Discrete Optimization

Linear Programming: Part I

Goals of the Lecture

- Linear programming
  - what is a linear program?
  - convexity
  - geometry

What is a Linear Program?

$$\min c_1x_1 + \ldots + c_nx_n$$

subject to

$$a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1$$

$$\ldots$$

$$a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m$$

$$x_i \geq 0 \quad (1 \leq i \leq n)$$

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- inequality constraints

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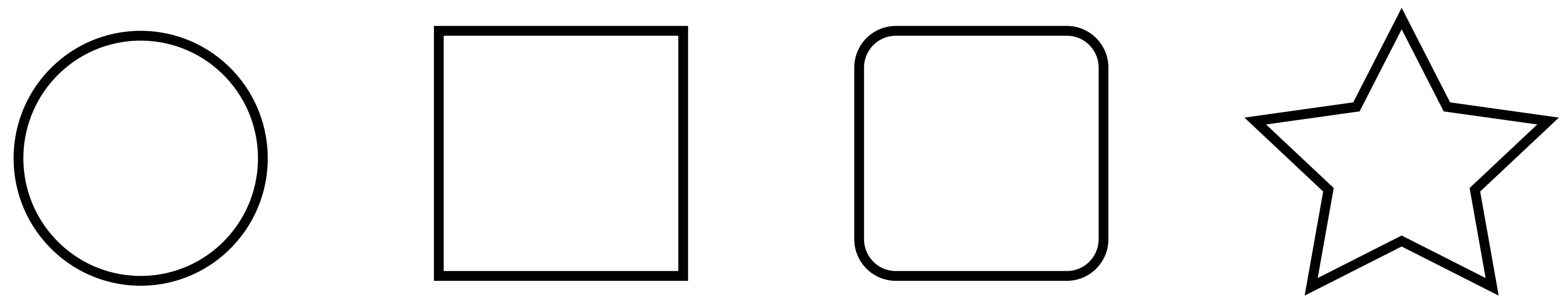
- What if I have a nonlinear constraint?
  - this is called a linear program :-)

Geometry of Linear Programs

- Convex sets



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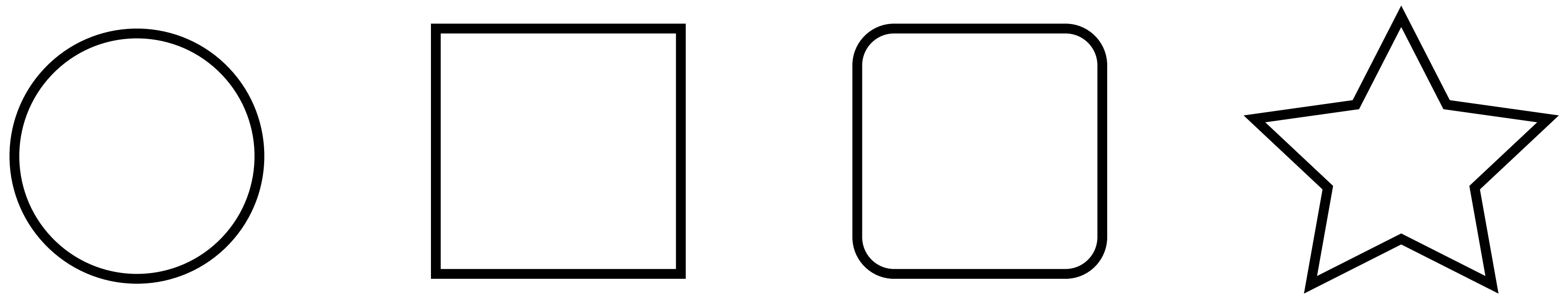

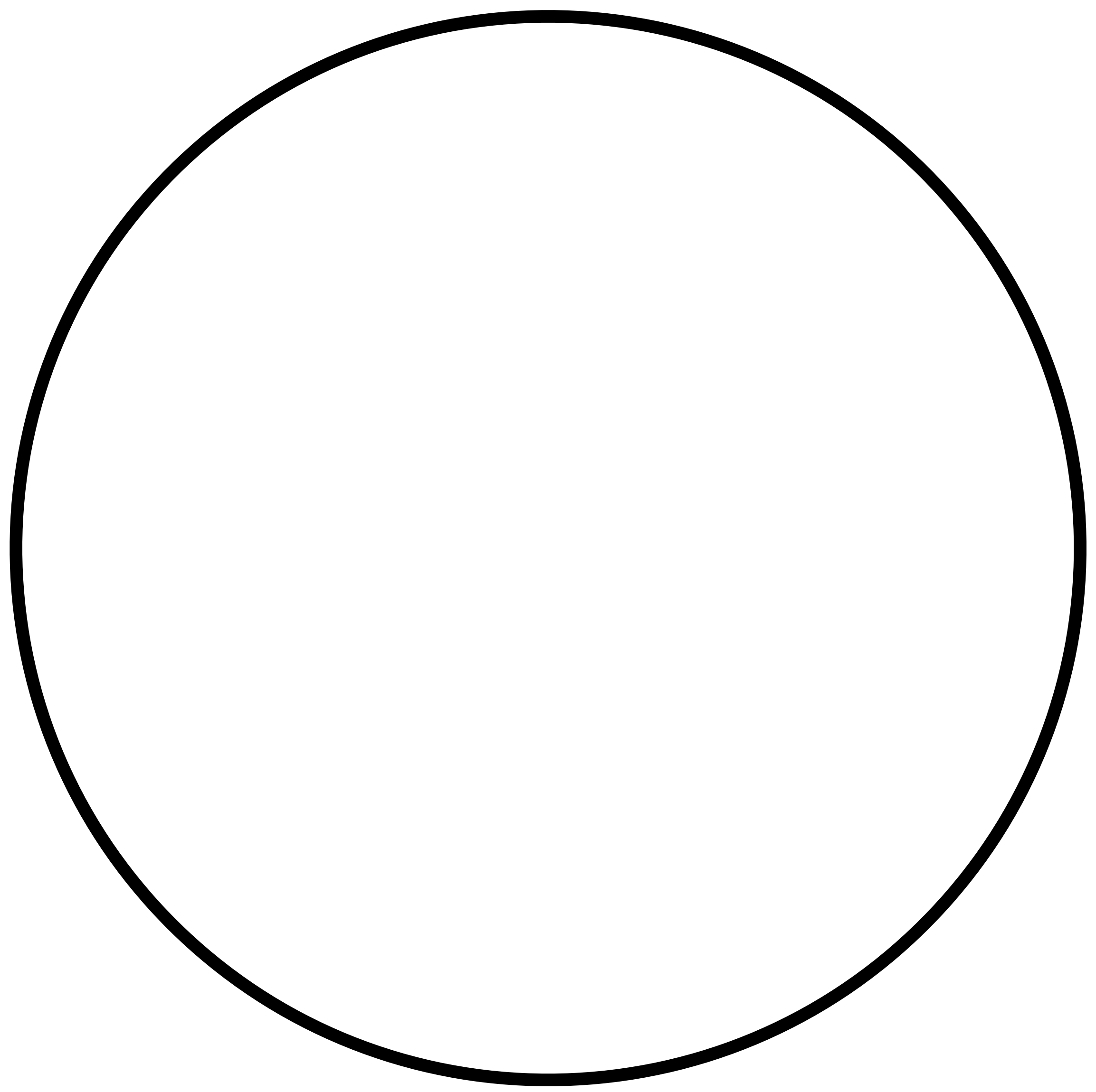

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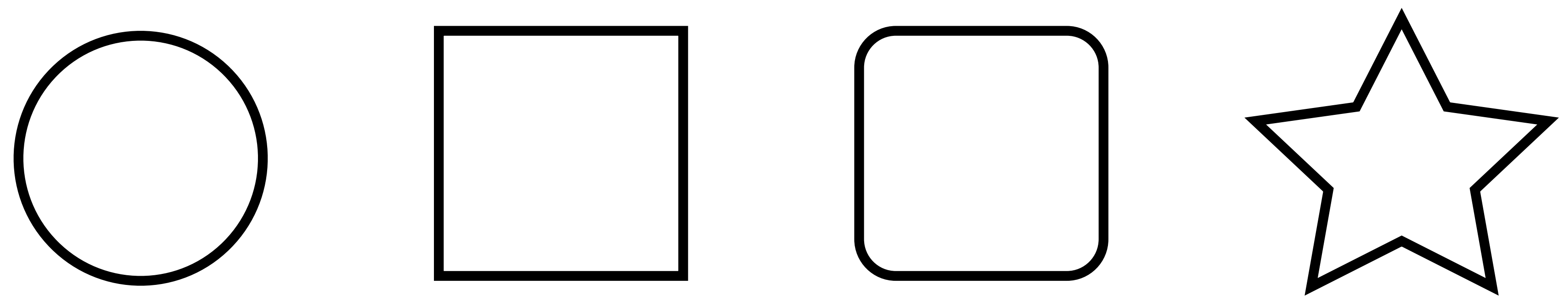

