Statistical estimation

Notebook:	INIAD Statistics		
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https://dominhhai.github.io/vi/2017/10/sampling-parameters-estimation/

- # MLE (Maximum Likelihood Estimation)
  - **Idea:** Picking a θ that f(x|θ) has highest chance t happen (x is an already happen circumstance)
  - **Example:** p (0<= p <=1) is the chance that a lottery piece is the winning one
    - If you bought N lottery and win Y of them. Then find p^ that with p=p^, this circumstance have highest change to happen
      - ANSWER: Change to this circumstance happen is **f(x|p^) = p^^{Y}*(1-p^)$^{N-Y}$**. So we have to maximum **f(x|p^)**
        - **f(x|p^) max <=> F = ln[f(x|p^)] max <=> F' = 0**
        - Continue on that formula, we can find that p^ = Y/N

$$L(\theta) = \prod_{i=1}^{n} f(X_i|\theta) = \prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i}$$

Phiên bản $\log$:

$$\begin{aligned} LL(\theta) &= \sum_{i=1}^{n} \log\left(p^{X_i}(1-p)^{1-X_i}\right) \\ &= \sum_{i=1}^{n} \log\left(p^{X_i}\right) + \log\left((1-p)^{1-X_i}\right) \\ &= \sum_{i=1}^{n} X_i \log(p) + (1-X_i)\log(1-p) \end{aligned}$$

Đặt $Y = \sum_{i=1}^{n} X_i$, ta có:

$$LL(\theta) = Y\log(p) + (n-Y)\log(1-p)$$

Giờ ta cần chọn $\hat{p}$ sao cho hàm trên đạt giá trị lớn nhất:

$$\hat{p} = \underset{p}{\operatorname{argmax}}\left(Y\log(p) + (n-Y)\log(1-p)\right)$$

Như ta đã biết hàm này đạt cực trị tại điểm có đạo hàm bằng 0, tức là:

$$\begin{aligned} & LL(p)' = 0 \\ \iff & Y\frac{1}{p} + (n-Y)\frac{-1}{1-p} = 0 \\ \iff & p = \frac{Y}{n} \end{aligned}$$

- **Exercise**

Q. 1

Recall that the pdf of the exponential distribution is as follows.

$$f(x;\lambda) = \lambda e^{-\lambda x}$$

i) Under the observed data of X_1, X_2, ...,X_n, find the log likelihood ln L(λ).

$$\ln L(\theta) = \sum_{i=1}^{n} \ln f(X_i;\theta)$$

ii) Find the maximum-likelihood estimator λ.

A.1

i)

$$\ln L(\theta) = \sum_{i=1}^{n} \ln f(X_i;\theta)$$

$$\log f(x|\theta) = \log \lambda - \lambda x$$

$$\log L(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i$$

A.1

ii)

$$\frac{\partial}{\partial \lambda}\Big(\log L(\lambda)\Big) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

Thus, $\frac{\partial}{\partial \lambda}\Big(\log L(\lambda)\Big) = 0$ yields

$$\boxed{\lambda = \frac{n}{\sum_{i=1}^{n} x_i}}$$

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Q.2

Recall that the pdf of the exponential distribution is as follows.

$$f(x;\lambda) = \lambda e^{-\lambda x}$$

Now, given the observed data of 0.30, 0.06, 0.05, 0.08, 0.12 That follow the exponential distribution, find the ML estimator λ.

Hint; You can apply the result of Q.1

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A.2

0.30, 0.06, 0.05, 0.08, 0.12

Since

$$\lambda = \frac{n}{\sum_{i=1}^{n} x_i}$$

and n=5,

$$\sum_{i=1}^{n} x_i = 0.30 + 0.06 + 0.05 + 0.08 + 0.12 = 0.61$$

$$\lambda = 5/0.61 = \boxed{8.2}$$

Q.3

As we observed the customer arrival interals in a certain amusement park, the observed data were:

1.51, 0.13, 0.21, 2.29, 0.11, 0.79, 0.65, 1.10, 1.08, 2.11 [sec].

