Introduction to Monte Carlo in Finance

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Outline

Valuation of American Option by Simulation

- As we have seen Monte Carlo simulation is a flexible and powerful numerical method to value financial derivatives of any kind.
- However being a forward evolving technique, it is per se not suited to address the valuation of American or Bermudan options which are valued in general by backwards induction.
- Longstaff and Schwartz provide a numerically efficient method to resolve this problem by what they call Least-Squares Monte Carlo.
- The problem with Monte Carlo is that the decision to exercise an American option or not is dependent on the continuation value.

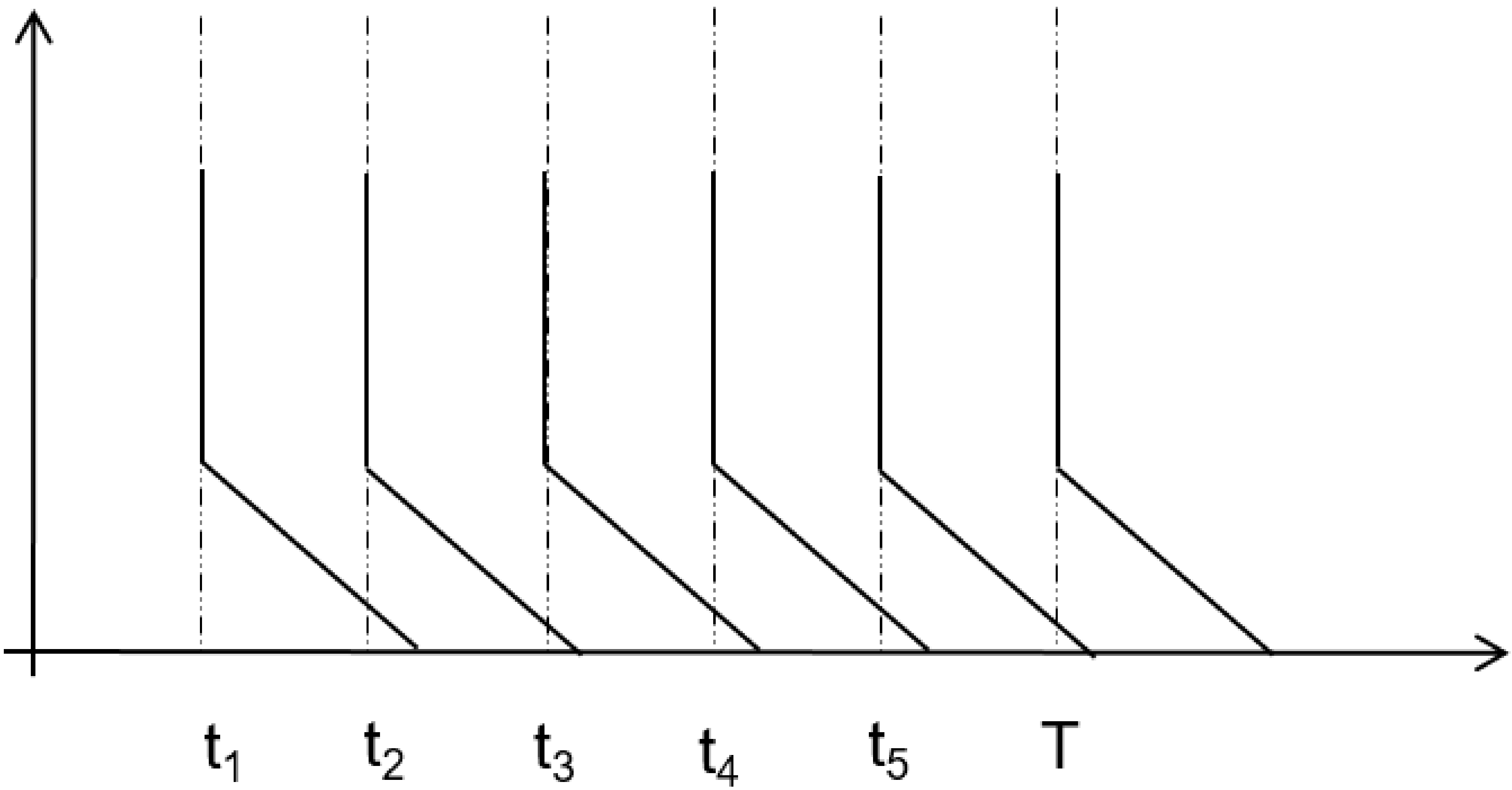
Valuation of American Option by Simulation

- Consider a simulation with $M + 1$ points in time and I paths.
- Given a simulated index level $S_{t,i}, t \in \{0, ..., T\}, i \in \{1, ..., I\}$, what is the continuation value $C_{t,i}(S_{t,i})$, i.e. the expected payoff of not exercising the option?
- The approach of Longstaff-Schwartz approximates continuation values for American options in the backwards steps by an ordinary least-squares regression.
- Equipped with such approximations, the option is exercised if the approximate continuation value is lower than the value of immediate exercise. Otherwise it is not exercised.

Valuation of American Option by Simulation

- In order to explain the metodology, let's start from a simpler problem.
- Consider a bermudan option which is similar to an american option, except that it can be early exercised once only on a specific set of dates.
- In the next figure, we can represent the schedule of a put bermudan option with strike K and maturity in 6 years. Each year you can choose whether to exercise or not ...

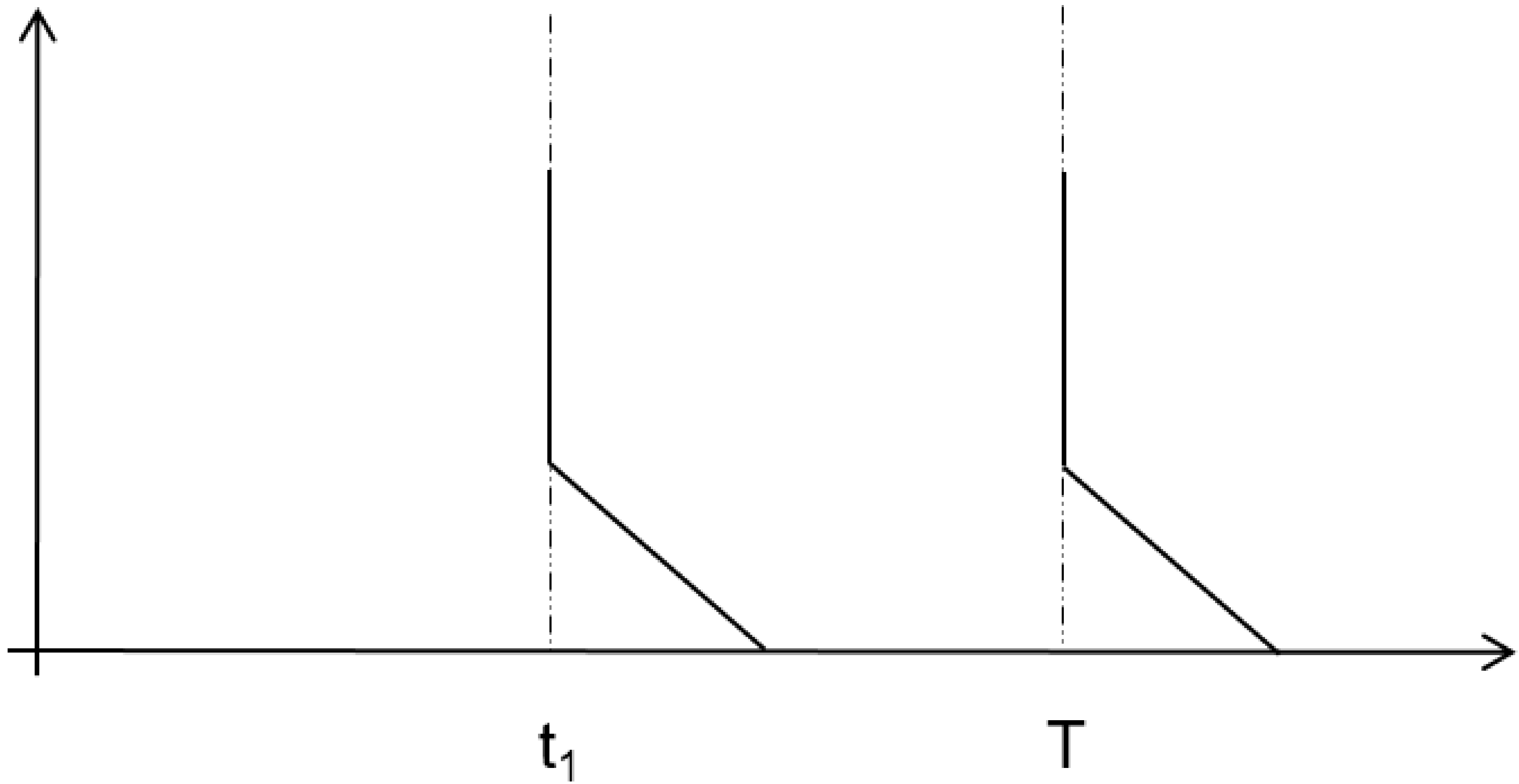
Valuation of American Option by Simulation



Valuation of American Option by Simulation

- Let's consider a simpler example: a put option which can be exercised early only once ...

Valuation of American Option by Simulation



Valuation of American Option by Simulation

- Can we price this product by means of a Monte Carlo? Yes we can! Let's see how.
- Let's implement a MC which actually simulates, besides the evolution of the market, what an investor holding this option would do (clearly an investor who lives in the risk neutral world). In the following example we will assume the following data, $S(T) =$, $K =$, $r =$, $\sigma =$, $t_1 = 1y$, $T = 2y$.
- We simulate that 1y has passed, computing the new value of the asset and the new value of the money market account

$$S(t_1 = 1y) = S(t_0)e^{(r-\frac{1}{2}\sigma^2)(t_1-t_0)+\sigma\sqrt{t_1-t_0}N(0,1)}$$

$$B(t_1 = 1y) = B(t_0)e^{r(t_1-t_0)}$$

Valuation of American Option by Simulation

- At this point the investor could exercise. How does he know if it is convenient?
- In case of exercise he knows exactly the payoff he's getting.
- In case he continues, he knows that it is the same of having a European Put Option.
- So, in mathematical terms we have the following payoff in t_1

$$\max\left[K - S(t_1), P(t_1, T; S(t_1), K)\right]$$

where $P(t_1, T; S(t_1), K)$ is the price of a Put which we compute analytically! In the jargon of american products, P is called the continuation value, i.e. the value of holding the option instead of early exercising it.

Valuation of American Option by Simulation

- So the premium of the option is the average of this discounted payoff calculated in each iteration of the Monte Carlo procedure.

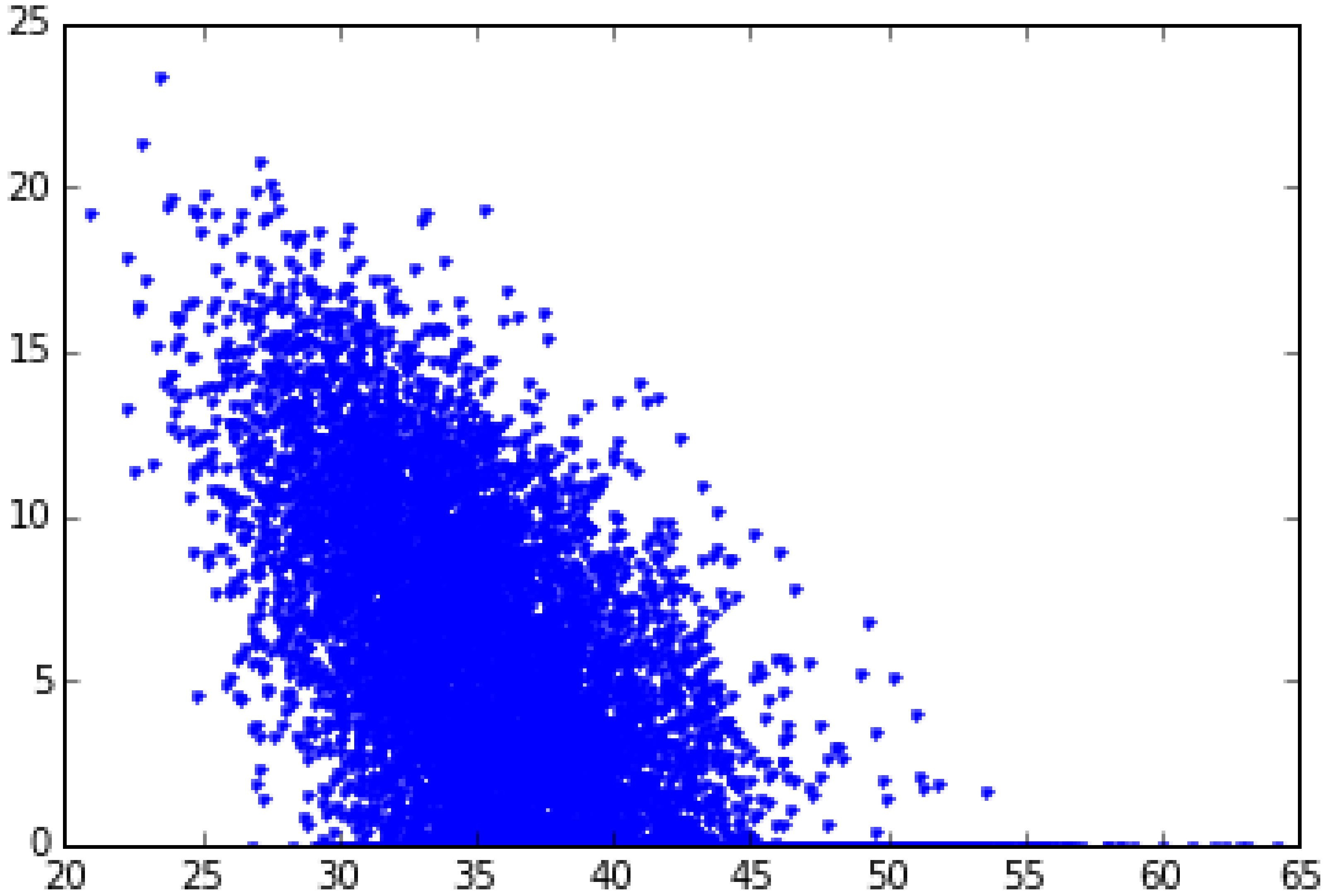
$$\frac{1}{N}\sum_i \max\left[K - S_i(t_1), P(t_1, T; S_i(t_1), K)\right]$$