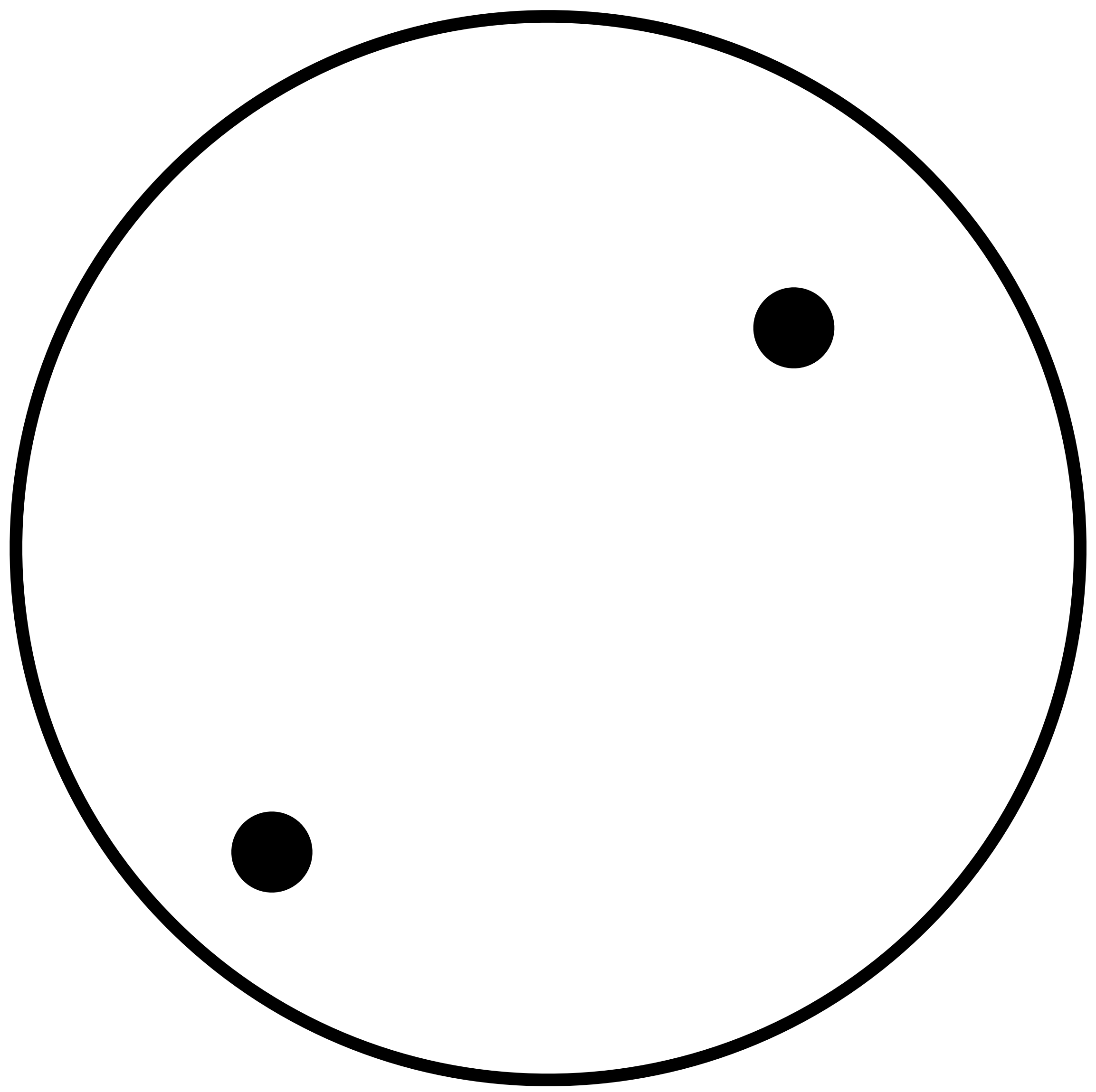

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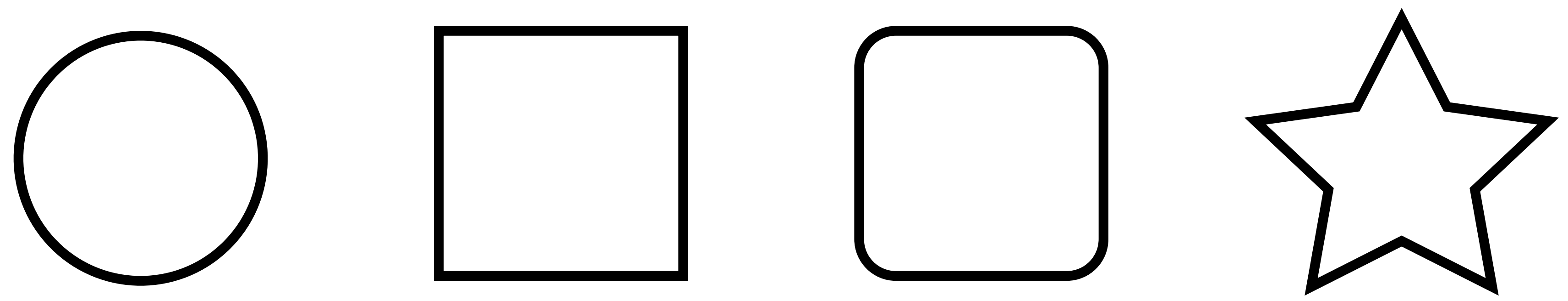

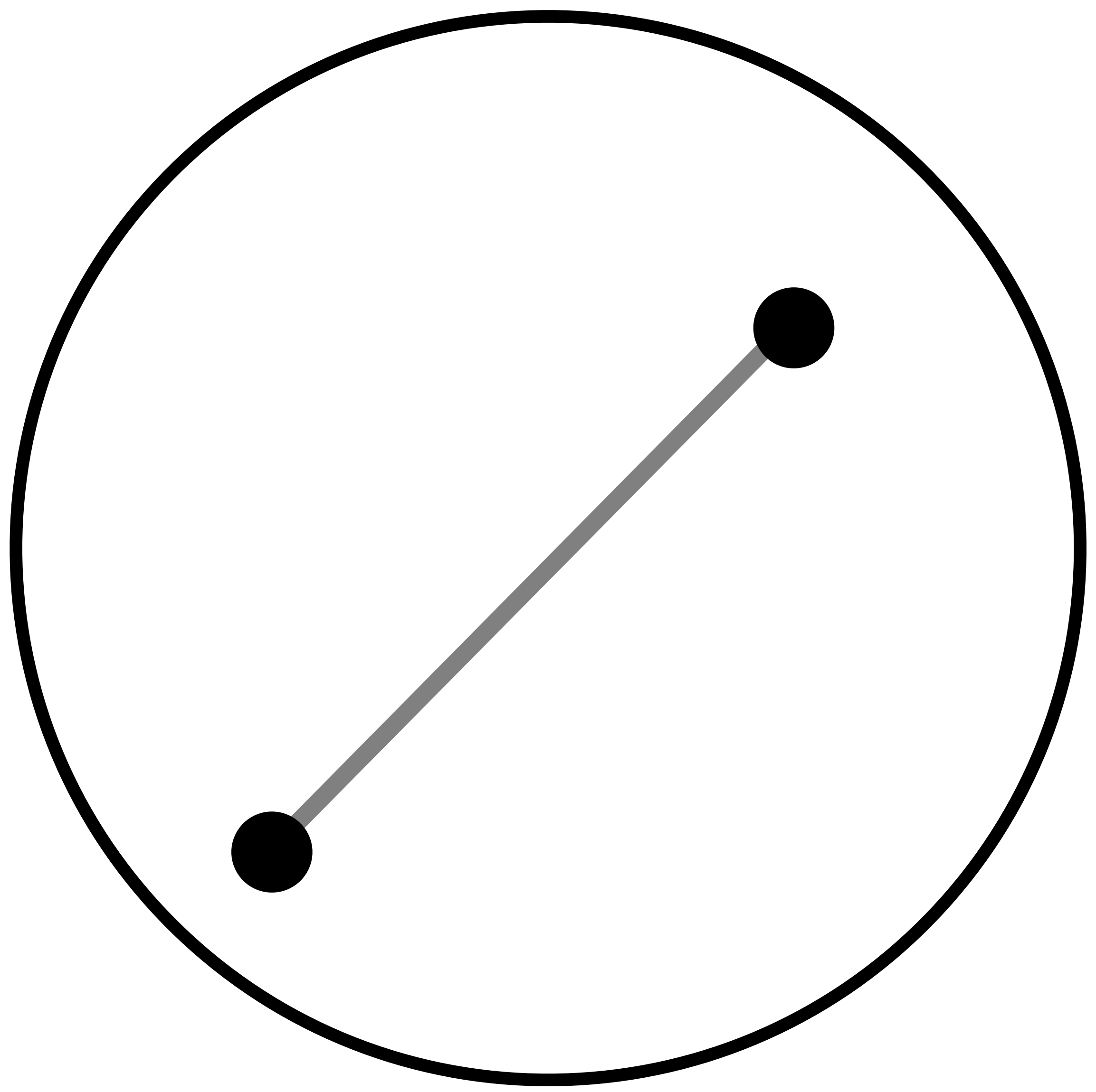

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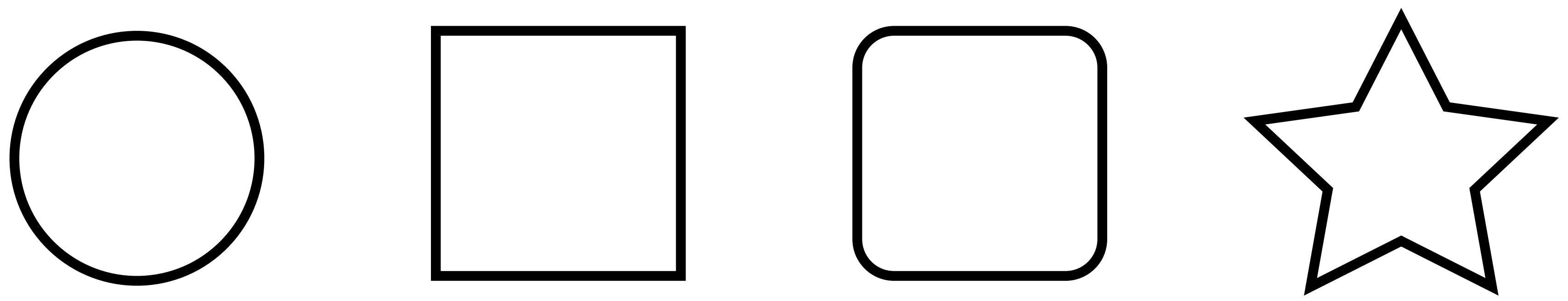

