Warm dark matter chills out: constraints on the halo mass function and the free-streaming length of dark matter with 11 quadruple-image strong gravitational lenses

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ABSTRACT

The free-streaming length of dark matter depends on fundamental dark matter physics, and determines the abundance and concentrations of dark matter halos on sub-galactic scales. We use the flux-ratios from eleven quadruply-imaged quasars to constrain the free-streaming length of dark matter, the amplitude of the subhalo mass function (SHMF), and the logarithmic slope of the SHMF. We model both main deflector subhalos and halos along the line of sight, and account for warm dark matter (WDM) free-streaming effects on the halo mass function and the mass-concentration relation. We calibrate the evolution of the SHMF with halo mass and redshift using a suite of simulated halos to extract from the data a global scaling factor of the SHMF. Our analysis also accounts for finite-size background sources, and marginalizes over the global mass profile of the main deflector. Parameterizing dark matter free-streaming effects through half-mode mass $m_{\rm hm}$, we constrain dark matter warmth and the corresponding thermal relic particle mass $m_{\rm DM}$. At 2σ : $m_{\rm hm} < 10^{8.2} M_{\odot} \ (m_{DM} > 4.0 \ {\rm keV})$. Assuming CDM, we infer a projected mass in substructure between $10^6 - 10^9 M_{\odot}$ near lensed images of $1.6 - 3.4 \times 10^7 M_{\odot} {\rm kpc}^{-2}$ at 1σ , and an upper limit $5.3 \times 10^7 M_{\odot} {\rm kpc}^{-2}$ at 2σ . These values correspond to mean projected mass fractions in substructure of $\bar{f}_{\rm sub} = 0.023^{+0.007}_{-0.009}$ and $\bar{f}_{\rm sub} < 0.052$, respectively. In addition to $\bar{f}_{\rm sub}$, we also constrain the logarithmic slope of the SHMF $\alpha = -1.892^{+0.010}_{-0.014}$ at 1σ . These results are in excellent agreement with the predictions of cold dark matter.

Key words: [gravitational lensing: strong - cosmology: dark matter - galaxies: structure - methods: statistical]

1 INTRODUCTION

The theory of cold dark matter (CDM) has withstood numerous tests on scales spanning individual galaxies to the large scale structure of the universe and the cosmic microwave background (Tegmark et al. 2004; de Blok et al. 2008; Hinshaw et al. 2013). The next frontier for this highly successful theory lies on sub-galactic scales, where CDM makes two unique predictions: First, the theory predicts a scale-free halo mass function, possibly down to halo masses comparable to that of a planet (Hofmann et al. 2001; Angulo et al. 2017). Second, in CDM models halo concentrations decrease monotonically with halo mass, a result of hierar-

chical structure formation (Moore et al. 1999; Avila-Reese et al. 2001; Zhao et al. 2003; Diemer & Joyce 2019). A confirmation of these predictions through a measurement of the mass function and halo concentrations on mass scales below $10^9 M_{\odot}$ would at once constitute a resounding success for CDM and rule out entire classes of alternative dark matter theories.

The abundance of small-scale dark matter depends on the matter power spectrum at early times. If the velocity distribution of the dark matter particles causes them to diffuse out of small peaks in the density field, this will prevent the direct collapse of over-densities below a characteristic scale referred to as the free-streaming length (Benson et al. 2013; Schneider et al. 2013). The delay in structure formation in these scenarios also suppresses the central den-

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sities of the smallest collapsed halos, changing the mass-concentration relation for low-mass objects (Avila-Reese et al. 2001; Schneider et al. 2012; Macciò et al. 2013; Bose et al. 2016; Ludlow et al. 2016). By definition, free-streaming effects are negligible in CDM, while models with cosmologically relevant free-streaming lengths are collectively referred to as warm dark matter (WDM). As the free-streaming length depends on the dark matter particle(s) mass and formation mechanism, an inference on the small-scale structure of dark matter on mass scales where some halos are expected to be completely dark directly constrains fundamental dark matter physics and the viability of specific WDM particle candidates, including sterile neutrino models (Dodelson & Widrow 1994; Shi & Fuller 1999; Abazajian & Kusenko 2019) and keV-mass thermal relics.

Interest in alternatives to the canonical CDM paradigm, such as WDM, were motivated in part by apparent failures of the CDM model on small scales (see Bullock & Boylan-Kolchin 2017, and references therein). Two challenges in particular dominate scientific discourse, and provide illustrative examples of the complexity associated with testing CDM's predictions on sub-galactic scales. The 'missing satellites problem' (MSP), first pointed out by Moore et al. (1999), refers to the paucity of observed satellite galaxies around the Milky Way, in stark contrast to dark-matter-only N-body simulations that predict hundreds of dark matter subhalos hosting a luminous satellite galaxy. Invoking freestreaming effects in WDM to remove these small subhalos would resolve the problem, and hence WDM models gained traction. A second challenge to the CDM picture emerged with the 'too big to fail' (TBTF) problem (Boylan-Kolchin et al. 2011), which points out that the subhalos housing the largest Milky Way satellites are either under-dense or too small. Self-interacting dark matter, which results in lower central densities in dark matter subhalos (see Tulin & Yu 2018, and references therein), gained traction in part as a resolution to the TBTF problem.

Today, new astrophysical solutions to the MSP and TBTF problems diminish the immediate threat to CDM, but the resolutions to these issues are riddled with assumptions regarding complicated physical processes on subgalactic scales. The inclusion of baryonic feedback and tidal stripping in N-body simulations results in the destruction of subhalos, pushing the surviving number down to observed levels (Kim et al. 2017), although recently it has been suggested that the role of tidal stripping in N-body simulations is artificially exaggerated by resolution effects (van den Bosch et al. 2018; Errani & Peñarrubia 2019). The continuous discovery of new dwarf galaxies seems to resolve the MSP, and might even suggest a 'too-many-satellites problem' (Kim et al. 2018; Homma et al. 2019), but the number of expected satellite galaxies in CDM itself rests on assumptions regarding the process of star formation in low mass halos, which can introduce uncertainties larger than the differences between CDM and WDM on these scales (Nierenberg et al. 2016; Dooley et al. 2017; Newton et al. 2018). The inclusion of baryonic feedback from star formation processes and supernova in low-mass halos can reduce halo central densities, and at least partially alleviates the issues associated with the TBTF problem (Tollet et al. 2016). However, the degree to which baryonic feedback resolves the problem depends on the manner in which this feedback is implemented in simulations.

Regarding constraints on WDM models, analysis of the Lyman- α forest (Viel et al. 2013; Iršič et al. 2017) and the luminosity function of distant galaxies (Menci et al. 2016; Castellano et al. 2019), while robust to the systematics associated with examining Milky Way satellites, to some degree rely on luminous matter to trace dark matter structure. Constraints from the Lyman- α forest also invokes certain assumptions for the relevant thermodynamics. The common theme is that disentangling the role of baryons and dark matter physics on sub-galactic scales is difficult and it would be ideal to test the predictions of matter theories irrespective of baryonic physics.

Strong gravitational lensing by galaxies provides a means of testing the predictions of dark matter theories directly, without relying on baryons to trace the dark matter. As photons emitted from distant background sources traverse the cosmos, they are subject to deflections by the gravitational potential of dark matter halos along the entire line of sight and by subhalos around the a main lensing galaxy. Each warped image produced by a strong lens contains a wealth of information regarding the dark matter structure in the universe. The aim of this work is to extract that information.

When the lensed background source is spatially extended – for example, a galaxy – the lensed image becomes an arc that partially encircles the main deflector. Dark matter halos near the arc produce small surface brightness distortions, which allows for the localization of the perturbing halo and enables constraints on its mass down to scales somewhere between $10^8 - 10^9 M_{\odot}$ (Vegetti et al. 2014; Hezaveh et al. 2016). Analysis of the surface brightness fluctuations over the entirety of the arc can also constrain the abundance of small halos too diminutive to be detected individually, and results in a 2 keV lower bound on the mass of thermal relic WDM (Birrer et al. 2017b). A joint analysis of individual detections and non-detections in a sample of arclenses can constrain certain models of dark matter and test the predictions of CDM (Vegetti et al. 2018; Ritondale et al. 2018). Recently, several works have proposed measuring the substructure convergence power spectrum in strong lenses (Cyr-Racine et al. 2018; Díaz Rivero et al. 2018; Brennan et al. 2018), and Bayer et al. (2018) applied this method to a strong lens system.

We focus on a second kind of lens system, quadruply imaged quasars (quads). Rather than extended arcs, the observables in quads are four image positions and three magnification ratios, or flux ratios (the observable is the flux ratio, not the intrinsic flux, because the intrinsic source brightness is unknown) with unresolved sources. Flux ratios depend on non-linear combinations of second derivatives of the lensing potential near an image, providing localized probes of small-scale structure down to scales of $10^7 M_{\odot}$. These systems have been used in the past to constrain the presence of dark matter halos near lensed images (Metcalf & Madau 2001; Metcalf & Zhao 2002; Amara et al. 2006; Nierenberg et al. 2014, 2017) and measure the subhalo mass function (Dalal & Kochanek 2002). Recently, Hsueh et al. (2019) improved on previous analyses of quadruply imaged quasars by including halos along the line of sight, which can contribute a significant signal in flux ratio perturbations (Xu et al. 2012; Gilman et al. 2018). They found results consistent with CDM, ruling out WDM models to a degree comparable to that of the Lyman- α forest (Viel et al. 2013; Iršič et al. 2017).

In the case of quadruple-image lenses, the luminous source is a compact background object, such as the ionized medium around a background quasar. Broad-line emission from the accretion disk is subject to microlensing by stars, whereas light that scatters off of the more spatially extended narrow-line region is immune to microlensing while retaining sensitivity to the milli-arcsecond scale deflection angles produced by dark matter halos in the range $10^7-10^{10}M_{\odot}$ (Moustakas & Metcalf 2003; Nierenberg et al. 2014, 2017). Likewise, radio emission from the background quasar, while generally expected to be more compact that the narrow-line emission based on certain quasar models (Elitzur & Shlosman 2006; Combes et al. 2019), is extended enough to absorb micro-lensing effects.

We carry out a an analysis of eleven quads using a forward modeling approach we have tested and verified with mock data sets (Gilman et al. 2018, 2019). The sample of lenses we consider contains seven systems with flux ratios measured with narrow-line emission presented in ?, and three with radio emission, so we expect these systems are robust to microlensing effects and yield reliable data with which to constrain dark matter models. None of the quads show evidence for morphological complexity in the form of stellar disks, which require more detailed lens modeling (Hsueh et al. 2016; Gilman et al. 2017; Hsueh et al. 2017).

This paper is organized as follows: In Section 2 we describe our forward modeling analysis method and our implementation of a rejection algorithm in Approximate Bayesian Computing. Section 3 describes our parameterizations for the dark matter structure in the main lens plane and along the line of sight, and our modeling of free-streaming effects in WDM. Section 4 contains a brief description of the data used in our analysis and the relevant references for each system. In Section 5 we describe in detail each physical assumption we make and the modeling choices and prior probabilities attached to these assumptions. In Section 6, we present our inferences on the free-streaming length of dark matter and the amount of lens plane substructure. We discuss the implications of our results and our general conclusions in Section 7.

All lensing computations are performed using lenstronomy¹ (Birrer & Amara 2018). Cosmological computations involving the halo mass function and the matter power spectrum are performed with colossus (Diemer 2018). We assume a standard cosmology using the parameters from WMAP9 (Hinshaw et al. 2013) $(\Omega_m = 0.28, \sigma_8 = 0.82, h = 0.7)$.

2 BAYESIAN INFERENCE IN SUBSTRUCTURE LENSING

In this section we frame the substructure lensing problem in a Bayesian context, and describe our analysis method which relies on a forward-generative model to sample the target posterior distribution through an implementation of Approximate Bayesian Computing. This is an analysis method we have tested using simulated data (Gilman et al. 2018, 2019).

2.1 The Bayesian inference problem

Our goal is to obtain samples from the posterior distribution

$$p\left(\mathbf{q}_{\mathrm{s}}|\boldsymbol{D}\right) \propto \pi\left(\mathbf{q}_{\mathrm{s}}\right) \prod_{n=1}^{N} \mathcal{L}\left(\mathbf{d}_{\mathrm{n}}|\mathbf{q}_{\mathrm{s}}\right)$$
 (1)

where \mathbf{q}_s is a set of hyper-parameters describing the subhalo and line of sight halo mass functions, \boldsymbol{D} denotes the set of positions and flux ratios from a set of N lenses with each lens denoted by the subscript n, and where π represents the prior on \mathbf{q}_s . A certain dark matter model makes predictions for the parameters in \mathbf{q}_s , which may include quantities such as the normalization of the subhalo mass function, a free streaming cutoff in the mass function, etc.

Computing $\mathcal{L}(\mathbf{d}_n|\mathbf{q}_s)$ requires marginalizing over nuisance parameters \mathbf{M} , which include the background source size $\sigma_{\rm src}$, and the lens model that describes the main lensing galaxy (hereafter the macromodel). From each proposal of \mathbf{q}_s , we may render realizations $\mathbf{m}_{\rm sub}$ that specify individual realizations of dark matter structure in the lensing volume. Marginalizing over these quantities the likelihood is therefore

$$\mathcal{L}\left(\mathbf{d}_{n}|\mathbf{q}_{s}\right) = \int p\left(\mathbf{d}_{n}|\mathbf{m}_{sub}, \mathbf{M}\right) p\left(\mathbf{m}_{sub}, \mathbf{M}|\ \mathbf{q}_{s}\right) d\mathbf{m}_{sub}\ d\mathbf{M}. \quad (2)$$

Note that we write the joint distribution $p(\mathbf{m}_{\text{sub}}, \mathbf{M}|\mathbf{q}_{\text{s}})$, and do not assume the parameters in \mathbf{M} and \mathbf{q}_{s} are independent.

Evaluating Equation 2 is a daunting task. We highlight two main reasons:

- M is a high-dimensional space, where the overwhelming majority of volume does not result in observables that resemble the data, and in particular does not predict the correct image positions. Thus the overwhelming majority of samples drawn from M, and the corresponding samples qs (even if they described the 'true' nature of dark matter) would not contribute to the integral. Exploring this space efficiently through traditional MCMC methods is extremely inefficient for this reason.
- The parameters M describing the lens macromodel may depend on the dark matter parameters \mathbf{q}_s , necessitating the simultaneous sampling of these quantities in the inference. It is therefore difficult to impose an informative prior on M since the 'true' parameters in \mathbf{q}_s are unknown. Recognizing this and using a very uninformative prior on M, most samples from M will be rejected since they do not resemble the data, which alludes back to the issue of dimensionality described in the first bullet point

To address these challenges, we use a statistical method that bypasses the direct computation of the integral in Equation 2

2.2 Forward modeling the data

Rather than compute the likelihood function, we recognize that by generating simulated observables \mathbf{d}_n'

https://github.com/sibirrer/lenstronomy

 $\mathbf{d}_n'(\mathbf{m}_{\mathrm{sub}}, \mathbf{M})$ from the model \mathbf{q}_s , and accepting the proposed \mathbf{q}_s if they satisfy $\mathbf{d}_n' = \mathbf{d}_n$, the accepted \mathbf{q}_s samples will be direct draws from the posterior in Equation 1 (Rubin 1984). In this forward-generative framework, simulating the relevant physics in substructure lensing replaces the task of evaluating the likelihood function in Equation 2. We propagate photons from a finite-size background source through lines of sight populated by dark matter halos, a lensing galaxy and its subhalos, and finally into a simulated observation with statistical measurement errors added. Provided the forward model contains all of the relevant physics, the simulated data \mathbf{d}_n' will express the same potentially complex covariances present in the observed data.

The 'curse of dimensionality' that prohibits direct evaluation of Equation 2 also afflicts the criterion of exact matching between \mathbf{d}_n and \mathbf{d}'_n . In particular, most draws of macromodel parameters \mathbf{M} will not yield the observed image positions, and would therefore be rejected from the posterior. To deal with this, our strategy will be to ensure that the \mathbf{M} parameters we sample in the forward model, when combined with the full line of sight and subhalo populations specified by $\mathbf{m}_{\mathrm{sub}}$, yield a lens model that predicts the same image positions as observed in the data.