- Some considerations are in order.
- We could have priced this product because we have an analytical pricing formula for the put. What if we didn't have it?

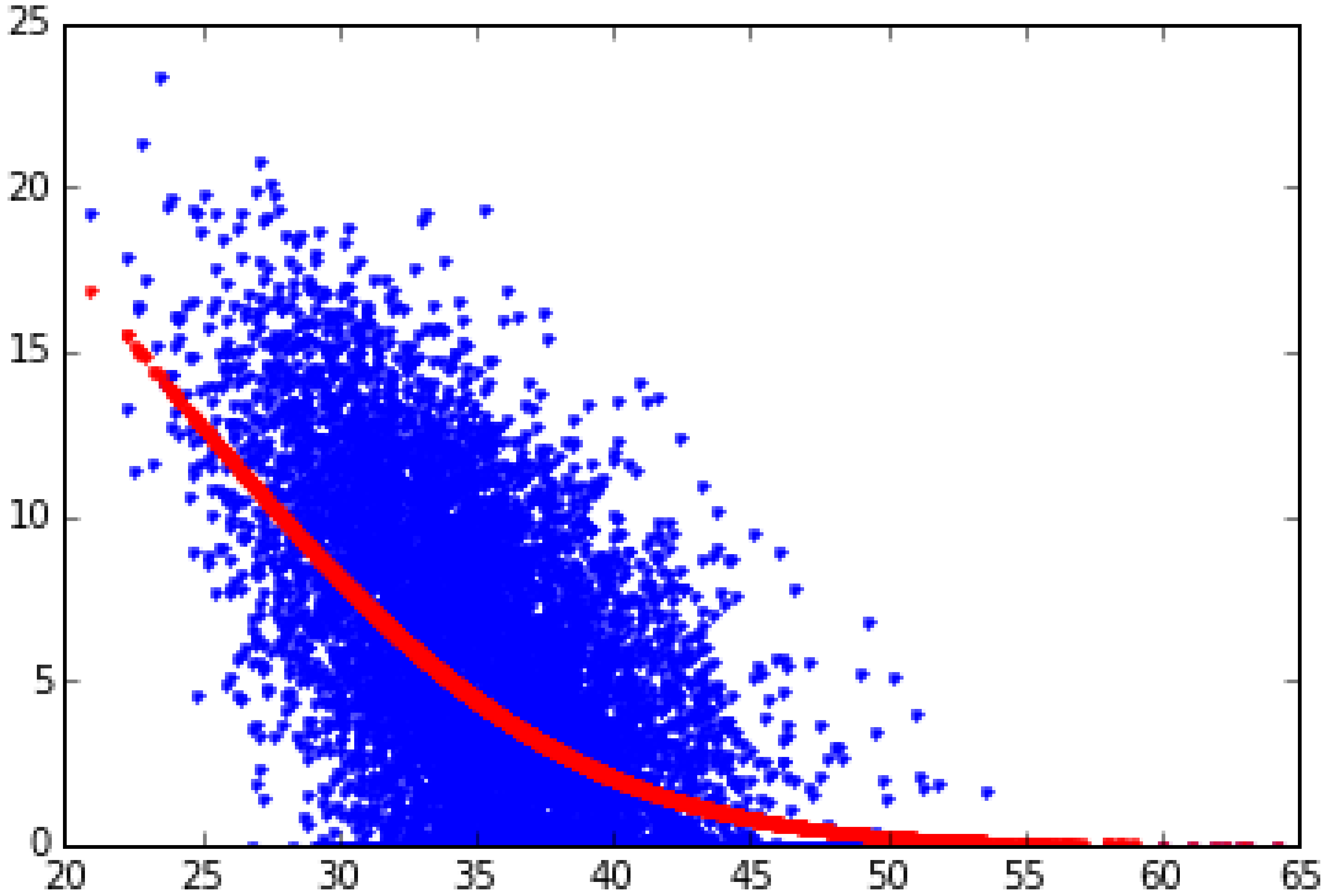
Valuation of American Option by Simulation

- Brute force solution: for each realization of $S(t_1)$ we run another Monte Carlo to price the put.
- This method (called Nested Monte Carlo) is very time consuming. For this very simple case it's time of execution grows as N^2, which becomes prohibitive when you deal with more than one exercise date!
- Let's search for a finer solution analyzing the relationship between the continuation value (in this very simple example) and the simulated realization of S at step t_1.
- let's plot the discounted payoff at maturity, P_i, versus $S_i(t_1)$ ...

Valuation of American Option by Simulation

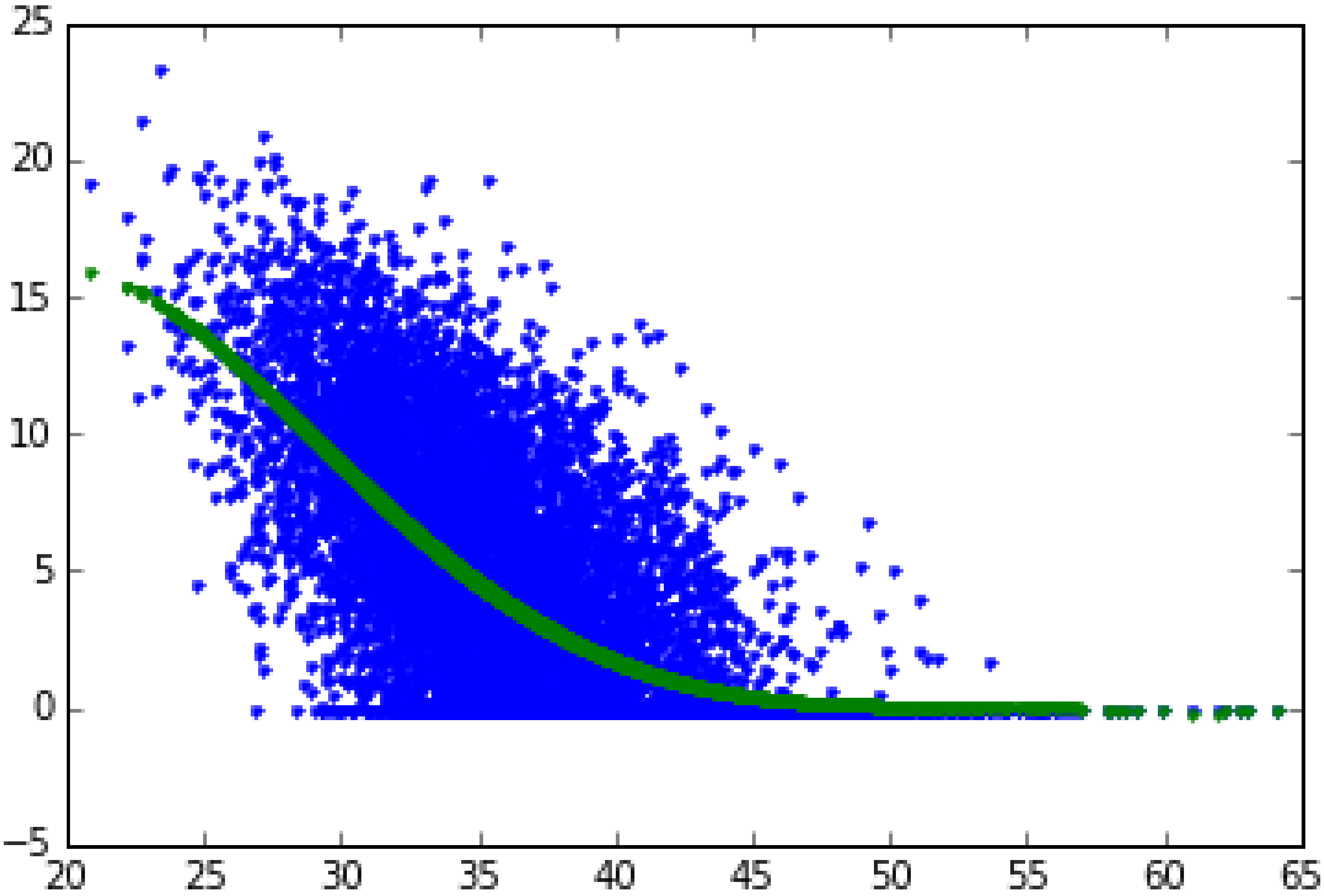


Valuation of American Option by Simulation

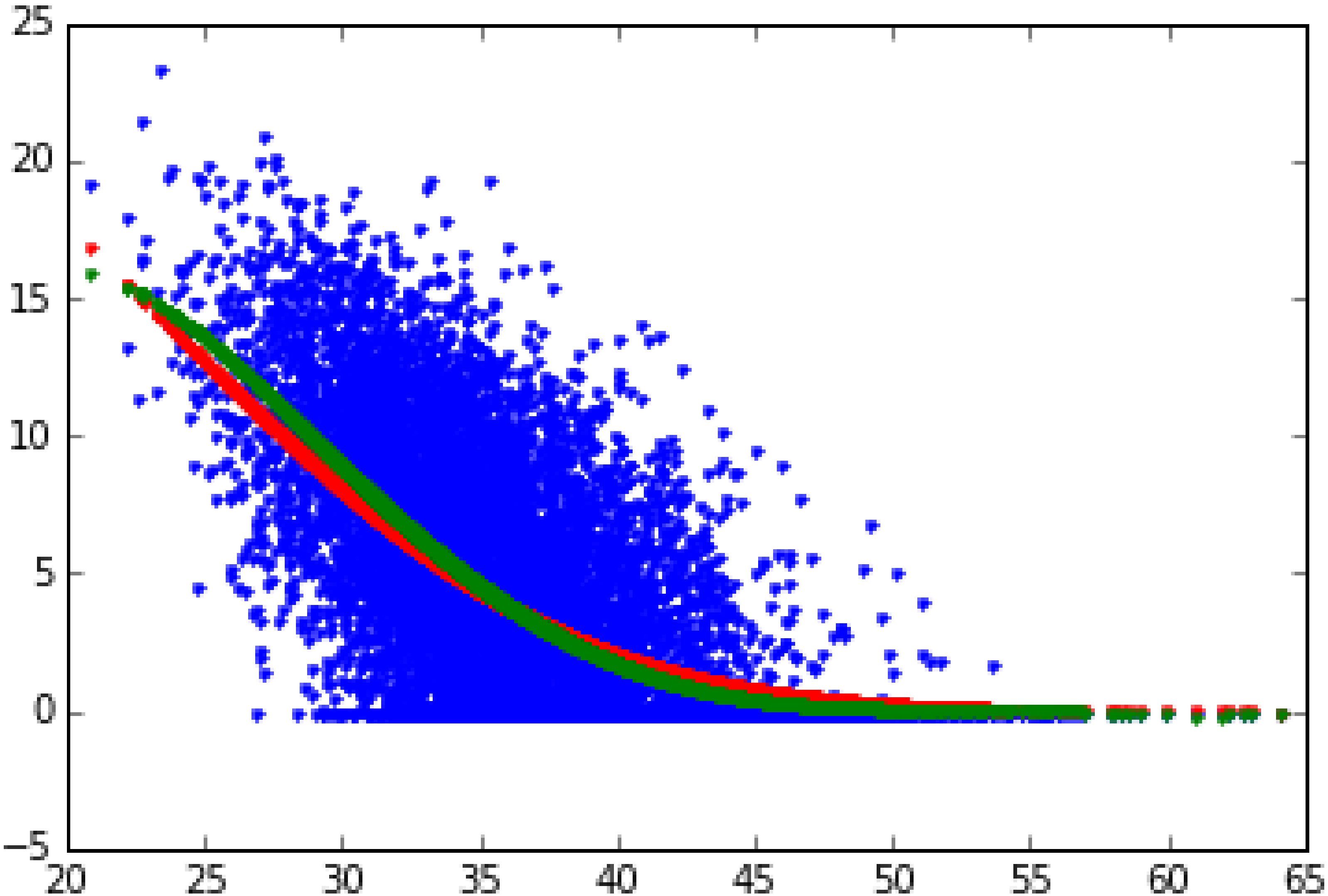


- As you can see, the analytical price of the put is a curve which kinds of interpolate the cloud of Monte Carlo points.
- This suggest us that the price at time t_1 can be computed by means of an average on all discounted payoff (i.e. the barycentre of the cloud made of discounted payoff)
- So maybe... the future value of an option can be seen as the problem of finding the curve that best fits the cloud of discounted payoff (up to date of interest)!!!
- In the next slide, for example, there is a curve found by means of a linear regression on a polynomial of 5th order...

