

Note to readers:
Please ignore these
sidenotes; they're just
hints to myself for
preparing the index,
and they're often flaky!

KNUTH

THE ART OF COMPUTER PROGRAMMING

VOLUME 4 PRE-FASCICLE 8A

HAMILTONIAN PATHS AND CYCLES

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ADDISON-WESLEY



February 8, 2023

Internet page <http://www-cs-faculty.stanford.edu/~knuth/taocp.html> contains current information about this book and related books.

See also <http://www-cs-faculty.stanford.edu/~knuth/sgb.html> for information about *The Stanford GraphBase*, including downloadable software for dealing with the graphs used in many of the examples in Chapter 7.

See also <http://www-cs-faculty.stanford.edu/~knuth/mmixmap.html> for downloadable software to simulate the MMIX computer.

See also <http://www-cs-faculty.stanford.edu/~knuth/programs.html> for various experimental programs that I wrote while writing this material (and some data files).

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Zeroth printing (revision -78), 08 Feb 2023

February 8, 2023

PREFACE

*But that is not my point.
I have no point.*

— DAVE BARRY (2002)

THIS BOOKLET contains draft material that I'm circulating to experts in the field, in hopes that they can help remove its most egregious errors before too many other people see it. I am also, however, posting it on the Internet for courageous and/or random readers who don't mind the risk of reading a few pages that have not yet reached a very mature state. *Beware:* This material has not yet been proofread as thoroughly as the manuscripts of Volumes 1, 2, 3, and 4A were at the time of their first printings. And alas, those carefully-checked volumes were subsequently found to contain thousands of mistakes.

Given this caveat, I hope that my errors this time will not be so numerous and/or obtrusive that you will be discouraged from reading the material carefully. I did try to make the text both interesting and authoritative, as far as it goes. But the field is vast; I cannot hope to have surrounded it enough to corral it completely. So I beg you to let me know about any deficiencies that you discover.

To put the material in context, this portion of fascicle 8 previews Section 7.2.2.4 of *The Art of Computer Programming*, entitled “Hamiltonian paths and cycles.” I haven't had time to write much of it yet — not even this preface!

* * *

The explosion of research in combinatorial algorithms since the 1970s has meant that I cannot hope to be aware of all the important ideas in this field. I've tried my best to get the story right, yet I fear that in many respects I'm woefully ignorant. So I beg expert readers to steer me in appropriate directions.

Please look, for example, at the exercises that I've classed as research problems (rated with difficulty level 46 or higher), namely exercises ...; I've also implicitly mentioned or posed additional unsolved questions in the answers to exercises 65, Are those problems still open? Please inform me if you know of a solution to any of these intriguing questions. And of course if no solution is known today but you do make progress on any of them in the future, I hope you'll let me know.

I urgently need your help also with respect to some exercises that I made up as I was preparing this material. I certainly don't like to receive credit for

things that have already been published by others, and most of these results are quite natural “fruits” that were just waiting to be “plucked.” Therefore please tell me if you know who deserves to be credited, with respect to the ideas found in exercises 11, 12, 36, 37, 41, 42, 53, 55, 62, 63, 65, 71, 73, 100, 199, 200, 251, Furthermore I’ve credited exercises . . . to unpublished work of Have any of those results ever appeared in print, to your knowledge?

Knuth
KNUTH
Ewing

* * *

Special thanks are due to . . . for their detailed comments on my early attempts at exposition, as well as to numerous other correspondents who have contributed crucial corrections.

* * *

I happily offer a “finder’s fee” of \$2.56 for each error in this draft when it is first reported to me, whether that error be typographical, technical, or historical. The same reward holds for items that I forgot to put in the index. And valuable suggestions for improvements to the text are worth 32¢ each. (Furthermore, if you find a better solution to an exercise, I’ll actually do my best to give you immortal glory, by publishing your name in the eventual book:—)

Cross references to yet-unwritten material sometimes appear as ‘00’; this impossible value is a placeholder for the actual numbers to be supplied later.

Happy reading!

Stanford, California
99 Umbruary 2016

D. E. K.

*I have twenty years’ work ahead of me
to finish The Art of Computer Programming.*

— DONALD E. KNUTH, letter to John Ewing (04 September 1990)

A long train of consistent calculations opens itself out, for every result of which there is found a corresponding geometrical interpretation, in the theory of two of the celebrated solids of antiquity, alluded to with interest by Plato in the Timæus; namely, the Icosaedron, and the Dodecaedron.

— WILLIAM ROWAN HAMILTON (1856)

The total number of possible [knight's] tours that can be made is so vast that it is safe to predict that no mathematician will ever succeed in counting up the total.

— ERNEST BERGHOLT (1915)

I'll show that this problem is susceptible to a very special analysis, which merits extra attention because it involves reasoning of a kind rarely used elsewhere. The excellence of Analysis is easy to see, but most people think that it's limited to traditional questions about Mathematics; hence it will always be quite important to apply Analysis to subjects that seem to make it out of reach, for it incorporates the art of reasoning in the highest degree.

One cannot then extend the bounds of Analysis without justifiably expecting great advantages.

— LEONHARD EULER (1759)

Plato
HAMILTON
BERGHOLT
EULER
Hamilton
quaternions
Platonic solids
icosaedron
Icosian Game
planar graph
dual of a planar graph
dodecahedron

7.2.2.4. Hamiltonian paths and cycles. A path or cycle that touches every vertex of a graph is called “Hamiltonian” in honor of W. R. Hamilton, who began to ponder and publicize such questions shortly after discovering the quaternions. Hamilton was fascinated by Platonic solids such as the icosahedron, with its 20 triangular faces; and he introduced what he called the Icosian Game, based on paths that go from face to face in that solid. Equivalently (see Fig. 200), his game was based on paths from vertices to vertices along the edges of a dodecahedron.

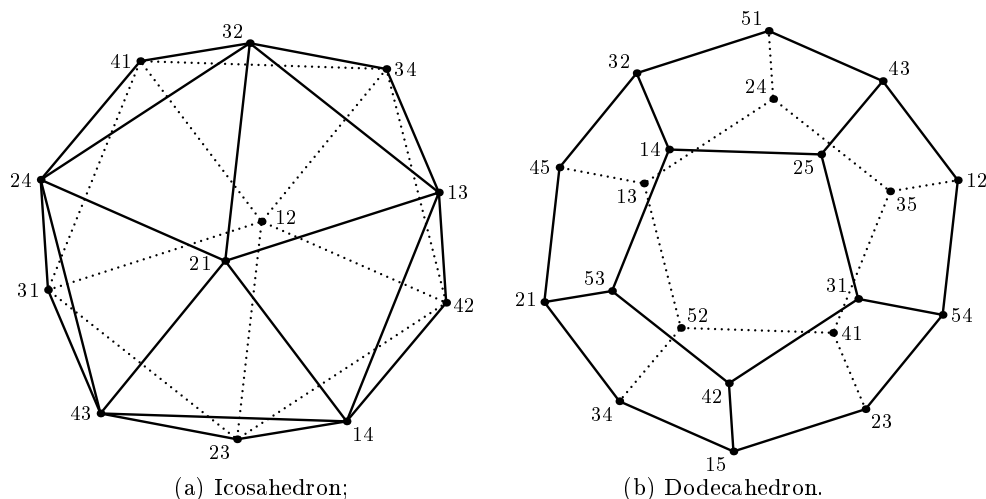


Fig. 200. The icosahedron and dodecahedron, whose vertices, edges, and faces define “dual” planar graphs: The faces of one solid correspond to the vertices of the other. (The vertices have been named with two-digit codes that are discussed in exercise 2.)

It's convenient to redraw Fig. 200(b) as three concentric rings, without crossing edges, as shown in Fig. 201(a). Then it's easy to find a Hamiltonian cycle, such as the one indicated by bold edges in Fig. 201(b). (In fact, Hamilton proved that *every* such cycle on the dodecahedron is essentially the same as this one; see exercise 8.) Thus we can also redraw the dodecahedron's graph as shown in Fig. 201(c). From that diagram it's *obviously* Hamiltonian—that is, it obviously has a spanning cycle; but it's not obviously planar at first glance.

concentric rings
Hamilton
spanning cycle
chords
3-regular graph
trivalent graphs
cubic graph
NP
Ham paths, history of—
Graeco-Roman icosahedra
Greek alphabet
Michon
Louvre
Perdrizet
British Museum
author

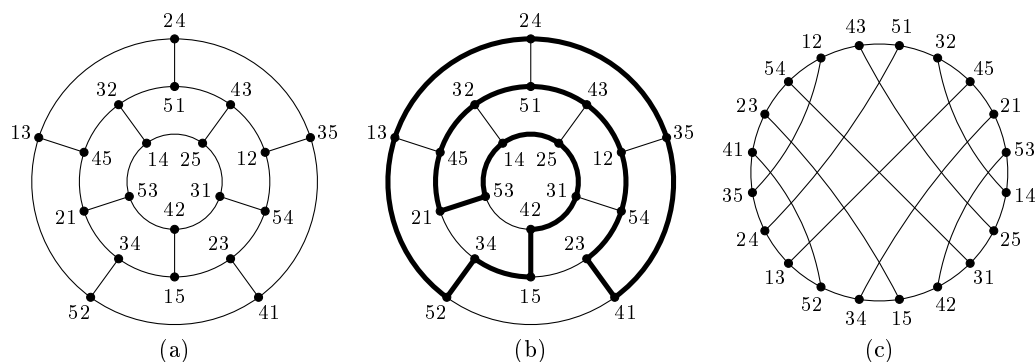


Fig. 201. Alternative views of a dodecahedron's vertices and edges.

Every Hamiltonian graph can clearly be drawn as a great big cycle, together with “chords” between certain pairs of vertices that aren't neighbors in the cycle. Thus a 3-regular graph can be specified compactly by listing only a third of its edges, if it is Hamiltonian. (On the other hand, many trivalent graphs are *not* Hamiltonian. In fact, the task of deciding whether or not a given cubic graph is Hamiltonian turns out to be NP-complete; see exercise 14.)

Hamiltonian paths in antiquity. Let's take a moment to discuss the rich history of the subject before we consider techniques by which Hamiltonian paths and cycles can be found. A strong case can actually be made for the assertion that questions of this kind represent the birth of graph theory, in the sense that they were the first nontrivial graph problems to be investigated.

For example, museums in many parts of the world contain specimens of ancient icosahedral objects whose 20 faces are inscribed with the first twenty letters of the Greek alphabet. In most of these cases the alphabetical sequence A, B, Γ, Δ, ..., T, Y on such artifacts forms a Hamiltonian path between adjacent triangles. [E. Michon, in *Bulletin de la Société nationale des Antiquaires de France* (1897), 310 and (1904), 327–329, described an example in the Louvre, catalog number I1532; P. Perdrizet, in *Bulletin de l'Institut français d'archéologie orientale* **30** (1930), 1–16, illustrated several others.]

In 2015, curators of the Egyptian antiquities at the British Museum kindly allowed the author to inspect the four icosahedra in their collection (EA 29418, EA 49738, EA 59731, EA 59732), of which the first three are Hamiltonian. The experience of rotating them by hand, slowly and systematically according to

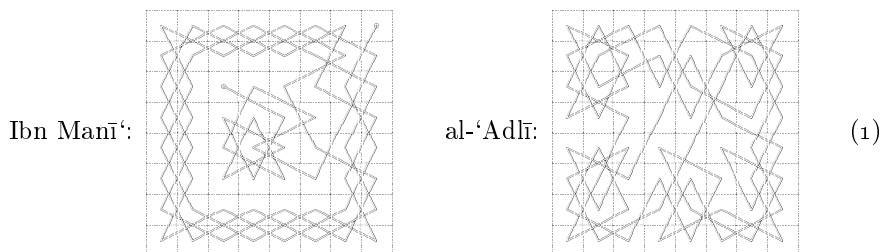
alphabetic order, turned out to be unexpectedly delightful. Here are views of the largest one, EA 49738, centered at each of its twelve vertices:



alphabetic order
 Pi, as written in Greece
 University College London
 Petrie
 coincidence
 Hamilton
 Chaturanga
 Shaṭranj
 knights
 al-‘Adlī ar-Rūmī
 Abū Zakarya Yaḥyā
 knight’s tour
 Ibn Manī’
 open versus closed tour
 closed versus open tour
 Murray

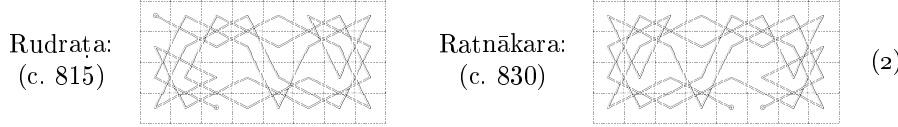
It is made of steatite, 5.8 centimeters in diameter and 228 grams in weight, and was acquired in 1911. A similar example, smaller and with more beautiful letterforms, is object number UC 59254 in the nearby Petrie Museum of University College London [see W. M. F. Petrie, *Objects of Daily Use* (1927), #288]. What a pleasant coincidence that W. R. Hamilton himself would independently come up with the same concept some 1800 years later, and would proceed to find a closed cycle instead of just a path!

Now fast forward to the ninth century, when Hamiltonian paths and cycles of quite a different kind came into play. The game of Chaturanga or Shaṭranj—a predecessor of chess, having different rules for certain pieces, but with knights moving just as they do today—was becoming popular in Asia. And in A.D. 842 the current world champion, al-‘Adlī ar-Rūmī, published a book about Shaṭranj. Complete copies of that work are lost; but we know from a subsequent treatise by Abū Zakarya Yaḥyā ibn Ibrāhīm al-Ḥakīm that al-‘Adlī had presented a closed *knight’s tour*: a Hamiltonian cycle on the chessboard. That same treatise also recorded an “open” knight’s tour (a Hamiltonian path that can’t be completed to a cycle), which al-Ḥakīm credited to an otherwise unknown author Ibn Manī’.



[See H. J. R. Murray, *History of Chess* (Oxford: 1913), 175–176, 336.] These remarkable constructions are the earliest known solutions to what was destined to become a classic combinatorial problem. It seems likely that the first path was discovered before the first cycle, because there are so many more of the former.

Remarkably, knight's tours on *half* of a chessboard, 4×8 , had been published even earlier, by Kashmiri poets who were famous for their wordsmithing skills:



Rudraṭa
Ratnākara
slokas
fractured English

Two copies of Rudraṭa's half-tour will make an open tour on the full board. And two copies of Ratnākara's will make a closed tour, if we rotate one copy by 180° .