Fitting the image positions amounts to demanding that the the four images seen by the observer on the sky at positions $\boldsymbol{\theta}$ map to the same position on the source plane $\boldsymbol{\beta}_{K}$. This requires the use of the full multi-plane lens equation describing the path of deflected light rays (e.g. Schneider 1997), see also Blandford & Narayan (1986))

$$\beta_{K} = \theta - \frac{1}{D_{s}} \sum_{k=1}^{K-1} D_{ks} \alpha_{k} (D_{k} \beta_{k}), \qquad (3)$$

where the quantities $D_{\rm s}$, $D_{\rm k}$ and $D_{\rm ks}$ denote angular diameter distances to the source plane, to the kth lens plane, and from the kth lens plane to the source plane, respectively. Equation 3 is a recursive equation for the $\beta_{\rm k}$ that couples deflection angles from objects at different redshifts, similar to looking through potentially thousands of magnifying glasses in series. Throughout this process, we account for uncertainties in the measured image positions by sampling astrometric perturbations δ_{xy} , and applying them to the observed image positions during the forward modeling.

To solve for macromodel parameters M, for each realization $m_{\rm sub}$ we sample the power-law slope of the main deflector mass profile $\gamma_{\rm macro}$ and the external shear strength $\gamma_{\rm ext}$. If the lens system in question has satellite galaxies or nearby deflectors, we sample priors for their masses and positions. The remaining parameters describing the lens macromodel² are allowed to vary freely until a lens model that fits the image positions is found³.

This method for simultaneously sampling ${\bf M}$ and ${\bf q}_{\rm s}$

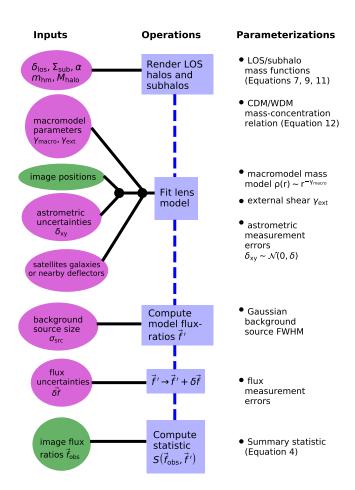


Figure 1. A graphical representation of the forward modeling procedure. Purple colors correspond to the action of sampling from a prior, blue represents an operation performed using the parameters sampled from a prior, and green colors indicate the use of observed information from the lenses. The arrow of time points from top to bottom: The first step is the rendering of dark matter structure, while the use of the information from observed flux ratios happens only at the very end.

does not involve lens model optimizations because we do not use the information from the observed image fluxes at this stage of the analysis. This avoids the potential biases incurred by optimizing, rather than marginalizing, over the macromodel parameters \mathbf{M} . As we will show in Section 6.1, by sampling \mathbf{M} and \mathbf{q}_s simultaneously we obtain joint posterior distributions that account for potential covariance between these quantities, recognizing that the addition of substructure may affect the distributions for the parameters \mathbf{M} .

With a lens model that fits the image positions in hand, we draw a background source size and ray-trace on a finely sampled grid around each image position using Equation 3 to compute the image fluxes f'. To account for statistical measurement errors in image fluxes, we sample flux uncertainties δf from M and render them onto the model fluxes $f' \to f' + \delta f$.

 $^{^2}$ The full set of macromodel parameters for a power-law ellipsoid are the overall normalization $b_{\rm macro},$ the mass centroid g_x and g_y , the ellipticity and ellipticity position angle ϵ and $\theta_{\epsilon},$ the external shear and shear angle $\gamma_{\rm ext}$ and $\theta_{\rm ext},$ and the power-law slope $\gamma_{\rm macro}.$ Nearby galaxies are modeled as Singular Isothermal spheres.

³ The four image positions provide $4 \times 2 = 8$ constraints, and the macromodel parameters that are allowed to vary freely, plus the source position, give 8 degrees of freedom.

2.3 Deriving posteriors from the forward model samples

We compute a summary statistic between the three observed flux ratios f_{obs} and those computed in the forward model

$$S_{\text{lens}}\left(\boldsymbol{f'}, \boldsymbol{f_{\text{obs}}}\right) \equiv \sqrt{\sum_{i=1}^{3} \left(f'_{i} - f_{\text{obs(i)}}\right)^{2}},$$
 (4)

and assign this statistic to the draw of \mathbf{q}_s . This summary statistic contains the full information content of the data, as the simultaneous matching of the three ratios requires that the forward model samples that minimize this statistic contain the same correlations present in the data. We repeat this procedure between 300,000 and 1,200,000 times for each quad, depending on with frequency with which the realizations match the data to within 1%.

We select the q_s parameters corresponding to the 800 lowest summary statistics $S_{\rm lens}$. The exact matching criterion between \mathbf{d}_{n} and \mathbf{d}'_{n} , which guarantees that the accepted samples q_s form the desired posterior, is replaced by selecting the realizations that look most like the data through the summary statistic S_{lens} . The resulting distribution of \mathbf{q}_s is therefore an approximation to the posterior distribution for each lens, with the approximation converging to the true posterior as the number of forward model samples increases while keeping the number of accepted samples fixed. The quality of the approximation can be quantified through a convergence test, in which we verify that the posteriors are unchanged as one removes realizations from the forwardmodeled data (see Appendix A). This method is an implementation of a rejection algorithm in Approximate Bayesian Computing (Rubin 1984; Marin et al. 2011; Lintusaari et al. 2017), a technique applied to problems where it is possible to generate simulated data from the model, but difficult to compute the likelihood (see also Akeret et al. 2015; Birrer et al. 2017b; Hahn et al. 2017).

To obtain the final posterior distribution $p(\mathbf{q_s}|\boldsymbol{D})$ (Equation 1), we multiply together the likelihoods obtained for each lens ⁴. This procedure is only possibly when using uniform priors in the forward model sampling, as the use of non-uniform priors would effectively move $\pi(\mathbf{q_s})$ inside the product in Equation 1 and over-use this information. We may, however, impose any prior we wish a-posteriori by reweighting the forward model samples accordingly. The full forward modeling procedure is illustrated in Figure 1, and the relevant parameters are summarized in Table 1.

3 THE SUBHALO AND LINE OF SIGHT HALO POPULATIONS

In this section, we describe the models we implement for the line of sight and subhalo mass functions in cold and warm dark matter that we sample in the forward model. We also describe the density profiles for individual halos, including their truncation radii and their distribution both along the line of sight and in the main lens plane. We begin with the

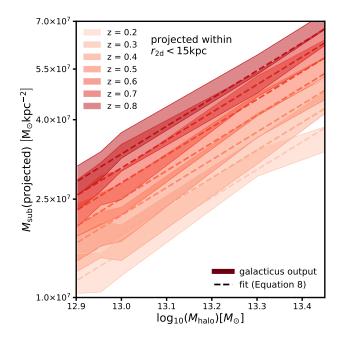


Figure 2. Output from the galacticus semi-analytic simulations of substructure within halos used to calibrate the evolution of the subhalo mass function with halo mass and redshift. While on the y-axis we plot the actual projected surface mass density in substructure output by galacticus, we only use the scaling with halo mass in redshift in our modeling, treating the overall normalization of the subhalo mass function as a free parameter. The projected mass density in substructure on the y-axis corresponds to a mass range $10^6-10^{10}M_{\odot}$, where we have extrapolated the mass function from the smallest resolved subhalo (10^8M_{\odot}) to 10^6M_{\odot} to compute the projected mass.

parameterizations used for the halo and subhalo density profiles and the spatial distribution of subhalos in Section 3.1. In Sections 3.2 and 3.3 we describe the parameterizations of the subhalo and line of sight halo functions, respectively, and in Section 3.4 describe how we model WDM free-streaming effects.

3.1 Subhalo density profiles and spatial distribution

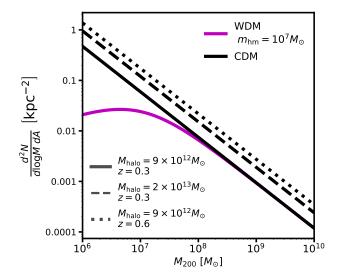
We model subhalos as tidally truncated NFW profiles (Baltz et al. 2009)

$$\rho(r) = \frac{\rho_s}{x(1+x)^2} \frac{\tau^2}{x^2 + \tau^2}$$
 (5)

where $x = \frac{r}{r_s}$, $\tau = \frac{r_t}{r_s}$, and r_t is a truncation radius and r_s is the NFW profile scale radius. We use the mass definition of M_{200} computed with respect to the critical density at z = 0, and a concentration mass relation that accounts for free-streaming effects in WDM (see Section 3.4).

In the main lens plane, we truncate halos according to their three-dimensional position inside the main lens halo $r_{\rm 3D}$ through a Roche-limit approximation that assumes a roughly isothermal global mass profile. The relevant scaling is $r_t \propto \left(M_{200} r_{\rm 3D}^2\right)^{\frac{1}{3}}$ (Tormen et al. 1998; Cyr-Racine et al.

⁴ We use a kernel density estimator with a first order boundary correction to approximate the continuous likelihood function used to compute the product.



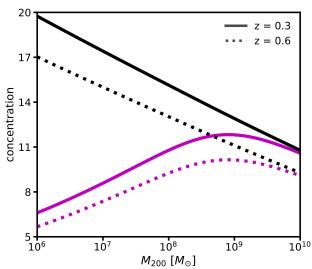


Figure 3. Top: The subhalo mass function as a function of halo mass, redshift, and the half-mode mass $m_{\rm hm}=10^7 M_{\odot}$ with $\Sigma_{\rm sub}=0.012{\rm kpc}^{-2}$. The line of sight halo mass function looks similar, but evolves differently with redshift. Bottom: The mass-concentration-relation for CDM and the same WDM model with $m_{\rm hm}=10^7 M_{\odot}$. Free-streaming affects the concentration of halos over one order of magnitude above $m_{\rm hm}$.

2016), which we implement as

$$r_t = 1.4 \left(\frac{M_{200}}{10^7 M_{\odot}}\right)^{\frac{1}{3}} \left(\frac{r_{3D}}{50 \text{kpc}}\right)^{\frac{2}{3}} [\text{kpc}].$$
 (6)

This results in truncation radii of $\sim 4-10r_s$. We note that the truncation radius depends implicitly on the parent halo mass $M_{\rm halo}$ through $r_{\rm 3D}$, which depends on the scale radius and the virial radius of $M_{\rm halo}$ at the lens redshift (see Figure 4).

We render subhalos out to a maximum projected radius $3R_{\rm Ein}$ and assign a three-dimensional z-coordinate between $-r_{200}$ and r_{200} , where r_{200} is the virial radius of the parent halo. Inside this volume, we distribute the subhalos assuming the spatial distribution follows the mass profile of the parent dark matter halo outside an inner tidal radius, which

we fix to half the scale radius of the parent halo. Inside this radius, we distribute halos with a uniform distribution in three dimensions. This choice is motivated by simulations that predict tidal disruption of subhalos near the lensing galaxy, resulting in an approximately uniform number of halos per unit volume in the inner regions of the halo (Jiang & van den Bosch 2017). The spatial distribution of subhalos that results from this procedure is approximately uniform in projection, which agrees with the predictions from N-body simulations (Xu et al. 2015).

3.2 The subhalo mass function

In principle, the projected mass in subhalos near the Einstein radius should depend on the parent halo mass, redshift, and the severity of tidal stripping by the main lensing galaxy. We will ultimately combine the inferences from multiple lenses at different redshifts and with different halo masses, so we parameterize the subhalo mass function in such a way that a single parameter $\Sigma_{\rm sub}$ can be used to simultaneously describe the projected mass density in substructure for each quad, regardless of halo mass or redshift.

We use the functional form for the subhalo mass function $% \left(x\right) =\left(x\right)$

$$\frac{d^2 N_{\text{sub}}}{dm dA} = \frac{\Sigma_{\text{sub}}}{m_0} \left(\frac{m}{m_0}\right)^{\alpha} \mathcal{F}\left(M_{\text{halo}}, z\right),\tag{7}$$

where scaling function $\mathcal{F}(M_{\rm halo},z)$ encodes the differential evolution of the projected number density with redshift and halo mass, such that $\Sigma_{\rm sub}$ can be interpreted as a common parameter for all the lenses. We choose the normalization such that $\mathcal{F}(M_{\rm halo}=10^{13}M_{\odot},z=0.5)=1$, anchoring $\Sigma_{\rm sub}$ at z=0.5 with a halo mass of $10^{13}M_{\odot}$. We use a pivot mass $m_0=10^8M_{\odot}$. We will marginalize over $\Sigma_{\rm sub}$ and α when quoting constraints on dark matter warmth to account for tidal stripping of subhalos and halo-to-halo scatter.

To determine the scaling function $\mathcal{F}(M_{\text{halo}}, z)$, we run a suite of simulations using the semi-analytic modeling code galacticus⁵ (Benson 2012; Pullen et al. 2014), simulating parent halos and their substructure in the redshift range 0.2 < z < 0.8 and halo mass range $0.8 - 3 \times 10^{13} M_{\odot}$, with a subhalo mass resolution of $10^8 M_{\odot}$. In each redshift and mass bin we simulate 24 halos, resulting in 840 $\sim 10^{13} M_{\odot}$ halos in total⁶. We average over the projected number densities along each principle axis inside a 15 kpc aperture to obtain trends in the projected number density with halo mass and redshift in the vicinity of the Einstein radius, where lensed images appear. The galacticus simulations include tidal destruction of subhalos by the global dark matter mass profile, which affects the evolution of the projected mass density with halo redshift: at early times, subhalos are more concentrated in the parent halo, while at later times tidal stripping from the main dark matter halo depletes the population of subhalos at small radii and the projected number density at small radii decreases. In addition, the physical size of the parent halo at higher redshifts is smaller by a factor of $(1+z)^{-1}$, so the number of halos per square physical kpc is higher. We also note that early-type galaxy halos simulated

⁵ Code version 7175:2bd6b8d84a39

⁶ These simulations completed in 1,000 CPU hours.

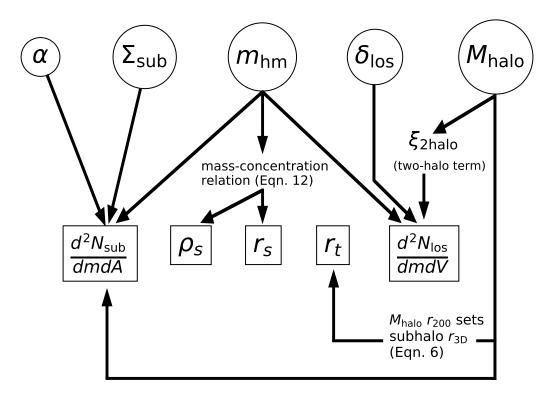


Figure 4. A graphical representation of the dark matter parameters in \mathbf{q}_s : α , the logarithmic slope of the subhalo mass function, Σ_{sub} , the overall scaling of the subhalo mass function, m_{hm} , the WDM half-mode mass, δ_{los} , the overall factor for the line of sight halo mass function, and M_{halo} , the main deflector's parent halo mass. ξ_{2halo} is implemented through Equation 9 (see Section 3.3). These parameters are linked to the physical dark matter quantities they affect. From left to right: the subhalo mass function $\frac{d^2N}{dmdA}$, the normalization ρ_s , scale radius r_s , and truncation radius r_t of individual halos (see Equation 5), and the line of sight halo mass function $\frac{d^2N}{dmdV}$. The priors for each of these parameters are summarized in Table 2, and discussed at length in Section 5.

