Discrete Optimization

Mixed Integer Programming: Part I

Goals of the Lecture

- Mixed Integer Linear Programming (MIP)
  - introduction
  - branch and bound

What is an Integer Program?

$$
\begin{array}{lllllll}
\min & c_1 x_1 & + & \dots & + & c_n x_n & \\
\text{subject to} & & & & & & \\
& a_{11} x_1 & + & \dots & + & a_{1n} x_n & \leq & b_1 \\
& \dots & & & & & \\
& a_{m1} x_1 & + & \dots & + & a_{mn} x_n & \leq & b_m \\
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& x_i \geq 0 & & & & & \\
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\end{array}
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- n variables, m constraints
- variables are nonnegative and possibly integral
- inequality constraints

Mixed Integer Versus Linear Programs?

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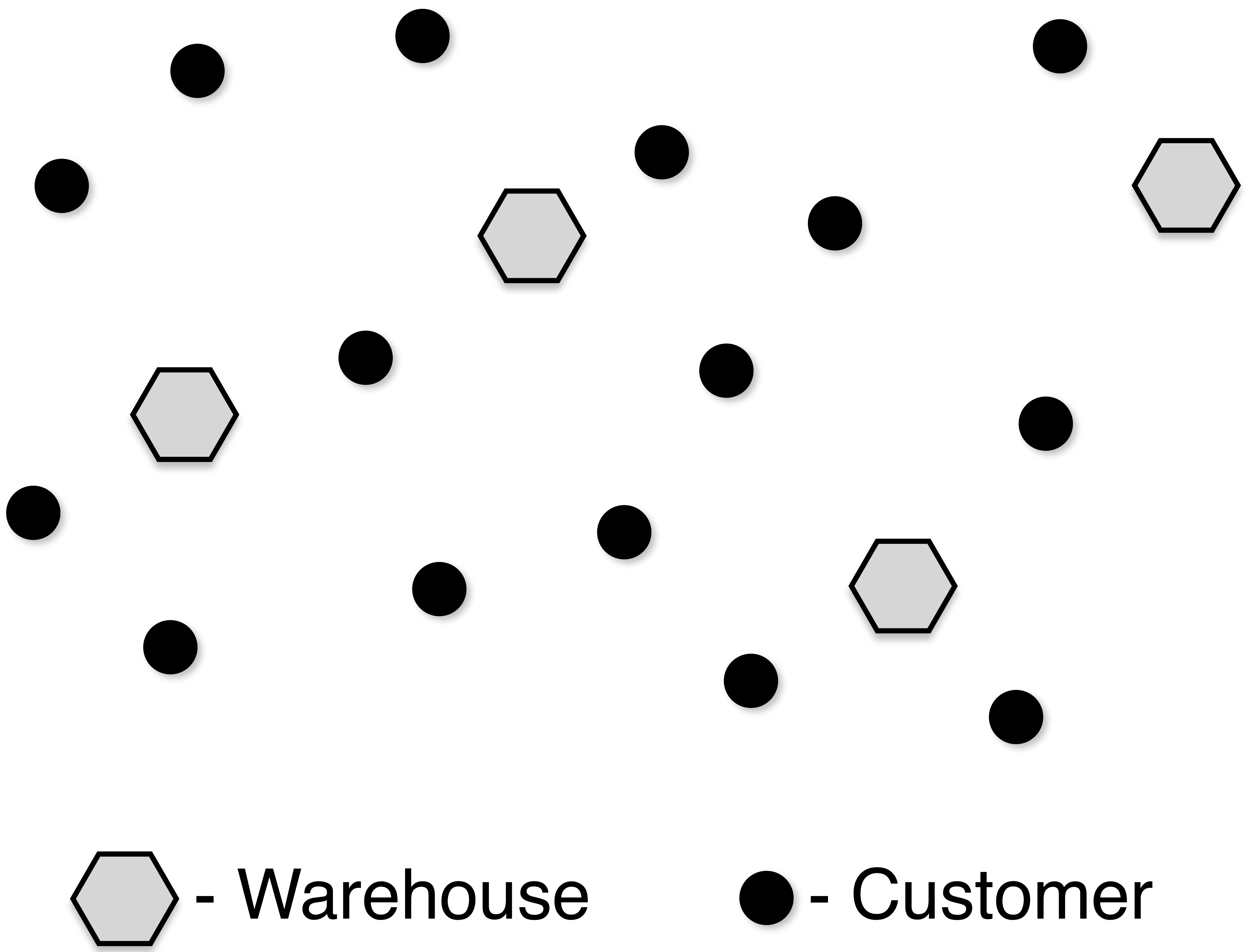
- Integrality constraints
  - the gap between P and NP

The Knapsack Problem

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i \in I} v_i \; x_i \\ \text{subject to} & \\ & \displaystyle\sum_{i \in I} w_i x_i \leq K \\ & x_i \in \{0, 1\} \quad (i \in I) \end{array}$$