Sanskrit poems traditionally consisted of verses called *slokas*, containing 32 syllables each. Here is sloka number 15 in chapter 5 of Rudraṭa's *Kāvyaṭaṅkāra*:

सेना लीलीलीना नाली लीनाना नानालीलीली । *senā līlīlīnā nālī līnānā nānālīlīlī*
नालीनालीले नालीना लीलीली नानानानाली ॥१५॥ *nālīnālīlē nālīnā līlīlī nānānālī* [15]

This enigmatic text, which speaks of military leadership, sounds almost like gibberish. But it cleverly represents a knight's tour, in the same way that his sloka 14 had represented a rook's tour: *When we read those 32 syllables in order of the left tour in (2), we get exactly the same words!*

More precisely, consider the following two 4×8 arrays of syllables σ_j :

$$\begin{array}{cccccccc}
 \sigma_1 & \sigma_{30} & \sigma_9 & \sigma_{20} & \sigma_3 & \sigma_{24} & \sigma_{11} & \sigma_{26} & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 \\
 \sigma_{16} & \sigma_{19} & \sigma_2 & \sigma_{29} & \sigma_{10} & \sigma_{27} & \sigma_4 & \sigma_{23} & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} \\
 \sigma_{31} & \sigma_8 & \sigma_{17} & \sigma_{14} & \sigma_{21} & \sigma_6 & \sigma_{25} & \sigma_{12} & \sigma_{17} & \sigma_{18} & \sigma_{19} & \sigma_{20} & \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\
 \sigma_{18} & \sigma_{15} & \sigma_{32} & \sigma_7 & \sigma_{28} & \sigma_{13} & \sigma_{22} & \sigma_5 & \sigma_{25} & \sigma_{26} & \sigma_{27} & \sigma_{28} & \sigma_{29} & \sigma_{30} & \sigma_{31} & \sigma_{32}
 \end{array} \quad (3)$$

The subscripts on the left correspond to the first sequence of knight moves in (2), while the subscripts on the right have their natural order. Rudraṭa composed a verse with the amazing property that both arrays agree (with $\sigma_1 = \sigma_1$, $\sigma_{30} = \sigma_2$, $\sigma_9 = \sigma_3$, \dots , $\sigma_5 = \sigma_{32}$), by choosing $\sigma_1 = \text{से}$, $\sigma_2 = \text{ना}$, $\sigma_3 = \sigma_4 = \sigma_5 = \text{ली}$, etc.

Notice that the constraints forced him to use at most four different symbols, thereby throwing away most of the tour's structure. It turns out, in fact, that there are *two* knight's tours consistent with his sloka. Therefore nobody knows whether he was thinking of the tour in (2) and (3) or the tour in exercise 36(i).

Thousands of 4×8 knight tours are possible, and if Rudraṭa had known more of them he could have written a much less ambiguous sloka that had twelve distinct syllables. For example, a "fractured English" verse that describes such a tour might go like this (see exercise 36(ii)):

Want a good, good time, lots of fun?
 Now not time so good; now not time.
 Foo. Ah, so! So now fun is lost.
 Time not now good, so time not now.

(4)

Ratnākara came up with a better idea a few years later. For his tour, illustrated at the right of (2), he composed two *different* slokas, both of which made sense as part of his overall poem. Their syllable patterns

$$\begin{array}{cccccccc}
 \sigma_{26} & \sigma_{11} & \sigma_{24} & \sigma_5 & \sigma_{20} & \sigma_9 & \sigma_{30} & \sigma_7 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 \\
 \sigma_{23} & \sigma_4 & \sigma_{27} & \sigma_{10} & \sigma_{29} & \sigma_6 & \sigma_{19} & \sigma_{16} & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} \\
 \sigma_{12} & \sigma_{25} & \sigma_2 & \sigma_{21} & \sigma_{14} & \sigma_{17} & \sigma_8 & \sigma_{31} & \sigma_{17} & \sigma_{18} & \sigma_{19} & \sigma_{20} & \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\
 \sigma_3 & \sigma_{22} & \sigma_{13} & \sigma_{28} & \sigma_1 & \sigma_{32} & \sigma_{15} & \sigma_{18} & \sigma_{25} & \sigma_{26} & \sigma_{27} & \sigma_{28} & \sigma_{29} & \sigma_{30} & \sigma_{31} & \sigma_{32}
 \end{array} \quad (5)$$

would have allowed him to define a tour quite precisely using 32 *distinct* syllables. (See his *Haravijaya*, Chapter 43, slokas 145 and 146.) For example, here's an English rendition of his two-sloka scheme:

Have some fun, watch this or that word —
Great four lines, take out, each gives eight.
Left; then two black; and just here white.
Three rook steps make one knight move, right?

One, two, three, four! Watch each word here;
Or take some left steps and move eight.
Just right gives this black rook great fun,
Then have lines make out that white knight.

poetic license
Rudraṭa
Ratnākara
Bhoja
Deśika
Somesvara III
nonsense verse
doggerel
al-Ḥakīm
abjad numerals

We obtain the second verse by reading the first verse in knight's tour order, starting at the fifth syllable of the fourth line. (Ratnākara actually used only 24 different syllables. Furthermore, his choices for σ_5 and σ_{32} did not agree in the two slokas; this may be due to errors in transmission of the ancient text, or to "poetic license." In any case his remarkable poem clearly defined a knight's tour.)

Such wordplay had many devotees in medieval India. For example, Rudraṭa's tour of (2) was rendered in Ratnākara's two-sloka style by King Bhoja in his *Sarasvatī-kaṇṭhābharaṇa* (c. 1050), slokas 2.306 and 2.308; also by Vedānta Deśika in his devotional hymn *Pādukāsahasra* (1313), slokas 929 and 930.

A simpler scheme, capable of encoding knight's tours on the full 8×8 board, was used in slokas 5.623–632 of the encyclopedic Sanskrit work *Mānasollāsa* by King Someshvara III (c. 1130). He named each square of the board systematically by combining a consonant for the column with a vowel for the row; then an arbitrary tour was a nonsense verse of 64 syllables, which could be memorized if you wanted to impress your friends. For example, in English we could use the names

$$\begin{bmatrix} \text{bah} & \text{bay} & \text{bee} & \text{boe} & \text{boo} & \text{buh} & \text{bai} & \text{bao} \\ \text{dah} & \text{day} & \text{dee} & \text{doe} & \text{doo} & \text{duh} & \text{dai} & \text{dao} \\ \text{fah} & \text{fay} & \text{fee} & \text{foe} & \text{foo} & \text{fuh} & \text{fai} & \text{fao} \\ \text{hah} & \text{hay} & \text{hee} & \text{hoe} & \text{hoo} & \text{huh} & \text{hai} & \text{hao} \\ \text{lah} & \text{lay} & \text{lee} & \text{loe} & \text{loo} & \text{luh} & \text{lai} & \text{lao} \\ \text{mah} & \text{may} & \text{mee} & \text{moe} & \text{moo} & \text{muh} & \text{mai} & \text{mao} \\ \text{nah} & \text{nay} & \text{nee} & \text{noe} & \text{noo} & \text{nuh} & \text{nai} & \text{nao} \\ \text{sah} & \text{say} & \text{see} & \text{soe} & \text{soo} & \text{suh} & \text{sai} & \text{sao} \end{bmatrix}$$

to encode Someshvara's tour as the following (memorable?) quatrain:

Sah nee soo nai lao fai bao duh, foe boo dee bah fay lah nay soe;
nuh sao mai hao dai huh doo bee, dah hay mah say noe suh nao lai.
Fao bai fuh dao buh doe bay fah, lay nah see noo sai mao hai foo?
Luh moe hee loo mee hoe moo lee, hoo fee boe day hah may loe muh.

Incidentally, al-Ḥakīm had presented the two knight's tours in (1) by first stating two 64-word poems in Arabic, then copying the words of those poems into 8×8 diagrams, according to the knight's paths. Then he repeated the right-hand tour, using the Arabic words "first," "second," ..., together with Persian-style abjad numerals, in place of the words of the corresponding poem.

[His work is preserved in a rare manuscript belonging to the John Rylands Library in Manchester: *Arabic MS. 766*, folio 39.] The latter convention, which corresponds to

60	11	56	7	54	3	42	1
57	8	59	62	31	64	53	4
12	61	10	55	6	41	2	43
9	58	13	32	63	30	5	52
34	17	36	23	40	27	44	29
37	14	33	20	47	22	51	26
18	35	16	39	24	49	28	45
15	38	19	48	21	46	25	50

(9)

John Rylands Library
Path diagrams
dalla Volpe
greedy
heuristic
greedy algorithms
Warnsdorf–

in decimal notation, has been used by many subsequent authors to characterize particular knight’s tours in an easy-to-understand way. Path diagrams such as (1) and (2), which provide complementary insights, weren’t invented until much later, when Lelio dalla Volpe published a short book *Corsa del Cavallo per tutt’i scacchi dello Scacchiere* (Bologna, 1766), containing nineteen examples.

A greedy heuristic. Early in the 1800s, the knight’s tour problem inspired an important new approach to combinatorial problems, based on making a sequence of locally optimum decisions. Such techniques, now known as “greedy algorithms,” were unheard-of at the time. But H. C. von Warnsdorf, a high court official in Hesse who had challenged himself by spending many nights trying to construct long paths of a knight, hit on a simple idea that worked like magic: *At each step, move to a place that has the fewest remaining exits.* This principle has become famous as “Warnsdorf’s rule.”

For example, suppose we want to construct an open knight’s tour on a 5×5 board, starting in a corner. Numbering the cells ij for $0 \leq i, j < 5$, we can assume by symmetry that the first two steps are 00 — 12. From cell 12 we can move the knight to either 04, 24, 33, 31, or 20, from which it could then exit in either 1, 3, 3, 3, or 3 ways; Warnsdorf’s rule tells us to choose 04, because $1 < 3$. (Indeed, this is our last chance to visit 04, unless the tour will end at that cell.) After 04 the knight must proceed to 23; and again we have five choices, namely 44, 42, 31, 11, or 02. The rule takes us to 44, then 32; then to 40, then 21; and we’ve completed a partial tour that looks like this:

1	<i>3</i>	<i>2</i>	<i>3</i>	3
<i>3</i>	<i>2</i>	2	<i>2</i>	<i>3</i>
<i>2</i>	8	<i>8</i>	4	<i>2</i>
<i>3</i>	<i>2</i>	6	<i>2</i>	<i>3</i>
7	<i>3</i>	<i>2</i>	<i>3</i>	5

(**Bold** numbers are the visited cells.

Italicized numbers tell how many exits remain from the unvisited cells.) (10)

Four cells are now candidates for step **9**, and they’re all currently marked ‘2’. So there’s a four-way tie. In such cases, von Warnsdorf explicitly said that it’s OK to choose arbitrarily, among all cells that have the fewest exits. Let us therefore proceed boldly to cell 33 (between **4**, **5**, and **6**). That makes a two-way tie; and we might as well go next to 41 (just to the right of **7**). From here we *don’t* want to go to the middle square, which has just dropped from 8 to 7, because our

other choice is a *1*. And now it's plain sailing, as von Warnsdorf leads us on a merry chase—ending gloriously with move **25** in the center cell 22.

It's easy to implement Warnsdorf's rule, by representing the given graph in SGB format. (The reader should be familiar with this format; see, for example, Algorithm 7B and the remarks that precede it.) The node for each vertex v in Algorithm W below extends the basic format by including two utility fields, $\text{DEG}(v)$ and $\text{TAG}(v)$, which correspond to the *italic* and **bold** numbers in (10).

Algorithm W allows the user to specify “target” vertices t_1, \dots, t_r , which are to be visited only when no other vertices are available. A similar mechanism was, in fact, used by von Warnsdorf himself, in the advanced examples of his original booklet that introduced the idea [*Des Rösselsprunges einfachste und allgemeinste Lösung* (Schmalkalden, 1823); see also *Schachzeitung* **13** (1858), 489–492].

Algorithm W (*Warnsdorf's rule*). Given a graph g , a source vertex s , and optional target vertices t_1, \dots, t_r , this algorithm applies Warnsdorf's rule to find a (hopefully Hamiltonian) path v_1, v_2, \dots , that begins with s . Let $n = N(g)$ be the number of vertices of g ; let $v_0 = \text{VERTICES}(g)$ be g 's initial vertex in memory.

- W1.** [Initialize.] For $0 \leq k < n$ and $v \leftarrow v_0 + k$, do the following: Set $d \leftarrow 0$, $a \leftarrow \text{ARCS}(v)$; while $a \neq \Lambda$, set $d \leftarrow d + 1$ and $a \leftarrow \text{NEXT}(a)$; then set $\text{DEG}(v) \leftarrow d$ and $\text{TAG}(v) \leftarrow 0$. (Thus $\text{DEG}(v)$ is the degree of v .) Finally set $k \leftarrow 0$, $v \leftarrow s$, and $\text{DEG}(t_i) \leftarrow \text{DEG}(t_i) + n$ for $1 \leq i \leq r$.
- W2.** [Visit v .] Set $k \leftarrow k + 1$, $v_k \leftarrow v$, $\text{TAG}(v) \leftarrow k$, $a \leftarrow \text{ARCS}(v)$, and $\theta \leftarrow 2n$.
- W3.** [All arcs tested?] If $a = \Lambda$, go to W7. Otherwise set $u \leftarrow \text{TIP}(a)$, and go to W6 if $\text{TAG}(u) \neq 0$. (Vertex u is a neighbor of v_k and a candidate for v_{k+1} .)
- W4.** [Decrease $\text{DEG}(u)$.] Set $t \leftarrow \text{DEG}(u) - 1$ and $\text{DEG}(u) \leftarrow t$.
- W5.** [Is $\text{DEG}(u)$ smallest?] If $t < \theta$, set $\theta \leftarrow t$ and $v \leftarrow u$.
- W6.** [Loop over arcs.] Set $a \leftarrow \text{NEXT}(a)$ and return to W3.
- W7.** [Done?] If $\theta = 2n$, terminate with path $v_1 \dots v_k$. Otherwise go to W2. ■

Notice that the candidates for v_{k+1} are precisely the vertices u whose DEG needs to change when v_k leaves the active graph. Therefore this algorithm runs in linear time: Every arc is examined at most twice, once in step W1 and once in step W3.

The path chosen by Algorithm W depends on the ordering of arcs that lead out of each vertex in SGB format, because Warnsdorf's rule makes an arbitrary decision in case of ties. A simple change to step W5 will randomize the path properly, as if all orderings of the arcs were equally likely (see exercise 53).

Now that we understand Warnsdorf's rule, let's talk a little bit about greed. Greed is of course one of the seven deadly sins; hence we might well question the morality of ever using a greedy algorithm in our own work. However, greed is actually a *virtue*, when it enhances the environment and harms nobody.

In what sense is Algorithm W greedy? From the standpoint of *short-term* greed, also known as “instant satisfaction,” the best choice for v_{k+1} would seem to be a vertex with *maximum* degree, not minimum, because that vertex will give us the most flexibility when choosing v_{k+2} . But from the standpoint of *long-term* greed, also known as “risk management” or maximizing our chance

SGB format
linear time
greed
seven deadly sins
morality
virtue

of success, it's best to choose a vertex with *minimum* degree, as von Warnsdorf stipulated; that choice leaves us with the most arcs remaining for moves in the future. Indeed, short-term greed turns out to be very bad (see exercise 59).