Valuation of American Option by Simulation



Valuation of American Option by Simulation



Valuation of American Option by Simulation

- We now have an empirical pricing formula for the put to be used in my MCS

$$P(t_1, T, S(t_1), K) = c_0 + c_1 S(t_1) + c_2 S(t_1)^2 + c_3 S(t_1)^3 + c_4 S(t_1)^4 + c_5 S(t_1)^5$$

- The formula is obviously fast, the cost of the algorithm being the best fit.
- Please note that we could have used any form for the curve (not only a plynomial).
- This method has the advantage that it can be solved as a linear regression, which is fast.

The Longstaff-Schwartz Algorithm

- The major insight of Longstaff-Schwartz is to estimate the continuation value $C_{t,i}$ by ordinary least-squares regression, therefore the name "Least Square Monte Carlo" for their algorithm;
- They propose to regress the I continuation values $Y_{t,i}$ against the I simulated index levels $S_{t,i}$.
- Given D basis functions b with $b_1, \ldots, b_D : \mathbb{R}^D \rightarrow \mathbb{R}$ for the regression, the continuation value $C_{t,i}$ is according to their approach approximated by:

The Longstaff-Schwartz Algorithm

- Given D basis functions b with $b_1, \ldots, b_D : \mathbb{R}^D \to \mathbb{R}$ for the regression, the continuation value $C_{t,i}$ is according to their approach approximated by:

$$\hat{C}_{t,i} = \sum_{d=1}^{D} \alpha^{\star}_{d,t} b_d(S_{t,i}) \quad (1)$$

- The optimal regression parameters $\alpha^{\star}_{d,t}$ are the result of the minimization

$$\min_{\alpha_{1,t},\ldots,\alpha_{D,t}} \frac{1}{I} \sum_{i=1}^{I} \left(Y_{t,i} - \sum_{d=1}^{D} \alpha_{d,t} b_d(S_{t,i}) \right)^2$$

The Longstaff-Schwartz Algorithm

- Simulate I index level paths with $M+1$ points in time leading to index level values $S_{t,i}$, $t \in \{0, ..., T\}$, $i \in \{1, ..., I\}$;
- For $t = T$ the option value is $V_{T,i} = h_T(S_{T,i})$ by arbitrage
- Start iterating backwards $t = T - \Delta t, \ldots, \Delta t$:
  - regress the $T_{t,i}$ against the $S_{t,i}$, $i \in \{1, \ldots, I\}$, given D basis function b
  - approximate $C_{t,i}$ by $\hat{C}_{t,i}$ according to (1) given the optimal parameters $\alpha^\star_{d,t}$ from (2)
  - set

$$V_{t,i} = \begin{cases} h_t(S_{t,i}) & \text{if } h_t(S_{t,i}) > \hat{C}_{t,i} \quad \text{exercise takes place} \\ Y_{t,i} & \text{if } h_t(S_{t,i}) \leq \hat{C}_{t,i} \quad \text{no exercise takes place} \end{cases}$$

  repeat iteration steps until $t = \Delta t$;
- for $t = 0$ calculate the LSM estimator

$$\hat{V}_0^{LSM} = e^{-r\Delta t} \frac{1}{I} \sum_{i=1}^{I} V_{\Delta t,i}$$

Notebook

- **GitHub :** polyhedron-gdl;
- **Notebook :** mcs_american;

Outline

Hull-White Model

- Described by the SDE for the short rate

$$dr = (\theta(t) - ar)dt + \sigma dw \tag{1}$$

- See Brigo-Mercurio ...
- Our version simplified: a and σ constant;
- AKA Extended Vasicek (Note: $r(t)$ is Gaussian);
- θ determined uniquely by term structure;

Hull-White Model: Solving for r(t)

$$d(e^{at}r) = e^{at}dr + ae^{at}rdt = \theta(t)e^{at} + e^{at}\sigma dw$$

integrating both sides we obtain

$$e^{at}r(t) = r(0) + \int_0^t \theta(s)e^{as}ds + \sigma \int_0^t e^{as}dw(s)$$

simplify

$$r(t) = r(0)e^{-at} + \int_0^t \theta(s)e^{-a(t-s)}ds + \sigma \int_0^t e^{-a(t-s)}dw(s)$$

Hull-White Model: Solving for P(t,T)

- $P(t,T) = V(t, r(t))$ where V solves the PDE

$$V_t + (\theta(t) - ar)V_r + \frac{1}{2}\sigma^2 V_{rr} - rV = 0$$

- Final-time condition $V(T,r) = 1$ for all r at $t = T$;
- Ansatz:

$$V = A(t,T)e^{-B(t,T)r(t)}$$

- A and B must satisfy:

$$A_t - \theta(t)AB + \frac{1}{2}\sigma^2(AB)^2 = 0, \quad \textit{and} \quad B_t - aB + 1 = 0$$

- Final-time conditions

$$A(T,T) = 1 \quad \textit{and} \quad B(T,T) = 0$$

Hull-White Model: Solving for P(t,T)

- B independent of θ so

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right) \tag{2}$$

- Solving for A requires integration of θ

$$A(t,T) = exp\left[-\int_t^T \theta(s)B(s,T)ds - \frac{\sigma^2}{2a^2}\left(B(t,T) - T + t\right) - \frac{\sigma^2}{4a}B(t,T)^2\right]$$

Hull-White Model: Determining θ

- Determining θ from the term structure at time 0;
- Goal: demonstrate the relation

$$\theta(t) = \frac{\partial f}{f \partial T}(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{3}$$

- Recall

$$f(t,T) = -\partial \log P(t,T)/\partial T$$

Hull-White Model: Determining θ

- We have

$$-\log P(0,T) = \int_0^T \theta(s)B(s,T)ds + \frac{\sigma^2}{2a^2}[B(0,T) - T] + \frac{\sigma^2}{4a}B(0,T)^2 + B(0,T)r_0$$

- Differentiating and using that $B(T,T) = 1$ and $\partial_T B - 1 = -aB$ we get

$$f(0,T) = \int_0^T \theta(s)\partial_T B(s,T)ds - \frac{\sigma^2}{2a^2}B(0,T) + \frac{\sigma^2}{2a^2}B(0,T)\partial_T B(0,T) + \partial_T B(0,T)r_0$$

Hull-White Model: Determining θ

- Differentiating again, get:

$$
\begin{aligned}
\partial_T f(0,T) = \theta(T) + \int_0^T \theta(s)\partial_{TT} B(s,T)ds \\
- \frac{\sigma^2}{2a^2}\partial_T B(0,T) \\
+ \frac{\sigma^2}{2a^2}[(\partial_T B(0,T))^2 + B(0,T)\partial_{TT} B(0,T)] \\
+ \partial_{TT} B(0,T) r_0
\end{aligned}
\tag{4}
$$

Hull-White Model: Determining θ

- Combine these equations, and use $a\partial_T B + \partial_{TT} B = 0$;
- Get:

$$af(0,T) + \partial_T f(0,T) = \theta(T) - \frac{\sigma^2}{2a}(aB + \partial_T B) + \frac{\sigma^2}{2a}[aB\partial_T B + (\partial_T B)^2 + B\partial_{TT} B]$$

- Substitute formula for B and simplify to get

$$af(0,T) + \partial_T f(0,T) = \theta(T) - \frac{\sigma^2}{2a}(1 - e^{-2aT})$$

QED

Additive Factor Gaussian Model

- The model is given by dynamics (Brigo-Mercurio p. 143):

$$r(t) = x(t) + \phi(t)$$

where

$$dx(t) = -ax(t)dt + \sigma dW_t \quad x(0) = 0$$

and ϕ is a deterministic shift which is added in order to fit exactly the initial zero coupon curve

Additive Factor Gaussian Model

- So the short rate $r(t)$ is distributed normally with mean and variance given by (Brigo-Mercurio p.144 equations 4.6 with $\eta = 0$)

$$E(r_t|r_s) = x(s)e^{-a(t-s)} + \phi(t)$$

$$Var(r_t|r_s) = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right)$$

where $\phi(T) = f^M(0,T) + \frac{\sigma^2}{2a}\left(1 - e^{-aT}\right)^2$ and $f^M(0,T)$ is the market instantaneous forward rate at time t as seen at time 0.

Additive Factor Gaussian Model

- Model discount factors are calculated as in Brigo-Mercurio (section 4.2):

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left(\mathcal{A}(t,T)\right)$$

$$\mathcal{A}(t,T) = \frac{1}{2}\left[V(t,T) - V(0,T) + V(0,t)\right] - \frac{1-e^{-a(T-t)}}{a}x(t)$$

where

$$V(t,T) = \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]$$

Outline

Counterparty Risk

- In general a given company, say a financial institution A, will have portfolios with many other counterparties, varying among sovereign entities, corporates, hedge funds, insurance companies;
- Counterparty credit exposure is the amount a company, say A, could potentially lose in the event of one of its counterparties defaulting.
- As we'll see in more details, it can be computed by simulating in different scenarios and at different times in the future, the price of the transactions with the given counterparty, and then by using some chosen statistic to characterise the price distributions that have been generated.

CVA Context

- Counterparty risk can be considered broadly from two different points of view;
- The first point of view is risk management, leading to capital requirements revision, trading limits discussions and so on;
- The second point of view is valuation or pricing, leading to amounts called Credit Valuation Adjustment (CVA) and extensions thereof, including netting, collateral, re-hypothecation, close-out specification, wrong way risk and funding cost;

Example: CVA of plain vanilla Interest-Rate Swap

- Consider counterparties A and B who enter into an interest-rate swap where A receives every six months the 6-month Libor rate on a notional of 100 million euro, while paying to B a fixed amount equal to the par 10-year swap rate on the same notional observed at inception.
- This is a typical swap contract with value zero at inception.
- As time passes and market conditions change, the value of the swap changes accordingly. Thus, if the swap rate decreases (resp. increases), the transaction will be out of the money (resp. in the money) as seen from A's point of view.

