

Algorithms for Planar Maximum Covering Location by Ellipses Problems^{*}

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Abstract

Planar Maximum Covering Location by Ellipses is an optimization problem where one wants to place fixed shape ellipses on the plane to cover demand points maximizing a function depending on the value of covered points. We propose new exact algorithms for two versions of this problem, one where the ellipses have to be parallel to the coordinate axis, and another where they can be freely rotated. Besides finding optimal solutions for previously published instances, including the ones where no optimal solution was known, both algorithms proposed by us were able to obtain optimal solutions for some new larger instances having with up to seven hundred demand points and five ellipses.

Keywords: Optimization, Covering, Combinatorial Optimization

1. Introduction

The Planar Maximum Covering Location Problem (PMCLP) was first introduced in [1], and can be seen as a category of problems where the coverage of a demand set, a collection of subsets of \mathbb{R}^2 , is to be maximized by determining the location of facilities in \mathbb{R}^2 , with coverage being determined by a distance function. In [1], methods for Euclidean and Rectilinear distances versions of the problem were proposed. In [2, 3], exact algorithms for the Euclidean PMCLP with only one facility are proposed; and in [4] an approximation algorithm is proposed for the version with multiple unit disks as facilities. A property of the Euclidean PMCLP, which is utilized in the algorithms developed in [2, 3, 4], and in the method proposed in [1], is that there is an optimal solution which every facility is located in the demand points, or in the intersection of two circles centered at two demand points; we will prove a similar property for ellipses in our work.

It is fair to say that PMCLP with elliptical coverage has not been vastly studied as only two articles have been found on it. In [5], a mixed-integer non-linear programming method was proposed as a first approach to the problem.

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For some instances, the method took too long and did not find an optimal solution. Because of that, a heuristic method was developed using a technique called Simulated Annealing, which was able to obtain solutions for the instances proposed in that study. The problem was further explored in [6], which introduced the version where the ellipses can be freely rotated, to which an exact and a heuristic method was proposed, and developed a new method for the axis-parallel version of the problem, which was able to obtain optimal solutions for instances that the method proposed by [5] could not. The exact method for the version with rotation could not obtain optimal solutions within a predefined time limit for several instances, the heuristic method though returned solutions for every instance, and impressively enough, obtained optimal solutions for every verifiable instance.

We study two versions of PMCLP with elliptical coverage facilities in this work. For both of them, each ellipse is defined to have a fixed shape and an undefined location, which is part of the solution. In the first version, introduced in [5], all the ellipses are restricted to be axis-parallel, while in the second version, introduced in [6], this constraint is dropped, and all the ellipses can be freely rotated. The first version will be referred to as Planar Maximum Covering Location by Ellipses Problem (MCE) and the second one as Planar Maximum Covering Location by Ellipses with Rotation Problem (MCER).

2. Problem Definition

An instance of MCE and MCER is given by n demand points $\mathcal{P} = \{p_1, \dots, p_n\}$, $p_j \in \mathbb{R}^2$; n weights $\mathcal{W} = \{w_1, \dots, w_n\}$, with $w_j \in \mathbb{R}$, $w_j > 0$ being the weight of the j -th point; and m shape parameters $\mathcal{R} = \{(a_1, b_1), \dots, (a_m, b_m)\}$, with (a_j, b_j) being the semi-major and semi-minor of the j -th ellipse, with $a_j > b_j$. We define a list of functions $\mathcal{E} = \{E_1, \dots, E_m\}$ representing the coverage area of each facility. For MCE $E_j: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined as

$$E_j(q) = \{p \in \mathbb{R}^2: (p_x - q_x)^2/a_j^2 + (p_y - q_y)^2/b_j^2 \leq 1\}. \quad (1)$$

For MCER, we define $E_j: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \times [0, \pi)$ as

$$E_j(q, \theta) = \left\{ p \in \mathbb{R}^2: \left\| \begin{pmatrix} \cos \theta/a_j & \sin \theta/a_j \\ \sin \theta/b_j & -\cos \theta/b_j \end{pmatrix} \begin{pmatrix} p_x - q_x \\ p_y - q_y \end{pmatrix} \right\|_2 \leq 1 \right\}. \quad (2)$$

Let $w: A \subset \mathcal{P} \rightarrow \mathbb{R}$ be a function which takes a subset of the demand set and returns the sum of the weights of the points in A . Then, we define MCE as the optimization problem

$$\max_{q_1, \dots, q_m} \sum_{j=1}^m w(\mathcal{P} \cap E_j(q_j)), \quad (3)$$

and similarly MCER as

$$\max_{(q_1, \theta_1), \dots, (q_m, \theta_m)} \sum_{j=1}^m w(\mathcal{P} \cap E_j(q_j, \theta_j)). \quad (4)$$

