Đề 1

a. $\begin{cases} y_0(x) = y_0 \\ y_{k+1}(x) = y_0 + \int_{x_0}^{x} f(s, y_k(s))\,ds. \end{cases}$

b. Euler

$\begin{cases} y(x) \approx \sum_{k=0}^{m} \dfrac{y^{(k)}}{k!}(x - x_0)^k \\ y_{k+1} \approx y_k + (x_{k+1} - x_k) f(x_k, y_k) \text{ với } m = 1. \end{cases}$

c. Euler cải tiến

$\begin{cases} \tilde{y}_{k+1} \approx y_k + (x_{k+1} - x_k) f(x_k, y_k) \\ y_{k+1} \approx y_k + \dfrac{x_{k+1} - x_k}{2}\left(f(x_k, y_k) + f(x_{k+1}, \tilde{y}_{k+1}\right. \end{cases}$

d. Runge-Kutta

Bậc 2: $\begin{cases} k_1 = h_i f(x_i, y_i) \\ k_2 = h_i f(x_{i+1}, y_1 + k_1) \\ y_{i+1} \approx y_i + \frac{1}{2}(k_1 + k_2) \end{cases}$

Bậc 3: $\begin{cases} k_1 = h_i f(x_i, y_i) \\ k_2 = h_i f(x_i + \frac{1}{2}h_i, y_i + k_1) \\ k_3 = h_i f(x_i + h_i, y_i - k_1 + 2k_2) \\ y_{i+1} \approx y_i + \frac{1}{6}(k_1 + 4k_2 + k_3) \end{cases}$

7.2

a. $\begin{cases} y' = xy \\ y(0) = 2 \end{cases}$ $x \in [0, 1]$ $\begin{cases} x_0 = 0 \\ y_0 = 2 \end{cases}$

$y_0(x) = 2$

$y_1(x) = 2 + \int_0^x s.2\, ds = x^2 + 2$

$y_2(x) = 2 + \int_0^x s(s^2 + 2)\, ds = \frac{x^4}{4} + x^2 + 2$

$y_3(x) = 2 + \int_0^x s\left(\frac{s^4}{4} + s^2 + 2\right) ds = \frac{s^6}{24} + \frac{s^4}{4} + s^2 + 2$

b. $\begin{cases} y' = (x+1)y \\ y(1) = 0 \end{cases}$ $\begin{cases} x_0 = 1 \\ y_0 = 0 \end{cases}$

$y_0(x) = 0$

$y_1(x) = 0 + \int_1^x (s+1).0\, ds = 0$

$y_2(x) = 0 + \int_1^x (s+1).0\, ds = 0$

$y_3(x) = 0 + \int_1^x (s+1).0\, ds = 0$

c. $\begin{cases} y' = x + xy^2 \\ y(0) = 1. \end{cases}$ $x \in [-2, 2]$ $\begin{cases} x_0 = 0 \\ y_0 = 1 \end{cases}$

$y_0(x) = 1$

$y_1(x) = 1 + \int_0^x (s + s.1)\, ds = x^2 + 1$

$y_2(x) = 1 + \int_0^x \left[s + s(s^2 + 1)\right] ds = \frac{s^4}{4} + s^2 + 1$

$y_3(x) = 1 + \int_0^x \left[s + s\left(\frac{s^4}{4} + s^2 + 1\right)\right] ds = \frac{s^6}{24} + \frac{s^4}{4} + s^2 + 1$

d. $\begin{cases} y' = x^2 + \frac{y}{x} \\ y(1) = 1 \end{cases}$ $x \in [1; 3]$ $\begin{cases} x_0 = 1 \\ y_0 = 1 \end{cases}$

$y_0(x) = 1$

$y_1(x) = 1 + \int_1^x \left(s^2 + \frac{1}{s}\right) ds = 1 + \frac{x^3}{3} + \ln(x) - \frac{1}{3}$

$= \frac{x^3}{3} + \ln x + \frac{2}{3}$

$y_2(x) = 1 + \int_1^x \left(s^2 + \frac{\frac{s^3}{3} + \ln s + \frac{2}{3}}{s}\right) ds$