Warehouse Location



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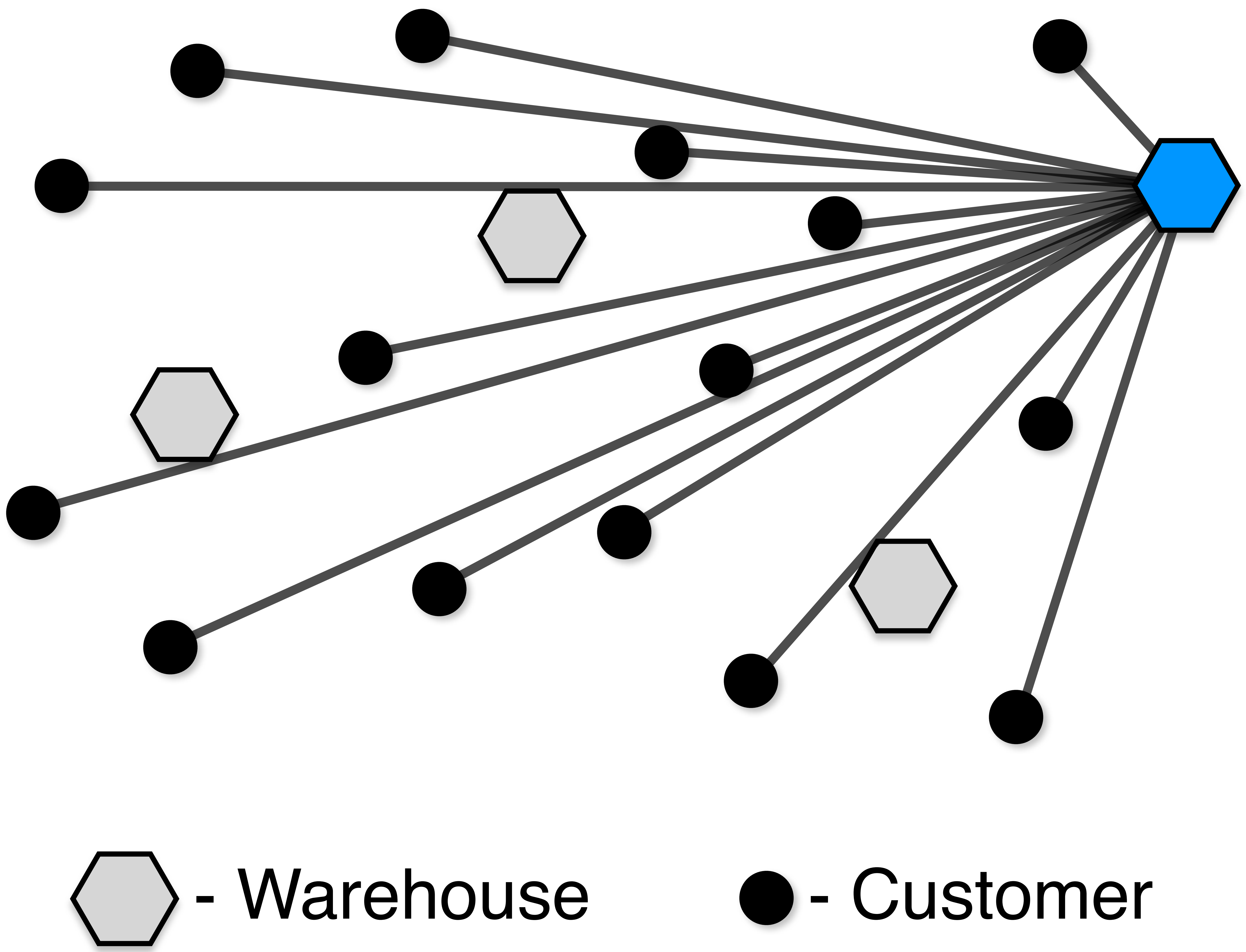