How good is Warnsdorf's rule? It works so well for knight moves that von Warnsdorf naïvely believed it to be infallible, except perhaps on $m \times n$ boards with $m < 6$ or $n < 6$. He even thought that he had a proof of guaranteed success. His booklet exhibited many examples: 6×6 , 6×7 , \dots , up to 10×10 . Experiments by C. F. de Jaenisch [*Traité des applications de l'analyse mathématique au jeu des échecs* **2** (1862), 59] showed in fact that, on an ordinary 8×8 chessboard, one can basically choose the first 40 moves at random, and obtain a complete knight's tour by applying Warnsdorf's rule only to the last 24 steps!

The rule can fail, however. On a 6×6 board, it gives a complete tour about 97.2% of the time, yet it sometimes stops after only 32 or 34 steps if the starting position is one of the eight interior diagonal squares. On the 8×8 board it succeeds even more often (about 97.9%). Yet with probability 0.0000038 it might stop with a path of length 39, as shown in the answer to exercise 59.

Hamilton's dodecahedron graph (Fig. 201) is quite different from a graph of knight moves, because it is 3-regular. A partial path in a 3-regular graph can be extended in at most two ways, after we've selected the first two points, while a knight can have up to seven choices at every step. (Furthermore, all starting edges of the dodecahedron are equivalent.) Nevertheless, Algorithm W handles that graph well: It finds a Hamiltonian path $v_1 v_2 \dots v_{20}$ with probability $\frac{31}{32} = .96875$. Furthermore, it finds a path with $v_{20} - v_1$ (hence a Hamiltonian cycle) with probability $\frac{15}{128} \approx .117$. That probability rises to $\frac{139}{256} \approx .543$ if we set t_1 to a neighbor of s ; it's exactly $1/2$ if we set $\{t_1, t_2, t_3\}$ to the *three* neighbors of s .

It's not difficult to see that Algorithm W always works perfectly when g is the graph of a rectangular grid and s is a corner vertex (see exercise 62). With a bit more thought, we can even prove that it always succeeds when g is an n -cube, thereby finding many examples of the generalized Gray binary codes that we studied in Section 7.2.1.1 (see exercise 63). When g is the SGB graph *perms* $(-4, 0, 0, 0, 0, 0, 0)$ —whose vertices are the permutations of $\{0, 1, 2, 3, 4\}$, related by swapping adjacent digits—Warnsdorf's rule finds “change ringing” paths of length $5! - 1 = 119$ about 29% of the time. (See Algorithm 7.2.1.2P. This probability drops to less than 2%, however, with permutations of 6 elements, and to near zero with permutations of 7.) Another instructive example is the SGB graph *binary* $(10, 0, 0)$, whose vertices are the 16796 binary trees with 10 nodes, related by “rotation.” Starting at the tree with all-null left links, Algorithm W finds a Hamiltonian path about 5.6% of the time. (See Algorithm 7.2.1.6L.)

Of course Algorithm W isn't a panacea. We can't expect any algorithm to solve the NP-complete Hamiltonian path problem in linear time! Warnsdorf's rule certainly has difficulty in critical cases; indeed, it can fail spectacularly even on small graphs (see exercise 65). But it's often a good first thing to try, when presented with a graph that we haven't seen before.

Ira Pohl [*CACM* **10** (1967), 446–449] has suggested breaking ties in Warnsdorf's rule by looking at the *sum of the degrees* of v_k 's neighbors.

de Jaenisch
Hamilton
dodecahedron graph
3-regular
 n -cube
Gray binary codes
perms
change ringing
binary
binary trees
rotation
NP-complete
Pohl

Path flipping. Long before Warnsdorf’s time, the great mathematician Leonhard Euler had already published a classic paper about knight’s tours [*Mémoires de l’académie des sciences de Berlin* **15** (1759), 310–337], in which he showed how to discover long paths by a completely different method. (Euler credited this idea, at least in part, to his friend Louis Bertrand.) Instead of Warnsdorf’s “greedy” algorithm, his approach might be called a “breedy” method, because it proceeded by simple mutations and adaptations of paths already known.

Suppose, for example, that we want to find a 3×10 knight’s tour, and that Warnsdorf has already told us how to reach 28 of the 30 cells:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 4 & 7 & 2 & 27 & 24 & 13 & 10 & 19 & a & 17 \\ \hline 1 & 28 & 5 & 14 & 9 & 22 & 25 & 16 & 11 & 20 \\ \hline 6 & 3 & 8 & 23 & 26 & 15 & 12 & 21 & 18 & b \\ \hline \end{array} . \quad (11)$$

We can’t go from position 28 to an unvisited cell; but we needn’t despair, because 28 is just one knight’s move away from cell 23. Similarly, cell 1 is adjacent to 8. Therefore we can immediately deduce that two more equally long paths exist:

$$1 \dots 23, 28 \dots 24; \quad 7 \dots 1, 8 \dots 28. \quad (12)$$

(Here ‘ $x \dots y$ ’ stands for the path from x to y that proceeds by unit steps ± 1 .) Operating in the same fashion on the first of these yields three more,

$$1 \dots 5, 24 \dots 28, 23 \dots 6; \quad 1 \dots 15, 24 \dots 28, 23 \dots 16; \quad 7 \dots 1, 8 \dots 23, 28 \dots 24. \quad (13)$$

And, aha, one of these can be extended to a full tour $1 \dots 15, 24 \dots 28, 23 \dots 16, b, a$:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 4 & 7 & 2 & 19 & 16 & 13 & 10 & 25 & 30 & 27 \\ \hline 1 & 20 & 5 & 14 & 9 & 22 & 17 & 28 & 11 & 24 \\ \hline 6 & 3 & 8 & 21 & 18 & 15 & 12 & 23 & 26 & 29 \\ \hline \end{array} . \quad (14)$$

Now the same subpath-flipping technique leads from (14) to additional tours

$$1 \dots 17, 30 \dots 18; \quad 1 \dots 23, 30 \dots 24; \quad 7 \dots 1, 8 \dots 30; \quad (15)$$

and we can continue to find tours galore:

$$\begin{aligned} &1 \dots 13, 18 \dots 30, 17 \dots 14; \quad 1 \dots 5, 18 \dots 30, 17 \dots 6; \quad 7 \dots 1, 8 \dots 17, 30 \dots 18; \\ &7 \dots 1, 8 \dots 23, 30 \dots 24; \quad 13 \dots 8, 1 \dots 7, 14 \dots 30; \quad 1 \dots 7, 14 \dots 17, 30 \dots 18, 13 \dots 8; \end{aligned}$$

etc. Indeed, the latter is a Hamiltonian *cycle* — a *closed* tour — because 1 is adjacent to 8! A Hamiltonian cycle represents 30 different Hamiltonian *paths*, each of which leads to further flips, hence further paths and cycles.

If we start with (14) and keep flipping until no new paths arise, it turns out that we will have discovered all 16 of the Hamiltonian cycles of the 3×10 knight graph, as well as 2472 of its 2568 noncyclic Hamiltonian paths.

One of the 96 noncyclic Hamiltonian paths *not* derivable from (14) is

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 12 & 3 & 22 & 15 & 10 & 7 & 26 & 29 & 18 \\ \hline 4 & 23 & 14 & 11 & 6 & 25 & 20 & 17 & 8 & 27 \\ \hline 13 & 2 & 5 & 24 & 21 & 16 & 9 & 28 & 19 & 30 \\ \hline \end{array} . \quad (16)$$

It leads via flips only to three others, namely $1 \dots 17, 30 \dots 18, 13 \dots 1, 14 \dots 30$, and $13 \dots 1, 14 \dots 17, 30 \dots 18$. So we *wouldn’t* have found a cycle, if starting with (16).

Let's formulate Euler's approach more precisely:

Algorithm F (*Long paths by flipping*). Given a simple path $v_1 \text{---} v_2 \text{---} \dots \text{---} v_t$ in a connected n -vertex graph, this algorithm repeatedly obtains new paths by reversing subpaths as explained above, until either exhausting all possibilities or finding a path that can be extended by a vertex $\notin \{v_1, v_2, \dots, v_t\}$. An auxiliary table of vertex labels $w[v]$ is used to discover potential flips.

Euler
breadth-first search
update
canonical form
breadth-first search

- F1.** [Initialize for breadth-first search.] Prepare a dictionary, initially empty, for storing paths of length t . Set $w[v] \leftarrow 0$ for each of the n vertices v of the graph. Set $q \leftarrow 0$ and perform *update*(v_1, \dots, v_t), where *update* is the subroutine defined below. Then set $d \leftarrow p \leftarrow p_1 \leftarrow p_2 \leftarrow 0$ and $p_0 \leftarrow q$.
- F2.** [Done with distance d ?] (At this point we've entered q paths into the dictionary, and we've explored the successors of the first p paths. Exactly p_i of those paths were obtained by making $\leq d - i$ flips, for $0 \leq i \leq 2$.) Go to F6 if $p = p_0$; otherwise set $p \leftarrow p + 1$.
- F3.** [Explore path p .] Let $u_1 \text{---} u_2 \text{---} \dots \text{---} u_t$ be the p th path that entered the dictionary, and set $w[u_k] \leftarrow k$ for $1 \leq k \leq t$. Go to F5 if $u_t \text{---} u_1$.
- F4.** [Process a noncyclic path.] For each vertex v such that $u_t \text{---} v$, do the following: Set $k \leftarrow w[v]$; terminate the algorithm if $k = 0$; otherwise call *update*($u_1, \dots, u_k, u_t, \dots, u_{k+1}$). Then, for each v such that $u_1 \text{---} v$, do the following: Set $k \leftarrow w[v]$; terminate if $k = 0$; otherwise *update*($u_{k-1}, \dots, u_1, u_k, \dots, u_t$). Then return to F2.
- F5.** [Process a cyclic path.] (A cyclic path will be in the dictionary only if $t = n$; see below.) For $1 \leq j \leq t$ and for each v such that $u_j \text{---} v$, do the following: Set $k \leftarrow w[v]$ (which will be positive). If $k < j$, call *update*($u_{j+1}, \dots, u_t, u_1, \dots, u_k, u_j, \dots, u_{k+1}$) and *update*($u_{k-1}, \dots, u_1, u_t, \dots, u_j, u_k, \dots, u_{j-1}$); otherwise *update*($u_{j+1}, \dots, u_k, u_j, \dots, u_1, u_t, \dots, u_{k+1}$) and *update*($u_{k-1}, \dots, u_j, u_k, \dots, u_t, u_1, \dots, u_{j-1}$). Then return to F2.
- F6.** [Advance d .] Terminate if $p = q$ (we have found all the reachable paths). Otherwise set $d \leftarrow d + 1$, $p_2 \leftarrow p_1$, $p_1 \leftarrow p_0$, $p_0 \leftarrow q$, and go back to F2. ■

Algorithm F relies on a subroutine '*update*(v_1, \dots, v_t)', whose purpose is to put the path $v_1 \text{---} \dots \text{---} v_t$ into the dictionary unless it's already there. First the path is converted to a canonical form, so that equivalent paths are entered only once: If $v_t \not\text{---} v_1$, the canonical form is obtained by changing $(v_1, \dots, v_t) \leftarrow (v_t, \dots, v_1)$ if $v_1 > v_t$. On the other hand if $v_t \text{---} v_1$, the path is cyclic, and we terminate the algorithm if $t < n$. (The graph is connected, so there must be a vertex outside the cycle that is adjacent to a vertex of the cycle.) Finally, if $t = n$ and $v_n \text{---} v_1$, we obtain the canonical form by permuting the cycle cyclically so that v_1 is the smallest element; then we set $(v_1, v_2, \dots, v_n) \leftarrow (v_1, v_n, \dots, v_2)$ if $v_2 > v_n$. Once (v_1, \dots, v_t) is in canonical form, the *update* routine looks for it in the dictionary. If unsuccessful, *update* sets $q \leftarrow q + 1$ and inserts it as the q th path.

The theory of breadth-first search tells us that (v_1, \dots, v_t) cannot match any path in the dictionary that was obtained with fewer than $d - 1$ flips. (Otherwise the path (u_1, \dots, u_t) that led to it would have been seen before making d flips.)

Therefore step F6 can save dictionary space and lookup time by deleting all paths of index $\leq p_2$ from the dictionary whenever p_2 increases. Exercise 73 discusses a simple trick that makes this deletion painless.

Algorithm F is amazingly versatile. For example, there are 9862 closed knight's tours on a 6×6 board, and 2963928 open tours. All of them will be found by Algorithm F, when given any single instance.

We began our search for a 3×10 knight's cycle by using the Warnsdorf-inspired path (11). But we could have started Algorithm F with $t = 1$, thus presenting it with only a single vertex v_1 . Every time the algorithm finds a larger path, we can simply restart it, with t increased.

For example, the author tried the 3×10 problem 100 times, choosing v_1 at random and ordering the vertex neighbors randomly in steps F4 and F5. A Hamiltonian cycle was found in 82 cases, usually after making fewer than 100 calls on *update*. A stubborn Hamiltonian path like (16) was found in 6 cases. And the remaining 12 cases failed to reach $t = 30$; once t was even stuck at 22.

Of course that's a very small problem. When presented with the graph of permutations of $\{0, 1, 2, 3, 4, 5\}$, Algorithm F was able to find a "change ringing" cycle of length 720 in each of ten random trials, averaging less than 50,000 updates per trial. On the other hand it did *not* do well when trying to find a closed 3×100 knight's tour.

Warnsdorf
permutations
change ringing



Who knows what I might eventually decide to say next? Please stay tuned.

EXERCISES

1. [15] We could save ourselves three syllables and three letters by saying “spanning cycle” and “spanning path” instead of “Hamiltonian cycle” and “Hamiltonian path.” Textbooks on graph theory could save lots of paper. Why doesn’t everybody do that?

2. [M22] Reverse-engineer the rules by which Fig. 200’s vertices have been named.

3. [M30] The Hamiltonian cycle in Fig. 201(b) doesn’t look symmetrical. Show, however, that it has fourfold symmetry when drawn on an undistorted dodecahedron.

4. [M20] A *second* glance at the graph depicted in Fig. 201(c) reveals that it actually is obviously planar. Why?

5. [22] Draw the graph of the icosahedron in the style of Fig. 201(a), arranging the vertices in three concentric rings.

6. [20] Draw the graph of the 4-cube in the style of Fig. 201(c), using Gray binary code as the Hamiltonian cycle.

7. [HM25] Show that it’s possible to redraw the graph of the dodecahedron, Fig. 201, in such a way that all lines between adjacent vertices have the same length.

► 8. [M21] A Hamiltonian cycle on a planar cubic graph, such as the dodecahedron in Fig. 200(b), can be described as a sequence of Ls and Rs denoting “left turn” and “right turn” at each vertex encountered during the cyclic journey.

- Prove that no Hamiltonian cycle on the dodecahedron can contain any of the following subsequences: (i) LLLL; (ii) LRRL; (iii) LRLRLRL; (iv) LLRLRLL; (v) LLRLRR; (vi) LLRLL; (vii)–(xii), subsequences (i)–(vi) with $L \leftrightarrow R$ swapped.
- Therefore there is essentially only one cycle (and its dual obtained by $L \leftrightarrow R$).

9. [24] For which vertices v of Fig. 201(a) is there a Hamiltonian path from 12 to v ?

