Mục lục

1 Angles

1.1 Find the length of an arc of a circle of radius 5cm subtending a central angle measuring 15°

We have:

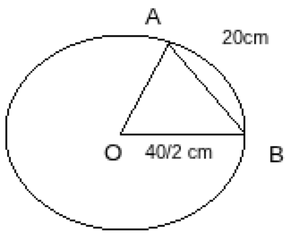
$$\theta = \frac{L}{r}$$
$$\rightarrow L = \theta r = \frac{15\pi 5}{180} = 1.31cm$$

1.2 Find in degrees the angle subtended at the center of a circle of diameter 50 cm by an arc of length 11 cm

We have (use with degree):

$$\theta = \frac{L}{r} = \frac{11}{50/2} . \frac{\pi}{180} = 25^o 12'$$

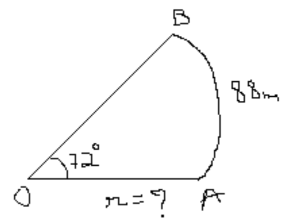
1.3 In a circle of diameter 40 cm the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.



From formular we have

$$l = \frac{\pi r \theta}{180} \rightarrow \theta^o = \frac{l.180}{\pi r}$$
$$\theta = \frac{L}{r} \rightarrow L_{AB} = \frac{l.180}{\pi r} . \frac{\pi}{180} .r = 20cm$$

1.4 A horse is tied to a post by a rope, if the horse moves along a circular path always keeping the rope tight and describe 88 meters when it has traced out 72° at the center, find the length of the rope



we have

$$l = \frac{\pi r \theta}{180} \rightarrow r = \frac{l.180}{\pi \theta^o}$$
$$r = \frac{88 \times 180}{\pi .72} = 70.03m$$

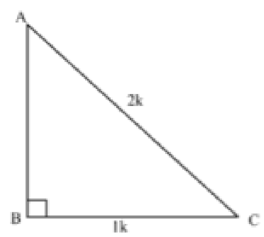
2 Trigonometry Ratio

2.1 If $\sqrt{3}\tan(\theta) = 3\sin(\theta)$, then find the value of $\sin^2(\theta) - \cos^2(\theta)$

We have

$$\sqrt{3}\tan(\theta) = 3\sin(\theta) \rightarrow \frac{\tan(\theta)}{\sin(\theta)} = \frac{3}{\sqrt{3}}$$
$$\rightarrow \frac{1}{\sin(\theta)} . \frac{\sin(\theta)}{\cos(\theta)} = \frac{3}{\sqrt{3}} \rightarrow \frac{1}{\cos(\theta)} = \frac{3}{\sqrt{3}}$$
$$\rightarrow \theta = \cos^{-1} \frac{\sqrt{3}}{3}$$
$$\rightarrow \sin^2(\theta) - \cos^2(\theta) = \frac{1}{3}$$

2.2 In triangle ABC, right angled at B if $\sin A = 1/2$, find the value of
(a) $\sin C \cos A - \cos C \sin A$
(b) $\cos A \cos C + \sin A \sin C$



$\sin A = BC/AC = 1/2. Set BC = 1k, AC = 2k.$
So BA $= \sqrt{(2k)^2 - k^2} = \sqrt{3k}$
$\sin C = \frac{AB}{AC} = \frac{\sqrt{3}}{2} \quad \cos C = \frac{AB}{AC} = \frac{\sqrt{3}}{2}$
$\cos A = \frac{AB}{AC} = \frac{\sqrt{3}}{2}$
$(a) \sin C \cos A - \cos C \sin A$

(b) $\cos A \cos C + \sin A \sin C$

$$\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} - \frac{1}{2}.\frac{1}{2} = \frac{1}{2}$$

$$\frac{\sqrt{3}}{2}.\frac{1}{2} + \frac{1}{2}.\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

