Solution 1

Sequence a_1, a_2, a_3, a_4, a_5

Proof

(a) By Contradiction. Suppose that there is no 3-chain in our sequence and $a_1 \leq a_3$. If $a_4 \geq a_3$ then we have 3-chain with a_1, a_3, a_4. This implies $a_4 < a_3$.

If $a_4 < a_3$ then $a_1 < a_2$ and $a_1 \leq a_3$ implies $a_3 < a_2$. So, $a_4 < a_3 < a_2$ is a 3-chain. This implies $a_4 \geq a_3$. Contradiction.

Thus $a_1 > a_3$.

Proof

(b) If there is no 3-chain then $a_1 > a_3$. So, $a_3 < a_2$. Now, $a_4 > a_3$ and $a_4 < a_2$ for no 3-chain to exist. Thus, $a_3 < a_4 < a_2$.

Proof

(c) We have $a_1 < a_2$ and $a_3 < a_4 < a_2$.

$a_5 \geq a_2$ will result in 3-chain a_1, a_2, a_5 $a_5 \geq a_4$ will result in 3-chain a_3, a_4, a_5 $a_5 \geq a_1$ will result in 3-chain a_3, a_4, a_5 or a_2, a_4, a_5. $a_5 \geq a_3$ will result in 3-chain a_2, a_4, a_5. $a_5 \leq a_3$ will result in 3-chain a_2, a_3, a_5.

Thus any value of a_5 produces a 3-chain.

Proof

(d) By Contradiction. Suppose $\exists a_1, a_2, a_3, a_4, a_5$ such that no 3-chain exists.

If $a_1 > a_2$ then, $a_2 < a_3 < a_1$ then any a_4 will create 3-chain.

$a_2 < a_1 < a_4 < a_3$ any a_5 will create 3-chain.

If $a_1 < a_2$ then, $a_1 > a_3$ and $a_3 < a_4 < a_2$. But now, any a_5 will create a 3-chain.

Contradiction.

Solution 2

By Induction.

Induction Hypothesis: $P(n)$ implies for all non negative integer n,
$\sum_{i=0}^{n} i^3 = ((n(n+1))/2)^2$

Base case: $n = 0$. $P(0)$ is $\sum_{i=0}^{0} i^3 = 0 = ((0(0+1))/2)^2$. Thus, $P(0)$ is true.

For induction, assume $P(n)$ is true. Now, $P(n+1)$ is $\sum_{i=0}^{n+1} i^3 = (((n+1)(n+2))/2)^2$.

Then, $\sum_{i=0}^{n+1} i^3 = \sum_{i=0}^{n} i^3 + (n+1)^3$.

$((n(n+1))/2)^2 + (n+1)^3 = (n+1)^2(n^2/4 + (n+1)) = (n+1)^2((n^2+4n+4)/4) = (n+1)^2((n+2)^2/4) = (((n+1)(n+2))/2)^2$.

Therefore $P(n) \implies P(n+1)$.

By the axiom of induction $P(n)$ is true.

Solution 3

By induction.

Suppose that the num edges can reach beyond the grid. Then for any $m < n$, num edges is atmost $4m$ which is less than $4n$. And for full grid num edges is exactly $4n$.

Let x denote the num edges.

States: If a new square is infected it will cancel at least two edges and add at most two edges. After any legal move the state of the grid will change in following ways:

(i) x will remain unchanged.

(ii) x will decrease by 1.

(iii) x will decrease by 2.

(iv) x will decrease by 4.

Induction Hypothesis: $P(m)$ implies after m time-steps, $x < 4n$ for $n \times n$ grid.
Base Case: $P(0)$ is true. Because, $m < n$ and $x \leq 4m < 4n$.
Assume it is true for all m for purposes of induction.
After $m+1$ steps the state changes. But the new state x' will be $x' \leq x$. Since $x < 4n$ (from induction hypothesis). $x' < 4n$.
It follows that $P(m) \implies P(m+1)$. Thus $P(m)$ is true.

Solution 4
The inductive hypothesis only covers a^k not a^{-1}. Assuming $a^{-1} = 1$ requires base case to consider $k = 1$. But the base case only considers $k = 0$. Simply assuming $a^{-1} = a^1 = 1$ assumes $P(k+1)$ is true implicitly. But the proposition $P(k+1)$ is the thing we are trying to prove.