Table 1. Free parameters sampled in the forward model. Notation $\mathcal{N}(\mu, \sigma)$ indicates a Gaussian prior with mean μ and variance σ , and $\mathcal{U}(u_1, u_2)$ indicates a uniform prior between u_1 and u_2 . Lens-specific priors are summarized in Table 2.

parameter	definition	prior
$\log_{10}\left(M_{\rm halo}\right)\left[M_{\odot}\right]$	main lens parent halo mass	(lens specific)
$\Sigma_{\rm sub} \left[{\rm kpc}^{-2} \right]$	normalization of subhalo mass function (Equation 7) (rendered between $10^6-10^{10}M_{\odot})$	$\mathcal{U}\left(0,0.1\right)$
α	logarithmic slope of the subhalo mass function	$\mathcal{U}(-1.95, -1.85)$
$\log_{10}\left(m_{\rm hm}\right)\left[M_{\odot}\right]$	half-mode mass (Equations 11 and 12) \propto to free streaming length and thermal relic mass $m_{\rm DM}$	$\mathcal{U}(4.8, 10)$
$\delta_{ m los}$	rescaling factor for the line of sight Sheth-Tormen mass function (Equation 9, rendered between $10^6-10^{10}M_\odot$)	$\mathcal{U}\left(0.8, 1.2\right)$
$\sigma_{ m src} \left[{ m pc} ight]$	source size parameterized as FWHM of a Gaussian	$\mathcal{U}\left(25,60 ight)_{\mathrm{NL}}$ $\mathcal{U}\left(5,25 ight)_{\mathrm{radio}}$
$\gamma_{ m macro}$	logarithmic slope of main deflector mass model	$\mathcal{U}\left(1.95, 2.2\right)$
$\gamma_{ m ext}$	external shear in the main lens plane	(lens specific)
δ_{xy} [m.a.s.]	image position uncertainties	(lens specific)
δf	image flux uncertainties	(lens specific)

by Fiacconi et al. (2016) also show significant evolution with redshift in the projected number density of halos by about a factor of two, very similar to the galacticus predictions.

We fit the evolution with halo mass and redshift predicted by galacticus with the relation

$$\log_{10}(\mathcal{F}) = k_1 \log_{10} \left(\frac{M_{\text{halo}}}{10^{13} M_{\odot}} \right) + k_2 \log_{10} (z + 0.5) \quad (8)$$

with $k_1 = 0.88$ and $k_2 = 1.7$. The galacticus output and the fit from Equation 8 are shown in Figure 2. We only extract information regarding the scaling of projected mass density with halo mass and redshift from the galacticus simulations, and treat the overall normalization as a free-parameter that absorbs effects to tidal destruction of subhalos by the main lens galaxy. We discuss our modeling assumptions in more detail in Section 5.4.

3.3 The line of sight halo mass function

We model line of sight structure by drawing halo masses from the Sheth-Tormen halo mass function (Sheth et al. 2001), with two modifications. First, we introduce an overall rescaling factor $\delta_{\rm los}$ which accounts for theoretical uncertainty in the predicted amplitude of the halo mass function (see e.g. Despali et al. 2016). The factor $\delta_{\rm los}$ accounts for the possibility of a section bias in the quads towards systematically over or under-dense lines of sight. The second modification we add is a contribution from the two-halo term $\xi_{\rm 2halo}$ ($M_{\rm halo}, z$), which accounts for the presence of correlated structure in the vicinity of main deflector parent dark matter halo ⁷ With these modifications the line of sight halo mass function takes the form

$$\frac{d^2 N_{\text{los}}}{dm dV} = \delta_{\text{los}} \left(1 + \xi_{\text{2halo}} \left(M_{\text{halo}}, z \right) \right) \frac{d^2 N}{dm dV} \big|_{\text{ShethTormen}}. \tag{9}$$

Halos along the line of sight are rendered in a double-cone geometry with opening angle $3R_{\rm Ein}$, where $R_{\rm Ein}$ is the Einstein radius of the main deflector, and a closing angle behind the main deflector such that the cone closes at the source redshift. Finally, we add negative convergence sheets to subtract the mean expected convergence from line of sight halos at each line of sight plane. Without this numerical procedure, lines of sight are systematically over-dense relative to the expected matter density of the universe, akin to lensing in a universe with positive curvature (Birrer et al. 2017a). This may bias results as the macromodel will attempt to compensate for the artificial focusing of light rays in this scenario.

3.4 Modeling free-streaming effects in WDM

Free-streaming refers to the diffusion of dark matter particles out of small peaks in the matter density field in the early universe. This has the effect of erasing structure on scales below a characteristic free-streaming length which depends

 7 In Appendix A of Gilman et al. (2019), we describe how this effect is implemented and show that this term contributes a $\sim 4\%$ increase in the frequency of flux ratio perturbations induced by objects outside the virial radius of the main deflector.

on the velocity distribution of the dark matter particles, and hence on their mass and formation mechanism. For a more in depth discussion, see Schneider et al. (2013).

It is convenient to express free-streaming effects in terms of the half-mode mass $m_{\rm hm}$, which is defined in terms of the length scale where the transfer function between the CDM and WDM power spectra drops to one-half. In the specific case that all of the dark matter exists in the form of thermal relics, a one-to-one mapping between the half-mode mass and the mass of the candidate particle $m_{\rm DM}$ exists, and has the scaling is $m_{\rm hm} \propto m_{\rm DM}^{-3.33}$ (Schneider et al. 2012)

$$m_{\rm hm} (m_{\rm DM}) = 3 \times 10^8 \left(\frac{m_{\rm DM}}{3.3 {\rm keV}}\right)^{-3.33} M_{\odot}.$$
 (10)

We have run galacticus models (Benson et al. 2013) with WDM mass functions corresponding to 3.3 and 5 keV thermal relics to investigate the effects of free-streaming on the trends with halo mass and redshift of the projected mass in substructure near the Einstein radius, and determine that the fit in Equation 8 is common to both CDM and WDM. We therefore use the same scaling function $\mathcal{F}(M_{\rm halo}, z)$ for WDM mass functions, and model the effects of free streaming using the fitting formula from (Lovell et al. 2014)

$$\frac{dN_{\text{WDM}}}{dm} = \frac{dN_{\text{CDM}}}{dm} \left(1 + \frac{m_{\text{hm}}}{m} \right)^{-1.3}.$$
 (11)

Since the parameter $m_{\rm hm}$ is related to the WDM transfer function, it should affect the subhalo and field halo mass functions in a similar manner. We therefore apply the same suppression factor in Equation 11 to both the subhalo mass function and the line of sight halo mass function in Equation 9. Lacking a theoretical prediction for the evolution of the turnover with redshift, we do not evolve the shape or position of the free-streaming cutoff in the mass function at higher redshifts.

In WDM scenarios, the delayed onset of structure formation affects the assembly history of dark matter halos and suppresses their concentrations on mass scales that extend above $m_{\rm hm}$ (Schneider et al. 2012; Bose et al. 2016). We use the functional form proposed by (Bose et al. 2016), and write the WDM concentration-mass relation as

$$\frac{c_{\text{wdm}}(m,z)}{c_{\text{cdm}}(m,z)} = (1+z)^{\beta(z)} \left(1 + 60 \frac{m_{\text{hm}}}{m}\right)^{-0.17}.$$
 (12)

with $\beta(z) = 0.026z - 0.04$, using the CDM mass-concentration model of Diemer & Joyce (2019) and a scatter of 0.1 dex (Dutton & Macciò 2014). The above model was calibrated for halos on mass scales below $M_{200} \sim 10^9 M_{\odot}$, and is accurate in the redshift range z = 0 - 3. We note that since flux ratios are particularly sensitive to the central density of perturbing halos, the suppression of halo concentrations far above $m_{\rm hm}$ (because of the factor of 60 in Equation 12) is possibly the dominant effect of dark matter free-streaming on lensing observables. We plot the subhalo mass function and the halo mass-concentration-redshift relation in Figure 3.

4 THE DATA

We apply the forward-modeling methodology outlined in Section 2 using the physical model described in Section 3

to eleven quadruply imaged quasars. In this Section, we describe the sample selection, and how the data for these eleven systems was obtained. In Table C1 in Appendix C, we summarize the data used in the analysis and provide the relevant references.

4.1 Narrow-line lenses

Eight quads in our sample have image fluxes measured using the narrow-line emission from the background quasar. Six of these (2038, WFI 2033, J0911, J1606, WGD J0405, and WFI 2026) have flux and astrometry presented by ?, while the data for B1422 and HE0435 are taken from Nierenberg et al. (2014) and Nierenberg et al. (2017), respectively. The narrow-line region spans several tens of parsecs (Müller-Sánchez et al. 2011), corresponding to angular source sizes on the order a few milli-arcseconds. Flux from this extended area is therefore immune to microlensing by stars, while retaining sensitivity to the milli-arcsecond deflection angles by dark matter halos produced by dark matter halos in the mass range $10^6 - 10^{10} M_{\odot}$. As the light crossing time for the narrow-line region is several thousand light days, much longer than the expected time delays between the images, the variability of the background quasar is also washed out in the narrow-line spectra. For additional details regarding the measurement methodology for the narrow-line flux ratios, we refer to ?.

Shajib et al. (2019) analyzed several systems in the narrow-line sample. They measured satellite galaxy location and provided the photometric information for the systems J1606 and WGD J0405, which we used to obtain photometric redshifts (see Appendix B). The lens and source redshifts, as well as the priors on the external shear and satellite galaxies, are summarized in Table 2.

4.2 Radio lenses

Three quads in the sample, MG0414, B0128, and PG1115 have fluxes measured using radio emission from the background quasar. In the case of MG0414, the stability of the image fluxes in multiple wavelengths over several years (see Figure 2 in Stacey & McKean (2018)) support the hypothesis that microlensing and variability of the background quasar are not seriously biasing the measurements. This hypothesis is also supported by the width of the observed CO line, which suggests a minimum size for the background radio source that is large enough to avoid microlensing effects. For this system, we inflate the flux uncertainties to 10% to account for the scatter in flux ratios measurements over several orders of magnitude in frequency (see Figure 2 in Stacey & McKean (2018)).

As pointed out by Koopmans et al. (2003), the time delays for the system B0128 are less than 1 day, such the variability of the background quasar should not affect the measured fluxes even if the background source is 0.1 parsecs across (0.1 parsecs ~ 1 light day). For this system, we therefore use the same flux measurements and uncertainties as quoted by Koopmans et al. (2003).

The system PG1115 does not have radio multi-epoch monitoring, and the time delays are longer than several days, so variability of the background quasar may affect the measurements if the background source is extremely compact.

For this system, we use conservative flux uncertainties of 10%.

4.3 Lenses omitted from our sample

We apply our analysis to a sample of eleven quads, although additional systems exist in the literature with measured flux ratios. We choose only a subset of the total number of possible lenses since the remaining systems either do not have reliable flux measurements (either from the narrow-line region or from radio emission), or have complicated deflector morphology that introduce significant uncertainties in the lens modeling. Some of these systems may be analyzed in a future work upon revision of our modeling strategy and new flux measurements.

Specifically, we do not include quads with main lensing galaxies that contain stellar disks, since accurate lens models for these systems require explicit modeling of the disk. This excludes the systems B1555 and B0712, and the system J1330 presented by ?. We also exclude HS 0810+2554, a system with narrow-line flux measurements presented by (?), and the quad B2045 with radio flux measurements (McKean et al. 2007), because in both cases the flux from the merging images becomes blended together for source sizes larger than 20 pc. This complicates our analysis, as it is unclear how to partition photons between the merging pair (or triplet) when the images blur together.

5 PHYSICAL ASSUMPTIONS AND PRIORS

The parameterizations we introduce in Section 3 and the priors use in the forward model reflect certain physical assumptions. In this section we describe these assumptions, and the prior probabilities attached to each parameter in the forward model for our sample of quads.

5.1 The extended background source

The effect of a dark matter halo of a given mass on the magnification of a lensed image is a function of the background source size (Dobler & Keeton 2006), see also Figure 14 in Amara et al. (2006) and Figure 8 in Xu et al. (2012). In general, more extended background sources are less sensitive dark matter halos (in terms of the image magnifications) on the mass scales relevant for substructure lensing.

Of the lenses in our sample, 8 (HE0435, B1422, WGD 2038, J1606, WFI 2033, J0911, WGD J0405, WFI 2026) have fluxes measured using emission from the narrow-line region of the background quasar (Nierenberg et al. 2014, 2017; ?), while 2 (B0128, and MG0414) have image fluxes measured in radio wavelengths using the Very Large Array (VLA). The remaining system, PG1115, was observed in both the mid-infrared and radio.

Both the narrow-line region and the radio emitting region are expected to subtend angular scales larger than a micro-arcsecond, corresponding to physical scales larger than $\sim 1 \mathrm{pc}$, such that they are immune to microlensing by stars. This physical extent also corresponds to a light-crossing time greater than the typical time delay between lensed images, such that variability in the background

Table 2. A summary of deflector z_d and source z_s redshifts, and satellite galaxies included in the lens model for the quads in our sample. Satellite position priors marked by * denote observed satellite locations, which may differ from the true physical location as the satellite's photons are deflected in the main lens plane (see Section 5.8). Satellite galaxy locations are quoted with respect to the light centroid of the main deflector (see Table C1). All priors on the satellite mass $G2_{\theta_E}$ are positive definite. The raised and lowered numbers around the deflector redshifts for J1606, WGD J0405, and WFI 2026 are the 68% confidence intervals on the estimated lens redshifts (see Appendix B), which we marginalize over.

lens	z_d	z_s	$\log_{10} M_{\mathrm{halo}}$	$\gamma_{ m ext}$	$\mathrm{G2}_{\mathrm{x}}$	$\mathrm{G2_y}$	$\mathrm{G2_{z}}$	$\mathrm{G2}_{ heta_{\mathrm{E}}}$
B0128+437	1.145	3.12	$\mathcal{N}(13.3, 0.3)$	U(0.2, 0.3)	-	-	-	-
WGD J0405-3308	$0.29_{0.25}^{0.32}$	1.71	$\mathcal{N}(13.3, 0.3)$	$\mathcal{U}\left(0.02,0.1\right)$	-	-	-	-
MG0414+0534	0.96	2.64	$\mathcal{N}(13.5, 0.3)$	U(0.02, 0.16)	$\mathcal{N}(-0.386, 0.05)$	$\mathcal{N}(1.46, 0.05)$	z_d	$\mathcal{N}\left(0.2,0.05\right)$
HE0435-1223	0.45	1.69	$\mathcal{N}(13.2, 0.3)$	$\mathcal{U}\left(0.02,0.13\right)$	$*\mathcal{N}(2.585, 0.05)^*$	$*\mathcal{N}(-3.637, 0.05)^*$	$z_d + 0.33$	$\mathcal{N}\left(0.37,0.03\right)$
RX J0911+0551	0.77	2.76	$\mathcal{N}(13.1, 0.3)$	see Section 5.9	$\mathcal{N}(-0.767, 0.05)$	$\mathcal{N}(0.657, 0.05)$	z_d	$\mathcal{N}\left(0.2,0.2\right)$
PG1115+080	0.31	1.72	$\mathcal{N}(13.0, 0.3)$	U(0.08, 0.24)	-	-	-	-
B1422+231	0.36	3.67	$\mathcal{N}(13.3, 0.3)$	$\mathcal{U}\left(0.12,0.35\right)$	-	-	-	-
PS J1606-2333	$0.31_{0.26}^{0.36}$	1.70	$\mathcal{N}(13.3, 0.3)$	$\mathcal{U}\left(0.1,0.28\right)$	$\mathcal{N}(-0.307, 0.05)$	$\mathcal{N}(-1.153, 0.05)$	z_d	$\mathcal{N}\left(0.27, 0.05\right)$
WFI 2026-4536	$1.04_{0.9}^{1.12}$	2.2	$\mathcal{N}(13.3, 0.3)$	U(0.03, 0.16)	-	-	-	-
WFI 2033-4723	0.66	1.66	$\mathcal{N}(13.4, 0.3)$	$\mathcal{U}\left(0.13,0.32\right)$	$\mathcal{N}(0.245, 0.025)$	$\mathcal{N}(2.037, 0.025)$	z_d	$\mathcal{N}\left(0.02,0.005\right)$
					$*\mathcal{N}(-3.965, 0.025)^*$	$*\mathcal{N}(-0.025, 0.025)^*$	$z_d + 0.085$	$\mathcal{N}(0.93, 0.05)$
WGD 2038-4008	0.23	0.78	$\mathcal{N}(13.4, 0.3)$	$\mathcal{U}\left(0.04,0.12\right)$	-	-	-	-

quasar should be washed out of the light curves if the source size is indeed large enough to avoid microlensing.

