The states input to the quantum dephasing channel obtained from Eq. (29) are

$$\begin{aligned}\rho_{S,1} &= \mathrm{Tr}_R(|\Phi_1\rangle\langle\Phi_1|) = \cos^2\theta|0\rangle\langle 0| + \sin^2\theta|1\rangle\langle 1|,\\ \rho_{S,2} &= \mathrm{Tr}_R(|\Phi_2\rangle\langle\Phi_2|) = \sin^2\theta|0\rangle\langle 0| + \cos^2\theta|1\rangle\langle 1|,\\ \rho_{S,3} &= \mathrm{Tr}_R(|\Phi_3\rangle\langle\Phi_3|) = \cos^2\theta|0\rangle\langle 0| + \sin^2\theta|1\rangle\langle 1|,\\ \rho_{S,4} &= \mathrm{Tr}_R(|\Phi_4\rangle\langle\Phi_4|) = \sin^2\theta|0\rangle\langle 0| + \cos^2\theta|1\rangle\langle 1|,\end{aligned} \tag{A2}$$

therefore, for all $\rho_{S,i}$

$$S(\rho_{S,i}) = -[\cos^2\theta\log_2\cos^2\theta + \sin^2\theta\log_2\sin^2\theta]. \tag{A3}$$

This results in the following expression for the first term in Eq. (8):

$$\begin{aligned}&\sum_i p_i(x_1,x_2)S(\rho_{S,i})\\ &= -(\cos^2 x_1\cos^2 x_2 + \sin^2 x_1\cos^2 x_2\\ &\quad + \cos^2 x_1\sin^2 x_2 + \sin^2 x_1\sin^2 x_2)\\ &\quad \times[\cos^2\theta\log_2\cos^2\theta + \sin^2\theta\log_2\sin^2\theta],\\ &= -[\cos^2\theta\log_2\cos^2\theta + \sin^2\theta\log_2\sin^2\theta],\end{aligned} \tag{A4}$$

where we have used the normalization $\sum_{i=1}^4 p_i(x_1,x_2) = 1, \forall x_1, x_2$. This yields the first term in Eq. (30).

Next we calculate the second term in Eq. (8). The output state is

$$\begin{aligned}&\mathcal{E}(\sum_i p_i(x_1,x_2)\rho_{S,i})\\ &= \sum_{i,j} K_{ij}(\sum_i p_i(x_1,x_2)\rho_{S,i})K_{ij}^\dagger,\end{aligned} \tag{A5}$$

where

$$\begin{aligned}&\sum_i p_i(x_1,x_2)\rho_{S,i}\\ &= [\cos^2\theta(\cos^2 x_1\cos^2 x_2 + \cos^2 x_1\sin^2 x_2)\\ &\quad + \sin^2\theta(\sin^2 x_1\sin^2 x_2 + \sin^2 x_1\cos^2 x_2)]|0\rangle\langle 0|\\ &\quad + [\cos^2\theta(\sin^2 x_1\sin^2 x_2 + \sin^2 x_1\cos^2 x_2)\\ &\quad + \sin^2\theta(\cos^2 x_1\cos^2 x_2 + \cos^2 x_1\sin^2 x_2)]|1\rangle\langle 1|\\ &= (\cos^2\theta\cos^2 x_1 + \sin^2\theta\sin^2 x_1)|0\rangle\langle 0|\\ &\quad + (\cos^2\theta\sin^2 x_1 + \sin^2\theta\cos^2 x_1)|1\rangle\langle 1|.\end{aligned} \tag{A6}$$

Since this state is diagonal ("classical") it is invariant under the dephasing channel with Kraus operators given by Eq. (18). Therefore the eigenvalues of the output state (A5) are

$$\begin{aligned}\upsilon_1 &= \cos^2\theta\cos^2 x_1 + \sin^2\theta\sin^2 x_1,\\ \upsilon_2 &= \cos^2\theta\sin^2 x_1 + \sin^2\theta\cos^2 x_1,\end{aligned} \tag{A7}$$

and

$$S[\mathcal{E}(\sum_i p_i(x_1,x_2)\rho_{S,i})] = -\sum_{i=1}^2 \upsilon_i\log_2\upsilon_i. \tag{A8}$$

Finally, we calculate the third term in Eq. (8):

$$\begin{aligned}&(\mathcal{E}\otimes I)(|\Phi_i\rangle\langle\Phi_i|)\\ &= \sum_{i,j}(K_{ij}\otimes I)(|\Phi_i\rangle\langle\Phi_i|)(K_{ij}^\dagger\otimes I),\end{aligned} \tag{A9}$$

which for all $|\Phi_i\rangle$ has eigenvalues $\omega_1 = \omega_2 = 0$ and ω_3, ω_4 are given in Eq. (31). The third term in Eq. (8) is thus

$$\begin{aligned}&\sum_i p_i(x_1,x_2)S[(\mathcal{E}\otimes I)(|\Phi_i\rangle\langle\Phi_i|)]\\ &= -\sum_{i=1}^4 \omega_i\log_2\omega_i,\end{aligned} \tag{A10}$$

where, as for Eq. (A4), we have used the normalization $\sum_{i=1}^4 p_i = 1$. This yields the third term in Eq. (30).

Thus, indeed only the second term in Eq. (8) depends on x_1, x_2, and for a given value of θ, the classical capacity assisted by limited entanglement is maximized by maximizing Eq. (A8). The maximum is attained when the output state (A5) is fully mixed, i.e., when its eigenvalues $\upsilon_1 = \upsilon_2 = 1/2$. This occurs when $x_1 = x_2 = \frac{\pi}{4}$, i.e., when we have an equiprobable ensemble of the states. This gives rise to the 1 in Eq. (30).

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