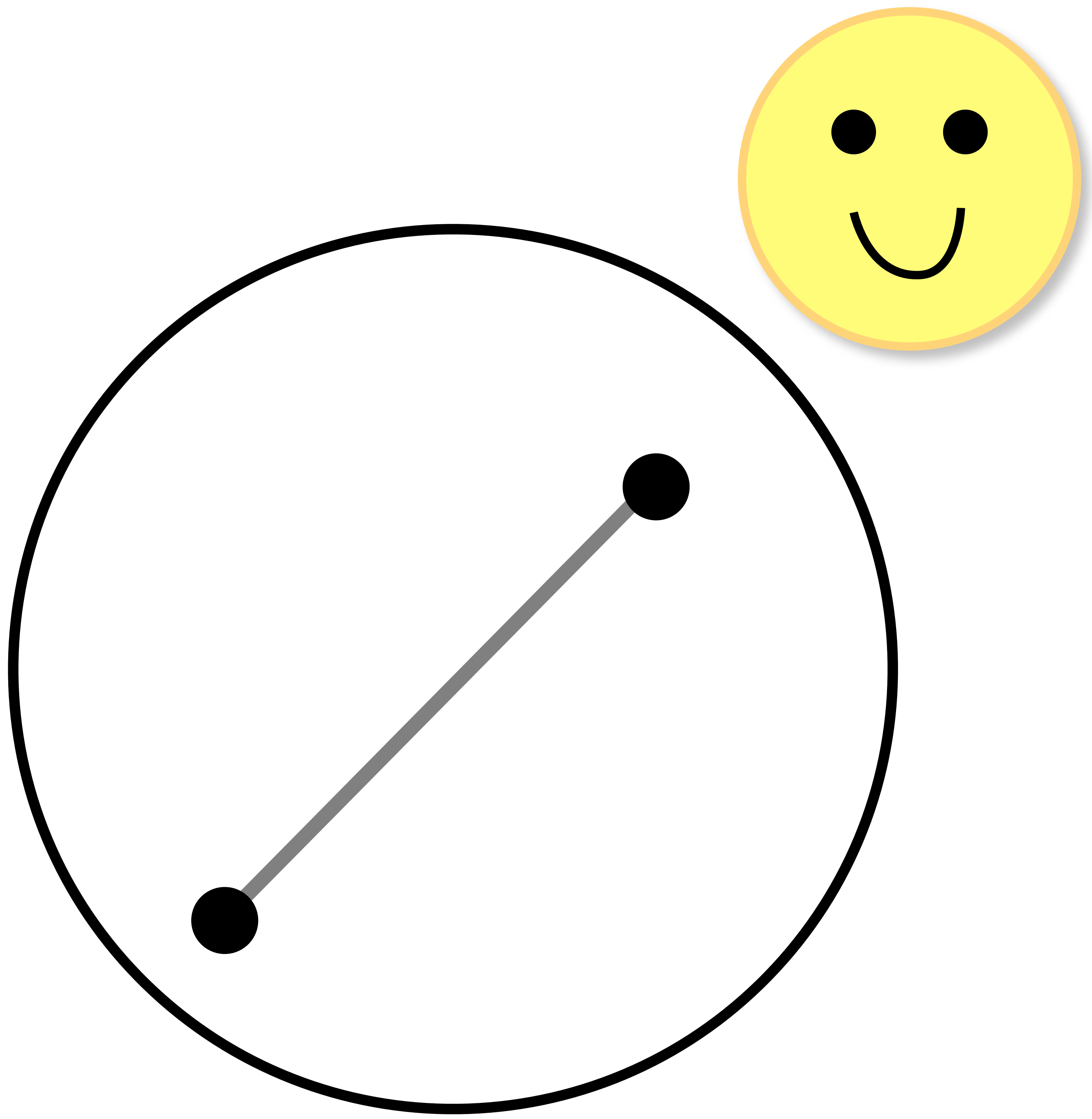

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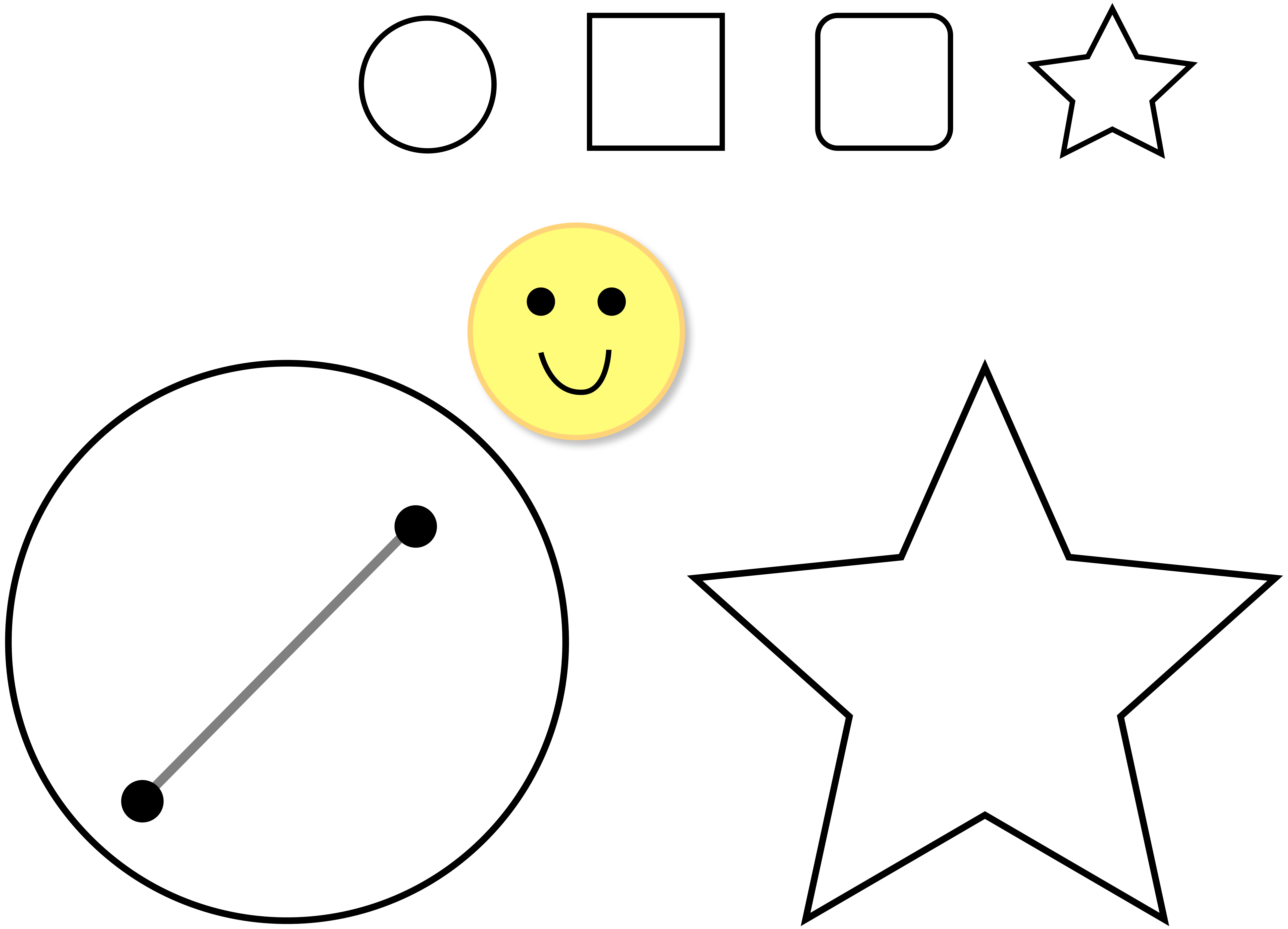

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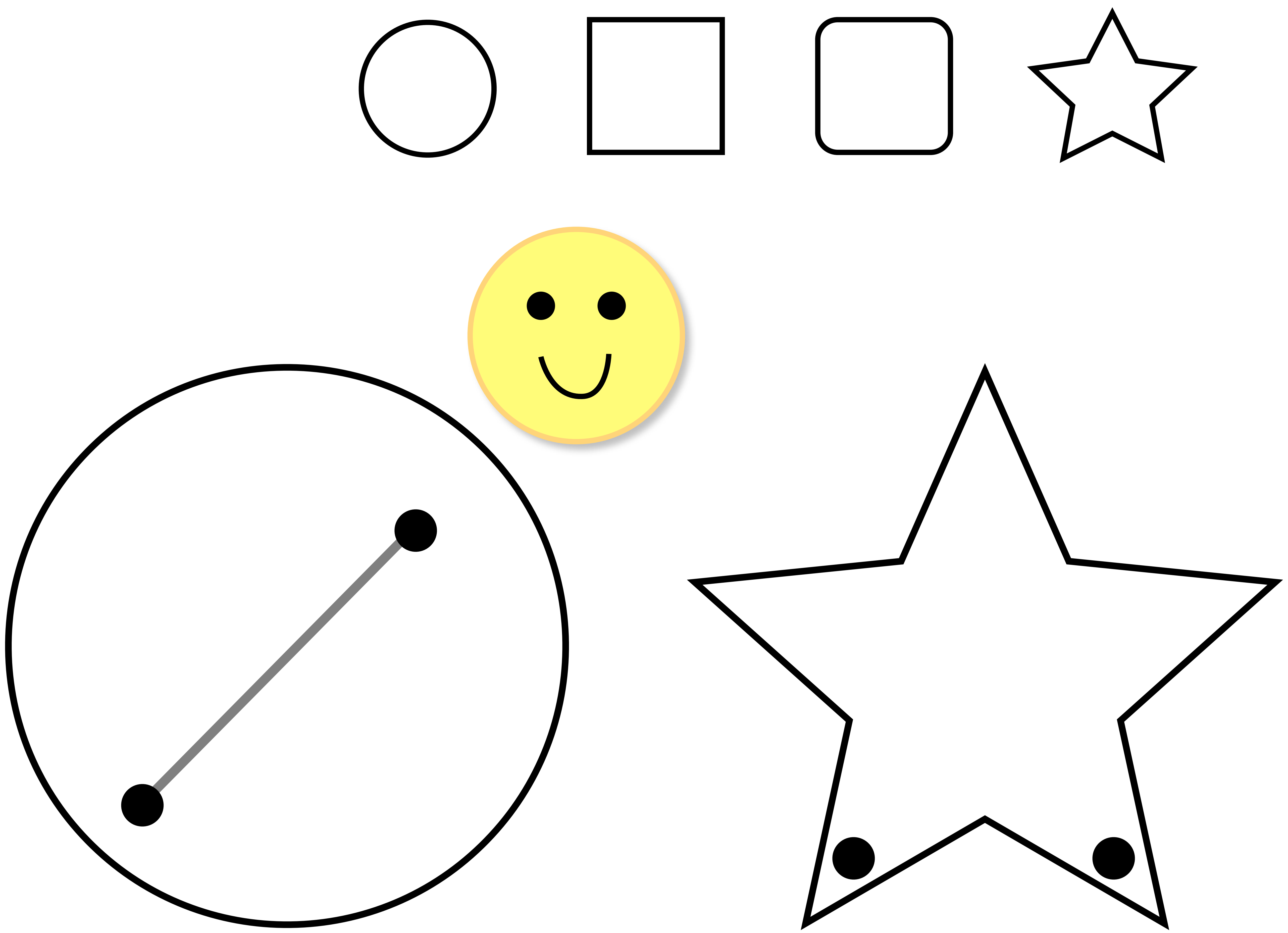

