Solution 1
Sequence a_1, a_2, a_3, a_4, a_5

Proof

(a) By Contradiction. Suppose that there is no 3-chain in our sequence and $a_1 \leq a_3$. If $a_4 \geq a_3$ then we have 3-chain with a_1, a_3, a_4. This implies $a_4 < a_3$.

If $a_4 < a_3$ then $a_1 < a_2$ and $a_1 \leq a_3$ implies $a_3 < a_2$. So, $a_4 < a_3 < a_2$ is a 3-chain. This implies $a_4 \geq a_3$. Contradiction.

Thus $a_1 > a_3$.

Proof

(b) If there is no 3-chain then $a_1 > a_3$. So, $a_3 < a_2$. Now, $a_4 > a_3$ and $a_4 < a_2$ for no 3-chain to exist. Thus, $a_3 < a_4 < a_2$.

Proof

(c) We have $a_1 < a_2$ and $a_3 < a_4 < a_2$.

$a_5 \geq a_2$ will result in 3-chain a_1, a_2, a_5 $a_5 \geq a_4$ will result in 3-chain a_3, a_4, a_5 $a_5 \geq a_1$ will result in 3-chain a_3, a_4, a_5 or a_2, a_4, a_5. $a_5 \geq a_3$ will result in 3-chain a_2, a_4, a_5. $a_5 \leq a_3$ will result in 3-chain a_2, a_3, a_5.

Thus any value of a_5 produces a 3-chain.

Proof

(d) By Contradiction. Suppose $\exists a_1, a_2, a_3, a_4, a_5$ such that no 3-chain exists.

If $a_1 > a_2$ then, $a_2 < a_3 < a_1$ then any a_4 will create 3-chain.

$a_2 < a_1 < a_4 < a_3$ any a_5 will create 3-chain.

If $a_1 < a_2$ then, $a_1 > a_3$ and $a_3 < a_4 < a_2$. But now, any a_5 will create a 3-chain.

Contradiction.

Solution 2
By Induction.
Induction Hypothesis: $P(n)$ implies for all non negative integer n,
$\sum_{i=0}^{n} i^3 = ((n(n+1))/2)^2$
Base case: $n = 0$. $P(0)$ is $\sum_{i=0}^{0} i^3 = 0 = ((0(0+1))/2)^2$. Thus, $P(0)$ is true.
For induction, assume $P(n)$ is true. Now, $P(n+1)$ is $\sum_{i=0}^{n+1} i^3 = (((n+1)(n+2))/2)^2$.
Then, $\sum_{i=0}^{n+1} i^3 = \sum_{i=0}^{n} i^3 + (n+1)^3$.
$((n(n+1))/2)^2 + (n+1)^3 = (n+1)^2(n^2/4 + (n+1)) = (n+1)^2((n^2+4n+4)/4) = (n+1)^2((n+2)^2/4) = (((n+1)(n+2))/2)^2$.
Therefore $P(n) \implies P(n+1)$.
By the axiom of induction $P(n)$ is true.