Solution 1

Sequence a_1, a_2, a_3, a_4, a_5

Proof

(a) By Contradiction. Suppose that there is no 3-chain in our sequence and $a_1 \leq a_3$. If $a_4 \geq a_3$ then we have 3-chain with a_1, a_3, a_4. This implies $a_4 < a_3$.

If $a_4 < a_3$ then $a_1 < a_2$ and $a_1 \leq a_3$ implies $a_3 < a_2$. So, $a_4 < a_3 < a_2$ is a 3-chain. This implies $a_4 \geq a_3$. Contradiction.

Thus $a_1 > a_3$.

Proof

(b) If there is no 3-chain then $a_1 > a_3$. So, $a_3 < a_2$. Now, $a_4 > a_3$ and $a_4 < a_2$ for no 3-chain to exist. Thus, $a_3 < a_4 < a_2$.

Proof

(c) We have $a_1 < a_2$ and $a_3 < a_4 < a_2$.

$a_5 \geq a_2$ will result in 3-chain a_1, a_2, a_5 $a_5 \geq a_4$ will result in 3-chain a_3, a_4, a_5 $a_5 \geq a_1$ will result in 3-chain a_3, a_4, a_5 or a_2, a_4, a_5. $a_5 \geq a_3$ will result in 3-chain a_2, a_4, a_5. $a_5 \leq a_3$ will result in 3-chain a_2, a_3, a_5.

Thus any value of a_5 produces a 3-chain.

Proof

(d) By Contradiction. Suppose $\exists a_1, a_2, a_3, a_4, a_5$ such that no 3-chain exists.

If $a_1 > a_2$ then, $a_2 < a_3 < a_1$ then any a_4 will create 3-chain.

$a_2 < a_1 < a_4 < a_3$ any a_5 will create 3-chain.

If $a_1 < a_2$ then, $a_1 > a_3$ and $a_3 < a_4 < a_2$. But now, any a_5 will create a 3-chain.

Contradiction.

Solution 2

By Induction.

Induction Hypothesis: $P(n)$ implies for all non negative integer n,
$\sum_{i=0}^{n} i^3 = ((n(n+1))/2)^2$
Base case: $n = 0$. $P(0)$ is $\sum_{i=0}^{0} i^3 = 0 = ((0(0+1))/2)^2$. Thus, $P(0)$ is true.
For induction, assume $P(n)$ is true. Now, $P(n+1)$ is $\sum_{i=0}^{n+1} i^3 = (((n+1)(n+2))/2)^2$.
Then, $\sum_{i=0}^{n+1} i^3 = \sum_{i=0}^{n} i^3 + (n+1)^3$.
$((n(n+1))/2)^2 + (n+1)^3 = (n+1)^2(n^2/4 + (n+1)) = (n+1)^2((n^2+4n+4)/4) = (n+1)^2((n+2)^2/4) = (((n+1)(n+2))/2)^2$.
Therefore $P(n) \implies P(n+1)$.
By the axiom of induction $P(n)$ is true.

Solution 3

By induction.

Suppose that the num edges can reach beyond the grid. Then for any $m < n$, num edges is atmost $4m$ which is less than $4n$. And for full grid num edges is exactly $4n$.

Let x denote the num edges.

States: If a new square is infected it will cancel at least two edges and add at most two edges. After any legal move the state of the grid will change in following ways:

(i) x will remain unchanged.

(ii) x will decrease by 1.

(iii) x will decrease by 2.

(iv) x will decrease by 4.

Induction Hypothesis: $P(m)$ implies after m time-steps, $x < 4n$ for $n \times n$ grid.
Base Case: $P(0)$ is true. Because, $m < n$ and $x \leq 4m < 4n$.
Assume it is true for all m for purposes of induction.
After $m+1$ steps the state changes. But the new state x' will be $x' \leq x$. Since $x < 4n$ (from induction hypothesis). $x' < 4n$.
It follows that $P(m) \implies P(m+1)$. Thus $P(m)$ is true.

Solution 4
The inductive hypothesis only covers a^k not a^{-1}. Assuming $a^{-1} = 1$ requires base case to consider $k = 1$. But the base case only considers $k = 0$. Simply assuming $a^{-1} = a^1 = 1$ assumes $P(k+1)$ is true implicitly. But the proposition $P(k+1)$ is the thing we are trying to prove.

Solution 5
By Strong Induction: $P(k)$ be $G_k = 3^k - 2^k$.
Base Case: For $k = 0$. $G_0 = 0$ which is true.
Inductive Step: Assume $P(0) \wedge P(1) \cdots P(k)$ is true.
$P(k+1)$ is $G_{k+1} = 3^{k+1} - 2^{k+1}$. So, $G_{k+1} = 5G_n - 6G_{n-1} = 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) = 5 \cdot 3^n - 5 \cdot 2^n - 2 \cdot 3^n + 3 \cdot 2^n = 3 \cdot 3^n - 2 \cdot 2^n = 3^{n+1} - 2^{n+1}$
which implies that $P(k+1)$ holds. If follows by induction that $P(k)$ holds for all $k \in \mathbb{N}$.