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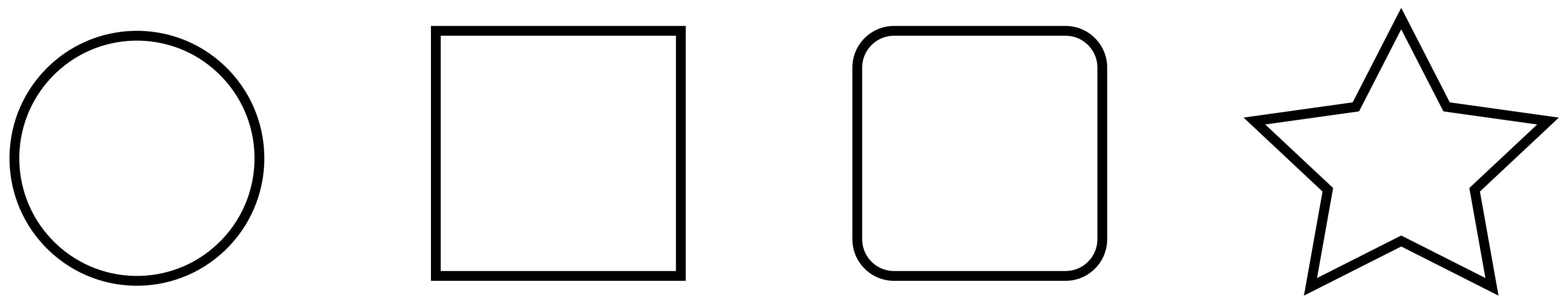