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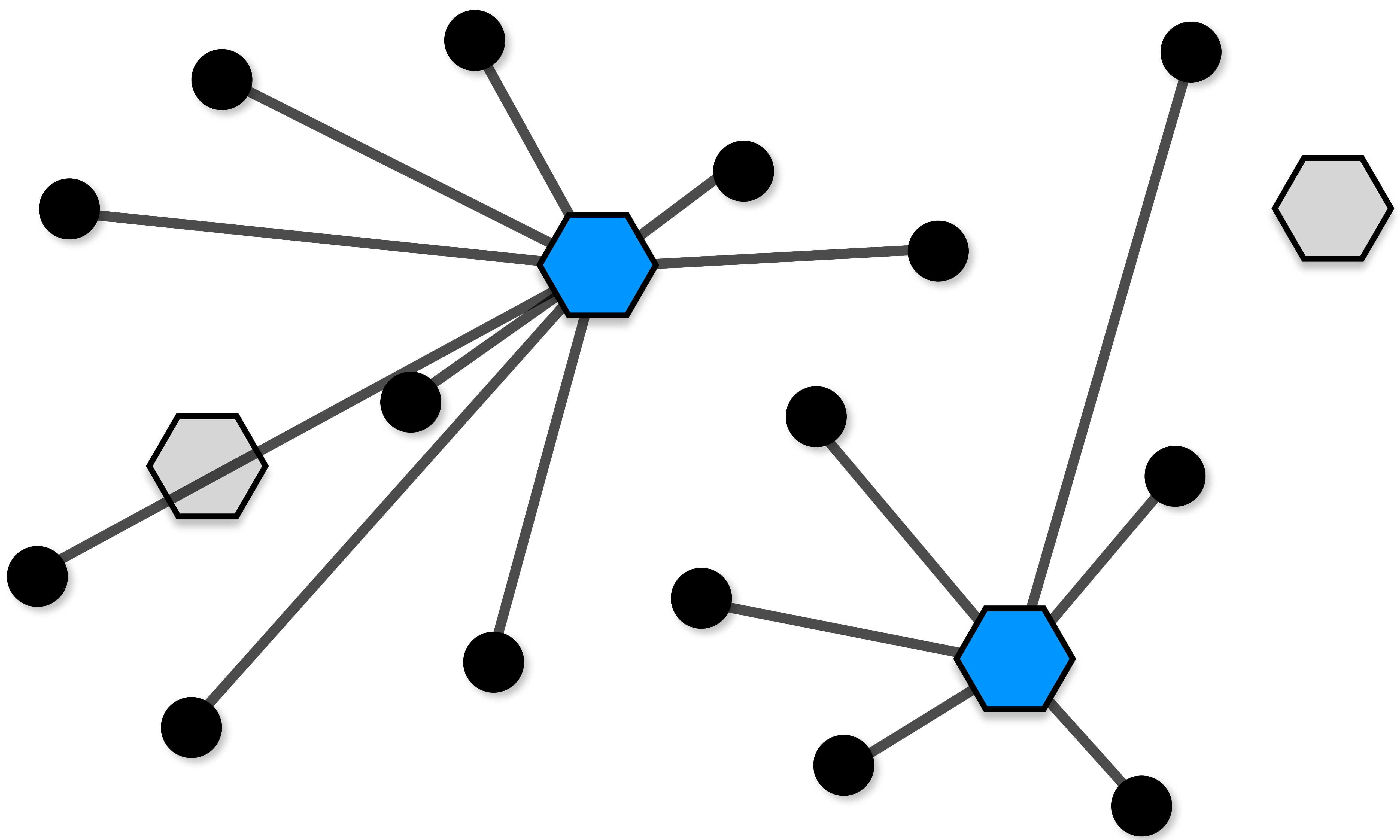
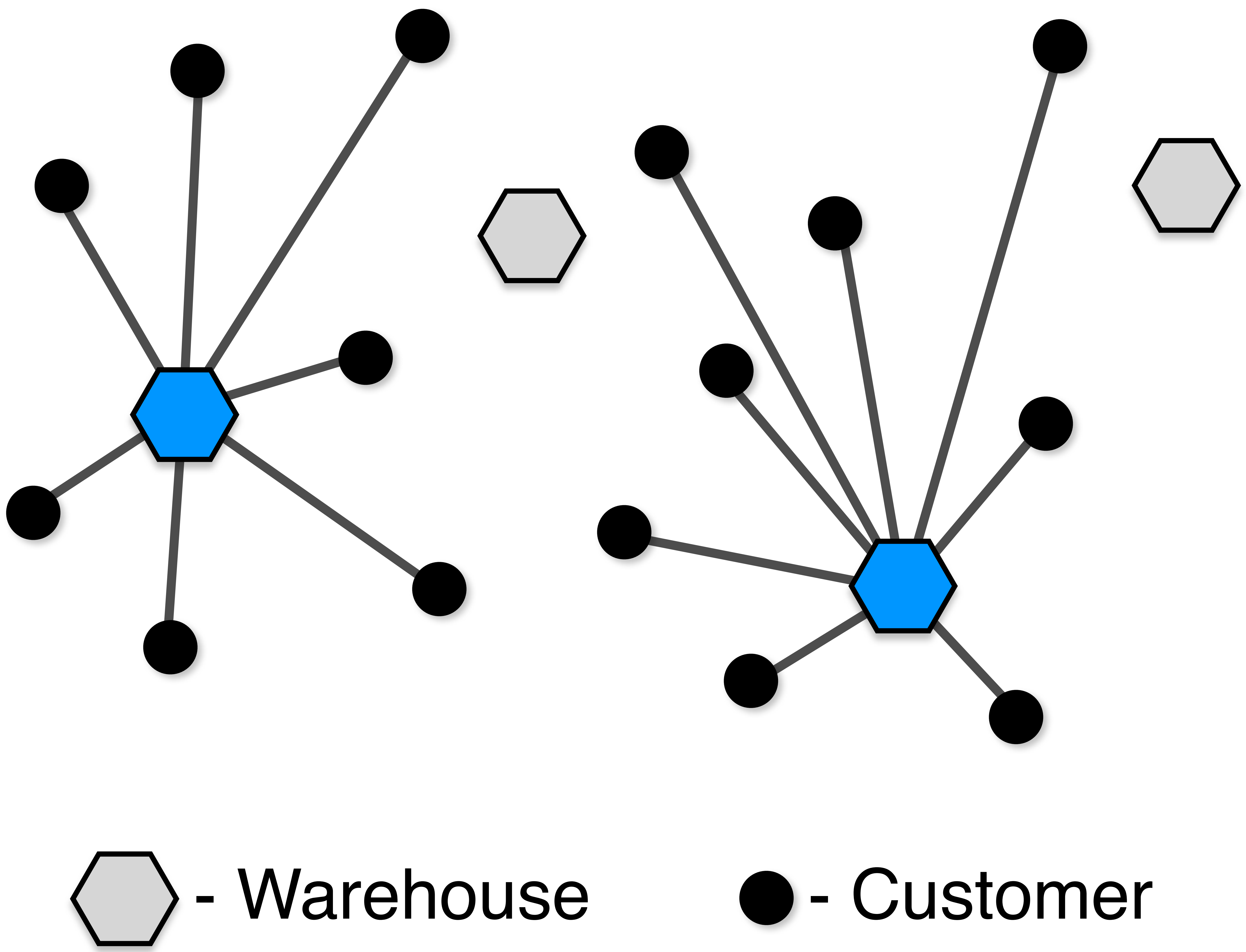


