Points:

$$\begin{aligned} p_i &= (x_i, y_i) \\ p_1 &= (0,0) \end{aligned} \tag{1}$$

If p_i are not co-linear and the ellipse with form:

$$x^2 + qy^2 = a^2 b^2 \tag{2}$$

can be determined by three points:

$$\frac{(x-x_1)(x-x_2)+q(y-y_1)(y-y_2)}{(y-y_1)(x-x_2)-(y-y_2)(x-x_1)} = \frac{(x_3-x_1)(x_3-x_2)+q(y_3-y_1)(y_3-y_2)}{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)} \tag{3}$$

q is fixed, we want to find θ, let

$$M(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

Now, lets define

$$u_i = M(\theta) p_i \tag{5}$$

we can rearrange 3

$$\frac{(X-u_1)^t (X-u_2)}{det(X-u_1, X-u_2)} = \frac{(u_3-u_1)^t (u_3-u_2)}{det(u_3-u_1, u_3-u_2)} \tag{6}$$

Analyzing the second part:

$$\begin{aligned} \frac{(u_3-u_1)^t (u_3-u_2)}{det(u_3-u_1, u_3-u_2)} &= \frac{u_3^t (u_3-u_2)}{det(u_3, u_3-u_2)} = \\ \frac{p_3^t M(\theta)^t (M(\theta)(p_3-p_2))}{det(u_3, u_3-u_2)} &= \frac{p_3^t (p_3-p_2)}{det(u_3, u_3-u_2)} \end{aligned} \tag{7}$$

Now looking at $det(u_3, u_3-u_2)$

$$\begin{aligned} det(u_3, u_3-u_2) = det(M(\theta)v, M(\theta)w) &= \\ \begin{vmatrix} M(\theta)_1 v & M(\theta)_2 v \\ M(\theta)_1 w & M(\theta)_2 w \end{vmatrix} &= \end{aligned} \tag{8}$$

$$\begin{aligned} (v_x \cos\theta - v_y \sin\theta)(w_x \sin\theta + w_y \cos\theta) - (\sin\theta v_x + \cos\theta v_y)(w_x \cos\theta - w_y \sin\theta) &= \\ \cos^2\theta (v_x w_y - v_y w_x) + \sin^2\theta(-v_y w_x + v_x w_y) + \sin\theta\cos\theta(v_x w_x - v_y w_y - v_x w_x + v_y w_y) &= \\ v_x w_y (\cos^2\theta + \sin^2\theta) - v_y w_x (\cos^2\theta + \sin^2\theta) &= \\ v_x w_y - v_y w_x = det(v, w) = det(u_3, u_3-u_2) &= det(p_3, p_3-p_2) \end{aligned}$$

Then, the second part is:

$$\frac{p_3^t(p_3 - p_2)}{det(p_3, p_3 - p_2)} \tag{9}$$

The main equation is:

$$\frac{(X - u_1)^t(X - u_2)}{det(X - u_1, X - u_2)} = \frac{p_3^t(p_3 - p_2)}{det(p_3, p_3 - p_2)} \tag{10}$$

Looking at the first part:

$$(X - u_1)^t(X - u_2) = X^t(X - u_2) = X^tX - X^tM(\theta)p_2 =$$
$$x^2 + qy^2 - x(p_{2x}\cos\theta - p_{2y}\sin\theta) - qy(p_{2x}\sin\theta + p_{2y}\cos\theta)$$

and

$$det(X, X - u_2) = \begin{vmatrix} x & y \\ x - M(\theta)_1 p_2 & y - M(\theta)_2 p_2 \end{vmatrix} =$$
$$\begin{vmatrix} x & y \\ x - (p_{2x}\cos\theta - p_{2y}\sin\theta) & y - (p_{2x}\sin\theta + p_{2y}\cos\theta) \end{vmatrix} =$$
$$y(p_{2x}\cos\theta - p_{2y}\sin\theta) - x(p_{2x}\sin\theta + p_{2y}\cos\theta)$$

