By the layer cake representation theorem, $\varphi_+(x)$ can be written as

$$\begin{aligned}\varphi_+(x) &= \int_0^{\varphi_+(x)} ds \\ &= \int_0^{\infty} \mathbf{1}\{\varphi_+(x) > s\}\, ds \\ &= \sum_{k=0}^{\infty} \int_0^1 \mathbf{1}\{\varphi_+(x) > s+k\}\, ds \\ &= \int_0^1 \sum_{k=0}^{\infty} \mathbf{1}\{\varphi_+(x) > s+k\}\, ds \\ &= \int_0^1 \sum_{k=0}^{\infty} \mathbf{1}\{\varphi(x) > s+k\}\, ds \\ &= \int_0^1 \sum_{k=0}^{\infty} \mathbf{1}_{A_k^+(\varphi,s)}(x)\, ds,\end{aligned} \tag{A.3}$$

where $A_k^+(f,s) = \{y \in \mathbb{R}; f(y) > s+k\}$ for any function $f$. The fourth equality in (A.3) follows from Fubini's theorem. Similarly, the nonpositive function $\varphi_-(x)$ can be represented as

$$\begin{aligned}\varphi_-(x) &= -\int_0^{\infty} \mathbf{1}\{\varphi_-(x) \leq -s\}\, ds \\ &= -\sum_{k=0}^{\infty} \int_0^1 \mathbf{1}\{\varphi_-(x) \leq -(s+k)\}\, ds \\ &= -\int_0^1 \sum_{k=0}^{\infty} \mathbf{1}\{\varphi_-(x) \leq -(s+k)\}\, ds \\ &= -\int_0^1 \sum_{k=0}^{\infty} \mathbf{1}\{\varphi(x) \leq -(s+k)\}\, ds \\ &= -\int_0^1 \sum_{k=0}^{\infty} \mathbf{1}_{A_k^-(\varphi,s)}(x)\, ds.\end{aligned}$$

where $A_k^-(f,s) = \{y \in \mathbb{R}; f(y) \leq -(s+k)\}$ for any function $f$. Similarly, $\psi_+(x)$ and $\psi_-(x)$ are written as follows:

$$\begin{aligned}\psi_+(x) &= \int_0^1 \sum_{k=0}^{\infty} \mathbf{1}_{A_k^+(\psi,s)}(x)\, ds, \\ \psi_-(x) &= -\int_0^1 \sum_{k=0}^{\infty} \mathbf{1}_{A_k^-(\psi,s)}(x)\, ds.\end{aligned}$$