The size of the narrow-line region typically spans up to $\sim 60 \mathrm{pc}$ (Müller-Sánchez et al. 2011) defined as the full-width at half maximum (FWHM) of the radially averaged luminosity profile. Upper limits of 50-60 pc may also be obtained by forward modeling the spectrum of the lensed images themselves (Nierenberg et al. 2014, 2017). We therefore model the background source as a circular Gaussian and impose a uniform prior on the FWHM between $25-60 \mathrm{pc}$.

Regarding the structure of the radio emitting region, a popular model (Elitzur & Shlosman 2006) posits that a dusty torus obscures the direct imaging of the accretion disk in the immediate vicinity of the black hole, and the observed radio flux is associated with the torus structure. Some evidence for this point of view comes from the specific case of 2045: consistency among different wavelengths suggest that microlensing is not present in this system, bounding the size of the emitting background source from below, while compact flux density maps of the individual images (McKean et al. 2007) suggest a background source much more compact than the typical narrow-line region. Based on these considerations and measurements of torus diameters (Elitzur & Shlosman 2006; Combes et al. 2019), we use a flat prior on the background source size for the radio lenses between 5-25 pc.

5.2 Halo and subhalo mass ranges

Both the line of sight and subhalo mass functions include halos in the range $10^6-10^{10}M_{\odot}$. Halos with masses below 10^6M_{\odot} do not leave imprints on lensing observables for the extended source sizes we consider, which we verify by comparing distributions of image flux ratios with different minimum subhalo masses. We model the rare objects more masses.

sive than $10^{10} M_{\odot}$ explicitly by including them in the lens model, since these halos typically host a satellite galaxy and are detected in the observations of the lenses themselves.

5.3 The line of sight halo mass function

We use the Sheth-Tormen (Sheth et al. 2001) halo mass function to model structure along the line of sight, with two modifications: First, we introduce a rescaling term $\delta_{\rm los}$ to account for a systematic shift in the predicted mean amplitude of the mass function. Second, we include a term $\xi_{\rm 2halo}$ ($M_{\rm halo}, z$) that rescales the amplitude of the mass function near the main deflector to account for the presence of correlated structure in the density field near the parent dark matter halo. This results in a 5–10% increase in the number halos near the main deflector.

Modulo uncertainty in the overall amplitude $\delta_{\rm los}$, we assume the halo mass function in the lens cone volume is well-described by the mean halo mass function in the universe. This is a reasonable approximation as lensing volumes span several Gpc, and we expect fluctuations in the dark matter density along the line of sight should average out over large distances. We note, however, that there is some scatter among the predictions from different parameterizations of the halo mass function below $10^{10} M_{\odot}$ (e.g. Despali et al. 2016) and cosmological model uncertainties, for instance associated with σ_8 and Ω_m . It is also possible that lenses are selected preferentially in over or under-dense lines of sight. We use a flat prior on $\delta_{\rm los}$ between 0.8 and 1.2 to account for these uncertainties.

5.4 The subhalo mass function

Our parameterization of the subhalo mass function is an improvement over previous modeling efforts in predicting strong lensing observables since it explicitly accounts for the evolution of the subhalo mass function with redshift and halo mass, and accounts for the tidal stripping of subhalos by the host dark matter halo. However, since the <code>galacticus</code> runs do not include a central galaxy we cannot predict the effects of tidal stripping on the projected mass in substructure near the Einstein radius, or the possible redshift and halo mass dependence of this effect. Since tidal destruction of substructures appears to be independent of subhalo mass (Garrison-Kimmel et al. 2017; Graus et al. 2018), we absorb the effects of tidal stripping into the normalization parameter $\Sigma_{\rm sub}$ in Equation 7.

To determine reasonable bounds on Σ_{sub} , we compare the predicted surface density in substructure obtained by integrating Equation 7 over mass with the output from $\ensuremath{\mathrm{N}\text{-}\mathrm{body}}$ simulations, and from the $\ensuremath{\mathtt{galacticus}}$ runs. At $z\,\sim\,0.7, {\rm the}\,\sim\,10^{13} M_{\odot}$ halos in Fiacconi et al. (2016) have projected substructure mass densities of $10^7 M_{\odot} \text{kpc}^$ at $0.02R_{\rm vir}$. Fiacconi et al. (2016) show that this value increases when accounting for baryonic contraction of the halo. The galacticus halos contain more substructure at the same redshift without accounting for baryonic contraction, corresponding to projected mass densities between $2.5 \times 10^7 M_{\odot} \mathrm{kpc}^{-2}$ and $6 \times 10^7 M_{\odot} \mathrm{kpc}^{-2}$. Both of these projected mass densities would likely decrease when account for tidal stripping. We note, however, that recent works call attention to possible numerical issues that can lead to the artificial fragmentation of subhalos in N-body simulations (van den Bosch et al. 2018; Errani & Peñarrubia 2019). For reference, $\Sigma_{\rm sub} = 0.012 {\rm kpc}^{-2}$ corresponds to a projected mass density of $10^7 M_{\odot} {\rm kpc}^{-2}$ at z = 0.5 in a $10^{13} M_{\odot}$ halo, using Equation 7.

With these considerations in mind, we use a wide, flat, prior on $\Sigma_{\rm sub}$ between 0 and 0.1 kpc⁻² that should encompass the theoretical uncertainties present in the literature. We reiterate that by factoring out the evolution with halo mass and redshift, we intend for the parameter $\Sigma_{\rm sub}$ to be common for all the lenses in our sample with scatter from different tidal stripping scenarios and halo-to-halo variance.

The power-law slope α of the subhalo mass function predicted by N-body simulations is consistently in the range -1.95 to -1.85 (Springel et al. 2008; Fiacconi et al. 2016), and because tidal stripping appears independent of mass the presence of a central galaxy should not cause significant deviations from this prediction. We therefore impose a flat prior on α between -1.95 and -1.85.

5.5 Free-streaming in WDM

The prior on $m_{\rm hm}$ needs to be chosen with care since statements using confidence intervals depend on the choice of prior. We specify the lower bound on the prior for $m_{\rm hm}$ with the WDM mass-concentration relation (Equation 12) in mind, since the factor of 60 in the denominator of Equation 12 results in suppressed halo concentrations nearly two orders of magnitude above the location of the turnover in the mass function (see Figure 3). We choose a lower bound

for $m_{\rm hm}$ at $10^{4.8} M_{\odot}$ that preserves the CDM-predicted halo concentrations down to $10^7 M_{\odot}$. At $10^6 M_{\odot}$, even the coldest mass function we model with $m_{\rm hm} = 10^{4.8} M_{\odot}$ result in halo concentrations for $10^6 M_{\odot}$ objects 25% lower than the CDM prediction, but we expect the signal from these very low-mass halos will be sub-dominant given that we model extended background sources which decrease sensitivity to low-mass halos.

5.6 The parent dark matter halo mass

We use information about the mean population of early-type galaxy lenses, as well as empirical relations between stellar mass, halo mass, and observable quantities such as the image separations and lens/source redshifts, to construct priors for the halo mass of each system.

First, we estimate the 'lensing' velocity dispersion from the Einstein radius and lens/source redshifts using the empirical relation between the stellar mass and velocity dispersion derived by Auger et al. (2010) for a sample of strong lens galaxies. We account for the scatter between spectroscopic velocity dispersion and the 'lensing' velocity dispersion (Treu et al. 2006), and uncertainties in the fit by Auger et al. (2010), and convert the estimated stellar mass into a halo mass using the halo-to-stellar mass ratio $\frac{M_{\rm halo}}{M_*} = 75^{+36}_{-27}$ inferred by Lagattuta et al. (2010). The typical uncertainty in the resulting prior for the halo mass is 0.3 dex.

We use this procedure to construct a prior for the halo mass of each quad, with the exceptions of B1422, J1606, WGD J0405, and B0128. The stellar velocity dispersions implied by the Einstein radii of the systems is significantly lower than the stellar velocity dispersion in the sample of quads used to calibrate the halo-to-stellar mass ratio in Lagattuta et al. (2010), and as such the estimate of the halo mass using the above procedure may not be accurate for these systems. For B1422, J1606, WGD J0405 and B0128, we therefore assume the population mean of $10^{13.3\pm0.3}M_{\odot}$ inferred by Lagattuta et al. (2010). We also assume the population mean halo mass for WFI 2026 since the lens redshift used to estimate the central velocity dispersion is very uncertain. Finally, the system J0911 is known to reside near a cluster of galaxies, and thus convergence from the cluster halo contributes to the mass within the Einstein radius. We approximate the contribution from the cluster convergence by noting that it should be approximately equal to the mean external shear we infer of 0.25. We then rescale the Einstein radius by $\sqrt{0.75}$, since the stellar mass scales as $R_{\rm Ein}^2$ and where we have used the fact that the mean convergence inside the Einstein radius is approximately equal to one for an isothermal deflector. The priors for the parent halo mass used for each quad are listed in Table 2.

Since we explicitly model the evolution with halo mass, we vary $\Sigma_{\rm sub}$ and $M_{\rm halo}$ independently. We note however, that $M_{\rm halo}$ and $\Sigma_{\rm sub}$ are not completely degenerate in our analysis. While the number of lens plane subhalos depends on both parameters, the truncation radius of the subhalos depends on $M_{\rm halo}$ through the distribution of subhalo z-coordinates, which in turn depends on the virial radius of the parent halo (see Equation 6), and the 2-halo term appearing in Equation 9 depends on the halo mass as a larger halo will have more correlated structure around it. Figure 4 provides

⁸ galacticus is capable of including the tidal stripping effects from a central galaxy, but we did not include them to minimize computation costs.

a visual representation of the link between $M_{\rm halo}$, $\Sigma_{\rm sub}$, α , $\delta_{\rm los}$, and $m_{\rm hm}$.

5.7 The main deflector lens model

The galaxies that dominate the lensing cross-section are typically massive early-types with stellar velocity dispersions $\sigma > 200$ km sec⁻¹ (Gavazzi et al. 2007; Lagattuta et al. 2010). The mass profiles of these systems are typically inferred to be isothermal, or close to isothermal (Treu et al. 2006, 2009; Auger et al. 2010; Shankar et al. 2017). These observations motivate a simple parameterization for the main deflector lens model, the singular isothermal ellipsoid (SIE) plus external shear. We generalize this model to a power-law ellipsoid with a variable logarithmic slope $\gamma_{\rm macro}$ to account for uncertainties associated with the mass profile of the lensing galaxy, and the model-predicted flux ratios. We assume a flat prior on the power-law slope $\gamma_{\rm macro}$ between 1.95 and 2.2 for each deflector (Auger et al. 2010).

In addition to the logarithmic slope of the main deflector mass profile, we sample values for the external shear strength $\gamma_{\rm ext}$. The prior for $\gamma_{\rm ext}$ is chosen on a lens-by-lens basis such that the posterior distribution of $\gamma_{\rm ext}$ obtained for each lens is contained well within the bounds of the prior. The specific priors used for each system are summarized in Table 2. Finally, we use a Gaussian prior for the mass centroid of each quad centered on the main deflector light with a variance of 0.05 arcseconds, a typical modeling uncertainty for quadruple-image systems (Shajib et al. 2019; ?).

Several studies (Hsueh et al. 2016; Gilman et al. 2017; Hsueh et al. 2017, 2018) explore the role of complicated main deflector morphologies on the model predicted flux ratios. As image magnifications are local probes of the gravitational potential, if there are fluctuations in the surface mass profile on scales comparable to the image separation these structures can affect the image magnifications. In particular, stellar disks, if they go unnoticed, can result in systematically inaccurate lens models. With deep HST images of the narrow-line quads in our sample, we can confirm that the galaxies in our sample do not contain disks, and indeed are representative of the massive elliptical galaxies that typically act as strong lenses. With the exception of B0128, the velocity dispersions (implied via the Einstein radii and lens/source redshifts) of the radio lenses suggest these are massive elliptical galaxies.

Three quads in our sample do not have measured spectroscopic redshifts. For two of these systems, we use photometry from Shajib et al. (2019) to compute photometric redshifts probability densities with the software eazy (Brammer et al. 2008), and sample the deflector redshift from these distributions in the forward model. For the third system (2026), we assume a typical velocity dispersion for a massive elliptical galaxy with a roughly isothermal mass profile, and derive a probability distribution for the lens redshift from measured quantities such as the source redshift and measured image separation. We give more details regarding this procedure in Appendix B.

5.8 Satellite galaxies and nearby deflectors

We model satellite galaxies and other deflectors near the main lens as Singular Isothermal Spheres (SIS), and assume they lie at the lens redshift unless they have measured redshifts that place them elsewhere. We marginalize over the position and Einstein radius of these objects using Gaussian priors on the positions centered on the light centroid with a variance of 0.05 arcseconds. We use a Gaussian prior on the Einstein radius which is estimated from lens model fitting, or in some cases by direct measurements on the central velocity dispersion (e.g. Wong et al. 2017; Rusu et al. 2019).

In the cases of HE0435 and WFI2033, the satellite galaxy lies at a higher redshift than the main lens plane. The light from the satellite is therefore subject to lensing by the main deflector, and its true physical location differs from its observed position. We estimate the true physical locations of these deflectors by sampling the macromodel parameter space using the image positions as constraints, and read out the physical position of background satellite given its observed (lensed) position. We then place the satellite at this derived physical location in the forward model sampling with uncertainties of ~ 0.05 arcseconds. This process significantly speeds up the lensing computations since it does not require the continuous reevaluation of the physical satellite location given its observed position during each lens model computation. The boost in speed comes at the cost of decoupling the satellite galaxy from the dark matter parameters \mathbf{q}_{s} in the inference, but we expect the covariance between these quantities will be negligible because the satellite galaxies, even when their locations are corrected for lensing effects, are relatively far from the images, introducing convergence at the main deflector light centroid of < 0.1in both cases.⁹

In the case of HE0435, we estimate the angular location without foreground lensing of the satellite to be (-2.37, 2.08), while for WFI 2033 we obtain (-3.63, -0.08), for observed (lensed) locations of (-2.911, 2.339) and (-3.965, -0.022), respectively. These coordinates are with respect to the galaxy light centroid (see Table C1). The angular locations of the lensed background satellites are closer to the mass centroid of the main deflector (which we place at the origin), just as the physical location of the lensed background quasar is closer to the origin.

The lens-specific priors on satellite galaxies are summarized in Table 2.

5.9 Lens-specific modeling for RX J0911+0551, B0128+437, and WGD 2038-4008

For system J0911, we alter the modeling strategy slightly to increase computational efficiency by allowing the external shear strength $\gamma_{\rm ext}$ to vary freely while solving for macromodel parameters that fit the observed image positions. For the systems B0128 and WGD 2038, we widen the prior on the power-law slope of the macromodel as the posterior using the default range for $\gamma_{\rm macro}$ between 1.95 - 2.2 is strongly

⁹ The default convention in lenstronomy is to place deflectors at their observed angular locations in the universe, but it is now possible (in code versions 0.8.0+) to specify which objects should be treated using the observed (lensed) position instead. We note that the default convention in lensmodel (Keeton et al. 1997) is to place objects at their observed (lensed) locations during multiplane ray-tracing.

