), \\
(0.\mathrm{b}) & y_j - b_j > y_i + b_i, & R_j(u_j) \text{ is above } R_i(u_i), \\
(0.\mathrm{c}) & x_j - a_j > x_i + a_i, & R_j(u_j) \text{ is right to } R_i(u_i), \\
(0.\mathrm{d}) & y_j + b_j < y_i - b_i, & R_j(u_j) \text{ is below } R_i(u_i).
\end{array}$$

Note that in order to uniquely define the sets C_{st} a more detailed partition is necessary for the cases (0.a)–(0.d). For instance, in case of (0.a) the demand $y_i + b_i \leq y_j + b_j$ leads to another configuration in comparison to $y_i + b_i \geq y_j + b_j$, etc.

In case of $R_i(u_i) \cap R_j(u_j) \neq \emptyset$ we have up to ten subcases which define different configurations. Among them, the cases

$$R_i(u_i) \subset R_j(u_j) \quad \text{and} \quad R_i(u_i) \supseteq R_j(u_j)$$

are not of interest when dealing with covering problems. Independent on the dimensions of R_i and R_j the following four subcases arise:

$$\begin{array}{lll}
(1.a) & (x_j + a_j, y_j - b_j) \in R_i(u_i), & (x_i - a_i, y_i + b_i) \in R_j(u_j), \\
(1.b) & (x_j - a_j, y_j - b_j) \in R_i(u_i), & (x_i + a_i, y_i + b_i) \in R_j(u_j), \\
(1.c) & (x_j - a_j, y_j + b_j) \in R_i(u_i), & (x_i + a_i, y_i - b_i) \in R_j(u_j), \\
(1.d) & (x_j + a_j, y_j + b_j) \in R_i(u_i), & (x_i - a_i, y_i - b_i) \in R_j(u_j).
\end{array}$$

In case of $b_j < b_i$ the following two situations occur:

$$\begin{array}{lll}
(2.a) & x_j - a_j < x_i - a_i, & x_i - a_i \le x_j + a_j, \\
 & y_i - b_i < y_j - b_j, & y_j + b_j < y_i + b_i, \\
(2.b) & x_j - a_j \le x_i + a_i, & x_i + a_i < x_j + a_j, \\
 & y_i - b_i < y_j - b_j, & y_j + b_j < y_i + b_i.
\end{array}$$

Similar cases arise when $a_j < a_i$:

$$\begin{array}{lll}
(2.c) & x_i - a_i < x_j - a_j, & x_j + a_j < x_i + a_i, \\
 & y_j - b_j \le y_i + b_i, & y_i + b_i < y_j + b_j, \\
(2.d) & x_i - a_i < x_j - a_j, & x_j + a_j < x_i + a_i, \\
 & y_j - b_j < y_i - b_i, & y_i - b_i \le y_j + b_j.
\end{array}$$

Finally, if $a_j < a_i$ and $b_j > b_i$ then a cross-wise overlapping of the two rectangles has to be considered:

$$\begin{array}{lll}
(3) & x_i - a_i < x_j - a_j, & x_i + a_i > x_j + a_j, \\
 & y_i + b_i < y_j + b_j, & y_i - b_i > y_j - b_j.
\end{array}$$

If $a_j = a_i$ or $b_j = b_i$ then several special situations occur which are not described here in detail.

Since these relative positions of two rectangles have to be considered for every pair R_i and R_j of rectangles with $i \neq j$, the number η of subsets $\mathbb{R}_q^{2n}$ is very huge (exponential in n), and it seems to be impossible to analyse each of the different configurations. Notice, the interaction of more than two rectangles can lead to empty subsets of $\mathbb{R}^{2n}$ because of contradictory relative positions (e.g. if it is demanded that r_i is left to R_j, R_j is left to R_k and R_k is left to R_i). With other words, the system of inequalities defining a subset $\mathbb{R}_q^{2n}$ can be inconsistent or infeasible.

Nevertheless, for our basic approach we suppose that we have a partition of $\mathbb{R}^{2n}$ into η non-empty convex subsets. For every subset $\mathbb{R}_q^{2n}$ we define a function

$\Gamma_q : \mathbb{R}_q^{2n} \to \mathbb{R}$ to be a Φ-function of Ω and $h(u)$, $u \in \mathbb{R}_q^{2n}$. Combining these functions we have

$$\Gamma(u) = \begin{cases} \Gamma_1(u), & \text{if } u \in \mathbb{R}_1^{2n}, \\ \Gamma_2(u), & \text{if } u \in \mathbb{R}_2^{2n}, \\ \vdots & \\ \Gamma_\eta(u), & \text{if } u \in R_\eta^{2n}. \end{cases} \tag{7}$$

Because of construction, if we identify a subset $\mathbb{R}_q^{2n}$ and a translation vector $u^* \in \mathbb{R}_q^{2n}$ with $\Gamma_q(u^*) \geq 0$ then the configuration $\Lambda(u^*)$ is a cover of Ω.

Notice, for numerical reasons we weaken the assumption to have a real partition of $\mathbb{R}^{2n}$ but maintain $\text{int}(\mathbb{R}_q^{2n}) \cap \mathbb{R}_p^{2n} = \emptyset$ for $q \neq p$ in order to have closed regions.

The relation between the Γ-function and Φ-functions can be used to construct Γ_q similar to the construction of Φ-functions. The essential difference is that only the principal structure of the configuration $h(u, w)$ is given, determined by $\mathbb{R}_q^{2n}$, but the translation vector u is still non-fixed.

In order to construct the function Γ_q for all $u \in \mathbb{R}_q^{2n}$ we consider first a convex subset $\Omega_k(0)$ $(k \in I_K)$ and a connected component $P_s(u, w)$ $(s \in I_\tau)$ of $P(u, w)$ and its complement $h_s(u, w)$. Because of (3), we can use a representation of $h_s(u, w)$ by means of convex subsets $C_{st}(u, w)$, $t \in I_{\lambda_s}$.

