Integrating a guitar string over time

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1 Third order solution

Starting with

$$\frac{\partial^2 y}{\partial t^2} = A\frac{\partial^2 y}{\partial x^2} + B\frac{\partial y}{\partial t}$$

discretise this over space to give

$$\frac{\partial^2 y_i}{\partial t^2} = \frac{A}{\Delta x^2}(y_{i-1} - 2y_i + y_{i+1}) + B\frac{\partial y_i}{\partial t} \tag{1}$$

Now take the third order Taylor approximation of the position of each point

$$y_i(t) = y_i(0) + \frac{\partial y_i(0)}{\partial t}t + \frac{1}{2}\frac{\partial^2 y_i(0)}{\partial t^2}t^2 + \frac{1}{6}\frac{\partial^3 y_i(0)}{\partial t^3}t^3$$

which gives

$$\frac{\partial y_i(t)}{\partial t} = \frac{\partial y_i(0)}{\partial t} + \frac{\partial^2 y_i(0)}{\partial t^2}t + \frac{1}{2}\frac{\partial^3 y_i(0)}{\partial t^3}t^2$$

and

$$\frac{\partial^2 y_i(t)}{\partial t^2} = \frac{\partial^2 y_i(0)}{\partial t^2} + \frac{\partial^3 y_i(0)}{\partial t^3}t$$

substituting back into equation 1 gives us an equation of the form

$$a\frac{\partial^3 y_{i-1}(0)}{\partial t^3} + b\frac{\partial^3 y_i(0)}{\partial t^3} + a\frac{\partial^3 y_{i+1}(0)}{\partial t^3} = d$$

where

$$a = \frac{At^3}{6\Delta x^2}$$

$$b = \frac{-2At^3}{6\Delta x^2} + \frac{Bt^2}{2} - t$$

$$d = a_i(0) - \frac{A}{\Delta x^2}(y_{i-1,2}(t) - 2y_{i,2}(t) + y_{i+1,2}(t)) - Bv_{i,2}(t)$$

where $y_{i,2}$ and $v_{i,2}$ are the second order approximations of the position and velocity of the i^{th} point respectively.

So we have a tridiagonal system of equations in the third derivative at each point, which we can solve with the tridiagonal algorithm. If we solve this at time Δt (the end of a timestep) then we ensure that position, velocity and acceleration are all continuous between timesteps; i.e. the simulated trajectory of each point is a cubic spline.