- Warehouse   - Customer

Warehouse Location



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$$y_{w,c} \leq x_w$$

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$$\sum_{w \in W} y_{w,c} = 1$$

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$$\sum_{w \in W} c_w \, x_w + \sum_{w \in W, c \in C} t_{w,c} \, y_{w,c}$$

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$$\underbrace{\sum_{w \in W} c_w}_{\text{fixed cost}} \; x_w + \sum_{w \in W, c \in C} t_{w,c} \; y_{w,c}$$

fixed cost

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$$\sum_{w \in W} \underbrace{c_w}_{\text{fixed cost}} \; x_w + \sum_{w \in W, c \in C} \underbrace{t_{w,c}}_{\text{transportation cost}} \; y_{w,c}$$

Warehouse Location

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\begin{aligned}
&\min \quad && \sum_{w \in W} c_w \; x_w + \sum_{w \in W, c \in C} t_{w,c} \; y_{w,c} \\
&\text{subject to} && \\
& && y_{w,c} \leq x_w && (w \in W, c \in C) \\
& && \sum_{w \in W} y_{w,c} = 1 && (c \in C) \\
& && x_w \in \{0,1\} && (w \in W) \\
& && y_{w,c} \in \{0,1\} && (w \in W, c \in C)
\end{aligned}
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  - for each warehouse, decide whether to open it
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- Why not use
  - y_c denotes the warehouse serving customer c?

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  - integer variables are typically 0/1 variables

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- Still many possible models to consider
  - decision variables
  - constraints
  - objectives

Branch and Bound

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  - active research area for many many decades

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  - Bounding: finding an optimistic relaxation
  - Branching: splitting the problem in subproblems

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- Branch and bound
  - Bounding: finding an optimistic relaxation
  - Branching: splitting the problem in subproblems

- MIP models have a natural relaxation
  - the linear relaxation
  - remove the integrality constraint on variables

Branch and Bound for MIP Models

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- Solve the linear relaxation

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  - If the linear relaxation is worse than the best solution found so far, prune this node
    - the associated problem is suboptimal

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- Solve the linear relaxation
  - If the linear relaxation is worse than the best solution found so far, prune this node
    - the associated problem is suboptimal
  - If the linear relaxation is integral, we have found a feasible solution
    - update the best feasible solution if appropriate
  - Otherwise, find an integer variable x that has a fractional value f in the linear relaxation
    - create two subproblems $\mathrm{x} \leq \lfloor f \rfloor$ and $\mathrm{x} \geq \lceil f \rceil$
    - repeat the algorithm on the subproblems

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  - the relaxation gives an optimistic bound

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- Pruning based on sub-optimality
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- Relax feasibility
  - relax the integrality constraints
- Global view of the relaxation
  - consider all problem constraints

The Knapsack Problem

$$\text{maximize} \quad \sum_{i \in I} v_i \; x_i$$

$$\text{subject to}$$

$$\sum_{i \in I} w_i x_i \leq K$$

$$x_i \in \{0, 1\} \quad (i \in I)$$

The Knapsack Problem: Linear Relaxation

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$$\text{subject to}$$

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$$0 \leq x_i \leq 1 \quad (i \in I)$$

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  - same as the greedy relaxation we used

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- How do we branch?
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- What do the subproblems mean?
  - do not take that item
    - what is the linear relaxation going to do?
  - take this item
    - what is the linear relaxation going to do?

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  - take this item
    - which item is now fractional?