To make the notation more clear, we denote an instance of MCE or MCER as the tuple $(\mathcal{P}, \mathcal{W}, \mathcal{R})$, and a solution of MCE as $Q := (q_1, \dots, q_m)$, and a solution of MCER as $Q := ((q_1, \theta_1); \dots; (q_m, \theta_m))$. We also use ∂ as the boundary operator, for example, $\partial E_1(q_1)$ denotes an ellipse with shape parameters (a_1, b_1) centered at q_1 given an instance of MCE.

3. An algorithm for MCE

We will develop a method which is similar to the one developed in [2] for only one euclidean disk, and the exact algorithm developed for multiple Euclidean disks in [4]. We first describe a Candidate List set (CLS) for each facility, which is finite set of possible locations for each ellipse, which we use to converting MCE into a discrete optimization problem. We, then prove that using the possible solutions obtained from the combination of every ellipse's CLS an optimal solution can be obtained.

Let $(\mathcal{P}, \mathcal{W}, \mathcal{R})$ be an instance of MCE, then for each $j = 1 \dots m$, consider n ellipses with shape parameters (a_j, b_j) centered at each one of the points in \mathcal{P} . If we have $q_j \in \mathbb{R}^2$ in a subset of the coverage areas of those ellipses, then q_j is a solution of MCE covering the centers of those ellipses. In other words, if $q_j \in \mathbb{R}^2$, and $X \subset \{1, \dots, n\}$, $X \neq \emptyset$, such that $q_j \in \cap_{i \in X} E_j(p_i)$, then $\mathcal{P} \cap E_j(q_j) = \{p_i : i \in X\}$. From that observation, we can constrain each q_j to be in $\cap_{i \in X} E_j(p_i)$, for some $X \subset \{1, \dots, n\}$, $X \neq \emptyset$.

In [7], an algorithm is proposed for the problem of determining the intersection of disks of fixed radii from a strictly convex normed plane. We say that $(\mathbb{R}^2, \|\cdot\|)$ is a strictly convex normed plane if the unit disk given by the norm $\|\cdot\|$ is strictly convex. Note that for any ellipse, there is a strictly convex normed plane whose unit circle is that ellipse. For that reason, we state some results from [7] here, which we use on the development of an algorithm for MCE.

Let $(\mathbb{R}^2, \|\cdot\|)$ be a strictly convex normed plane, and $\mathcal{D} = \{D_1, \dots, D_n\}$ be a set of n unit disks in that space centered at different points, with the condition that $\cap_{i=1}^n D_i \neq \emptyset$. In [7], an algorithm was developed to construct this intersection in $\mathcal{O}(n \lg n)$, some of its preliminary results are:

- $\partial \cap_{i=1}^n D_i$ is formed by arcs of $\partial D_1, \dots, \partial D_n$.
- The vertices of $\partial \cap_{i=1}^n D_i$ is contained in the set of pairwise-intersection of the circles $\partial D_1, \dots, \partial D_n$.
- $|\partial D_i \cap \partial D_j| \leq 2$.

Based on those, we introduce the next definition for the k -th ellipse's CLS, which we refer to as S_k .

Definition 1. Given an instance of MCE, for any $k \in \{1, \dots, m\}$, we define the CLS for the k -th ellipse as

$$S_k = \bigcup_{1 \leq i < j \leq n} \partial E_k(p_i) \cap \partial E_k(p_j) \bigcup \mathcal{P}. \quad (5)$$

The set of solutions S_k , can be computed in $\mathcal{O}(n^2)$ as determining the intersections between two axis-parallel ellipses can be done analytically.

Lemma 1. *Given an instance of MCE, and S_1, \dots, S_m as defined by 1, then the set $\Omega = \{(q_1, \dots, q_m) : \text{for all } q_k \in S_k\}$ contains an optimal solution of MCE and $|\Omega| \leq n^{2m}$.*

PROOF. Let Q^* be an optimal solution of MCE for the given instance. Then, we are going to prove that there exists $Q' \in \Omega$, which is also optimal.

For each $k = 1, \dots, m$, let $X_k = \{p_i \in \mathcal{P} : p_i \in E_k(q_k^*)\}$.