Given that they follow the exponential distribution,

i) Find the ML estimator λ;

ii) Find the probability that an interval is 1 sec or less.

A.3

i) Find the ML estimator λ;

$$\sum_{i=1}^{n} x_i = 1.51 + 0.13 + 0.21 + 2.29 + 0.11 + 0.79 + 0.65 + 1.10 + 1.08 + 2.11 = 0.998$$

$$\lambda = 10/0.998 = \boxed{1.0}$$

ii) Find the probability that an interval is 1 sec or less.

$$P(X \leqq x) = 1 - e^{-\lambda x}$$

$$P(X \leqq 1) = 1 - e^{-1.0 \times 1} = = \underline{0.63}$$

Q.4

The Weibull distribution is used to model the interval of system failures. A specific form of its pdf is:

$$f(x\ ;\ \lambda) = 2\lambda^{-2}xe^{-\frac{x^2}{\lambda^2}} \quad (x \geq 0, \lambda > 0)$$

Hint:
$$\ln(f(x)) = \ln(2\lambda^{-2}) + \ln x - \frac{x^2}{\lambda^2}$$

i) Given $X_1, X_2, \ldots, X_n$,
Find the log-likelihood;

ii) Find the ML estimator λ in general;

iii) Given the following data, find the ML estimator λ:
7.01, 7.72, 3.57, 2.56, 3.53

$$\ln(f(x)) = \ln(2\lambda^{-2}) + \ln x - \frac{x^2}{\lambda^2}$$

$$\ln L(\lambda) = n\ln(2\lambda^{-2}) + \sum \ln(x_i) - \frac{1}{\lambda^2}\sum x_i^2$$

A.4

$$L(\lambda) = f(x_1;\lambda) \times f(x_2;\lambda) \times \ldots f(x_n;\lambda) = \frac{2}{\lambda^2}x_1 \mathrm{e}^{-\frac{x_1^2}{\lambda^2}} \times \frac{2}{\lambda^2}x_2 \mathrm{e}^{-\frac{x_2^2}{\lambda^2}} \times \ldots \frac{2}{\lambda^2}x_n \mathrm{e}^{-\frac{x_n^2}{\lambda^2}}$$

$$\boxed{= \frac{2^n}{\lambda^{2n}}(\prod_{j=1}^{n} x_j)\mathrm{e}^{-\frac{\sum_{k=1}^{n} x_k^2}{\lambda^2}}}$$

$$= \frac{2^n}{\lambda^{2n}}(\prod_{j=1}^{n} x_j)\mathrm{e}^{-\frac{\sum_{k=1}^{n} x_k^2}{\lambda^2}}$$

①
$$\boxed{\log L(\lambda) = n \log 2 - 2n \log \lambda + \sum_{j=1}^{n} \log x_j - \frac{1}{\lambda^2}\sum_{k=1}^{n} x_k^2}$$

A.4

$$\log L(\lambda) = n \log 2 - 2n \log \lambda + \sum_{j=1}^{n} \log x_j - \frac{1}{\lambda^2}\sum_{k=1}^{n} x_k^2$$

$$\Downarrow$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\log L(\lambda)) = -\frac{2n}{\lambda} + \frac{2}{\lambda^3}\sum_{k=1}^{n} x_k^2 = 0$$

②
$$\boxed{\lambda = \sqrt{\frac{\sum_{k=1}^{n} x_k^2}{n}}}$$

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A.4

$$\lambda = \sqrt{\frac{\sum_{k=1}^{n} x_k^2}{n}}$$

③ $$\boxed{\hat{\lambda} = \sqrt{\frac{(7.01^2 + 7.72^2 + 3.57^2 + 2.56^2 + 3.53^2)}{5}} = 5.30}$$

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