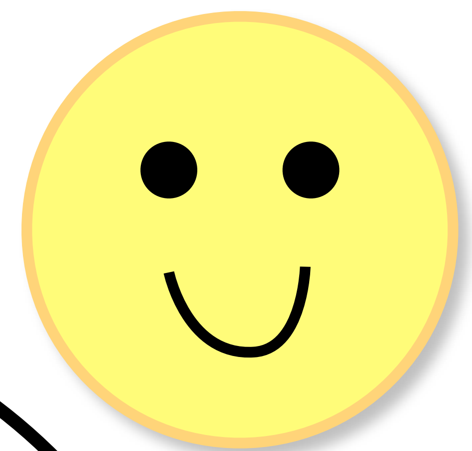

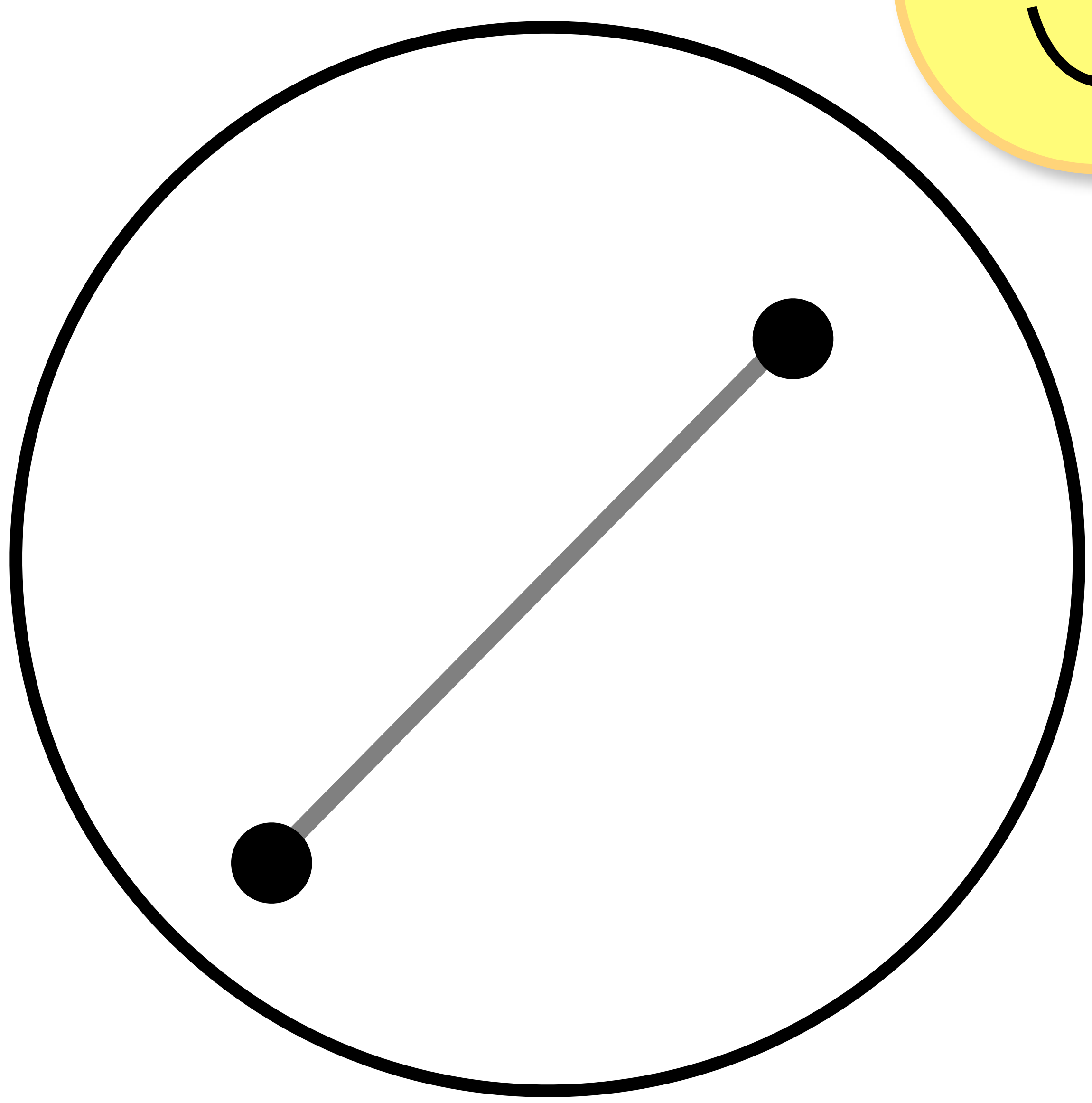

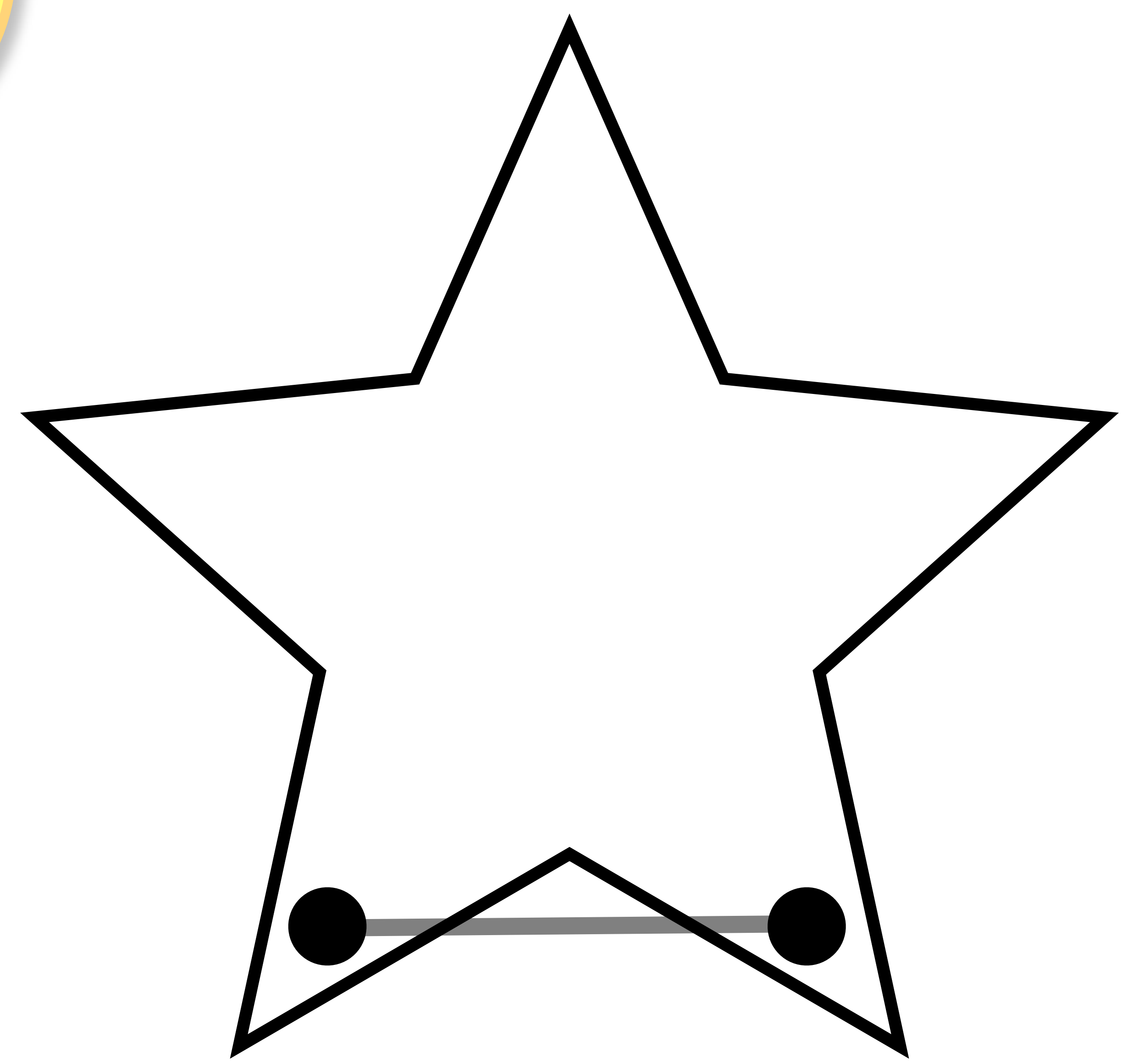

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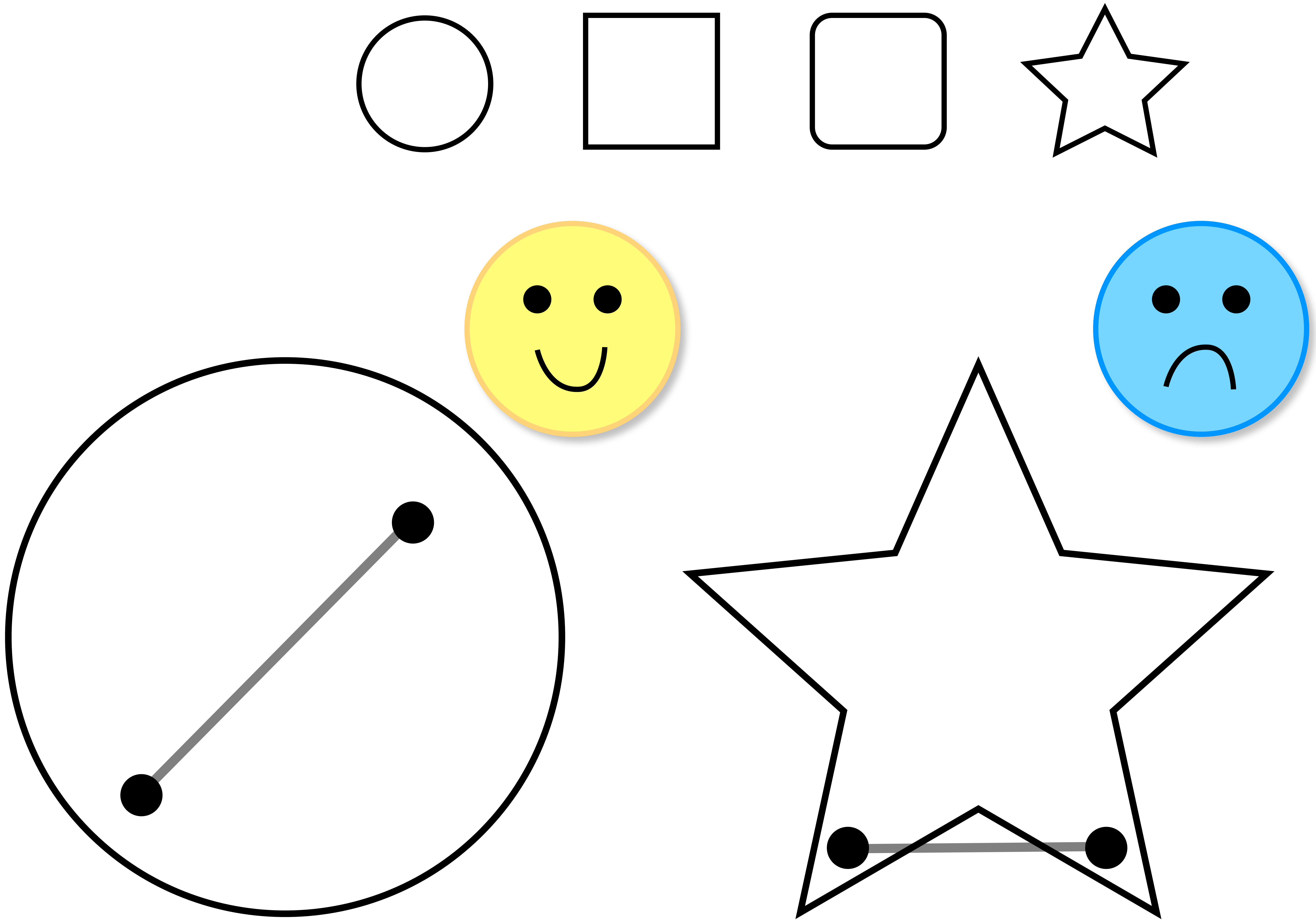
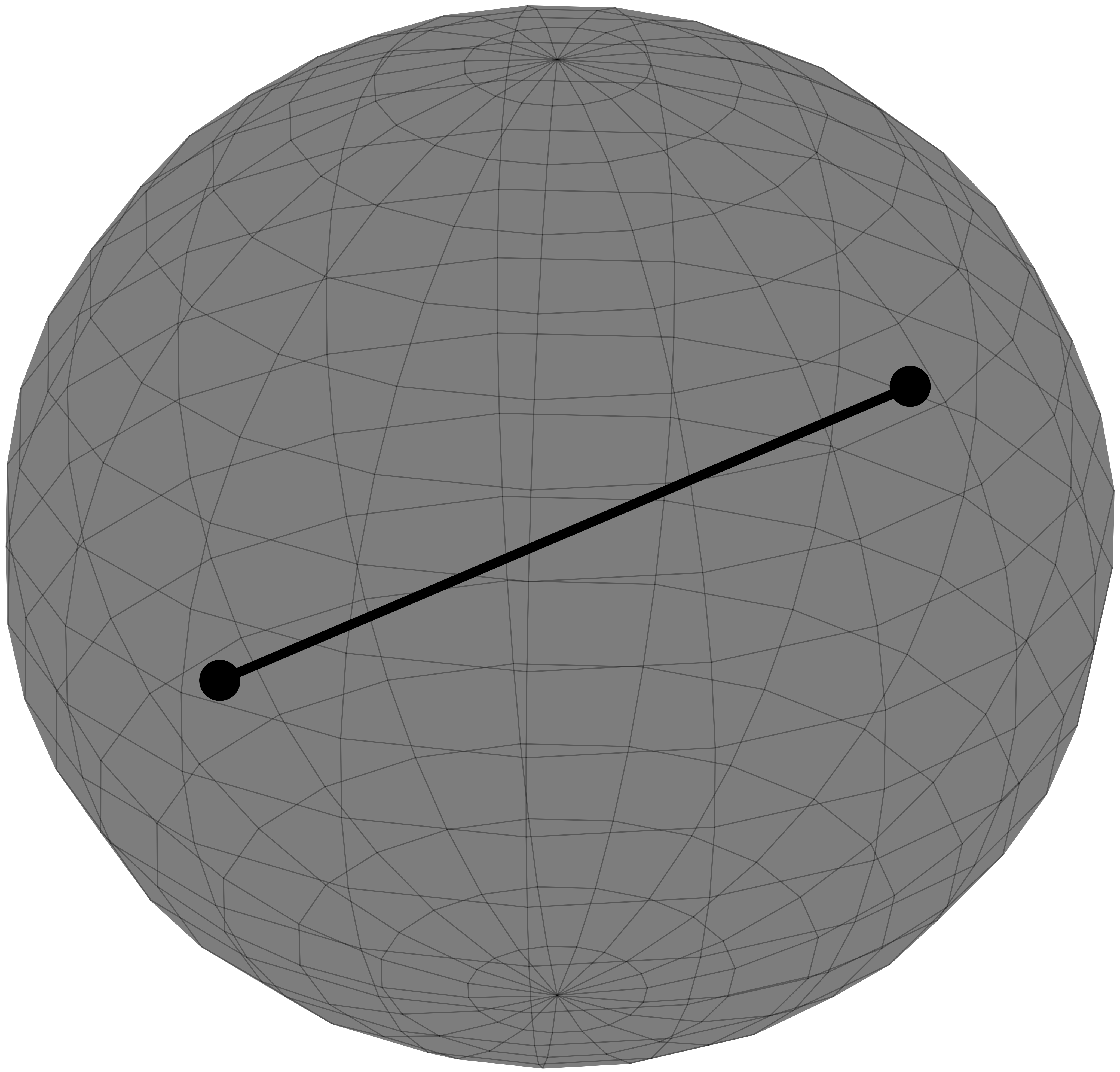


Geometry of Linear Programs



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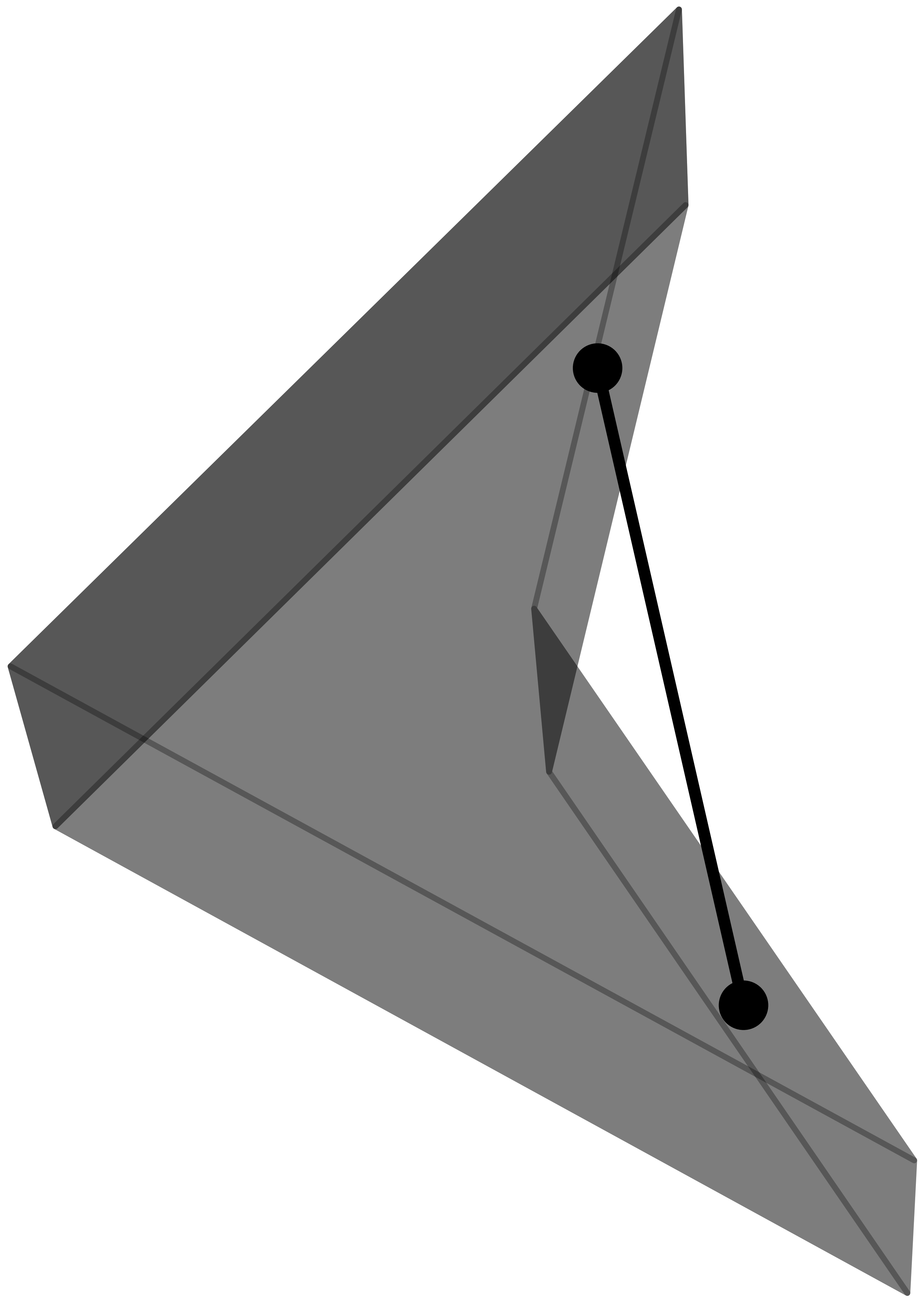
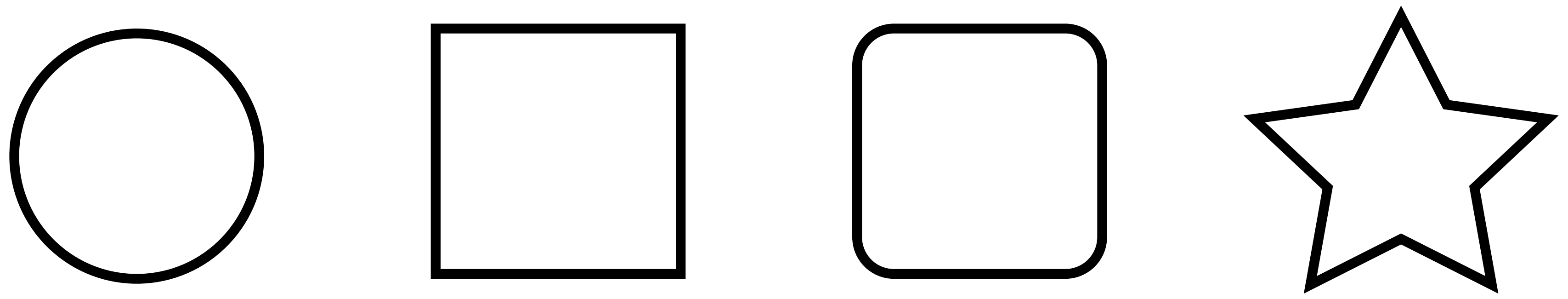


