Case 2: If $u \in [F(a_{j-1}), F(a_j-))$, similar arguments as in Case 1 yields

$$\mathbb{P}(U \leq u) = (u - F\{a_{j-1}\} - \ldots - F\{a_1\}) + (F\{a_1\} + \ldots + F\{a_{j-1}\}) = u.$$

Case 3: If $u \in [0, F(a_1))$, obviously,

$$\mathbb{P}(U \leq u) = u.$$

Case 4: If $u \in [F(a_J), 1]$, obviously,

$$\mathbb{P}(U \leq u) = (u - F\{a_J\} - \ldots - F\{a_1\}) + (F\{a_1\} + \ldots + F\{a_J\}) = u.$$

Thus, it follows that U is uniformly distributed on $[0,1]$. $\square$

Lemma A.2 *Under Assumptions 3.1 and 3.3, $\mathcal{F} = \{x \mapsto 1\{q(x,\theta) \leq u\} : u \in [0,1], \theta \in \Theta\}$ is a Donsker class of functions.*

Proof of Lemma A.2: From Lemma A.1, simply letting $Y = q(X,\theta)$, we have that $1\{q(X,\theta) \leq u\} = 1\{U \leq F_\theta(u)\}$ for all $u \in [0,1]$, i.e., we can exploit the quantile transformation and express each indicator function $1\{q(X,\theta) \leq u\}$ through U via the time transformation F_θ. In light of this quantile transformation, it suffices to study the class of functions $\mathcal{F}_{cdf} = \{\bar{u} \mapsto 1\{\bar{u} \leq F_\theta(u)\} : u \in [0,1], \theta \in \Theta\}$.

Let $N_{[\,]}(\bar{\epsilon}, \mathcal{F}_{cdf}, \mathcal{L}_2(\mathbb{P}))$ be the bracketing number of the class $\mathcal{F}_{cdf}$ with respect to the underlying probability $\mathbb{P}$, which by definition is the minimal number of $\bar{\epsilon}$-brackets under $\mathcal{L}_2(\mathbb{P})$ metric that cover $\mathcal{F}_{cdf}$. Henceforth, we define the underlying probability $\mathbb{P}$ as the probability measure of U. By Theorem 2.5.6 in van der Vaart and Wellner (1996), the Donsker property is implied by

$$\int_0^\infty \sqrt{\log N_{[\,]}(\bar{\epsilon}, \mathcal{F}_{cdf}, \mathcal{L}_2(\mathbb{P}))}\, d\bar{\epsilon} < \infty.$$

To show that such entropy result hold, we follow similar steps as the proof of Lemma 1 in Akritas and van Keilegom (2001) and of Lemma A.4 in Frazier et al. (2018).

Let $\bar{\epsilon} > 0$ be an arbitrarily small constant, and consider partitions $\{\Theta_l\}_{l=1}^L$ of Θ. Given that Θ is compact, under Assumption 3.3, there exists a finite constant $K \leq diam\,(\Theta/\bar{\epsilon})^k$ such that $diam\,(\Theta_l) \leq \bar{\epsilon}$ for every $l = 1, \ldots, K$. Fix $l \in \{1, \ldots, K\}$ and pick up some $\theta_l \in \Theta_l$. Then, for any fixed $u \in [0,1]$ and any $\theta \in \Theta_l$, it follows from the Lipschitz condition in Assumption 3.3 that

$$F_l^-(u) \leq F_\theta(u) \leq F_l^+(u), \tag{A.4}$$