Problem Set 1

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Problem 1

Statement: Using a proof by contradiction, prove for any integer n if $n^3 + 5$ is odd, then n is even.

Proof by Contradiction:
Assume for contradiction that $n^3 + 5$ is odd and n is odd.
Since n is odd, we can write $n = 2k + 1$ for some integer k.
Then:

$$\begin{align} n^3 &= (2k+1)^3 \tag{1} \\ &= 8k^3 + 12k^2 + 6k + 1 \tag{2} \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \tag{3} \end{align}$$

Let $m = 4k^3 + 6k^2 + 3k$. Since k is an integer, m is also an integer.
Therefore, $n^3 = 2m + 1$, which means n^3 is odd.
Now, $n^3 + 5 = (2m + 1) + 5 = 2m + 6 = 2(m + 3)$.
Since $m + 3$ is an integer, $n^3 + 5$ is even.
Any integer times 2 is even. Therefore, $n^3 + 5$ is even.
This contradicts the assumption that $n^3 + 5$ is odd.
Therefore, if $n^3 + 5$ is odd, then n must be even.

Problem 2

Statement:

$$\overline{A} \cap \overline{B} = \overline{A \cup B}$$

Example: Let the universal set be

$$U = \{1, 2, 3, 4, 5\}$$

and let

$$A = \{1, 2\}, \quad B = \{3, 4\}.$$

Step 1: Find complements:

$$\overline{A} = U - A = \{3, 4, 5\}, \quad \overline{B} = U - B = \{1, 2, 5\}$$

Step 2: Solve left-hand side:

$$\overline{A} \cap \overline{B} = \{3, 4, 5\} \cap \{1, 2, 5\} = \{5\}$$

Step 3: Solve right-hand side:

$$A \cup B = \{1, 2\} \cup \{3, 4\} = \{1, 2, 3, 4\}$$

$$\overline{A \cup B} = U - (A \cup B) = \{5\}$$

Step 4: Compare both sides:

$$\overline{A} \cap \overline{B} = \{5\} = \overline{A \cup B}$$

Conclusion: The equality holds in this example, showing that it's true

$$\overline{A} \cap \overline{B} = \overline{A \cup B}$$

Problem 3

Statement: Using a proof by induction, show that for all $n > 5$, $2^n > 5n$.

Proof by Induction:

Base Case: Let $n = 6$. $2^6 = 64 \quad \text{and} \quad 5 \cdot 6 = 30$ Since $64 > 30$, the base case holds.

Inductive Hypothesis: Assume that for some $k \geq 6$, $2^k > 5k$.

Inductive Step: We need to show that $2^{k+1} > 5(k+1)$.

$2^{k+1} = 2 \cdot 2^k > 2 \cdot 5k = 10k$ (by the inductive hypothesis)

We need to show that $10k > 5(k+1) = 5k + 5$.

This is equivalent to showing $10k > 5k + 5$, or $5k > 5$, or $k > 1$.

Since $k \geq 6 > 1$, we have $10k > 5k + 5$.

Therefore, $2^{k+1} > 10k > 5k + 5 = 5(k+1)$.

By mathematical induction, $2^n > 5n$ for all $n > 5$.

This also holds true because $2^n > 5n$ for all $n > 5$.

Problem 4

Statement: Let p and q be truth values. Using a truth table, prove or disprove the following statement:

$$(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow (p \wedge q)$$

Truth Table:

p	q	$\neg p$	$\neg p \vee q$	$p \wedge q$	$p \wedge (p \wedge q)$	LHS	LHS $\Leftrightarrow$ RHS
T	T	F	T	T	T	T	T
T	F	F	F	F	F	F	T
F	T	T	T	F	F	F	T
F	F	T	T	F	F	F	T

Where LHS $= (\neg p \vee q) \wedge (p \wedge (p \wedge q))$ and RHS $= (p \wedge q)$.
Step-by-step calculation:

- Row 1: $(\neg T \vee T) \wedge (T \wedge (T \wedge T)) = (F \vee T) \wedge (T \wedge T) = T \wedge T = T$
- Row 2: $(\neg T \vee F) \wedge (T \wedge (T \wedge F)) = (F \vee F) \wedge (T \wedge F) = F \wedge F = F$
- Row 3: $(\neg F \vee T) \wedge (F \wedge (F \wedge T)) = (T \vee T) \wedge (F \wedge F) = T \wedge F = F$
- Row 4: $(\neg F \vee F) \wedge (F \wedge (F \wedge F)) = (T \vee F) \wedge (F \wedge F) = T \wedge F = F$

The biconditional is true in all rows, therefore we end up with a tautology, therefore the statement is **true**.
$(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow (p \wedge q)$ is **true**.