Commutation relations in Fock space

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1 Commutation identities

$$AB = BA + [A, B] \tag{1}$$

$$[A, BC] = [A, B]C + B[A, C] \tag{2}$$

$$[AB, C] = A[B, C] + [A, C]B \tag{3}$$

$$[A, B + C] = [A, B] + [A, C] \tag{4}$$

$$[A, mB] = m[A, B] \tag{5}$$

where m is a scalar multiplicative constant.

$$[A, B] = -[B, A] \tag{6}$$

$$\begin{aligned} [ab, AB] &= a[b, A]B + [a, A]bB + Aa[b, B] + A[a, B]b & (7) \\ &= a[b, A]B + [a, A]Bb + aA[b, B] + A[a, B]b & (8) \end{aligned}$$

$$[A, B^n] = \sum_{q=1}^{n} B^{q-1}[A, B]B^{n-q} \tag{9}$$

$$[A, B^{n+1}] = [A, B]B^n + [A, B^n]B - [[A, B^n], B] \tag{10}$$

$$\begin{aligned} [[A, C], [B, D]] = [[[A, B], C], D] + [[[B, C], D], A] \\ + [[[C, D], A], B] + [[[D, A], B], C] \end{aligned} \tag{11}$$

(source Wikipedia "Commutator")

$$[[A,H],[B,H]] = [[[A,B],H],H] + [[[B,H],H],A] - [[[A,H],B],H] \tag{12}$$

so

$$\begin{aligned}[[^nA,H],[^mB,H]] = {} & [^2[[^{n-1}A,H],[^{m-1}B,H]],H] \\ & + [[^{m+1}B,H],[^{n-1}A,H]] - [[[^nA,H],[^{m-1}B,H]],H]\end{aligned} \tag{13}$$

If we define $[^nA,H] = [\ldots[[A,H],H]\ldots,H]$ to be the n-fold commutation:

$$[^nAB,H] = \sum_{m=0}^{n} \binom{n}{m} [^mA,H][^{n-m}B,H] \tag{14}$$

$$[^nABC,H] = \sum_{a+b+c=n} \frac{n!}{a!b!c!}[^aA,H][^bB,H][^cC,H] \tag{15}$$

2 Annihilation and creation operators

Beginning with

$$[a_\psi, a_\phi^\dagger] = a_\psi a_\phi^\dagger - a_\phi^\dagger a_\psi = \begin{cases} 1 & \text{if } \psi = \phi \\ 0 & \text{otherwise} \end{cases}$$

3 $a^\dagger$ and a operators

$$[a^m, a^\dagger] = m a^{m-1} \tag{16}$$

$$[a, a^{\dagger m}] = m a^{\dagger(m-1)} \tag{17}$$

the above holds for all m, even -ve.

$$[a^-, a^m] = ((a^- + a)^m - a^m)a^- \tag{18}$$

$$[a^n, a^{\dagger m}] = \sum_{q=1}^{\min(m,n)} \frac{m!n!}{q!(m-q)!(n-q)!} a^{\dagger m-q} a^{n-q} \tag{19}$$

$$[a^{\dagger p}a^m, a^{\dagger q}a^n] = a^{\dagger p}[a^m, a^{\dagger q}]a^n - a^{\dagger q}[a^n, a^{\dagger p}]a^m \tag{20}$$

4 a^- operator

Define the a^- operator such that

$$a^- a^\dagger = I$$

where I is the identity operator. Given this we can see immediately that $[a^\dagger, a^- a^\dagger] = 0$, so

$$a^\dagger a^- a^\dagger - a^- a^\dagger a^\dagger = [a^\dagger, a^-] a^\dagger = 0$$

So, for all states, S other than the ground state,

$$[a^\dagger, a^-] S = 0$$

For the ground state, $\emptyset$, in order to ensure $[a^\dagger, a^-]\emptyset = 0$ we define

$$a^\dagger (a^- \emptyset) = \emptyset$$

However, such terms as $a^- \emptyset$ will never arise through annihilation operators as they will always be multipllied by zero.

5 L_{ir} operator

Changes λ_i to $(1-r)\lambda_i$

$$[L_{ir}, a_i^\dagger] = 0 \tag{21}$$

$$[L_{ir}, a_i] = r\lambda_i L_{ir} \tag{22}$$

$$[L_{ir}, a_i^n] = L_{ir}(a_i^n - (a_i - r\lambda_i)^n) \tag{23}$$

6 g_{ir} operator

Multiplies each basis by $(1-r)^{\Delta_i}$

$$g_{ir} D_0 = 1$$

$$[a_i^\dagger, g_{ir}] = r a_i^\dagger g_{ir} \tag{24}$$

$$[a_i, g_{ir}] = \frac{r(\lambda - a_i)}{1-r} g_{ir} \tag{25}$$

so

$$g_{ir}a_i^{\dagger n} = (1-r)^n a_i^{\dagger n} g_{ir} \tag{26}$$

and

$$[g_{ir}, a_i^{\dagger n}] = ((1-r)^n - 1)a_i^{\dagger n} g_{ir} \tag{27}$$

also

$$g_{ir}a_i = \frac{a_i - r\lambda_i}{1-r} g_{ir} \tag{28}$$

so

$$g_{ir}a_i^n = \left(\frac{a_i - r\lambda_i}{1-r}\right)^n g_{ir} \tag{29}$$

$$g_{ir}a_i^n = (1-r)^{-n} \sum_{m=0}^{n} \binom{n}{m} a_i^m (-r\lambda_i)^{n-m} g_{ir} \tag{30}$$

and

$$[g_{ir}, a_i^n] = \left((1-r)^{-n} \sum_{m=0}^{n} \binom{n}{m} a_i^m (-r\lambda_i)^{n-m} - a_i^n\right) g_{ir} \tag{31}$$

so

$$[g_{ir}, a_i^{\dagger n} a_i^m] = \left((1-r)^{n-m} \sum_{l=0}^{m} \binom{m}{l} a_i^{\dagger n} a_i^l (-r\lambda_i)^{m-l} - a_i^{\dagger n} a_i^m\right) g_{ir} \tag{32}$$

7 Lg_{ir} operator

We can join the L and g operators into a compound operator that multiplies bases by $(1-r)_i^{\Delta}$ and multiplies λ_i by $(1-r)$. It has the properties:

$$[Lg_{ir}, a_i^{\dagger}] = -ra_i^{\dagger} Lg_{ir} \tag{33}$$

$$[Lg_{ir}, a_i] = \frac{ra_i}{1-r} Lg_{ir} \tag{34}$$

So

$$Lg_{ir}a_i^{\dagger n} a_i^m = (1-r)^{n-m} a_i^{\dagger n} a_i^m Lg_{ir} \tag{35}$$

and

$$[Lg_{ir}, a_i^{\dagger n} a_i^m] = ((1-r)^{n-m} - 1)a_i^{\dagger n} a_i^m Lg_{ir} \tag{36}$$