On model reduction and equivalence

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1 Poisson models and Markov processes

Let a Poisson model be a dynamic system on some discrete state space such that the probability that the process will transition from state x to state x' in time dt is given by $\rho('x|x)dt$. If we let ρ_{ij} be a transition matrix such that

$$\rho_{ij} = \begin{cases} \rho(i|j) & \text{if } i \neq j \\ \sum_{k \neq j} -\rho(k|j) & \text{if } i = j \end{cases}$$

and let X be a vector representing a probability distribution over the state space, then we can define the Poisson model as a continuous time dynamic system

$$\frac{dX}{dt} = \rho X$$

so that

$$X(t) = e^{\rho t}X(0) = e^{(\mu - I)rt}X(0) = \sum_{k=0}^{\infty} \frac{(rt)^k e^{-rt}}{k!} \mu^k X(0)$$

where $r = \max_j \sum_{k \neq j} \rho(k|j)$ is a scalar and $\mu = \frac{\rho}{r} + I$. It can be seen that each entry of μ is non-negative and the sum of entries in each column is 1, so $\mu^k X(0)$ is the state of a discrete time Markov process at time k.

So, each continuous-time Poisson model ρ has an associated discrete-time Markov process $\mu = \frac{\rho}{r} + I$, and state of the Poisson model at time t is the sum of states in a trajectory of the Markov process weighted by a Poisson distribution with rate rt.

This means, among other things, that as $t \to \infty$ the state distribution of a Poisson model tends to the attractor of its associated Markov model. So every Poisson model tends to a steady state distribution.

2 Model reduction of Poisson models