It is known [8], a Φ-function of $\Omega_k(v)$ and $C_{st}(u, w)$ has the form

$$\Phi_{kst}(w, v) = \max\{f_{kst}^j(w, v) : j \in I_{kst}\} \tag{8}$$

where the functions $f_{kst}^j(w, v)$ are affine-linear and their coefficients depend on u. The number $|I_{kst}|$ of such functions corresponds to the number of bounding sides of the polygonal set

$$\Omega_k(0) \oplus C_{st}(u, 0)$$

where $\Omega_k(0) \oplus C_{st}(u, 0)$ is the Minkovski sum of $\Omega_k(v)|_{v=0}$ and $C_{st}(u, w)|_{w=0}$. Hence, if $\Phi_{kst}(w, 0) \geq 0$ then $\text{int}(\Omega_k(0)) \cap C_{st}(u, w) = \emptyset$. Consequently, $\text{int}(\Omega_k(0)) \cap h_s(u, w) = \emptyset$ if $\Phi_{ks}(w, 0) \geq 0$ where

$$\Phi_{ks}(w, v) = \min_{t \in I_{n_s}} \Phi_{kst}(w, v). \tag{9}$$

Since $\text{int}(\Omega_k(0)) \cap h(u, w) = \emptyset$ if one of the connected components $h_s(u, w)$ of $h(u, w)$ fulfills $\text{int}(\Omega_k(0)) \cap h_s(u, w) = \emptyset$ we define

$$\Phi_k(w, v) = \max_{s \in I_\tau} \Phi_{ks}(w, v) = \max_{s \in I_\tau} \min_{t \in I_{n_s}} \Phi_{kst}(w, v). \tag{10}$$

Thus, $\text{int}(\Omega_k(0)) \cap h(u, w) = \emptyset$ if $\Phi_k(w, 0) \geq 0$.

Finally, since the target set Ω is a union of convex subsets we have $\text{int}(\Omega(0)) \cap h(u, w) = \emptyset$ if $\Phi(w, 0) \geq 0$ where

$$\Phi(w, v) = \min_{k \in I_K} \Phi_k(w, v) = \min_{k \in I_K} \max_{s \in I_\tau} \min_{t \in I_{n_s}} \max\{f_{kst}^j(w, v) : j \in I_{kst}\}. \tag{11}$$

We recall that formula (11) is derived for a subset $\mathbb{R}^{2n}_q$ and that the affine-linear functions f^j_{kst} depend on $u \in \mathbb{R}^{2n}_q$. In order to express this situation we set $v = 0$ and integrate w into u to obtain

$$\Gamma_q(u) = \min_{k \in I_K} \max_{s \in I_\tau} \min_{t \in I_{n_s}} \max\{\widetilde{f}^j_{kst}(u) : j \in I_{kst}\} \tag{12}$$

where $\widetilde{f}^j_{kst}(u)$, $j \in I_{kst}$ results from $f^j_{kst}(w - v, 0)$.

Consequently, because of construction, if

$$\max_{u \in \mathbb{R}^{2n}_q} \Gamma_q(u) < 0 \tag{13}$$

then the configuration of rectangles defining $\mathbb{R}^{2n}_q$ does not lead to any cover of the target set. The verification of criterion (13) can be very expensive because the max-term in (12) can lead to an exponential number of local extrema.

4 Mathematical model

Formally, a mathematical model of the covering problem under consideration can be formulated as the following optimization problem:

$$\text{Find } u^* \quad \text{with } \Gamma(u^*) = \max\{\Gamma(u) : u \in \mathbb{R}^{2n}\}. \tag{14}$$

But in fact we have to deal with the decision problem

$$\text{Find } u^* \in \mathbb{R}^{2n} \quad \text{with } \Gamma(u^*) \geq 0. \tag{15}$$

However, from the theoretical point of view, we can conclude: if

$$\max\{\Gamma(u) : u \in \mathbb{R}^{2n}\} < 0$$

then no cover exists for target region Ω and the family of rectangles $\Lambda(u)$.

On the other hand, if we identify a configuration, say $u^* \in \mathbb{R}^{2n}_q$, which yields a cover for Ω then, in general, there can be infinite many translation vectors $u \in \mathbb{R}^{2n}_q$ which also define a covering. In this case, our approach allows us to add some further criterion and to look for a best cover.

To make more clear, the numerical problems when solving (14) or (15) we state some basic characteristics of the two problems.

1. Objective function $\Gamma(u)$ is discontinuous.
2. Problems (14) and (15) can be separated into η independent subproblems.
3. The construction of the partition of $\mathbb{R}^{2n}$ can be realized on hands of a branching tree.
4. The evaluation of $\Gamma_q(u)$ for $u \in \mathbb{R}^{2n}_q$, based on Φ-functions, can also be realized along a searching tree. Different branches arise from different functions $\widetilde{f}^j_{kst}$ which define the maximum in the corresponding term.

5. Every node of the branching trees represents a system of inequalities for the (unknown) translation vector u.
6. The number of nodes is exponential. Therefore the verification that no cover exists is not possible in reasonable time for larger instances.
7. Problem (14) is multi-extremal and NP-hard [2, 4].

5 Solution strategy

In order to simplify the description we assume in the following that the target set Ω consists of only one component. Consequently, we need to find a translation vector $u \in \mathbb{R}^{2n}$ which defines a connected set $P(u) = \bigcup_{i=1}^{n} R_i(u_i)$. Therefore, $\tau = 1$. In this case, formula (12) simplifies to

$$\Gamma_q(u) = \min_{k \in I_K} \min_{t \in I_T} \max\{\widetilde{f}_{kt}^{j}(u) : j \in I_{kt}\} \tag{16}$$

where $\widetilde{f}_{kt}^{j}(u) = \widetilde{f}_{k1t}^{j}(u)$, $I_T = I_{n_1}$ and $I_{kt} = I_{k1t}$. It is obvious, if we find $u^* \in \mathbb{R}^{2n}$ such that for every $k \in I_K$ and every $t \in I_T$ there exists an index $j(k,t) \in I_{kt}$ with $\widetilde{f}_{kt}^{j(k,t)}(u) \geq 0$ then we get a cover.

In order to find these functions we have to apply a branching scheme. Let $L_l \subset I_K \times I_T$ with $|L_l| = l$ represent these index pairs of $I_K \times I_T$ in stage $l \in \{1, \ldots, K \cdot n_1\}$ for which the corresponding index j is fixed by $j = j(k,t)$. Furthermore, let

$$Q_l = \{(k_r, t_r, j_r) : r = 1, \ldots, l\} \quad \text{with } j_r = j(k_r, t_r),\ (k_r, t_r) \in L_l$$

represent a certain node of the branching tree.

To get in this way restrictions for u we consider level sets of Γ_q. For a non-empty subset $\mathbb{R}_q^{2n}$ and for $\chi \geq 0$ the set

$$\{u \in \mathbb{R}_q^{2n} : \Gamma_q(u) \geq \chi\} = \left\{u \in \mathbb{R}_q^{2n} : \min_{k \in I_K} \min_{t \in I_T} \max_{j \in I_{kt}} \widetilde{f}_{kt}^{j}(u) \geq \chi\right\}$$

which is called *level set of* Γ_q *to level* χ, contains translation vectors u which form covers.