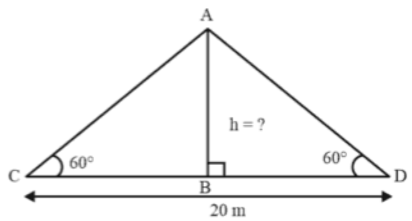
2.3 If θ is an acute angle and $\frac{\sin\theta+1}{\sin\theta+1} = \frac{\sqrt{3}+2}{\sqrt{3}-2}$

$$\frac{\sin\theta\cos 1 + \cos\theta\sin 1}{\sin\theta\cos 1 - \cos\theta\sin 1} = \frac{\sqrt{3}+2}{\sqrt{3}-2}$$
$$\left(\sin\theta\cos 1 + \cos\theta\sin 1\right)\left(\sqrt{3}-2\right) = \left(\sqrt{3}+2\right)\left(\sin\theta\cos 1 - \cos\theta\sin 1\right)$$
$$\sqrt{3}\cos 1\sin\theta + \sqrt{3}\sin 1\cos\theta - 2\cos 1\sin\theta - 2\sin 1\cos\theta = \sqrt{3}\cos 1\sin\theta - \sqrt{3}\sin 1\cos\theta + 2\cos 1\sin\theta - 2\sin 1\cos\theta$$
$$\left(\sqrt{3}\cos 1 - 2\cos 1\right)\sin\theta + \left(\sqrt{3}\sin 1 - 2\sin 1\right)\cos\theta = \left(\sqrt{3}\cos 1 + 2\cos 1\right)\sin\theta - \left(\sqrt{3}\sin 1 + 2\sin 1\right)\cos\theta$$
$$\left[\left(\sqrt{3}\cos 1 - 2\cos 1\right) - \left(\sqrt{3}\cos 1 + 2\cos 1\right)\right]\sin\theta = -\left[\left(\sqrt{3}\sin 1 + 2\sin 1\right) + \left(\sqrt{3}\sin 1 - 2\sin 1\right)\right]\cos\theta$$

We have: $ax = by \rightarrow \dfrac{x}{y} = \dfrac{b}{a}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\left(\sqrt{3}\sin 1 + 2\sin 1\right) - (\sqrt{3}\sin 1 - 2\sin 1)}{\left(\sqrt{3}\cos 1 - 2\cos 1\right) - \left(\sqrt{3}\cos 1 + 2\cos 1\right)}$$
$$\rightarrow \theta = 53^o 26'$$

2.4 Find the height of the triangle



We have

$$\tan(60) = \frac{h}{20/2}$$
$$\rightarrow h = \tan(60) * 10 = 10\sqrt{3}$$

3 Trigonometry Angles

3.1 If $\sin(\theta) - \cos(\theta) = 1$, then $\sin(\theta)\cos(\theta)$ equal

$$\frac{d}{h} - \frac{k}{h} = 1; \qquad \frac{d}{h}\frac{k}{h} = \frac{dk}{h^2} =?$$
$$\frac{d}{h} = 1 + \frac{k}{h} \text{ and } \frac{k}{h} = \frac{d}{h} - 1$$
$$\sin\theta\cos\theta = \left(1 + \frac{k}{h}\right)\left(\frac{d}{h} - 1\right) =?$$
$$\frac{d}{h} - 1 + \frac{kd}{h^2} - \frac{k}{h} =?$$
$$\frac{kd}{h^2} = \sin\theta\cos\theta = \frac{-d}{h} + \frac{k}{h} + 1 = -1 + 1 = 0$$

3.2 If $\sin(\theta) = \mathbf{e^x}$ then $\cos(\theta)$ equal

$$\sin(\theta) = e^x \rightarrow \cos^2(\theta) + e^{2x} = 1$$
$$\cos(\theta) = \sqrt{1 - e^{2x}}$$

4 Trigonometry Formula

4.1 If $\mathbf{x} = \mathbf{sec}(\theta) + \tan(\theta)$, then $\mathbf{x} + \frac{\mathbf{1}}{\mathbf{x}} =$?

$$\begin{aligned}
x + \frac{1}{x} &= sec(\theta) + \tan(\theta) + \frac{1}{sec(\theta) + \tan(\theta)} \\
&= \frac{sec(\theta)\Big(sec(\theta) + \tan(\theta)\Big) + \tan(\theta)\Big(sec(\theta) + \tan(\theta)\Big) + 1}{sec(\theta) + \tan(\theta)} \\
&= sec^2(\theta) + sec(\theta)\tan(\theta) + \tan^2(\theta) + sec(\theta)\tan(\theta) + 1 \\
&= sec^2(\theta) + sec(\theta)\tan(\theta) + \tan^2(\theta) + sec(\theta)\tan(\theta) + sec^2(\theta) - \tan^2(\theta) \\
&= 2sec^2(\theta) + 2sec(\theta)\tan(\theta) \\
&= \frac{2sec(\theta)\Big(sec(\theta) + \tan(\theta)\Big)}{sec(\theta) + \tan(\theta)} \\
&= 2sec(\theta)
\end{aligned}$$