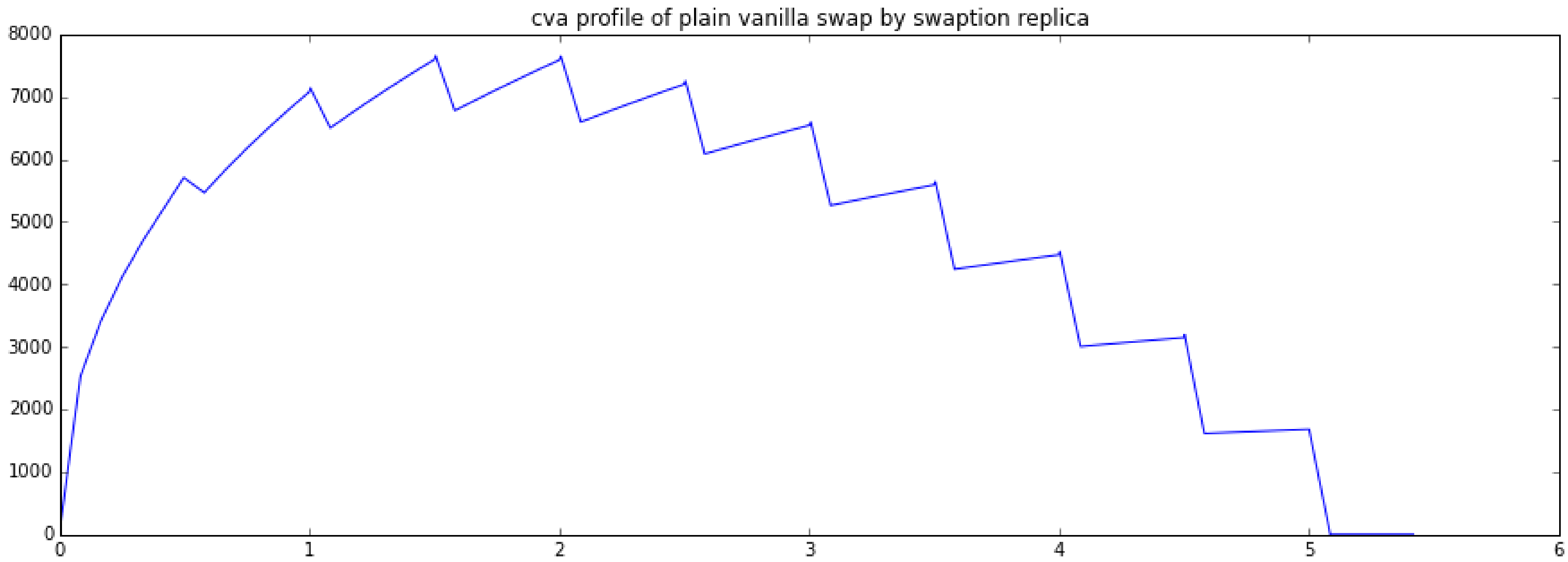
Example: CVA of plain vanilla Interest-Rate Swap

- Therefore, if B were to default at a point in the life of the trade when swap rates had increased, then A would need to replace in the market—at higher cost than the fixed amount being paid to B—the floating cashflows promised and not delivered by B.
- To compute the credit exposure for the swap, we would need to estimate the values the swap could take in different market scenarios at points in the future.
- For practical reason it can be useful to characterise the distribution of values with some quantity which can be useful for various risk controlling;
- For example we could compute the so called **Expected Positive Exposure** (more in the following) as

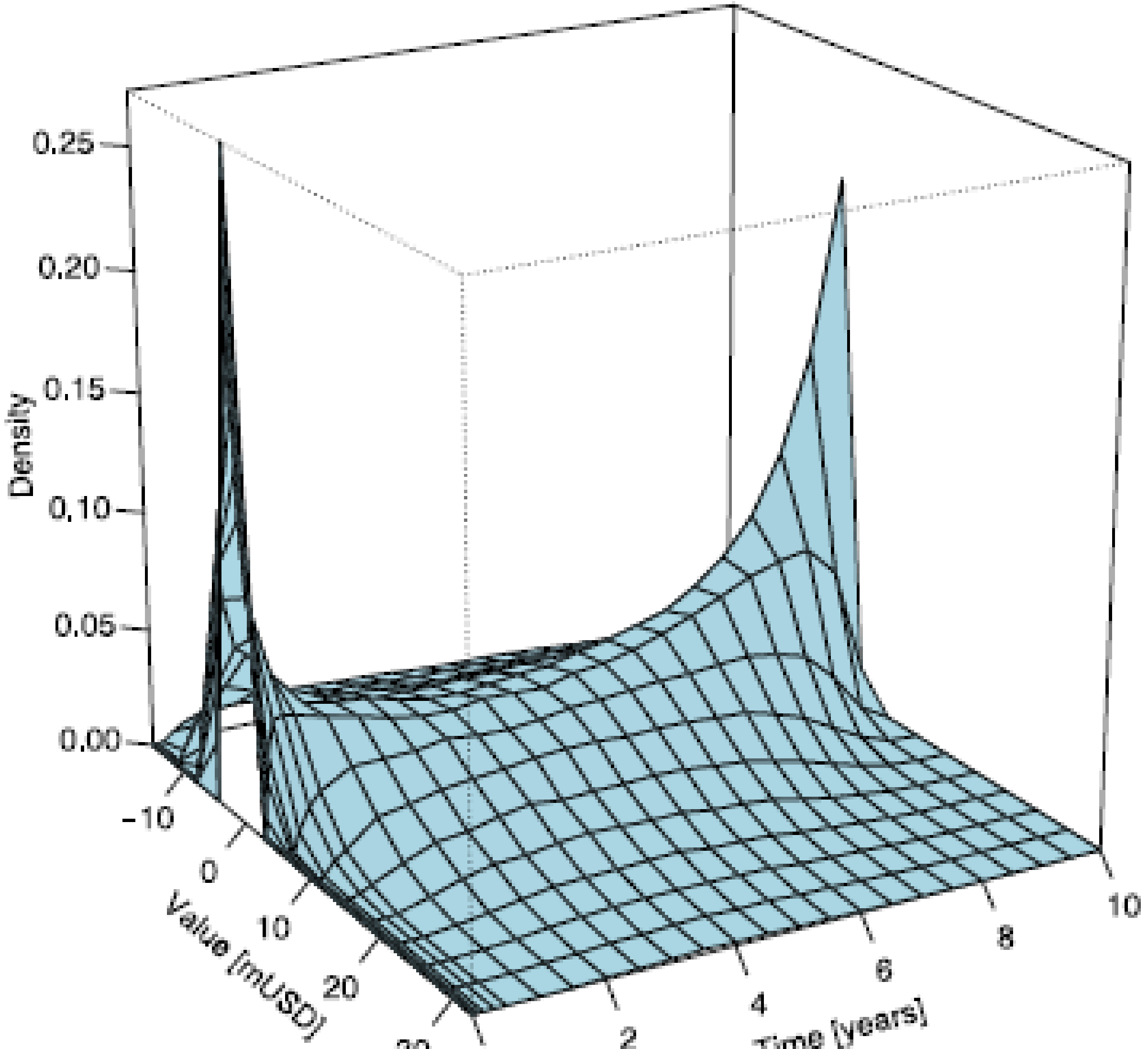
$$EPE_t = \mathbb{E}[V_t^+]$$

Exposure Profile: EPE

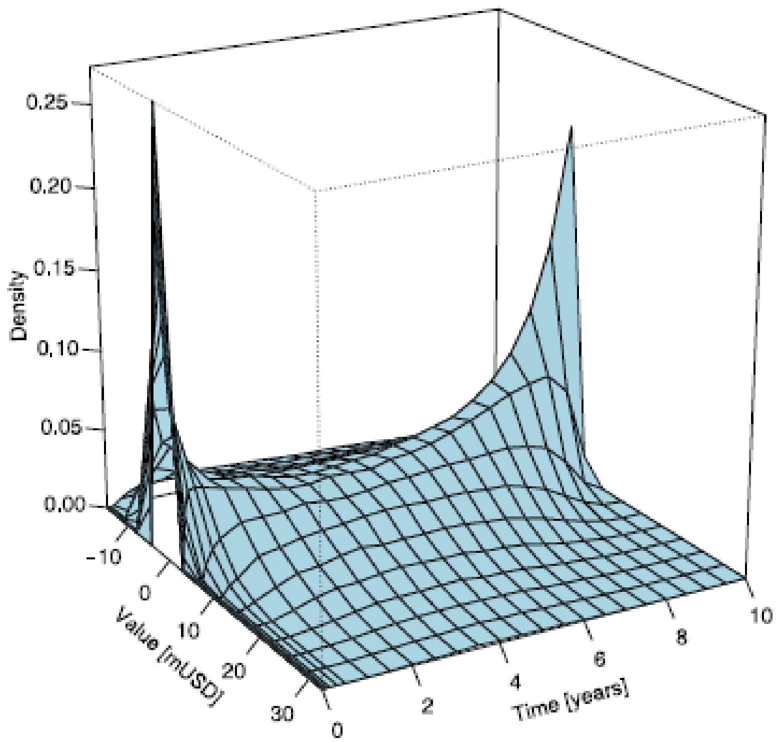
- Figure shows the EPE of the swap price distribution, over its entire life, as seen from party A's point of view.



Exposure Profile: Value Density

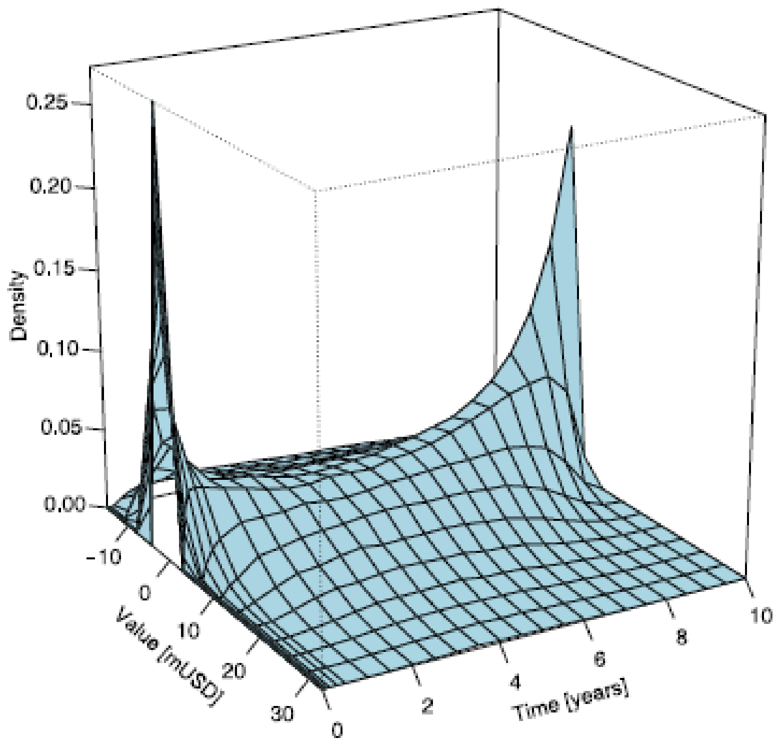


Exposure Profile



- Figure shows the full price distribution over time.
- The fair value for the swap, and this must therefore have value (and hence exposure level) identically equal to zero at inception.
- Similarly, towards the end of the transaction, when all payments but one due under the swap have been paid, the exposure remaining is that from only a single coupon exchange.

Exposure Profile



- This explains what happens at the right end of the profile.
- At intermediate times, the shape of the profile is the result of opposing effects.
- On the one hand, as the interest rates underlying the swap diffuse, there is more variability in the realised Libor rates, potentially leading to higher exposure.
- On the other hand, as time evolves there are fewer payments remaining under the swap, and this mitigates the effect of diffusing rates.

Exposure Profiles

- So there are two main effects that determine the credit exposure over time for a single transaction or for a portfolio of transactions with the same counterparty: diffusion and amortization;
- as time passes, the "diffusion effect" tends to increase the exposure, since there is greater variability and, hence, greater potential for market price factors to move significantly away fro current levels;
- the "amortization effect", in contrast, tends to decrease the exposure over time, because it reduces the remaining cash flows that are exposed to default;

Exposure Profiles

- For single cash flow products, such as FX Forwards, the potential exposure peaks at the maturity of the transaction, because it is driven purely by "diffusion effect";
- on the other hand, for products with multiple cash flows, such as interest-rate swaps, the potential exposure usually peaks at one-third to one-half of the way into the life of the transaction;

Credit Value Adjustment

- **Credit Value Adjustment (CVA)** is by definition the difference between the risk-free portfolio value and the true portfolio value that takes into account the possibility of a counterparty's default;
- in other words CVA is the market value of counterparty credit risk;

Credit Value Adjustment

- Assuming independence between exposure and counterparty's credit quality greatly simplifies the analysis.
- under this assumption we can write

$$CVA = (1 - R) \int_0^T EE^\star(t) dPD(0, t) \tag{5}$$

where $EE^\star(t)$ is the risk-neutral discounted expected exposure (EE) given by

$$EE^\star(t) = E^Q \left[\frac{B_0}{B_t} E(t) \right] \tag{6}$$

Credit Value Adjustment

- In general calculating discounted EE requires simulations;
- Exposure is simulated at a fixed set of simulation dates, therefore the integral in (5) has to be approximated by the sum:

$$CVA = (1 - R) \sum_{i=1}^{N} EE^{\star}(t_k) PD(t_{k-1}, t_k) \tag{7}$$

- Since expectation in (6) is risk-neutral, scenario models for all price factors should be arbitrage free;
- This is achieved by appropriate calibration of drifts and volatilities specified in the price-factor evolution model;

The General Unilateral Counterparty Risk Pricing Formula

- At valuation time t, and provided the counterparty has not defaulted before t, i.e. on $\{\tau > t\}$, the price of our payoff with maturity T under counterparty risk is

$$\begin{aligned}\mathbb{E}_t[\bar{\Pi}(t,T)] &= \mathbb{E}_t[\Pi(t,T)] - \underbrace{\mathbb{E}_t[LGD\,\mathbb{I}_{t\leq\tau\leq T}D(t,\tau)(NPV(\tau))^+]}_{\text{positive counterparty-risk adjustment}} \\ &= \mathbb{E}_t[\Pi(t,T)] - U_{CVA}(t,T)\end{aligned} \tag{8}$$

with

$$\begin{aligned}U_{CVA}(t,T) &= \mathbb{E}_t[LGD\,\mathbb{I}_{t\leq\tau\leq T}D(t,\tau)(NPV(\tau))^+] \\ &= \mathbb{E}_t[LGD\,\mathbb{I}_{t\leq\tau\leq T}D(t,\tau)EAD]\end{aligned} \tag{9}$$

Where $LGD = 1 - REC$ is the loss given default, and the recovery fraction REC is assumed to be deterministic.