11. [M32] The *generalized Petersen graph* $GP(n, k)$ is an interesting cubic graph with $2n$ vertices $\{0, 1, \dots, n-1, 0', 1', \dots, (n-1)'\}$ and $3n$ edges

$$\{i \text{ --- } (i+1) \bmod n, i \text{ --- } i', i' \text{ --- } (i+k)' \bmod n \mid 0 \leq i < n\}.$$

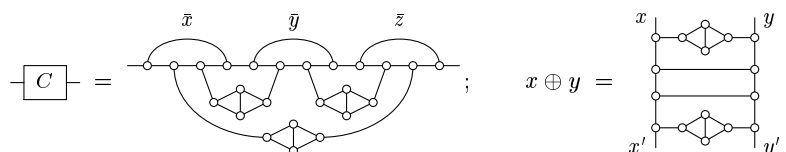
Figure 201(a) is the special case $q = 5$ of a general *concentric-ring graph* $GP(2q, 2)$.

For which vertices v does the graph $GP(2q, 2)$ have a Hamiltonian path from $0'$ to v ?

12. [HM28] How many Hamiltonian cycles exist in the graphs $GP(2q, 2)$?

14. [22] The one-in-three satisfiability problem of exercise 7.2.2.2–517 is NP-complete. For every such problem F , we shall construct a cubic graph G that is Hamiltonian if and only if F is satisfiable. Every edge of G corresponds to a Boolean variable; values of the variables for which the true edges form a Hamiltonian cycle will be called a *win*.

- A cubic graph that contains $K_{2,1,1} = \text{---} \circ \text{---} \circ \text{---} \circ \text{---}$ as an induced subgraph also contains the “Wheatstone bridge” $\text{---} \circ \text{---} \circ \text{---} \circ \text{---}$, which has two edges that connect to other vertices. Show that those connecting edges must be true in every win.
- For every clause $C = (x \vee y \vee z)$ of F , where x , y , and z are literals, include the “clause gadget” $\text{---} \boxed{C} \text{---}$ below as part of G . Show that $x + y + z = 1$ in every win.



spanning
Hamiltonian
fourfold symmetry
symmetry
planar
icosahedron
concentric rings
4-cube
Gray binary code
draw the graph
unit-distance graph
graph drawing
dodecahedron
Hamiltonian path
generalized Petersen graph
Petersen graph
 $GP(n, k)$
concentric-ring graph
enumeration of Ham cycles
one-in-three satisfiability problem
NP-complete
cubic graph
satisfiable
win
 $K_{2,1,1}$
induced subgraph
Wheatstone bridge
literals
clause gadget

- c) If two edges x and y of G are replaced by the “XOR gadget” $x \oplus y$ above, show that $x = x'$, $y = y'$ and $x = \bar{y}$ in every win.
- d) Suppose the clauses of F are $\{C_1, \dots, C_m\}$. Use the gadgets above to construct the desired graph G , starting with $\boxed{C_1} - \dots - \boxed{C_m}$.

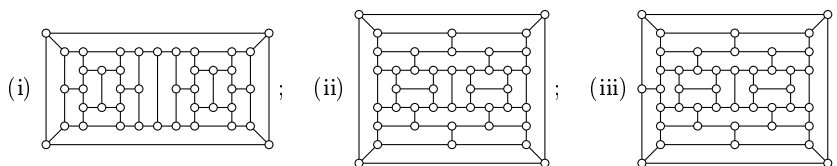
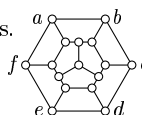
16. [29] What's the smallest connected cubic graph that is *not* Hamiltonian?

18. [M20] True or false: If a planar graph has a Hamiltonian cycle, so does its dual.

- 20. [M30] (T. P. Kirkman, 1855.) Let G be a planar graph with n vertices and with exactly α_k k -sided faces for $k \geq 3$ (including the unbounded exterior face). For example, the graph of the dodecahedron, Fig. 201, has $n = 20$ and $\alpha_k = 12$ [$k = 5$].

- a) If G is Hamiltonian, prove that integers a_k exist such that $0 \leq a_k \leq \alpha_k$ and $\sum_{k=3}^n (k-2)a_k = n-2$. (For example, the dodecahedron has $a_k = 6$ [$k = 5$].)
- b) In a similar way, prove that the dodecahedron has no cycle of length 19.
- c) Furthermore its vertices can't be completely covered by two *disjoint* cycles.
- d) Use (a) to prove that every Hamiltonian cycle in the planar 16-vertex cubic graph G shown here must include the edges $a-b$, $c-d$, $e-f$.
- e) Use (a) to decide whether any of the following graphs are Hamiltonian:

XOR gadget
gadgets
planar graph
dual
Kirkman
planar graph
faces
dodecahedron
cycle cover
benchmarks
perfectly Hamiltonian
edge coloring
cubic graphs
Tutte gadget
pentagonal prism
generalized Petersen graphs

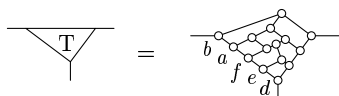


21. [M25] Large graphs that contain no Hamiltonian cycles can often be useful benchmarks. Construct infinitely many cubic planar graphs that fail to satisfy exercise 20(a).

24. [M28] A cubic graph is called *perfectly Hamiltonian* if its edges can be 3-colored in such a way that the edges of any two colors form a Hamiltonian cycle.

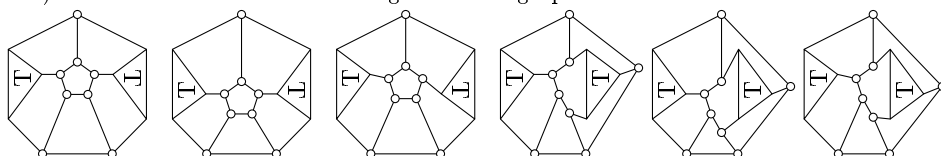
- a) Which of the cubic graphs on 8 vertices are Hamiltonian? Perfectly Hamiltonian?
- b) Prove that a *planar* cubic graph can be perfectly Hamiltonian only if there are nonnegative integers (a_k, b_k, c_k, d_k) for all $k \geq 3$ such that $a_k + b_k + c_k + d_k = \alpha_k$ is the number of k -faces as in exercise 20(a), and $\sum_k (k-2)a_k = \sum_k (k-2)b_k = \sum_k (k-2)c_k = \sum_k (k-2)d_k = (n-2)/2$, where n is the number of vertices.

27. [M21] The *Tutte gadget* is a useful 15-vertex graph fragment



that can be obtained by removing vertex c from the 16-vertex graph in exercise 20(d).

- a) Prove that every Hamiltonian path in a graph that contains the gadget must use the edge at the bottom of the T.
- b) Prove that no Hamiltonian path in the pentagonal prism $GP(5, 1)$, includes the edges of two nonconsecutive “spokes.”
- c) Therefore none of the following 38-vertex graphs are Hamiltonian:



d) Are any two of those six graphs isomorphic to each other?

30. [20] Each letter in a Græco-Roman icosahedron can be placed three ways within its triangular face, depending on the choice of “bottom edge” (except that Δ and O are symmetric). From this standpoint, the fact that Π and Y share the *same* bottom edge, in the text’s example from the British Museum, is a bit disconcerting.

Redesign that layout for the 21st century, so that (i) Roman letters A, B, \dots, T replace the Greek ones; (ii) the bottom edge of a letter’s successor is always the upper left or upper right edge of the current letter; and (iii) T is adjacent to A , completing a cycle.

- **33.** [M20] Suppose G is an n -vertex graph that has H Hamiltonian cycles and h Hamiltonian paths that aren’t cycles. (Thus, there are H sets of n edges whose union is a cycle, and h sets of $n - 1$ edges whose union is a path but not a cycle.) Let $G' = \{*\} \text{---} G$ be the $(n + 1)$ -vertex graph obtained from G by adjoining a new vertex that’s adjacent to all the others. How many Hamiltonian cycles does G' have?

35. [M25] A close look at (1) shows that al-‘Adlī’s closed tour is traced by a “thread” that weaves alternately over and under itself at each crossing, forming a “knot.”

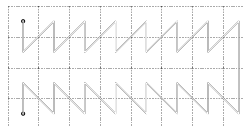
- Prove that *every* closed knight’s tour can be drawn as such a knot.
- On the other hand, the over-under rule is violated four times in Ibn Manī’s open tour. Prove that every drawing of his tour must necessarily have at least four such exceptions to the rule.

36. [22] Find 4×8 knight’s tours that (i) preserve the syllables of Rudraṭa’s sloka, but differ from (2); (ii) preserve the fractured English syllables of (4).

37. [22] How many 4×8 knight’s tours are possible?

38. [25] Write a two-verse English poem for Rudraṭa’s 4×8 tour, analogous to (6).

40. [25] The variant of Chaturanga played in Rudraṭa’s day used a curious piece called an *elephant* (gaja) instead of a chess bishop. This piece had only five moves: One step forward or one step diagonally, representing the elephant’s trunk and its four legs. For example, an elephant can tour a 4×8 board by following the path illustrated here.



Represent this half-tour with a two-verse poem in English.

- **41.** [M32] This exercise classifies all elephant’s tours on an $m \times n$ board, for $m, n \geq 2$.
- Let E_{mn} be the $m \times n$ elephant digraph. How many arcs does it have?
 - Does E_{mn} have a *closed* tour (a Hamiltonian *cycle*), for some values of m and n ?
 - The open elephant’s tour in exercise 40 begins at the bottom left corner of E_{48} . Show that there’s also an open tour that begins at the *top* left corner of E_{48} .
 - Prove that every elephant’s tour must begin or end in the top row, when $m > 2$.
 - Similarly, prove that every such tour must begin or end in the bottom row.
 - Characterize all $m \times n$ elephant’s tours that begin in the top row.
 - Characterize all $m \times n$ elephant’s tours that begin in the bottom row.
 - Explain how to compute the number of Hamiltonian paths of E_{mn} that begin at a given vertex s and end at a given vertex t .

42. [30] Find a cycle of elephant moves on the 8×8 chessboard that (i) visits all but two of the cells, and (ii) has the fewest “trunk moves” among all such cycles.

44. [18] Using the syllables (7), construct a knight’s tour quatrain that *rhymes*.

46. [19] Draw Someshvara’s tour (8) in the style of (1).

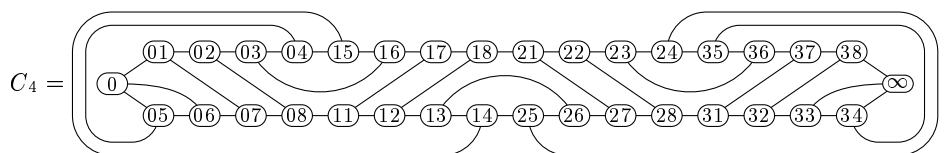
50. [19] The text describes only one scenario for moves **9**, **10**, \dots , that might extend the partial tour (10). What other paths are consistent with Warnsdorf’s rule?

isomorphic
Græco-Roman icosahedron
British Museum
Hamiltonian paths that aren’t cycles
al-‘Adlī
path diagrams
weaves
knot
alternating knot diagrams
Ibn Manī
Rudraṭa
sloka
fractured English
poem
Chaturanga
elephant
elephant’s tours
elephant digraph
trunk moves
quatrain
nonsense verse
Someshvara
path diagrams
Warnsdorf’s rule

51. [21] What paths does Algorithm W construct when g is the graph of knight moves on a 5×5 board, s is cell 00, $r = 1$, and t_1 is cell 44 (the corner opposite 00)?
52. [20] What is the behavior of Algorithm W if $t_i = t_j$ for some $i \neq j$?
- 53. [M21] *Randomize* Algorithm W, by changing step W5 so that each candidate u is chosen with probability $1/q$ when there's a q -way tie for the minimum number of exits. *Hint*: There's a nice way to do this "on the fly" without building a table of candidates.
55. [20] How many of the 63 moves in the historic knight's tour (1) by Ibn Manī agree with Warnsdorf's rule? Consider also the closed tour of al-'Adlī, with the same opening moves v_1 and v_2 , as well as the open tour of Someshvara in exercise 46.
56. [20] Algorithm W sometimes moves to a "dead end" vertex (from which there's no exit), even though it could prolong the path by making a different choice. Discuss.
- 57. [21] Design an algorithm to compute the tree of all possible paths that might be computed by Algorithm W, given g , s , and $\{t_1, \dots, t_r\}$. Also compute, for each path, the probability that it would be obtained by the algorithm of exercise 53.
59. [21] What are the longest and shortest paths obtainable in the 8×8 knight graph when the *anti-Warnsdorf* rule is used? (Move always to a cell with the *most* exits.) Compare those results to the behavior of Algorithm W.
60. [22] Study empirically the behavior of Algorithm W on the concentric-ring graphs $R_q = \text{GP}(2q, 2)$ of exercise 11, for $6 \leq q \leq 10$. What is the probability of obtaining (a) a Hamiltonian path? (b) a Hamiltonian cycle, when no target vertex is specified? (c) a Hamiltonian cycle, when there's a single target vertex with $s \text{ --- } t_1$?
62. [16] Prove that Algorithm W always finds a Hamiltonian path when $g = P_m \square P_n$ is the $m \times n$ grid graph, $s = (0, 0)$ is a corner vertex, and $r = 0$.
- 63. [M30] Prove that Algorithm W always finds a Hamiltonian path when $g = P_2 \square P_2 \square \dots \square P_2$ is an n -cube and $r = 0$.
- 65. [25] Is there a Hamiltonian graph for which Algorithm W always *fails* to find a Hamiltonian path, regardless of the starting point and the ordering of arcs?
70. [11] Show that step F4 sometimes calls ' $\text{update}(u_1, \dots, u_t)$ ', which does nothing.
71. [M20] Euler believed that his method for discovering tours was "safe" and "infallible"; but (16) is a case where it fails to find a cycle. Construct arbitrarily large Hamiltonian graphs for which Algorithm F can in fact get stuck with paths of length 10.
73. [21] Discuss implementing the dictionary of Algorithm F with a hash table based on linked lists. If the entry for each path links to the number of the previous path that belongs to the same list, step F6 can regard all links $\leq p_2$ as null.
75. [M23] (*One-sided flips*.) Simplify Algorithm F so that it flips subpaths only at the right, and doesn't distinguish between paths and cycles; call the resulting procedure "Algorithm F⁻." (More precisely, Algorithm F⁻ never goes to step F5; it omits the second *update* in step F4; and it doesn't put paths into canonical form.)
- Suppose Algorithm F⁻ is applied to a Hamiltonian path $v_1 \text{ --- } \dots \text{ --- } v_n$ in a *cubic* graph G . Show that it constructs a *cycle* of Hamiltonian paths, each beginning with v_1 , and illustrate this cycle when G is the 3-cube.
 - Furthermore the number of *cyclic* Hamiltonian paths in that cycle is even.
 - Moreover, the number of Hamiltonian cycles containing any given edge is even.
 - Every cubic Hamiltonian graph therefore has at least three Hamiltonian cycles.
 - Every cubic graph with exactly three Hamiltonian cycles is *perfectly* Hamiltonian.