The consideration of level sets allows a ranking of subproblems within the branching process when for certain subproblems, determined by Q_l, a linear optimization problem of the following form is applied:

$$\chi_l^* = \max\left\{\chi : \widetilde{f}_{kt}^{j(k,t)}(u) \geq \chi,\ u \in \mathrm{cl}(\mathbb{R}_q^{2n}),\ (k,t,j(k,t)) \in Q_l\right\} \tag{17}$$

where index $j(k,t) \in I_{kt}$ indicates that function $\widetilde{f}_{kt}^{j(k,t)}$ which yields the maximum in $\max_{j \in I_{kt}} \widetilde{f}_{kt}^{j}(u)$. For this scenario, let

$$W_l(\chi) = \{u \in \mathbb{R}^{2n} : \widetilde{f}_{kt}^{j}(u) \geq \chi,\ (k,t,j) \in Q_l\}.$$

Now the optimization problem (17) is

$$\chi_l^* = \max\{\chi : \mathrm{cl}(\mathbb{R}_q^{2n}) \cap W_l(\chi) \neq \emptyset\}.$$

Based on these ideas and the characteristics of the mathematical model (14), a strategy to solve the covering problem can be as follows:

1. First of all, a suitable starting configuration (starting point) $u^0 = (u_1^0, u_2^0, \ldots, u_n^0)$ has to be constructed by means of heuristics (cf. Sect. 8). u^0 does not yield, in general, a cover but determines the configuration of rectangles, i.e. $V_q = \mathbb{R}_q^{2n}$. If u^0 indeed defines a cover it remains the verification that this is true.
2. Construct the corresponding system of inequalities defining V_q.
3. In order to construct the system of inequalities corresponding to Γ_q-function a branching tree has to be introduced since the max-term in (16) requires the considerations of many situations. Along with the construction of the Γ_q-function a reduction of the set $W_l(0) \cap V_q$ of feasible translation vectors $u \in V_q$ is realised.
4. If no cover is found then information obtained in the searching process can be used to construct a new starting configuration.

In order to obtain a starting configuration u^0 several heuristics can be applied. We use a method based on the Voronoi polygons [6], a simple consecutive placement heuristic, and a heuristic which applies a partition of the smallest rectangle enveloping the target region into elementary rectangles.

6 The branching tree

According to formula (16), the choice of one index $j(k,t)$ from I_{kt} for every $k \in I_K$ and every $t \in I_T$ leads to another linear optimization problem of form

$$\chi^* = \max\left\{\chi : \widetilde{f}_{kt}^{j(k,t)}(u) \geq \chi,\ u \in \mathrm{cl}(\mathbb{R}_q^{2n}),\ k \in I_K,\ t \in I_T\right\}. \tag{18}$$

If its optimal value is greater than or equal to 0, or if there is known a feasible solution with value not smaller than 0 then a cover is found.

The total number of such systems of inequalities, and hence of optimization problems is therefore

$$\prod_{k \in I_K} \prod_{t \in I_T} |I_{kt}|. \tag{19}$$

In case that $\mathbb{R}_q^{2n} \neq \emptyset$, problem (18) is always solvable since box-constraints for $u = (u^1, \ldots, u^n) \in \mathbb{R}_q^{2n}$ can be derived by demanding $R_i(u^i) \cap R_0 \neq \emptyset$ where R_0 is the smallest axis-parallel rectangle containing Ω.

Since we have

$$\chi_l^* \geq \chi_{l+1}^* \quad \text{if } Q_l \subset Q_{l+1},$$

since $W_l(\chi) \supseteq W_{l+1}(\chi)$ for all χ, subproblems need not to be considered if the father problem has an optimal value less then 0.

In order to keep the branching tree thin it seems to be favourable to branch according to increasing cardinality of I_{kt}. Especially, if there are index sets with $|I_{kt}| = 1$ then every feasible solution (if any) of (18) has to fulfill the corresponding restrictions. Therefore, they should be used first. If all these inequalities, which define in fact the box constraints for every variable, are contained in L_l then problem (17) is solvable.

If for a certain subproblem determined by Q_l a real branching with respect to $(k_0, t_0) \notin L_l$ is necessary because of $|I_{k_0 t_0}| > 1$ then a ranking of the resulting subproblems,

$$Q_l^j = Q_l \cup \{(k_0, t_0, j)\}, \quad j = 1, \dots, |I_{k_0 t_0}|$$

is given by means of the optimal values according to (17). Since problems (17) with respect to Q_l and Q_l^j differ by only one additional restriction, the solution for Q_l^j can efficiently be done by the dual simplex method. In this way, restrictions $j \in I_{k_0 t_0}$ which do not reduce W_l can be identified easily.

The following fathoming rules can be applied to reduce the searching tree:

- *Rule 1* Let Q_r and Q_s with $r \neq s$ be two subproblem of problem Q and let $W_l(Q)$ denote the feasible region corresponding to (17). If $W_{l+1}(Q_r) \supseteq W_{l+1}(Q_s)$ then subproblem Q_s can be pruned since, if there is a cover in $W_{l+1}(Q_s)$ then it belongs also to $W_{l+1}(Q_r)$.
- *Rule 2* If there exists $r \in I_{kt}$ with $W_{l+1}(Q_r) = W_l(Q)$ then all subproblems Q_s, $s \neq r$ can be pruned because of rule 1.

7 Example

In this section we consider a small example in order to illustrate the proposed approach and to show some aspects.

Let the target region Ω consist of $K = 2$ convex subsets (Fig. 1)

$$\Omega = \Omega_1 \cup \Omega_2$$
$$\text{with } \Omega_1 = \text{conv}\{(0,0), (6,3), (0,3)\}, \ \Omega_2 = \text{conv}\{(0,3), (2,3), (0,5)\}.$$

Furthermore, let be given $n = 3$ rectangles

$$R_1 = R_2 = \{(x,y) : |x| \le 2, \ |y| \le 1\}, \qquad R_3(\alpha) = \{(x,y) : |x| \le \alpha, \ |y| \le \alpha\}$$

where the size of R_3 depends on $\alpha \in \mathbb{R}$ with $0 < \alpha < 3$. In fact, in the implementation of our approach the parameter α has to be fixed but its usage within this example allows the discussion of some further issues. As easily can be seen there is no cover if $\alpha < 1.5$. At first, suppose $\alpha = 2$.