The General Unilateral Counterparty Risk Pricing Formula

- It is clear that the value of a defaultable claim is the sum of the value of the corresponding default-free claim minus a positive adjustment.
- The positive adjustment to be subtracted is called (Unilateral) Credit Valuation Adjustment (CVA), and it is given by a call option (with zero strike) on the residual NPV at default, giving nonzero contribution only in scenario where $\tau \leq T$.

The General Unilateral Counterparty Risk Pricing Formula

- Counterparty risk thus adds and optionality level to the original payoff.
- This renders the counterparty risk payoff model dependent even when the original payoff is model independent.
- This implies, for example, that while the valuation of swaps without counterparty risk is model independent, requiring no dynamical model for the term structure (no volatility and correlations in particular), the valuation of swaps under counterpaty risk will require and interest rate model.

The General Unilateral Counterparty Risk Pricing Formula

- Now we explore the well known result that the component of the IRS price due to counterparty risk is the sum of swaption prices with different maturities, each weighted with the probability of defaulting around that maturity.
- Let us suppose that we are a default free counterparty "B" entering a payer swap with a defaultable counterparty "C", exchanging fixed for floating payments at times $T_{a+1}, \ldots, T_b$.
- Denote by β_i the year fraction between T_{i-1} and T_i, and by $P(t, T_i)$ the default-free zero coupon bond price at time t for maturity T_i. We take a unit notional on the swap.
- The contract requires us to pay a fixed rate K and to receive the floating rate L resetting one period earlier until the default time τ of "B" or until final maturity T if $\tau > T$.
- The fair forward-swap rate K at a given time t in a default-free market is the one which renders the swap zero-valued in t.

The General Unilateral Counterparty Risk Pricing Formula

- In the risk-free case the discounted payoff for a payer IRS is

$$\sum_{i=a+1}^{b} D(t, T_i)\beta_i(L(T_{i-1}, T_i) - K) \tag{10}$$

- and the forward swap rate rendering the contract fair is

$$K = S(t; T_a, T_b) = S_{a,b}(t) = \frac{P(t, T_a) - P(t, T_b)}{\sum_{i=a+1}^{b} P(t, T_i)\beta_i}$$

The General Unilateral Counterparty Risk Pricing Formula

- Of course if we consider the possibility that "C" may default, the correct spread to be paid in the fixed leg is lower as we are willing to be rewarded for bearing this default risk.
- In particular we have

$$
\begin{aligned}
U_{CVA}(t, T_b) &= LGD\, \mathbb{E}_t[\mathbb{I}_{\tau \le T_b} D(t,\tau)(NPV(\tau))^+] \\
&= LGD \int_{T_a}^{T_b} PS\left(t; s, T_b, K, S(t; s, T_b), \sigma_{s,T_b}\right)\, d_s\mathbb{Q}\{\tau \le s\}
\end{aligned}
\tag{11}
$$

being $PS\left(t; s, T_b, K, S(t; s, T_b), \sigma_{s,T_b}\right)$ the price in t of a swaption with maturity s, strike K underlying forward swap rate $S(t; s, T_b)$, volatility σ_{s,T_b} and underlying swap with final maturity T_b.

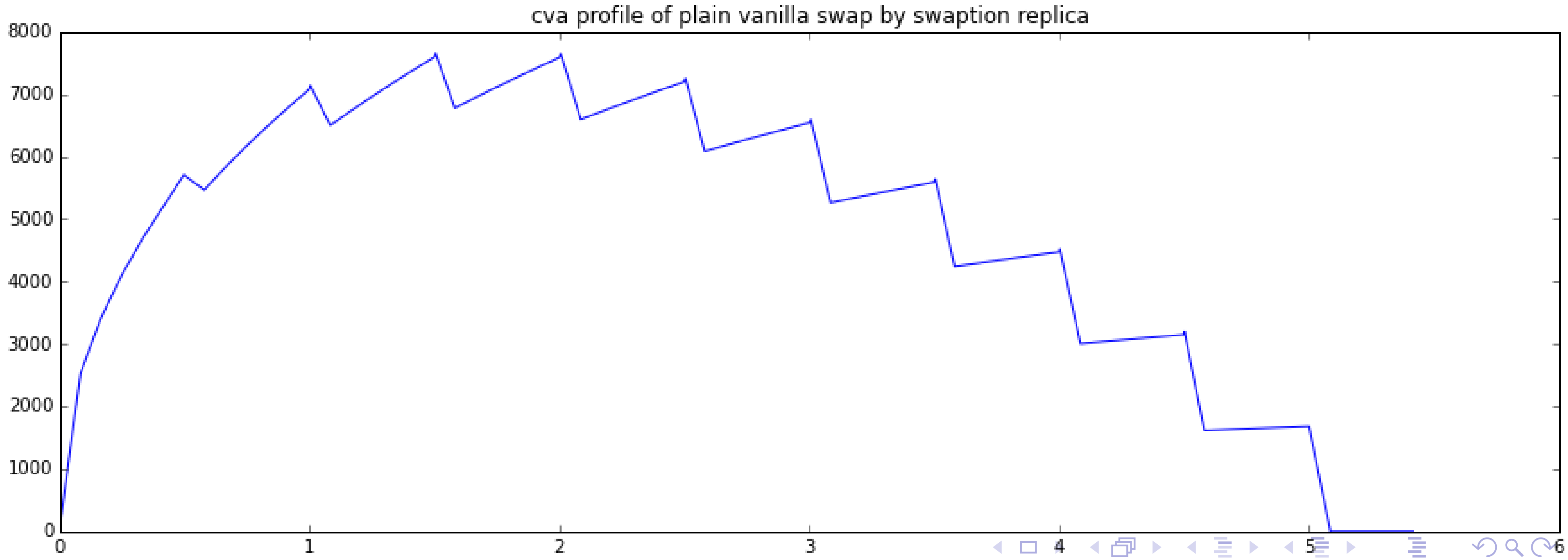
The General Unilateral Counterparty Risk Pricing Formula

- The proof is the following: given independence between τ and the interest rates, and given that the residual NPV is a forward start IRS starting at the default time, the option on the residual NPV is a sum of swaptions with maturities ranging the possible values of the default time, each weighted (thanks to the independence assumption) by the probabilities of defaulting around each time value.
- We can simplify formulas allow the default to happen only at points T_i of the fixed leg payment grid.
- In this way the expected loss part is simplified.

The General Unilateral Counterparty Risk Pricing Formula

- Indeed in the case of postponed (default occour to the first T_i following τ) payoff we obtain:

$$U_{CVA}(t, T_b) = LGD\,\mathbb{E}_t[\mathbb{I}_{\tau \le T_b} D(t,\tau)(NPV(\tau))^+]$$
$$= LGD \sum_{i=a+1}^{b-1} PS\left(t; s, T_b, K, S(t; s, T_b), \sigma_{s,T_b}\right) \left(\mathbb{Q}\{\tau \geq T_i\} - \mathbb{Q}\{\tau > T_i\}\right) \tag{12}$$



Notebook

- **GitHub :** polyhedron-gdl;
- **Notebook :** n09_cva_swap;

Monte Carlo Framework for CVA

The industry practice to compute exposure is to use a simple Monte Carlo framework implemented in three steps:

1. scenario generation,
2. pricing
3. aggregation.

Monte Carlo Framework for CVA

- The first step involves generating scenarios of the underlying risk factors at future points in time.
- Simple products can then be priced on each scenario and each time step, therefore generating empirical price distributions.
- From the price distribution at each time it is then possible to extract convenient statistical quantities.
- Exposure of portfolios can be computed by consistently pricing different products on the same underlying scenarios and aggregating the results taking into account possible netting and collateral agreement with the counterparty.

Monte Carlo Framework for CVA

- If taken literally, this approach works only for relatively simple products which can be priced analytically, or which can be approximated in analytical form, and which do not need complex calibrations depending on market scenarios.
- More exotic products requiring relatively complex pricing, cannot be treated in this way.

CVA of a Plain Vanilla Swap: the Simulation Approach

1. Simulate yield curve at future dates
2. Calculate your derivatives portfolio NPV (net present value) at each time point for each scenario
3. Calculate CVA as sum of Expected Exposure multiplied by probability of default at this interval

$$CVA = (1 - R) \int DF(t)EE(t)dQ_t$$

where R is the Recovery Rate (normally set to 40%) $EE(t)$ is the expected exposure at time t and dQ_t the survival probability density, $DF(t)$ is the discount factor at time t.

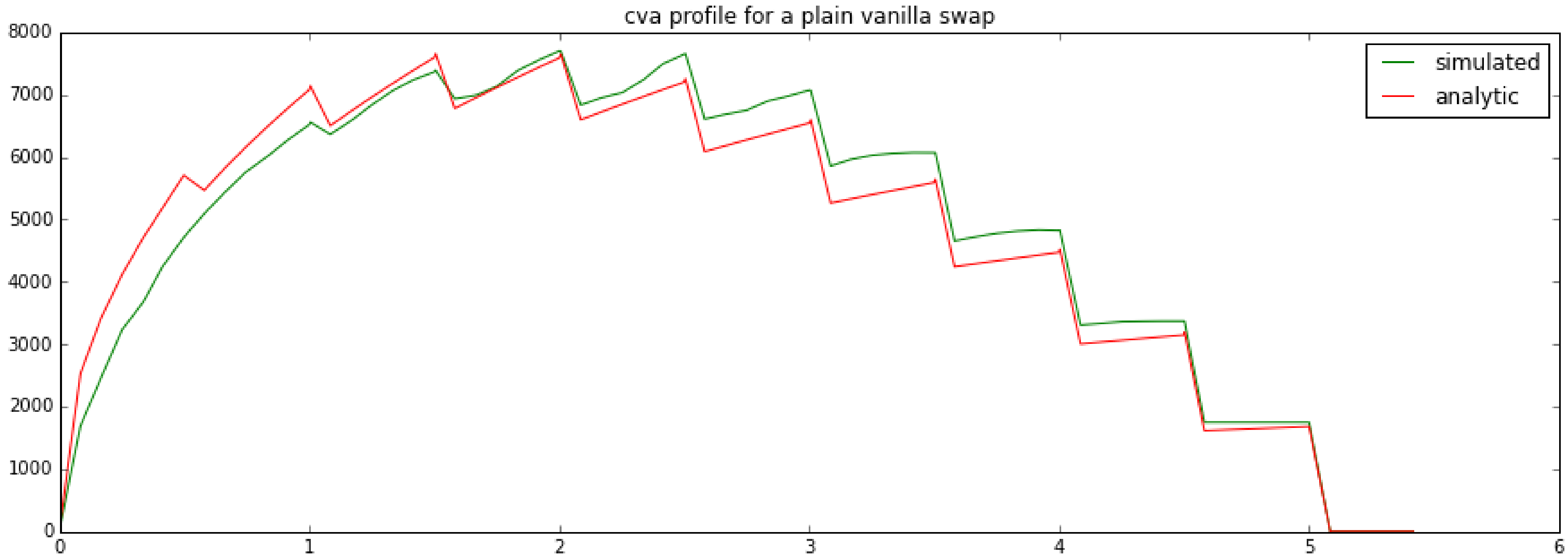
CVA of a Plain Vanilla Swap: the Simulation Approach

- In this simple example we will use a modified version of Hull White model to generate future yield curves. In practice many banks use some yield curve evolution models based on this model.
- For each point of time we will generate whole yield curve based on short rate. Then we will price our interest rate swap on each of these curves;
- To approximate CVA we will use BASEL III formula for regulatory capital charge approximating default probability [or survival probability] as $exp(-S_T/(1-R))$ so we get

$$CVA = (1-R)\sum_i \frac{EE(T_i)^\star + EE(T_{i-1}^\star}{2}\left(e^{-S(T_{i-1})/(1-R)} - e^{-S(T_i)/(1-R)}\right)$$

where $EE^\star$ is the discounted Expected Exposure of portfolio.