Convex Combinations

- Convex sets



- Convex combinations

$\lambda_1 v_1 + \ldots + \lambda_n v_n$ is a convex combination of $v_1, \ldots, v_n$ if

$$\lambda_1 + \ldots + \lambda_n = 1$$

and

$$\lambda_i \geq 0 \quad (1 \leq i \leq n).$$

Geometry of Linear Programs

- Convex sets
  - a set S in R^n is convex if it contains all the convex combinations of the points in S.

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  - a set S in R^n is convex if it contains all the convex combinations of the points in S.
- The intersection of convex sets is a convex set

Let $\bigcap S_i$ be the intersection of the convex sets

$$S_i \quad (1 \le i \le m).$$

Consider $v_1, \ldots, v_n \in \bigcap S_i$. It follows that

$$v_1, \ldots, v_n \in S_i \quad (1 \le i \le m)$$

All convex combinations of $v_1, \ldots, v_n$ are in S_i $(1 \le i \le m)$.
Hence all convex combinations of $v_1, \ldots, v_n$ are in $\bigcap S_i$.

Geometry of Linear Programs

- A half space is a convex set

$$a_1 x_1 + \ldots + a_n x_n \leq b$$

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- A half space is a convex set

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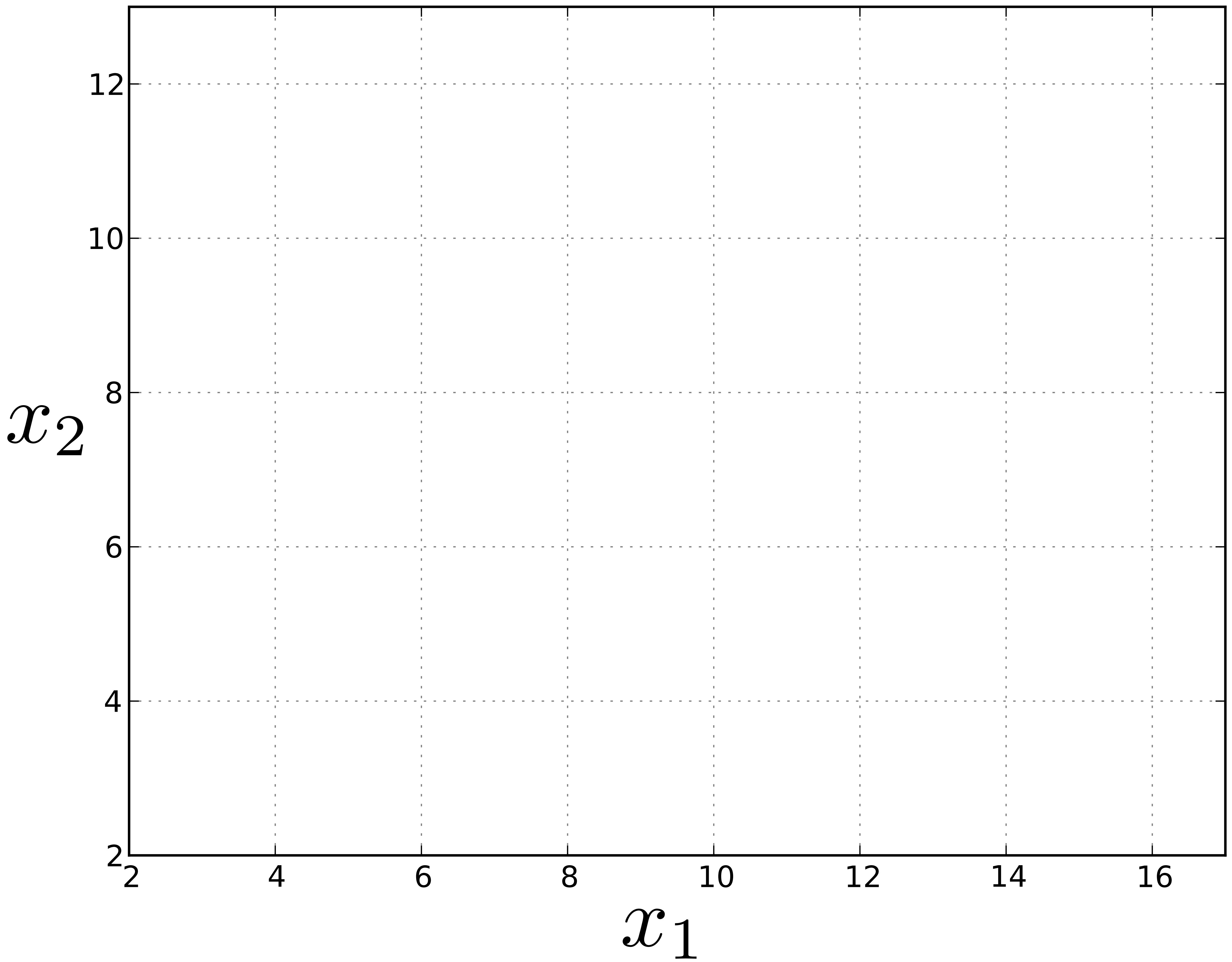
- The intersection of a set of half spaces is convex and is called of polyhedron. If it is finite, it is called a polytope.

$$a_{11} x_1 + \ldots + a_{1n} x_n \leq b_1$$
$$\ldots$$
$$a_{m1} x_1 + \ldots + a_{mn} x_n \leq b_m$$