Fig. 1 Target set Ω

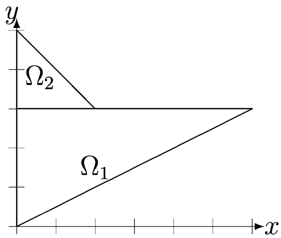
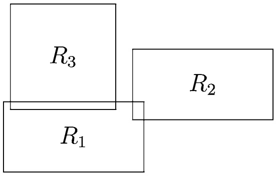


Fig. 2 First configuration



The first configuration which we consider is drawn in Fig. 2. That means, for the pair (i, j) of rectangles we have case (1.b) for pair $(1, 2)$, case (2.c) for $(1, 3)$ and case (0.a) for $(2, 3)$. The corresponding system of inequalities which defines subset $V_1 \subset \mathbb{R}^6$ is the following:

$$x_2 - x_1 \le 4, \qquad x_1 - x_2 \le 0, \qquad y_2 - y_1 \le 2, \qquad y_1 - y_2 \le 0,$$
$$x_1 - x_3 \le 2 - \alpha, \qquad x_3 - x_1 \le 2 - \alpha, \qquad y_1 - y_3 \le 1 - \alpha, \qquad y_3 - y_1 \le 1 + \alpha,$$
$$x_3 - x_2 \le -2 - \alpha, \qquad y_2 - y_3 \le \alpha - 1, \qquad y_3 - y_2 \le \alpha + 1.$$

Note that the relative position of R_2 and R_3 is specified by two additional restrictions in comparison to case (0.a). From these constraints it follows (necessary conditions) that only regions with maximal length size $2(a_1 + a_2) = 8$ and maximal width $2(b_1 + \alpha) = 2 + 2\alpha$ can be covered. From this, $\alpha \ge 1.5$ follows. Furthermore, from $a_3 \le a_1$ we have $\alpha \le 2$ for this configuration. Hence, only for $\alpha \in [1.5, 2]$ there can exist a cover of the target set having this configuration. We consider again $\alpha = 2$.

The complementary set $h(u)$ belonging to this configuration can be represented as union of eight polygonal sets $C_t(u)$, $t = 1, \ldots, 8$ with

$$C_1 = \{(x, y) : x \le x_1 - a_1\}, \qquad C_2 = \{(x, y) : y \ge y_3 + b_3\},$$
$$C_3 = \{(x, y) : x \ge x_2 + a_2\}, \qquad C_4 = \{(x, y) : y \le y_1 - b_1\},$$
$$C_5 = \{(x, y) : x \le x_3 - a_3, y \ge y_1 + b_1\},$$
$$C_6 = \{(x, y) : x \ge x_3 + a_3, y \ge y_2 + b_2\},$$
$$C_7 = \{(x, y) : x \ge x_1 + a_1, y \le y_2 - b_2\},$$
$$C_8 = \{(x, y) : x \ge x_3 + a_3, y \ge y_1 + b_1, x \le x_2 - a_2\}.$$

The convex subsets C_t, $t = 1, \ldots, 8$ are drawn in Fig. 3.

The construction of Φ-functions for Ω_1 and C_t yields the following functions $\widetilde{f}^j_{1t}$, $j \in I_{1t}$:

$$\widetilde{f}^1_{11}(u) = 2 - x_1, \qquad \widetilde{f}^1_{12}(u) = y_3 - 3 + \alpha,$$
$$\widetilde{f}^1_{13}(u) = x_2 - 4, \qquad \widetilde{f}^1_{14}(u) = 1 - y_1,$$
$$\widetilde{f}^1_{15}(u) = \alpha - x_3, \qquad \widetilde{f}^2_{15}(u) = y_1 - 2,$$
$$\widetilde{f}^1_{16}(u) = x_3 - 6 + \alpha, \qquad \widetilde{f}^2_{16}(u) = y_2 - 2,$$
$$\widetilde{f}^1_{17}(u) = x_1 - 4, \qquad \widetilde{f}^2_{17}(u) = 1 - y_2, \qquad \widetilde{f}^3_{17}(u) = x_1 - 2y_2 + 4,$$
$$\widetilde{f}^1_{18}(u) = x_3 - 6 + \alpha, \qquad \widetilde{f}^2_{18}(u) = y_1 - 2, \qquad \widetilde{f}^3_{18}(u) = 2 - x_2.$$

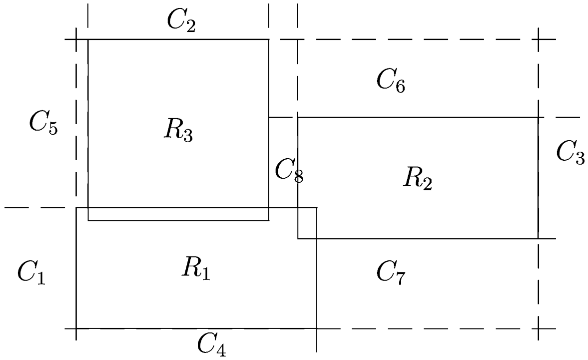


Fig. 3 Complementary set $h(u)$ of $P(u)$ and sets C_t, $t=1,\dots,8$

We obtain similar functions $\widetilde{f}_{2t}^{j}$, $j \in I_{2t}$ by the construction of Φ-functions for Ω_2 and C_t:

$$\widetilde{f}_{21}^{1}(u)=2-x_1, \quad \widetilde{f}_{22}^{1}(u)=y_3-5+\alpha, \quad \widetilde{f}_{23}^{1}(u)=x_2, \quad \widetilde{f}_{24}^{1}(u)=4-y_1,$$
$$\widetilde{f}_{25}^{1}(u)=\alpha-x_3, \quad \widetilde{f}_{25}^{2}(u)=y_1-4, \quad \widetilde{f}_{27}^{1}(u)=x_1, \quad \widetilde{f}_{27}^{2}(u)=4-y_2,$$
$$\widetilde{f}_{26}^{1}(u)=x_3-2+\alpha, \quad \widetilde{f}_{26}^{2}(u)=y_2-4, \quad \widetilde{f}_{26}^{3}(u)=x_3+y_2+\alpha-4,$$
$$\widetilde{f}_{28}^{1}(u)=x_3-2+\alpha, \quad \widetilde{f}_{28}^{2}(u)=y_1-4,$$
$$\widetilde{f}_{28}^{3}(u)=2-x_2, \quad \widetilde{f}_{28}^{4}(u)=x_3+y_1+\alpha-4.$$

