in (a) and (b), respectively. Therefore, from consideration of Case a, b and c,

$$\begin{aligned}
&\sup_{\{B_k\}_{k=-\infty}^{\infty}} \sum_{k=-\infty}^{\infty} \max\left\{\mu_{0,W}\left(B_k|w\right) - \mu_{1,W}\left(B_k^D|w\right), 0\right\} \\
&= \sup_{\{b_k\}_{k=-\infty}^{\infty}} \sum_{k=-\infty}^{\infty} \max\left\{F_{0,W}\left(b_k|w\right) - F_{1,W}\left(\frac{t_1-t_W}{t_0-t_W}b_{k+1} - \frac{t_1-t_0}{t_0-t_W}w|w\right), 0\right\}
\end{aligned}$$

where $\frac{t_0-t_W}{t_1-t_0}\delta + w \leq b_{k+1} \leq b_k$. Consequently, the sharp upper bound is written as follows: letting $F_{\Delta,W}^U\left(\delta|w\right)$ be the sharp upper bound on $\Pr\left(Y_1 - Y_0 \leq \delta | W = w\right)$,

$$\begin{aligned}
&F_{\Delta}^U\left(\delta\right) \\
&= \int F_{\Delta,W}^U\left(\delta|w\right) dF_W\left(w\right) \\
&= \int \left\{1 - \sup_{\{B_k\}_{k=-\infty}^{\infty}} \sum_{k=-\infty}^{\infty} \max\left\{\mu_{0,W}\left(B_k|w\right) - \mu_{1,W}\left(B_k^D|w\right), 0\right\}\right\} dF_W \\
&= 1 + \int \inf_{\{b_k\}_{k=-\infty}^{\infty}} \sum_{k=-\infty}^{\infty} \min\left\{F_{1,W}\left(\frac{t_1-t_W}{t_0-t_W}b_{k+1} - \frac{t_1-t_0}{t_0-t_W}w|w\right) - F_{0,W}\left(b_k|w\right), 0\right\} dF_W
\end{aligned}$$

where $\frac{t_0-t_W}{t_1-t_0}\delta + w \leq b_{k+1} \leq b_k$. $\blacksquare$

Appendix B

Appendix B presents the procedure used to compute the sharp lower bound under MTR in Section 4 and Section 5. The following lemma is useful for reducing computational costs:

Lemma B.1 Let

$$\{a_k\}_{k=-\infty}^{\infty} \in \underset{\{a_k\}_{k=-\infty}^{\infty} \in \mathcal{A}_\delta}{\arg\max} \sum_{k=-\infty}^{\infty} \max\left\{F_1\left(a_{k+1}\right) - F_0\left(a_k\right), 0\right\},$$

$$\text{where } \mathcal{A}_\delta = \left\{\{a_k\}_{k=-\infty}^{\infty}; 0 \leq a_{k+1} - a_k \leq \delta \text{ for each integer } k\right\}.$$

It is innocuous to assume that $\{a_k\}_{k=-\infty}^{\infty}$ satisfies $a_{k+2} - a_k > \delta$ for each integer k.

Proof. I will show that for any sequence $\{a_k\}_{k=-\infty}^{\infty} \in \mathcal{A}_\delta$ satisfying $a_{k+2} - a_k \leq \delta$ for some integer k, one can construct $\{\widetilde{a}_k\}_{k=-\infty}^{\infty} \in \mathcal{A}_\delta$ with $\widetilde{a}_{k+2} - \widetilde{a}_k > \delta$ for each integer k and

$$\sum_{k=-\infty}^{\infty} \max\left\{F_1\left(a_{k+1}\right) - F_0\left(a_k\right), 0\right\} \leq \sum_{k=-\infty}^{\infty} \max\left\{F_1\left(\widetilde{a}_{k+1}\right) - F_1\left(\widetilde{a}_k\right), 0\right\}.$$