CVA of a Plain Vanilla Swap: the Simulation Approach



Notebook

- **GitHub :** polyhedron-gdl;
- **Notebook :** n10_mcs_cva_swap;

Implementation Challenges

- The Monte Carlo framework we have shown in the previous section, seems to give a good implementation recipe.
- For a given portfolio of transactions we could identify the underlying risk factors and simulate forward (or spot) prices, taking into account correlations if required, use functions already implemented to price each product, and then derive statistical quantities.
- As we have mentioned already, this could be the approach followed by a financial institution to assess the counterparty credit risk of its OTC derivatives portfolios.

Implementation Challenges

- In the implementation phase, however, there can be issues which need to be addressed.
- The generation of correlated scenarios is not trivial, as there can be thousands of different risk factors driving the dynamics of products in the portfolio.
- Consider for example an equity portfolio, where each underlying stock needs, at least in principle, a specific simulation.

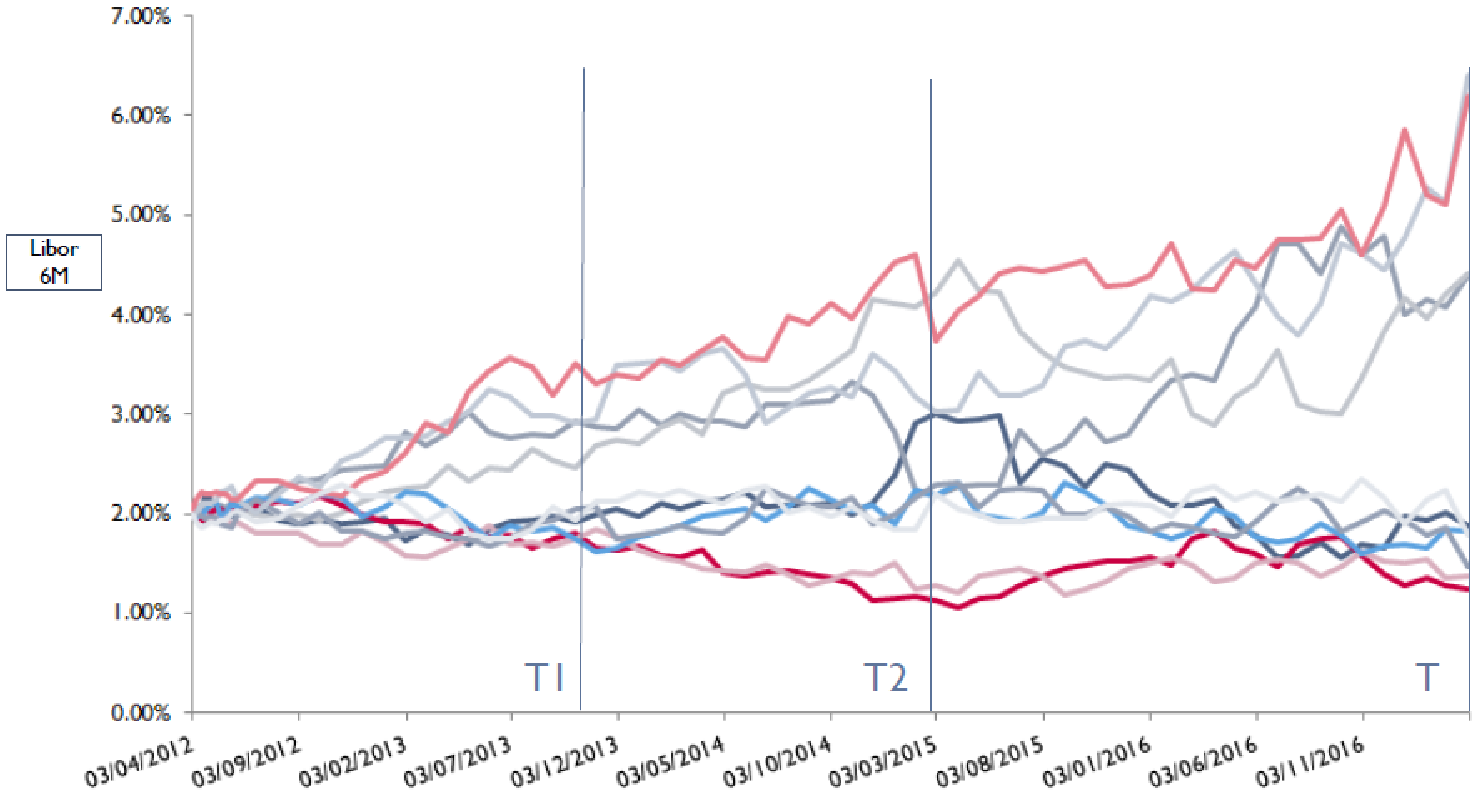
Implementation Challenges

- Not all products can be computed in analytical form.
- Most exotics are priced on grids using PDEs or using Monte Carlo approaches.
- In these cases the exposure computation would require a Monte Carlo simulation for scenarios and a Monte Carlo simulation, or a PDE computation, for each scenario and time step to price the instrument.
- This becomes quickly unfeasible from a computational point of view.
- In addition, depending on the model used for pricing, calibration could also become problematic, as it has to be performed at each scenario.

American Monte Carlo

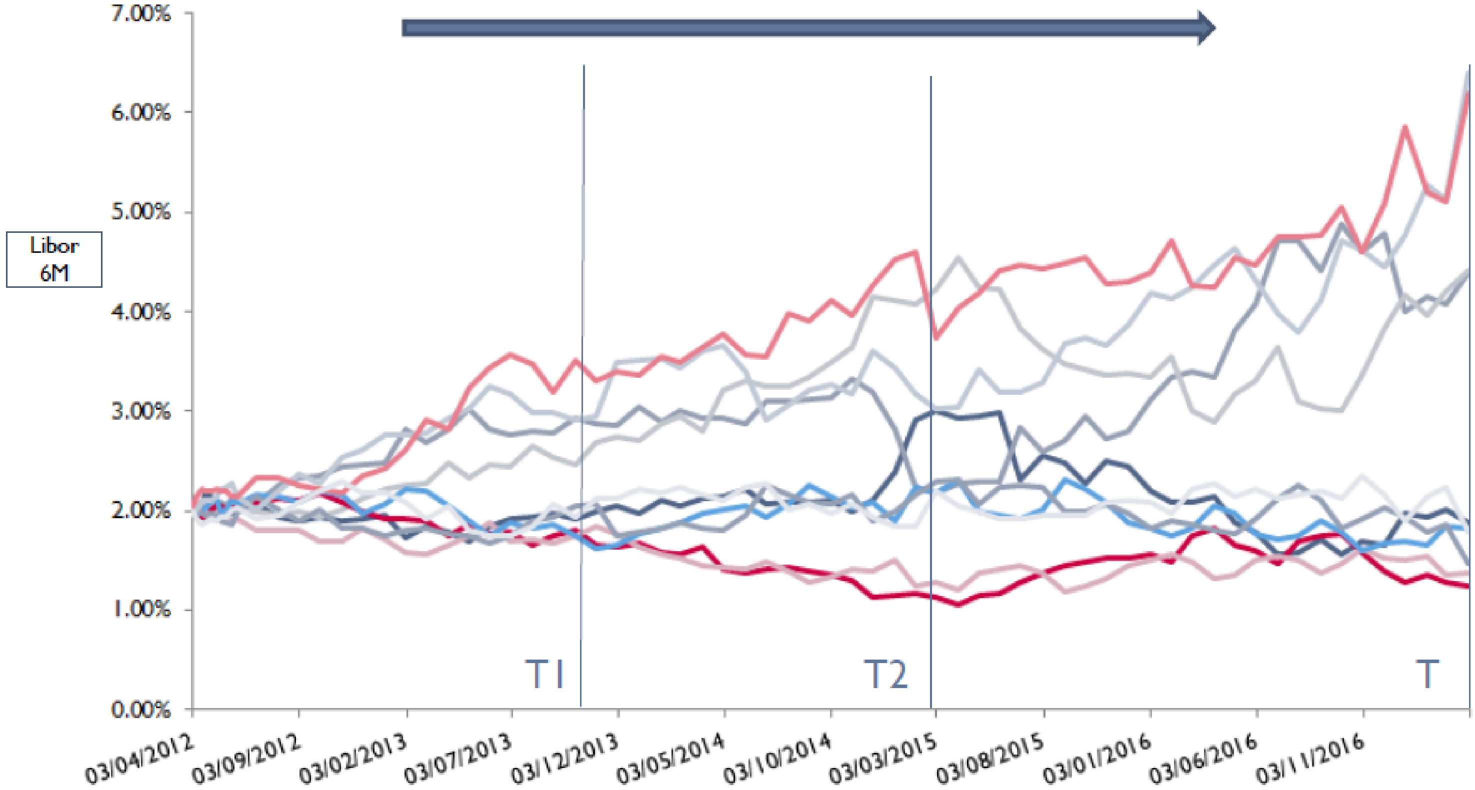
- The points highlighted in the previous section clearly show that the classical Monte Carlo scheme has intrinsic limitations and that we need an alternative approach.
- There are possibilities to circumvent in a systematic way some of the problems related to valuation and architecture.
- The basic idea is to approach the counterparty exposure problem as a pricing problem, and thus to use pricing algorithms, which generate not just the value of a trade at inception, but rather a price distribution at predetermined time steps.
- One possibility is to use the so called American Monte Carlo algorithm, which we will refer to as, simply, the AMC algorithm.
- The main feature of this algorithm is that, instead of building a price moving forward in time, it starts from maturity, where the value of the transaction is known, and goes backwards, till the inception.

AMC CVA Simulation



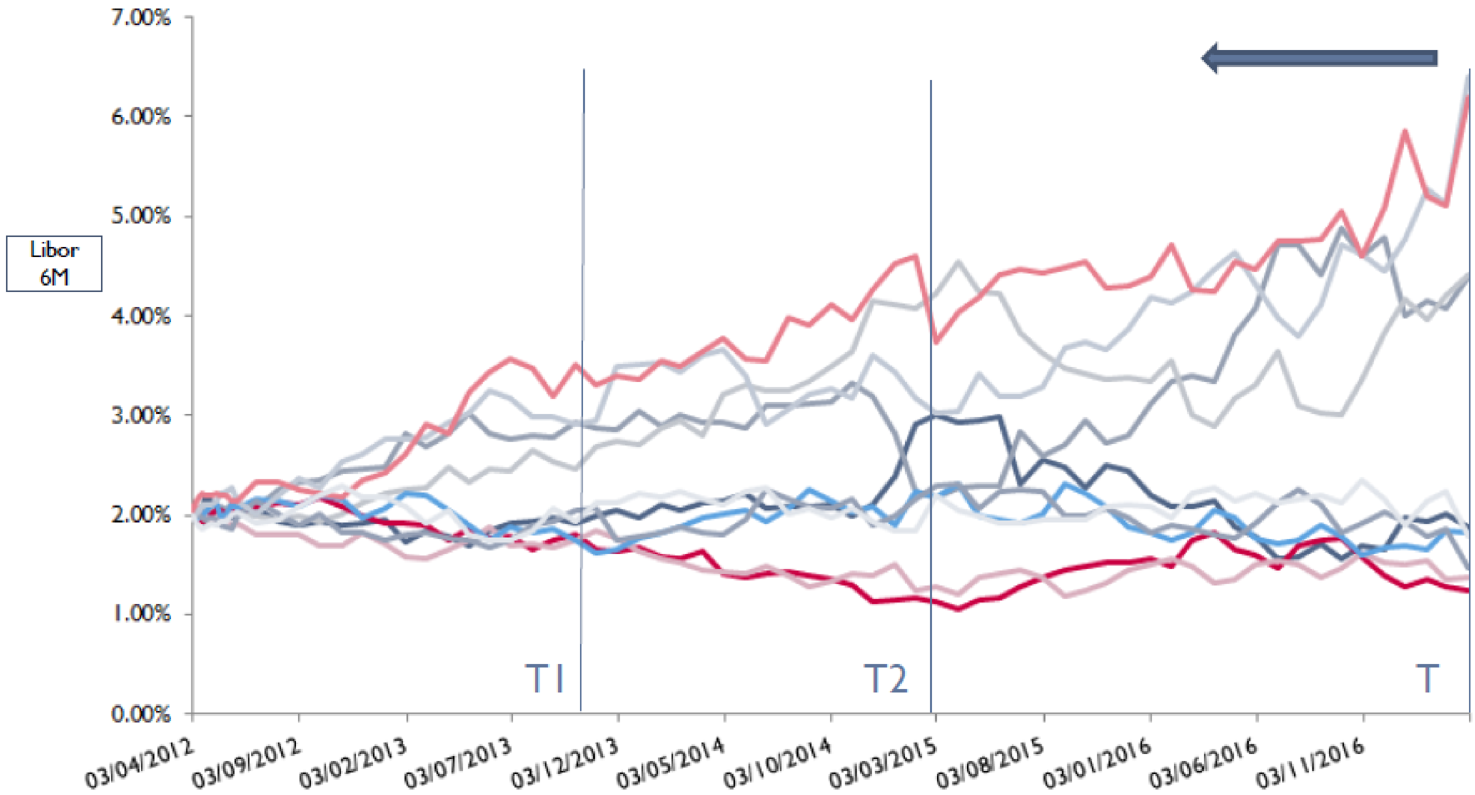
- Let's illustrate with a Bermuda Swaption which can be exercised at 2 dates, T1 and T2.