Depth-First Branch and Bound

i	V_i	W_i
1	45	5
2	48	8
3	35	3

K = 10

$0
10
$92

Value
Room
Estimate

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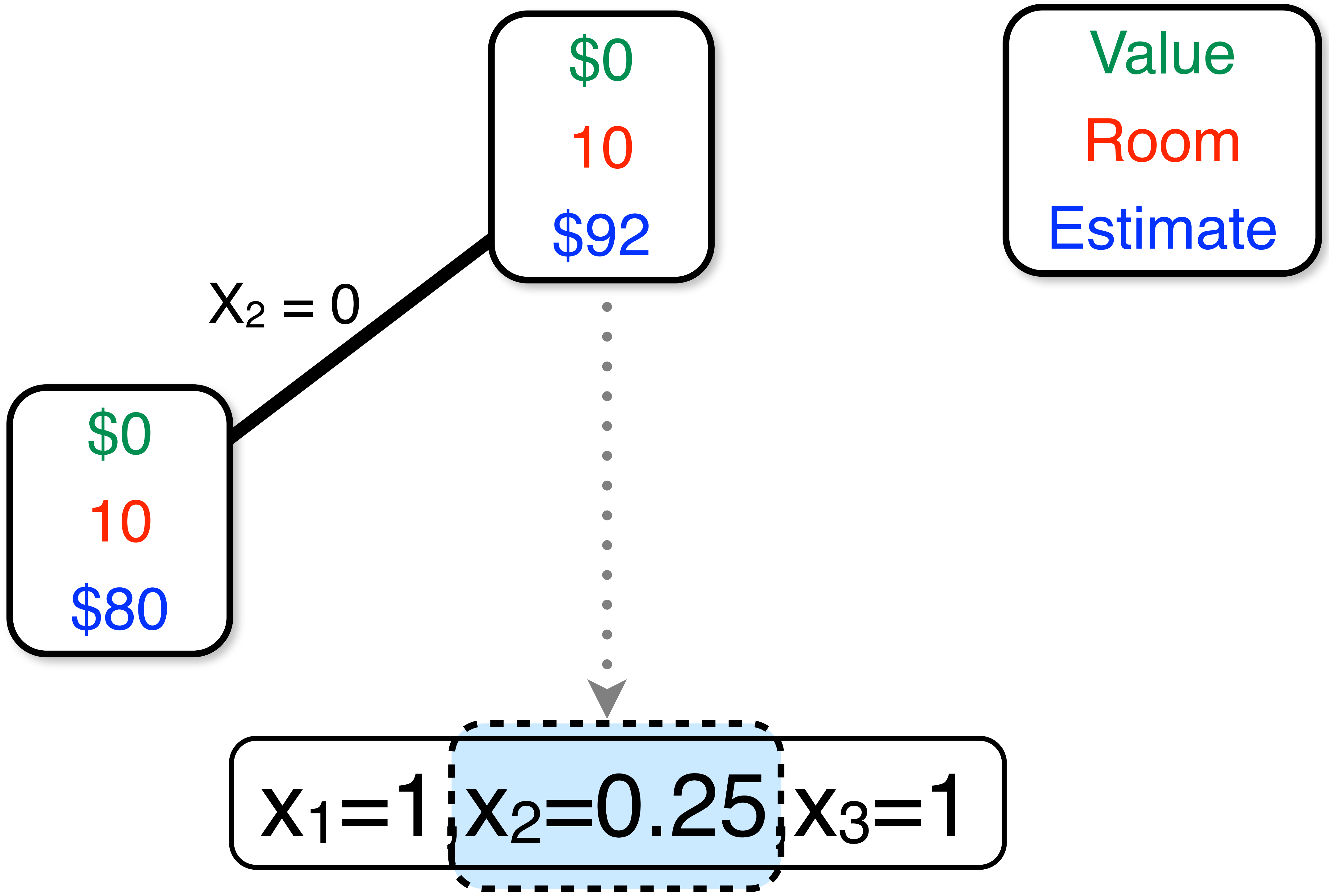

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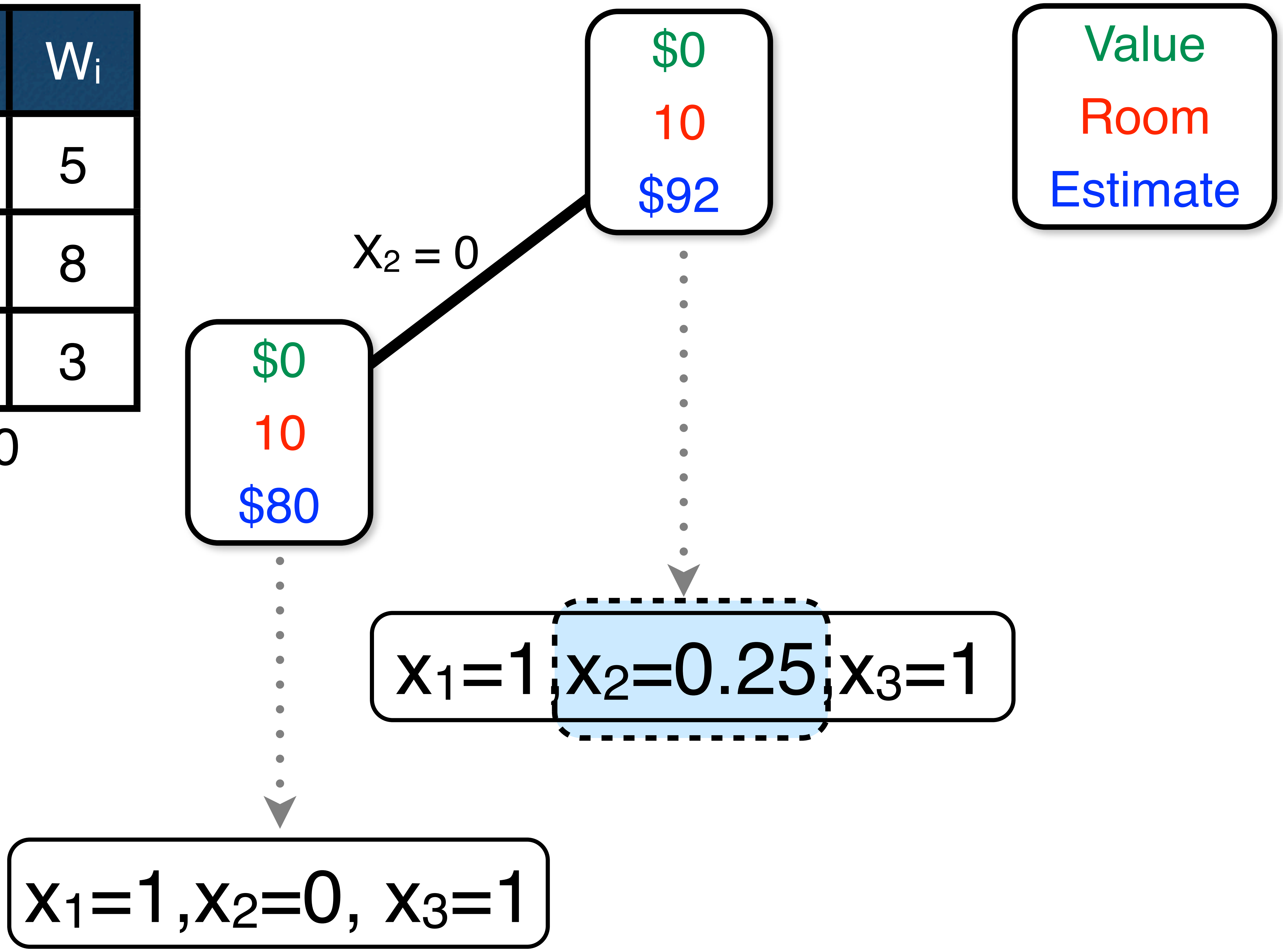