$= \frac{4x^3}{9} + \frac{\ln^2 x}{2} + \frac{1}{3}\ln x + \frac{5}{9}$

$$\boxed{y_3(x) = 1 + \int_1^x \left(s^2 + \frac{\frac{4s^3}{9} + \frac{\ln^2 x}{2} + \frac{2}{3}\ln x + \frac{5}{9}}{s}\right) dx}$$

đ3

a. $\begin{cases} y' = \sqrt{x^2 + 2y + 1} + y = f(x,y) & x \in [0; 0,8] \to [0 \quad 0,2 \quad 0,4 \quad 0,6 \quad 0,8] \\ y(0) = 1 \Rightarrow y_0 = 1; \ x_0 = 0; \ h = 0,2 \end{cases}$

$y_1 = y_0 + h f(x_0, y_0) = 1 + h. f(0; 1) = 1,4$

$y_2 = y_1 + h f(x_1, y_1) = 1 + h. f(0,2; 1,4) =$

$= 1,4 + \frac{1}{5}. 2,5489 = 1,9098.$

$y_3 = y_2 + h. f(x_2, y_2) = 1,9098 + h f(0,4; 1,9098)$

$= 2,5691$

$y_4 = y_3 + h. f(x_3, y_3) = 2,5691 + h f(0,6; 2,5691)$

$= 3,4332.$

→ Euler cải tiến.

$\tilde{y}_1 = y_0 + h f(0, 1) = 1,4$

$y_1 = y_0 + \frac{h}{2}\left(f(t_0, y_0) + f(t_1, \tilde{y}_1)\right) = 1,4549.$

$\tilde{y}_2 = y_1 + h f(x_1, y_1) = 1,4549 + h f(0,2; 1,4549) = 1,9866$

$y_2 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, \tilde{y}_2)\right) = 1,4549 + \frac{h}{2}\left(f(0,2; 1,4549) + f(0,2; 1,986\ldots)\right)$

$= 2,0531$

$\tilde{y}_3 = y_2 + h f(x_2, y_2) = 2,0531 + h f(0,4; 2,0531) = 2,7452$

$y_3 = y_2 + \frac{h}{2}\left(f(x_2, y_2) + f(x_3, \tilde{y}_3)\right) = 2,0531 + \frac{h}{2}\left(f(0,4; 2,0531) + f(0,6; 2,745\ldots)\right)$

$= 2,8421$

$\tilde{y}_4 = y_3 + h f(x_3, y_3) = 2,8421 + h f(0,6; 2,8421) = 3,86\ldots$

$y_4 = y_3 + \frac{h}{2} f\left(f(x_3, y_3) + f(x_4, \tilde{y}_4)\right) = \boxed{3,8993}$

(5) $\begin{cases} y' = x \ln 2x^2 + y^2 - 1 \\ y(0,5) = 1 \end{cases}$

— o — Euler:

$y_1 = y_0 + h f(x_0; y_0) = 1 + h f(0,5; 1) = 1,3302$

$y_2 = y_1 + h f(x_1; y_1) = 1,3302 + h f(0,7; 1,3302) = 1,882$

$y_3 = y_2 + h f(x_2; y_2) = 1,882 + h f(0,9; 1,882) = \boxed{2,8172}$

— o — Euler cải tiến:

$\tilde{y}_1 = y_0 + h.\ f(x_0, y_0) = 1 + h.f(0{,}5 ; 1) = 1{,}3302.$

$y_1 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, \tilde{y}_1)\right) = 1 + \frac{h}{2}\left(f(0{,}5 ; 1) + f(0{,}7 ; 1{,}330)\right)$
$= 1{,}441$

$\tilde{y}_2 = y_1 + hf(x_1, y_1) = 1{,}441 + hf(0{,}7 ; 1{,}441) = 2{,}0535$

$y_2 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, \tilde{y}_2)\right) = 1{,}441 + \frac{h}{2}\left(f(0{,}7 ; 1{,}441) + f(0{,}9 ; 2{,}0535)\right)$
$= 2{,}3123$