AMC CVA Simulation



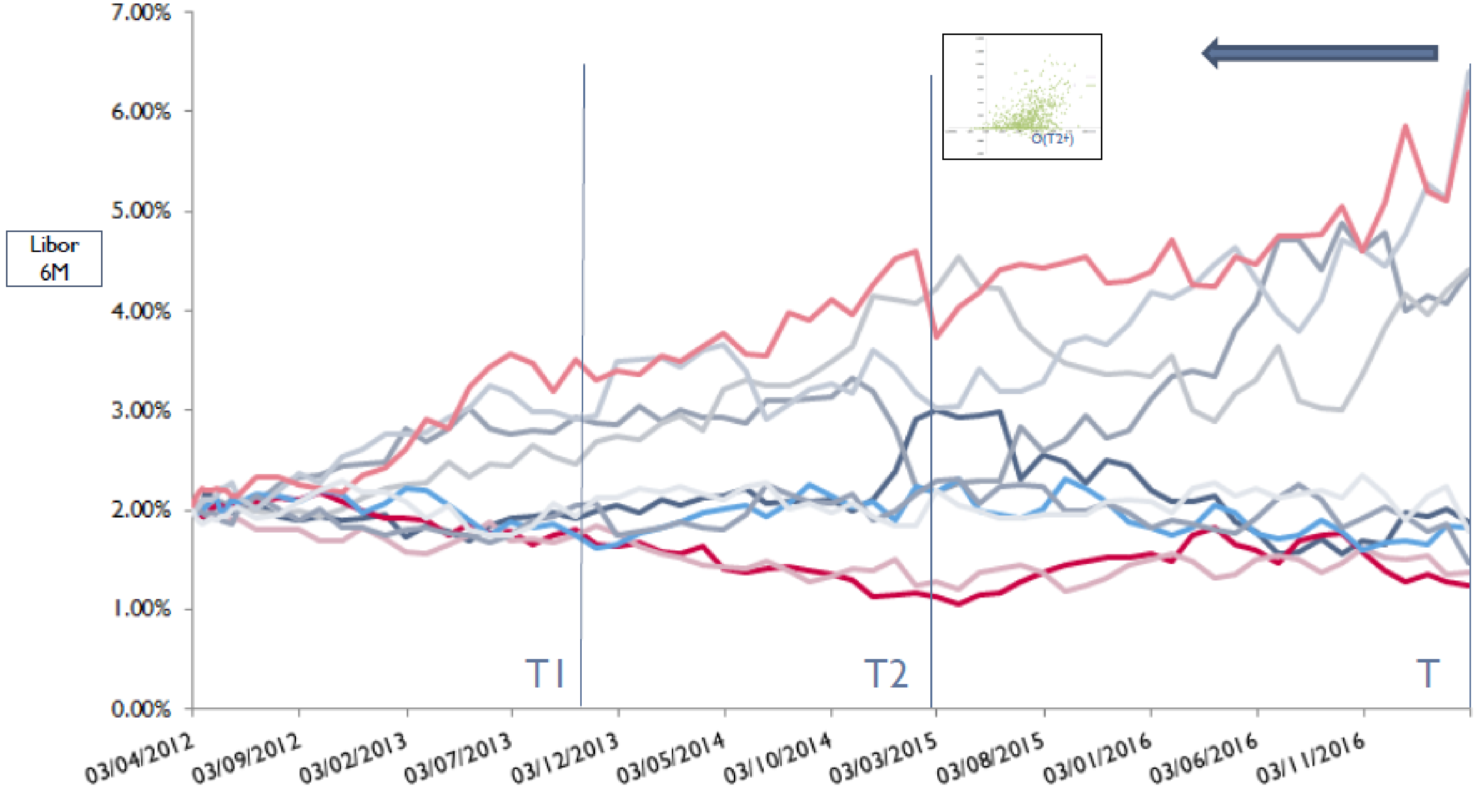
- Forward Phase: we diffuse risk factors and store in memory contract cash flows and payoff variables as the will be used in the Backward Phase.

AMC CVA Simulation



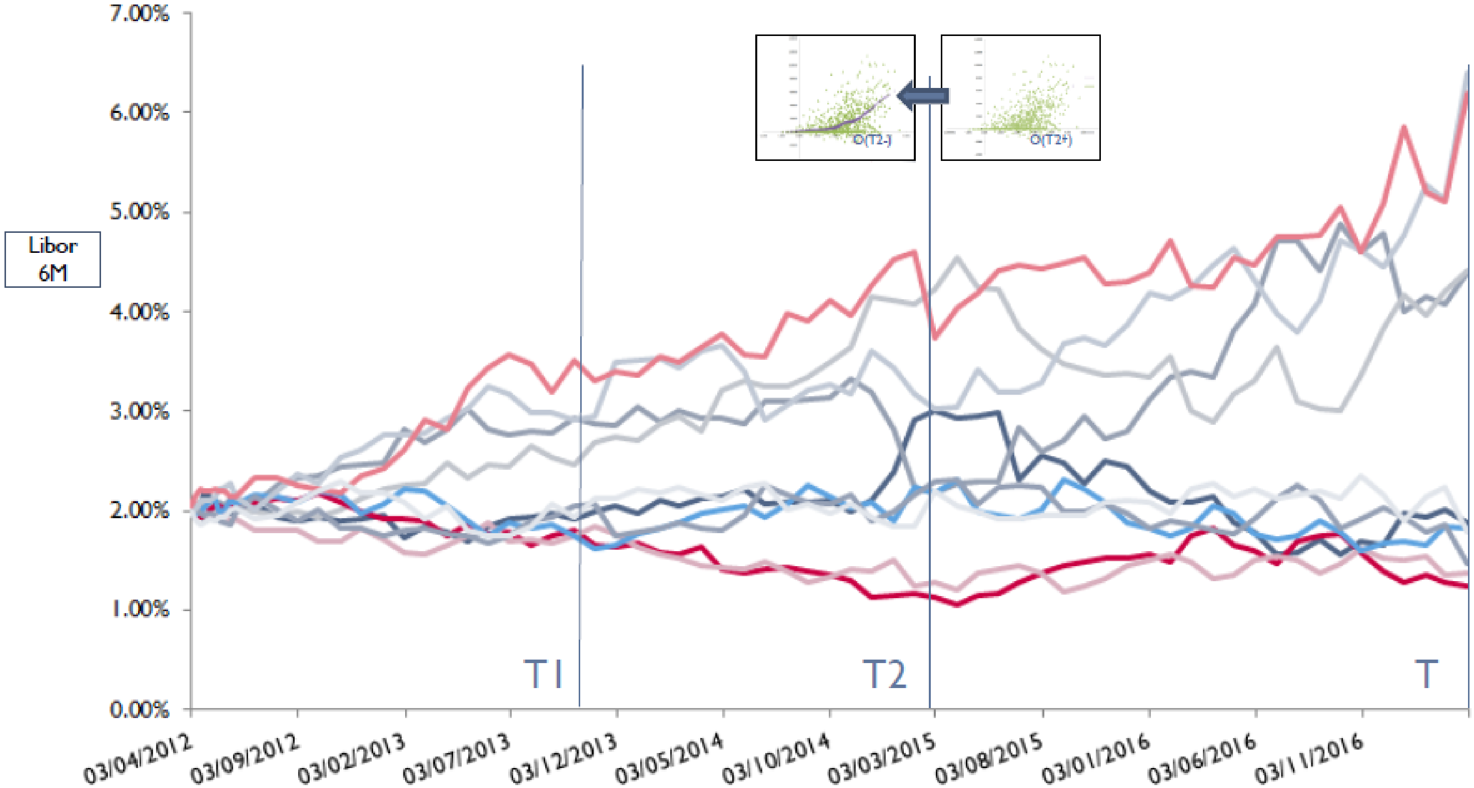
- Backward Phase: we discount payoff contract and add cash flows up to T2.

AMC CVA Simulation



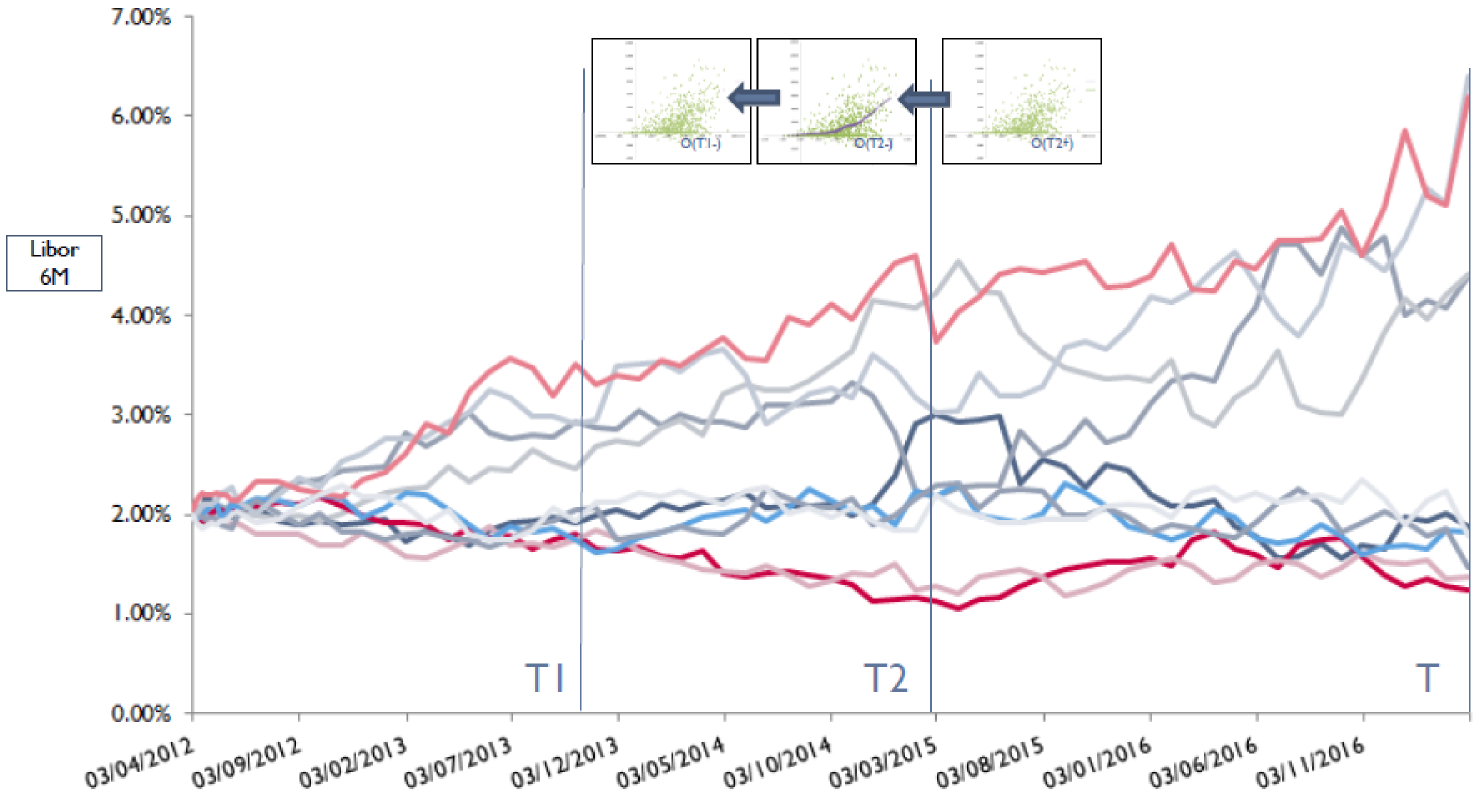
- Backward Phase: At T2 we obtain a cloud of points...

