is $\omega(E)$ given by Eq. (18). In particular, the low energy levels have spacing ω_J, while the high energy levels are doubly-degenerate with spacing ω_+. In the vicinity of the separatrix we get

$$\omega(E) \;\approx\; \left[\frac{1}{\pi}\log\left|\frac{NK}{E-E_x}\right|\right]^{-1}\omega_J. \tag{29}$$

Using the WKB quantization condition, we find that the level spacing at the vicinity of the separatrix ($E \sim E_{\text{x}}$) is *finite* and given by the expression

$$\omega_{\text{x}} \;=\; \left[\log\left(N/\sqrt{u}\right)\right]^{-1}\omega_J. \tag{30}$$

Using an iterative procedure one finds that at the same level of approximation the near-separatrix energy levels are

$$E_\nu = E_{\text{x}} + \left[\frac{1}{\pi}\log\left|\frac{N/\sqrt{u}}{\nu-\nu_{\text{x}}}\right|\right]^{-1}(\nu-\nu_{\text{x}})\;\omega_J, \tag{31}$$

where $\nu_{\text{x}} = A(E_{\text{x}})/h$. Fig. 1 demonstrates the accuracy of the WKB quantization, and of the above approximations.

IV. THE INITIAL PREPARATION AND ITS PHASE-SPACE REPRESENTATION - THE WIGNER FUNCTION

Our approach for investigating the dynamics of various initial preparations relies on the Wigner-function formalism for spin variables, developed in Refs. [43, 44]. Each initial preparation is described as a Wigner distribution function over the spherical phase space. The dynamics is deduced from expanding the initial state in terms of the semiclassical eigenstates described in Sec III. In this section we specify the Wigner distribution for the preparations under study whereas the following section presents the eigenstate expansion of each of these four initial wavepackets, evaluated semiclassically.

To recap the phase-space approach to spin [43, 44], the Hilbert space of the BHH has the dimension $\mathcal{N} = 2j{+}1$, and the associated space of operators has the dimensionality $\mathcal{N}^2$. According to the Stratonovich-Wigner-Weyl correspondence (SWWC) [50], any observable A in this space, as well as the probability matrix of a spin$^{(j)}$ entity, can be represented by a real sphere$^{(2j)}$ function $A_{\text{W}}(\Omega)$. The sphere$^{(2j)}$ is spanned by the $Y^{\ell m}(\Omega)$ functions with $\ell \le 2j$, and the practical details regarding this formalism can be found in Refs. [43, 44]. The SWWC allows to do exact quantum calculation in a classical-like manner. A few examples for Wigner functions pertinent to this work, are displayed in Fig. 2. Expectation values are calculated as in classical statistical mechanics:

$$\text{trace}[\hat{\rho}\;\hat{A}] \;=\; \int \frac{d\Omega}{h}\rho_{\text{W}}(\Omega)A_{\text{W}}(\Omega) \tag{32}$$

In particular the Wigner-Weyl representation of the identity operator is 1, and that of J_x is as expected $[(j{+}1)j]^{1/2}\sin(\theta)\cos(\varphi)$ [44]. We adopt the convention that ρ_{W} is normalized with respect to the measure $d\Omega/h$, allowing to handle on equal footing a cylindrical phase space upon the re-identification $d\Omega = d\varphi dn$ and $h = 2\pi$.

Within this phase-space representation, the Fock states $|n\rangle$ are represented by stripes along constant θ contours (see e.g. left panel of Fig. 2). The $|n{=}N\rangle$ state (all particles in one site) is a Gaussian-like wave packet concentrated around the NorthPole. From this state, we can obtain a family of spin coherent states (SCS) $|\theta, \varphi\rangle$ via rotation.

In what follows, we explore the dynamics of the following experimentally-accessible preparations (see Fig. 2), the first being a Fock state, whereas the last three are spin coherent states:

- *TwinFock preparation:* The $n{=}0$ Fock preparation. Exactly half of the particles are in each side of the double well. The Wigner function is concentrated along the equator $\theta = \pi/2$.
- *Zero preparation:* Coherent $(\theta{=}\pi/2, \varphi{=}0)$ preparation, located entirely in the (linear) sea region. Both sites are equally populated with definite 0 relative phase. The minimal wave-packet is centered at $(n = 0, \varphi = 0)$.
- *Pi preparation:* Coherent $(\theta{=}\pi/2, \varphi{=}\pi)$ on-separatrix preparation. The sites are equally populated with π relative phase. The minimal wave-packet is centered at $(n = 0, \varphi = \pi)$.
- *Edge preparation:* Coherent $\varphi \neq \pi$ on-separatrix preparation. The minimal wave-packet is centered on the separatrix but away from the saddle point on the φ=0 side.

The Wigner function of an SCS resembles a *minimal* Gaussian wave-packet, and it should satisfy

$$\int \rho_{\text{W}}(\theta,\varphi)\frac{d\Omega}{h} = \int [\rho_{\text{W}}(\theta,\varphi)]^2\frac{d\Omega}{h} = 1. \tag{33}$$

This requirement helps to determine the phase space spread without the need to use the lengthy algebra of Refs. [43, 44]. For the Fock coherent state $|n{=}N\rangle$, that is centered at the NorthPole ($\theta = 0$), one obtains:

$$\rho_{\text{W}}^{(\psi)}(\theta,\varphi) \approx 2\mathrm{e}^{-\frac{\mathcal{N}}{2}\theta^2}\;. \tag{34}$$

For the coherent states centered around the Equator, it is more convenient to use (φ, n) coordinates, e.g. the $\varphi = 0$ coherent state is well approximated as,

$$\rho_{\text{W}}^{(\psi)}(n,\varphi) \;\approx\; \frac{1}{ab}\mathrm{e}^{-\frac{\varphi^2}{2a^2}-\frac{n^2}{2b^2}}\;, \tag{35}$$

with $a = 1/\sqrt{2\mathcal{N}}$ and $b = \sqrt{\mathcal{N}/2}$. Shifted versions of these expressions describe the $\varphi = \pi$ and the "Edge" preparations. The Wigner function of a Fock state $\psi = |\mathsf{n}\rangle$ is semiclassically approximated as

$$\rho_{\text{W}}^{(\psi)}(n,\varphi) \;\approx\; \delta(n-\mathsf{n}), \tag{36}$$