the physical quark masses and the continuum. Although we label the formulae with the subscript "B", we note that these functions can also be used to extrapolate D-meson decay constants and mixing matrix elements with the caveat that the low-energy constants (except for the light-light meson tree-level parameters f and B) are different for the case of B-mesons and D-mesons. We first show the $SU(3)$ HMχPT formulae in Sec. D 1; we then take the appropriate limits of the $SU(3)$ expressions to obtain those in $SU(2)$ HMχPT in Sec. D 2.

1. $SU(3)$ HMχPT expressions

The tree-level mass-squared of a meson composed of two domain-wall valence quarks with flavors x and y is

$$m_{xy}^2 = B(m_x + m_y + 2m_{\rm res}), \tag{D1}$$

where B is a continuum low-energy constant and $m_{\rm res}$ is the residual quark mass.

The NLO result for $\Phi_{B_x} = f_{B_x}\sqrt{m_{B_x}}$ in the partially-quenched domain-wall theory with 2+1 flavors of sea quarks is [81, 82]:

$$\begin{aligned}\Phi_{B_x} = \phi_0\Bigg\{&1 - \frac{1}{16\pi^2 f^2}\frac{1+3g^2_{B^*B\pi}}{2}\sum_{f=l,l,h}\ell(m_{xf}^2)\\ &+\frac{1}{16\pi^2 f^2}\frac{1+3g^2_{B^*B\pi}}{6}\left[R_X^{[2,2]}(\{M_X\};\{\mu\})\widetilde{\ell}(m_X^2) - \sum_{j\in\{M_X\}}\frac{\partial}{\partial m_X^2}\left(R_j^{[2,2]}(\{M_X\};\{\mu\})\right)\ell(m_j^2)\right]\\ &+c_{\rm sea}(2m_l+m_h)+c_{\rm val}m_x+c_a a^2\Bigg\},\end{aligned} \tag{D2}$$

where $f \approx 130.4$ MeV is the tree-level pion decay constant. The NLO expression for $M_{B_x} = 8/3\ m_{B_x} f_{B_x}^2 B_{B_x}$ is similar [81, 83]:

$$\begin{aligned}M_{B_x} \ = \ \beta_0\Bigg\{&1 - \frac{1+3g^2_{B^*B\pi}}{16\pi^2 f^2}\sum_{f=l,l,h}\ell(m_{xf}^2) - \frac{1-3g^2_{B^*B\pi}}{16\pi^2 f^2}\ \ell(m_X^2)\\ &+\ \frac{1}{24\pi^2 f^2}\left[R_X^{[2,2]}(\{M_X\};\{\mu\})\widetilde{\ell}(m_X^2) - \sum_{j\in\{M_X\}}\frac{\partial}{\partial m_X^2}\left(R_j^{[2,2]}(\{M_X\};\{\mu\})\right)\ell(m_j^2)\right]\\ &+\ d_{\rm sea}(2m_l+m_h)+d_{\rm val}m_x+d_a a^2\Bigg\}\ .\end{aligned} \tag{D3}$$

In both the decay constant and the mixing matrix element, the only effect of the nonzero lattice spacing is a new analytic term proportional to a^2. These results agree with the