If $|X_k| = 0$, then any $q_k \in S_k$ makes $X_k \subset \mathcal{P} \cap E_k(q_k)$.

if $|X_k| = 1$, then there is at least one element in S_k that makes $X_k \subset \mathcal{P} \cap E_k(q_k)$.

if $|X_k| > 1$, then let $Y_k = \cap_{p \in X_k} E_k(p)$, by the results of [7], we have that the boundary of Y_k has vertices in the pairwise intersections of $\{\partial E_k(p) : p \in X_k\}$. Therefore, at least one vertex of Y_k is in S_k , and any of those vertices produce a solution that covers at least the same points covered by Q^* .

Lastly, we have that $|S_k| \leq 2\binom{n}{2} + n = n(n+1)/2 \leq n^2$ hence $|\Omega| \leq n^{2m}$.

With all this in hand, we can go ahead and define an algorithm for MCE.

Algorithm 1 Algorithm for MCE

Input: A set of points $\mathcal{P} = \{p_1, \dots, p_n\}$, a list of weights $\mathcal{W} = \{w_1, \dots, w_n\}$, and a list of shape parameters $\mathcal{R} = \{(a_1, b_1), \dots, (a_m, b_m)\}$.

Output: An optimal solution for MCE.

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1: procedure  $MCE(\mathcal{P}, \mathcal{W}, \mathcal{R})$ 
2:   return  $MCE_{bt}(\mathcal{P}, \mathcal{W}, \mathcal{R}, 1)$ 
3: end procedure
4:
5: procedure  $MCE_{bt}(Z, \mathcal{W}, \mathcal{R}, j)$ 
6:   if  $j = m + 1$  then
7:     return 0
8:   end if
9:    $(q_j^*, \dots, q_m^*) \leftarrow (0, \dots, 0)$ 
10:  Let  $S_j$  be the CLS for the  $j$ -th ellipse as defined by 1.
11:  for  $q_j \in S_j$  do
12:     $Cov \leftarrow \mathcal{P} \cap E_j(q_j)$ 
13:     $(q_{j+1}, \dots, q_m) \leftarrow MCE_{bt}(Z \setminus Cov, \mathcal{W}, \mathcal{R}, j + 1)$ 
14:    if  $w(\cup_{k=j}^m Z \cap E_k(q_k)) > w(\cup_{k=j}^m Z \cap E_k(q_k^*))$  then
15:       $(q_j^*, \dots, q_m^*) \leftarrow (q_j, \dots, q_m)$ 
16:    end if
17:  end for
18:  return  $(q_j^*, \dots, q_m^*)$ 
19: end procedure

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1 can be proved to have a runtime complexity of $\mathcal{O}(mn^{2m+1})$.

4. Determining every location of an ellipse given its shape and three points

In this section, we introduce the problem of determining every location, here defined as the center and angle of rotation, of an ellipse with fixed shape parameters, such that it contains three given points. This problem comes up in the development of an algorithm for MCER in the next section. It is important to point out that no prior studies were found on it, or even on related problems. We propose an algorithm for it that involves determining the eigenvalues of a 6×6 complex matrix. We also analyze its efficiency in terms of numerical accuracy and display some solutions that it was able to obtain.

4.1. Problem definition

Given the shape parameters of an ellipse (a, b) , $a > b > 0$, and three points $u, v, w \in \mathbb{R}^2$, let $E: \mathbb{R}^2 \times [0, \pi) \rightarrow \mathbb{R}^2$ be the coverage region of an ellipse with shape parameters (a, b) , we refer to the problem of obtaining $(q, \theta) \in \mathbb{R}^2 \times [0, \pi)$, such that $\{u, v, w\} \subset \partial E(q, \theta)$ as Ellipse by Three Points Problem (E3P). Because of its application here in our work, we are only interested in a method that can obtain every solution of E3P.

4.2. Transforming E3P into a circle problem

Initially, E3P is a problem with three unknown variables: (q_x, q_y) , and θ . However, we can reduce this number to only one, as it will be shown, it is possible to obtain q from θ . Let us assume that point u is at the origin. If it is not, a simple translation by $-u$ applied to the three points can be made to put u at the origin. Assume as well that (q, θ) is a solution of E3P.

Applying a rotation of $-\theta$ to the coordinate system makes the ellipse in the original solution become axis-parallel. Then, that ellipse can be transformed into a circle of radius b by squeezing the x -axis by b/a . This two-step transformation can be written as a function $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as

$$\varphi(p, \theta) = \begin{bmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}. \quad (6)$$

An example of this transformation can be seen in Figure 1. As φ^{-1} is well-defined, instead of solving E3P, we can work with the univariate problem of determining an angle of rotation $\theta \in [0, \pi)$ that makes the triangle with vertices $\varphi(u, \theta), \varphi(v, \theta), \varphi(w, \theta)$ be circumscribed in a circle of radius b . To make the notation less cluttered, we denote by $\Lambda(\theta)$ the triangle with vertices $\varphi(u, \theta), \varphi(v, \theta), \varphi(w, \theta)$.

