in a given band is compensated by an appropriate change of the spectral weight in other bands such that the total spectral weight, integrated over all bands, is conserved, as in Eq. (1). Still, non-conservation of the spectral weight within a given band is an interesting phenomenon as the degree of non-conservation is an indicator of relevant energy scales in the problem. Indeed, when relevant energy scales are much smaller than the Fermi energy, i.e., changes in the conductivity are confined to a near vicinity of a Fermi surface (FS), one can expand ε_k near k_F as $\varepsilon_k = v_F(k-k_F) + (k-k_F)^2/(2m_B) + O(k-k_F)^3$ and obtain $\nabla^2_{\vec{k}_x} \varepsilon_{\vec{k}} \approx 1/m_B$ [this approximation is equivalent to approximating the density of states (DOS) by a constant]. Then W_K becomes $\pi n e^2/(2m_B)$ which does not depend on temperature. The scale of the temperature dependence of W_K is then an indicator how far in energy the changes in conductivity extend when, e.g., a system evolves from a normal metal to a superconductor. Because relevant energy scales increase with the interaction strength, the temperature dependence of W_K is also an indirect indicator of whether a system is in a weak, intermediate, or strong coupling regime.

In a conventional BCS superconductor the only relevant scales are the superconducting gap Δ and the impurity scattering rate Γ. Both are generally much smaller than the Fermi energy, so the optical integral should be almost T-independent, i.e., the spectral weight lost in a superconducting state at low frequencies because of gap opening is completely recovered by the zero-frequency δ-function. In a clean limit, the weight which goes into a $\delta-$function is recovered within frequencies up to 4Δ. This is the essence of FGT sum rule [2,3]. In a dirty limit, this scale is larger, $O(\Gamma)$, but still W_K is T-independent and there was no "violation of sum rule".

The issue of sum rule attracted substantial interest in the studies of high T_c cuprates[5–18,21–26] in which pairing is without doubts a strong coupling phenomenon. From a theoretical perspective, the interest in this issue was originally triggered by a similarity between W_K and the kinetic energy $K = 2\sum \varepsilon_{\vec{k}} n_{\vec{k}}$.[18–20] For a model with a simple tight binding cosine dispersion $\varepsilon_k \propto (\cos k_x + \cos k_y)$, $\frac{d^2 \varepsilon_{\vec{k}}}{d k_x^2} \sim -\varepsilon_{\vec{k}}$ and $W_K = -K$. For a more complex dispersion there is no exact relation between W_K and K, but several groups argued [17,27,28] that W_K can still be regarded as a good monitor for the changes in the kinetic energy. Now, in a BCS superconductor, kinetic energy increases below T_c because n_k extends to higher frequencies (see Fig.2). At strong coupling, K not necessary increases because of opposite trend associated with the fermionic self-energy: fermions are more mobile in the SCS due to less space for scattering at low energies than they are in the NS. Model calculations show that above some coupling strength, the kinetic energy decreases below T_c[29]. While, as we said, there is no one-to-one correspondence between K and W_K, it is still likely that, when K decreases, W_K increases.

A good amount of experimental effort has been put into addressing the issue of the optical sum rule in the $c-$axis[7] and in-plane conductivities [8–16] in overdoped, optimally doped, and underdoped cuprates. The experimental results demonstrated, above all, outstanding achievements of experimental abilities as these groups managed to detect the value of the optical integral with the accuracy of a fraction of a percent. The analysis of the change of the optical integral between normal and SCS is even more complex because one has to (i) extend NS data to $T < T_c$ and (ii) measure superfluid density with the same accuracy as the optical integral itself.

The analysis of the optical integral showed that in overdoped cuprates it definitely decreases below T_c, in consistency with the expectations at weak coupling[11]. For underdoped cuprates, all experimental groups agree that a relative change of the optical integral below T_c gets much smaller. There is no agreement yet about the sign of the change of the optical integral : Molegraaf *et al.*[8] and Santander-Syro *et al.*[9] argued that the optical integral increases below T_c, while Boris *et al.*[10] argued that it decreases.

Theoretical analysis of these results[21,22,25,28,30] added one more degree of complexity to the issue. It is tempting to analyze the temperature dependence of W_K and relate it to the observed behavior of the optical integral, and some earlier works[25,28,30] followed this route. In the experiments, however, optical conductivity is integrated only up to a certain frequency ω_c, and the quantity which is actually measured is

$$W(\omega_c) = \int_0^{\omega_c} Re\,\sigma(\Omega)\,d\Omega = W_K + f(\omega_c)$$
$$f(\omega_c) = -\int_{\omega_c}^{'\infty'} Re\,\sigma(\Omega)\,d\Omega \tag{4}$$

The Kubo formula, Eq. (3) is obtained assuming that the second part is negligible. This is not guaranteed, however, as typical $\omega_c \sim 1 - 2eV$ are comparable to the bandwidth.

The differential sum rule ΔW is also a sum of two terms

$$\Delta W(\omega_c) = \Delta W_K + \Delta f(\omega_c) \tag{5}$$

where ΔW_K is the variation of the r.h.s. of Eq. 3, and $\Delta f(\omega_c)$ is the variation of the cutoff term. Because conductivity changes with T at all frequencies, $\Delta f(\omega_c)$ also varies with temperature. It then becomes the issue whether the experimentally observed $\Delta W(\omega_c)$ is predominantly due to "intrinsic" ΔW_K, or to $\Delta f(\omega_c)$. [A third possibility is non-applicability of the Kubo formula because of the close proximity of other bands, but we will not dwell on this.]

For the NS, previous works[21,22] on particular models for the cuprates indicated that the origin of the temperature dependence of $W(\omega_c)$ is likely the T dependence of the cutoff term $f(\omega_c)$. Specifically, Norman *et. al.*[22] approximated a fermionic DOS by a constant (in which