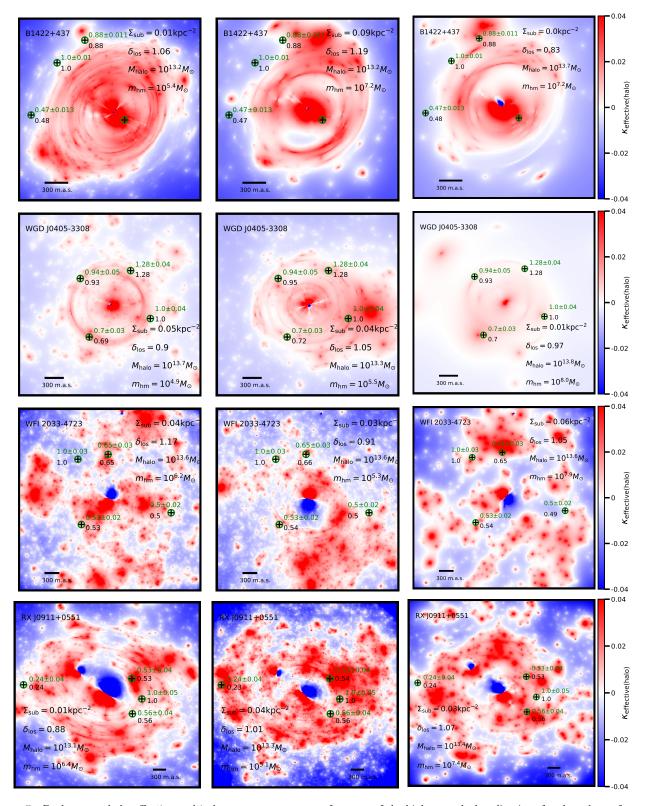


Figure 5. Dark matter halo effective multi-plane convergence maps for some of the highest-ranked realizations for the subset of quads B1422, WGD J0405, WFI 2033, and J0911, each of which has flux ratios inconsistent with smooth lens models. The defintion of the effective multi-plane convergence takes into account the non-linear effects present in multi-plane lensing, and is defined with respect to the mean dark matter density in the universe such that some regions are underdense (blue), while other regions (specifically, dark matter halos) are over-dense (red). The subhalo mass function normalization, line of sight normalization, halo mass and half-mode mass are displayed for each realization. Green text/circles denote observed image positions and fluxes, while black text/crosses denote the model positions and fluxes. The forward-model data sets fit the image positions and fluxes to within the measurement uncertainties.

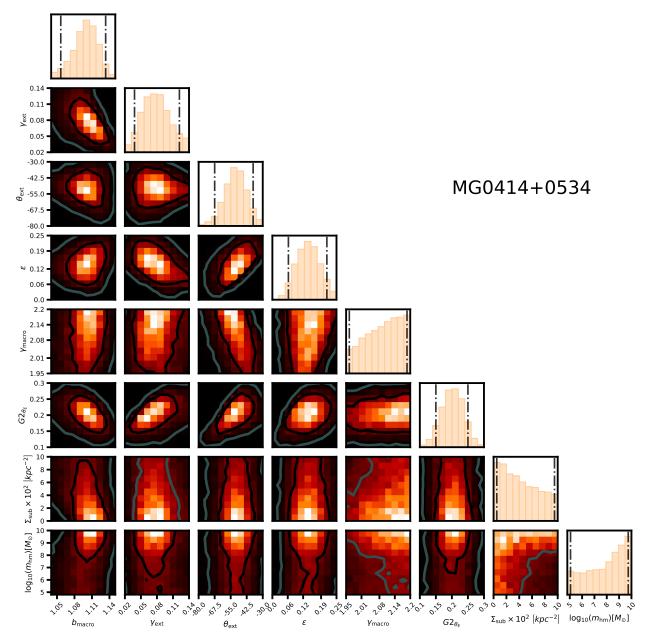


Figure 6. Joint posterior distribution for a subset of M and \mathbf{q}_{s} parameters for the system MG0414. We display the normalization of the main deflector lens model b_{macro} , the external shear strength and position angle γ_{ext} and θ_{ext} , the deflector ellipticity ϵ , the power-law slope of the main deflector mass profile γ_{macro} , the Einstein radius of the satellite galaxy $G2_{\theta_E}$, the normalization of the subhalo mass function Σ_{sub} , and the half-mode mass m_{hm} . We simultaneously sample the distributions of these parameters to account for covariance between the macromodel and the dark matter hyper-parameters \mathbf{q}_{s} .

biased towards higher values of $\gamma_{\rm macro}$. For WGD 2038, the posterior peaks at $\gamma_{\rm macro} \sim 2.25$, while for B0128 the peak is at $\gamma_{\rm macro} \sim 2.28$.

6 RESULTS

In this section we present the results of our analysis. We begin in Section 6.1 by showing dark matter halo convergence maps for some of the top-ranked realizations drawn in the forward model. We then display the posterior distributions for a few individual lenses, showing the simultaneous

inference of parameters describing the macro lens model and the dark matter hyper-parameters. In Section 6.2 we present the constraints on the abundance of substructure and dark matter warmth for the full sample of 11 quads.

6.1 Top-ranked realizations and posteriors for individual lenses

Minimizing the summary statistic in Equation 4 is equivalent to enforcing equality between the simulated datasets and the observed datasets. This guarantees the sets of ac-

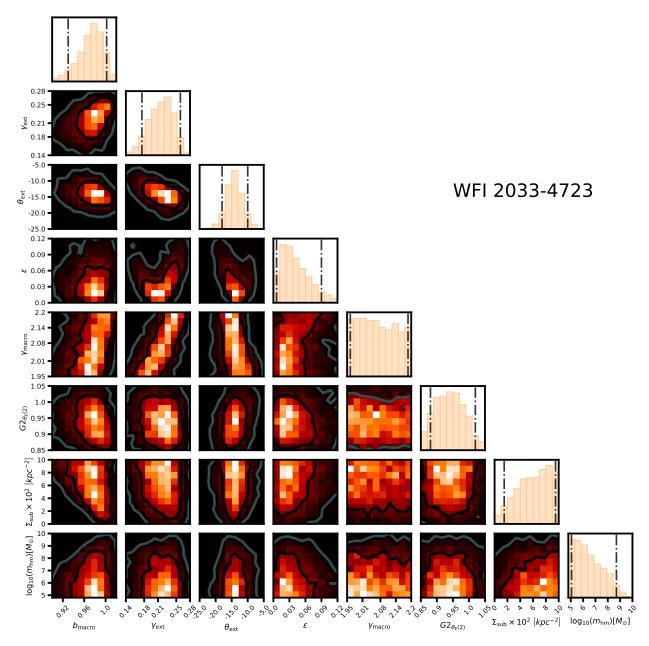


Figure 7. Joint posterior distribution for a subset of M and \mathbf{q}_s parameters for the system WFI 2033. The parameters are the same as in Figure 6. In addition to the main deflector we model two additional nearby galaxies, with Einstein radii $G2_{\theta_E(1)}$ and $G2_{\theta_E(2)}$. We show the distributions of the Einstein radius for the larger nearby galaxy $(G2_{\theta_E(2)})$, whose position we correct for foreground lensing effects (see Section 5.8).

cepted dark matter hyper-parameters that correspond to the accepted realizations are direct draws from the posterior of \mathbf{q}_{s} for each individual lens with data d_n : $p(\mathbf{q}_{\mathrm{s}}|\mathbf{d}_{\mathrm{n}})$. For visualization purposes, and to reinforce the fact that the topranked realizations look like the data and satisfy $S_{\mathrm{lens}} \approx 0$ (Equation 4), in Figure 5 we display the dark matter halo effective multi-plane convergence maps for some of the topranked realizations for a subset of quads in our sample. The effective multi-plane convergence is defined as half the divergence of the full deflection field α

$$\kappa_{\text{effective}} \equiv \frac{1}{2} \, \boldsymbol{\nabla} \cdot \boldsymbol{\alpha}$$
(13)

This definition of the multi-plane convergence accounts for the non-linear effects present in multi-plane lensing, and satisfies the single-plane definition of convergence as second derivatives of a lensing potential in the absence of multiple lens planes.

To visualize individual realizations of dark matter structure, we define $\kappa_{\rm effective(halo)} \equiv \kappa_{\rm effective} - \kappa_{\rm macro}$, where $\kappa_{\rm macro}$ is the convergence from the lens macromodel, including satellite galaxies and nearby deflectors. In the resulting convergence maps, halos located behind the main lens plane appear sheared tangentially around the Einstein radius due

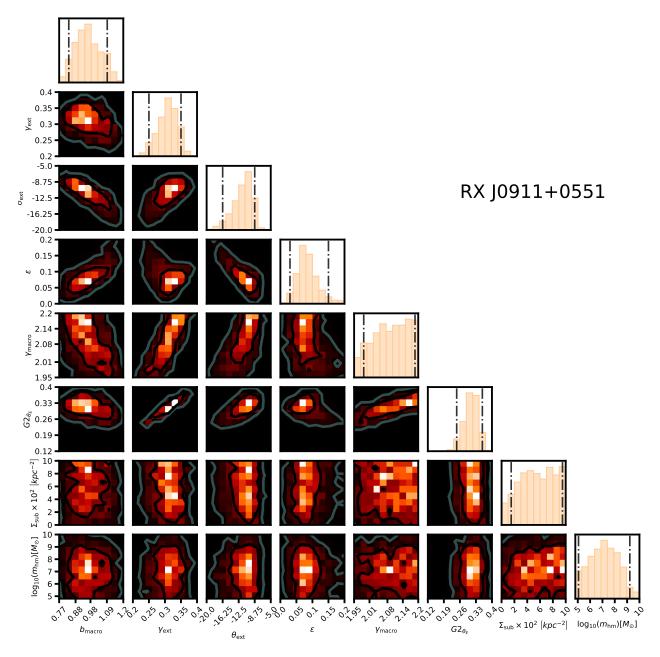


Figure 8. Joint posterior distribution for a subset of M and q_s parameters for the system J0911. The parameters are the same as in Figure 6.

to coupling to the large deflections produced by the macro-model

In Figure 5, we show the $\kappa_{\rm effective(halo)}$ maps for the systems B1422, WGD J0405, WFI 2033, and J0911. These four systems are selected since they have flux ratios inconsistent with smooth lens models ?. The addition of dark matter clumps in the main lens plane and along the line of sight brings the model-predicted flux ratios into agreement with the observed flux ratios. We construct the posterior distributions for each quad from the 800 best-fitting realizations, which typically corresponds to statistics $S_{\rm lens} \sim 0.02$.

The top-ranked realizations for B1422 each have a relatively massive dark matter halo, or several smaller ones,

located near the top right merging triplet image with flux ratio 0.88. This is in agreement with the analysis by Nierenberg et al. (2014), who find that a blob of dark matter near this image brings the model-predicted flux ratios into agreement with a smooth lens model.

Although not obvious from examining Figure 5, the underlying macromodels for each accepted realization are unique, with different external shears, power-law slopes, lens ellipticities, etc. We marginalize over different macromodel configurations by simultaneously sampling the macromodel parameters and the dark matter hyper-parameters in the forward model. To illustrate, in Figures 6, 7, and 8 we show the posterior distributions for several parameters in the lens

macromodel, along with the dark matter hyper-parameters $\Sigma_{\rm sub}$ and $m_{\rm hm}$ for MG0414, WFI 2033, and J0911. The lensing signal in MG0414, due to the high lens redshift, is dominated by line of sight halos, and favors models with low subhalo mass function normalizations (low $\Sigma_{\rm sub}$), or a turnover the mass function which can mimic a lower overall normalization by removing some subhalos. The system WFI 2033 is the opposite, with a posterior favoring CDM-like mass functions with many lens plane subhalos. The system J0911 lies somewhere in between, with a peak in the posterior distribution of $m_{\rm hm}$ near $10^{7.5} M_{\odot}$.

For each of these systems, in particular WFI 2033, there is a visibly obvious covariance between the overall normalization of the main deflector mass profile $b_{\rm macro}^{10}$, and the parameters $\Sigma_{\rm sub}$ and $m_{\rm hm}$. This covariance is readily understood: To reproduce the observed image positions, the macromodel responds to the addition of mass in the form of subhalos in main lens plane by decreasing the overall normalization of the main deflector mass profile, and hence these quantities are anti-correlated. Similarly, WDM models correspond to macromodels with larger $b_{\rm macro}$ because WDM realizations contain fewer subhalos. In the case of MG0414, there is also correlation between the main deflector mass profile $\gamma_{\rm macro}$, $m_{\rm hm}$, $\Sigma_{\rm sub}$, and $M_{\rm halo}$. Interestingly, there is some structure in the posterior distribution for the lens ellipticity ϵ in WFI 2033, and both $m_{\rm hm}$ and $\Sigma_{\rm sub}$.

By simultaneously sampling the lens macromodel and dark matter hyper-parameters, we obtain posterior distributions that account for covariance between M and q_s . We do not use lens model priors from more sophisticated lens modeling efforts (e.g. Wong et al. (2017); Shajib et al. (2019)) because these analyses did not include substructure in the lens models and therefore do not account for covariances between the macromodel parameters and the dark matter parameters of interest. For the same reason, we do not decouple the lens macromodel parameters from the dark matter hyper-parameters by first sampling the macromodel parameter space that fits the image positions, and using these distributions as priors in the forward modeling.

6.2 Constraints on the free-streaming length of dark matter

For each quad, we obtain a joint likelihood between the macromodel parameters \mathbf{M} and the dark matter-hyper parameters \mathbf{q}_{s} . We project this 20+ dimensional space into the four dimensional space of \mathbf{q}_{s} parameters that includes logarithmic slope of the subhalo mass function α , the scaling of the line of sight halo mass function δ_{los} , the overall scaling of the subhalo mass function Σ_{sub} , and the half-mode mass m_{hm} . We reiterate that these four parameters describe universal properties of dark matter and should therefore be common to all the lenses, while the parameters \mathbf{M} and the halo mass M_{halo} are lens-specific. After projecting, we compute the product of the resulting likelihoods and obtain the desired posterior distribution in Equation 1, which we display in Figure 9.

The marginalized constraints on $m_{\rm hm}$ rule out $m_{\rm hm} >$

 $10^{8.2} M_{\odot}$ at 2σ , and $m_{\rm hm} > 10^{7.1} M_{\odot}$ at 1σ , corresponding to thermal relic particle masses of 4.0 and 8.5keV, respectively. It is apparent from Figure 9 that $m_{\rm hm}$ and $\Sigma_{\rm sub}$ are correlated, since halos added by increasing the normalization can be subsequently removed by increasing $m_{\rm hm}$ such that the total amount of lensing substructure remains relatively constant.

Since $m_{\rm hm}$ and $\Sigma_{\rm sub}$ are covariant, the constraints on dark matter warmth in terms of confidence intervals are Σ_{sub} prior-dependent. Similarly, the confidence interval on $m_{\rm hm}$ depends on the lower bound of this parameter that is set by the prior on $m_{\rm hm}$. As discussed in Section 5.5, we have chosen the prior on $m_{\rm hm}$ to encompass the region of parameter space where the data can constrain $m_{\rm hm}$, keeping in mind that the WDM mass concentration relation affects the central densities of subhalos 60 times above $m_{\rm hm}$ (Equation 12), and the upper bound of $\Sigma_{\text{sub}} = 0.1 \text{kpc}^{-2}$ is a conservative choice as most N-body simulations and the galacticus runs predict values below 0.05kpc⁻². In light of these complications, we also quote likelihood ratios which do not depend on the choice of prior: Relative to the peak of the $m_{\rm hm}$ posterior at $10^{7.1} M_{\odot}$, we obtain likelihood ratios for WDM with $m_{\rm hm} = 10^{8.2} M_{\odot} \ (m_{\rm hm} = 10^9 M_{\odot})$ of 3:1 (20:1). With only eleven quads we cannot simultaneously measure $m_{\rm hm}$ and $\Sigma_{\rm sub}$, although our previous forecasts indicate this is possible with more lenses (Gilman et al. 2018).

The posterior for $\delta_{\rm los}$ indicates the data favors more line of sight structure, but the preference is not statistically significant. The parameters $\delta_{\rm los}$ and $\Sigma_{\rm sub}$ are anti-correlated, as one would expect as one can, to a certain degree, remove lens plane subhalos and replace them with line of sight halos while keeping the total amount of flux perturbation constant. This is not a perfect covariance, however, since lensing efficiency and the relative number of subhalos and line of sight halos changes with redshift. Thus, lenses at different redshifts could break the covariance between $\Sigma_{\rm sub}$ and $\delta_{\rm los}$.

Finally, the 1σ contour in the joint posterior between $\log_{10} (m_{\rm hm})$ and α suggests the data prefers $\alpha \sim -1.88$, but the peak is not statistically significant. In the next section we will show that we can simultaneously constrain α and $\Sigma_{\rm sub}$ by assuming CDM.

6.3 Constraints on the subhalo mass function assuming CDM

We discard samples from the forward model with $m_{\rm hm} > 10^{5.5} M_{\odot}$ to construct a joint posterior distribution of $\Sigma_{\rm sub}$, α , and $\delta_{\rm los}$ considering only CDM-like mass functions. While $\delta_{\rm los}$ remains unconstrained, we are able to simultaneously measure $\Sigma_{\rm sub}$ and α .