Randomize
 Ibn Manī
 al-'Adlī
 Someshvara
 anti-Warnsdorf
 $m \times n$ knight graph: The SGB graph *board*(m)
 concentric-ring graphs
 generalized Petersen graphs
 grid graph
 n -cube
 $\text{update}(u_1, \dots, u_t)$
 Euler
 hash table
 linked lists
 null
 One-sided flips
 canonical form
 lollipop method, see one-sided flips
cubic
 3-cube
 perfectly

77. [M26] The *Cameron graph* C_n of order n is a planar cubic graph on the $8n + 2$ vertices $\{ij \mid 0 \leq i < n, 1 \leq j \leq 8\} \cup \{0, \infty\}$ defined by the relations $i7 - i1 - i2 - i3 - i4 - i5 - i6 - i7 - i8 - i2$, $i3 - (i+1)6$, $i4 - (i+1)5$, and $i8 - (i+1)1$ for all integers i ; replace all vertices ij for $i < 0$ by 0, and all ij for $i \geq n$ by ∞ . For example,



- Prove that the involution $ij \leftrightarrow (n-1-i)(9-j)$ is an automorphism of C_n .
- Prove that C_n has exactly three Hamiltonian cycles (one of which is the “obvious” cycle $\alpha_n = 0 - 01 - 02 - 03 - \dots - \infty - \dots - 07 - 06 - 05 - 0$).
- Compute the number c_n of one-sided flips needed to go from α_n to its mate β_n with respect to $0 - 01$, in the sense of answer 75(c), for $1 \leq n \leq 9$.
- Surprise! Exactly $c_{n-2} + 10$ flips go from α_n to β_n with respect to $01 - 0$.
- How many flips go from α_n to its mate γ_n with respect to (i) $0 - 05$? (ii) $05 - 0$?

78. [22] Study empirically the behavior of Algorithm F on the concentric-ring graphs $R_q = \text{GP}(2q, 2)$ of exercise 11, for $6 \leq q \leq 10$ and $q = 100$. Let $t = 1$, and choose v_1 at random; also randomize the order in which a vertex's neighbors are examined. Estimate the probability of obtaining (a) a Hamiltonian path; (b) a Hamiltonian cycle. How many nontrivial calls of *update* are typically needed, before succeeding?

79. [M32] (N. Beluhov.) Say that two Hamiltonian paths or cycles are *equivalent* if they can be transformed into each other by Algorithm F.

- Find a graph with two inequivalent cycles.
- Can a graph have arbitrarily many pairwise inequivalent cycles?

- **100.** [HM30] Analyze the cycle covers of the flower snark graph J_q , for $q > 2$ (see exercise 7.2.2.2–176). How many of them have exactly k cycles?

198. [20] Given a bipartite graph G with n vertices in each part, construct an exact cover problem with $3n$ primary items u^- , u^+ , v : two for each vertex u in the first part, and one for each vertex v in the second part. Let the options be ‘ $u^- v w^+$ ’, for all triples with $u - v - w$ and $u \neq w$.

- What do the solutions to this exact cover represent?
- Experiment with this construction when G is the 6×6 knight graph.

- **199.** [27] How many 8×8 closed knight's tours have the property that moves k and $32 + k$ occupy the same column, for $1 \leq k \leq 32$? *Hint:* Define an exact cover problem.

200. [24] Find a knight's tour whose step matrix has

$$a_{11} = 1, a_{16} = 16, a_{32} = 64, a_{52} = 32, a_{71} = 33, a_{83} = 34,$$

and such that $1 \leq a_{ij} \leq 18$ implies $a_{(9-i)(9-j)} = 50 - a_{ij}$. (The latter condition means that moves 1, 2, ..., 18 are rotated 180° from moves 49, 48, ..., 32.)

- **201.** [24] (G. E. Carpenter, 1881.) Find a knight's tour for which the top row of the step matrix is ‘1 4 9 16 25 36 49 64’.

Cameron graph
planar cubic graph
involution: perm of order 2
flips
mate
concentric-ring graphs
generalized Petersen graphs
Beluhov
equivalent
cycle covers
flower snark graph
bipartite
exact cover problem
knight graph
step matrix
rotated 180°
near symmetry
Carpenter

- **250.** [*M27*] Exactly how many Hamiltonian cycles are possible in the *Sierpiński gasket graph* $S_n^{(3)}$? (See Fig. 113, near 7.2.2.3–(69).) *Hint:* There is a fairly simple formula.
- 999.** [*M00*] this is a temporary exercise (for dummies)

Sierpiński gasket graph

*After [this] way of Solving Questions, a man may steale a Nappe,
and fall to worke again afresh where he left off.*

— JOHN AUBREY, *An Idea of Education of Young Gentlemen* (c. 1684)

AUBREY
Kowalewski
threefold symmetries
complex plane
golden ratio ϕ
Gerbracht
multigraph

SECTION 7.2.2.4

1. Established conventions promote communication, so they outweigh convenience.

2. The 12 vertices of Fig. 200(a) are named ij for $i \neq j$ and $1 \leq i, j \leq 4$. If $ij — i'j'$ then $ji — j'i'$ and $(i\alpha)(j\alpha) — (i'\alpha)(j'\alpha)$, where α is the permutation (123). We also have $12 — 23$, $12 — 34$, $12 — 41$, $12 — 42$, $14 — 42$. These rules define all edges.

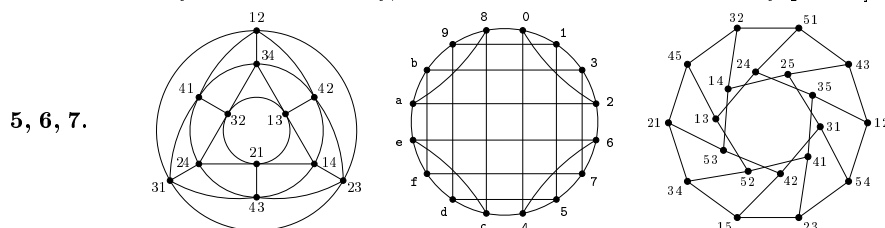
The 20 vertices of Fig. 200(b) are named ij for $i \neq j$ and $1 \leq i, j \leq 5$. If $ij — i'j'$ then $ji — j'i'$ and $(i\sigma)(j\sigma) — (i'\sigma)(j'\sigma)$, where σ is the permutation (12345). We also have $12 — 35$, $12 — 43$, and $13 — 24$. These rules define all of the edges.

[See A. Kowalewski, *Sitz. Akad. Wiss. Wien* (IIa), **126** (1917), 67–90, 963–1007. See exercise 7.2.2.1–136 for a 3D-geometry-based representation scheme.]

3. It remains unchanged after 180° rotation about any of the following lines: (i) from $\frac{13+45}{2}$ to $\frac{31+54}{2}$; (ii) from $\frac{14+53}{2}$ to $\frac{41+35}{2}$; (iii) from $\frac{15+34}{2}$ to $\frac{51+43}{2}$.

There also are remarkable threefold symmetries of a different kind: Color the edges of the cycle alternately red and green; color the other edges blue. Then a 120° rotation about any of the lines from 21 to 12, 23 to 32, 24 to 42, or 25 to 52 will permute the colors cyclically(!). That will yield green-blue and blue-red cycles (see exercise 24).

4. We can redraw the edges $\{12 — 35, 51 — 24, 45 — 13, 21 — 34, 53 — 42\}$ so that they lie outside the circle. [A cubic planar graph with a Hamiltonian cycle can always be drawn as a circle, with some of the unused $n/2$ noncrossing edges inside and the others entirely outside. Similarly, the 4-cube in answer 6 is obviously planar.]



To determine the distances and angles needed for the third drawing above, assuming that vertex '12' is point 1 in the complex plane and that '35' is point $re^{i\theta}$, solve $|1 - re^{i\theta}|^2 = |1 - e^{i\pi/5}|^2 = |re^{i\theta} - re^{i(\theta-2\pi/5)}|^2$. The answer (see exercise 1.2.8–19) is $r = 5^{-1/4}\phi^{-1/2} \approx .5257$, $\theta = \frac{\pi}{4} - \frac{1}{2} \arctan \frac{1}{2} \approx .5536$. [E. H.-A. Gerbracht, *Kolloquium über Kombinatorik*, Universität Magdeburg (15 November 2008).]

8. (a) We can assume by symmetry that the cycle begins at vertex 12, having just come from 35. Case (i) takes us to 54, 23, 41, 35; oops! In case (ii) it's 54, 31, 25, 14; now 43 is stranded. In case (iii) the moves to 54, 31, 42, 53, 21, 45, 13 force the cyclic path $51 — 43 — 25 — 14 — 32 — 51$. The opening moves 54, 23, 15, 34 in cases (iv)–(vi) force the ending to be $\dots, 51, 24, 13, 52, 41, 35$; so those cases are ruled out.

(b) The only remaining possibilities are $(LLRRRLRLR)^2$ and $(RRLLLLRLRL)^2$.

9. All but 23, 24, 25, 31, 41, and 51. (There are 20 Hamiltonian paths from 12 to 35, in spite of the "uniqueness" of exercise 8. There are only six such paths from 12 to 21.)

11. Let $a_j = (2j)'$, $b_j = 2j$, $c_j = 2j + 1$, and $d_j = (2j + 1)'$. Notice that $GP(2q, 2)$ is a graph for $q \geq 3$, a multigraph for $q = 1$ and 2.

A Hamiltonian path P can be characterized by its endpoints and its 3-bit “states”

$$s_j = [a_j \text{---} a_{j'} \in P][c_j \text{---} b_{j'} \in P][d_j \text{---} d_{j'} \in P], \quad 0 \leq j < q, \quad j' = (j+1) \bmod q.$$

For example, with endpoints $\{a_0, a_1\}$ and $q = 3$, the states $(s_0, s_1, s_2) = (011, 111, 010)$ can arise only from the path $a_0 \text{---} b_0 \text{---} c_2 \text{---} d_2 \text{---} d_1 \text{---} d_0 \text{---} c_0 \text{---} b_1 \text{---} c_1 \text{---} b_2 \text{---} a_2 \text{---} a_1$. Moreover, the states $(s_0, s_1, \dots, s_{q-2}, s_{q-1}) = (011, 111, \dots, 111, 010)$ yield a path from a_0 to a_1 whenever $q \geq 3$. (Adding the edge $a_1 \text{---} a_0$ then gives a Hamiltonian cycle.) Those same states also define a Hamiltonian path from a_0 to c_1 .

Only certain state transitions $s_j \rightarrow s_{j'}$ are possible. For example, parity is preserved if $\{a_{j'}, b_{j'}, c_{j'}, d_{j'}\}$ contains no endpoint; and the only such legal transitions are

$$\begin{aligned} 001 \rightarrow 100, 001 \rightarrow 111, 010 \rightarrow 111, 100 \rightarrow 001, 111 \rightarrow 010, 111 \rightarrow 100, 111 \rightarrow 111; \\ 000 \rightarrow 101, 011 \rightarrow 110, 101 \rightarrow 000, 101 \rightarrow 011, 110 \rightarrow 011, 110 \rightarrow 101. \end{aligned}$$

Certain additional restrictions also apply. For example, $110 \rightarrow 011$ can be used at most once, or it will “disconnect” the path. The sequence $001 \rightarrow 111 \rightarrow 010$ forces a 5-cycle.

Parity is preserved also when *both* endpoints lie in $\{a_{j'}, b_{j'}, c_{j'}, d_{j'}\}$. For example, we get a path from a_0 to d_0 for all $q \geq 2$ from the sequence $(111, \dots, 111, 010)$.

Transition rules at parity changes are also easy to work out. For example, if $a_{j'}$ is an endpoint the legal transitions are

$$\begin{aligned} 000 \rightarrow 001, 011 \rightarrow 010, 011 \rightarrow 100, 011 \rightarrow 111, 110 \rightarrow 001; \\ 001 \rightarrow 000, 001 \rightarrow 011, 010 \rightarrow 011, 010 \rightarrow 101, 111 \rightarrow 000, 111 \rightarrow 011. \end{aligned}$$

It turns out that Hamiltonian paths from a_0 to v exist except when v lies in B_q , where $B_q = \{a_j \mid j \bmod 3 = 0, 0 < j < q\}$ when $q \bmod 3 = 0$; $B_q = \{a_j \mid j \bmod 3 = 2, 0 < j < q\}$ when $q \bmod 3 = 1$; and $B_q = \{a_j \mid j \bmod 3 \neq 1, 0 < j < q\} \cup \{c_0, c_{q-1}\} \cup \{b_j \mid j \bmod 3 = 1, 0 < j < q\}$ when $q \bmod 3 = 2$. (Unless $q < 4$: $B_3 = \{b_1, b_2\}$.)

12. Consider the state transitions in answer 11. The cycles of odd-parity states are of two kinds, namely $(010, 111^*, 111)^k$ and $(001, 111^*, 100)^k$; here ‘ 111^* ’ stands for zero or more repetitions of 111. Two cycles of even-parity states exist when $q = 3k + 2$, namely $(000, 101, (011, 110, 101)^k)$ and $(110, (011, 110, 101)^k, 011)$.

With generating functions we can enumerate the number of Hamiltonian cycles that begin with, say, a_0 ; this turns out to be $X_q = 2L_q - 2 + q[q \bmod 3 = 2]$, where $L_q = F_{q+1} + F_{q-1}$ is the q -th Lucas number. When q is prime the number of different cycles is X_q/q . (Otherwise we need to consider cycles whose period length divides q .)

14. (a) In fact, let H be *any* induced subgraph whose vertices all have degree 3 except for exactly two vertices of degree 2. The other edges from those two must be true in every win, or we’d have a cycle entirely within H .

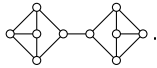
(b) The connecting edges are 1, and so are many of the internal edges. Thus internal cycles will appear when $x + y + z$ is 0, 2, or 3. (But if $x + y + z = 1$ we easily have a path through all the internal vertices.)

(c) The long horizontal edges must be true, because consecutive true vertical edges would yield a short cycle. Hence the edge values at the left and right are $x, \bar{x}, x, \bar{x}, x$ and $y, \bar{y}, y, \bar{y}, y$. If $x = y$ we’d have a 4-cycle or two 8-cycles.

(d) Hook C_m to C_1 . Then insert XOR gadgets to ensure that all appearances of the same variable have consistent values. [This construction can be extended so that G is not only cubic but planar, and triconnected, with at least five sides on every face. See M. R. Garey, D. S. Johnson, and R. E. Tarjan, *SICOMP* 5 (1976), 704–714.]

parity
generating functions
Lucas number
Fibonacci numbers
planar
triconnected
Garey
Johnson
Tarjan

16. (1, 2, 5, 19) connected cubic graphs on (4, 6, 8, 10) vertices are essentially distinct (not isomorphic); we'll study how to generate them in Section 7.2.3. They all are Hamiltonian except for two of order 10. One of the latter is the famous Petersen graph (Fig. 2(e) near the beginning of Chapter 7), which also is nonplanar.