Because $|I_{kt}|=1$ for $k=1,2$ and $t=1,\dots,4$ we use

$$L_8=\{(k,t): k=1,2,\ t=1,\dots,4\}$$

and obtain with $\chi=0$

$$W_8(0)=\{u\in V_1: x_1\le 2,\ y_3\ge 5-\alpha,\ x_2\ge 4,\ y_1\le 1\}.$$

Considering $(k,t)=(1,5)$ we obtain a contradiction when demanding $\widetilde{f}_{15}^{2}(u)=y_1-2\ge 0$, i.e. the linear optimization problem

$$\chi \to \max$$
$$\text{s.t.} \quad 2-x_1\ge\chi, \quad y_2+\alpha-5\ge\chi, \quad x_3+4\ge\chi, \quad 1-y_1\ge\chi, \quad y_1-2\ge\chi$$

has optimal value $\chi^*\le -0.5<0$. Hence, if there exists a solution u then it has to fulfill $\widetilde{f}_{15}^{1}(u)=\alpha-x_3\ge 0$. In a similar way, for $(k,t)=(1,6)$ we get a contradiction when demanding $\widetilde{f}_{16}^{1}(u)=x_3-6+\alpha\ge 0$ since $x_3+\alpha\le x_1+2\le 4$ because of demands in V_1. Therefore $\widetilde{f}_{16}^{2}(u)=y_2-2\ge 0$ has to hold true. In two of the three cases for $(k,t)=(1,7)$ contradictions arise, only $\widetilde{f}_{17}^{3}(u)=x_1-2y_2+4\ge 0$ does not lead to an empty set $W_{11}(0)$. Note that up to now no branching is necessary, only pruning of potential subproblems took place.

Finally, we find for $(k,t)=(1,8)$ that in every of the three cases contradictions (empty level sets for $\chi\ge 0$) result. Therefore we can conclude, there does not exists

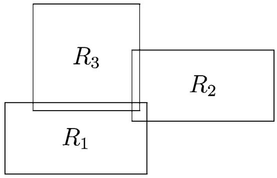


Fig. 4 Second configuration

a cover for configuration 1. Note that there was no need to consider Ω_2 to obtain this intermediate result.

It is remarkable, although theoretically 1728 terminal nodes of the branching tree can be constructed (number of different sets Q_l with $l = 16$) only 18 systems of inequalities with $l \leq 12$ are considered till termination. Moreover, incorporating the restriction $\chi \geq 0$ leads sometimes to empty sets which can be identified rapidly by the dual simplex method.

Moreover, the above termination of the branching process gives a hint that C_8 could be the causation.

For that reason we allow in configuration 2 (Fig. 4) the overlapping of R_2 with R_3. Therefore

$$x_3 - x_2 \geq -2 - \alpha, \qquad x_3 - x_2 \leq 2 - \alpha$$

is demanded instead of $x_3 - x_2 \leq -2 - \alpha$. We denote the resulting set by V_2. For this configuration $\alpha \geq 1.5$ has also to be. (We consider again $\alpha = 2$.)

The complementary set $h(u)$ to the second configuration can be represented as union of the polygonal sets $C_t(u)$, $t = 1, \dots, 7$. Starting with $Q_{11} = Q_8 \cup \{(1,5,1), (1,6,2), (1,7,3)\}$ we obtain the intermediate result

$$W_{11}(0) = \{u \in W_8(0) : \alpha \geq x_3, y_2 \geq 2, 2y_2 \leq x_1 + 4\}.$$

For $(k,t) = (2,5)$ we find that $y_1 \geq 4$ is impossible. Hence, $x_3 \leq \alpha$ has to be valid in any case. Furthermore, for $(k,t) = (2,7)$ we find that $y_2 \leq 4$ does not reduce the current set W; this restriction is always satisfied. Moreover, for $(k,t) = (2,6)$ the demand $y_2 \geq 4$ yields a contradiction so that only two subproblems arise. Thus, we obtain two systems of inequalities which all are feasible for $\chi = 0$. These are

$$\begin{array}{llll}
x_2 - x_1 \leq 4, & x_1 - x_2 \leq 0, & y_2 - y_1 \leq 2, & y_1 - y_2 \leq 0, \\
x_1 - x_3 \leq 2 - \alpha, & x_3 - x_1 \leq 2 - \alpha, & y_1 - y_3 \leq 1 - \alpha, & y_3 - y_1 \leq 1 + \alpha, \\
x_3 - x_2 \geq -2 - \alpha, & x_3 - x_2 \leq 2 - \alpha, & y_2 - y_3 \leq 1 - \alpha, & y_2 - y_3 \leq \alpha - 1, \\
2 - x_1 \geq \chi, & 1 - y_1 \geq \chi, & x_2 - 4 \geq \chi, & \alpha - 5 + y_3 \geq \chi, \\
\alpha - x_3 \geq \chi, & y_2 - 2 \geq \chi, & x_1 - 2y_2 + 4 \geq \chi, & \chi \geq 0, \\
\multicolumn{4}{l}{(x_3 - 2 + \alpha \geq \chi) \vee (x_3 + y_2 - 4 + \alpha \geq \chi).}
\end{array}$$

The first 12 inequalities define V_2 whereas the remaining result from the Φ-function for Ω_k and C_t for all k and t. Obviously, for every $\alpha \in [1.5, 3]$ one can separately verify the feasibility of the inequality systems, or because of dominance considerations, if one of the systems is consistent for $\alpha = 1.5$ then also for $\alpha \in (1.5, 3]$.

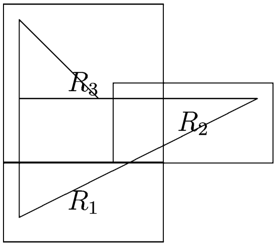


Fig. 5 Best covering for $\alpha = 2$

Every feasible solution of one of the systems of inequalities represents a cover of the target region. This fact allows to ask for a*best possible* cover. For instance, we can consider the optimization problem $\chi^* = \max \chi$ with respect to a certain configuration, or at all.

In order to obtain a best cover with respect to the chosen configuration the Hessian form of the inequalities should be used when solving problem (17), or equivalently, instead of inequality $ax + by + c \geq \chi$ the normalized inequality $ax + by + c \geq \sqrt{a^2 + b^2}\chi$ has to be taken. In case of $\alpha = 1.5$ one obtains $\chi^* = 0$ since the vertical position of R_1 and R_3 is uniquely determined.

For $\alpha = 2$ we obtain as maximal χ-value (with respect to this configuration) $\chi^* = 0.382$ and $u* = (1.618, 0.382, 4.382, 2.382, 1.618, 3.382)$ (Fig. 5). This solution is in the following sense optimal: the overall minimum of minimal distances between any point of the border of the union of rectangles and any point of the target set is maximal.

8 Numerical experiments

In this section we present some results of computational experiments. Hereby different forms of the target set as well as a varying number of rectangles are considered. The tests were done using Microsoft Windows 9x/2000/XP, IBM PC/AT, Pentium-166.