4.2 $\sqrt{\frac{\mathbf{1}-\sin(\theta)}{\mathbf{1}+\sin(\theta)}}$ equal?

$$\begin{aligned}
\sqrt{\frac{1-\sin(\theta)}{1+\sin(\theta)}} &= \sqrt{\frac{1-\sin(\theta)}{1+\sin(\theta)}} \times 1 = \sqrt{\frac{1-\sin(\theta)}{1+\sin(\theta)}} \times \sqrt{\frac{1-\sin(\theta)}{1-\sin(\theta)}} \\
&= \frac{\Big[1-\sin(\theta)\Big]^{1/2} \times \Big[1-\sin(\theta)\Big]^{1/2}}{\Big[1+\sin(\theta)\Big]^{1/2} \times \Big[1-\sin(\theta)\Big]^{1/2}} = \frac{1-\sin(\theta)}{\Bigg[\Big(1+\sin(\theta)\Big)\Big(1-\sin(\theta)\Big)\Bigg]^{1/2}} \\
&= \frac{1-\sin(\theta)}{\Big[1-\sin(\theta)+\sin(\theta)-\sin^2(\theta)\Big]} \\
&= \frac{1-\sin(\theta)}{\sqrt{1-\sin^2(\theta)}} = \frac{1-\sin(\theta)}{\sqrt{\cos^2(\theta)}} = \frac{1-\sin(\theta)}{\cos(\theta)} \\
&= \frac{1}{\cos(\theta)} - \tan(\theta) = \sec(\theta) - \tan(\theta)
\end{aligned}$$

4.3 $\frac{\sin(\theta)}{\mathbf{1}-\cot(\theta)} + \frac{\cos(\theta)}{\mathbf{1}-\tan(\theta)}$ equal?

$$\begin{aligned}
\frac{\sin(\theta)}{1-\cot(\theta)} + \frac{\cos(\theta)}{1-\tan(\theta)} &= \frac{\sin(\theta)[1-\tan(\theta)] + \cos(\theta)[1-\cot(\theta)]}{[1-\cot(\theta)][1-\tan(\theta)]} \\
&= \frac{\sin(\theta) - \sin(\theta)\tan(\theta) + \cos(\theta) - cos(\theta)\cot(\theta)}{[1-\cot(\theta)][1-\tan(\theta)]} \\
&= \frac{\sin(\theta)[1-\tan(\theta)] + \cos(\theta)[1-\cot(\theta))]}{[1-\cot(\theta)][1-\tan(\theta)]} \\
&= \sin(\theta) + \cos(\theta)
\end{aligned}$$

5 Differentiation

5.1 $\mathbf{f(x) = 6x^3 - 9x + 4}$

$$f'(x) = 18x^2 - 9$$

5.2 $\mathbf{f(x)} = \sqrt{\mathbf{x}} + \mathbf{8}\sqrt[3]{x} - 2\sqrt[4]{x}$

$$\begin{aligned}
f'(x) &= \frac{1}{2}x^{-1/2} + \frac{8}{3}x^{-2/3} - \frac{1}{2}x^{-3/4} \\
&= \frac{1}{2\sqrt{x}} + \frac{8}{3\sqrt[3]{x^2}} - \frac{1}{2\sqrt[4]{x^3}}
\end{aligned}$$

5.3 $\mathbf{f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3}$

$$\begin{aligned} f'(x) &= 6x^{-2/5} - \frac{7}{2}x^{5/2} + 16x^{5/3} \\ &= \frac{6}{\sqrt{x^5}} - \frac{7\sqrt{x^5}}{2} + 16\sqrt[3]{x^5} \end{aligned}$$

5.4 $\mathbf{g(x) = \frac{4x^3-7x+8}{x}}$

Way 1: $(\frac{u}{v})^2 = \frac{u'.v-v'u}{v^2}$

$$\begin{aligned} &u = 4x^3 - 7x + 8 \quad u' = 12x^2 - 7 \\ &v = x \quad v' = 1 \end{aligned}$$

$$\begin{aligned} g'(x) &= \frac{(12x^2-7)(x) - (4x^3-7x+8)}{x^2} \\ &= \frac{12x^3 - 7x - 4x^3 + 7x - 8}{x^2} \\ &= \frac{8x^3-8}{x^2} = \frac{8x^3}{x^2} - \frac{8}{x^2} = 8x - \frac{8}{x^2} \end{aligned}$$

Way 2:

$$\begin{aligned} g(x) &= 4x^2 - 7 + 8x^{-1} \\ g'(x) &= 8x - \frac{8}{x^2} \end{aligned}$$