$\tilde{y}_3 = y_2 + hf(x_2, y_2) = 2{,}3123 + hf(0{,}9 ; 2{,}3123) = 3{,}6685$

$y_3 = y_2 + \frac{h}{2}\left(f(0{,}9 ; 2{,}3123) + f(1{,}1 ; 3{,}6685)\right) = \boxed{4{,}5335}$

c) $\begin{cases} y' = xy\cos x^2 + y^2 \\ y(0, 1) = 1 \end{cases}$

— o — Euler

$y_1 = y_0 + hf(x_0, y_0) = 1 + hf(0{,}1 ; 1) = 1{,}101$

$y_2 = y_1 + hf(x_1, y_1) = 1{,}11 + hf(0{,}2 ; 1{,}11) = 1{,}2554$

$y_3 = y_2 + hf(x_2, y_2) = 1{,}2554 + hf(0{,}3 ; 1{,}2554) = 1{,}4505$

$y_4 = y_3 + hf(x_3, y_3) = 1{,}4505 + hf(0{,}4 ; 1{,}4505) = 1{,}7182$

— o — Euler cải tiến

$\tilde{y}_1 = y_0 + h.f(x_0, y_0) = 1 + hf(0{,}1 ; 1) = 1{,}11$

$y_1 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, \tilde{y}_1)\right) = 1 + \frac{h}{2}\left(f(0{,}1 ; 1{,}11) + f(0{,}2 ; 1{,}11)\right)$
$= 1{,}1277.$

$\tilde{y}_2 = y_1 + hf(x_1, y_1) = 1{,}1277 + hf(0{,}2 ; 1{,}1277) = 1{,}2774$

$y_2 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, \tilde{y}_2)\right) = 1{,}3032$

$\tilde{y}_3 = y_2 + hf(x_2, y_2) = 1,3032 + hf(0,3; 1,3032) = 1,512$

$y_3 = y_2 + \frac{h}{2}\left(f(x_2, y_2) + f(x_3, \tilde{y}_3)\right) = 1,5518.$

$\tilde{y}_4 = y_3 + hf(x_3, y_3) = 1,5518 + hf(0,4; 1,5518) = 1,8538$

$y_4 = y_3 + \frac{h}{2}\left(f(x_3, y_3) + f(x_4, \tilde{y}_4)\right) = 1,9195$

d. $\begin{cases} y' = (x+1)/y^2 \\ y(0) = 1 \end{cases}$

$y_1 = y_0 + hf(x_0, y_0) = 1 + hf(0, 1) = 1,2$

$y_2 = y_1 + hf(x_1, y_1) = 1,2 + hf(0,2; 1,2) = 1,3667$

$y_3 = y_2 + hf(x_2, y_2) = 1,3667 + hf(0,4; 1,3667) = 1,5166$

$y_4 = y_3 + hf(x_3, y_3) = 1,5166 + hf(0,6; 1,5166) = 1,6558$

$y_5 = y_4 + hf(x_4, y_4) = \cancel{1,5166}\ 1,787$

— o — Euler cải tiến.

$\tilde{y}_1 = y_0 + hf(x_0, y_0) = 1,2$

$y_1 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, \tilde{y}_1)\right) = 1 + \frac{1/5}{2}(1 + 0,8333)$
$= 1,1833$

$\tilde{y}_2 = y_1 + hf(x_1, y_1) = 1,3548$

$y_2 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, \tilde{y}_2)\right) = 1,1833 + \frac{1/5}{2}(0,857 + 0,2628) = 1,3$

$\tilde{y}_3 = y_2 + hf(x_2, y_2) = 1,5$

$y_3 = y_2 + \frac{h}{2}\left(f(x_2, y_2) + f(x_3, \tilde{y}_3)\right) = 1,3453 + \frac{1/5}{2}(0,7735 + 0,2\ldots$
$= 1,4938.$

$\bar{y}_4 = y_3 + hf(x_3, y_3) = 1,4938 + \frac{1}{5} f(0,6; 1,4938) = 1,6322$

$y_4 = y_3 + \frac{h}{2}\left(f(x_3, y_3) + f(x_4, \bar{y}_4)\right) = 1,4938 + \frac{1/5}{2}(0,2212 + 0,6215)$

$= 1,6326$

$\bar{y}_5 = y_4 + h f(x_4, y_4) = 1,2622$

$y_5 = y_4 + \frac{h}{2}\left(f(x_4, y_4) + f(x_5, \bar{y}_5)\right) = 1,2642$.