The other “smallest” non-Hamiltonian example is actually planar: . [Arun Giridhar verified in 2015 that a 16-vertex variant of this graph, consisting of three 5-vertex diamonds joined to a central vertex, is (uniquely) the smallest cubic graph that has no Hamiltonian *path*.]

18. False. For example, consider $0 \text{ --- } 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 4 \text{ --- } 5 \text{ --- } 0$, $0 \text{ --- } 2 \text{ --- } 4 \text{ --- } 0$.

20. (a) The condition holds when a_k is the number of k -sided faces *inside* the n -cycle. For it's certainly true when there's just one such face ($a_k = [k = n]$). And if a new chord is added to the graph, breaking an inner p -face into a q -face and an r -face where $q + r = p + 2$, $\sum_k (k - 2)a_k$ changes by $(q - 2) + (r - 2) - (p - 2) = 0$.

[The number $a'_k = \alpha_k - a_k$ of k -faces *outside* the cycle is also a solution to Kirkman's conditions. Indeed, we always have $\frac{1}{2} \sum_k (k - 2)\alpha_k = \frac{1}{2} \sum_k k\alpha_k - \sum_k \alpha_k = \langle \text{edges} \rangle - \langle \text{faces} \rangle = \langle \text{vertices} \rangle - 2$ in a connected planar graph.]

(b) We can assume that the missing vertex is outside the cycle; and $3a_5$ can't equal $19 - 2$. (The dodecahedron does have cycles of lengths $\{5, 8, 9, 10, \dots, 17, 18\}$.)

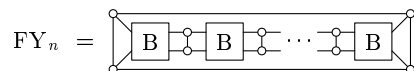
(c) We can assume that neither cycle is inside the other. A cycle that contains exactly a pentagons has length $3a + 2$; and $(3a + 2) + (3b + 2)$ can't equal 20.

(d) Let G' be G without the edge $a \text{ --- } b$, and with two additional vertices of degree 2: one inserted between a and f , another between b and c . Any Hamiltonian cycle in G that omits $a \text{ --- } b$ corresponds to a Hamiltonian cycle in G' . But G' isn't Hamiltonian, because $\alpha'_k = [k = 4] + 7[k = 5] + [k = 11]$ and $2a'_4 + 3a'_5 + 9a'_{11} \neq 18 - 2$.

(e) Graph (i) has $\alpha_k = [k = 4] + 20[k = 5] + 2[k = 11]$; graph (ii) has $\alpha_k = [k = 4] + 18[k = 5] + 4[k = 8]$. So both of them fail Kirkman's test (a). Graph (iii), with $\alpha_k = 18[k = 5] + 3[k = 6] + 3[k = 8]$, passes the test only if $a_6 = 0$ or 3. But the three 6-edged faces can't all be inside or outside the cycle, since they share a common point.

[*Historical notes:* As noted near the beginning of this chapter, Kirkman actually studied full-length cycles in convex polyhedra in *Philosophical Transactions* **146** (1856), 413–418, before Hamilton began to toy with such ideas. Graphs (ii) and (iii) in part (e) are due to É. Ya. Grinberg, *Latviškie Matematičeskij Ezhegodnik* **4** (1968), 51–58, who rediscovered Kirkman's long-forgotten criterion. Graph (i) was found as part of an exhaustive computer search by G. B. Faulkner and D. H. Younger, *Discrete Math.* **7** (1974), 67–74, who also established that Grinberg's (iii) is the unique smallest cubic planar graph that is *cyclically 5-connected*: It cannot be broken into two components each containing a cycle unless at least five edges are removed. (Graph (ii) clearly has four automorphisms; and graph (iii), obtained by adding a single edge, actually has six, although that isn't obvious from the diagram. If we add *another* edge at the right, in the mirror-image position, we get a 46-vertex graph with 36 Hamiltonian cycles. Of course each of those cycles uses both of the edges that were added to (ii).)]

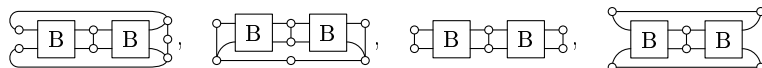
21. Among many possibilities, the simplest are perhaps the $(20n + 2)$ -vertex graphs



made from n copies of an 18-vertex gadget \square_B , where graph (i) in exercise 20(e) is FY_2 and \square_B is illustrated there. In general, FY_n has $\alpha_k = [k = 4] + 10n[k = 5] +$

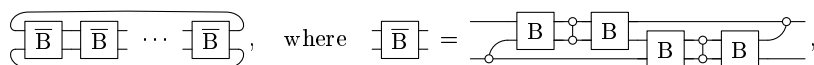
isomorphic
Petersen graph
Giridhar
Hamiltonian *path*
Historical notes
Hamilton
Grinberg
Faulkner
Younger
cyclically 5-connected
automorphisms
gadget

$2[k = 5n + 1]$; so it fails Kirkman's test whenever $n \bmod 3 = 2$. Further analysis, based on the fact that each of the four graphs



also fails Kirkman's test, shows that FY_n is actually non-Hamiltonian for *all* $n > 1$.

(Faulkner and Younger went on to show that the $78m$ -vertex graphs



are not only non-Hamiltonian, they can't be covered by fewer than m disjoint cycles.)

Grinberg's paper of 1968 described non-Hamiltonian cubic planar graphs having $(14 \cdot 4^s - 10 \cdot 3^s)(3t - 1)$ vertices, for any $s, t > 0$. His graphs are noticeably harder than FY_n for a computer to analyze; even the case $(s, t) = (2, 1)$ is quite a challenge.

24. (a) $K_4 \oplus K_4$ is disconnected (and K_4 is perfectly Hamiltonian). The others are at least Hamiltonian, and we can number the vertices so that $0 - 1 - \dots - 7 - 0$. There are five nonisomorphic possibilities: *Case 1*, $0 - 2, 1 - 3, 4 - 6, 5 - 7$: 16 auts, 4H. [Translation: 16 automorphisms and 4 Hamiltonian cycles.] *Case 2*, $0 - 2, 1 - 5, 3 - 6, 4 - 7$: 12 auts, 6H, perfect (two sets of three). *Case 3*, $0 - 2, 1 - 5, 3 - 7, 4 - 6$: 4 auts, 3H, planar, perfect. *Case 4*, $0 - 3, 1 - 6, 2 - 5, 4 - 7$: 48 auts, 6H, planar [the 3-cube]. *Case 5*, $0 - 4, 1 - 5, 2 - 6, 3 - 7$: 16 auts, 5H.

(b) Let a_k be the number of k -faces inside none of the three Hamiltonian cycles; let b_k, c_k, d_k be the number that are inside cycles $\{1, 2\}, \{1, 3\}, \{2, 3\}$, respectively. Then Kirkman's criterion for cycle 1 is satisfied by $b_k + c_k$ and $a_k + d_k$, the number of faces respectively inside or outside. Similarly, it's satisfied for cycle 2 by $b_k + d_k$ and $a_k + c_k$; for cycle 3 by $c_k + d_k$ and $a_k + b_k$. Let $A = \sum_k (k-2)a_k, \dots, D = \sum_k (k-2)d_k$; we've shown that $A+B = A+C = A+D = B+C = n-2$. [See Grinberg's paper in answer 20.]

[Similarly, an r -regular graph is perfectly Hamiltonian if its edges can be r -colored in a such a way that all $\binom{r}{2}$ pairs of colors yield Hamiltonian cycles. (The 4-regular 6-vertex "Star of David" graph $L(K_4)$ is a good example; so is the 5-regular K_6 .) Such graphs are also called P1F, "perfectly 1-factorable," because two 1-factors — also known as perfect matchings — are called perfect if they yield a Hamiltonian cycle, and because an r -regular graph with $\chi(L(G)) = r$ is called 1-factorable. The pioneering explorations of A. Kotzig (see *Theory of Graphs and its Applications*, ed. by M. Fiedler (1964), 63–82) have led to a large literature with many provocative problems still unsolved (see A. Kotzig and J. Labelle, *Annales des Sciences Math. du Québec* **3** (1979), 95–106); for an excellent survey see A. Rosa, *Mathematica Slovaca* **69** (2019), 479–496.]

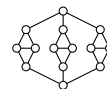
27. (a) This follows directly from exercise 20(d). (In general, we get a "forcing" gadget from *any* graph that has a "forced" edge, by removing any vertex of that edge.)

(b) Insert a degree-2 vertex into each of those spokes and apply 20(a).

(c) Two nonconsecutive spokes are forced to be in any Hamiltonian path.

(d) No. One of the five 4-faced regions touches the unique 9-faced region. Its other neighbors have respectively $\{5, 8, 8\}, \{5, 7, 7\}, \{5, 7, 8\}, \{7, 8, 8\}, \{7, 7, 7\}, \{7, 7, 8\}$ faces.

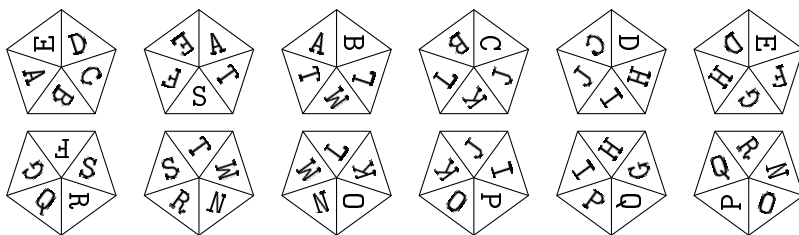
Historical notes: A cubic graph that can be disconnected by removing one edge is clearly non-Hamiltonian. Many cyclically 2-connected planar cubic graphs, such as the example shown, also have no Hamiltonian cycle. However, P. G. Tait investigated numerous cyclically 3-connected planar cubic graphs — the "true" polyhedra — and found Hamiltonian cycles easily. So he conjectured that



Kirkman
Faulkner
Younger
Grinberg
3-cube
Grinberg
Historical notes
 r -regular graph
"Star of David" graph
P1F
1-factors
perfect matchings
Kotzig
Fiedler
Labelle
strongly H graphs, see perfectly H graphs
Rosa
Historical notes
isthmus
bridge
cyclically 2-connected
bridge, wheatstone
Tait
cyclically 3-connected

such cycles always exist, and he pointed out that the famous “Four Color Theorem” would then follow. (See §15 and §16 of his paper cited in answer 35.) Tait’s conjecture was believed for many years, until W. T. Tutte [J. London Math. Soc. (2) **21** (1946), 98–101] found a 46-vertex counterexample by putting together three Tutte gadgets. The smaller graphs in (c) were found independently in 1964 by D. Barnette, J. Bosák, and J. Lederberg; those graphs are the *only* counterexamples with fewer than 40 vertices [see D. A. Holton and B. D. McKay, J. Combinatorial Theory **B45** (1988), 305–319]. Every counterexample has a face with more than 6 vertices [F. Kardoš, SIAM J. Discrete Math. **34** (2020), 62–100]; in particular, all “fullerene graphs” are Hamiltonian.

30. We can use the Ls and Rs of answer 8 as a guide:



(The author cherishes a 3D-printed object like this, received as a surprise gift in 2016.)

33. $nH + h$ — one for every Hamiltonian path in G (cyclic or not). (Thus an algorithm that finds all Hamiltonian cycles can readily be adapted to find all Hamiltonian paths.)

35. (a) The tour lines divide the plane into regions. Every such region can be assigned a rank, representing its distance from the outside. (More precisely, the rank is the minimum number of tour lines crossed by any path in the plane that goes from a point in the region to a point outside the chessboard, without passing through any of the tour’s intersection points.) Then, as you walk along the tour, make your thread go on top at an intersection if and only if the region on your left has odd rank. [See P. G. Tait, London, Edinburgh, and Dublin Philosophical Magazine **17** (1884), 30–46, §19.]

(b) One endpoint is in the outside region, but the other is in a region of rank 4. Any artificial path that connects the two, and crosses k tour lines, will lead to a drawing with k exceptions when the artificial path is removed. Conversely, the exceptional tour segments in any drawing can be crossed by an artificial connection path together with zero or more artificial cycles; so there must be at least 4 exceptions.

(There’s no problem in (2), because each endpoint lies in the outer region.)

36. In (i), $\sigma_{22} \leftrightarrow \sigma_{24}$ and $\sigma_{25} \leftrightarrow \sigma_{27}$. In (ii), ‘lots’ matches ‘lost’:



37. Starting at cells (1, 2, 3, 4) of row 1 we obtain respectively (7630, 2740, 2066, 3108) tours. Starting at cells (1, 2, 3, 4) of row 2 we obtain none. Thus there are exactly $4 \cdot (7630 + 2740 + 2066 + 3108) = 62,176$ tours, all of which are open because they begin and end in the top or bottom row. (Among all such tours, 1904 cannot be represented by a single Rudraṭa-style sloka because all 32 syllables of such a sloka would have to be identical! Only the example in answer 36(ii), and its reversal after 180° rotation, are representable by a sloka that has 12 distinct syllables.)

Four Color Theorem
Tutte
Barnette
Bosák
Lederberg
Holton
McKay
Kardoš
fullerene graphs
author
3D-printed
all Hamiltonian paths
rank
Tait
open
Rudraṭa

38. One knight jumps like three rookwise steps.

Past sore too mean; so, just for free,
 Hops here, turns there, flies each goose now.
 Can't place last word? Won't find the sea.
 One, two, three, four! See each word here:
 Jumps so wise now find their place passed.
 Terns can't soar, like flies the free rook;
 Goose steps just won't mean knight hops last.

poetic license
 Murray
 silver general
 shōgi
 Japanese chess

40. Meet me, you fool; trip some word up;

Eat, see if autumn is a mess.
 To forgo this ordeal, I cheat:
 Won three games like dice, card-trick, chess.
 One, two, three, four! Games go like this:
 Dice or card deal, tricky chess cheat,
 Mess up; a word is some dumb trip.
 Awful if you see me eat meat.





[Sloka 16 in Rudraṭa's *Kāvyālaṅkāra* can be interpreted as two poorly joined quarter-tours of an elephant. See Murray's *History of Chess*, pages 54 and 55.]

(A “silver general” in shōgi (Japanese chess) has the same moves as an elephant.)

41. (a) There are $(m-1)n$ “trunk” arcs from (i, j) to $(i-1, j)$, plus $4(m-1)(n-1)$ “leg” arcs from (i, j) to $(i \pm 1, j \pm 1)$; total $(m-1)(5n-4)$.



(b) Yes: If and only if $m = 2$ and n is even. (Use just two trunk moves.)

(c) In fact, the solution in Fig. A-15(a) is the *only* such Hamiltonian path on E_{48} .




(d) If not, every vertex of row 1 is of type A () or B () or C () or D () within the path. There's a B at the left and an A at the right. The adjacent pairs AD, BD, CA, CB are not permitted, nor are the near-adjacent pairs B◦A, B◦C, C◦D, D◦A, D◦C (with one vertex intervening). Furthermore the substrings B(CD)*A = {BA, BCDA, BCDCDA, ...} are forbidden, because these are closed cycles and $m > 2$.