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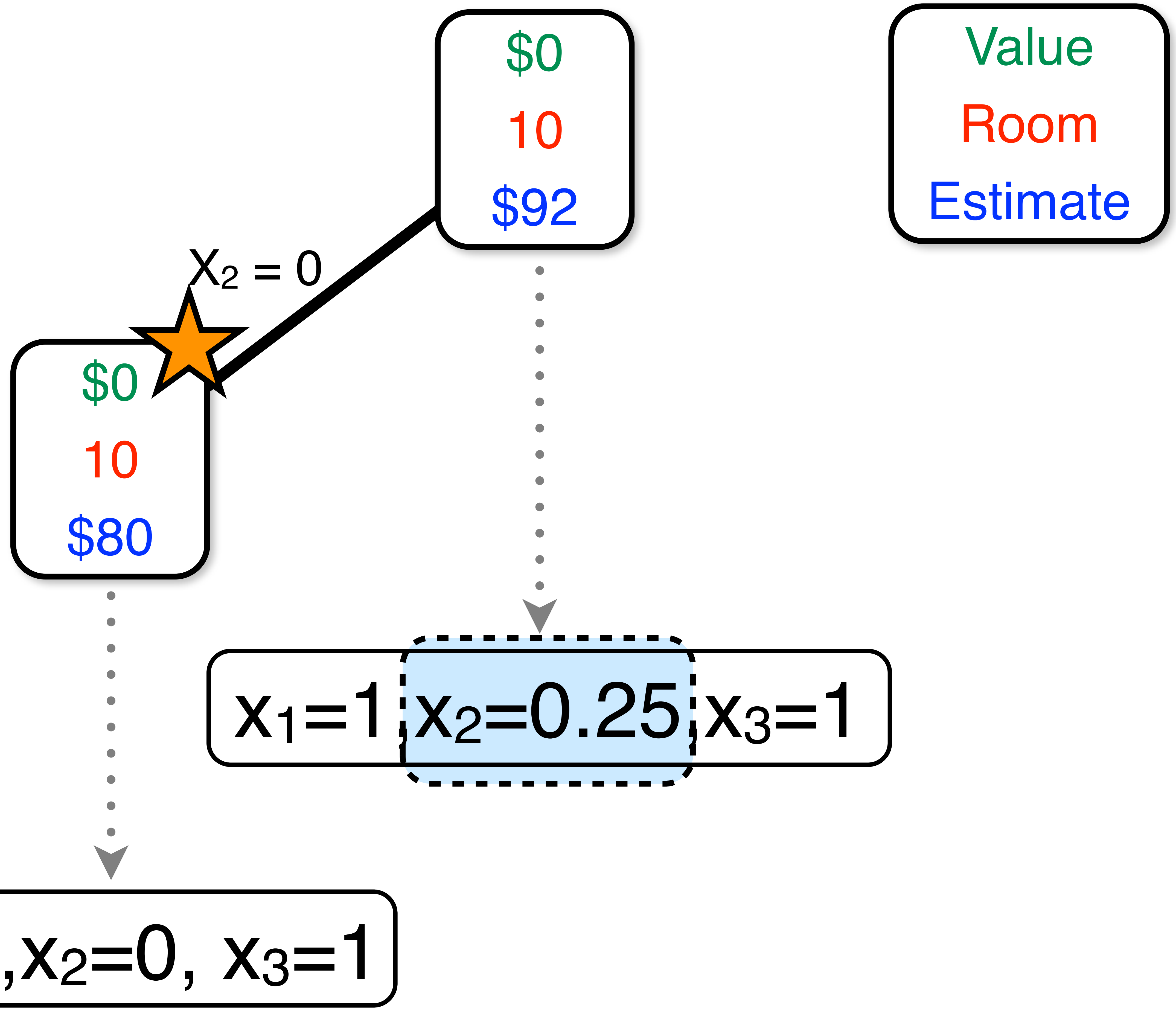

