Solution 1

Sequence a_1, a_2, a_3, a_4, a_5

Proof

(a) By Contradiction. Suppose that there is no 3-chain in our sequence and $a_1 \leq a_3$. If $a_4 \geq a_3$ then we have 3-chain with a_1, a_3, a_4. This implies $a_4 < a_3$.

If $a_4 < a_3$ then $a_1 < a_2$ and $a_1 \leq a_3$ implies $a_3 < a_2$. So, $a_4 < a_3 < a_2$ is a 3-chain. This implies $a_4 \geq a_3$. Contradiction.

Thus $a_1 > a_3$.

Proof

(b) If there is no 3-chain then $a_1 > a_3$. So, $a_3 < a_2$. Now, $a_4 > a_3$ and $a_4 < a_2$ for no 3-chain to exist. Thus, $a_3 < a_4 < a_2$.

Proof

(c) We have $a_1 < a_2$ and $a_3 < a_4 < a_2$.

$a_5 \geq a_2$ will result in 3-chain a_1, a_2, a_5 $a_5 \geq a_4$ will result in 3-chain a_3, a_4, a_5 $a_5 \geq a_1$ will result in 3-chain a_3, a_4, a_5 or a_2, a_4, a_5. $a_5 \geq a_3$ will result in 3-chain a_2, a_4, a_5. $a_5 \leq a_3$ will result in 3-chain a_2, a_3, a_5.

Thus any value of a_5 produces a 3-chain.

Proof

(d) By Contradiction. Suppose $\exists a_1, a_2, a_3, a_4, a_5$ such that no 3-chain exists.

If $a_1 > a_2$ then, $a_2 < a_3 < a_1$ then any a_4 will create 3-chain.

$a_2 < a_1 < a_4 < a_3$ any a_5 will create 3-chain.

If $a_1 < a_2$ then, $a_1 > a_3$ and $a_3 < a_4 < a_2$. But now, any a_5 will create a 3-chain.

Contradiction.