Homework 1 - CPSC 326
Solutions

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Question 1 (10 points)

Let $\Sigma = \{0, 1, a, b, c, d\}$ be an alphabet.

- What is $|\Sigma|$?
- What is $|\Sigma^*|$?
- What is $|\Sigma^+|$?
- What is $|\Sigma^* - \Sigma^+|$?

Answer:

- $|\Sigma| = 6$ (the alphabet contains 6 symbols)
- $|\Sigma^*| = \infty$ (the Kleene closure contains all finite strings including the empty string)
- $|\Sigma^+| = \infty$ (the positive closure contains all finite non-empty strings)
- $|\Sigma^* - \Sigma^+| = 1$ (the only element in Σ^* but not in Σ^+ is λ)

Question 2 (10 points)

Let the set $A = \{w \in \{a, b\}^* \mid |w| \leq 2\}$. List the elements of the set A.
Answer: $A = \{\lambda, a, b, aa, ab, ba, bb\}$

- Strings of length 0: λ
- Strings of length 1: a, b
- Strings of length 2: aa, ab, ba, bb

Question 3 (10 points)

Let $A = \{1, 2, 3, \ldots, n\}$. What is $\bigcup_{i=1}^{n} A$?

Answer: $\bigcup_{i=1}^{n} A = A = \{1, 2, 3, \ldots, n\}$

Since we are taking the union of A with itself n times, the result is just A.

Question 4 (10 points)

What is $\mathcal{P}(\{a, b, c\})$? What is $|\mathcal{P}(\{a, b, c\}) - \mathcal{P}(\{a, b\})|$?

Answer: $\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

$\mathcal{P}(\{a, b, c\}) - \mathcal{P}(\{a, b\}) = \{\{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Therefore, $|\mathcal{P}(\{a, b, c\}) - \mathcal{P}(\{a, b\})| = 4$

Question 5 (10 points)

For sets A and B prove or disprove the following: $A - B = B - A$.

Answer: This statement is **false**. We disprove it with a counterexample:

Let $A = \{1, 2\}$ and $B = \{2, 3\}$.

Then:

- $A - B = \{1\}$ (elements in A but not in B)
- $B - A = \{3\}$ (elements in B but not in A)

Since $\{1\} \neq \{3\}$, we have $A - B \neq B - A$.

The equality $A - B = B - A$ holds if and only if $A = B$.

Question 6 (10 points)

What is $\mathcal{P}(\emptyset)$? What about $\mathcal{P}(\{\emptyset\})$?

Answer:

- $\mathcal{P}(\emptyset) = \{\emptyset\}$ (the power set of the empty set contains only empty set)
- $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$ (the power set contains the empty set and the set containing the empty set)

Question 7 (10 points)

Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined as follows: $f(x) = 2x$.

- Is f a bijection? Why or why not?

- Let E denote the set of even natural numbers. Now consider f with the same definition but now $f : \mathbb{N} \to E$. Is f a bijection?

Answer:

- $f : \mathbb{N} \to \mathbb{N}$ is **not a bijection**. While f is one to one, it is not surjective because odd numbers like 1, 3, 5, etc., are never mapped to by f. For example, there is no $x \in \mathbb{N}$ such that $f(x) = 1$.
- $f : \mathbb{N} \to E$ **is a bijection**.
  - Injective: If $f(x_1) = f(x_2)$, then $2x_1 = 2x_2$, so $x_1 = x_2$.
  - Surjective: For any even number $y \in E$, we have $y = 2k$ for some $k \in \mathbb{N}$, and $f(k) = 2k = y$.

Question 8 (10 points)

What is the error in the following proof that $1 = 2$? Let $a = b$, for some a and b. Multiply both sides of the equation by a to get $a \cdot a = a \cdot b$. Now subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now apply factoring and get $(a + b)(a - b) = b(a - b)$. Divide each side of the equality by $(a - b)$ to get $a + b = b$. Finally, let a and b both be 1 and thus $2 = 1$.

Answer: The error is when you divide both sides by $(a-b)$. Since we started with $a = b$, we have $a - b = 0$. Division by zero is undefined, making this step invalid. The algebraic steps up to $(a + b)(a - b) = b(a - b)$ is correct, but we can't divide $(a - b) = 0$ to show $a + b = b$.

Question 9 (10 points)

Let Σ be an alphabet.

- Prove or disprove the following: $\Sigma^+ \cup \emptyset = \Sigma^*$
- Is $\lambda \in \emptyset$?

Answer:

- The statement $\Sigma^+ \cup \emptyset = \Sigma^*$ is **false**.

  $\Sigma^+ \cup \emptyset = \Sigma^+$ (because union with the empty set doesn't change a set).

  Also, $\Sigma^* = \Sigma^+ \cup \{\lambda\}$ (the Kleene closure includes the empty string).

  Therefore, $\Sigma^+ \cup \emptyset = \Sigma^+ \neq \Sigma^*$ (unless $\Sigma = \emptyset$, which is a contradiction for it being an alphabet).

  The correct statement would be: $\Sigma^+ \cup \{\lambda\} = \Sigma^*$
- No, $\lambda \notin \emptyset$. The empty set $\emptyset$ contains no elements at all which also doesn't include λ.

Question 10 (10 points)

What is $\sum_{x \in \mathcal{P}(\{a,b,c\})} |x|$?

Answer:

- $\emptyset$: $|\emptyset| = 0$
- $\{a\}$: $|\{a\}| = 1$
- $\{b\}$: $|\{b\}| = 1$
- $\{c\}$: $|\{c\}| = 1$
- $\{a, b\}$: $|\{a, b\}| = 2$
- $\{a, c\}$: $|\{a, c\}| = 2$
- $\{b, c\}$: $|\{b, c\}| = 2$
- $\{a, b, c\}$: $|\{a, b, c\}| = 3$

Sum $= 0 + 1 + 1 + 1 + 2 + 2 + 2 + 3 = 12$