The joint posterior for $\Sigma_{\rm sub}$ and α is shown in Figure 10. We infer $\Sigma_{\rm sub} = 0.036 {\rm kpc}^{-2}$, with a 1σ confidence interval $0.022 < \Sigma_{\rm sub} < 0.046 {\rm kpc}^{-2}$. At the 2σ level we cannot exclude $\Sigma_{\rm sub} = 0$, but we can rule out $\Sigma_{\rm sub} > 0.08 {\rm kpc}^{-2}$. To put these numbers in physical units, the mean value of $\Sigma_{\rm sub}$ corresponds to a projected mass in substructure between $10^6 - 10^{10} M_{\odot}$ of $4.0 \times 10^7 M_{\odot} {\rm kpc}^{-2}$, and the 1σ confidence interval corresponds to $2.4 - 5.1 \times 10^7 M_{\odot} {\rm kpc}^{-2}$. At 2σ , we find $\Sigma_{\rm sub} < 8.9 \times 10^7 M_{\odot} {\rm kpc}^{-2}$. To convert into the average projected mass, we have computed the average of the projected masses for each of the eleven lenses in our sample, using the scaling of the halo mass function with redshift in Equation 8

 $^{^{10}}$ $b_{
m macro}$ has units of convergence, or projected mass density divided by the critical surface mass density for lensing.

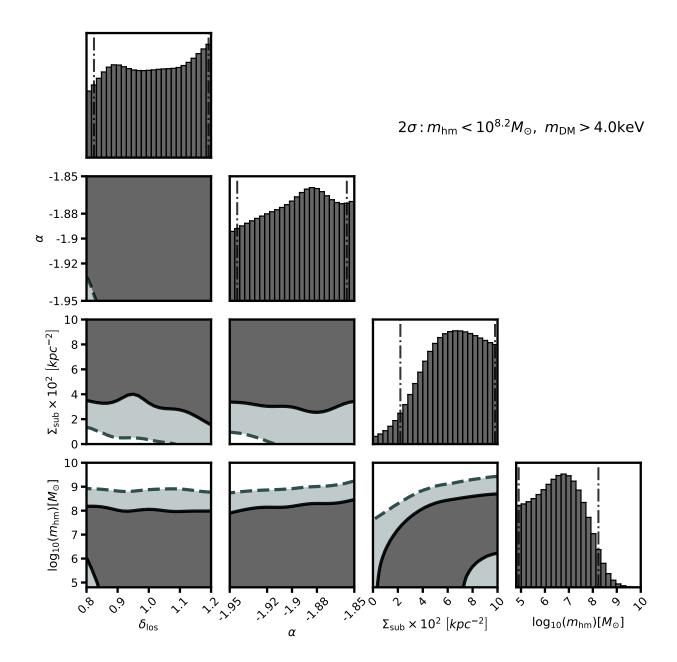
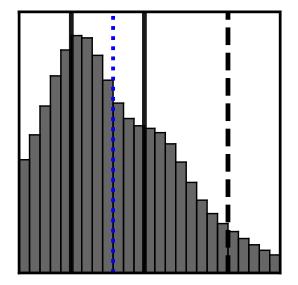


Figure 9. Marginal and joint posterior distributions for the dark matter hyper-parameters δ_{los} , α , Σ_{sub} , and m_{hm} , which represent the overall scaling of the line of sight halo mass function, the logarithmic slope of the subhalo mass function, the global normalization of the subhalo mass function that accounts for evolution with halo mass and redshfit (see Equation 7), and the half-mode mass m_{hm} relevant to WDM models. Contours show 68% and 95% confidence intervals, while the dot-dashed lines on the marginal distributions show the 95% confidence intervals.

while assuming a halo mass of $10^{13} M_{\odot}$. At the specific lens redshifts z=0.2 (z=0.6), the mean $\Sigma_{\rm sub}=0.036 {\rm kpc}^{-2}$ value we infer corresponds to a projected mass in substructure between $10^6-10^{10} M_{\odot}$ of $1.9\times 10^7 M_{\odot}$ ($4.0\times 10^7 M_{\odot}$), where we have again used Equation 7 assuming a halo mass of $10^{13} M_{\odot}$.

Other works (Vegetti et al. 2014; Hsueh et al. 2019) instead quote projected masses in the range $10^6-10^9 M_{\odot}$. In this mass range $\Sigma_{\rm sub}=0.036 {\rm kpc}^{-2}$ corresponds to a projected mass in substructure of $1.2\times 10^7 M_{\odot}$ at z=0.2, and $2.6\times 10^7 M_{\odot} {\rm kpc}^{-2}$ at z=0.6 for a $10^{13} M_{\odot}$ halo.

As flux ratios probe a dynamic range spanning several decades in halo mass, in addition to the overall amplitude of the subhalo mass function $\Sigma_{\rm sub}$ we are able to simultaneously constrain the logarithmic slope α . The marginal distribution for α in Figure 10 has a mean $\alpha=-1.892$ and a 1σ confidence interval between -1.906 and -1.882.



Assuming CDM:

 1σ : 0.022 < Σ_{sub} < 0.046kpc⁻

 2σ : $\Sigma_{sub} < 0.08 \text{kpc}^{-2}$

 1σ : $-1.906 < \alpha < -1.882$

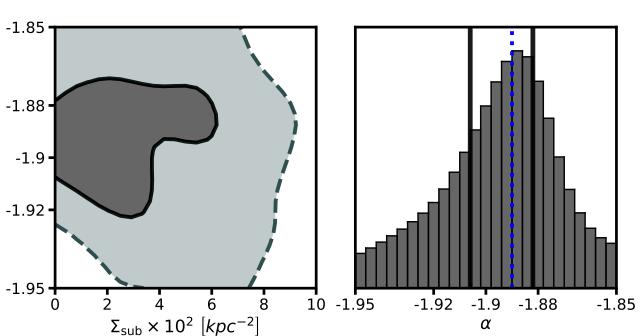


Figure 10. Joint constraints on the global normalization of the subhalo mass function $\Sigma_{\rm sub}$ and the logarithmic slope α . Blue dashed lines show the means of the marginal distributions, while black solid (black dashed) represent 68% and 95% confidence intervals. The contours in the joint distribution also represent 68% and 95% confidence intervals. These results are marginalized over the normalization of the line of sight halo mass function $\delta_{\rm los}$, which is unconstrained.

7 DISCUSSION AND CONCLUSIONS

In this section we review the main results of this work and discuss the implications for cold and warm dark matter. In Section 7.1 we summarize our main results, and in Section 7.2 we compare our results with those obtained in previous works. In Section 7.3 we discuss the sources of systematic uncertainty in our analysis, and we conclude in Section 7.4 by discussing the implications of our result for cold and warm dark matter.

7.1 Summary of the analysis and main results

We have carried out a measurement of the free-streaming length of dark matter and subhalo mass function in the dark matter halos hosting early-type galaxies using a sample of eleven quadruply-imaged quasars. The methodology we use to constrain the dark matter parameters of interest has been tested and verified with simulated datasets (Gilman et al. 2019). We model halos both in the main deflector and along the line of sight, and account for evolution of the projected subhalo mass function with redshift and halo mass using a suite of simulations using the semi-analytic modeling code galacticus. We compute image flux ratios by ray-tracing to

finite-size background sources, which correctly accounts for the sensitivity of image flux ratios to perturbing halos. We also marginalize over the macromodel parameters for each system, including the power-law slope of the main deflector, and simultaneously constrain the lens macromodel and dark matter hyper-parameters for the first time in a lensing analysis with image flux ratios and full realizations of dark matter structure. In addition to the turnover in the halo mass function, our analysis models the WDM mass-concentration relation and accounts for the effect of reduced central densities of WDM halos on lensing observables.

The main results of this analysis are summarized as follows:

- We constrain the half-mode mass $m_{\rm hm}$ (thermal relic dark matter particle mass) to $m_{\rm hm} < 10^{8.2} M_{\odot} \ (m_{\rm DM} > 4.0 {\rm keV})$ at 2σ . Since the confidence intervals depend on the prior used for both $m_{\rm hm}$ and $\Sigma_{\rm sub}$, we quote likelihood ratios relative to the peak of the posterior distribution for $m_{\rm hm}$: with $m_{\rm hm} = 10^{8.2} M_{\odot} \ (m_{\rm DM} = 4.0 {\rm keV})$ we obtain a likelihood ratio of 3:1, and with $m_{\rm hm} = 10^9 M_{\odot} \ (m_{\rm DM} = 2.3 {\rm keV})$ the relative likelihood is 20:1. These bounds are marginalized over the amplitude of the subhalo mass function, the amplitude of the line of sight halo mass function, the power-law slope of the subhalo mass function, the parent halo mass, the background source size, and the parameters describing the main deflector mass profile.
- Assuming cold dark matter, we infer a value of the global amplitude of the subhalo mass function $\Sigma_{\rm sub}=0.036^{0.010}_{-0.014}{\rm kpc}^{-2}$ at 1σ , and $\Sigma_{\rm sub}<0.08{\rm kpc}^{-2}$ at 2σ . In our lens sample, these values correspond to an average projected mass density in substructure between $10^6-10^9M_{\odot}$ of $2.6^{+0.7}_{-1.0}M_{\odot}{\rm kpc}^{-2}$, and an upper limit of $5.8\times10^7M_{\odot}{\rm kpc}^{-2}$. At fixed redshifts, for a $10^{13}M_{\odot}$ halo at z=0.2 (z=0.6) the 1σ constraint corresponds to a projected mass in substructure of $1.2^{+0.4}_{-0.4}\times10^7M_{\odot}{\rm kpc}^{-2}$ ($2.6^{+0.6}_{-1.0}\times10^7M_{\odot}{\rm kpc}^{-2}$), while at 2σ we obtain an upper limit on the projected mass of $2.7\times10^7M_{\odot}{\rm kpc}^{-2}$ ($5.9\times10^7M_{\odot}{\rm kpc}^{-2}$) in the subhalo range $10^6-10^9M_{\odot}$.
- Again assuming cold dark matter, we constrain the logarithmic slope of the subhalo mass function $\alpha=-1.892^{+0.010}_{-0.014}$ at 1σ .

7.2 Discussion and comparison with previous work

7.2.1 Constraints on dark matter warmth and the CDM halo mass function normalization

The first comprehensive analysis of multiply-imaged quasars was carried out by Dalal & Kochanek (2002) (hereafter DK2), who inferred a projected mass fraction in substructure $\bar{f}_{\rm sub}^{-11}$ between $0.006 < \bar{f}_{\rm sub} < 0.07$ at 2σ modeling only lens-plane substructure, and assuming CDM. Recently, Hsueh et al. (2019) (hereafter H19) improved on the analysis of DK2 by including the effects of line of sight halos,

measuring $0.006 < \bar{f}_{\rm sub} < 0.018$ at 1σ with a mean of 0.011, and also constrained the free-streaming length of dark matter to $m_{\rm hm} < 10^{8.4}~(m_{\rm DM} > 3.8 {\rm keV})$ assuming CDM. The 2σ bound from H19 of $m_{\rm hm} < 10^{8.4} M_{\odot}$ is similar to the constraint from this work $m_{\rm hm} < 10^{8.2} M_{\odot}$. We note that the analysis codes are completely independent, thus the good agreement between the two measurements is reassuring.

To facilitate direct comparison between this analysis and that of DK2 and H19 regarding the constraints on the subhalo mass function assuming CDM, we convert our $\Sigma_{\rm sub}$ values into estimates of $\bar{f}_{\rm sub}$ by computing the projected mass density $\Sigma,$ and then using the fact that $\frac{\Sigma}{\Sigma_{\rm crit}}=0.5$ near the Einstein radius, where $\Sigma_{\rm crit}$ is the critical surface mass density for lensing. In these conversions, we also assuming a halo mass of $10^{13} M_{\odot}$, and take care to compute $\bar{f}_{\rm sub}$ using the same mass range $10^6-10^9 M_{\odot}$ used by H19. Our 1σ bounds on $\Sigma_{\rm sub}$ correspond to an average mass fraction in substructure $0.014 < \bar{f}_{\rm sub} < 0.030$ with a mean of $\bar{f}_{\rm sub} = 0.023$, while at 2σ $\bar{f}_{\rm sub} < 0.052$. This result is statistically consistent with the constraints from H19, and also with those of DK2.

There are several key differences between our analysis and that of H19 that pull in opposite directions in terms of constraining power over dark matter models. First, our sample size is $\sim 40\%$ larger, which should result in stronger constraints. Second, we model free-streaming effects on the mass-concentration relation, and include the contribution from the two-halo term to account for correlated structure near the main deflector. These pieces of additional physics add information and increase our constraining power over WDM models, which should lead to tighter constraints. On the other hand, the use of finite-size background source decreases our sensitivity to perturbing halos on essentially all the relevant mass scales, and we expect to infer a higher normalization of the subhalo mass function as more substructure is needed to produce the same degree of flux perturbation. The tidal truncation of lens plane subhalos that we account for may also reduce the overall impact of subhalos on lensing observables. We also marginalize over the power-law slope of the main deflector and simultaneously sample the macromodel parameters and the dark matter hyper-parameters. These processes introduce additional covariances that should lead to weaker constraints.

Other lensing studies, primarily those using the technique of gravitational imaging, have also sought to measure the subhalo mass function. Vegetti et al. (2014) inferred $\bar{f}_{\rm sub} = 0.0064^{0.0080}_{-0.0042}$ at 1σ in the mass range $4\times10^6-4\times10^9 M_{\odot}$ assuming a prior on the slope of the subhalo mass function centered on $\alpha=-1.9$, while Hezaveh et al. (2016) constrained the normalization of subhalo mass function assuming $\alpha=-1.9$, but with significantly wider uncertainties.

To compare with the analysis of Vegetti et al. (2014), we take our mean inferred value of $\Sigma_{\rm sub} = 0.036 {\rm kpc}^{-2}$ and assume a halo mass of $10^{13} M_{\odot}$ at a lens redshift $z_d = 0.25$ and a source at $z_{\rm src} = 0.7$, characteristic values for the lens sample analyzed by Vegetti et al. (2014). Using these values with our expression for the subhalo mass function in Equation 7, we obtain $f_{\rm sub} = 0.009^{+0.003}_{-0.003}$ between 4×10^6 and $6 \times 10^9 M_{\odot}$, the same mass range used by Vegetti et al. (2014). This result is consistent with that of Vegetti et al.

 $^{^{11}}$ Throughout this section, we will use $\bar{f}_{\rm sub}$ to refer to the average mass fraction in substructure inferred from a sample of multiple lenses at different redshifts, and $f_{\rm sub}$ to refer to the mass fraction in substructure implied by a certain $\Sigma_{\rm sub}$ value at a specific redshift.

 $(2014)^{12}$. We caution that conclusions derived from f_{sub} constraints should be interpreted with care, given the dependence on the subhalo mass range and assumptions regarding the lensing convergence near the Einstein radius.

Comparing our results with semi-analytic simulations of massive $10^{13} M_{\odot}$ hosts, our results in terms of the projected mass in substructure is consistent with the galacticus simulations used to calibrate the evolution of the subhalo mass function with halo mass and redshift. We stress that our model was not tuned to match the normalization predicted by galacticus, it only utilized the evolution with redshift and halo mass from the simulations.

Our results are also consistent with N-body simulations of $10^{13} M_{\odot}$ halos by Fiacconi et al. (2016), who predict projected substructure mass densities of $2.0-2.8\times10^7 M_{\odot}{\rm kpc}^{-2}$ after accounting for baryonic contraction of the halo. We infer roughly double the predicted mass in substructure than the amount predicted by Xu et al. (2015), who simulated $10^{13} M_{\odot}$ halos by rescaling Milky Way size hosts to higher masses. We note that our results arrive on the heels of several works that examine numerical features of N-body simulations that may result in the artificial fragmentation of subhalos (van den Bosch et al. 2018; Errani & Peñarrubia 2019). Taken at face value, these results suggest that N-body simulations may underpredict substructure abundance in dark matter halos.

