Homework 4

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Problem 1. Prove $P(X_1|X_1 + X_2 = n)$ is binomially distributed if each X_i occurs at a rate λ_i

Let $X_1 + X_2 = Z$

We want to find $P(X_1|Z = n)$. By definition:

$$P(X_1 = k|Z = n) = P(X_1 = k, Z = n)/P(Z = n) \tag{1}$$

Since X_i are Poisson distributed with λ_i, the probability $P(Z = n)$ can be calculated straighforwardly, since the sum of poisson variables is itself poisson with rate equal to the sum of the rates.

$$P(Z = n) = \frac{(\lambda_1 + \lambda_2)^n}{n!} e^{(\lambda_1+\lambda_2)} \tag{2}$$

The term $P(X_1 = k, Z = n)$ can be rephrased:

$$P(X_1 = k, Z = n) = P(X_1 = k, X_2 = n - k) \tag{3}$$

Since the X_i are independent, the above expression factorizes:

$$P(X_1 = k, X_2 = n - k) = P(X_1 = k)P(X_2 = n - k) = \frac{{\lambda_1}^k}{k!} e^{\lambda_1} \cdot \frac{{\lambda_2}^{n-k}}{n - k!} e^{\lambda_2} \tag{4}$$

This can be simplified:

$$\frac{{\lambda_1}^k}{k!} e^{\lambda_1} \cdot \frac{{\lambda_2}^{n-k}}{n - k!} e^{\lambda_2} = \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n - k)!} e^{\lambda_1+\lambda_2} \tag{5}$$

Putting numerator and denominator together, we obtain:

$$\begin{aligned}
P(X_1 = k|Z = n) &= \frac{\frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} e^{\lambda_1+\lambda_2}}{\frac{(\lambda_1+\lambda_2)^n}{n!} e^{(\lambda_1+\lambda_2)}} \\
&= \frac{\frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} e^{\lambda_1+\lambda_2}}{\frac{(\lambda_1+\lambda_2)^{n-k}(\lambda_1+\lambda_2)^k}{n!} e^{\lambda_1+\lambda_2}} \\
&= \frac{n!}{k!(n - k)!} \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^{n-k}(\lambda_1 + \lambda_2)^k} \\
&= \frac{n!}{k!(n - k)!} (\frac{\lambda_1}{(\lambda_1 + \lambda_2)})^k (\frac{\lambda_2}{(\lambda_1 + \lambda_2)})^k \\
&= Bin(n, k, p = \frac{\lambda_1}{\lambda_1 + \lambda_2})
\end{aligned} \tag{6}$$