5.5 $\mathbf{y = e^{\sqrt{2x+17}}}$

We have:

$$(e^u)' = u'e^u \qquad (\sqrt[n]{u})' = \frac{u'}{n\sqrt[n]{u^{n-1}}}$$

So we have:

$$\begin{aligned} \frac{dx}{dy} &= (\sqrt{2x+17})'e^{\sqrt{2x+17}} = \frac{2}{2\sqrt{2x+17}}e^{\sqrt{2x+17}} \\ &= \frac{e^{\sqrt{2x+17}}}{\sqrt{2x+17}} \end{aligned}$$

5.6 $\mathbf{y = e^x \ln(5x^3 + x^2)}$

We have:

$$(uv)' = u'v + v'u \qquad (e^u)' = u'e^u \qquad (\ln(u))' = \frac{u'}{u}$$

So,

$$\frac{dy}{dx} = e^x \ln(5x^3 + x^2) + \frac{(15x^2 + 2x)e^x}{5x^3 + x^2}$$

5.7 $\mathbf{y = \frac{x^2}{\ln(1-4x^2)}}$

We have:

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{u^2} \qquad (x^n)' = nx^{n-1} \qquad (\ln(u))' = \frac{u'}{u}$$

So,

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x.\ln(1-4x^2) - \frac{-8x.x^2}{(1-4x^2)}}{\ln^2(1-4x^2)} \\ &= \frac{2x(1-4x^2)\ln(1-4x^2) + 8x^3}{(1-4x^2)\ln^2(1-4x^2)} \end{aligned}$$

5.8 $\mathbf{y = \frac{\tan(x)+\cot(x)}{\tan(x)-\cot(x)}}$

We have:

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} \qquad (\tan(u))' = \frac{u'}{\cos^2(u)} \qquad (\cot(u))' = \frac{-u'}{\sin^2(u)}$$

So, now we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{\Big(\frac{1}{\cos^2(x)} + \frac{-1}{\sin^2(x)}\Big)\Big(\tan(x) - \cot(x)\Big) - \Big(\frac{1}{\cos^2(x)} - \frac{-1}{\sin^2(x)}\Big)\Big(\tan(x) + \cot(x)\Big)}{\Big(\tan(x) - \cot(x)\Big)^2} \\ &= \frac{\Big(\sec^2(x) - cosec^2(x)\Big)\Big(\tan(x) - \cot(x)\Big) - \Big(\sec^2(x) + cosec^2(x)\Big)\Big(\tan(x) + \cot(x)\Big)}{\Big(\tan(x) - \cot(x)\Big)^2} \\ &= \frac{-2\tan(x)cosec^2(x)}{\Big(\tan(x) - \cot(x)\Big)^2} \end{aligned}$$

5.9 Find $\mathbf{\frac{dy}{dx}}$ for $x^2 + y^3 = 3$

Differentiationg both size and aplly chain rule

$$\begin{aligned} &\frac{dy}{dx}x^2 + \frac{dy}{dx}y^3 = \frac{dy}{dx}3 \\ &\to 2x + \frac{dy}{dx}3y^2 = 0 \\ &\to \frac{dy}{dx} = -\frac{2x}{3y^2} \end{aligned}$$

5.10 Find $\frac{\mathbf{dy}}{\mathbf{dx}}$ for $x^2 + y^2 = 2$

Differentiationg both size and aplly chain rule

$$\begin{aligned}
&(x^2 + y^2 = 2)' \\
&\rightarrow 2x + 2y\frac{dy}{dx} = 0 \\
&\rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}
\end{aligned}$$

5.11 Find $\frac{\mathbf{dy}}{\mathbf{dx}}$ for $2y^3 + 4x^2 - y = x^6$

Differentiationg both size and aplly chain rule

$$\begin{aligned}
&(2y^3 + 4x^2 - y = x^6)' \\
&\rightarrow 6y^2\frac{dy}{dx} + 8x - \frac{dy}{dx} = 6x^5 \\
&\rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}
\end{aligned}$$

5.12 Find $\frac{\mathbf{dy}}{\mathbf{dx}}$ for $7y^2 + \sin(3x) = 12 - y^4$

Differentiationg both size and aplly chain rule

$$\begin{aligned}
&(7y^2 + \sin(3x) = 12 - y^4)' \\
&\rightarrow 14y\frac{dy}{dx} + 3\cos(3x) = -4y^3\frac{dy}{dx} \\
&\rightarrow \frac{dy}{dx}(14y + 4y^3) = -3\cos(3x) \\
&\rightarrow \frac{dy}{dx} = \frac{-3\cos(3x)}{14y + 4y^3}
\end{aligned}$$

