Understanding Analysis Solutions

Bijaya Dangol

Preface

Fun

Contents

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Solution

(a) Suppose for contradiction that p/q is in lowest terms such that $(p/q)^2 = 3$. Then $p^2 = 3q^2$ implying that p^2 is a multiple of 3. Since 3 is not a perfect square, this implies that p is a multiple of 3. Therefore $p = 3r$ for some r, substituting p we get, $(3r)^2 = 3q^2$ and $3r^2 = q^2$ implying that q is also a multiple of 3 contradicting the assumption
that p/q is already at its lowest terms.

The similar argument works for $\sqrt{6}$.

(b) $\sqrt{4}$ is a perfect square. This is where the proof for Theorem 1.1.1 breaks down.

Solution

If $r = 0$ then $2^r = 1 \neq 3$. If $r \neq 0$ then set $p/q = r$ to get $2^p = 3^q$ which is not possible since 2 and 3 share no factors.

Solution

(a) It does not hold for $A_n = \{m \mid m \geq n : \forall n \in \mathbb{N}\}$

(b) It does not hold for $A_n = \{m \mid n \leq m \leq n + 100 \forall n \in \mathbb{R}\}$.

Update: The initial reasoning is wrong because in that $A_n \not\supseteq A_m forn \leq m$. It is true because, eventually there must exist A_m for some m and the pattern continues $A_1, \ldots, A_{m-1}, A_m, A_m, A_m, \ldots, A_m$.

(c) $x \in A \cap (B \cup C)$ means $x \in A$ and $x \in B$ or $x \in C$. If $x \in C$ then $x \in (A \cap B) \cup C$. If $x \in B$ then $x \in A \cap B$, which implies $x \in (A \cap B) \cup C$. This shows that $(A \cap B) \cup C \subseteq A \cap (B \cup C)$.

Again, $x \in (A \cap B) \cup C$ means $x \in$ A and $x \in B$ or $x \in C$.

If $x \in C$ then, $x \in (A \cap B) \cup C$. If $x \in A$ and $x \in B$ then, $x \in A \cap (B \cup C)$. This shows that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Thus, $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) Similar as above

(e) $x \in A \cap (B \cup C)$ means $x \in A$ and $x \in B$ or $x \in C$. This means $x \in A \cap B$ or $x \in A \cap C$. So, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

$x \in (A \cap B) \cup (A \cap C)$ means $x \in A \cap B$ or $x \in A \cap C$. This means, $x \in A$ and $x \in B$ or $x \in C$. So, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution

$$
\begin{array}{llllll}
1 & 3 & 6 & 10 & 15 & \cdots \\
2 & 5 & 9 & 14 & \cdots & \\
4 & 8 & 13 & \cdots & & \\
7 & 12 & \cdots & & & \\
11 & \ldots & & & & \\
\vdots & & & & &
\end{array}
$$

A_i is the i^{th} row.

Solution

(a) $x \in (A \cap B)^c$ means $x \notin A$ or $x \notin B$. This implies, $(A \cap B)^c \subseteq A^c \cup B^c$

(b) $x \in A^c \cup B^c$ means $x \in A^c$ or $x \in B^c$. This means, $x \notin A \cap B$, which implies $x \in (A \cap B)^c$.

$(A \cap B)^c \supseteq A^c \cup B^c$

(c) $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \supseteq A^c \cup B^c$. So $(A \cap B)^c = A^c \cup B^c$

Solution

(a) $|a+b| \leq |a|+|b|$. Squaring, we get, $(a+b)^2 \leq (|a|+|b|)^2$ $a^2+2ab+b^2 \leq |a|^2+2\,|a|\,|b|+|b|^2$ $a^2+2ab+b^2 \leq a^2+2\,|a|\,|b|+b^2$. For $a, b > 0$ or $a, b < 0$, $ab = |a|\,|b|$. Therefore, $a^2+2ab+b^2 = a^2+2\,|a|\,|b|+b^2$. So the triangle ineqaulity holds.

(b) The inequality reduces to, $ab \leq |a|\,|b|$ which is true when $ab < 0$ as $|a|\,|b|$ is always greater than 0. since squaring preserves inequality this implies that $|a+b| \leq |a|+|b|$.

(c) $|a-b| = |a-c+c-b| \leq |a-c|+|c-b|$, $absa-c+|c-b| = |a-c|+|c-d+d-b| \leq |a-c|+|c-d|+|d-b|$.

(d) Then inequality reduces to, $-|a|\,|b| \leq -ab$ which is true. Alternatively, we can also prove it using the triangle inequality and $a = a-b+b$. We know $||a|-|b|| = ||b|-|a||$, so we can assume $|a| > |b|$ without the loss of generality. Then $||a|-|b|| = |a|-|b| = |(a-b)+b|-|b| \leq |a-b|+|b|-|b| = |a-b|$.

Solution

(a) $f(A) = [0,4]$ and $f(B) = [1,16]$. In this case, $A \cap B = [1,2]$, $f(A) \cap f(B) = [1,4] = f(A \cap B)$. $A \cup B = [0,4]$, $f(A) \cup f(B) = [0,16] = f(A \cup B)$.

(b) For $A = [-b, -a]$ and $B = [a, b]$, $A \cap B = \emptyset$ which implies $f(A \cap B) = \emptyset$. But $f(A) = [a^2, b^2]$ and $f(B) = [a^2, b^2]$ with $f(A) \cap f(B) = [a^2, b^2] \neq \emptyset = f(A \cap B)$.

