tor, this leads to discretization errors in the ratio f_{B_s}/f_{B_d} of $\mathcal{O}(\alpha_s \times (am_s - am_d)) \sim 1.2\%$ plus $\mathcal{O}(\alpha_s^2 a\Lambda_{\rm QCD} \times (\widetilde{m}_s - \widetilde{m}_d)/\Lambda_{\rm QCD}) \sim 0.6\%$ plus $\mathcal{O}(a^2\Lambda_{\rm QCD}^2 \times (\widetilde{m}_s - \widetilde{m}_d)/\Lambda_{\rm QCD}) \sim 1.7\%$. Although we do not improve the heavy-light four fermion operator used to compute the B-mixing matrix element, the operator does not have any tree-level $\mathcal{O}(a)$ errors [38]. Thus the leading discretization errors in the ratio ξ from the four-fermion operator are of $\mathcal{O}(\alpha_s \times (am_s - am_d)) \sim 1.2\%$ plus $\mathcal{O}(\alpha_s a\Lambda_{\rm QCD} \times (\widetilde{m}_s - \widetilde{m}_d)/\Lambda_{\rm QCD}) \sim 1.9\%$ plus $\mathcal{O}(a^2\Lambda_{\rm QCD}^2 \times (\widetilde{m}_s - \widetilde{m}_d)/\Lambda_{\rm QCD}) \sim 1.7\%$.

Adding the contributions from light-quark and gluon discretization errors, heavy-quark discretization errors, and discretization errors in the heavy-light current or four-fermion operator in quadrature, we estimate the error in f_{B_s}/f_{B_d} to be $\sim 3.2\%$ and the error in ξ to be $\sim 3.7\%$.

D. Heavy-light current and four-fermion operator renormalization

We compute the renormalization factors needed to match the lattice axial current and four-fermion operator to the continuum using one-loop lattice perturbation theory. This leaves a residual error due to the omission of higher-order terms. Based on power-counting, we estimate the truncation error in the coefficients to be of $\mathcal{O}(\alpha_s^2)$, which is the size of the first neglected term in the series. As we noted earlier in Sec. III C, however, the matching coefficient Z_Φ cancels in the ratio of decay constants f_{B_s}/f_{B_d}; thus its contribution to the error in f_{B_s}/f_{B_d} is zero. Although such an exact cancellation does not occur for the ratio of mixing matrix elements ξ, the error in ξ due to the uncertainty in the ratio of matching coefficients Z_{SP}/Z_{VA} is suppressed by the $SU(3)$-breaking factor $(\widetilde{m}_s - \widetilde{m}_d)/\Lambda_{\rm QCD}$. This is because, in the $SU(3)$ limit, the four-fermion operator matrix elements would be equal in the numerator and denominator, so the error in ξ from the renormalization factor uncertainty would be zero. We therefore expect the error in f_{B_s}/f_{B_d} to be 0% and the error in ξ to be of $\mathcal{O}(\alpha_s^2 \times (\widetilde{m}_s - \widetilde{m}_d)/\Lambda_{\rm QCD}) \sim 2.2\%$. This error will decrease with the inclusion of data at a finer lattice spacing because the smaller coupling constant will improve the convergence of the series.

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to $am_{\rm res}$. These effects, however, are expected to be sub-percent level in the matrix elements [75], and therefore negligible in the $SU(3)$-breaking ratios.