Finally, we may compare our constraints with the projections from Gilman et al. (2019). With a sample of ten quads, they projected a 2σ bound on $m_{\rm hm}$ with $\Sigma_{\rm sub}=0.02{\rm kpc}^{-2}$ of $10^{7.7}M_{\odot}$ with 2% uncertainties in image fluxes, and $10^{8.6}M_{\odot}$ with 6% uncertainties. Our constraint of $m_{\rm hm}<10^{8.2}M_{\odot}$ is broadly consistent with these predictions¹³, given the higher $\Sigma_{\rm sub}$ value of $0.036{\rm kpc}^{-2}$ we infer in this analysis, and the flux uncertainties in the lens sample which are $\sim 5\%$ on average.

7.2.2 Constraints on the slope of the subhalo mass function

Due to the large dynamic range in halo masses probed by image flux ratios (we estimate most signal comes from halos between $10^7-10^9 M_{\odot}$ for the source sizes we model) we are able to simultaneously constrain the normalization and the slope of the subhalo mass function. At 1σ , we obtain $\alpha=-1.892^{+0.014}_{-0.014}$. This result is in slight tension with the constraint on the logarithmic slope from Vegetti et al. (2012) ($\alpha=-1.1^{+0.4}_{-0.6}$ at 1σ), but entirely consistent with a later constraint Vegetti et al. (2014) ($\alpha>-2.93$ at 2σ). Given our choice of prior, which is significantly narrower than those used by Vegetti et al. (2012, 2014), we do not make statements on α at the level of 2σ . However, the posterior clearly indicates the data favors logarithmic slopes consistent with

those predicted in N-body simulations (Springel et al. 2008; Fiacconi et al. 2016). This result is therefore in excellent agreement with a fundamental prediction of cold dark matter.

7.3 Sources of systematic uncertainties

7.3.1 The lens macromodels

Several works (Gilman et al. 2017; Hsueh et al. 2018) have investigated the ability of smooth isothermal mass models plus external shear to fit the smooth mass component of galaxy scale strong lenses. These works reach similar conclusions, determining that isothermal models predict image flux ratios to better than 10% unless a stellar disk is present, in which case explicit modeling of the disk is required (e.g. Hsueh et al. 2017, 2018). Each of these analysis restricted the smooth lens models to exactly isothermal mass density profiles.

In this work, we exclude all lens systems with known stellar disks to avoid any bias they may introduce in the inference. To account for remaining uncertainties associated with the lens macromodel, we highlight two features of our lens modeling implemented in an effort to mitigate this source of systematic uncertainty. First, we note that flux ratios are highly localized probes of the surface mass density in the immediate vicinity of the lensed images, and therefore the main requirement for this work is to accurately predict the mass profile in these four small isolated regions. By relaxing the strictly isothermal mass profile assumption and marginalizing over the logarithmic slope of the main deflector mass profile, we allow for the local mass profile in the vicinity of the lensed images to vary. The additional degree of freedom added in the lens macromodel increases our uncertainties, but accounts for deviations from exact $\rho(r) \propto r^{-2}$ power-law ellipsoids.

Second, we note that smooth power-law models predict a distribution of flux ratios, rather than single values (for example, see Figures A1-A8 in ?). Following common practice, Gilman et al. (2017) and Hsueh et al. (2018) identified flux ratio 'anomalies' with respect to a single smooth model fit to lensed images, a procedure that does not account for the distribution of flux ratios predicted by smooth lens models that is marginalized over in the full forward modeling analysis we perform. In this work, we also take care to explore the macromodel parameter space and the dark matter hyper-parameter space simultaneously, which accounts for additional covariances that contribute to the model-predicted flux uncertainties.

7.3.2 Modeling of the dark matter content

We assume specific functional forms for the halo and subhalo mass functions (Equations 7 and 9), and the massconcentration-redshift relation (Equation 12). We acknowledge that there are other parameterizations in the literature for both of these quantities (e.g. Schneider et al. 2012; Benson et al. 2013), but in this work we implement only one parameterization of WDM effects on the mass function (Equation 11) and halo concentrations (Equation 12), which corresponds to one specific WDM model. We note that additional physics, such as the velocity dispersion of dark mat-

¹² Although Vegetti et al. (2014) did not model line of sight halos, the low lens/source redshifts their sample lessen the impact of line of sight halos on the inferred subhalo mass fraction such that we may compare our results, which include line of sight halos, with theirs.

¹³ The conversion between the half-mode mass and the mass of the corresponding thermal relic dark matter particle used by Gilman et al. (2019) is off by a factor of h=0.7, but the comparison between the half-mode masses is robust.

ter particles in the early universe, can alter the shape of the mass function, but with the current sample size of lenses it is unlikely we have enough information to constrain these additional features if they were included in the model.

It is possible that free-streaming effects on the halo mass function near the half-mode mass scale may become more pronounced at high redshifts. This could affect both the location and shape of the turnover in the mass function. However, in the absence of a specific prediction for the evolution of the turnover with redshift, we apply the parameterization in Equation 11 through the relevant redshift range z=0-3.5. We note that since the lensing efficiency of halos decreases approaching source redshift, systematic errors from possible redshift evolution of the WDM turnover will be correspondingly down-weighted. We note that the mass-concentration-redshift relation for WDM calibrated by Bose et al. (2016) that we implement does evolve with redshift, as does the CDM mass-concentration relation from (Diemer & Joyce 2019).

7.4 Implications for WDM models

Galaxy-galaxy strong lensing provides a useful compliment to the strongest existing probe of the free-streaming length of dark matter from the Lyman- α forest (Viel et al. 2013; Iršič et al. 2017). Our 2σ bound on the thermal relic mass of $m_{\rm DM} > 4.0 {\rm keV}$ surpasses than the 3.3 keV constraint from Viel et al. (2013). It is weaker than the 5.4keV constrain from Iršič et al. (2017) who invoked additional assumptions regarding the thermodynamics relevant to the Lyman- α forest. With that said, the exact number of keV's excluded is of less importance in the context of investigating the nature of dark matter than the fact that lensing depends on completely different systematics and measures the halo mass function directly (as opposed to the matter power spectrum). In spite of these differences, the results from lensing and the Lyman- α forest support the following statement: the halo mass function extends down in a scale-free manner to mass scales of $\sim 10^8 M_{\odot}$, where halos are mostly, if not completely, dark. There appears to be little room left for a viable warm dark matter solution to the small-scale issues of cold dark matter.

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REFERENCES

Abazajian K. N., Kusenko A., 2019, arXiv e-prints, p. arXiv:1907.11696

Agnello A., et al., 2018, MNRAS, 479, 4345

Akeret J., Refregier A., Amara A., Seehars S., Hasner C., 2015, JCAP, 8, 043

Amara A., Metcalf R. B., Cox T. J., Ostriker J. P., 2006, MNRAS, 367, 1367

Anguita T., et al., 2018, MNRAS, 480, 5017

Angulo R. E., Hahn O., Ludlow A. D., Bonoli S., 2017, MNRAS, 471, 4687

Auger M. W., Treu T., Bolton A. S., Gavazzi R., Koopmans L. V. E., Marshall P. J., Moustakas L. A., Burles S., 2010, ApJ, 724, 511

Avila-Reese V., Colín P., Valenzuela O., D'Onghia E., Firmani C., 2001, ApJ, 559, 516

Bade N., Siebert J., Lopez S., Voges W., Reimers D., 1997, AA, 317, L13

Baltz E. A., Marshall P., Oguri M., 2009, JCAP, 1, 015

Bayer D., Chatterjee S., Koopmans L. V. E., Vegetti S., McKean J. P., Treu T., Fassnacht C. D., 2018, arXiv e-prints, p. arXiv:1803.05952

Benson A. J., 2012, NewA, 17, 175

Benson A. J., et al., 2013, MNRAS, 428, 1774

Birrer S., Amara A., 2018, Physics of the Dark Universe, 22, 189

Birrer S., Welschen C., Amara A., Refregier A., 2017a, JCAP, 4, 049

Birrer S., Amara A., Refregier A., 2017b, JCAP, 5, 037 Blackburne J. A., Pooley D., Rappaport S., Schechter P. L., 2011, ApJ, 729, 34

Blandford R., Narayan R., 1986, ApJ, 310, 568

Bose S., Hellwing W. A., Frenk C. S., Jenkins A., Lovell M. R., Helly J. C., Li B., 2016, MNRAS, 455, 318

Boylan-Kolchin M., Bullock J. S., Kaplinghat M., 2011, MNRAS, 415, L40

Brammer G. B., van Dokkum P. G., Coppi P., 2008, ApJ, $686,\,1503$

Brennan S., Benson A. J., Cyr-Racine F.-Y., Keeton C. R., Moustakas L. A., Pullen A. R., 2018, arXiv e-prints, p. arXiv:1808.03501

Bullock J. S., Boylan-Kolchin M., 2017, Annual Review of Astronomy and Astrophysics, 55, 343

Castellano M., Menci N., Grazian A., Merle A., Sanchez N. G., Schneider A., Totzauer M., 2019, arXiv e-prints, p. arXiv:1903.12580

Chiba M., Minezaki T., Kashikawa N., Kataza H., Inoue K. T., 2005, ApJ, 627, 53

Combes F., et al., 2019, AA, 623, A79

- Cyr-Racine F.-Y., Moustakas L. A., Keeton C. R., Sigurdson K., Gilman D. A., 2016, PhysRevD, 94, 043505
- Cyr-Racine F.-Y., Keeton C. R., Moustakas L. A., 2018, arXiv e-prints, p. arXiv:1806.07897
- Dalal N., Kochanek C. S., 2002, ApJ, 572, 25
- Despali G., Giocoli C., Angulo R. E., Tormen G., Sheth R. K., Baso G., Moscardini L., 2016, MNRAS, 456, 2486 Díaz Rivero A., Dvorkin C., Cyr-Racine F.-Y., Zavala J., Vogelsberger M., 2018, PhysRevD, 98, 103517
- Diemer B., 2018, ApJS, 239, 35
- Diemer B., Joyce M., 2019, ApJ, 871, 168
- Dobler G., Keeton C. R., 2006, MNRAS, 365, 1243
- Dodelson S., Widrow L. M., 1994, Phys. Rev. Lett., 72, 17Dooley G. A., Peter A. H. G., Carlin J. L., Frebel A., Bechtol K., Willman B., 2017, MNRAS, 472, 1060
- Dutton A. A., Macciò A. V., 2014, MNRAS, 441, 3359
- Elitzur M., Shlosman I., 2006, ApJ, 648, L101
- Errani R., Peñarrubia J., 2019, arXiv e-prints, p. arXiv:1906.01642
- Falco E. E., Lehar J., Shapiro I. I., 1997, ApJ, 113, 540
 Fiacconi D., Madau P., Potter D., Stadel J., 2016, ApJ, 824, 144
- Garrison-Kimmel S., et al., 2017, MNRAS, 471, 1709
- Gavazzi R., Treu T., Rhodes J. D., Koopmans L. V. E., Bolton A. S., Burles S., Massey R. J., Moustakas L. A., 2007, ApJ, 667, 176
- Gilman D., Agnello A., Treu T., Keeton C. R., Nierenberg A. M., 2017, MNRAS, 467, 3970
- Gilman D., Birrer S., Treu T., Keeton C. R., Nierenberg A., 2018, MNRAS, 481, 819
- Gilman D., Birrer S., Treu T., Nierenberg A., Benson A., 2019, MNRAS, p. 1618
- Graus A. S., Bullock J. S., Boylan-Kolchin M., Nierenberg A. M., 2018, MNRAS, 480, 1322
- Hahn C., Vakili M., Walsh K., Hearin A. P., Hogg D. W., Campbell D., 2017, MNRAS, 469, 2791
- Hewitt J. N., Turner E. L., Lawrence C. R., Schneider D. P., Brody J. P., 1992, AJ, 104, 968
- Hezaveh Y. D., et al., 2016, ApJ, 823, 37
- Hinshaw G., et al., 2013, The Astrophysical Journal Supplement Series, 208, 19
- Hofmann S., Schwarz D. J., Stöcker H., 2001, PhysRevD, 64, 083507
- Homma D., et al., 2019, arXiv e-prints,
- Hsueh J.-W., Fassnacht C. D., Vegetti S., McKean J. P., Spingola C., Auger M. W., Koopmans L. V. E., Lagattuta D. J., 2016, MNRAS, 463, L51
- Hsueh J. W., et al., 2017, MNRAS, 469, 3713
- Hsueh J.-W., Despali G., Vegetti S., Xu D., Fassnacht C. D., Metcalf R. B., 2018, MNRAS, 475, 2438
- Hsueh J.-W., Enzi W., Vegetti S., Auger M., Fassnacht C. D., Despali G., Koopmans L. V. E., McKean J. P., 2019, arXiv e-prints, p. arXiv:1905.04182
- Iršič V., et al., 2017, PhysRevD, 96, 023522
- Jiang F., van den Bosch F. C., 2017, MNRAS, 472, 657
 Katz C. A., Moore C. B., Hewitt J. N., 1997, ApJ, 475, 512
 Keeton C. R., Kochanek C. S., Seljak U., 1997, ApJ, 482,
- Kim S. Y., Peter A. H. G., Wittman D., 2017, MNRAS, 469, 1414
- Kim S. Y., Peter A. H. G., Hargis J. R., 2018, Phys. Rev. Lett., 121, 211302

- Koopmans L. V. E., et al., 2003, ApJ, 595, 712 Lagattuta D. J., et al., 2010, ApJ, 716, 1579
- Lemon C. A., Auger M. W., McMahon R. G., Ostrovski F., 2018, MNRAS, 479, 5060
- Lintusaari J., Gutmann M. U., Dutta R., Kaski S., Corander J., 2017, Systematic Biology, 66, e66
- Lovell M. R., Frenk C. S., Eke V. R., Jenkins A., Gao L., Theuns T., 2014, MNRAS, 439, 300
- Ludlow A. D., Bose S., Angulo R. E., Wang L., Hellwing W. A., Navarro J. F., Cole S., Frenk C. S., 2016, MNRAS, 460, 1214
- Macciò A. V., Ruchayskiy O., Boyarsky A., Muñoz-Cuartas J. C., 2013, MNRAS, 428, 882
- Marin J.-M., Pudlo P., Robert C. P., Ryder R., 2011, arXiv e-prints, p. arXiv:1101.0955
- McKean J. P., et al., 2007, MNRAS, 378, 109
- Menci N., Sanchez N. G., Castellano M., Grazian A., 2016, ApJ, 818, 90
- Metcalf R. B., Madau P., 2001, ApJ, 563, 9
- Metcalf R. B., Zhao H., 2002, ApJ, 567, L5
- Moore B., Ghigna S., Governato F., Lake G., Quinn T., Stadel J., Tozzi P., 1999, ApJL, 524, L19
- Morgan N. D., Caldwell J. A. R., Schechter P. L., Dressler A., Egami E., Rix H.-W., 2004, AJ, 127, 2617
- Morgan C. W., Kochanek C. S., Dai X., Morgan N. D., Falco E. E., 2008, ApJ, 689, 755
- Moustakas L. A., Metcalf R. B., 2003, MNRAS, 339, 607 Müller-Sánchez F., Prieto M. A., Hicks E. K. S., Vives-Arias H., Davies R. I., Malkan M., Tacconi L. J., Genzel R., 2011, ApJ, 739, 69
- Newton O., Cautun M., Jenkins A., Frenk C. S., Helly J. C., 2018, MNRAS, 479, 2853
- Nierenberg A. M., Treu T., Wright S. A., Fassnacht C. D., Auger M. W., 2014, MNRAS, 442, 2434
- Nierenberg A. M., Treu T., Menci N., Lu Y., Torrey P., Vogelsberger M., 2016, MNRAS, 462, 4473
- Nierenberg A. M., et al., 2017, MNRAS, 471, 2224
- Patnaik A. R., Browne I. W. A., Walsh D., Chaffee F. H., Foltz C. B., 1992, MNRAS, 259, 1P
- Phillips P. M., et al., 2000, MNRAS, 319, L7
- Pullen A. R., Benson A. J., Moustakas L. A., 2014, ApJ, 792, 24
- Ritondale E., Vegetti S., Despali G., Auger M. W., Koopmans L. V. E., McKean J. P., 2018, arXiv e-prints, p. arXiv:1811.03627
- Ros E., Guirado J. C., Marcaide J. M., Pérez-Torres M. A., Falco E. E., Muñoz J. A., Alberdi A., Lara L., 2000, AA, 362, 845
- Rubin D. B., 1984, The Annals of Statistics, 12, 1151
- Rusu C. E., et al., 2019, arXiv e-prints, p. arXiv:1905.09338 Schneider P., 1997, MNRAS, 292, 673
- Schneider A., Smith R. E., Macciò A. V., Moore B., 2012, MNRAS, 424, 684
- Schneider A., Smith R. E., Reed D., 2013, MNRAS, 433, 1573
- Shajib A. J., et al., 2019, MNRAS, 483, 5649
- Shankar F., et al., 2017, ApJ, 840, 34
- Sheth R. K., Mo H. J., Tormen G., 2001, MNRAS, 323, 1Shi X., Fuller G. M., 1999, Phys. Rev. Lett., 82, 2832
- Springel V., et al., 2008, MNRAS, 391, 1685
- Stacey H. R., McKean J. P., 2018, MNRAS, 481, L40 Tegmark M., et al., 2004, PhysRevD, 69, 103501