In order to get starting points for the search of a feasible configuration we apply three approaches depending on the shape of target region and sizes of rectangles. Algorithm 1 applies a straightforward consecutive placement of the rectangles according to a random sequence. Algorithm 2 places the rectangles in a random way on the grid points of a regular grid which covers the target region. Finally, in Algorithm 3 a modification of the Voronoi polygons method from [6] is applied which looks for the minimum radius needed to cover the target region with n identical circles. The latter method is in particular useful when the given rectangles are nearly squares and of same size. More detailed investigations are needed for selecting a best algorithm (in dependence of the input data) to get good starting points. Notice, more sophisticated algorithms can be developed but even pure Monte-Carlo placement is applicable.

In the following we give results of six examples. The detailed data information can be found in Appendix including the sizes of the rectangles and the coordinates (in a truncated form) of starting points and final solutions which always represent feasible coverings.

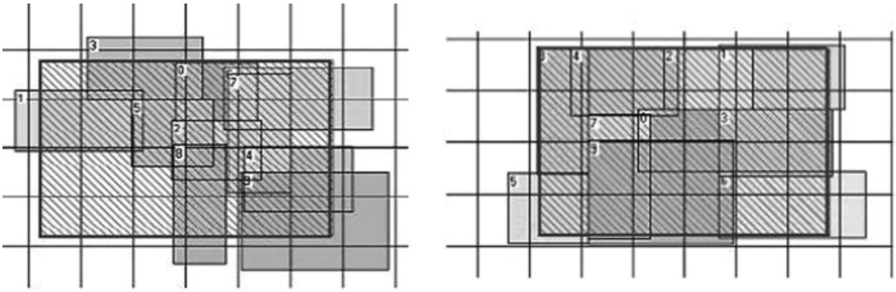


Fig. 6 Covering a rectangle with 10 rectangles: initial and final solution

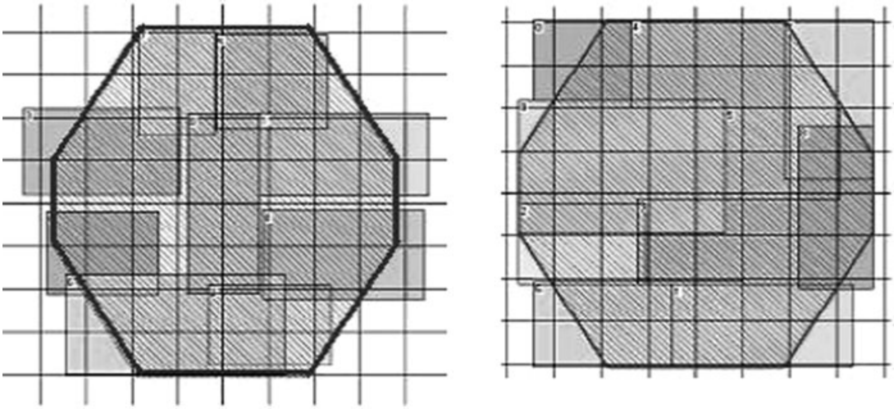


Fig. 7 Covering a convex polygon with 9 rectangles: initial and final solution

In *Example 1* the target set Ω is a rectangle defined by

$$\Omega = \mathrm{conv}\{(14,9), (-14,9), (-14,-9), (14,-9)\}$$

and 10 rectangles are available to cover Ω (cf. Table 1). Figure 6 shows the starting configuration and the final one which is a cover. The starting point was computed with Algorithm 2 and the total time to find the cover was 1 s.

Next, in *Example 2*, the target set Ω is a convex polygon (Fig. 7) defined by

$$\Omega = \mathrm{conv}\{(6,13), (-6,13), (-12,3), (-12,-3), \\ (-6,-13), (6,-13), (12,-3), (12,3)\}.$$

There are nine rectangles (cf. Table 2) to cover Ω. The starting point was computed with Algorithm 3 and the total time to find the cover was about 54 s.

In *Example 3* a non-convex polygon is considered as target region. It is the union of two rectangles (Fig. 8):

$$\Omega = \mathrm{conv}\{(18,0), (-9,14), (-9,-14)\} \cup \mathrm{conv}\{(-18,0), (9,-14), (9,14)\}.$$

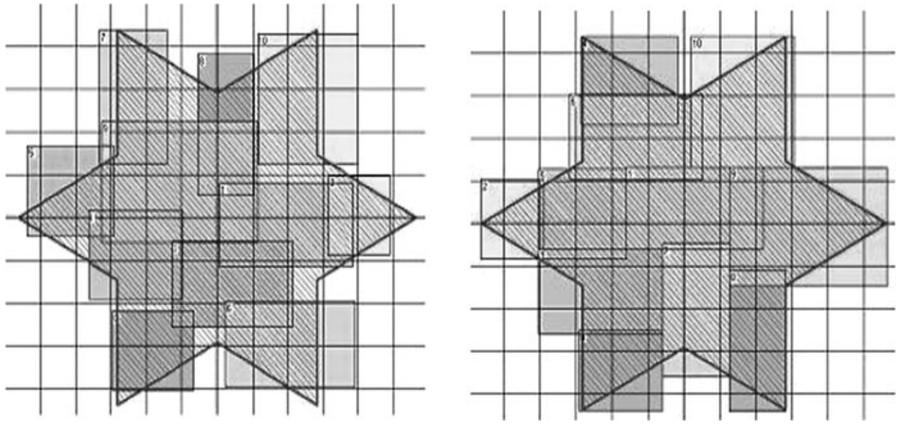


Fig. 8 Covering a non-convex polygon with 11 rectangles: initial and final solution

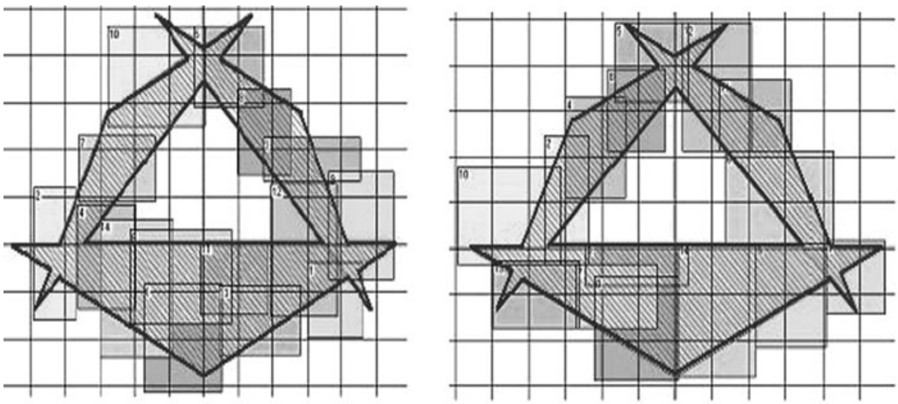


Fig. 9 Covering a two-connected polygonal set with 15 rectangles: initial and final solution

There are 11 rectangles available given in Table 3. Using Algorithm 3 the starting point was computed. The total consumption of time to find the cover was about 211 s.

