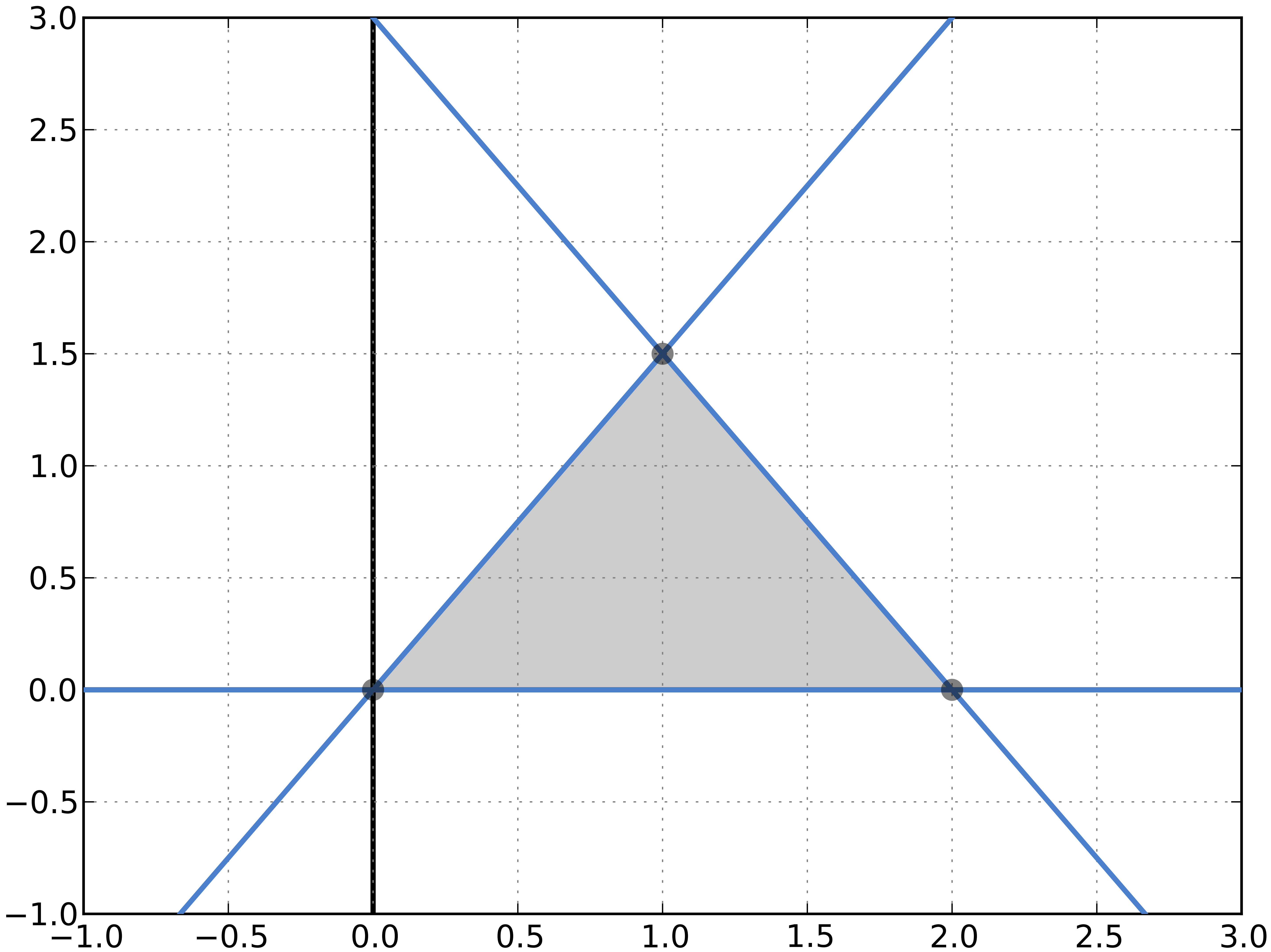
Discrete Optimization

Mixed Integer Programming: Part IV

Goals of the Lecture

- Polyhedral cuts
  - warehouse location
  - node covering

Convex Hull



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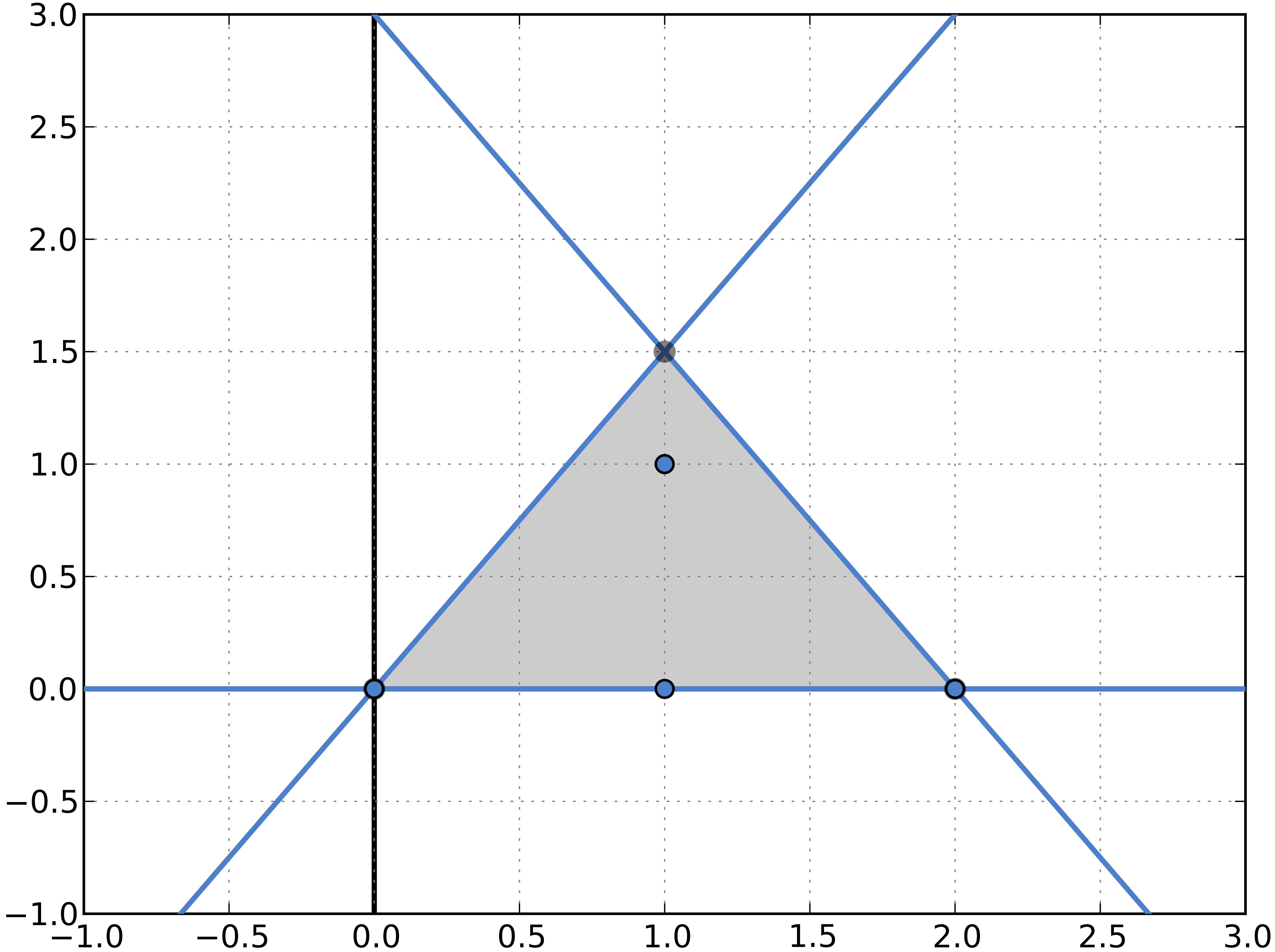

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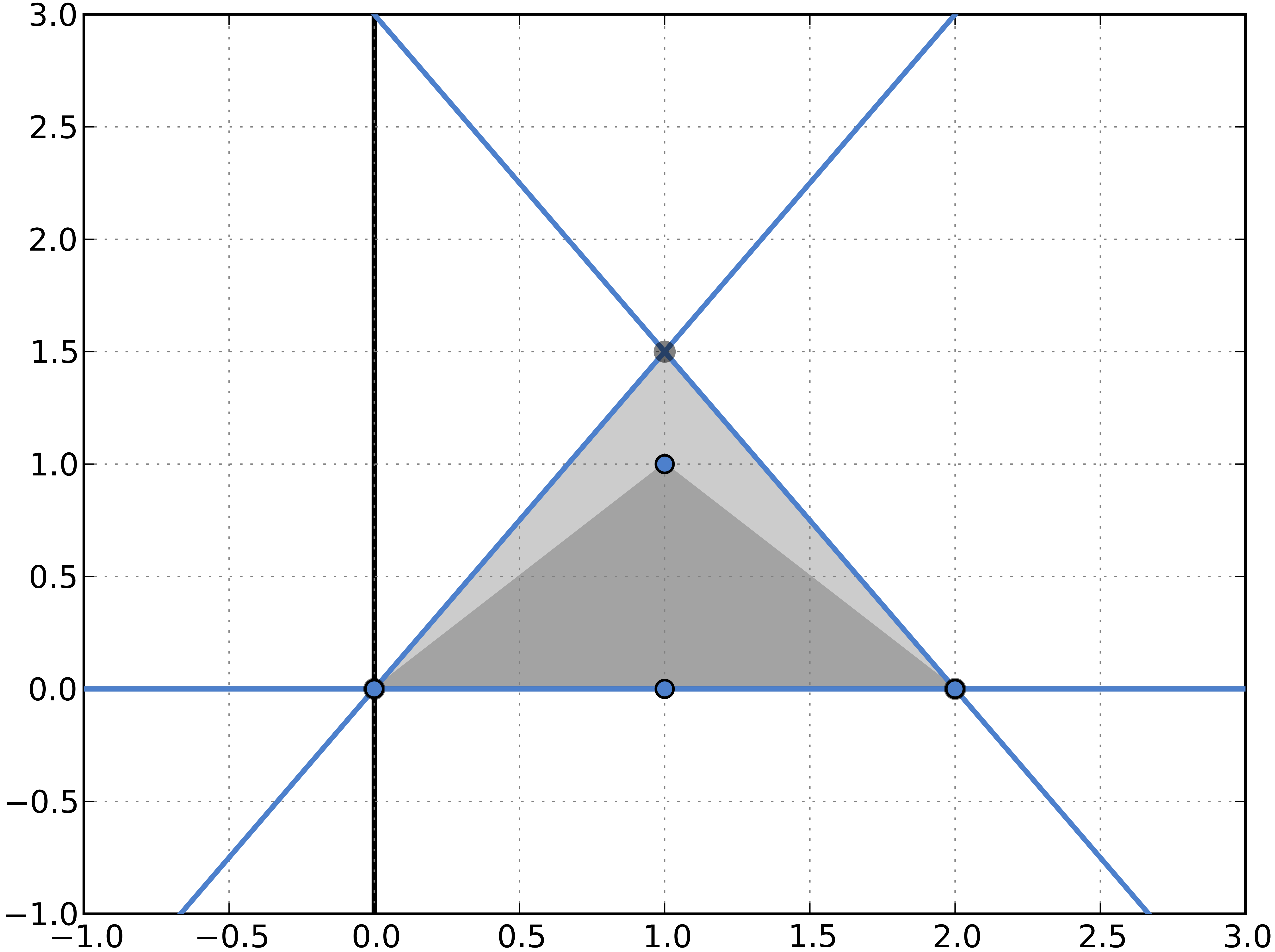