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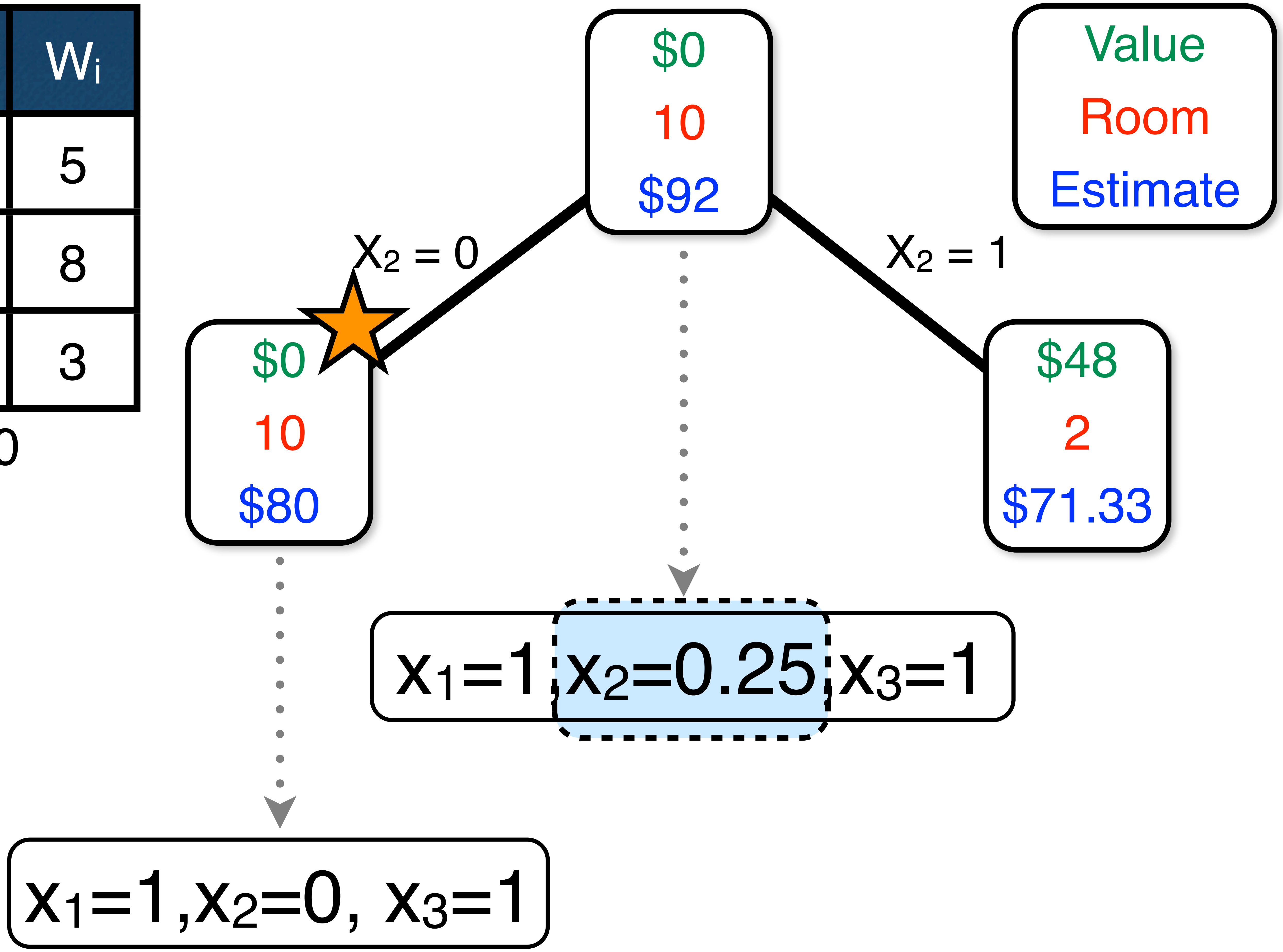

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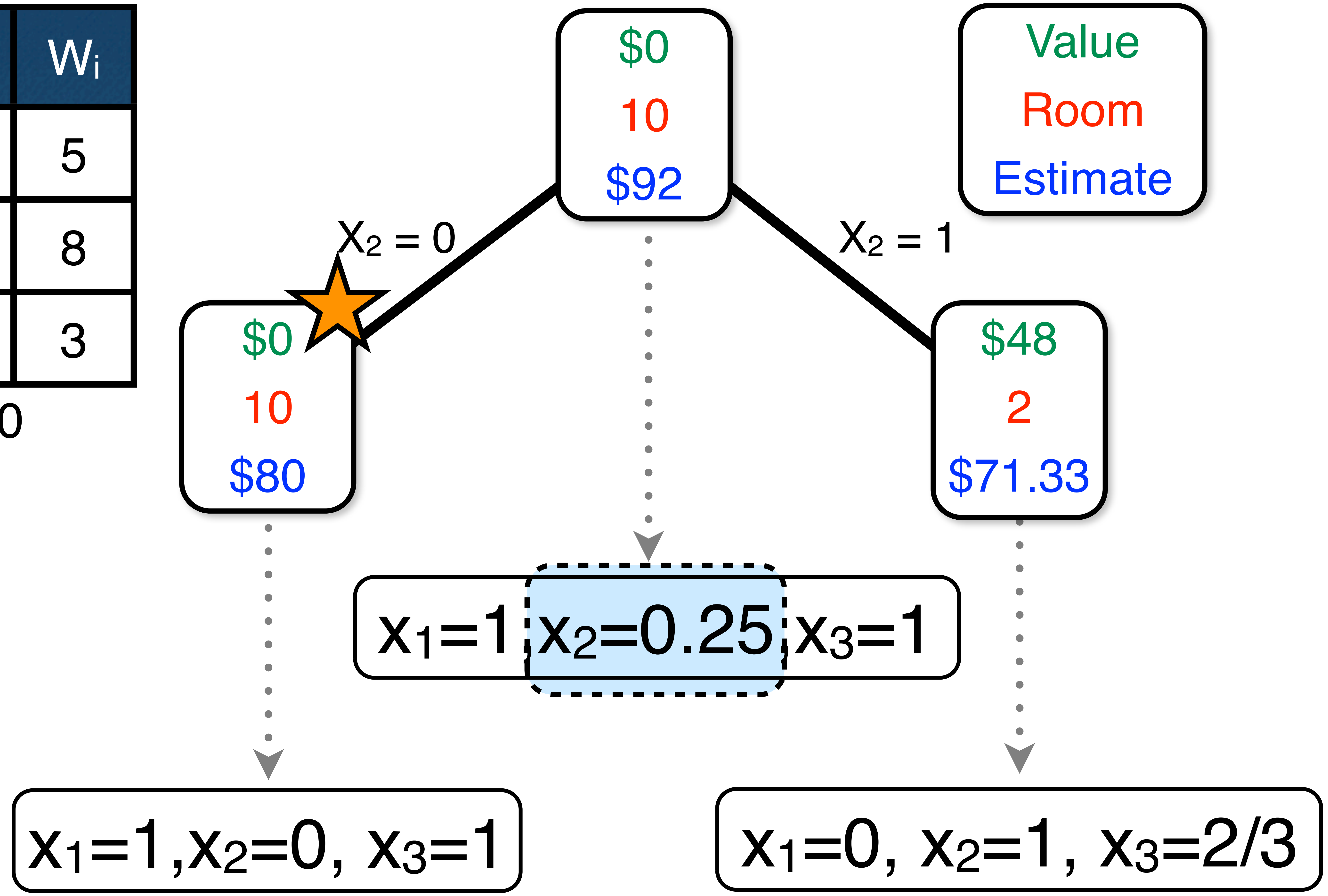