With *Example 4* we consider as target set a two-connected polygonal set composed by three convex subsets:

$$\Omega = \bigcup_{r=1}^{3} \Omega_r$$

where $\Omega_1 = \mathrm{conv}\{(24.5, -7),\ (14, 14),\ (-7, 24.5)\}$, $\Omega_2 = \mathrm{conv}\{(-28, 0),$ $(0, -14.092), (28, 0)\}$, $\Omega_3 = \mathrm{conv}\{(-14, 14), (-24.5, -7), (7, 24.5)\}$. 15 rectangles are available given in Table 4. The starting point was computed with Algorithm 3 and the total time to find the cover was about 50 minutes (Fig. 9).

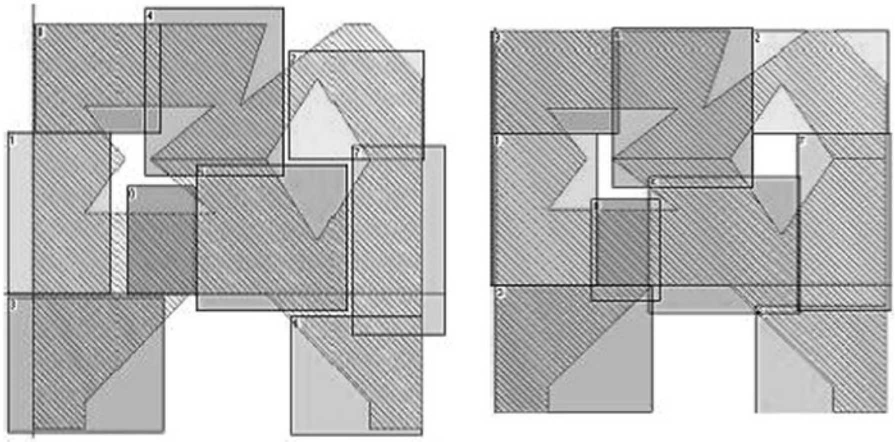


Fig. 10 Covering a two-connected polygonal set with 9 rectangles: initial and final solution

A more complex two-connected polygonal set forms the target region Ω in *Example 5* (Fig. 10):

$$\Omega = \bigcup_{k=1}^{11} \Omega_k$$

where $\Omega_r = \text{conv}\{(0,100), (0,70), (80,70), (90,100)\}$, $\Omega_2 = \text{conv}\{(0,70), (0,30), (20,30), (35,50), (20,70)\}$, $\Omega_3 = \text{conv}\{(0,30), (0,-50), (20,-50), (20,30)\}$, $\Omega_4 = \text{conv}\{(20,30), (20,-40), (60,0), (70,30)\}$, $\Omega_5 = \text{conv}\{(0,70), (45,50), (70,30), (90,50), (80,70)\}$, $\Omega_6 = \text{conv}\{(90,50), (70,30), (60,0), (90,0), (110,20)\}$, $\Omega_7 = \text{conv}\{(110,20), (90,0), (130,-40)\}$, $\Omega_8 = \text{conv}\{(150,50), (130,50), (110,20), (130,-40)\}$, $\Omega_9 = \text{conv}\{(150,50), (130,-40), (130,-50), (150,-50)\}$, $\Omega_{10} = \text{conv}\{(130,100), (120,100), (110,80), (130,50), (150,50), (150,80)\}$, $\Omega_{11} = \text{conv}\{(120,100), (80,70), (110,80), (90,50)\}$. The data of the nine rectangles are given in Table 5. The starting point was computed with Algorithm 3 and the total time to find the cover was about 7 s.

Finally, in *Example 6* we consider again a convex polygonal target region. (Fig. 11)

$$\Omega = \text{conv}\{(6,13),\ (-6,13),\ (-12,0),\ (-6,-13),\ (6,-13),\ (12,0)\}$$

but now 27 rectangles are given in Table 6. The starting point was computed with Algorithm 3 and the total time to find the cover was 13 min 43 s.

As can be seen by these examples the total time varies in a large range which is not surprisingly for decision problems. Moreover, since one has to work with a given time-limit it can happen that the computation is terminated without a final decision. In this respect further investigations are necessary, e.g. formulation of a mixed-integer linear programming model to obtain stronger bounds and possibly better starting points.

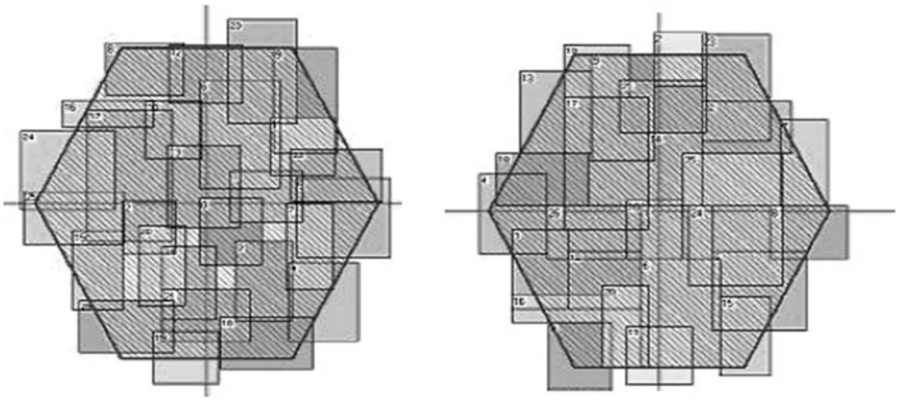


Fig. 11 Covering a convex polygon with 27 rectangles: initial and final solution

9 Conclusions

The proposed solution strategy allows the verification that a given configuration is a cover of the target set and, if not, to search for a cover maintaining the relative positions of the rectangles. Moreover, a basis is given to construct alternative configurations. Furthermore, the proposed approach operates with local and global optimization methods and allow in principle to apply parallel computing. In addition, we can apply the strategy to 3D covering problem with a family of parallelepipeds and it is possible to extend the approach to a covering problem with a family of convex polygons.

Future research has also to consider criteria to decide fast that there does not exist any cover for the target set using the given rectangles.

Appendix

Within the tables there are given the dimension parameters of the rectangles available to cover the target region as well as the coordinates x_i^0 and y_i^0 of the used starting point and the coordinates x_i^* and y_i^* of the obtained covering.