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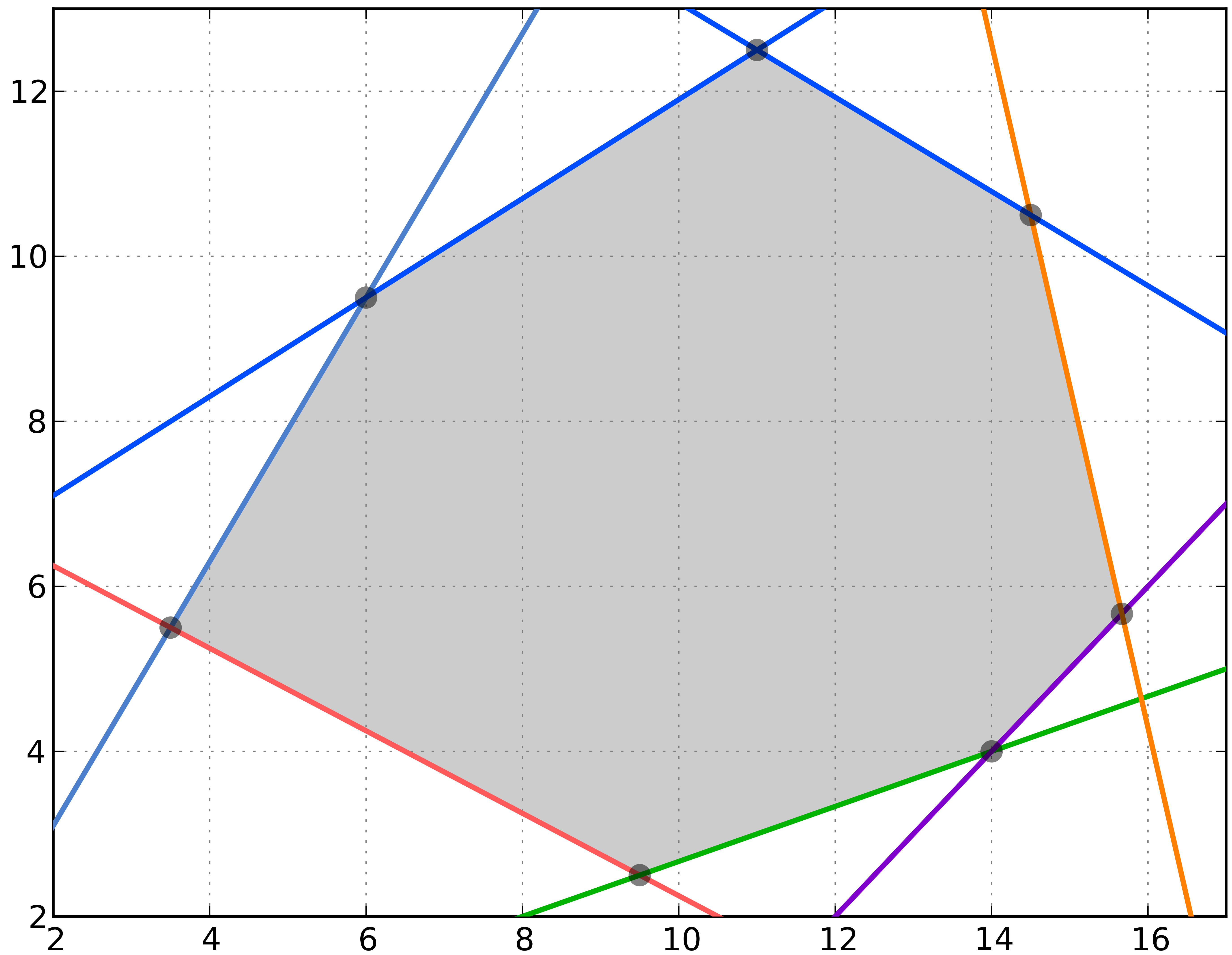

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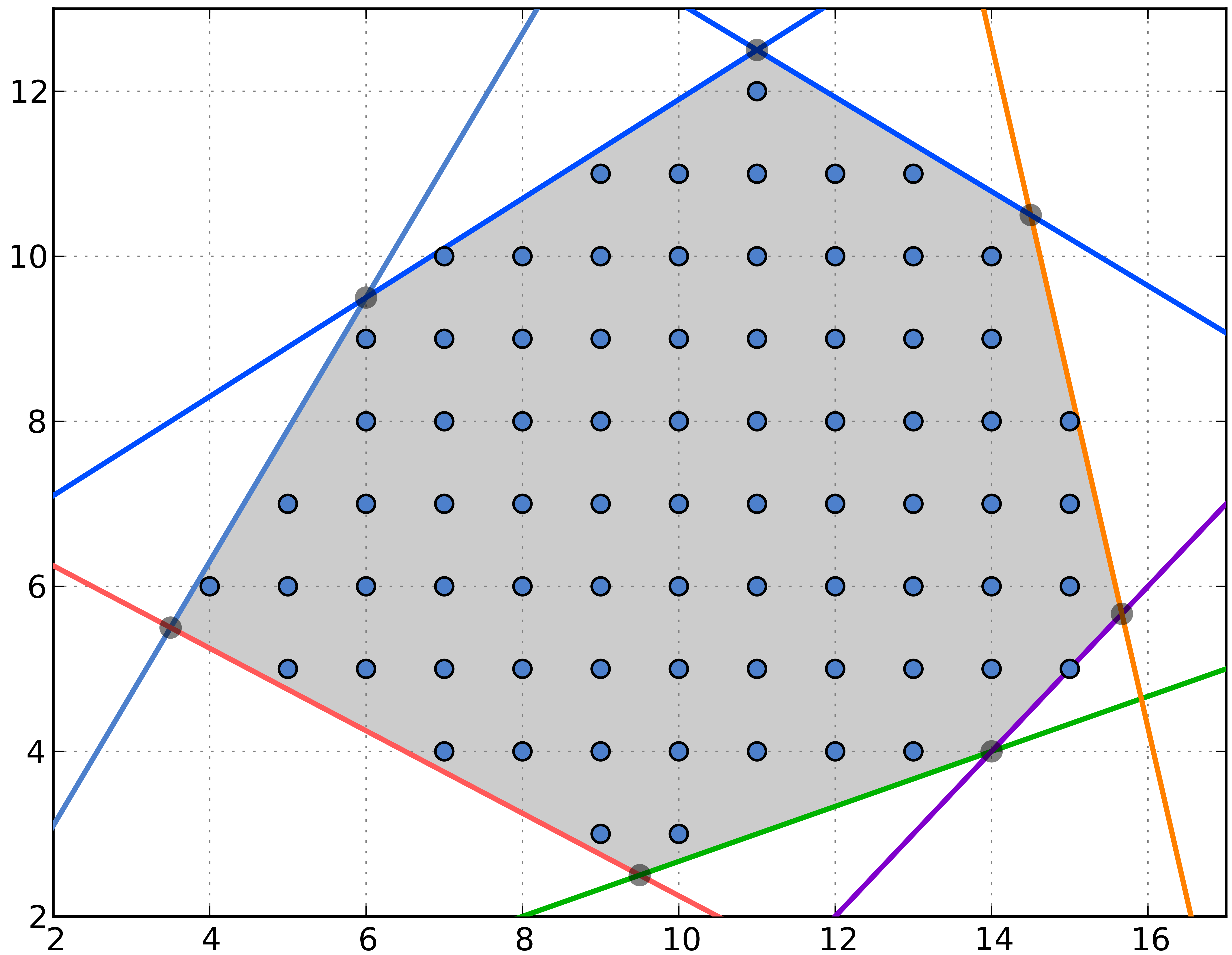

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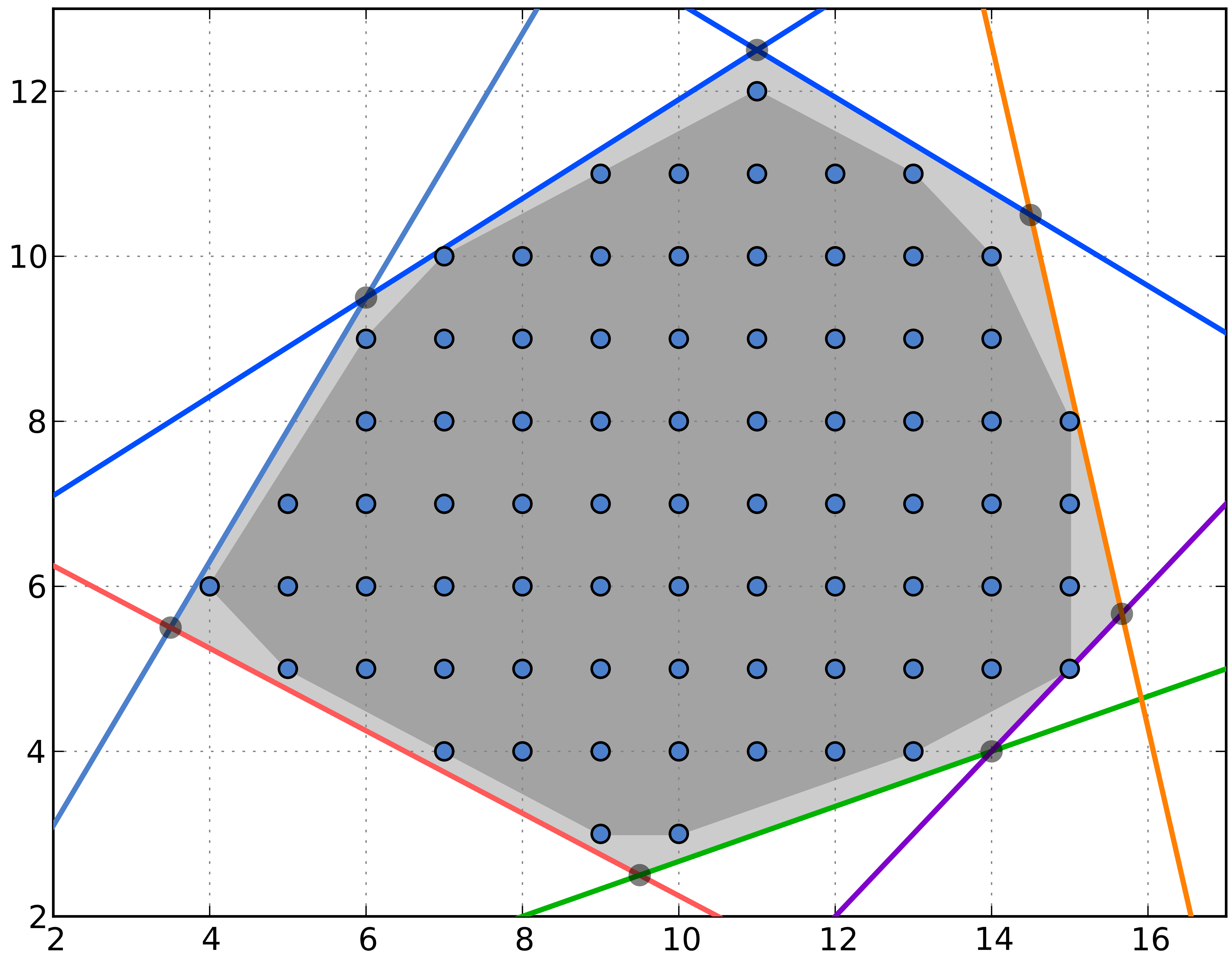

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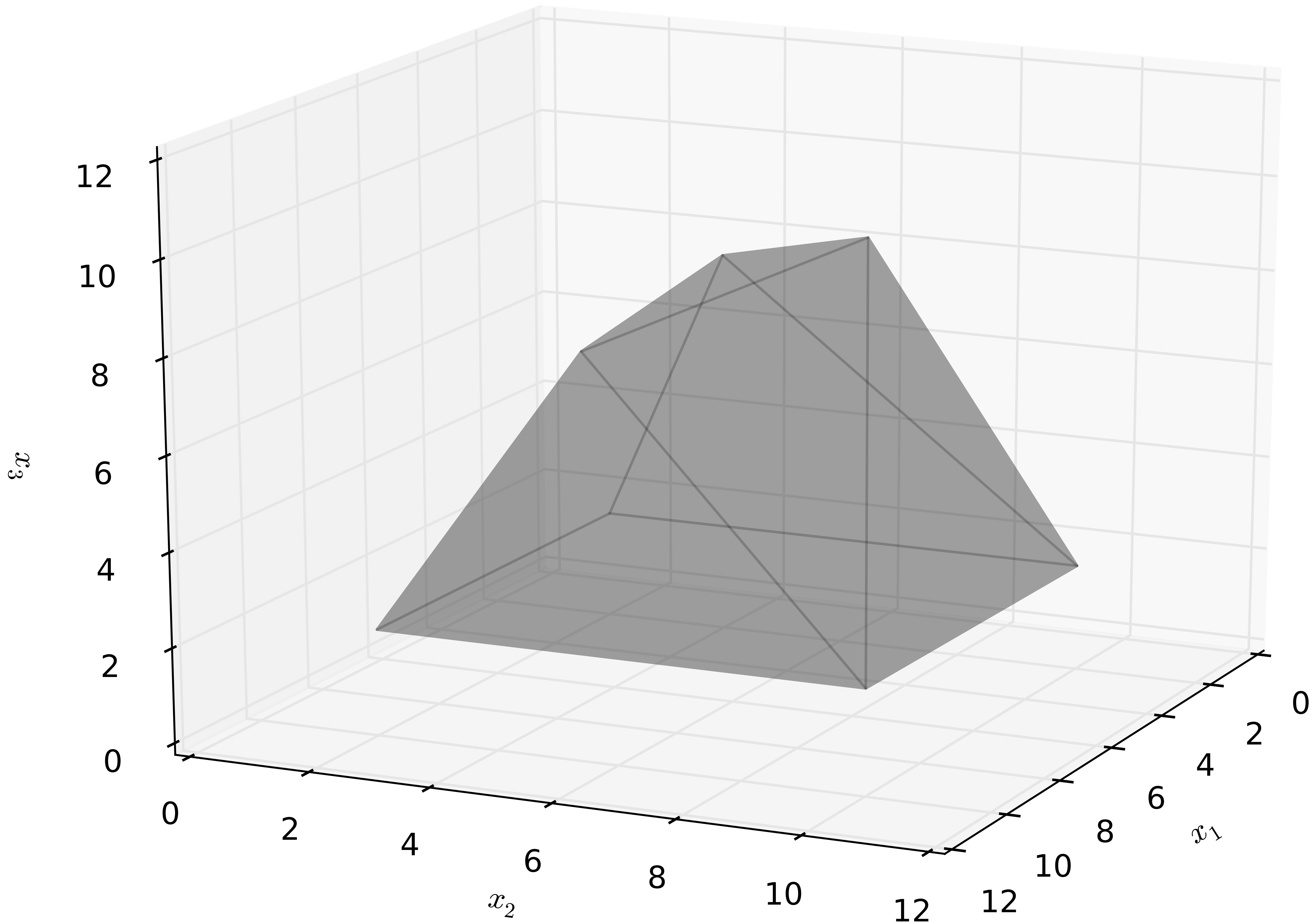