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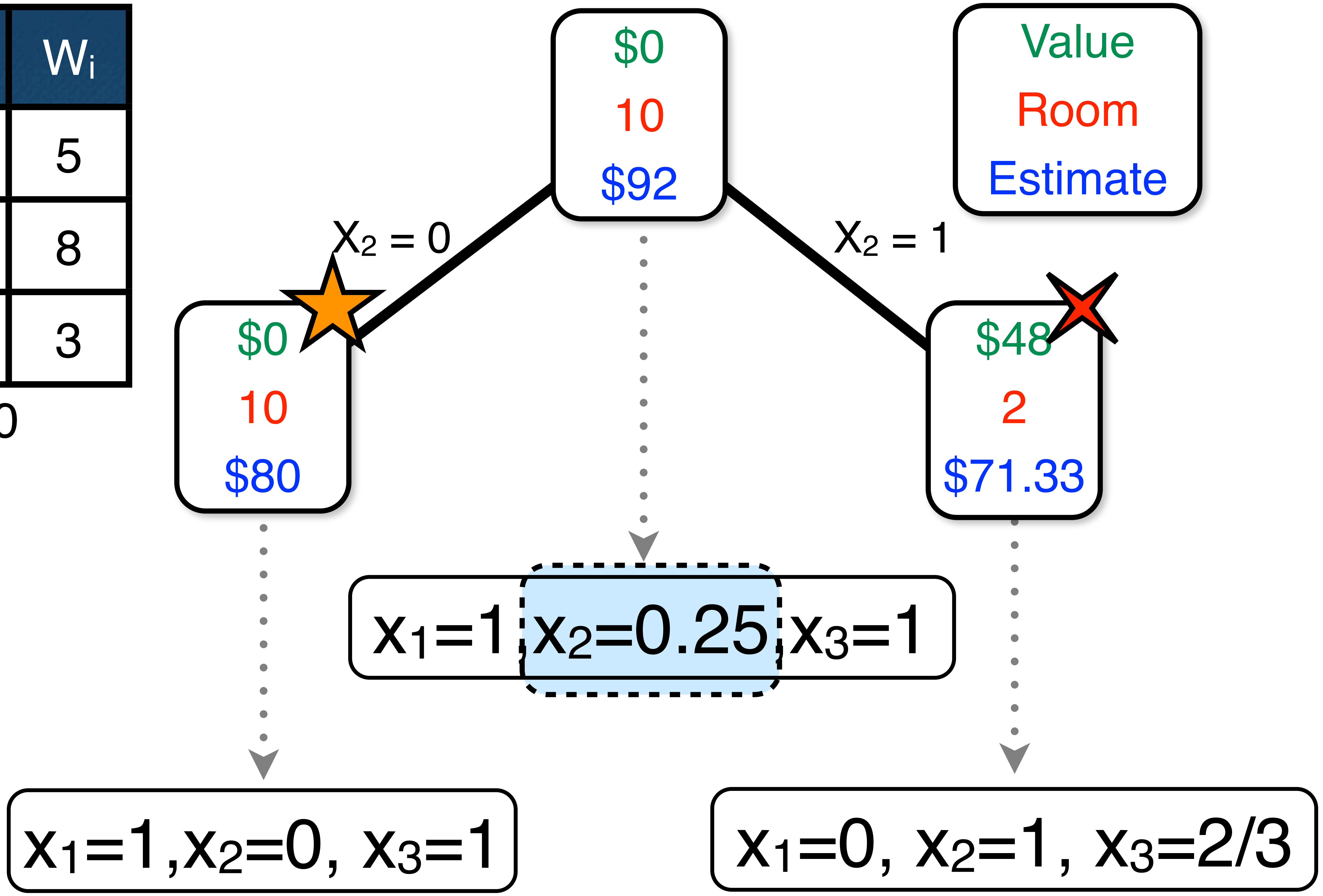

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- When is Branch and Bound effective?
  - necessary condition: the linear relaxation is strong
    - is it sufficient?
- What is a good MIP model?
  - one with a good linear relaxation
- Which variable should one branch on?
  - most fractional value
    - why? exaggerate ...

Until Next Time