AMC CVA Simulation



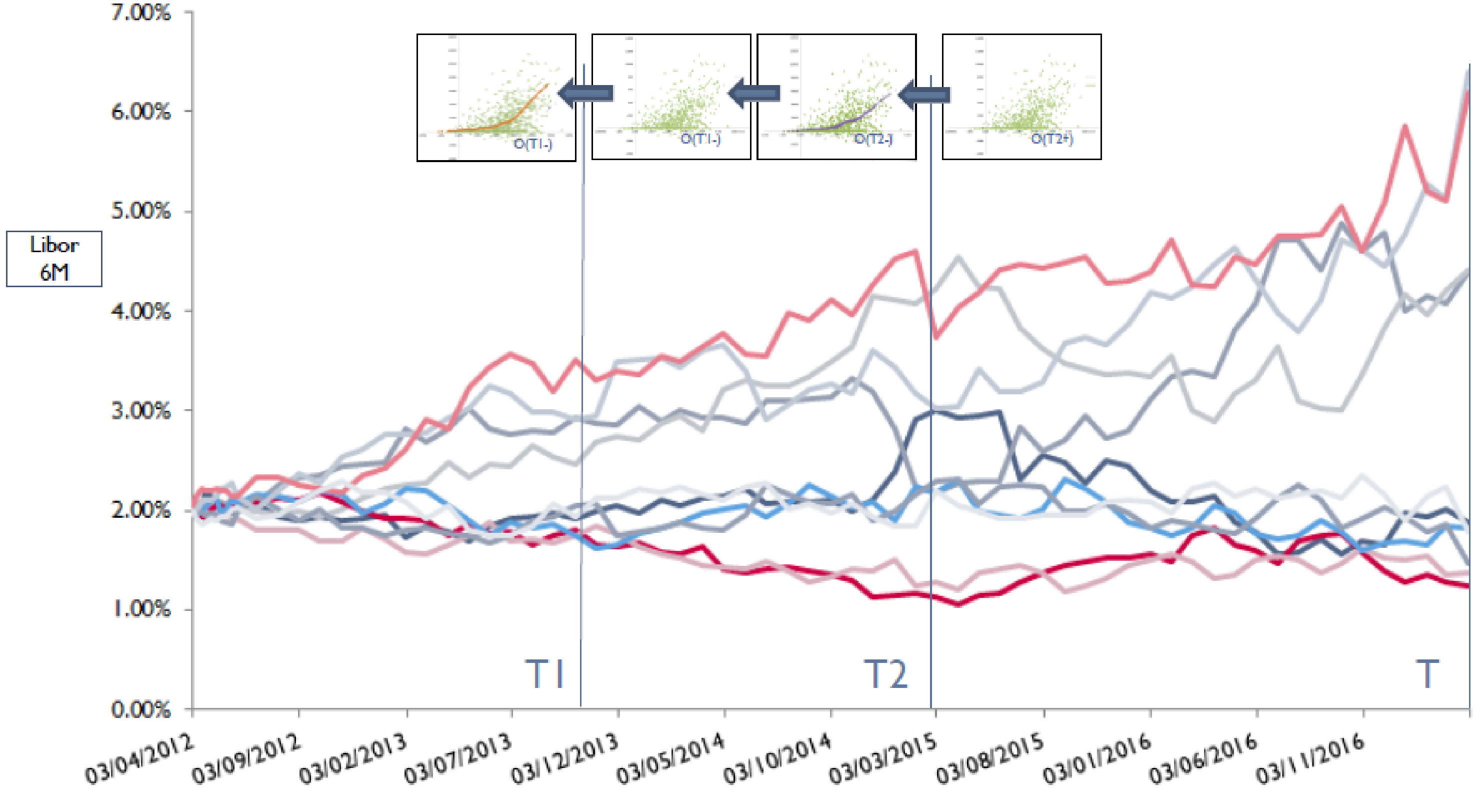
- ... regression is performed to obtain product fair value at T2 as a function of regression basis.

AMC CVA Simulation



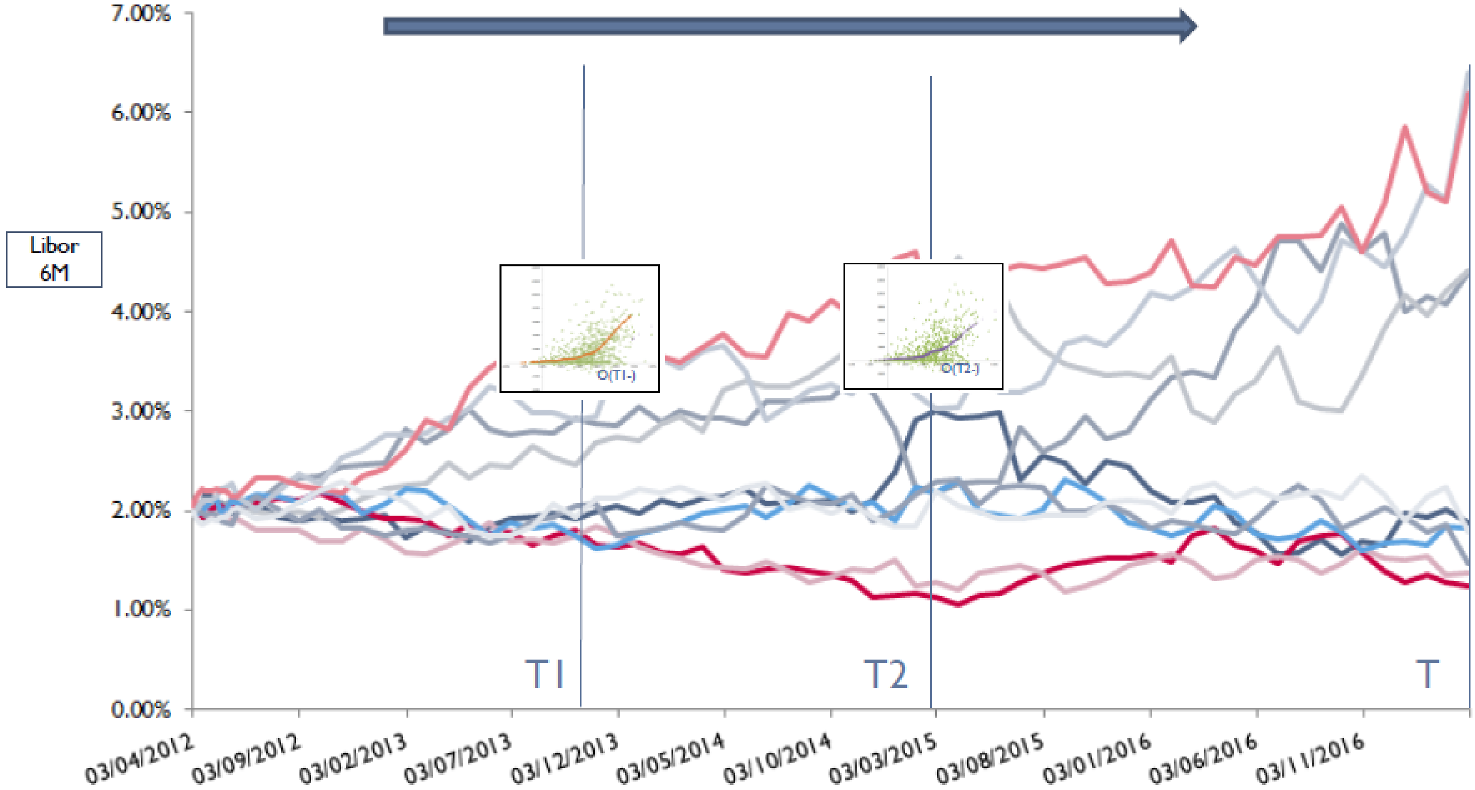
- We discount to T1 obtaining a cloud of points (adding present cash flow from forward phase).

AMC CVA Simulation



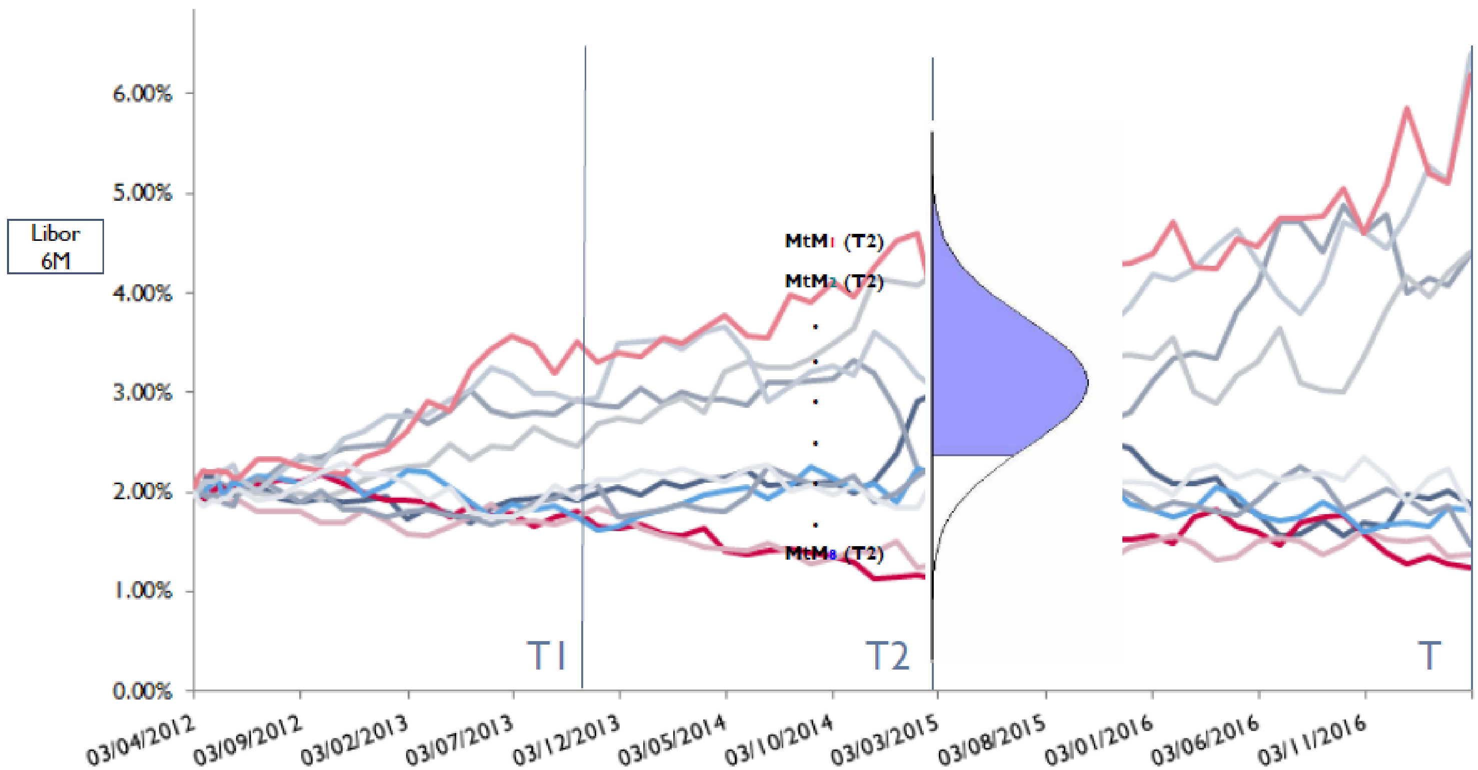
- Regression is performed at T1 as a function of regression basis.

AMC CVA Simulation



- Forward Phase: we then move forward again and invoke regression functions with the new set of MC paths ...

AMC CVA Simulation



- We can compute distribution of MTMs, exposure, EPE, PFE, etc...

AMC CVA Simulation

Recap

1 Forward Phase	2 Backward Phase	3 Forward Phase (CVA specific)
▪ Model Diffusion ▪ Storage of deal cash flows ▪ Construction of regression basis ▪ $\boxed{\text{N Monte Carlo paths}}$	▪ Payoff raw evaluation ▪ Regression function building ▪ N MC paths	▪ Payoff evaluation using regression functions and path dependency ▪ $\boxed{\text{P Monte Carlo paths}}$

- **Important Remark**: In Phase 3, MC diffusion can be different from Phase 1.

Credits

- Damiano Brigo, Fabio Mercurio "Interest Rate Models — Theory and Practice" Springer Finance (2006)
- Damiano Brigo, Massimo Morini, Andrea Pallavicini "Counterparty Credit Risk, Collateral and Funding" Wiley Finance (2013)
- Umberto Cherubini, Elisa Luciano, Walter Vecchiato "Copula Methods in Finance" Wiley Finance (2004)
- Tommaso Gabbriellini "American Options with Monte Carlo" Presentation on Slideshare
- Yves Hilpisch "Derivatives Analytics with Python" Wiley Finance (2015)
- Don L. McLeish "Monte Carlo Simulation and Finance" Wiley Finance (2005)