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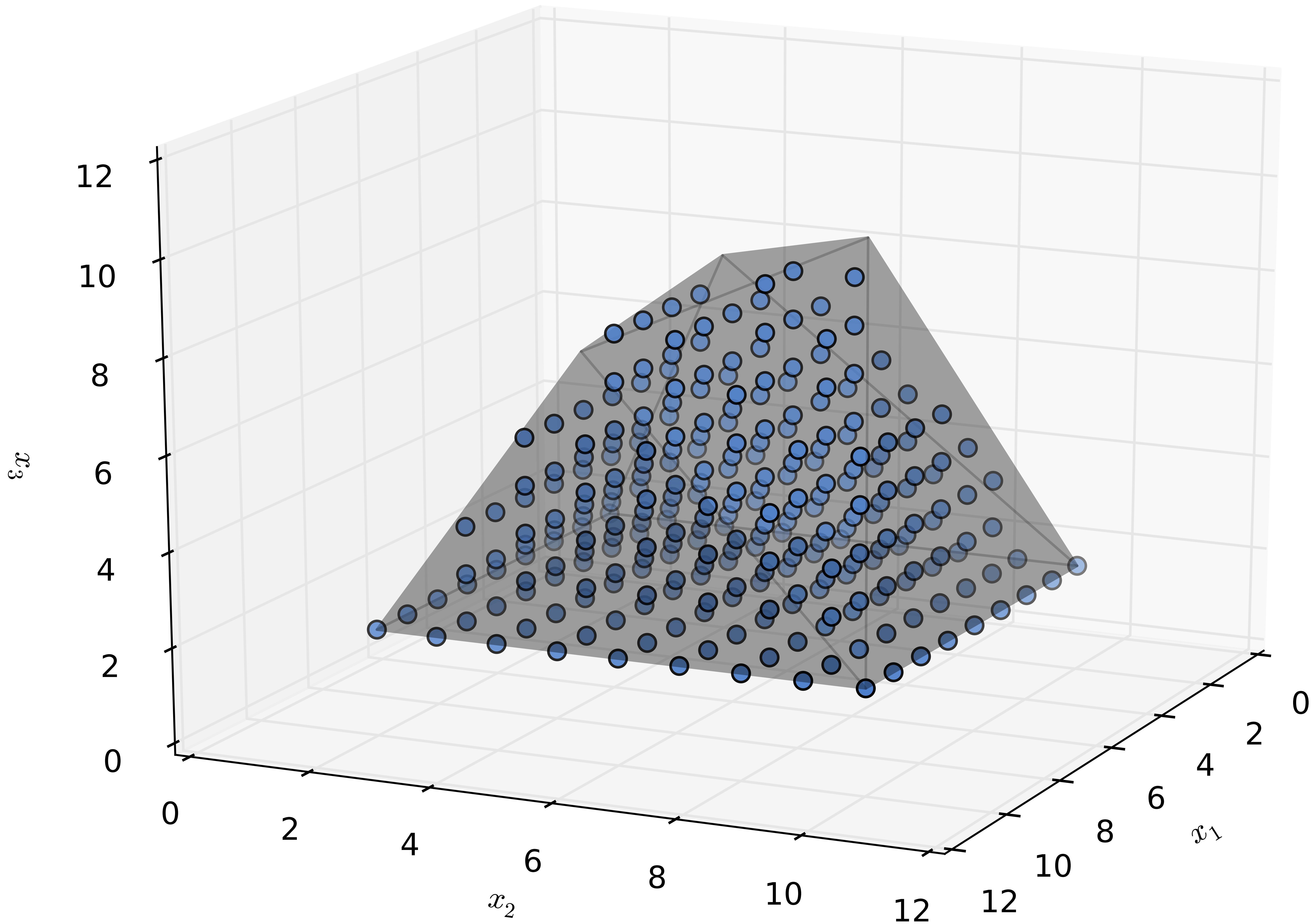

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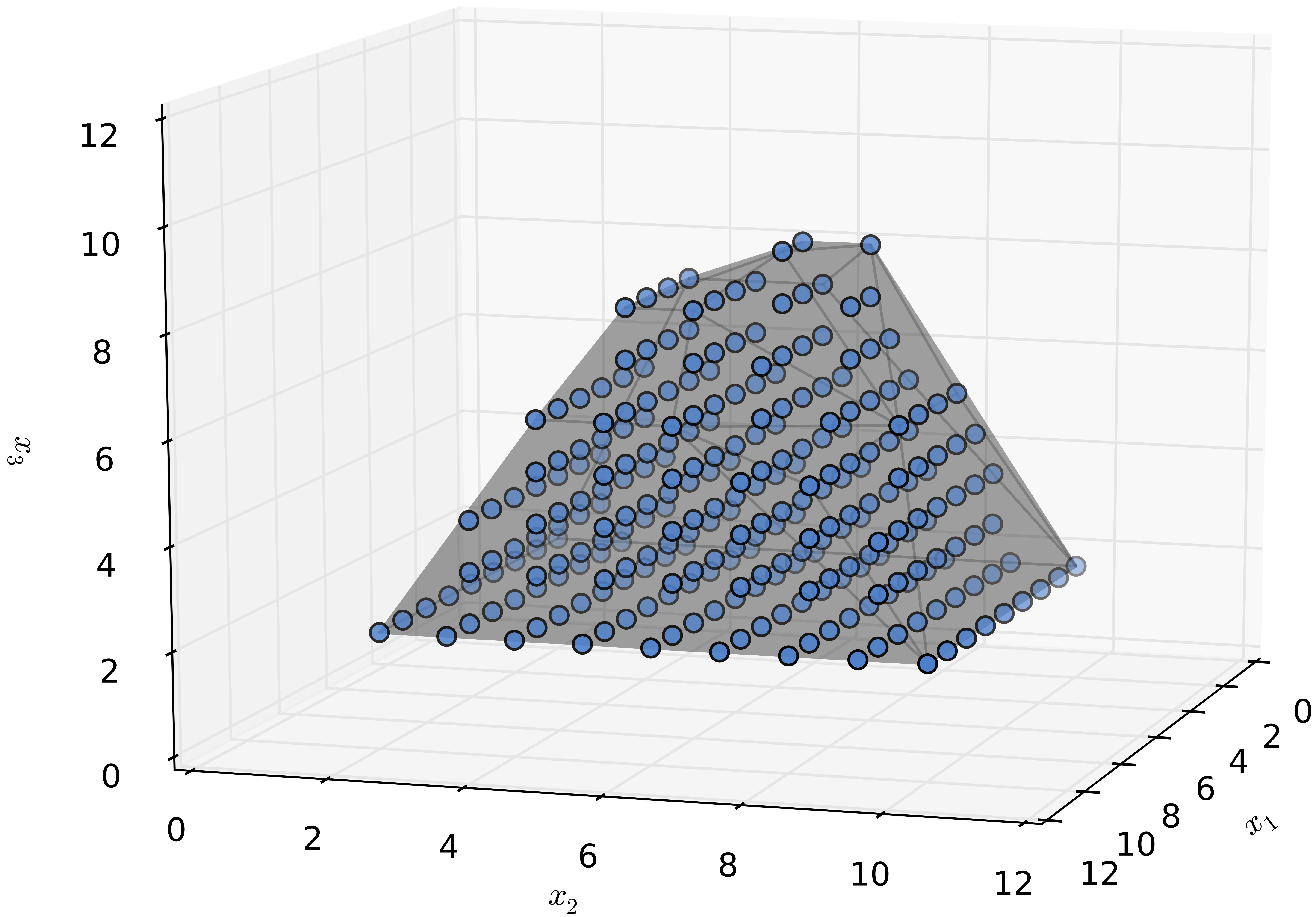

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  - cuts that represent the facets of the convex hull of the integer solution
- These cuts are valid
  - they do not remove any solution
- The cuts are as strong as possible
  - if we have all of them, we could use linear programming to solve the problem

Polyhedral Cuts

- They exploit the problem structure
  - they are derived from the structure of constraints
  - not based on information in the tableau

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- They share some of the spirit of syntactic cuts
  - validity
  - must cut the current basic feasible solution
  - do not need to generate all of them
- An application may use multiple cut types
  - exploit different substructure

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  $x_1, \ldots, x_n$ are affinely independent iff $(x_1, 1), \ldots, (x_n, 1)$ are linearly independent.

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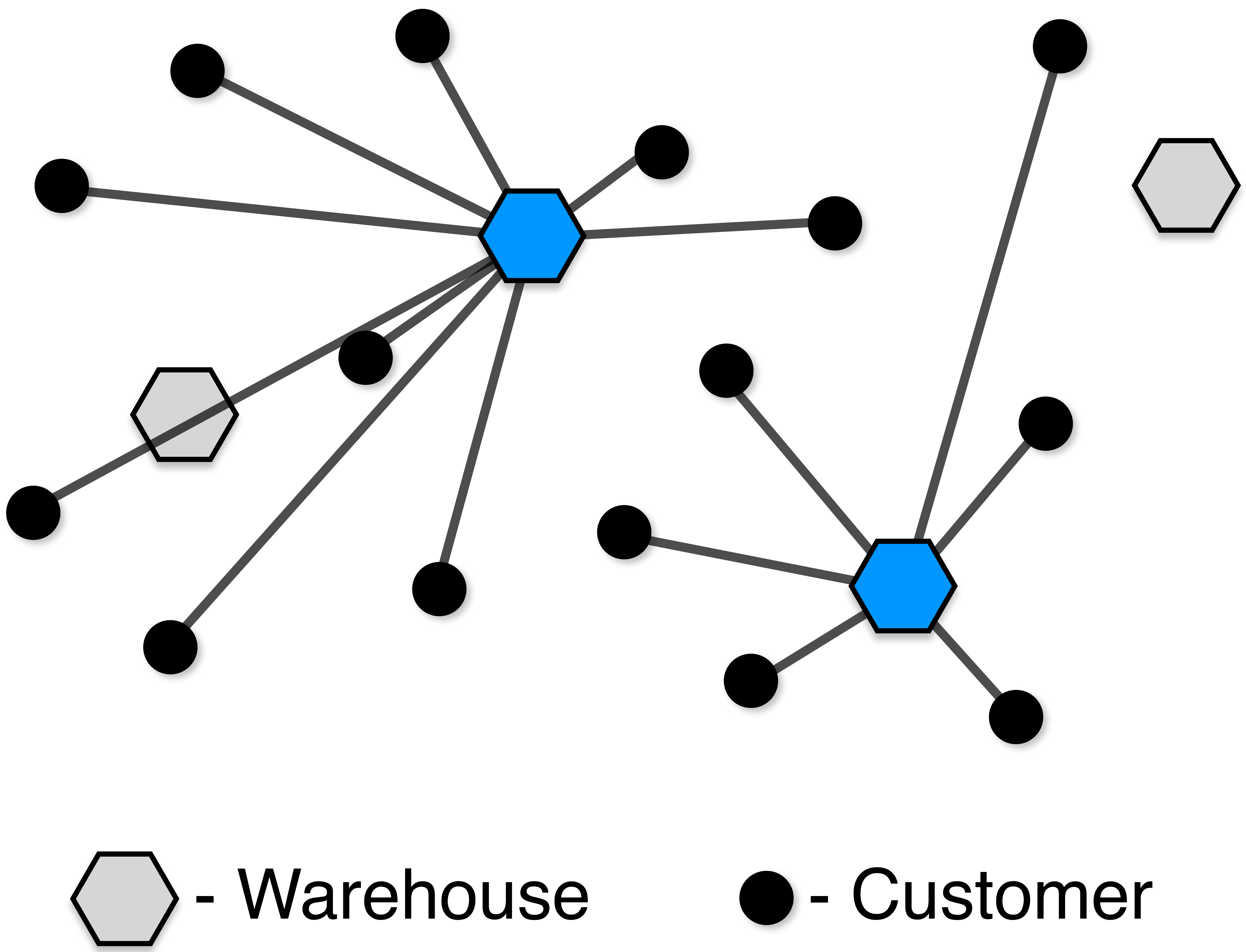
- Linear independence

  $x_1, \ldots, x_n$ are linearly independent iff $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ implies that $\alpha_i = 0$ for all i.

