Plug in the expressions (4) and (6) into (7), the Hamiltonian reads explicitly as

$$H = \sum_j (J_{\text{cluster}}/2)(\mathbf{S}_{j1}+\mathbf{S}_{j2}+\mathbf{S}_{j3}+\mathbf{S}_{j4})^2 - \sum_{z-\text{links } <jk>} J_z\,(16/9)[\mathbf{S}_{j2}\cdot(\mathbf{S}_{j3}\times\mathbf{S}_{j4})][\mathbf{S}_{k2}\cdot(\mathbf{S}_{k3}\times\mathbf{S}_{k4})]$$
$$- \sum_{x-\text{links } <jk>} J_x\,(2\mathbf{S}_{j1}\cdot\mathbf{S}_{j2}+1/2)(2\mathbf{S}_{k1}\cdot\mathbf{S}_{k2}+1/2) - \sum_{y-\text{links } <jk>} J_y\,(4/3)[\mathbf{S}_{j1}\cdot(\mathbf{S}_{j3}-\mathbf{S}_{j4})][\mathbf{S}_{k1}\cdot(\mathbf{S}_{k3}-\mathbf{S}_{k4})] \tag{8}$$

While by the represenation (4) and (5), the Hamiltonian becomes

$$H = \sum_j (J_{\text{cluster}}/2)(\mathbf{S}_{j1}+\mathbf{S}_{j2}+\mathbf{S}_{j3}+\mathbf{S}_{j4})^2$$
$$- \sum_{x-\text{links } <jk>} J_x\,(2\mathbf{S}_{j1}\cdot\mathbf{S}_{j2}+1/2)(2\mathbf{S}_{k1}\cdot\mathbf{S}_{k2}+1/2) - \sum_{y-\text{links } <jk>} J_y\,(4/3)[\mathbf{S}_{j1}\cdot(\mathbf{S}_{j3}-\mathbf{S}_{j4})][\mathbf{S}_{k1}\cdot(\mathbf{S}_{k3}-\mathbf{S}_{k4})]$$
$$- \sum_{z-\text{links } <jk>} J_z\,(-4/3)(2\mathbf{S}_{j3}\cdot\mathbf{S}_{j4}+1/2)[\mathbf{S}_{j1}\cdot(\mathbf{S}_{j3}-\mathbf{S}_{j4})](2\mathbf{S}_{k3}\cdot\mathbf{S}_{k4}+1/2)[\mathbf{S}_{k1}\cdot(\mathbf{S}_{k3}-\mathbf{S}_{k4})] \tag{9}$$

This model, in terms of physical spins $\mathbf{S}$, has full spin rotation symmetry and time-reversal symmetry. A pseudo-magnetic field term $\sum_j \vec{h}\cdot\vec{\tau}_j$ term can also be included under this mapping, however the resulting Kitaev model with magnetic field is not exactly solvable. It is quite curious that such a formidably looking Hamiltonian (8), with biquadratic and six-spin(or eight-spin) terms, has an exactly solvable low energy sector.

We emphasize that because the first intra-cluster term $\sum_{\text{cluster}} H_{\text{cluster}}$ commutes with the latter Kitaev terms independent of the representation used, the Kitaev model is realized as the *exact* low energy Hamiltonian of this model without truncation errors of perturbation theories, namely no $(|J_{x,y,z}|/J_{\text{cluster}})^2$ or higher order terms will be generated under the projection to low energy cluster singlet space. This is unlike, for example, the t/U expansion of the half-filled Hubbard model[22,23], where at lowest t^2/U order the effective Hamiltonian is the Heisenberg model, but higher order terms (t^4/U^3 etc.) should in principle still be included in the low energy effective Hamiltonian for any finite t/U. Similar comparison can be made to the perturbative expansion studies of the Kitaev-type models by Vidal *et al.*[9], where the low energy effective Hamiltonians were obtained in certian anisotropic (strong bond/triangle) limits. Although the spirit of this work, namely projection to low energy sector, is the same as all previous perturbative approaches to effective Hamiltonians.

Note that the original Kitaev model (1) has three-fold rotation symmetry around a honeycomb lattice site, combined with a three-fold rotation in pseudo-spin space (cyclic permutation of τ^x, τ^y, τ^z). This is not apparent in our model (8) in terms of physical spins, under the current representation of $\tau^{x,y,z}$. We can remedy this by using a different set of pseudo-spin Pauli matrices $\tau'^{x,y,z}$ in (7),

$$\tau'^x = \sqrt{1/3}\tau^z + \sqrt{2/3}\tau^x,$$
$$\tau'^y = \sqrt{1/3}\tau^z - \sqrt{1/6}\tau^x + \sqrt{1/2}\tau^y,$$
$$\tau'^z = \sqrt{1/3}\tau^z - \sqrt{1/6}\tau^x - \sqrt{1/2}\tau^y$$

With proper representation choice, they have a symmetric form in terms of physical spins,

$$\begin{aligned}\tau'^x &= -(4/3)\mathbf{S}_2\cdot(\mathbf{S}_3\times\mathbf{S}_4) + \sqrt{2/3}(2\mathbf{S}_1\cdot\mathbf{S}_2+1/2)\\ \tau'^y &= -(4/3)\mathbf{S}_3\cdot(\mathbf{S}_4\times\mathbf{S}_2) + \sqrt{2/3}(2\mathbf{S}_1\cdot\mathbf{S}_3+1/2)\\ \tau'^z &= -(4/3)\mathbf{S}_4\cdot(\mathbf{S}_2\times\mathbf{S}_3) + \sqrt{2/3}(2\mathbf{S}_1\cdot\mathbf{S}_4+1/2)\end{aligned} \tag{10}$$

So the symmetry mentioned above can be realized by a three-fold rotation of the honeycomb lattice, with a cyclic permutation of $\mathbf{S}_2$, $\mathbf{S}_3$ and $\mathbf{S}_4$ in each cluster. This is in fact the three-fold rotation symmetry of the physical spin lattice illustrated in FIG. 2. However this more symmetric representation will not be used in later part of this paper.