Tollet E., et al., 2016, MNRAS, 456, 3542

Tormen G., Diaferio A., Syer D., 1998, MNRAS, 299, 728
 Treu T., Koopmans L. V., Bolton A. S., Burles S., Moustakas L. A., 2006, ApJ, 640, 662

Treu T., Gavazzi R., Gorecki A., Marshall P. J., Koopmans L. V. E., Bolton A. S., Moustakas L. A., Burles S., 2009, ApJ, 690, 670

Tulin S., Yu H.-B., 2018, Phys. Rept., 730, 1

Vegetti S., Lagattuta D. J., McKean J. P., Auger M. W., Fassnacht C. D., Koopmans L. V. E., 2012, Nat, 481, 341 Vegetti S., Koopmans L. V. E., Auger M. W., Treu T., Bolton A. S., 2014, MNRAS, 442, 2017

Vegetti S., Despali G., Lovell M. R., Enzi W., 2018, preprint, (arXiv:1801.01505)

Viel M., Becker G. D., Bolton J. S., Haehnelt M. G., 2013, PhysRevD, 88, 043502

Vuissoz C., et al., 2008, AA, 488, 481

Weymann R. J., Latham D., Angel J. R. P., Green R. F., Liebert J. W., Turnshek D. A., Turnshek D. E., Tyson J. A., 1980, Nat, 285, 641

Wisotzki L., Schechter P. L., Bradt H. V., Heinmüller J., Reimers D., 2002, AA, 395, 17

Wong K. C., et al., 2017, MNRAS, 465, 4895

Xu D. D., Mao S., Cooper A. P., Gao L., Frenk C. S., Angulo R. E., Helly J., 2012, MNRAS, 421, 2553

Xu D., Sluse D., Gao L., Wang J., Frenk C., Mao S., Schneider P., Springel V., 2015, MNRAS, 447, 3189

Zhao D. H., Mo H. J., Jing Y. P., Börner G., 2003, MNRAS, 339. 12

de Blok W. J. G., Walter F., Brinks E., Trachternach C., Oh S. H., Kennicutt R. C. J., 2008, AJ, 136, 2648 van den Bosch F. C., Ogiya G., Hahn O., Burkert A., 2018, MNRAS, 474, 3043

APPENDIX A: CONVERGENCE OF THE POSTERIOR DISTRIBUTIONS

The approximation of the true posterior obtained in Approximate Bayesian Computing (ABC) algorithms converges to the true posterior distribution as the acceptance criterion becomes increasingly more stringent. In our framework, changing the acceptance criterion is equivalent to reducing the number of forward model samples while keeping the number of total accepted realizations fixed. We exploit this property to test for convergence of the posteriors.

In Figure A1, we compare the posterior constructed from the full set of forward model samples (the same as Figure 9) with a second posterior derived from a depleted set of forward model samples, where we have discarded half of the realizations and accepted the same rejection criterion (accept the realizations corresponding to the 800 lowest values of $S_{\rm lens}$) to those that remain. The mass of the posterior distributions remains relatively unchanged, and the 1σ and 2σ contours are nearly identical. We conclude we have generated enough realizations of dark matter structure to reliably construct posterior distributions using the ABC rejection algorithm described in Section 2.

APPENDIX B: OBTAINING DEFLECTOR REDSHIFTS

The quads J1606 and WGD J0405 do not have measured spectroscopic redshifts, so we use photometry from Shajib et al. (2019) to obtain photometric redshift estimates. The photometry from Shajib et al. (2019) comes in three bands: F160W, F814W, and F475X with magnitude uncertainties of 0.1–0.3 dex. We use the software package eazy (Brammer et al. 2008), and restrict the templates to only consider the SEDs for early-type galaxies, which are 90% of galaxies acting as strong lenses. We verify this procedure is accurate by applying it to other deflectors in sample analyzed by (Shajib et al. 2019) that have measured spectroscopic redshifts, and then proceed to derive PDFs for deflector redshifts in the systems J1606 and WGD J0405.

The results are shown in Figure B1. The top row shows four quads from the sample analyzed in Shajib et al. (2019) with measured spectroscopic redshifts, and the bottom row shows the pdfs output by eazy for the systems J1606 and WGD J0405.

The system WFI 2026 does not have a photometric redshift, and the photometry available in the literature comes in only one or two bands with larger uncertainties. For this system, we use the equation for isothermal mass profiles relating the Einstein radius $R_{\rm Ein}$, source redshift z_s , lens redshift z_d , velocity dispersion σ and speed of light c

$$R_{\rm Ein} = 4\pi \left(\frac{\sigma}{c}\right)^2 \frac{D_{\rm ds}\left(z_d, z_s\right)}{D_{\rm s}\left(z_s\right)} \tag{B1}$$

where $D_{\rm ds}$ and $D_{\rm s}$ are angular diameter distances between the lens and the source, and the observer and the source, respectively.

We sample a Gaussian distribution of velocity dispersions typical of early-type galaxies $240 \pm 30 \mathrm{kms}^{-1}$, evaluate the right hand side of Equation B1, and numerically solve for the lens redshift that yields the resulting angular diameter distance. The resulting PDF showin the bottom right panel of Figure B2 peaks around $z_d=1$, for the measured values $R_{\rm Ein}=0.67$ ", $z_s=2.2$. We have experimented with placing WFI 2026 at various specific redshifts, but find the posteriors for $\Sigma_{\rm sub}$, $\delta_{\rm los}$, α , and $m_{\rm hm}$ are unchanged within the uncertainties.

APPENDIX C: DATA

We summarize the data used in this analysis, and the references for the astrometry, fluxes or flux ratios, and the corresponding uncertainties, and satellite galaxies or nearby nearby deflectors in Figure C1.

Table C1. The data used in this analysis. Letters A-D correspond to the lensed images, while G is the galaxy light centroid. The centroid for MG0414 was measured by Falco et al. (1997) to lie (+0.9, +0.32), relative to image D, but the astrometry differs at the 10 m.a.s. level from the more precise measurements from Katz et al. (1997), so we inflate the uncertainty on the deflector centroid to 100 m.a.s. The priors sampled for the satellite galaxies or nearby deflectors are quoted in Table 2. Discovery papers are marked with a † .

Lens	Image	dRA	dDec	NL flux	Radio flux ratio
B0128+437	A	0 ± 0.003	0 ± 0.003	-	$\equiv 1$
†Phillips et al. (2000)	В	0.098 ± 0.003	0.094 ± 0.003	-	0.584 ± 0.029
Koopmans et al. (2003)	$^{\mathrm{C}}$	0.520 ± 0.003	-0.172 ± 0.003	-	0.520 ± 0.029
	D	0.108 ± 0.003	-0.250 ± 0.003	-	0.506 ± 0.029
	G	0.205 ± 0.05	-0.107 ± 0.05	-	-
WGD J0405-3308	A	1.066 ± 0.003	0.323 ± 0.003	1.00 ± 0.04	-
?	В	0 ± 0.003	0 ± 0.003	0.65 ± 0.04	-
[†] Anguita et al. (2018)	$^{\mathrm{C}}$	0.721 ± 0.003	1.159 ± 0.003	1.25 ± 0.03	-
	D	-0.157 ± 0.003	1.021 ± 0.003	1.17 ± 0.04	-
	G	0.358 ± 0.05	0.567 ± 0.05	-	-
MG0414+0534	A	0.588 ± 0.003	-1.934 ± 0.003	-	$\equiv 1$
Katz et al. (1997)	В	0.721 ± 0.003	-1.530 ± 0.003	-	0.90 ± 0.09
Ros et al. (2000)	\mathbf{C}	0 ± 0.003	0 ± 0.003	-	0.39 ± 0.039
Falco et al. (1997)	D	-1.361 ± 0.003	-1.635 ± 0.003	-	0.15 ± 0.015
†Hewitt et al. (1992)	G	-0.460 ± 0.1	-1.33 ± 0.1	-	-
HE0435-1223	A	2.424 ± 0.008	0.792 ± 0.008	0.97 ± 0.05	-
Nierenberg et al. (2017)	В	1.458 ± 0.008	-0.456 ± 0.008	0.98 ± 0.049	-
Wong et al. (2017)	$^{\mathrm{C}}$	0 ± 0.008	0 ± 0.008	1 ± 0.048	-
[†] Wisotzki et al. (2002)	D	0.768 ± 0.008	1.662 ± 0.008	0.54 ± 0.056	-
	G	1.152 ± 0.05	0.636 ± 0.05	-	
RX J0911+0551	A	0 ± 0.003	0 ± 0.003	0.56 ± 0.04	-
?	В	0.258 ± 0.003	0.405 ± 0.003	1.00 ± 0.05	-
Blackburne et al. (2011)	$^{\rm C}$	-0.016 ± 0.003	0.959 ± 0.003	0.53 ± 0.04	-
†Bade et al. (1997)	D	-2.971 ± 0.003	0.791 ± 0.003	0.24 ± 0.04	-
	G	-0.688 ± 0.05	0.517 ± 0.05	-	-
PG1115+080	A	1.328 ± 0.003	-2.034 ± 0.003	-	$\equiv 1$
Chiba et al. (2005)	В	1.477 ± 0.004	-1.576 ± 0.004	-	0.93 ± 0.093
Morgan et al. (2008)	С	-0.341 ± 0.003	-1.961 ± 0.003	-	0.16 ± 0.016
[†] Weymann et al. (1980)	D G	0 ± 0.003	0 ± 0.003	-	0.21 ± 0.021
	G	0.381 ± 0.05	1.344 ± 0.05	-	_
B1422+231	A	0.387 ± 0.005	0.315 ± 0.005	0.88 ± 0.01	-
Nierenberg et al. (2014)	В	0 ± 0.005	0 ± 0.005	1.00 ± 0.01	-
†Patnaik et al. (1992)	C	-0.362 ± 0.005	-0.728 ± 0.005	0.474 ± 0.006	-
	D	0.941 ± 0.01	-0.797 ± 0.01	-	-
	G	0.734 ± 0.01	-0.649 ± 0.01	-	-
PS J1606-2333	A	1.622 ± 0.003	0.589 ± 0.003	1.00 ± 0.03	-
?	В	0 ± 0.003	0 ± 0.003	1.00 ± 0.03	-
Shajib et al. (2019)	С	0.832 ± 0.003	-0.316 ± 0.003	0.59 ± 0.02	-
†Lemon et al. (2018)	D	0.495 ± 0.003	0.739 ± 0.003	0.79 ± 0.02	-
	G	0.784 ± 0.05	0.211 ± 0.05	-	
WFI 2026-4536	A	0.164 ± 0.003	-1.428 ± 0.003	1.00 ± 0.02	-
?	В	0.417 ± 0.003	-1.213 ± 0.003	0.75 ± 0.02	-
†Morgan et al. (2004)	С	0 ± 0.003	0 ± 0.003	0.31 ± 0.02	-
	D	-0.571 ± 0.003	-1.044 ± 0.003	0.28 ± 0.01	-
	G	-0.023 ± 0.05	-0.865 ± 0.05	-	-
WFI 2033-4723	A	-2.196 ± 0.003	1.260 ± 0.003	1.00 ± 0.03	-
? V.: (2000)	В	-1.484 ± 0.003	1.375 ± 0.003	0.65 ± 0.03	-
Vuissoz et al. (2008)	С	0 ± 0.003	0 ± 0.003	0.50 ± 0.02	-
†Morgan et al. (2004)	D G	-2.113 ± 0.003 -1.445 ± 0.05	-0.278 ± 0.003 2.344 ± 0.05	0.53 ± 0.02	-
	A	-2.306 ± 0.003	1.708 ± 0.003	1.00 ± 0.01 1.16 ± 0.02	-
	P	0 ± 0.009			
?	В	0 ± 0.003	0 ± 0.003		-
WGD 2038-4008 ? †Agnello et al. (2018)	B C D	0 ± 0.003 -1.518 ± 0.003 -0.126 ± 0.003	0.029 ± 0.003 2.089 ± 0.003	0.92 ± 0.02 0.46 ± 0.01	-

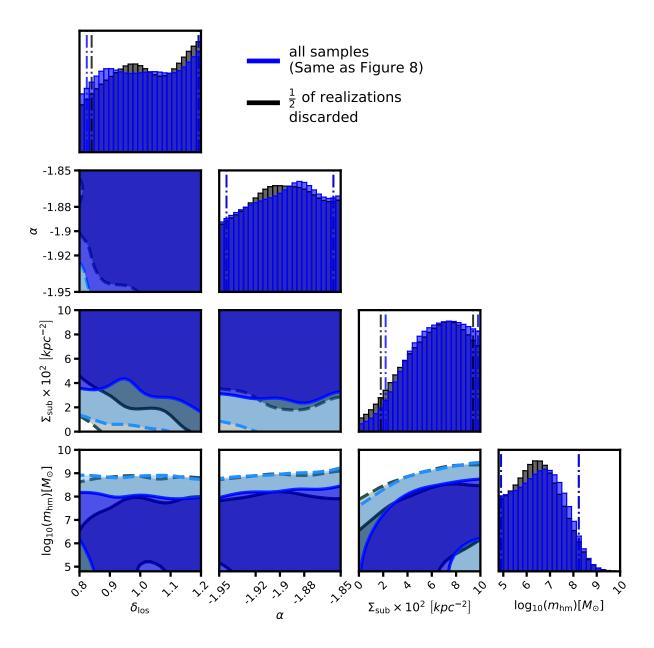


Figure A1. A convergence test of the posterior distributions. By discarding half of the forward model samples and applying the same rejection criterion to those that remain, we verify the inference obtained through the ABC rejection algorithm is robust.

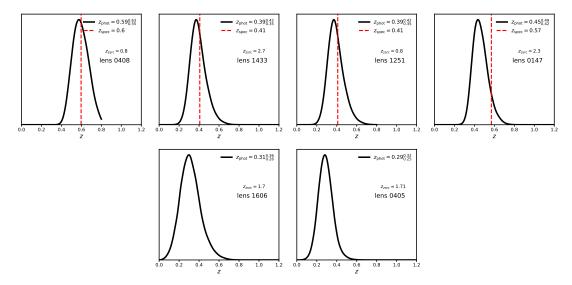


Figure B1. PDFs for main deflector redshifts computed with the software eazy and photometry from Shajib et al. (2019), restricting the photometry templates to those of early-type galaxies. The top shows four applications of this procedure to quads with measured spectroscopic redshifts (red dotted lines). The bottom row shows the results of this procedure, using the same photometry and template assumptions, applied to the quads J1606 and WGD J0405, which do not have spectroscopic redshift measurements.

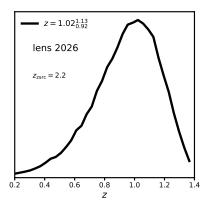


Figure B2. The PDF for the deflector redshift of WFI 2026 obtained by assuming a velocity dispersion of $240\pm30~\rm km~s^{-1}$ and a roughly isothermal mass profile.