Warehouse Location



Warehouse Location

$$
\begin{aligned}
\min \quad & \sum_{w \in W} c_w \; x_w + \sum_{w \in W, c \in C} t_{w,c} \; y_{w,c} \\
\text{subject to} \quad & \\
& y_{w,c} \leq x_w & (w \in W, c \in C) \\
& \sum_{w \in W} y_{w,c} = 1 & (c \in C) \\
& x_w \in \{0, 1\} & (w \in W) \\
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Facets for Warehouse Location

- Are these inequalities facets?

$$y_{w,c} \leq x_w$$

Facets for Warehouse Location

- Are these inequalities facets?

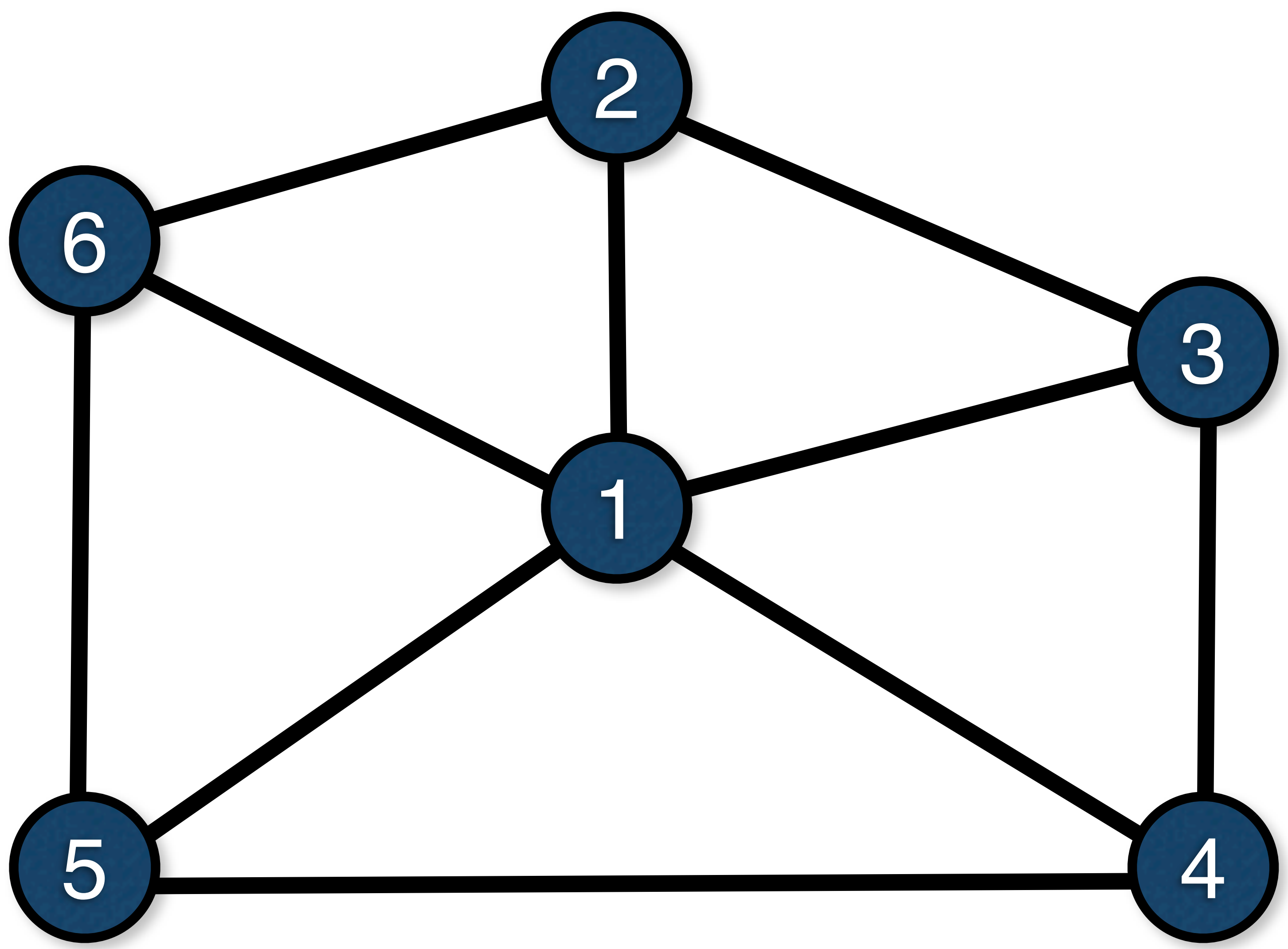
$$y_{w,c} \leq x_w$$

- Consider $y_{w,1} \leq x_w$ and the following n points

w	1	2	3	$\ldots$	n	
0	0	0	0	$\ldots$	0	1
1	0	0	0	$\ldots$	0	1
1	1	0	0	$\ldots$	0	1
1	1	1	0	$\ldots$	0	1
1	1	0	1	$\ldots$	0	1
				$\ldots$		
1	1	0	0	$\ldots$	1	1

Node Packing

- Let G=(V,E) be a graph.
  - A node packing is a subset W of V such that no two nodes in W are connected by an edge. The goal is to find the node packing of maximal size.



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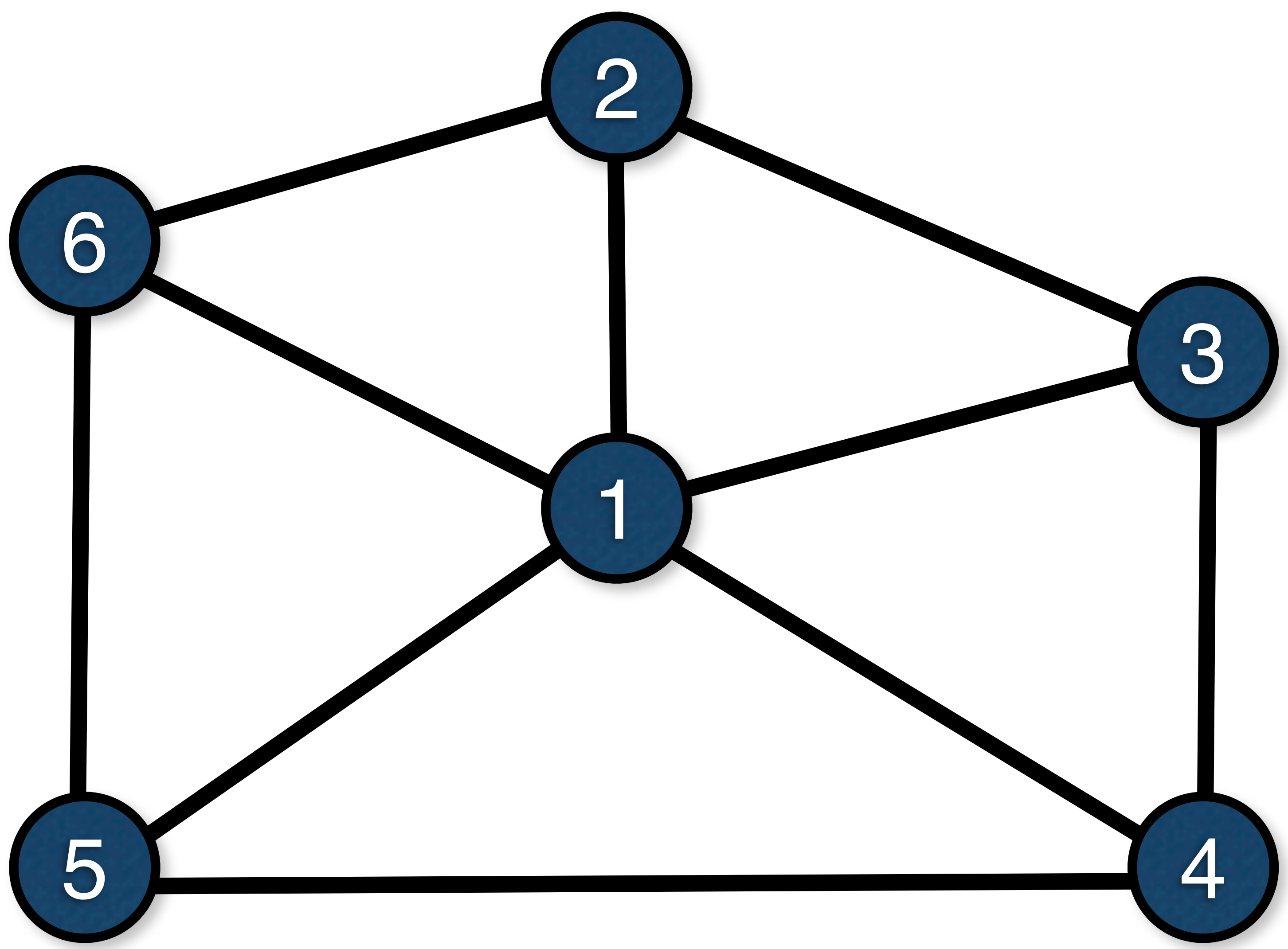

