Finite difference scheme for a Kirchoff thin plate with free edges

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The vibrations in a thin, isotropic, homogeneous plate can be described as

$$\frac{\partial^2 u}{\partial t^2} = -\kappa^2 \nabla^2 \nabla^2 u$$

where

$$\kappa = \frac{EH^2}{12\rho(1-\nu^2)}$$

where E is the Youngs modulus of the material, H is the thickness of the plate, ρ is its density and ν is its Poisson ratio.

For free edges, we have the boundary conditions (along a left-hand edge):

$$\frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\frac{\partial^3 u}{\partial x^3} + (2-\nu)\frac{\partial^3 u}{\partial x y^2} = 0$$

while at a corner

$$\frac{\partial^2 u}{\partial x y} = 0$$

Boundary conditions for other edge orientations can be inferred by symmetry.

Using the notation ∂_x, ∂_{xx} etc to mean the second order, central finite difference approximation of $\frac{\partial}{\partial x}$, $\frac{\partial^2}{\partial x^2}$ etc. and ∂_{x+} and ∂_{x-} to mean the second order, forward and backward finite difference respectively, we can express the thin plate equation as a finite difference scheme:

$$\partial_{tt} u = -\kappa(\partial_{xx} + \partial_{yy})(\partial_{xx} + \partial_{yy})u$$

The boundary conditions along a left-hand edge can be expressed

$$\partial_{xx} u + \nu \partial_{yy} u = 0 \tag{1}$$

$$\partial_{x-}(\partial_{xx} u + (2-\nu)\partial_{yy} u) = 0 \tag{2}$$

and at a bottom-left corner as

$$\partial_{x-y-} u = 0 \tag{3}$$

0.1 Laplacian finite difference at an edge

At a left-hand edge, from equation 1 we have

$$\partial_{xx} = -\nu\partial_{yy}$$

so

$$\nabla^2 = \partial_{xx} + \partial_{yy} = (1-\nu)\partial_{yy} \tag{4}$$

Rearranging equation 2 gives

$$\partial_{x-}(\nabla^2 + (1-\nu)\partial_{yy}) = 0$$

so, at a left-hand edge

$$\partial_{x-}\nabla^2 = (1-\nu)\partial_{x-yy} \tag{5}$$

but

$$\partial_{xx}\nabla^2 = \frac{\partial_{x+}\nabla^2 - \partial_{x-}\nabla^2}{\Delta x} = \frac{\partial_{x+}\nabla^2 - (1-\nu)\partial_{x-yy}}{\Delta x} \tag{6}$$

where Δx is the grid spacing in the x direction.

Differentiating equation 1 with respect to y gives

$$\partial_{xxyy} = \frac{\partial_{x+yy} - \partial_{x-yy}}{\Delta x} = -\nu\partial_{yyyy}$$

so

$$\partial_{x-yy} = \partial_{x+yy} + \Delta x\nu\partial_{yyyy}$$

substituting back into equation 6

$$\partial_{xx}\nabla^2 = \frac{\partial_{x+}\nabla^2 - (1-\nu)(\partial_{x+yy} + \Delta x\nu\partial_{yyyy})}{\Delta x}$$

but, differentiating equation 4 by y gives

$$\partial_{yy}\nabla^2 = (1-\nu)\partial_{yyyy}$$

so, at a left-hand edge

$$\nabla^2\nabla^2 = \frac{\partial_{x+}\nabla^2 - (1-\nu)\partial_{x+yy}}{\Delta x} - (1-\nu)^2\partial_{yyyy} \tag{7}$$

simplifying gives

$$\nabla^2\nabla^2 = \frac{\partial_{x+}(\partial_{xx} + \nu\partial_{yy})}{\Delta x} - (1-\nu)^2\partial_{yyyy} \tag{8}$$

this is convenient to calculate since, on the left-hand edge,

$$\partial_{xx} + \nu\partial_{yy} = 0$$

0.2 Laplacian finite difference at a corner

At the bottom-left corner we have, from equation 1,

$$\partial_{xx} = -\nu\partial_{yy}$$

and

$$\partial_{yy} = -\nu\partial_{xx}$$

Since ν is positive, this implies $\partial_{xx} = 0$ and $\partial_{yy} = 0$ so $\nabla^2 = 0$

At the corner, we cannot differentiate past the corner, so we need to calculate ∂_{x-yy} in a different way:

$$\partial_{x-yy} = \frac{\partial_{x-}(\partial_{y+} - \partial_{y-})}{\Delta y}$$

so, from equation 3

$$\partial_{x-yy} = \frac{\partial_{x-y+}}{\Delta y}$$

However, along the left-hand edge

$$\frac{\partial_{x+} - \partial_{x-}}{\Delta x} = -\nu\partial_{yy}$$

so

$$\partial_{x-} = \Delta x\nu\partial_{yy} + \partial_{x+}$$

so

$$\partial_{x-yy} = \frac{\partial_{y+}(\Delta x\nu\partial_{yy} + \partial_{x+})}{\Delta y}$$

substituting this into equation 6

$$\partial_{xx}\nabla^2 = \frac{\Delta y\partial_{x+}\nabla^2 - (1-\nu)\partial_{y+}(\Delta x\nu\partial_{yy} + \partial_{x+})}{\Delta x\Delta y}$$

by symmetry

$$\partial_{yy}\nabla^2 = \frac{\Delta x\partial_{y+}\nabla^2 - (1-\nu)\partial_{x+}(\Delta y\nu\partial_{xx} + \partial_{y+})}{\Delta x\Delta y}$$

so, at the bottom-left corner

$$\nabla^2\nabla^2 = \frac{\Delta x\partial_{y+}\nabla^2 + \Delta y\partial_{x+}\nabla^2 - (1-\nu)(\partial_{y+}(\Delta x\nu\partial_{yy} + \partial_{x+}) + \partial_{x+}(\Delta y\nu\partial_{xx} + \partial_{y+}))}{\Delta x\Delta y} \tag{9}$$

simplifying gives

$$\nabla^2\nabla^2 = \frac{\partial_{y+}(\nabla^2 - \nu(1-\nu)\partial_{yy})}{\Delta y} + \frac{\partial_{x+}(\nabla^2 - \nu(1-\nu)\partial_{xx})}{\Delta x} - \frac{2(1-\nu)\partial_{x+y+}}{\Delta x\Delta y} \tag{10}$$

which is convenient to calculate as, on the corner, $\nabla^2 - \nu(1-\nu)\partial_{yy} = 0$ and $\nabla^2 - \nu(1-\nu)\partial_{xx} = 0$