Solution 5
By Induction: $P(n)$ be $G_n = 3^n - 2^n$.
Special Case: For $n = 0$. $G_0 = 0$ which is true.
Base Case: For $n = 1$. $G_1 = 1$ which is true.
Inductive Step: Assume $\forall n \geq 1, P(n)$ is true.
$P(n+1)$ is $G_{n+1} = 3^{n+1} - 2^{n+1}$. So, $G_{n+1} = 5G_n - 6G_{n-1} = 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) = 5 \cdot 3^n - 5 \cdot 2^n - 2 \cdot 3^n + 3 \cdot 2^n = 3 \cdot 3^n - 2 \cdot 2^n = 3^{n+1} - 2^{n+1}$
which implies that $P(k+1)$ holds. If follows by induction that $P(k)$ holds for all $k \in \mathbb{N}$.

Solution 6

(a) No. A row move moves a tile from cell i to either $i+1$ or $i-1$. Nothing else moves. If you're going from i to $i+1$, everything leq than $i-1$ and geq than $i+2$ stays the same and relative order does not change.

If you're going from i to $i-1$, everything leq than $i-2$ and geq than $i+1$ stays the same and relative order does not change.

(b) A column move changes relative order of 2 pairs of letters.

A column move moves a tile from cell i to either $i+3$ or $i-3$. When an item moves 3 positions it changes order with 2 items $i+1, i+2$ or $i-1, i-2$.

(c) A row move does not change the relative order of items in the grid. Hence it does not affect the parity of the number of inversions.

(d) A column move changes the relative order of 2 pairs of letters. The number of inversions either stays the same or increases by 2 or decreases by 2 during a column move. Adding/subtracting 2 does not change parity. Thus a column move doesn't change parity of inversions.

(e) In every state reachable from the given configuration of the grid the parity of number of inversions is odd.

Lemma 1 (A row move does not change the relative ordering)

(f) A row move moves a tile from cell i to either $i+1$ or $i-1$. Nothing else moves. If you're going from i to $i+1$, everything leq than $i-1$ and geq than $i+2$ stays the same and relative order does not change.

If you're going from i to $i-1$, everything leq than $i-2$ and geq than $i+1$ stays the same and relative order does not change.

Lemma 2 (A column move changes the relative ordering of 3 pairs of letters)
A column move moves a tile from cell i to either $i+4$ or $i-4$. When an item moves 4 positions it changes order with 3 items $i+1, i+2, i+3$ or $i-1, i-2, i-3$.

Proof
By induction: $P(n)$ be the proposition that after n steps the parity of the number of inversions is different from the parity of the row containing the blank sqaure.

Base Case: $P(0)$ is true because parity of number of inversions (which is 1) is odd. The parity of row containing the blank square (which is 4) is even.

Inductive Step: For the purposes of induction, assume $P(n)$ is true for all n. Consider a sequence of $n+1$ moves, $m_1, m_2, \cdots m_{n+1}$.

Now by the inductive hypothesis, we know that after $m_1 \cdots m_n$ is the parity of the number of inversions, inv is different from the parity of the row containing the blank sqaure, i.

On the m_{n+1} move,

A row move changes nothing so the property holds. A column move has the effect $inv + 3$ or $inv - 3$ for number of inversions inv and moves the blank square from row i to either $i+1$ or $i-1$.

The parity of both i and inv are changed.

Since the parity was different from $m_1 \cdots m_n$ it will stay different on m_{n+1}. $P(n+1)$ holds.

Proof

(g) By Induction: $P(n)$ be the proposition that after n steps the parity of number of inversions is odd.

Base Case: For $n = 0$. $P(0)$ is true. Since, we begin with odd parity and do nothing.

Inductive Step: Assume it is true for all n. To show $P(n) \implies P(n+1)$ Consider a sequence of $n+1$ moves, $m_1, m_2, \cdots m_{n+1}$.

Now by the inductive hypothesis, we know that after $m_1 \cdots m_n$ is odd.

Now, we know the parity of number of inversions does not change during m_{n+1}. This implies that parity after $m_1, m_2, \cdots m_{n+1}$ is still odd. Thus, $P(n+1)$ holds.

Solution 7

By induction: Let $P(n)$ be the proposition that in n-th generation, $z_n \leq b_n$ where z_n is number of Z-lings and b_n is number of B-lings in generation n.

Base Case: $P(1)$ is true since $z_1 = 200 \leq 800 = b_1$.

Inductive Step: Assume $P(n)$ is true.

For $n+1$ generation, we know $z_n \leq b_n$ by assumption.

Now, if $z_n = b_n, z_{n+1} = z_n$ and $b_{n+1} = b_n$.

If $z_n < b_n$ then, $z_{n+1} = z_n + \lfloor (b_n - z_n)/2 \rfloor$ and $b_{n+1} = b_n + 2 \cdot \lfloor (b_n - z_n)/2 \rfloor$.

$z_{n+1} < b_{n+1}$.

This implies that $P(n+1)$ holds. Hence $P(n)$ is true for all n.

Since $z_n \leq b_n$ for all n, $z_n \leq 2b_n$ for all n.