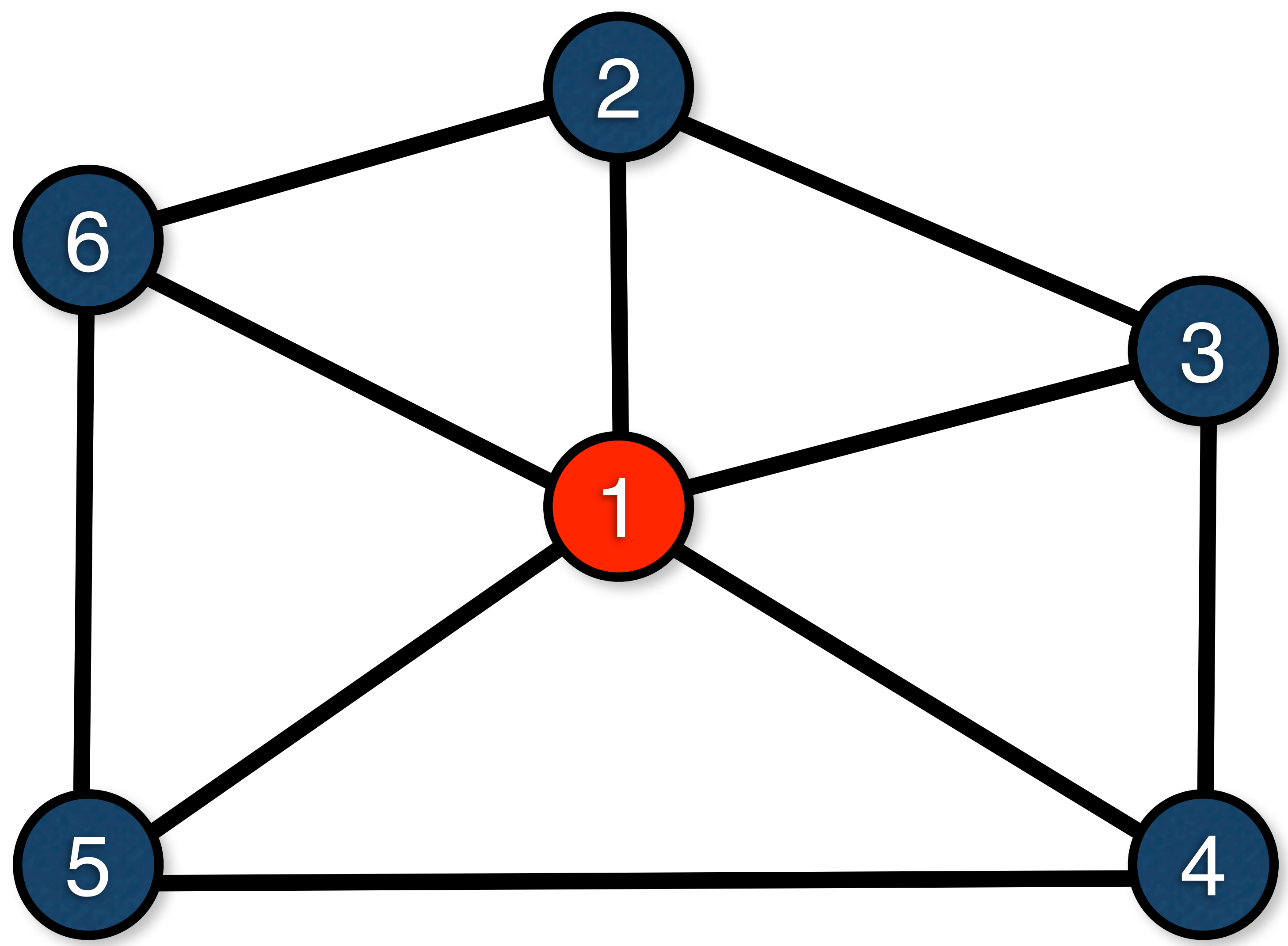

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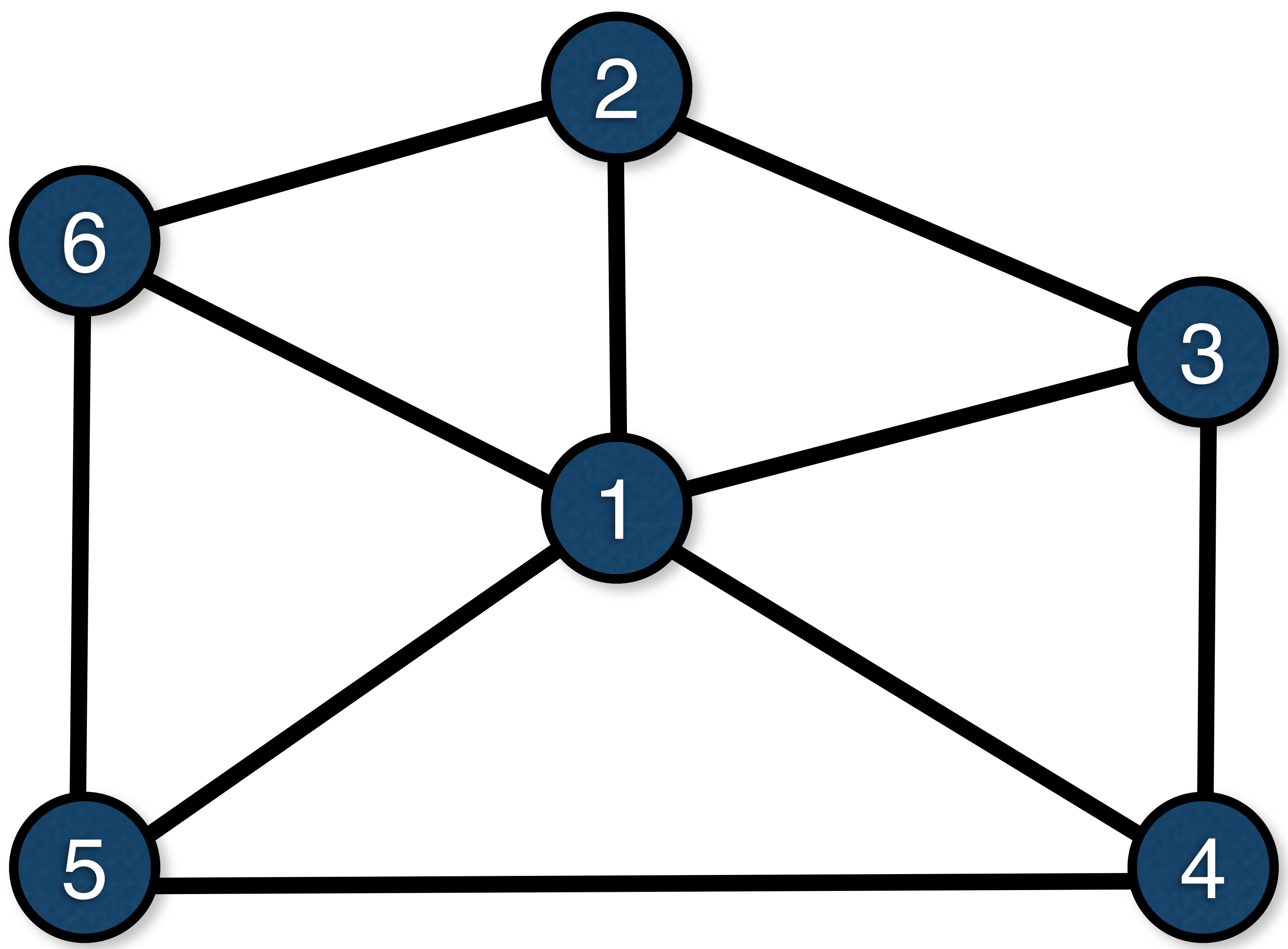

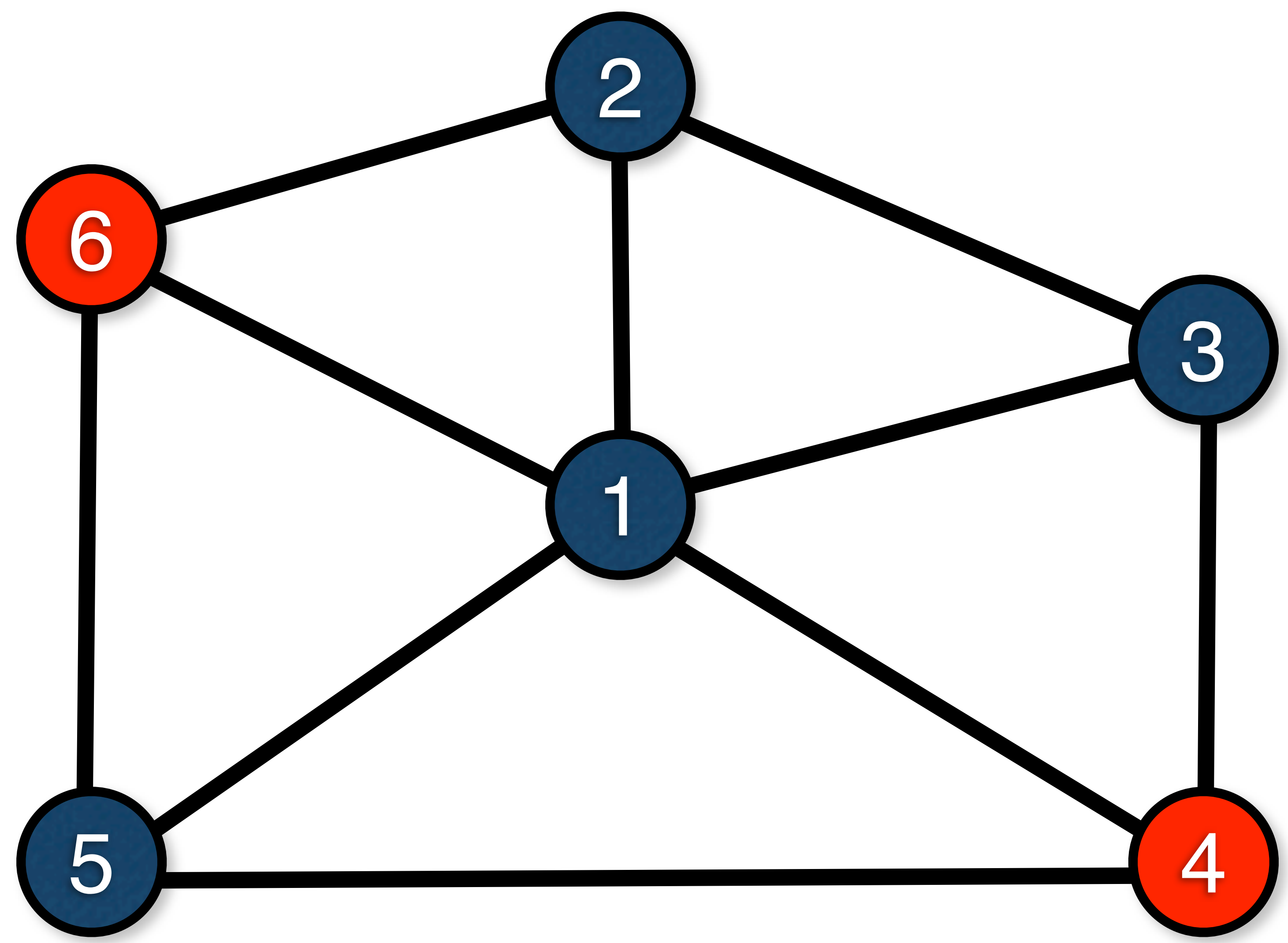

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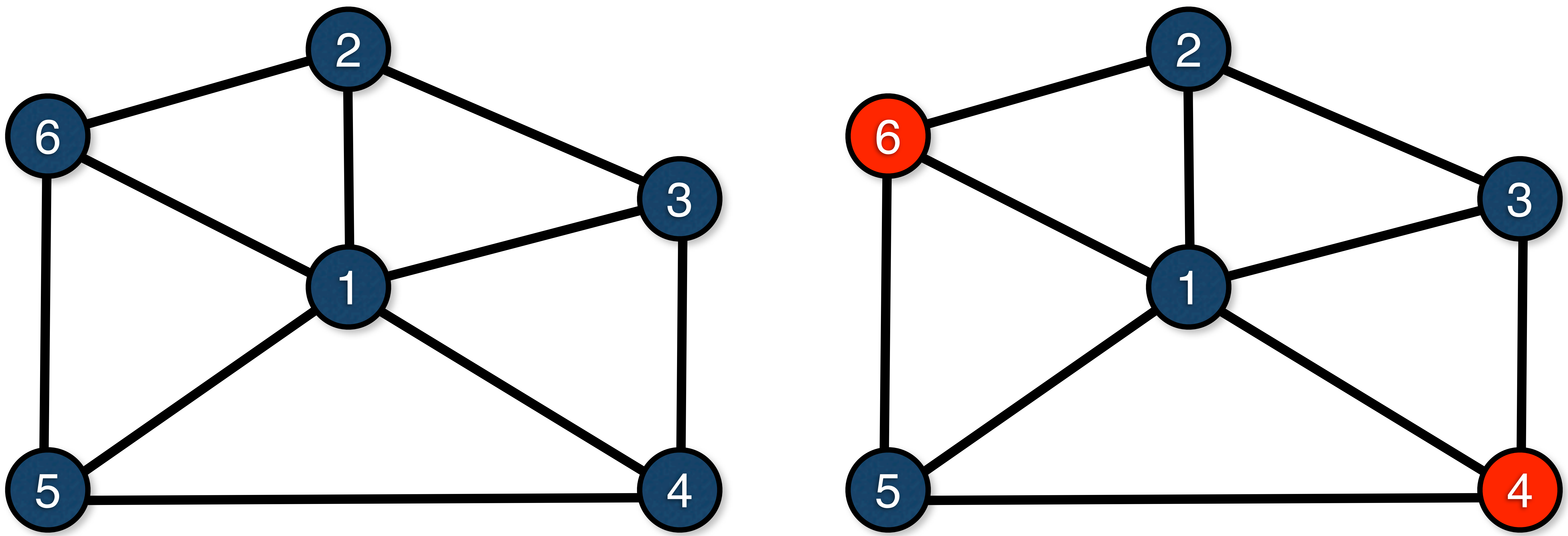

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Node Packing

$$
\begin{array}{llllll}
\max & x_1 & + & \dots & + & x_6 \\
\text{s.t.} & & & & & \\
 & x_1 & + & x_2 & \leq & 1 \\
 & x_1 & + & x_3 & \leq & 1 \\
 & x_1 & + & x_4 & \leq & 1 \\
 & x_1 & + & x_5 & \leq & 1 \\
 & x_1 & + & x_6 & \leq & 1 \\
 & x_2 & + & x_3 & \leq & 1 \\
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- What does the linear relaxation produce?

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$$

- What does the linear relaxation produce?

$$x_1 = \frac{1}{2}, \ldots, x_6 = \frac{1}{2}$$

How do we Find Facets?