Table 1 Example 1: Input-data and coordinates of starting and final configuration

i	1	2	3	4	5	6	7	8	9	10
a_i	3.9	6.1	4.3	5.5	5.2	3.9	7.1	3.0	2.5	7
b_i	3.0	3.1	3.0	3.2	3.3	3.4	3.2	6.0	6.1	5
x_i^0	2.86	−10.16	2.86	3.910	10.68	−1.30	10.68	7.03	1.30	12.25
$y_i^{=}$	5.80	2.82	−0.16	8.12	−3.15	1.49	5.14	1.49	−5.80	−7.465
x_i^*	−0.60	9.40	2.34	8.80	−5.80	−13.20	10.40	−6.30	11.80	−2.30
y_i^*	0.20	6.30	6.20	0.00	5.90	−6.30	−6.00	−3.30	3.10	−4.80

Table 2 Example 2: Input-data and coordinates of starting and final configuration

i	1	2	3	4	5	6	7	8	9
a_i	3.9	5.9	4.3	5.5	5.7	3.9	7.7	2.7	2.5
b_i	3.1	3.1	3.0	3.2	3.3	3.5	3.7	4.0	6.7
x_i^0	−8.46	8.47	3.27	−8.49	8.46	3.37	−3.36	−3.26	0.02
y_i^0	−3.67	3.65	−9.10	3.81	−3.82	9.07	−9.05	9.04	−0.01
x_i^*	−5.15	−3.90	−3.00	6.51	−1.35	8.14	2.34	−7.93	−9.50
y_i^*	9.90	9.90	3.80	3.60	−2.49	−2.90	−9.30	−9.00	1.35

Table 3 Example 3: Input-data and coordinates of starting and final configuration

i	1	2	3	4	5	6	7	8	9	10	11
a_i	3.7	6.1	2.8	5.5	4.3	3.9	5.9	3.0	2.5	7.0	4.6
b_i	3.0	3.1	3.0	3.2	3.3	3.4	3.2	5.0	5.3	4.5	4.9
x_i^0	−5.79	6.27	12.8	1.31	−7.39	−13.2	6.63	−7.61	0.77	−3.41	8.23
y_i^0	−9.9566	−0.52	0.15	−4.97	−2.76	1.99	−9.4	9.02	6.94	2.63	8.87
x_i^*	−5.65	0.94	−15.25	−7.45	−4.84	−9.05	−4.25	1.04	6.54	11.05	5.22
y_i^*	−11.05	1.14	0.32	−5.15	10.75	0.72	6.52	−6.51	−8.83	0.250	9.149

Table 4 Example 4: Input-data and coordinates of starting and final configuration

i	1	2	3	4	5	6	7	8	9
a_i	5.7	4.0	3.0	7.0	4.1	5.0	7.3	5.5	3.9
b_i	5.7	4.0	7.0	2.3	5.5	4.3	5.0	3.5	4.5
x_i^0	3.00	19.0	−21.5	15.5	−13.9	3.62	−3.28	−12.4	8.83
y_i^0	−9.92	−5.91	−0.95	8.95	−1.30	18.8	−3.40	8.02	11.9
x_i^*	−5.28	25.4	−14.8	−5.17	−10.8	−3.17	14.0	−7.98	−5.25
y_i^*	−8.71	−3.10	5.32	−1.67	11.0	20.3	5.60	−5.17	15.1
i	10	11	12	13	14	15			
a_i	4.7	7.0	4.9	4.7	5.9	5.3			
b_i	5.7	5.3	3.9	7.0	3.7	7.3			
x_i^0	22.7	−6.77	4.48	14.4	8.11	−9.68			
y_i^0	1.97	17.7	−3.53	−0.60	−7.98	−4.64			
x_i^*	15.7	−22.7	10.9	5.70	−18.9	5.71			
y_i^*	−5.09	3.62	14.5	17.6	−4.97	−6.69			

Table 5 Example 5: Input-data and coordinates of starting and final configuration

i	1	2	3	4	5	6	7	8	9
a_i	13	20	26	30	27	29	25	18	24
b_i	20	30	20	25	31	27	22	35	20
x_i^0	49	10	125	20	70	92	125	141	25
y_i^0	20	30	70	−26	75	21	−30	20	80
x_i^*	50	19	125	30	72	88	125	134	23
y_i^*	14	30	80	−25	70	16	−30	25	80

Table 6 Example 6: Input-data and coordinates of starting and final configuration

i	1	2	3	4	5	6	7	8	9
a_i	2.2	2.53	1.87	1.98	2.42	3.3	2.805	1.65	2.75
b_i	2.75	2.2	2.2	3.3	3.3	3.3	4.51	3.63	2.2
x_i^0	3.89	−1.25	9.55	−3.16	2.29	−9.36	7.19	9.10	3.97
y_i^0	11.0	−6.66	−1.25	−5.25	5.81	−1.19	−3.68	2.35	−6.50
x_i^*	5.57	1.21	1.48	−8.46	−10.3	7.04	1.49	10.2	10.5
y_i^*	−12.0	−1.82	12.6	−8.461	−10.3	7.04	1.49	4.00	−1.82

i	10	11	12	13	14	15	16	17	18
a_i	2.2	1.98	2.31	2.585	2.53	1.87	1.76	3.19	2.97
b_i	4.4	2.42	2.42	2.42	3.41	3.08	3.3	1.1	5.5
x_i^0	−9.74	6.70	−4.02	−0.21	−7.66	1.66	6.86	−5.40	8.19
y_i^0	2.80	4.69	−2.02	1.44	−5.65	−2.35	8.60	2.28	−8.20
x_i^*	−2.58	−0.33	−0.00	−3.89	−7.31	1.10	6.05	−7.25	−3.73
y_i^*	8.64	−1.60	−12.0	−5.72	8.18	3.4	−10.4	−8.15	3.96

i	19	20	21	22	23	24	25	26	27
a_i	3.245	2.31	1.65	3.08	3.19	2.42	3.3	3.52	3.3
b_i	2.2	2.2	3.41	2.2	2.2	4.4	3.3	2.2	2.2
x_i^0	4.18	−0.05	−0.08	−2.28	−4.35	−6.95	−1.48	−5.62	4.14
y_i^0	0.58	−9.50	10.7	6.11	11.2	7.45	−12.9	−10.3	−11.7
x_i^*	−8.28	−4.39	−2.41	0.27	6.16	5.39	5.38	5.16	−4.61
y_i^*	2.57	11.6	−9.63	8.73	6.97	10.2	−2.92	2.57	−1.82

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