Hyperplanes



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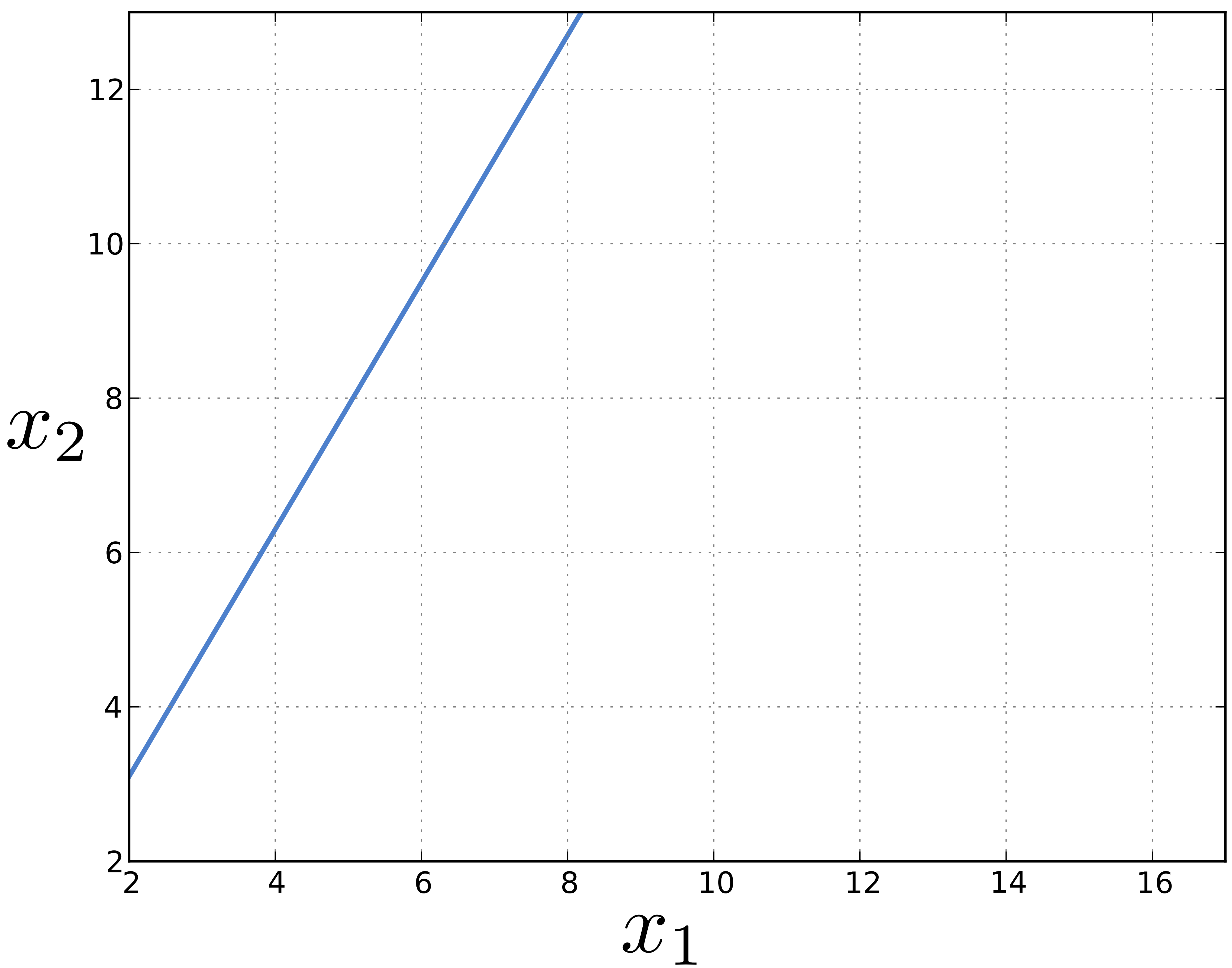
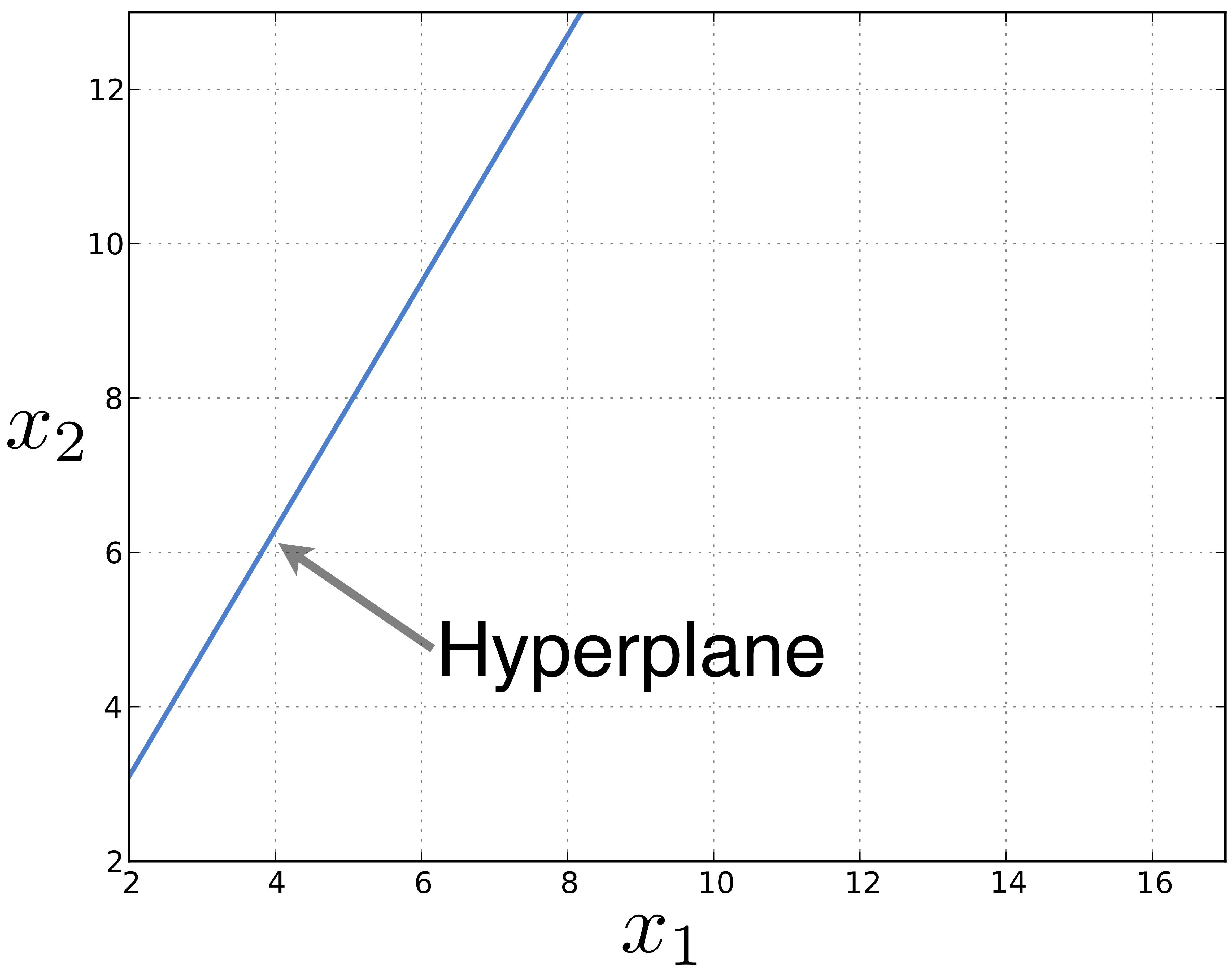


$$-4.0x_1 \quad + \quad 2.5x_2 \quad \leq \quad -0.25$$

Hyperplanes



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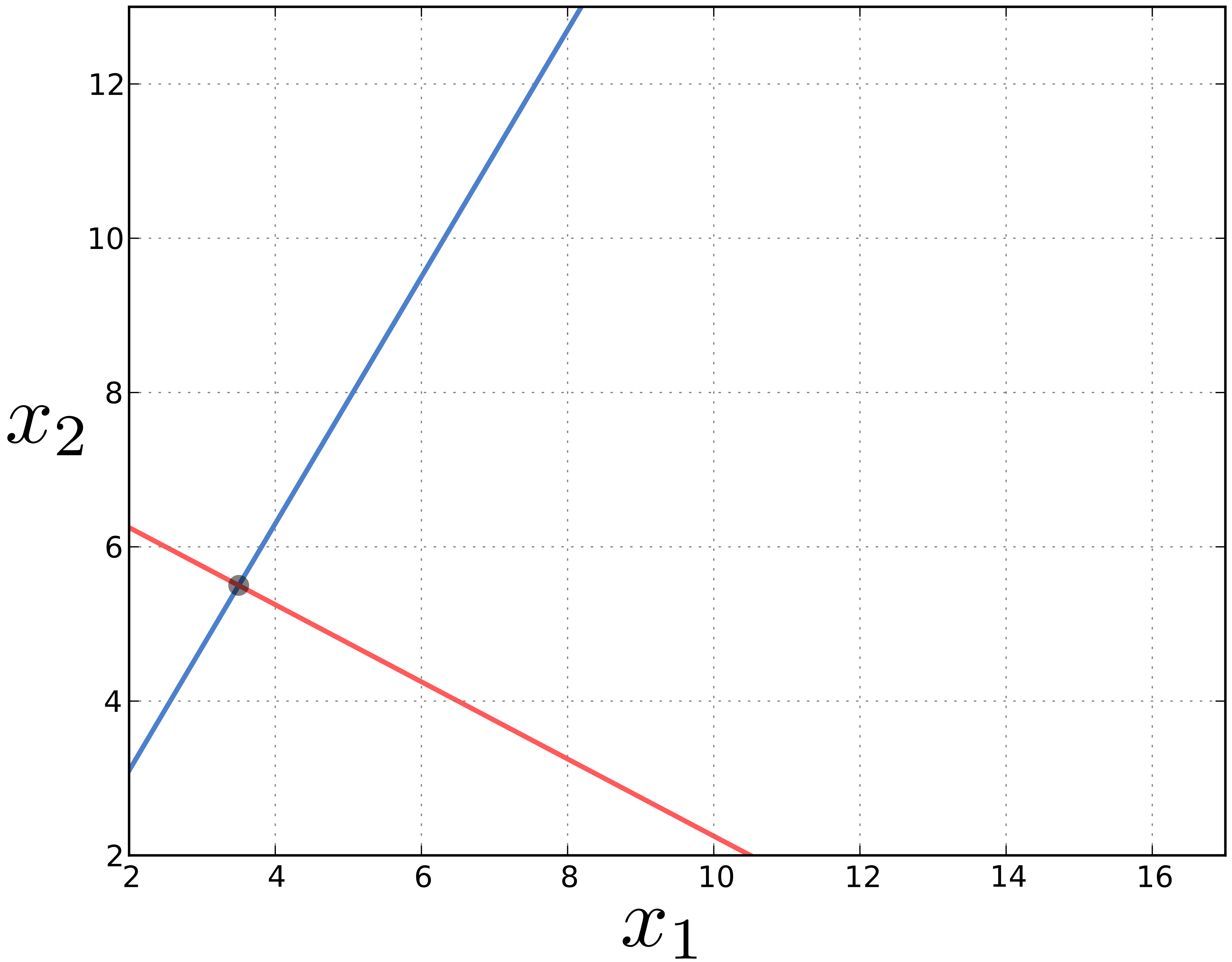
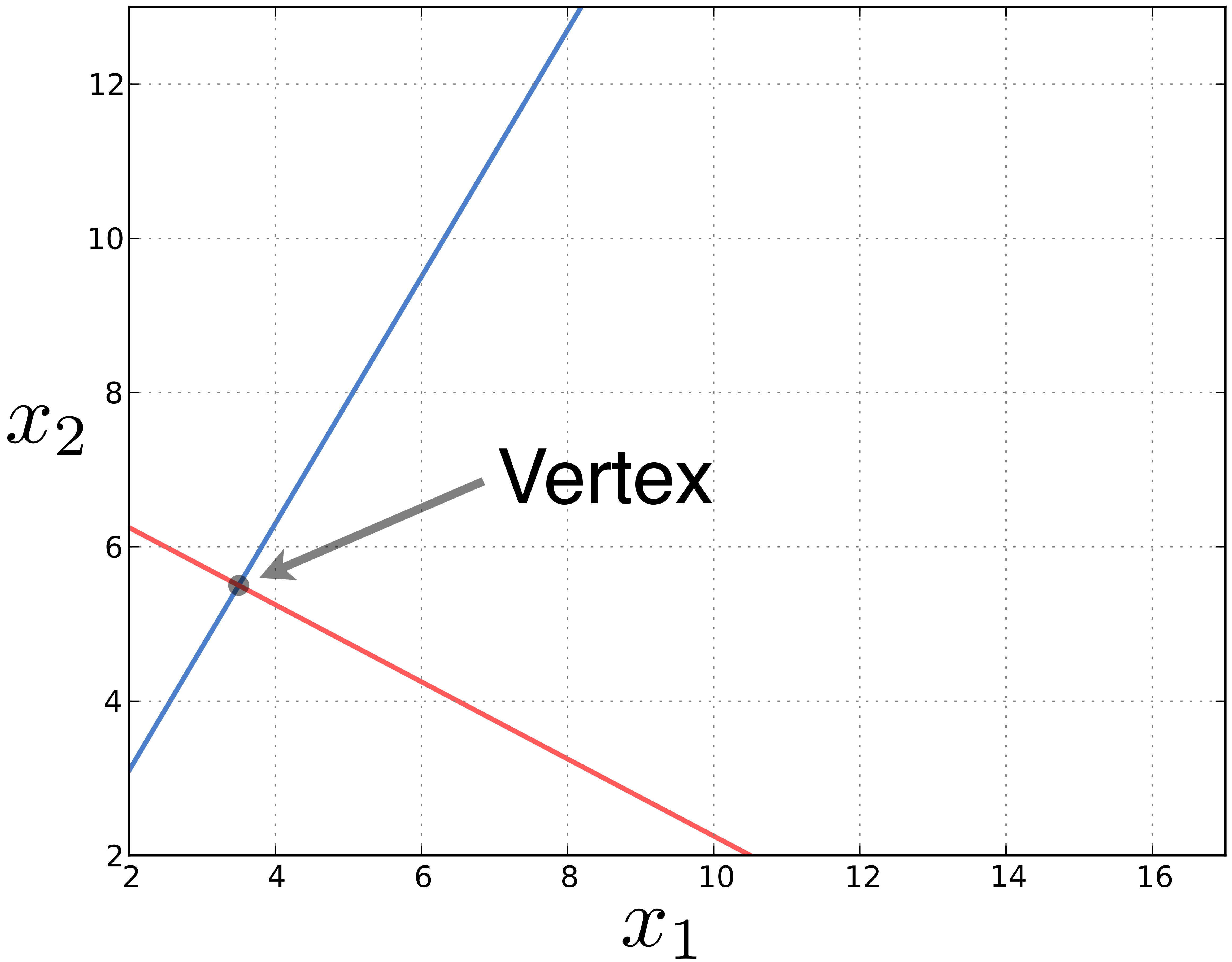