- Find a property of the solution
  - constraints satisfied by all solutions

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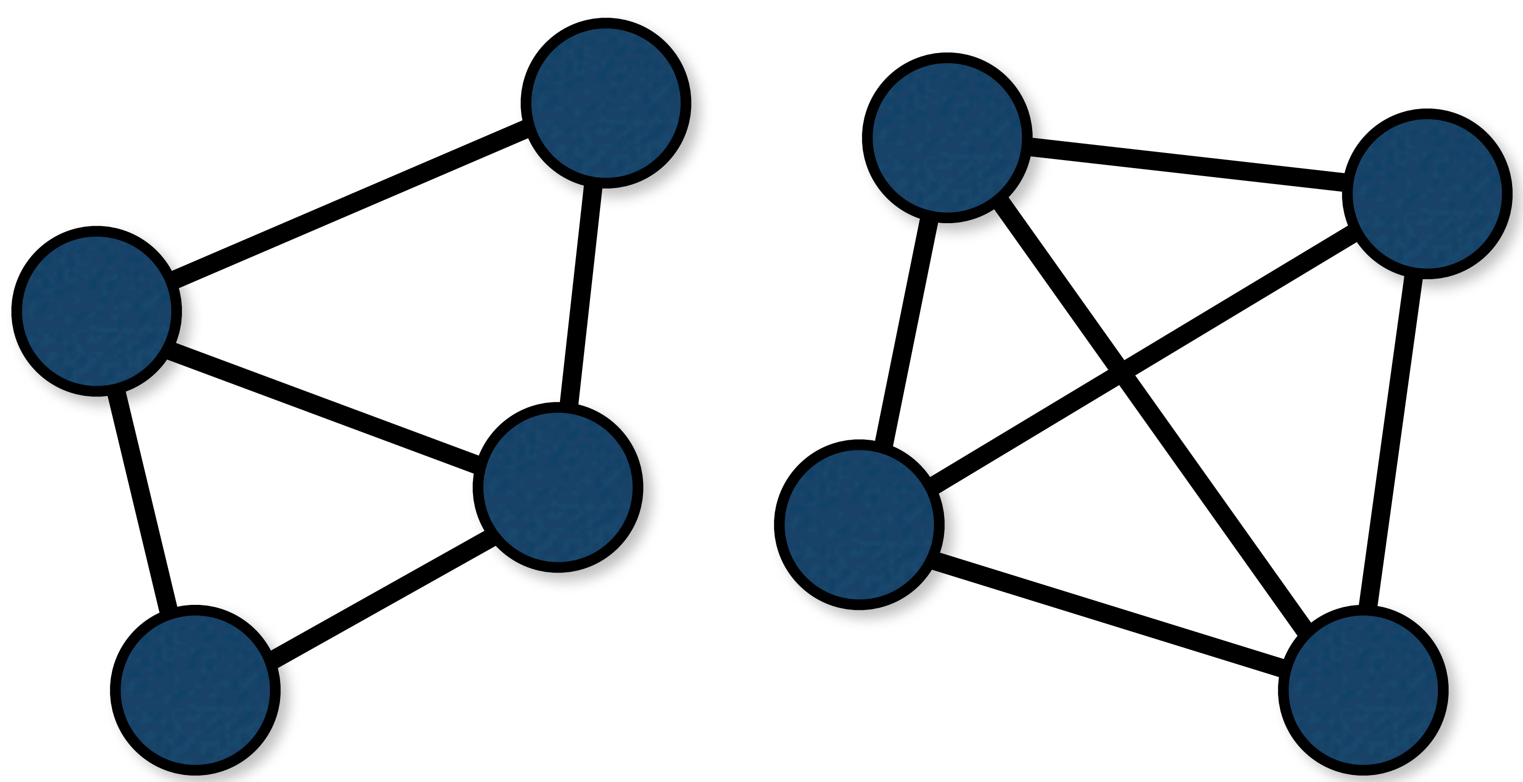
- Consider a clique



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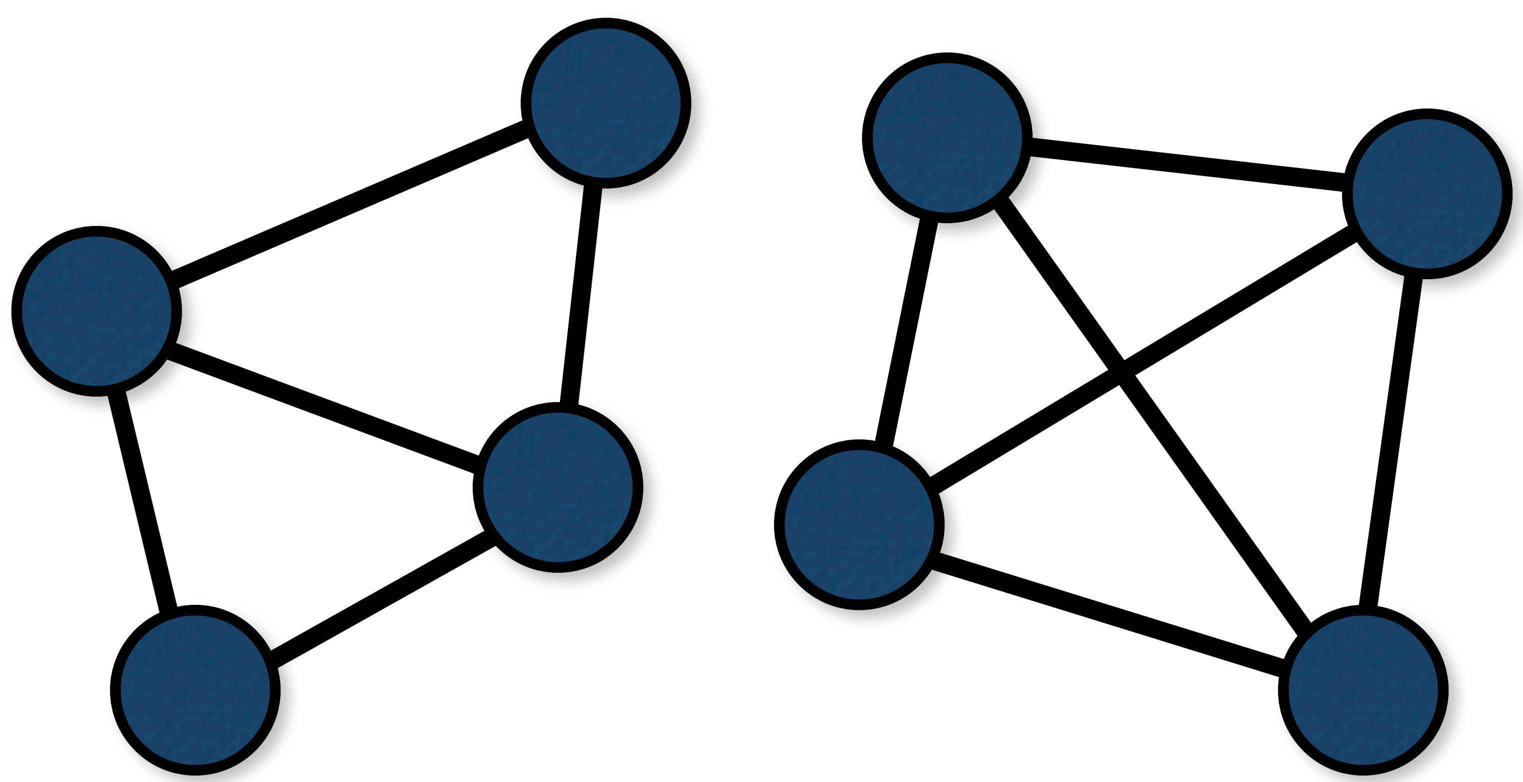
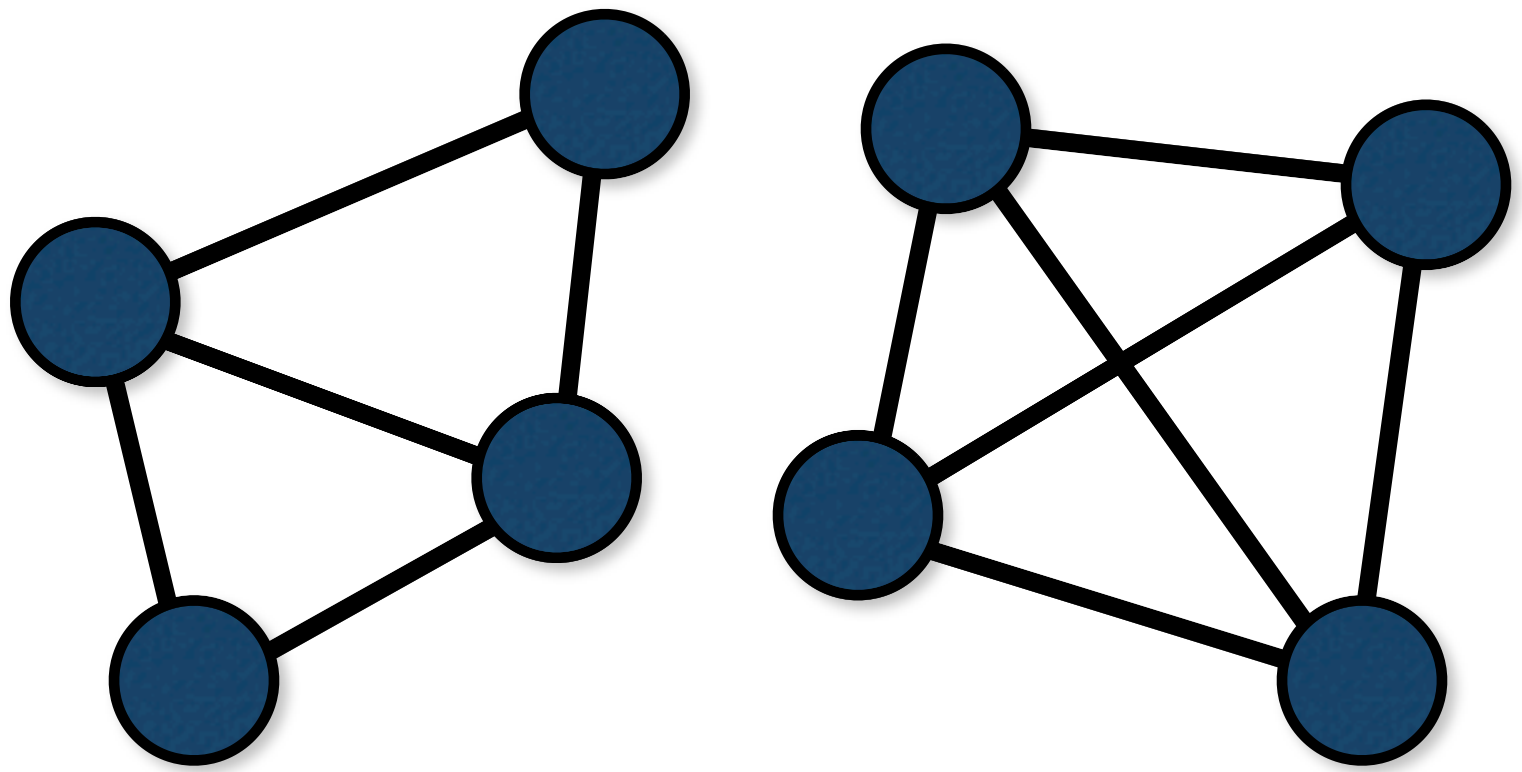


- How many nodes can be in a packing?

Clique Facets

- Consider a clique



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  - a clique that cannot be extended further

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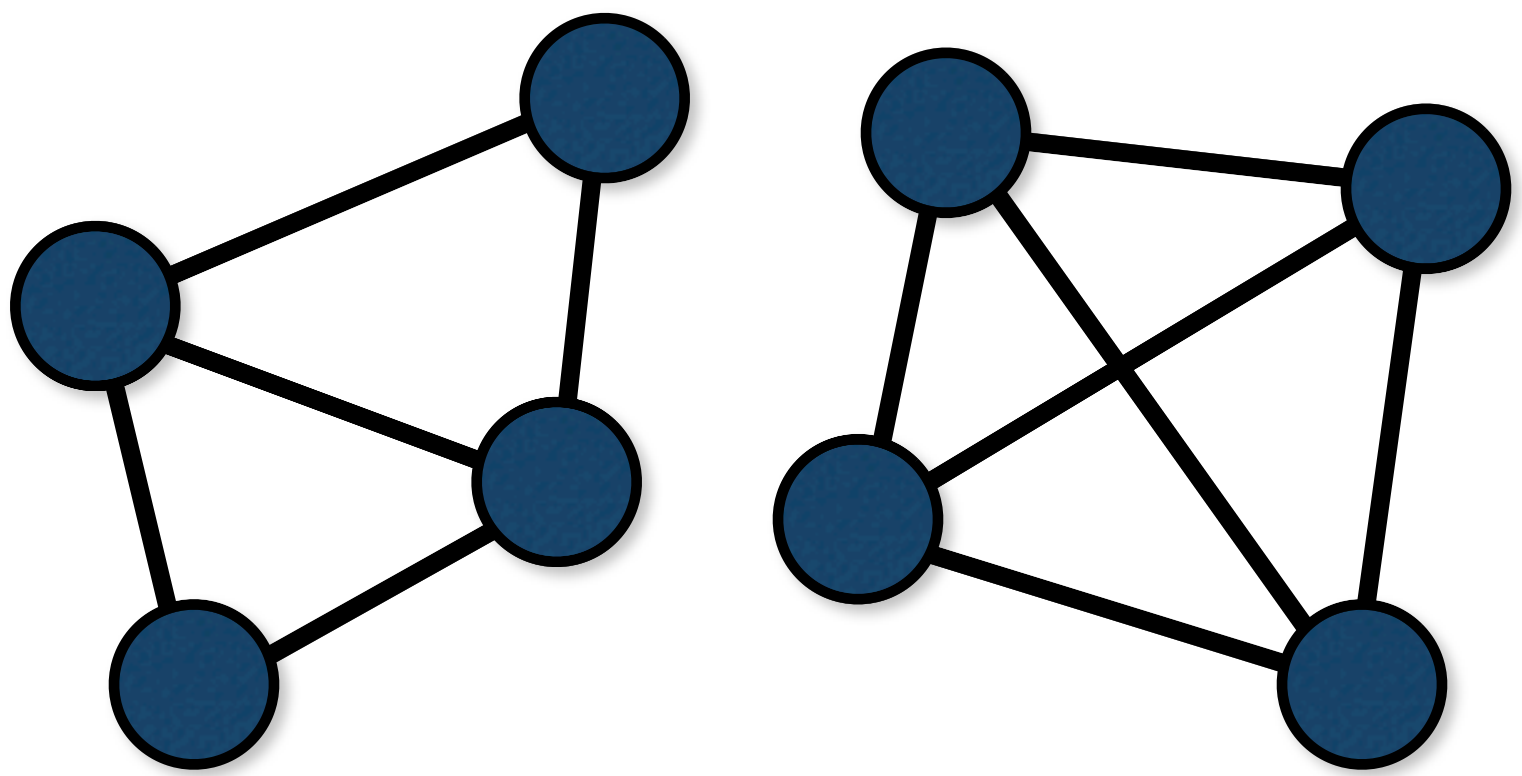

