Thus, the value of $\sup_{(\varphi,\psi)\in\Phi_c} J(\varphi,\psi)$ is unchanged even if one restricts the supremum to pairs of the form $\left(\sum_{k=0}^{\infty}\left(\mathbf{1}_{A_k^+(\varphi,s)}-\mathbf{1}_{A_k^-(\varphi,s)}\right), \sum_{k=0}^{\infty}\left(\mathbf{1}_{A_k^+(\psi,s)}-\mathbf{1}_{A_k^-(\psi,s)}\right)\right)$. Hence for all $(y_0,y_1)\in\mathbb{R}^2$,

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(\mathbf{1}_{A_k^+(\varphi,s)}(y_0)-\mathbf{1}_{A_k^-(\varphi,s)}(y_0)\right)+\sum_{k=0}^{\infty}\left(\mathbf{1}_{A_k^+(\psi,s)}(y_1)-\mathbf{1}_{A_k^-(\psi,s)}(y_1)\right)\\
&\leq \mathbf{1}\{y_1-y_0<\delta\}+\lambda\left(1-\mathbf{1}_C(y_0,y_1)\right),
\end{aligned}$$

which implies that for each $y_1\in\mathbb{R}$,

$$\begin{aligned}
-\infty &< \sup_{y_0\in\mathbb{R}}\left[\sum_{k=0}^{\infty}\left(\mathbf{1}_{A_k^+(\varphi,s)}(y_0)-\mathbf{1}_{A_k^-(\varphi,s)}(y_0)\right)-\mathbf{1}(y_1-y_0<\delta)-\lambda\left(1-\mathbf{1}_C(y_0,y_1)\right)\right]\\
&\leq -\sum_{k=0}^{\infty}\left(\mathbf{1}_{A_k^+(\psi,s)}(y_1)-\mathbf{1}_{A_k^-(\psi,s)}(y_1)\right).
\end{aligned}$$

Define $\left\{A_{k,D}^+(\varphi,s)\right\}_{k=0}^{\infty}$, $\left\{A_{k,D}^-(\varphi,s)\right\}_{k=0}^{\infty}$ as follows:

$$\begin{aligned}
A_{k,D}^+(\varphi,s) = {} & \left\{y_1\in\mathbb{R}|\exists y_0\in A_k^+(\varphi,s) \text{ s.t. } y_1-y_0\geq\delta \text{ and } (y_0,y_1)\in C\right\}\\
& \cup\left\{y_1\in\mathbb{R}|\exists y_0\in A_{k+1}^+(\varphi,s) \text{ s.t. } y_1-y_0<\delta \text{ and } (y_0,y_1)\in C\right\}\\
& \text{for any integer } k\geq 0,
\end{aligned}\tag{A.6}$$

$$\begin{aligned}
A_{0,D}^-(\varphi,s) = {} & \left\{y_1\in\mathbb{R}|\forall y_0\leq y_1-\delta \text{ s.t. } (y_0,y_1)\in C,\ y_0\in A_0^-(\varphi,s)\right\}\\
& \cap\left\{y_1\in\mathbb{R}|\forall y_0> y_1-\delta \text{ s.t. } (y_0,y_1)\in C,\ y_0\in \left(A_0^+(\varphi,s)\right)^c\right\},\\
A_{k,D}^-(\varphi,s) = {} & \left\{y_1\in\mathbb{R}|\forall y_0\leq y_1-\delta \text{ s.t. } (y_0,y_1)\in C,\ y_0\in A_k^-(\varphi,s)\right\}\\
& \cap\left\{y_1\in\mathbb{R}|\forall y_0> y_1-\delta \text{ s.t. } (y_0,y_1)\in C,\ y_0\in A_{k-1}^-(\varphi,s)\right\}\\
& \text{for any integer } k> 0.
\end{aligned}$$

Also, according to the definitions above and Figure A.1, if $y_1\in A_{\rho,D}^+(\varphi,s)$ for some $\rho\geq 0$, then

$$\begin{aligned}
&\sup_{y_0\in\mathbb{R}}\left[\sum_{k=0}^{\infty}\left(\mathbf{1}_{A_k^+(\varphi,s)}(y_0)-\mathbf{1}_{A_k^-(\varphi,s)}(y_0)\right)-\mathbf{1}\{y_1-y_0<\delta\}-\lambda\left(1-\mathbf{1}_C(y_0,y_1)\right)\right]\\
&\geq \rho+1,
\end{aligned}$$