But no string of A's, B's, C's, and D's obeys all of those restrictions.

(e) The same proof works, with types A () , B () , C () , and D () .

(f) The vertices of (d) are joined by one of type E () or F () . The leftmost is either B or F; the rightmost is either A or E. Cases B◦E, D◦E, F◦A, F◦C are excluded, in addition to those of (d). Exactly $n[n \text{ even}]$ such sequences are possible, having the forms F(CD)*A, B(CD)*ED(CD)*A, B(CD)*CF(CD)*A, or B(CD)*E.

Each of these has exactly one unsaturated vertex in row 2. Thus there are one or two possible moves to row 3, and we've effectively reduced m to $m-2$.

(g) Now the vertices of (e) are joined by one of type E () or F () or G () . The leftmost is either B, F, or G; the rightmost is A, E, or G. The new forbidden substrings are AE, BE, CG, FA, FB, GD, C◦E, F◦D. Six species of solutions exist, namely GA*(CD)*A, B(CD)*B*G, B*FD(CD)*A, BCD(CD)*B*FD(CD)*A, B(CD)*CEA*, and B(CD)*CEA*(CD)*CDA. The solutions containing A* or B* work when n is odd.

Again we reduce m to $m-2$ and continue. (By induction, m must be even.)

(h) Let $A^{(m)}$ be the $n \times n$ matrix where $a_{ij}^{(m)}$ is the number of Hamiltonian paths from $(1, i)$ to (m, j) ; let $B^{(m)}$ be analogous, where $b_{ij}^{(m)}$ counts paths from (m, i) to $(1, j)$. (These matrices are symmetric about both diagonals, because the left-right and top-

bottom reflections of any elephant path are elephant paths, possibly reversed.) We have

$$a_{ij}^{(2)} = ([i = j = 1] + [i = j = n] + [|i - j| = 1 \text{ and } \max(i, j) \text{ odd}])[n \text{ even}],$$

$$b_{ij}^{(2)} = \begin{cases} [i \text{ odd}][j \text{ even}][i < j] + [j \text{ odd}][i \text{ even}][j < i], & \text{if } n \text{ is even,} \\ [i \text{ odd}][j \text{ odd}] \max([i = 1], [i = n], [i \neq j]), & \text{if } n \text{ is odd,} \end{cases}$$

by (f) and (g). Moreover, by considering moves between two-row subgraphs,

$$A^{(m+m')} = A^{(m)} X A^{(m')} \text{ and } B^{(m+m')} = B^{(m)} (X + I) B^{(m')}, \text{ where } x_{ij} = [|i - j| = 1].$$

For example, there are $\sum A^{(4)} + \sum B^{(4)} = 14 + 120 = 134$ tours on a 4×8 board.

42. The technology of exercise 7.1.4–226 can be extended to directed graphs in a straightforward way. It constructs a ZDD of about 1.3 meganodes for all oriented cycles in the 8×8 elephant digraph, and shows that there are exactly 277,906,978,347,470 of them. The generating function by cycle length is $98z^2 + 205z^4 + 698z^6 + 3853z^8 + \cdots + 50128559z^{60} + 6544z^{62}$. If we say that trunk moves have weight 2 while other moves have weight 3, we find (by computing maximum-weight cycles) that exactly four of the 6544 62-cycles have only eight trunk moves. All four of these solutions are equivalent under reflection to Fig. A–15(b).

(Fig. A–15(c) shows an interesting symmetrical 60-cycle that omits the corners and has just *four* trunk moves.)

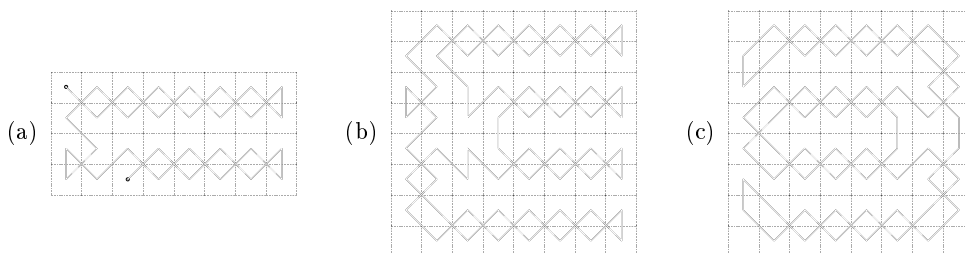
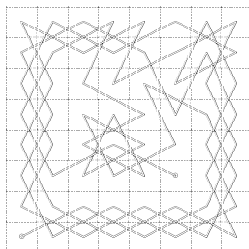


Fig. A–15. Noteworthy paths and cycles for elephants.

44. We can use Rudraṭa's half-tour twice:

Bah dee boo dai hao fuh hoe fay, bee doo bai fao huh foe hay dah?
 Fee hah day boe foo hai dao buh, fai bao duh hoo doe bay fah hee.
 Lah mee loo mai sao nuh soe nay, lee moo lai nao suh noe say mah?
 Nee sah may loe noo sai mao luh, nai lao muh soo moe lay nah see!

(This is an open tour. See exercise 199 for *closed* tours that rhyme just as perfectly.)



46. This tour is rather like that of Ibn Manī' in (1):

50. In fact, the text's "merry chase" need not end at cell 22; by swapping **23** \leftrightarrow **25** we get a tour that ends at 02. The other seven choices of **9** and **10** can lead, similarly, to either 22 or one of $\{02, 11, 13, 20, 24, 31, 33\}$. Each case completes an open tour.

[See page 280 of de Jaenisch's book, for his analysis of 5×5 paths.]

de Jaenisch
backtrack algorithm

51. Again $v_1 v_2 v_3 v_4 = 00\ 12\ 04\ 23$ without loss of generality. Now $t_1 = 44$ forces $v_5 = 31$, and there's a tie for v_6 . If $v_6 = 43$, the path continues $v_7 \dots v_{15} = 24\ 03\ 11\ 30\ 42\ 34\ 13\ 01\ 20$; and v_{16} is 32 or 41. The former case forces $v_{17} = 40$, hence "shutting out" 44; it leads to four paths, each ending at v_{24} . But the latter case leads to three paths, two of which end with $v_{25} = 44$ (yea) and one that ends with $v_{21} = 44$ (boo).

On the other hand if $v_6 = 10$ we get $v_7 \dots v_{13} = 02\ 14\ 33\ 41\ 20\ 01\ 13$ and then $v_{14} v_{15} v_{16} = 21\ 40\ 32$ or $32\ 40\ 21$. Either case shuts 44 out. Ten continuations are possible, each of which involves 24 vertices — all but cell 44 (close but no cigar).

(The randomized algorithm of exercise 53 will yield a Hamiltonian path with probability $\frac{3}{16}$. If we set $\{t_1, t_2, t_3\} = \{23, 32, 44\}$ and $r = 3$, this probability rises to $\frac{3}{8}$.)

52. The algorithm acts just as if a double-target vertex t has been entirely removed from the graph, because $\text{DEG}(t)$ will never be less than $2n$ in step W5.

53. W5'. [Is $\text{DEG}(u)$ smallest?] If $t < \theta$, set $\theta \leftarrow t$, $v \leftarrow u$, $q \leftarrow 1$. Otherwise, if $t = \theta$, set $q \leftarrow q + 1$, then set $v \leftarrow u$ with probability $1/q$.

55. Ibn Manī' broke Warnsdorf's rule first when choosing $v_{14} = 41$ instead of 06. His choices for v_{26} , v_{27} , v_{38} , v_{39} , v_{45} , v_{48} , and v_{54} also broke the rule. But altogether, his "hug the edge" strategy followed it $\frac{55}{63} \approx 87\%$ of the time, so he probably had some of the same intuition that von Warnsdorf acquired later. Similarly, Someshvara deviated only eight times. But al-'Adlī broke the rule 15 times, clearly thinking other thoughts.

56. If there's no remaining exit from u , there's no remaining entrance to u . Therefore Algorithm W will not find a Hamiltonian path unless it moves to u . We might as well do that, if our goal is simply to find a Hamiltonian path. But maybe we really want to find as long a path as possible, via Warnsdorf-like rules; then we can do better.

Vertex u is a dead end if and only if $\text{DEG}(u)$ is 0 or n . Suppose v_k has just two neighbors, u and u' , where $\text{DEG}(u) = 0$ and $\text{DEG}(u') = n$. We presumably should choose $v_{k+1} = u'$ in such a case, because u' is one of the designated targets.

Thus we can improve the algorithm by changing ' $t < \theta$ ' in step W5 to ' $f(t) < f(\theta)$ ', where $f(0) = 2n + 1$, $f(n) = 2n$, $f(t) = t + 2$ for $t \geq 2n$, and $f(t) = t$ otherwise.

57. This is a simple backtrack algorithm, following the outline of Algorithm 7.2.2B. Leaves of the search tree correspond to the paths of Algorithm W. The probability of each node is the probability of its parent, divided by the family size.

59. Build 64 anti-Warnsdorf trees, as in exercise 57. There are 8 paths $v_1 v_2 \dots v_{20}$ of length 19 (equivalent to each other via rotation/reflection), each occurring with probability $2^{-9} 3^{-6} \approx .0000027$; they actually turn out to be cycles of length 20. At the other extreme, there are 24 anti-Warnsdorf paths as long as 53, of which 8 have probability $p = 2^{-20} 3^{-5} \approx .0000000039$ and 16 have probability $3p$; representatives are shown below. The mean path length is ≈ 34.8 , and the standard deviation is ≈ 5.3 .

By contrast, Algorithm W will find a path that's longer than *every* anti-Warnsdorf path, more than 99.8% of the time. Its worst case has length 39; there are eight such paths, equivalent to the one shown, each obtained with probability 2^{-21} . (And if we use the modification of answer 56, the worst case length rises to 46; there are 40 such

71. Let the edges be $1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n \text{ --- } 1$ and $1 \text{ --- } 5, 3 \text{ --- } (n-2), (n-4) \text{ --- } n$; consider the path $2 \text{ --- } 1 \text{ --- } 5 \text{ --- } 4 \text{ --- } 3 \text{ --- } (n-2) \text{ --- } (n-3) \text{ --- } (n-4) \text{ --- } n \text{ --- } (n-1)$.

73. Assign a random 32-bit weight to each edge of the graph, and let each path have a “long hash code” H that’s the sum of its edge weights (modulo 2^{32}). Let there be 2^b hash lists; a path with long hash H will go into list $H \bmod 2^b$. If c vertex names can be packed into an octabyte, the dictionary entry for (v_1, \dots, v_t) will occupy $B = 1 + \lceil t/c \rceil$ octabytes: one for the link and H , the others for the vertices (which are examined in detail during a search only when the long hash code is correct). Store the q th path in positions $(qB + s) \bmod M$ of a large array of octabytes, for $0 \leq s < B$, where M is a large power of 2. (Overflow occurs if we try to write into position $(p_2B + B) \bmod M$.)

75. (a) Let H be the graph whose vertices are the Hamiltonian paths of G that start with v_1 , adjacent if they differ only by flipping a subpath. Since G is cubic, every vertex of H has degree 2. So H consists of cycles, and Algorithm F[−] constructs the cycle that begins with the given path. For example, when $G = P_2 \square P_2 \square P_2$, we can represent the vertices (000, 001, ..., 111) by (0, 1, ..., 7), and H has two cycles: 01326754 — 01326457 — 01375462 — 02645731 — 02645137 — 02673154 — 04513762 — 04513267 — 04576231 — 01326754; 04623751 — 01573264 — 01546247 — 01546732 — 02376451 — 02315467 — 02315764 — 04675132 — 04623157 — 04623751.

(b) Let $\alpha = v_1 \text{ --- } v_2 \text{ --- } \cdots \text{ --- } v_n$ be a Hamiltonian path with $v_n \text{ --- } v_1$. One of its two neighbors in H is $v_1 \text{ --- } v_n \text{ --- } \cdots \text{ --- } v_2$, which is the reflected cycle α^R . The other neighbor will have v_2 unchanged. So the cyclic paths come in pairs.

(c) Consider maximal segments of the cycle in H whose paths begin with $v_1 \text{ --- } v_2$. The first and last of these paths are Hamiltonian cycles, which we can regard as *mates* of each other. (For example, the mate of 01326754 with respect to $0 \text{ --- } 1$ is 01375462.)

(d) If α is a Hamiltonian cycle containing the edge e , its mate β is a Hamiltonian cycle containing an edge $e' \notin \alpha$. Hence γ , the mate of β with respect to e' , is a third.

(e) Color the n edges of α alternately red and green; color the other $n/2$ edges blue. The blue edges of β and γ are the same; suppose there are $n/2 - x$ of them. Let β have r red and g green; hence γ has $n/2 - r$ red and $n/2 - g$ green. We have $r + g + n/2 - x = (n/2 - r) + (n/2 - g) + (n/2 - x) = n$; hence $x = 0$. Therefore no two consecutive edges of α can appear in β or γ ; they must all be the same color.

[*Historical notes:* The theorem in (c) was discovered via algebraic reasoning by C. A. B. Smith, about 1940, but not published until later. See *Combinatorial Mathematics and its Applications* (Oxford conference, 1969), 259–283; W. T. Tutte, *Graph Theory As I Have Known It* (1998), 18, 48, 94. A. G. Thomason, in *Annals of Discrete Mathematics* **3** (1978), 259–268, introduced one-sided flips and used them to give an algorithmic proof of Smith’s theorem.]

77. (a) Easily verified. (This is the *only* automorphism, when $n > 1$.)

(b) Any Hamiltonian cycle containing $05 \text{ --- } 0 \text{ --- } 01$ mustn’t contain $0 \text{ --- } 06$; hence $07 \text{ --- } 06 \text{ --- } 05$ but not $05 \text{ --- } 04$ or $01 \text{ --- } 07$; hence $03 \text{ --- } 04 \text{ --- } 15, 01 \text{ --- } 02, 08 \text{ --- } 07 \text{ --- } 06$, not $02 \text{ --- } 08$; hence $11 \text{ --- } 08$. Replacing all ‘0j’ by ‘1’ yields a Hamiltonian cycle containing $15 \text{ --- } 1 \text{ --- } 11$ in a graph isomorphic to C_{n-1} . By induction, it’s α_n . Similarly, the only Hamiltonian cycles containing $06 \text{ --- } 0 \text{ --- } 01$ and $05 \text{ --- } 0 \text{ --- } 06$ are

$$\begin{aligned}\beta_n &= 0 \text{ --- } 01 \text{ --- } 07 \text{ --- } 08 \text{ --- } 02 \text{ --- } 03 \text{ --- } 16 \text{ --- } \cdots \text{ --- } 15 \text{ --- } 04 \text{ --- } 05 \text{ --- } 06 \text{ --- } 0; \\ \gamma_n &= 0 \text{ --- } 06 \text{ --- } 07 \text{ --- } 01 \text{ --- } 02 \text{ --- } 08 \text{ --- } 11 \text{ --- } \cdots \text{ --- } 16 \text{ --- } 03 \text{ --- } 04 \text{ --- } 05 \text{ --- } 0.\end{aligned}$$

(c) (11, 65, 265, 1005, 3749, 13927, 51683, 191735, 711243). [But an appropriate sequence of only $4n$ *two-sided* flips will take us from α_n to β_n , or β_n to γ_n , or γ_n to α_n .]

random 32-bit weight
long hash code
mates
Historical notes
Smith
Tutte
Thomason

(d) When $n > 2$, the first five flips yield $01 \text{ --- } 0 \text{ --- } 05 \text{ --- } 06 \text{ --- } 07 \text{ --- } 08 \text{ --- } 02 \text{ --- } 03 \text{ --- } 04 \text{ --- } 15 \text{ --- } 16 \text{ --- } 17 \text{ --- } 11 \text{ --- } 12 \text{ --- } 18$ followed by a sequence $21 \text{ --- } 22 \text{ --- } \dots$ that's the same as the suffix $01 \text{ --- } 02 \text{ --- } \dots$ of the second path obtained with respect to $0 \text{ --- } 01$, *except* that all entries are increased by 20, and '14 --- 13' appears between 25 and 26. The next $c_{n-2} - 1$ flips mimic (c); then six more flips give the reverse of β_n .