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$$x_1 + \ldots + x_5 \leq 1$$

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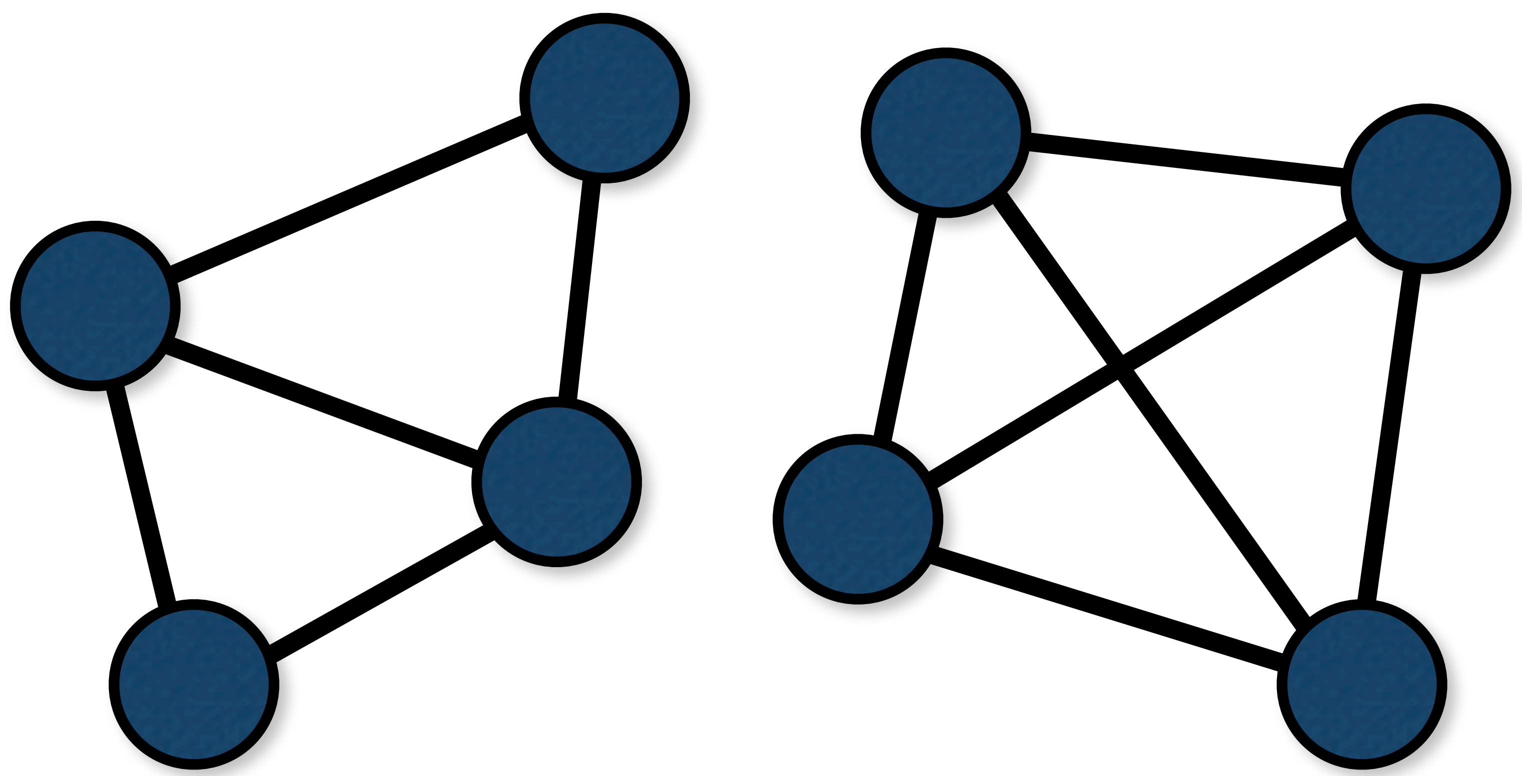

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- A maximal clique constraint is a facet!

MIP with Clique Facets

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\begin{array}{lllllllllllll}
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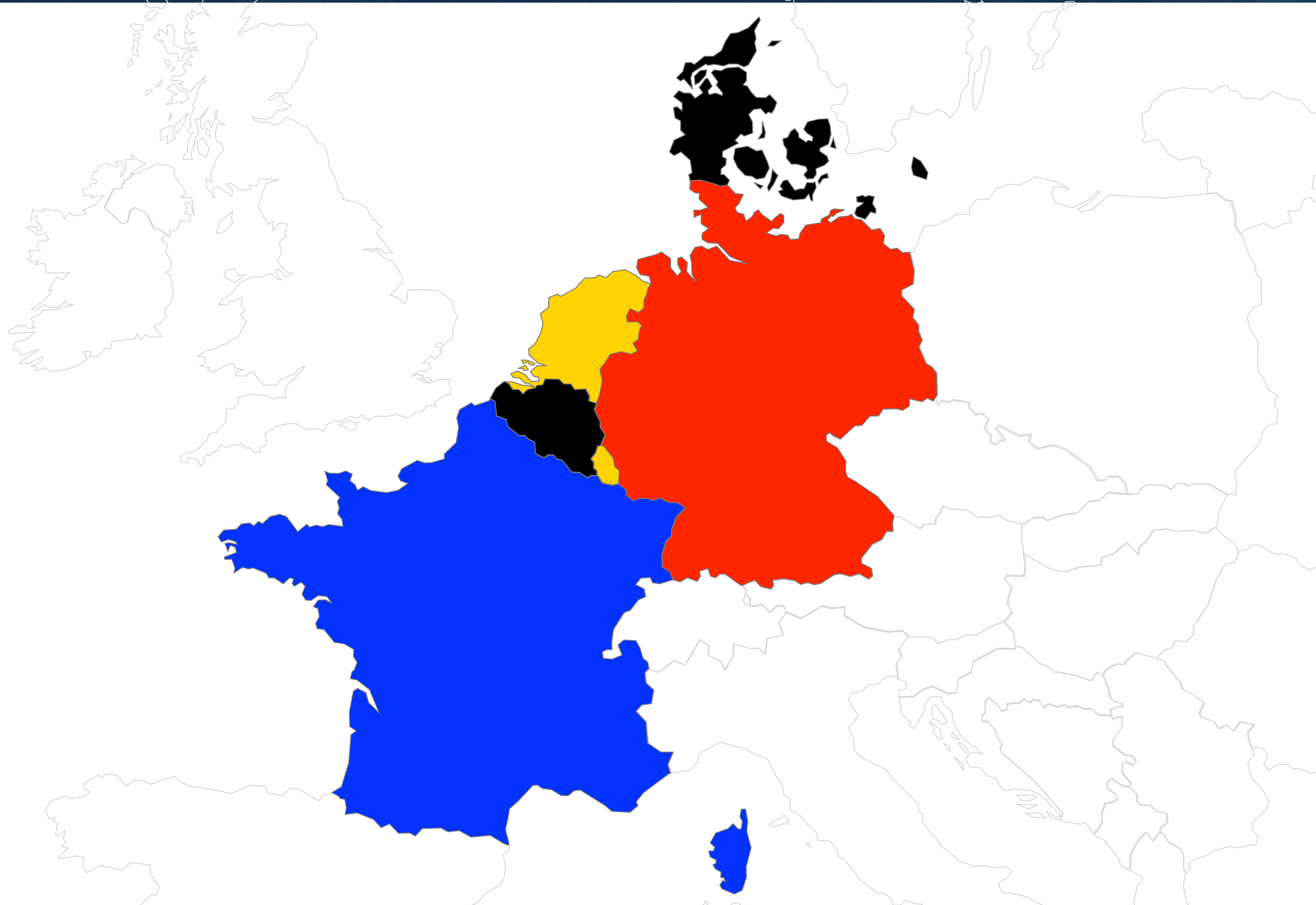
$$x_1 = 0, x_2 = \frac{1}{2}, \ldots, x_6 = \frac{1}{2}$$

Coloring a Map



Coloring a Map with 0/1 Variables

MIP:

Optimal - 9 nodes

Proof - 41 nodes

$$obj \in \{0, 1, 2, 3\}$$
$$\text{color}_{c,v} \in \{0, 1\}$$

$$\min \quad obj$$

$$\text{subject to} \quad obj \geq \sum_{v=0}^{3} v \times \text{color}_{c,v} \qquad (c \in C)$$

$$\sum_{v=0}^{3} \text{color}_{c,v} = 1 \qquad (c \in C)$$

$$\text{color}_{c_1,v} + \text{color}_{c_2,v} \leq 1 \quad (c_1, c_2 \in C \text{ and adjacent}, v \in 0..3)$$

LP:

$$obj = 0.5$$
$$\text{color}_{c,0} = 0.5$$
$$\text{color}_{c,1} = 0.5$$
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Need at least 2 colors!