(c) $y \in g(A \cap B)$ means $y = g(x)$ for some $x \in A \cap B$. This implies $x \in A$ and $x \in B$ which means, $y \in g(A)$ and $y \in g(B)$. So, $g(A \cap B) \subseteq g(A) \cap g(B)$.

The reverse inclusion does not apply when g is not 1-1 function because it is possible to get $y = g(x) = g(x')$ where $x \neq x'$

(d) **Conjecture:** $g(A \cup B) = g(A) \cup g(B)$

$y \in g(A \cup B)$ means $\exists x \in A \cup B$ such that $y = g(x)$. This means, $x \in A$ or $x \in B$, hence $y \in g(A) \cup g(B)$. So, $g(A \cup B) \subseteq g(A) \cup g(B)$.

Now, $y \in g(A) \cup g(B)$ means $\exists x \in A$ or $\exists x' \in B$ such that $y = g(x) = g(x')$. Regardless of the equality of x and x', $x, x' \in A \cup B$ meaning, $g(A \cup B) \supseteq g(A) \cup g(B)$.

Hence, $g(A \cup B) = g(A) \cup g(B)$

Solution

(a) $f(x) = 2n$

(b) $f(1) = 1$ and $f(n) = n - 1$

(c) $f(n) = n/2$ when n is even and $f(n) = -(n+1)/2$ when n is even.

Solution

(a) $A = [0, 4]$ $B = [-1, 1]$ $f^{-1}(A) = [-2, 2]$ $f^{-1}(B) = [-1, 1]$

$A \cap B = [0, 1]$ $f^{-1}(A \cap B) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$

$A \cup B = [-1, 4]$ $f^{-1}(A \cup B) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$

(b) $x \in g^{-1}(A \cap B)$ means that $g(x) \in A \cap B$. which means $g(x) \in A$ and $g(x) \in B$, this implies $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$. So, $g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B)$ Now, $x \in g^{-1}(A) \cap g^{-1}(B)$ means that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$. Which is same as $g(x) \in A$ and $g(x) \in B$. Then we get, $g(x) \in A \cap B$ this implies that $x \in g^{-1}(A \cap B)$. Therefore, $g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B)$.

Hence, $g^{-1}(A) \cap g^{-1}(B) = g^{-1}(A \cap B)$.

Same logic for the union case. Alternatively, we can also use the fact we proved in 1.2.7 (d).

Solution

enum

The reverse inclusion does not hold for $a = b$.

Same reason as above.

$b < b + \epsilon$ for every $\epsilon > 0$, this implies, $a < b + \epsilon$.

For backward inclusion, $a < b + \epsilon$ for every $\epsilon > 0$. $b < a$ is not possible because we will reach a contradiction for $\epsilon_0 = a - b$. So either $a < b$ or $a = b$.

Solution

(a) The claim is true. Negation: For all real numbers satisfying $a > b$, $a + 1/n \geq b$ for all $n \in \mathbb{N}$.

(b) The claim is false. Negation: For all real numbers $x > 0$, there exists $n \in \mathbb{N}$ such that $x \geq n$.

(c) The claim is true. Negation: There exists two distinct real numbers a and b such that $a < b$ with $r < a$ and $r > b$ for all $r \in \mathbb{Q}$.

Solution

(a) For $n = 1$, $y_1 = 6 > -6$. It is true for the base case. Suppose inductively that $y_n > -6$. Now, $y_{n+1} = (2y_n - 6)/3$ we get, $y_{n+1} = 2/3y_n - 2$ We know, $2/3y_n > -4$, from this we get, $y_{n+1} = 2/3y_n - 2 > -6$.

(b) For $n = 1$, $y_2 = 2 < 6 = y_1$. Suppose inductively that $y_{n+1} < y_n$. Then we get, $2y_{n+1} < 2y_n$,$2y_{n+1} - 6 < 2y_n - 6$, ,$(2y_{n+1} - 6)/3 < (2y_n - 6)/3$, $y_{n+2} < y_{n+1}$.

Solution

(a) We know, $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$. So it is true for the base case. Suppose for induction that, $(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots A_n^c$.

Then, $(A_1 \cup A_2 \cup \cdots \cup A_{n+1})^c = (A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1})^c$

$(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1})^c = (\bigcup_{i=1}^n A_i \cup A_{n+1})^c$

$(\bigcup_{i=1}^n A_i \cup A_{n+1})^c = (\bigcup_{i=1}^n A_i)^c \cap A_{n+1}^c$

$(\bigcup_{i=1}^n A_i)^c \cap A_{n+1}^c = \bigcap_{i=1}^n A_i^c \cap A_{n+1}^c = \bigcap_{i=1}^{n+1} A_i^c$.

(b) $B_n = \{m \in \mathbb{N} : m \geq n\}$

(c) $x \in (\bigcup_{i=1}^n A_i)^c$ means that $x \notin A_i$ so $x \in A_i^c$ for every i. Then, $x \in \bigcap_{i=1}^n A_i^c$ So, $(\bigcup_{i=1}^n A_i)^c \subseteq \bigcap_{i=1}^n A_i^c$

Now, $x \in \bigcap_{i=1}^n A_i^c$ means that $x \in A_i^c$ so $x \notin A_i$ for every i. Then, $x \notin \bigcup_{i=1}^n A_i$ which implies $x \in (\bigcup_{i=1}^n A_i)^c$. So, $(\bigcup_{i=1}^n A_i)^c \supseteq \bigcap_{i=1}^n A_i^c$.

Thus, $(\bigcup_{i=1}^n A_i)^c = \bigcap_{i=1}^n A_i^c$