(e) When $n > 1$, the number is $c_{n-1} + 5$ in both cases(!), proved as in (d).

[The graphs C_n were introduced by K. Cameron, *Discrete Math.* **235** (2001), 69–77, who simplified a similar construction by A. Krawczyk and proved that $c_n \geq 2^n$. In 2020, Filip Stappers discovered that the generating function $c(z) = \sum_n c_n z^n$ is $p(z)/q(z)$, where $q(z) = (1-z)(1-3z-2z^2-2z^3-z^4-z^5)$ and $p(z) = z(1+z)(11+10z+6z^2+4z^3+z^4)$. He also proved that the number of one-sided flips to go from β_n to its mate γ_n , with respect to either $0 \text{ --- } 06$ or $06 \text{ --- } 0$, is \hat{c}_n , where $\sum_n \hat{c}_n z^n = \hat{p}(z)/q(z)$ and $\hat{p}(z) = 2z(3+2z+z^2-2z^3)$. Consequently the actual limiting ratio c_{n+1}/c_n is $\rho \approx 3.709398$, the real root of $z^5 = 3z^4 + 2z^3 + 2z^2 + z + 1$. Asymptotically, $c_n \sim c\rho^n - 8$ and $\hat{c}_n \sim \hat{c}\rho^n$, where $c \approx 5.349$ and $\hat{c}_n/c_n \sim \hat{p}(\rho)/p(\rho) \approx 0.3959$. In *Bull. Aust. Math. Soc.* **98** (2018), 18–26, L. Zhong introduced a family of graphs on $16n$ vertices for which the number of flips to get from a certain Hamiltonian path to its mate with respect to $0 \text{ --- } 1$ is *exactly* $6 \cdot 2^n - 10$. However, that number with respect to $1 \text{ --- } 0$ is only 4 (!).]

78. (a, b) In contrast to exercise 60, success occurs with probability $\approx 100\%$ when $q \leq 10$. Furthermore, a Hamiltonian cycle is usually found soon after finding the first Hamiltonian path. The average number of updates before that first cycle, observed in 100 runs for each q , was $\approx (81, 141, 146, 240, 295)$.

But $q = 100$ was a different story. Here a 400-cycle was successfully found in only six of ten cases—sometimes after as few as 18 thousand updates, sometimes after as many as 8.3 million. In one of the other cases, millions of Hamiltonian paths (not cycles) were found; but memory overflow, with more than 2 million paths in the dictionary, aborted the run. Memory overflow also arose in the three other cases, once before achieving any paths longer than 365.

We conclude that Algorithm F can have wildly eccentric behavior, and it should probably be restarted if it spins its wheels too long.

79. (a) Let there be 12 vertices $\{0, \dots, 11\}$, with $k \text{ --- } (k+1)$ for $0 \leq k < 12$ and $3k \text{ --- } (3k+5)$ for $0 \leq k < 4$ (modulo 12). This graph has two equivalence classes, each containing one Hamiltonian cycle and 12 Hamiltonian paths that aren't cycles.

(b) Let there be $13n$ vertices ij for $0 \leq i < n$ and $-1 \leq j < 12$, with $ik \text{ --- } ik'$ whenever $k \text{ --- } k'$ in (a); also $i\bar{1} \text{ --- } i0$ and $i1 \text{ --- } ((i+1) \bmod n)\bar{1}$. This graph has 2^n equivalence classes, each containing one cycle and $17n$ noncycles.

Cameron
Krawczyk
Stappers
generating function
Zhong

100. (Using ideas of N. Beluhov.) When C is a cycle cover, let $s_j = 4[t_j - t_{j+1} \in C] + 2[v_j - w_{j+1} \in C] + [w_j - v_{j+1} \in C]$ encode its edges between indices j and $j + 1$ modulo q . A simple case analysis shows that $s_j \neq 0$; $s_j \in \{1, 2, 4\} \implies s_{j+1} = 7$; $s_j = 3 \implies s_{j+1} \in \{5, 6\}$; $s_j = 5 \implies s_{j+1} \in \{3, 5\}$; $s_j = 6 \implies s_{j+1} \in \{3, 6\}$; $s_j = 7 \implies s_{j+1} \in \{1, 2, 4\}$; and that the sequence $s_1 s_2 \dots s_q$ completely determines C .

Thus there are two kinds of covers: Type A, where s_j is alternately 7 and an element of $\{1, 2, 4\}$; or type B, where each s_j is an element of $\{3, 5, 6\}$. Type A covers arise only when q is even, and they have $k + 1$ cycles when there are k occurrences of $s_j = s_{j+2} \neq 7$. Type B covers always have exactly 2 cycles.

Let $g(w, z) = \sum w^{[a_0=a_1]+\dots+[a_{n-1}=a_n]} z^n [a_0=a_n]$, summed over all ternary sequences $a_0 a_1 \dots a_n$, and let $h(w, z)$ be similar but requiring $a_0 \neq a_n$. Then $g(w, z) = 3 + wzg(w, z) + zh(w, z)$ and $h(w, z) = 2zg(w, z) + (1+w)zh(w, z)$. So we find $g(w, z) = 3(1 - (1+w)z) / ((1 - (w-1)z)(1 - (w+2)z)) = 2 / (1 - (w-1)z) + 1 / (1 - (2+w)z)$. Consequently the number of type A covers with k cycles is $2[w^{k-1}z^{q/2}]g(w, z) = 4\binom{q/2}{k-1}(2^{q/2-k} - (-1)^{q/2-k})$ when q is even. (In particular, the number of Hamiltonian cycles is $4(2^{q/2-1} + (-1)^{q/2})$.)

Turning to type B, let there be f_{xyn} sequences $a_0 \dots a_n$ with $a_0 = x$, $a_n = y$, and each $a_j \in \{3, 5, 6\}$, having no consecutive 33 or 56 or 65. We find by induction that $f_{xyn} = (2^n - (-1)^n)/3 + \delta_{xyn}$, where $\delta_{xyn} = 1$ when n is even and $x = y$, $\delta_{xyn} = -1$ when n is odd and $xy \in \{33, 56, 65\}$, otherwise $\delta_{xyn} = 0$. Hence there are $f_{33q} + f_{55q} + f_{66q} = 2^q + 2[q \text{ even}]$ covers of type B.

Beluhov
generating functions
ternary sequences

198. (a) There's one solution for every way to cover the vertices of G by disjoint oriented cycles of length ≥ 4 . A cycle $u_0 \rightarrow v_0 \rightarrow u_1 \rightarrow v_1 \rightarrow u_2 \rightarrow \cdots \rightarrow v_{k-1} \rightarrow u_0$ corresponds to choosing the options ' $u_0^- v_0 u_1^+$ ', ' $u_1^- v_1 u_2^+$ ', \dots , ' $u_{k-1}^- v_{k-1} u_0^+$ '.

(b) From the 332 options, Algorithm 7.2.2.1X needs about $180 \text{ M}\mu$ to find 185868 solutions, of which $2 \cdot 9862$ are the closed knight's tours (without removing symmetries).

199. Set up an exact cover problem as in exercise 198, where $n = 32$ and the vertices of the "first part" are the cells ij with $1 \leq i, j \leq 8$ and $i + j$ odd. Also add primary items ij^\times for $i + j$ odd and $i > 2$. Each option now contains at least *six* items, not three: ' $u_1^- v_1 w_1^+ u_2^- v_2 w_2^+$ ' where $u_1 - v_1 - w_1$ and $u_2 - v_2 - w_2$, the six vertices are distinct, the i -coordinate of u_1 is less than the i -coordinate of u_2 , and the j coordinates of (u_1, u_2) , (v_1, v_2) , (w_1, w_2) are equal. (The u 's and w 's belong to the "first part." This option represents a pair of moves with matching columns.) Furthermore, append ij^\times to each option for which $\{w_1, w_2\} = \{mj, ij\}$ or for which $\{u_1, u_2\} = \{mj, kj\}$ and $k \neq i$, where $m \in \{1, 2\}$. This trick forces the pairs of paths to "hook up" properly. For example, two of the options are ' $12^- 24 16^+ 52^- 44 56^+ 32^\times 72^\times 56^\times$ ' and ' $41^- 22 43^+ 61^- 42 23^+ 43^\times$ '. Exploit symmetry by removing options with $v_1 = 11$ and $w_1 = 32$.

That makes a total of 1998 options, and Algorithm 7.2.2.1X finds 383080 solutions in $14\text{G}\mu$. Each solution chooses 16 options, and a good one yields a cycle $(v_0 v_1 \dots v_{63})$ in which the chosen options involve $v_k^-, v_{k+1}^+, v_{k+2}^-, v_{k+3}^+, v_{k+4}^-, v_{k+5}^+, v_{k+6}^-, v_{k+7}^+$ for $k = 0, 2, \dots, 30$. Most solutions define short cyclic paths; but 5264 of them yield correct tours, such as the one shown.

200. Notice that we must have $a_{23} = 2$, $a_{28} = 17$, $a_{47} = 18$, $a_{67} = 50$, $a_{76} = 48$, $a_{88} = 49$. To find such a tour, we can begin by finding a knight's path of length 14 from step 2 to step 16 that doesn't interfere with 180° rotation, nor does it involve any of the "reserved" cells. All paths of length 14 are efficiently found by pasting together compatible paths of length 7. Useful paths also have the property that each vertex in the set U of cells available for steps (18..30) and (50..64) has degree ≥ 2 in the graph restricted to $U \cup I$, where $I = \{47, 52, 67, 32\}$ is the set of endpoints for future paths. The endpoints must also have degree ≥ 1 in that graph. A similar method finds 14-step paths for steps 18 through 32 and 50 through 64. One of the 46,596 solutions is shown.

201. Adapting the method in the previous exercise to paths of other lengths, we find that there are respectively (2, 47, 3217, 280244, 1205980, 259230, 41366) feasible solutions for the first (4, 9, 16, 25, 36, 49, 64) steps. The first full solution is shown. [Brentano's *Chess Monthly* 1,1 (May 1881), 36; 1,5 (September 1881), 248–249. See George P. Jelliss, *Mathematical Spectrum* 25 (1992), 16–20, for information about many similar "figured tours."]

250. Let the number be X_n , and let u, v, w be the "middle" vertices on the boundary. A Hamiltonian cycle on $S_{n+1}^{(3)}$ has the form $u - \cdots - v - \cdots - w - \cdots - u$, where the portions from u to v , v to w , and w to u are Hamiltonian *paths* from corner to corner of $S_n^{(3)}$. Consequently $X_{n+1} = Y_n^3$, where Y_n is the number of such paths.

Write uv for the corner between u and v . A Hamiltonian path from uw to vw has the form $uw - \cdots - u - \cdots - v - \cdots - vw$; and there are two cases, depending on whether w appears before u or after v . Thus $Y_{n+1} = Z_n Y_n Y_n + Y_n Y_n Z_n$, where

disjoint oriented cycles
Jelliss
figured tours
Hamiltonian *paths*

1 32 57 30 3 12 55 16
58 29 2 11 56 15 52 13
33 64 31 4 35 54 17 50
28 59 34 41 10 51 14 53
7 40 63 36 5 22 49 18
60 27 6 9 42 19 46 21
39 8 25 62 37 44 23 48
26 61 38 43 24 47 20 45

1 30 9 6 3 16 61 24
10 7 2 15 62 25 4 17
31 64 29 8 5 38 23 60
28 11 14 63 26 59 18 37
13 32 27 58 51 36 39 22
54 57 12 45 42 21 50 19
33 46 55 52 35 48 43 40
56 53 34 47 44 41 20 49

1 4 9 16 25 36 49 64
8 15 24 3 10 17 26 37
5 2 7 14 35 50 63 48
56 13 34 23 18 11 38 27
33 6 55 12 51 40 47 62
54 57 22 19 46 43 28 39
21 32 59 52 41 30 61 44
58 53 20 31 60 45 42 29

there are Z_n Hamiltonian paths from corner to corner in a graph that's like $S_n^{(3)}$ but with the third corner removed. Similarly, $Z_{n+1} = Z_n Z_n Y_n + Y_n Z_n Z_n + [n=1]$.

We have $(X_1, Y_1, Z_1) = (1, 1, 1)$ and $(X_2, Y_2, Z_2) = (1, 2, 3)$. Hence, for $n \geq 3$, the formulas $X_n = 2^{3^{n-2}} 3^{3^1+3^2+\dots+3^{n-3}}$, $Y_n = 3X_n$, $Z_n = \frac{3}{2}Y_n$ hold by induction.

We can in fact write $X_n = 8 \cdot 12^{(3^{n-2}-3)/2}$. [This problem was first solved by R. M. Bradley, *J. de Physique* **47** (1986), 9–14. Sierpiński's original curve, in *Comptes Rendus Acad. Sci.* **160** (Paris, 1915), 302–305, was the limit of long *paths*, not cycles. See also A. M. Teguia and A. P. Godbole, *Australasian J. Combinatorics* **35** (2006), 181–192, who showed among other things that $S_n^{(3)}$ is *pancyclic*: It has cycles of every length, from 3 to $(3^n + 3)/2$.]

999. ...

Bradley
Teguia
Godbole
pancyclic

INDEX AND GLOSSARY

HUNT

*Index-making has been held to be the driest
as well as lowest species of writing.
We shall not dispute the humbleness of it;
but the task need not be so very dry.*
— LEIGH HUNT, in *The Indicator* (1819)

When an index entry refers to a page containing a relevant exercise, see also the *answer* to that exercise for further information. An answer page is not indexed here unless it refers to a topic not included in the statement of the exercise.

2-factor, *see* Cycle cover.

Barry, David McAlister (= Dave), iii.

Cycle cover: A covering of the vertices
by disjoint cycles (a 2-regular
spanning subgraph).

Hamiltonian graph: A graph with a
spanning cycle, 2.

Nothing else is indexed yet (sorry).

Preliminary notes for indexing appear in the
upper right corner of most pages.

If I've mentioned somebody's name and
forgotten to make such an index note,
it's an error (worth \$2.56).