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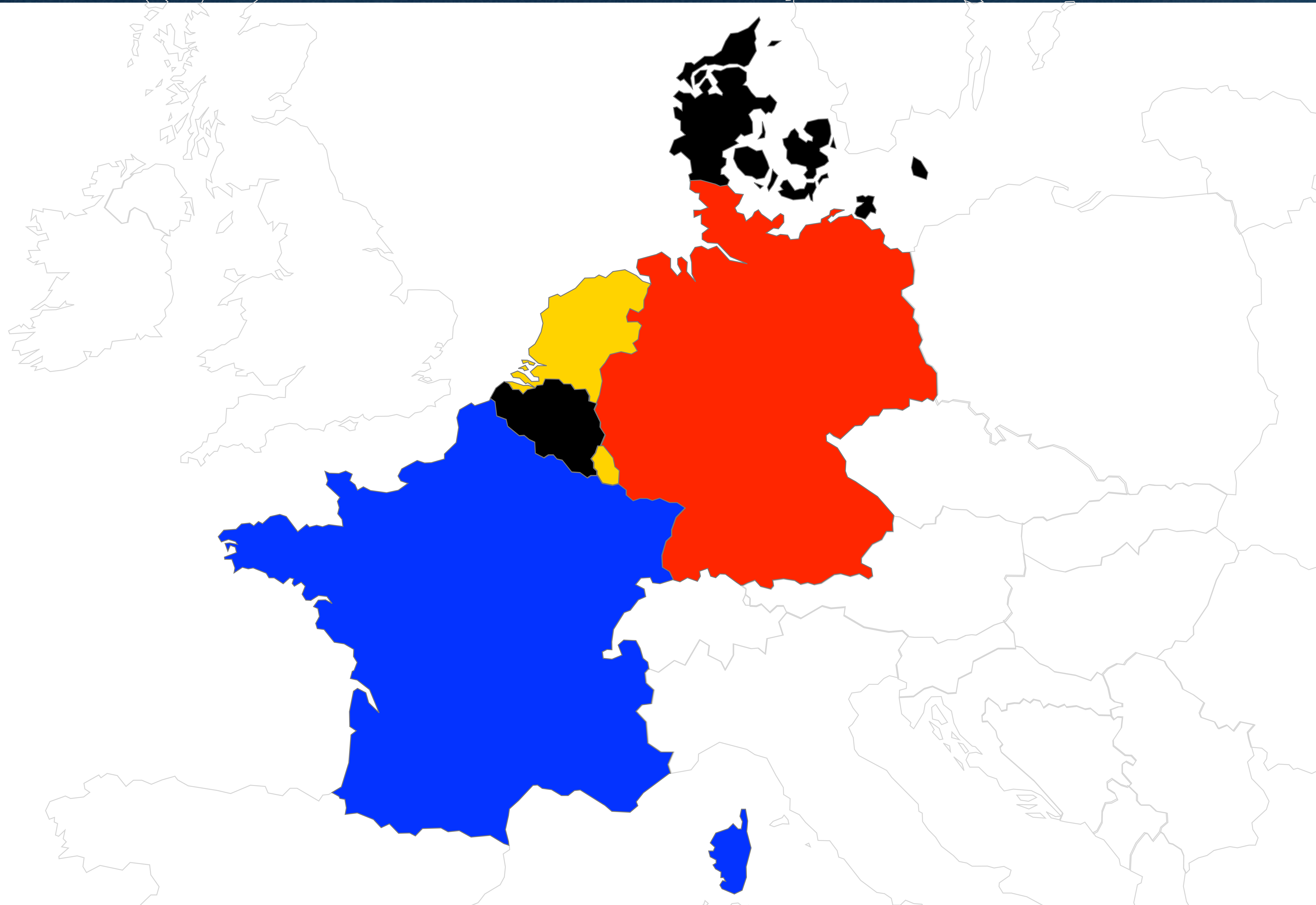
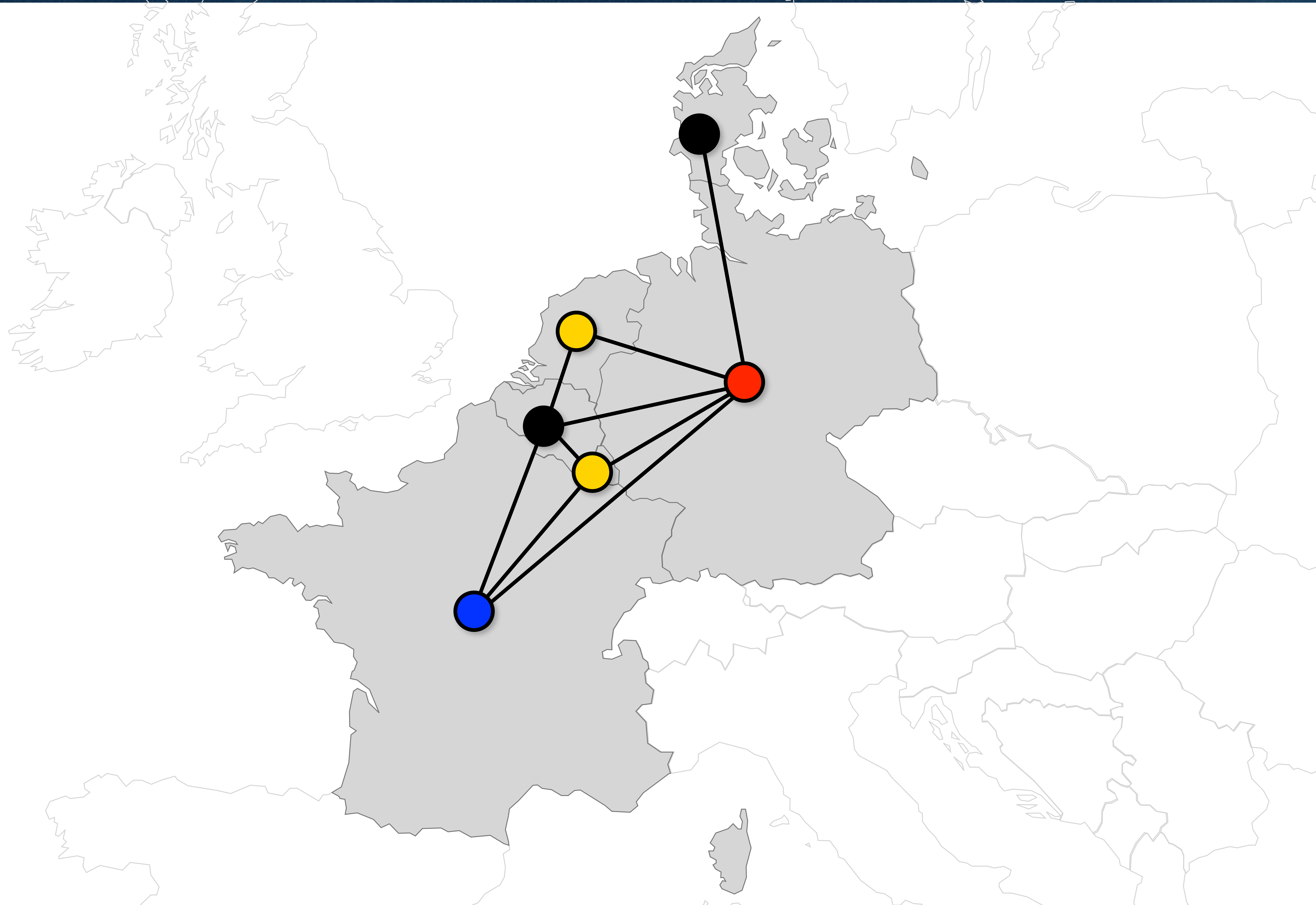


Coloring a Map as a Graph



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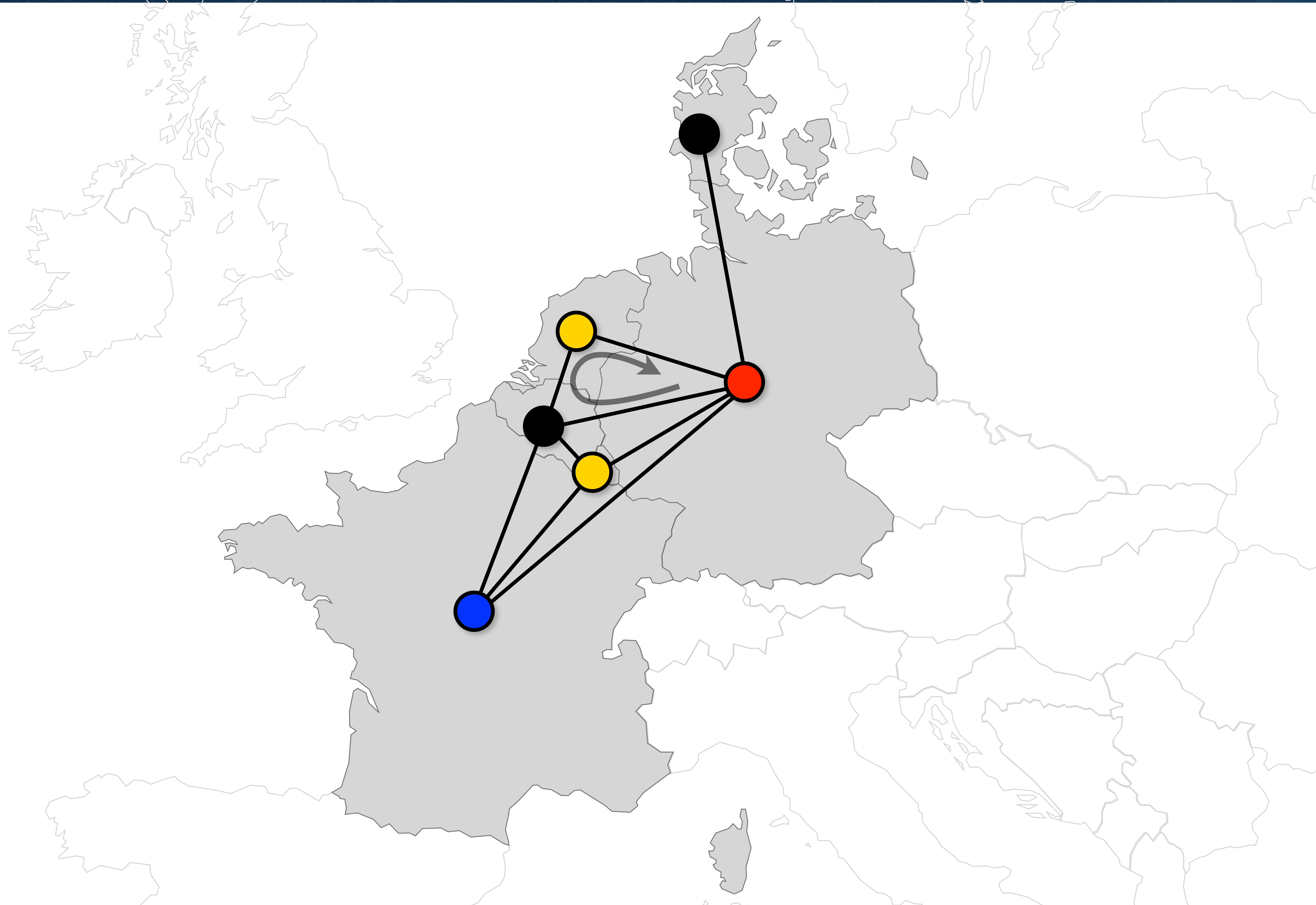

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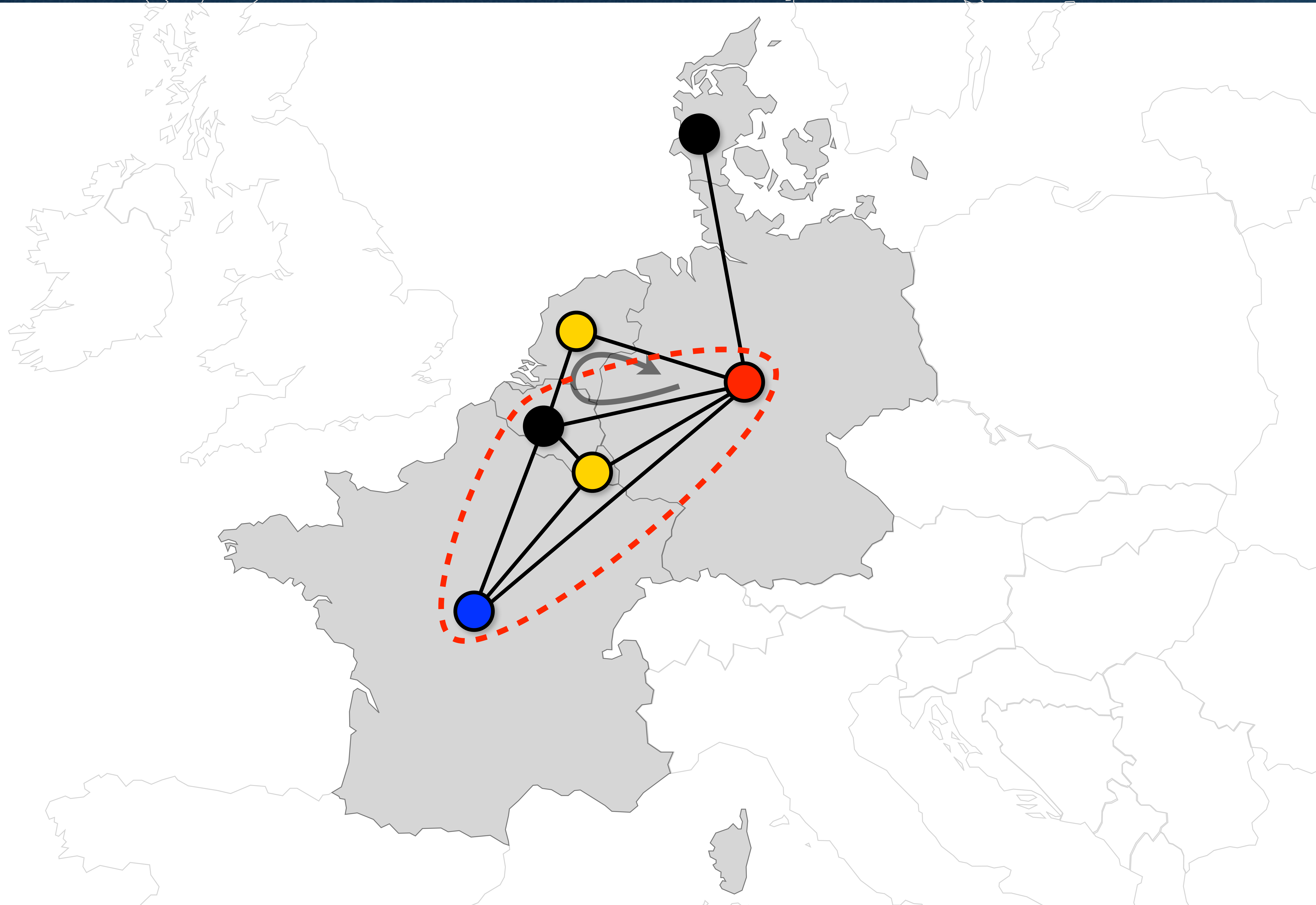

Improving the Coloring Relaxation

- With the best model from before

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$$\sum_{c \in \{0,3,4\}}^{3} \sum_{v=0}^{3} v \times \mathrm{color}_{c,v} \geq 3$$

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- Add the 4-Clique

$$\sum_{c \in \{0,2,3,5\}}^{3} \sum_{v=0}^{3} v \times \text{color}_{c,v} \geq 6$$

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$$obj = 1.0$$

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- Add the 4-Clique

$$\sum_{c \in \{0,2,3,5\}}^{3} \sum_{v=0}^{3} v \times \text{color}_{c,v} \geq 6$$

MIP:
Optimal - 5 nodes
Proof - 9 nodes

$$obj = 1.5$$ Need at least 3 colors!!!

Until Next Time