Discrete Optimization

Linear Programming: Part V

Goals of the Lecture

- Linear programming
  - duality theory

Duality

$$\begin{array}{ll} \min & c\,x \\ \text{subject to} & \\ & Ax \geq b \\ & x_j \geq 0 \end{array}$$

primal

$$\begin{array}{ll} \max & y\,b \\ \text{subject to} & \\ & yA \leq c \\ & y_i \geq 0 \end{array}$$

dual

Duality

$$
\begin{array}{llllllll}
\min & 3x_1 & + & 2x_2 & + & 4x_3 & & \\
\text{subject to} & & & & & & & \\
 & 2x_1 & + & x_2 & + & & \geq & 2 \\
 & 2x_1 & - & x_2 & + & x_3 & \geq & 5
\end{array}
$$

$$
\begin{array}{llllll}
\max & 2y_1 & + & 5y_2 & & \\
\text{subject to} & & & & & \\
 & 2y_1 & + & 2y_2 & \leq & 3 \\
 & y_1 & - & y_2 & \leq & 2 \\
 & & & y_2 & \leq & 4
\end{array}
$$

How do We Obtain this Dual?

$$\begin{array}{llllllll}
\min & 3x_1 & + & 2x_2 & + & 4x_3 & & \\
\text{subject to} & & & & & & & \\
& 2x_1 & + & x_2 & + & & \geq 2 & y_1 \\
& 2x_1 & - & x_2 & + & x_3 & \geq 5 & y_2
\end{array}$$

$$\begin{array}{lllll}
\max & 2y_1 & + & 5y_2 & \\
\text{subject to} & & & & \\
& 2y_1 & + & 2y_2 & \leq 3 \\
& y_1 & - & y_2 & \leq 2 \\
& & & y_2 & \leq 4
\end{array}$$

Duality

$$
\begin{array}{llllllll}
\text{min} & 3x_1 & + & 2x_2 & + & 4x_3 & & \\
\text{subject to} & & & & & & & \\
& 2x_1 & + & x_2 & + & & \geq & 2 \\
& 2x_1 & - & x_2 & + & x_3 & \geq & 5
\end{array}
$$

$$
\begin{array}{llllll}
\text{max} & 2y_1 & + & 5y_2 & & \\
\text{subject to} & & & & & \\
& 2y_1 & + & 2y_2 & \leq & 3 \\
& y_1 & - & y_2 & \leq & 2 \\
& & & y_2 & \leq & 4
\end{array}
$$

Duality

$$\min \quad \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\max \quad \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

subject to

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Duality

primal

$$\begin{array}{ll} \text{min} & c\,x \\ \text{subject to} & \\ & Ax \geq b \\ & x_j \geq 0 \end{array}$$

dual

$$\begin{array}{ll} \text{max} & y\,b \\ \text{subject to} & \\ & yA \leq c \\ & y \geq 0 \end{array}$$

- Theorem: If the primal has an optimal solution, the dual has an optimal solution with the same cost

  Let x and Π be feasible solutions to the primal and dual respectively.

  We have that $cx \geq \Pi A x \geq \Pi b$.

  – since the primal has a feasible solution, the dual cannot be unbounded.

Testing if a Basis is Optimal

- What are the costs in the basic feasible solution?

$$cx = c_B A_B^{-1} b + \boxed{(c - c_B A_B^{-1} A)}x$$

$$cx = \Pi b + \boxed{(c - \Pi A)}x$$

- The basis is optimal if these costs are non-negative
- So the simplex multiplier are a feasible solution to the dual

Duality

- Theorem: If the primal has an optimal solution, the dual has an optimal solution with the same cost

Consider the optimal solution x^*.
It has an associated basis B

$$x^*_B = A_B^{-1} b.$$

The dual has a feasible solution

$$y^* = c_B A_B^{-1}$$

by the optimality of the primal. Hence,

$$y^* b = c_B A_B^{-1} b = c_B x^*.$$

General Form of the Dual

$$
\begin{array}{lrcll}
\min & c\;x & & & \\
\text{subject to} & & & & \\
 & a_i\;x & = & b_i & (i \in E) \\
 & a_i\;x & \geq & b_i & (i \in I) \\
 & x_j & \geq & 0 & (j \in P) \\
 & x_j & \in & \mathcal{R} & (j \in O)
\end{array}
$$

Primal

$$
\begin{array}{lrcll}
\max & y\;b & & & \\
\text{subject to} & & & & \\
 & y_i & \in & \mathcal{R} & (i \in E) \\
 & y_i & \geq & 0 & (i \in I) \\
 & yA_j & \leq & c_j & (j \in P) \\
 & yA_j & = & c_j & (j \in O)
\end{array}
$$

