an energy of interband transitions, which is roughly $2eV$. This would be consistent with Refs. 8,9.

We begin with formulating our calculational basis in the next section. Then we take up the four cases and consider in each case the extent to which the Kubo sum is satisfied up to the order of bandwidth and the functional form and the sign of $\Delta W(\omega_c)$. The last section presents our conclusions.

II. OPTICAL INTEGRAL IN NORMAL AND SUPERCONDUCTING STATES

The generic formalism of the computation of the optical conductivity and the optical integral has been discussed several times in the literature[21–23,26,29] and we just list the formulas that we used in our computations. The conductivity $\sigma(\Omega)$ and the optical integral $W(\omega_c)$ are given by (see for example Ref. 35).

$$\sigma'(\Omega) = Im\left[-\frac{\Pi(\Omega)}{\Omega+i\delta}\right] = -\frac{\Pi''(\Omega)}{\Omega} \;+\; \pi\delta(\Omega)\,\Pi'(\Omega) \tag{7a}$$

$$W(\omega_c) = \int_0^{\omega_c} \sigma'(\Omega)\,d\Omega = -\int_{0+}^{\omega_c} \frac{\Pi''(\Omega)}{\Omega}\,d\Omega \;+\; \frac{\pi}{2}\Pi'(0) \tag{7b}$$

where 'X'' and 'X''' stand for real and imaginary parts of X. We will restrict with $T = 0$. The polarization operator $\Pi(\Omega)$ is (see Ref. 36)

$$\Pi(i\Omega) = T\sum_{\omega}\sum_{\vec{k}} (\nabla_{\vec{k}}\varepsilon_{\vec{k}})^2 \,\Big(G(i\omega,\vec{k})G(i\omega+i\Omega,\vec{k}) \;+\; F(i\omega,\vec{k})F(i\omega+i\Omega,\vec{k})\Big) \tag{8a}$$

$$\Pi''(\Omega) = -\frac{1}{\pi}\sum_{\vec{k}} (\nabla_{\vec{k}}\varepsilon_{\vec{k}})^2 \int_{-\Omega}^{0} d\omega \,\Big(G''(\omega,\vec{k})G''(\omega+\Omega,\vec{k}) \;+\; F''(\omega,\vec{k})F''(\omega+\Omega,\vec{k})\Big) \tag{8b}$$

$$\Pi'(\Omega) = \frac{1}{\pi^2}\sum_{\vec{k}} (\nabla_{\vec{k}}\varepsilon_{\vec{k}})^2 \int' \int' dx\,dy \,\Big(G''(x,\vec{k})G''(y,\vec{k}) \;+\; F''(x,\vec{k})F''(y,\vec{k})\Big)\,\frac{n_F(y)-n_F(x)}{y-x} \tag{8c}$$

where $\int'$ denotes the principal value of the integral, $\sum_{\vec{k}}$ is understood to be $\frac{1}{N}\sum_{\vec{k}}$,($N$ is the number of lattice sites), $n_F(x)$ is the Fermi function which is a step function at zero temperature, G and F are the normal and anomalous Greens functions. given by[37]

$$\text{For a NS, } G(\omega,\vec{k}) = \frac{1}{\omega-\Sigma(k,\omega)-\varepsilon_{\vec{k}}+i\delta} \tag{9a}$$

$$\text{For a SCS, } G(\omega,\vec{k}) = \frac{Z_{k,\omega}\omega+\varepsilon_{\vec{k}}}{Z^2_{k,\omega}(\omega^2-\Delta^2_{k,\omega})-\varepsilon^2_{\vec{k}}+i\delta sgn(\omega)} \tag{9b}$$

$$F(\omega,\vec{k}) = \frac{Z_{k,\omega}\Delta_{k,\omega}}{Z^2_{k,\omega}(\omega^2-\Delta^2_{k,\omega})-\varepsilon^2_{\vec{k}}+i\delta sgn(\omega)} \tag{9c}$$

where $Z_{k,\omega} = 1-\frac{\Sigma(k,\omega)}{\omega}$, and $\Delta_{k,\omega}$, is the SC gap. Following earlier works[31,33], we assume that the fermionic self-energy $\Sigma(k,\omega)$ predominantly depends on frequency and approximate $\Sigma(k,\omega) \approx \Sigma(\omega)$ and also neglect the frequency dependence of the gap, i.e., approximate $\Delta_{k,\omega}$ by a $d-$wave Δ_k. The lattice dispersion $\varepsilon_{\vec{k}}$ is taken from Ref. 38. To calculate W_K, one has to evaluate the Kubo term in Eq.3 wherein the distribution function $n_{\vec{k}}$, is calculated from

$$n(\varepsilon_{\vec{k}}) = -2\int_{-\infty}^{0} \frac{d\omega}{2\pi}\,G''(\omega,\vec{k}) \tag{10}$$

The 2 is due to the trace over spin indices. We show the distribution functions in the NS and SCS under different circumstances in Fig 2.

The $\vec{k}$-summation is done over first Brillouin zone for a 2-D lattice with a 62x62 grid. The frequency integrals are done analytically wherever possible, otherwise performed using Simpson's rule for all regular parts. Contributions from the poles are computed separately using Cauchy's theorem. For comparison, in all four cases we also calculated FGT sum rule by replacing $\int d^2k = d\Omega_k d\epsilon_k \nu_{\epsilon_k,\Omega_k}$ and keeping ν constant. We remind that the FGT is the result when one assumes that the integral in $W(\omega_c)$ predominantly comes from a narrow region around the Fermi surface.

We will first use Eq 3 and compute W_K in NS and SCS. This will tell us about the magnitude of $\Delta W(\omega_c = \infty)$. We next compute the conductivity $\sigma(\omega)$ using the equations listed above, find $W(\omega_c)$ and $\Delta W(\omega_c)$ and compare $\Delta f(\omega_c)$ and ΔW_K.

For simplicity and also for comparisons with earlier studies, for BCSI, EB, and MFLI models we assumed that the gap is just a constant along the FS. For CB model, we used a $d-$wave gap and included into consideration the fact that, if a CB is a spin fluctuation, its propagator develops a resonance when the pairing gap is $d-$wave.