2.7

a. $\begin{cases} y' = x\sin(x + 2y) \\ y(0) = 1 \end{cases}$

- $x_1 = 0,2$: $k_1 = hf(x_0, y_0) = 0$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0,0123$

$k_3 = hf(x_0 + h, y_0 - k_1 + 2k_2) = 0,0366$

$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3) = 1,0166$

$x_2 = 0,4$: $k_1 = h f(x_1, y_1) = 0,0315$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1\right) = 0,0421$

$k_3 = hf(x_1 + h, y_1 - k_1 + 2k_2) = 0,0459$

$y_2 = y_1 + \frac{1}{6}(k_1 + 4k_2 + k_3) = 1,0585$

$x_3 = 0,6$: $k_1 = 0,0396$

$k_2 = 0,0241$

$k_3 = 0,0169$

$y_4 = 1,1303$

b. $\begin{cases} y' = x\ln(1+2y) \\ y(0) = 1 \end{cases}$

$x_1 = 0,4$: $K_1 = hf(x_0, y_0) = 0$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}K_1\right) = 0,0879$

$K_3 = hf\left(x_0 + h, y_0 - K_1 + 2K_2\right) = 0,1935$

$y_1 = y_0 + \frac{1}{6}\left(K_1 + 4K_2 + K_3\right) = 1,0908$

$x_2 = 0,8$: $K_1 = hf(x_1, y_1) = 0,1852$

$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}K_1\right) = 0,2914$

$K_3 = hf\left(x_1 + h, y_1 - K_1 + 2K_2\right) = 0,4918$

$y_2 = y_1 + \frac{1}{6}\left(K_1 + 4K_2 + K_3\right) = 1,8896$

c. $\begin{cases} y' = \dfrac{xy}{x^2 + y^2} \\ y(0) = 1 \end{cases}$

$x_1 = 0,25$: $K_1 = hf(x_0, y_0) = 0$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}K_1\right) = 0,0308$

$K_3 = hf\left(x_0 + h, y_0 - K_1 + 2K_2\right) = 0,0558$

$y_1 = y_0 + \frac{1}{6}\left(K_1 + 4K_2 + K_3\right) = 1,0298$

$x_2 = 0,5$: $K_1 = hf(x_1, y_1) = 0,0558$

$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}K_1\right) = 0,0787$

$K_3 = hf\left(x_1 + h, y_1 - K_1 + 2K_2\right) = 0,0925$

$y_2 = y_1 + \frac{1}{6}\left(K_1 + 4K_2 + K_3\right) = 1,1072$.

d. $\begin{cases} y' = (x+y)^2 \\ y(0) = 1 \end{cases}$

$x_1 = 0,1$: $K_1 = hf(x_0, y_0) = 0,1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}K_1\right) = 0,121$

$K_3 = hf(x_0 + h, y_0 - K_1 + 2K_2) = 0,1593$

$y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3) = 1,123$

$x_2 = 0,2$: $K_1 = hf(x_1, y_1) = 0,1496$

$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}K_1\right) = 0,1818$

$K_3 = hf(x_1 + h, y_1 - K_1 + 2K_2) = 0,2362$

$y_2 = y_1 + \frac{1}{6}(K_1 + 4K_2 + K_3) = 1,3089$

8.6. $T'(t) = -2,2067 \cdot 10^{-12}(T^4(t) - 81 \cdot 10^8)$; $T(0) = 1200$

$y(8) = y_0 + hf(t_0, y_0) = 1200 + 8(-4,6693) = 1162,644$

8.7. $R \cdot Q'(t) + \frac{Q}{C} = E(t)$; $Q(0) = 0$

$\Rightarrow 5 \cdot Q'(t) + \frac{Q(t)}{0,05} = 60$

$\Leftrightarrow Q'(t) + 4Q(t) = 12 \Leftrightarrow Q'(t) = 12 - 4Q(t)$

$Q(3) = Q(0) + 3f(0,0) = 3(12 - 4 \cdot 0) = 36$