Let $A(\theta)$ denote the area of the triangle $\Lambda(\theta)$. Using the formula given in [8, p. 189] for the radius of the circumscribed circle of a triangle, we can obtain a function $\xi: [0, \pi) \rightarrow \mathbb{R}$ given by

$$\xi(\theta) = 16b^2 A(\theta)^2 - \|\varphi(v, \theta)\|_2^2 \|\varphi(w, \theta)\|_2^2 \|\varphi(v, \theta) - \varphi(w, \theta)\|_2^2, \quad (7)$$

whose roots determine solutions of E3P.

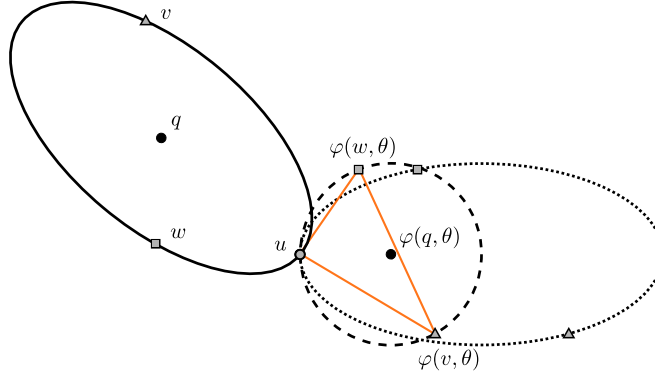


Figure 1: Transforming a solution of E3P into a solution of the circumradius problem.

Lemma 2. *E3P has at most six solutions.*

PROOF. The first thing to notice is that ξ is a real trigonometric polynomial of degree 6. Its term of highest degree is the multiplication of the norms $\|\varphi(v, \theta)\|_2^2 \|\varphi(w, \theta)\|_2^2 \|\varphi(v, \theta) - \varphi(w, \theta)\|_2^2$. In [9, p. 150], where a definition of real trigonometric polynomial is also given, it is stated that a n -degree real trigonometric polynomial can have up to $2n$ roots in $[0, 2\pi)$. Therefore, E3P has at most 12 solutions in $[0, 2\pi]$. Half of these solutions, though, are duplicated as ellipses are symmetric to their axis.

4.3. Converting ξ into a polynomial

In [10, p. 195], a theorem is presented stating that for every univariate polynomial of degree n , there exists a companion matrix which is a $n \times n$ matrix, such that its eigenvalues are the zeros of that polynomial. Finding every eigenvalue of a matrix can be done using the QR algorithm, which runs in $\mathcal{O}(n^3)$ and uses $\mathcal{O}(n^2)$ memory (a very complete introduction to it can be found in [11]). Therefore, finding the roots of a univariate polynomial of degree n can be done in $\mathcal{O}(n^3)$. Because of that, we use an approach published in [12] to convert real trigonometric polynomials into complex polynomials to convert ξ into a complex polynomial of degree 12.

Replacing $\cos \theta$ and $\sin \theta$ by their identities defined as follows

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad (8)$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \quad (9)$$

It is possible to show that with that substitution, plus changing the variable to $z = e^{i\theta}$, we obtain the following function $g : \mathbb{S} \mapsto \mathbb{C}$, with \mathbb{S} being the unit complex circle ($\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$), and $g(z) = \sum_{k=0}^{12} c_k z^{k-6}$. Multiplying g by z^6 , and extending the domain to \mathbb{C} , we obtain the degree-12 complex polynomial $h(z) = z^6 g(z)$.

We can cut in half the degree of the complex polynomial by observing that $\text{angle}(-z) = \pi + \text{angle}(z)$, which implies, because of the symmetry of ellipses, that $g(z) = g(-z)$. Therefore, every odd degree coefficient of h is zero. Using the substitution $y = z^2$, we obtain a degree-6 complex polynomial $f(y) = \sum_{k=0}^6 c_{2k}y^k$. Then, from the roots of f we obtain the set of solutions of E3P: $\{(u + \varphi^{-1}(\hat{q}, \text{angle}(\hat{y})/2), \text{angle}(\hat{y})/2) : f(\hat{y}) = 0\}$.

5. An Algorithm for MCER

6. References

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