Making:

$$\gamma = p_{2x}\cos\theta - p_{2y}\sin\theta \tag{11}$$
$$\delta = p_{2x}\sin\theta + p_{2y}\cos\theta \tag{12}$$
$$D = \frac{p_3^t(p_3 - p_2)}{det(p_3, p_3 - p_2)} \tag{13}$$

Then, we get:

$$\frac{(X - u_1)^t(X - u_2)}{det(X - u_1, X - u_2)} = \frac{x^2 + qy^2 - x\gamma - qy\delta}{y\gamma - x\delta} = D \tag{14}$$

Rearranging

$$x^2 + qy^2 - x(\gamma - D\delta) - qy(\delta + \frac{D\gamma}{q}) = 0$$
$$(x - \frac{\gamma - D\delta}{2})^2 + q(y - \frac{q\delta + D\gamma}{2q})^2 - \frac{(\gamma - D\delta)^2}{4} - \frac{(q\delta + D\gamma)^2}{4q} = 0$$

Then, as q is fixed, we need to impose:

$$\frac{(\gamma - D\delta)^2}{4} + \frac{(q\delta + D\gamma)^2}{4q} = 1$$

$$q(\gamma^2 - 2D\gamma\delta + D^2\delta^2) + q^2\delta^2 + 2q\delta D\gamma + D^2\gamma^2 = 4q =$$

$$q\gamma^2 + qD^2\delta^2 + q^2\delta^2 + D^2\gamma^2 = 4q$$

$$\gamma^2(q + D^2) + \delta^2(qD^2 + q^2) = 4q$$

Then,

$$\gamma^2 = p_{2x}^2 \cos^2\theta - p_{2x}p_{2y}\sin 2\theta + p_{2y}^2 \sin^2\theta \tag{15}$$
$$\delta^2 = p_{2x}^2 \sin^2\theta + p_{2x}p_{2y}\sin 2\theta + p_{2y}^2 \cos^2\theta \tag{16}$$

We have

$$\begin{aligned} \cos^2\theta((q + D^2)p_{2x}^2 + (qD^2 + q^2)p_{2y}^2) + \\ \sin^2\theta((q + D^2)p_{2y}^2 + (qD^2 + q^2)p_{2x}^2)) + \\ \sin 2\theta((qD^2 + q^2)p_{2x}p_{2y} - (q + D^2)p_{2x}p_{2y}) = 4q \end{aligned}$$

Which can be written as

$$A\cos^2\theta + 2B\sin\theta\cos\theta + C\sin^2\theta = D \tag{17}$$
$$A, B, C, D > 0 \tag{18}$$

Which can be rearranged as:

$$\frac{A - C}{2}\cos(2\theta) + B\sin(2\theta) + \frac{A + C - 2D}{2} = 0$$

Rewritting everything (keep up!):

$$\hat{A}\cos(2\theta) + \hat{B}\sin(2\theta) + \hat{C} = 0$$

Then,

$$\hat{A}\cos(t) + \hat{B}\sin(t) + \hat{C} = \hat{C} + \sqrt{\hat{A}^2 + \hat{B}^2}\sin(t + \alpha) = 0 \tag{19}$$
$$t + \alpha = \arcsin\frac{-\hat{C}}{\sqrt{\hat{A}^2 + \hat{B}^2}} \tag{20}$$

where

$$\tan\alpha = \frac{\hat{A}}{\hat{B}} \Rightarrow \alpha = \arctan\frac{\hat{A}}{\hat{B}}$$

Then, finally

$$\theta = \arcsin \frac{-\hat{C}}{2\sqrt{\hat{A}^2 + \hat{B}^2}} - \frac{1}{2} \arctan \frac{\hat{A}}{\hat{B}} \tag{21}$$

Similirarly we can achieve the other result:

$$\theta = \arccos \frac{-\hat{C}}{2\sqrt{\hat{A}^2 + \hat{B}^2}} + \frac{1}{2} \arctan \frac{\hat{A}}{\hat{B}} \tag{22}$$