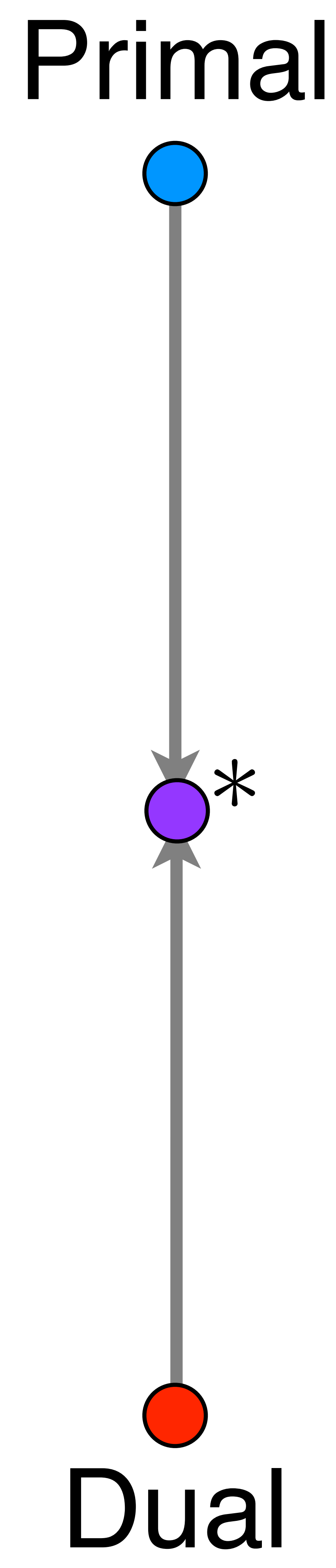
Dual

Properties of Duality

- The dual of the dual is the primal

	Finite Primal	Unbounded Primal	Infeasible Primal
Finite Dual	Yes	?	?
Unbounded Dual	?	?	?
Infeasible Dual	?	?	?

Primal and Dual



Properties of Duality

- The dual of the dual is the primal

	Finite Primal	Unbounded Primal	Infeasible Primal
Finite Dual	Yes	?	?
Unbounded Dual	?	?	?
Infeasible Dual	?	?	?

Primal and Dual

Let x and Π be feasible solutions to the primal and dual respectively. We have that $cx \geq \Pi A x \geq \Pi b$.

Properties of Duality

- The dual of the dual is the primal

	Finite Primal	Unbounded Primal	Infeasible Primal
Finite Dual	Yes	?	?
Unbounded Dual	?	?	?
Infeasible Dual	?	?	?

Primal / Dual Relationships

$$
\begin{array}{ll}
\text{min} & x_1 \\
\text{subject to} & \\
& \begin{array}{rcrcl} x_1 & + & x_2 & \geq & 1 \\ -x_1 & - & x_2 & \geq & 1 \end{array}
\end{array}
$$

infeasible primal

$$
\begin{array}{ll}
\text{max} & y_1 + y_2 \\
\text{subject to} & \\
& \begin{array}{rcrcl} y_1 & - & y_2 & = & 1 \\ y_1 & - & y_2 & = & 0 \\ y_i & \geq & 0 & & \end{array}
\end{array}
$$

infeasible dual

Primal / Dual Relationships

$$
\begin{array}{ll}
\min & x_1 \\
\text{subject to} & \\
& \begin{array}{rcrcr} x_1 & + & x_2 & \geq & 1 \\ -x_1 & - & x_2 & \geq & 1 \\ x_j & \geq & 0 & & \end{array}
\end{array}
$$

infeasible primal

$$
\begin{array}{ll}
\max & y_1 + y_2 \\
\text{subject to} & \\
& \begin{array}{rcrcr} y_1 & - & y_2 & \leq & 1 \\ y_1 & - & y_2 & \leq & 0 \\ y_i & \geq & 0 & & \end{array}
\end{array}
$$

unbounded dual

Certificate of Optimality

- NP-Complete Problems
  - certificate of feasibility

- Can you provide
  - a certificate of optimality?

- Consider now a linear program.
  - can you convince me that you have found an optimal solution?

Certificate of Optimality

primal

$$\begin{array}{ll} \min & c\,x \\ \text{subject to} & \\ & Ax \geq b \\ & x_j \geq 0 \end{array}$$

dual

$$\begin{array}{ll} \max & y\,b \\ \text{subject to} & \\ & yA \leq c \\ & y \geq 0 \end{array}$$

- Give me a x* that satisfies A x* ≥ b
- Give me a y* that satisfies y* A ≤ c
- Show me that c x* = y* b.

Until Next Time

Citations

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