5.13 Find $\frac{\mathbf{dy}}{\mathbf{dx}}$ for $xy = 1$

Apply product rule

$$\begin{aligned}
&(xy = 1)' \\
&\rightarrow x'y + xy' = 0 \\
&\rightarrow \frac{dy}{dx}y = -x \\
&\rightarrow \frac{dy}{dx} = -\frac{x}{y}
\end{aligned}$$

5.14 Find $y' = \frac{dy}{dx}$ for each of the follwing

a) $x^3y^5 + 3x = 8y^3 + 1$
b) $x^2\tan(y) + y^{10}\sec(x) = 2x$
c) $e^{2x+3y} = x^2 - \ln(xy^3)$

<Solution>
a) $x^3y^5 + 3x = 8y^3 + 1$
We have

$$\begin{aligned}
&(u.v)' = u'v + v'u \\
&\rightarrow 3x^2y^5 + 5x^3y^4\frac{dy}{dx} + 3 = 24y^2\frac{dy}{dy} \\
&\leftrightarrow \frac{dy}{dy}(5x^3y^4 - 24y^2) = -3 - 3x^2y^5 \\
&\rightarrow \frac{dy}{dx} = \frac{-3 - 3x^2y^5}{5x^3y^4 - 24y^2}
\end{aligned}$$

b) $x^2\tan(y) + y^{10}\sec(x) = 2x$
We have

$$\begin{aligned}
&(uv)' = u'v + v'u \\
&\Big(\tan(u)\Big)' = \frac{u'}{\cos^2(u)} \\
&\Big(\sec(x)\Big)' = \sec(x).\tan(x) \\
\rightarrow &\Big(x^2\tan(y)\Big)' = 2x\tan(y) + \frac{dy/dx}{\cos^2(y)}x^2 \\
\rightarrow &\Big(y^{10}\sec(x)\Big)' = 10y^9\frac{dy}{dx}\sec(x) + y^{10}\sec(x)\tan(x)
\end{aligned}$$

So,

$$
\begin{aligned}
2x\tan(y)+\frac{dy/dx}{\cos^2(y)}x^2+10y^9\frac{dy}{dx}\sec(x)+y^{10}\sec(x)\tan(x)=0\\
\rightarrow \frac{dy/dx}{\cos^2(y)}x^2+10y^9\frac{dy}{dx}\sec(x)=-y^{10}\sec(x)\tan(x)-2x\tan(y)\\
\rightarrow \frac{dy}{dx}=\frac{2-y^{10}\sec(x)\tan(x)-2x\tan(y)}{x^2/\cos^2(y)+10y^9\sec(x)}\\
\rightarrow \frac{dy}{dx}=\frac{2-2x\tan(y)-y^{10}\sec(x)\tan(x)}{x^2\sec^2(y)+10y^9\sec(x)}
\end{aligned}
$$

c) $e^{2x+3y}=x^2-\ln(xy^3)$
We have

$$
\begin{aligned}
\Big(e^{2x+3y}\Big)' &= \Big(2+3\frac{dy}{dx}\Big)e^{2x+3y}\\
&= 2e^{2x+3y}+3e^{2x+3y}\frac{dy}{dx}\\
\Big(\ln(xy^3)\Big)' &= \frac{1}{xy^3}\Big(y^3+3xy^2\frac{dy}{dx}\Big)\\
&= \frac{y^3}{xy^3}+\frac{3xy^2(dy/dx)}{xy^3}\\
&= \frac{1}{x}+\frac{3}{y}\frac{dy}{dx}
\end{aligned}
$$

Now we have

$$
\begin{aligned}
2e^{2x+3y}+3e^{2x+3y}\frac{dy}{dx}=2x-\frac{1}{x}-\frac{3}{y}\frac{dy}{dx}\\
\frac{dy}{dx}\Big(3e^{2x+3y}-3y\Big)=2x-\frac{1}{x}-2e^{2x+3y}\\
\rightarrow \frac{dx}{dy}=\frac{2x-1/x-2e^{2x+3y}}{3e^{2x+3y}-3/y}
\end{aligned}
$$

6 Expert Implicit Differentiation

6.1 If $x=exp\left[\tan^{-1}\left(\frac{y-x^2}{x^2}\right)\right]$ then $\frac{dy}{dx}$ equals