$$-3.0x_1 \quad - \quad 6.0x_2 \quad \leq \quad -43.50$$

Hyperplanes



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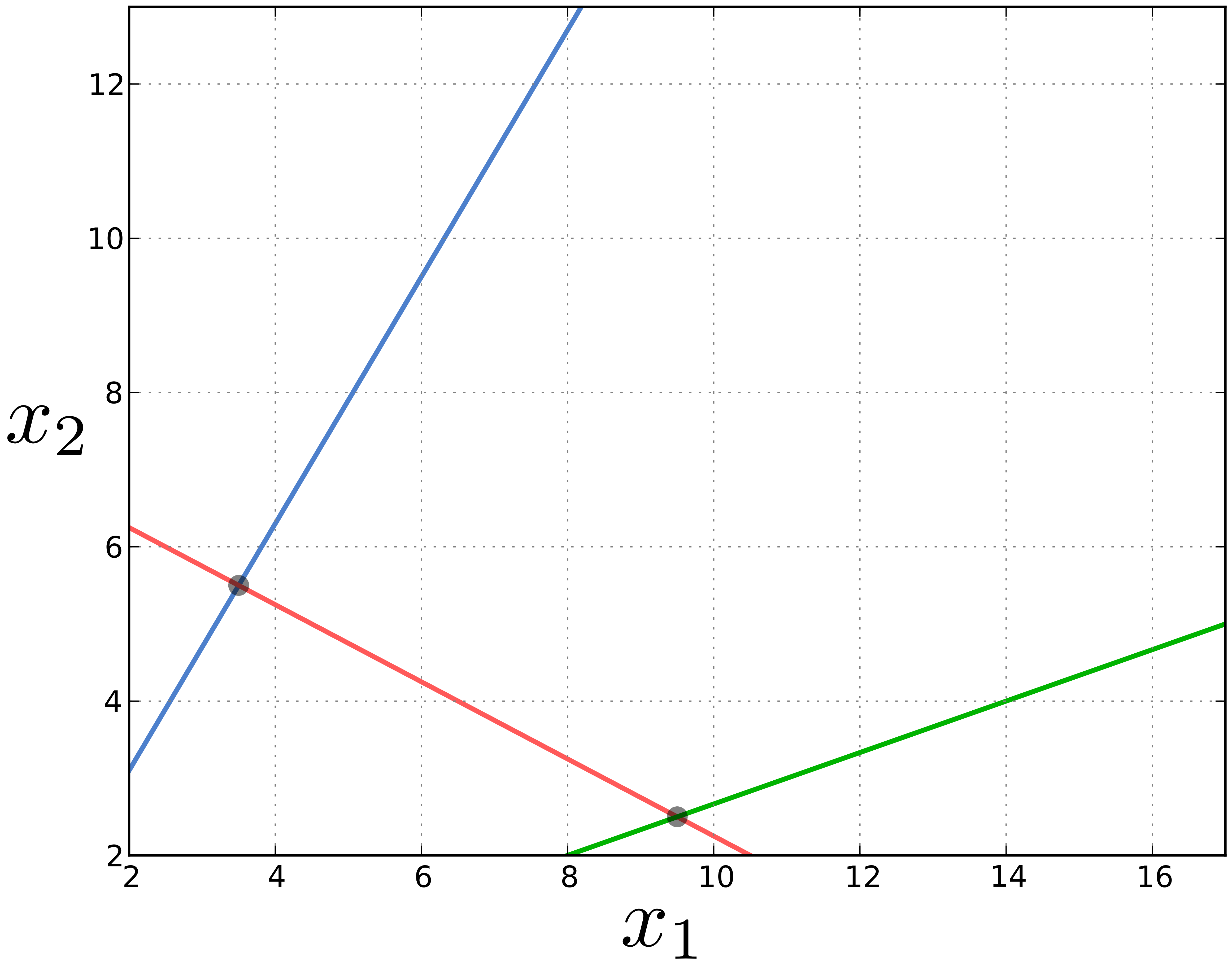
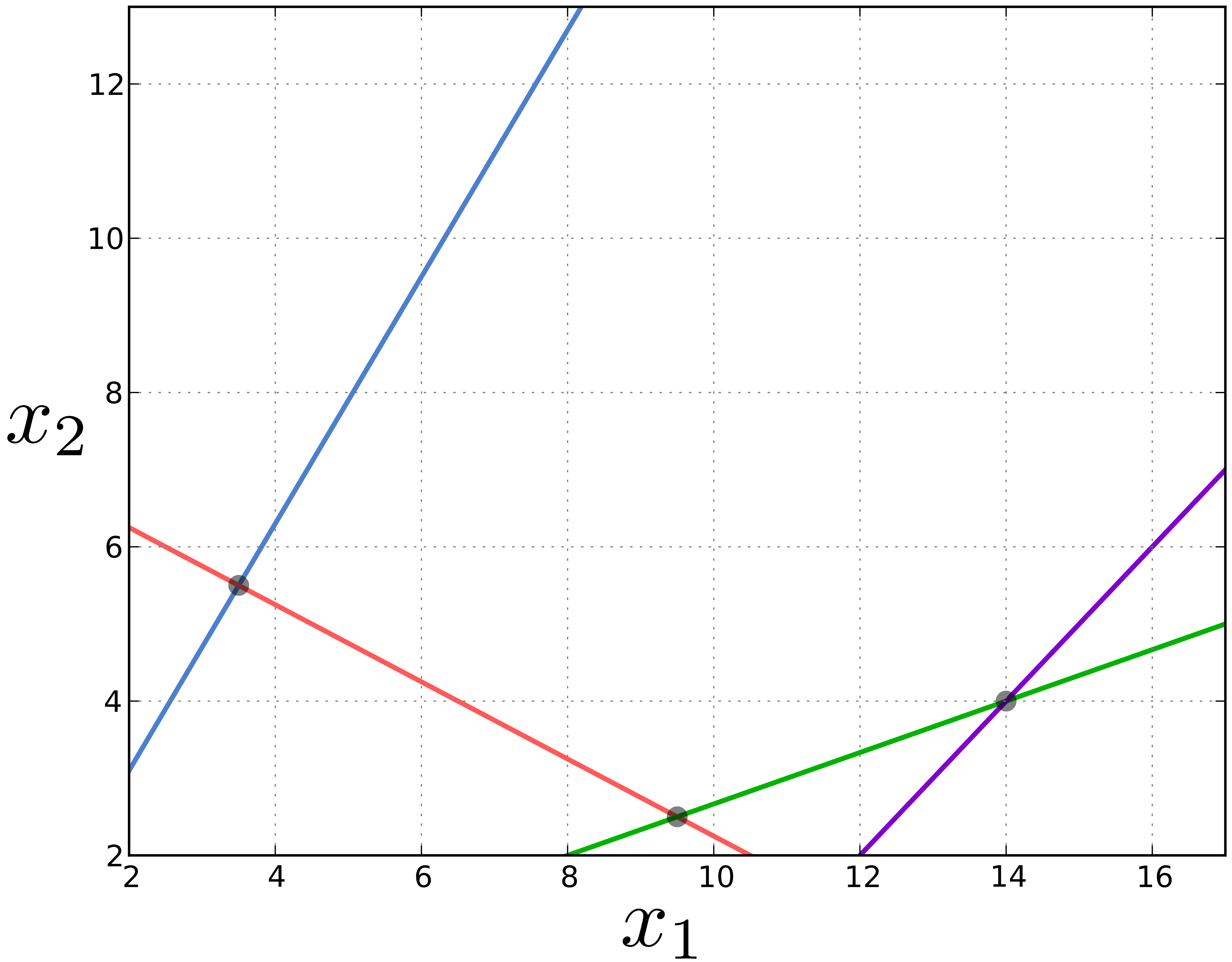


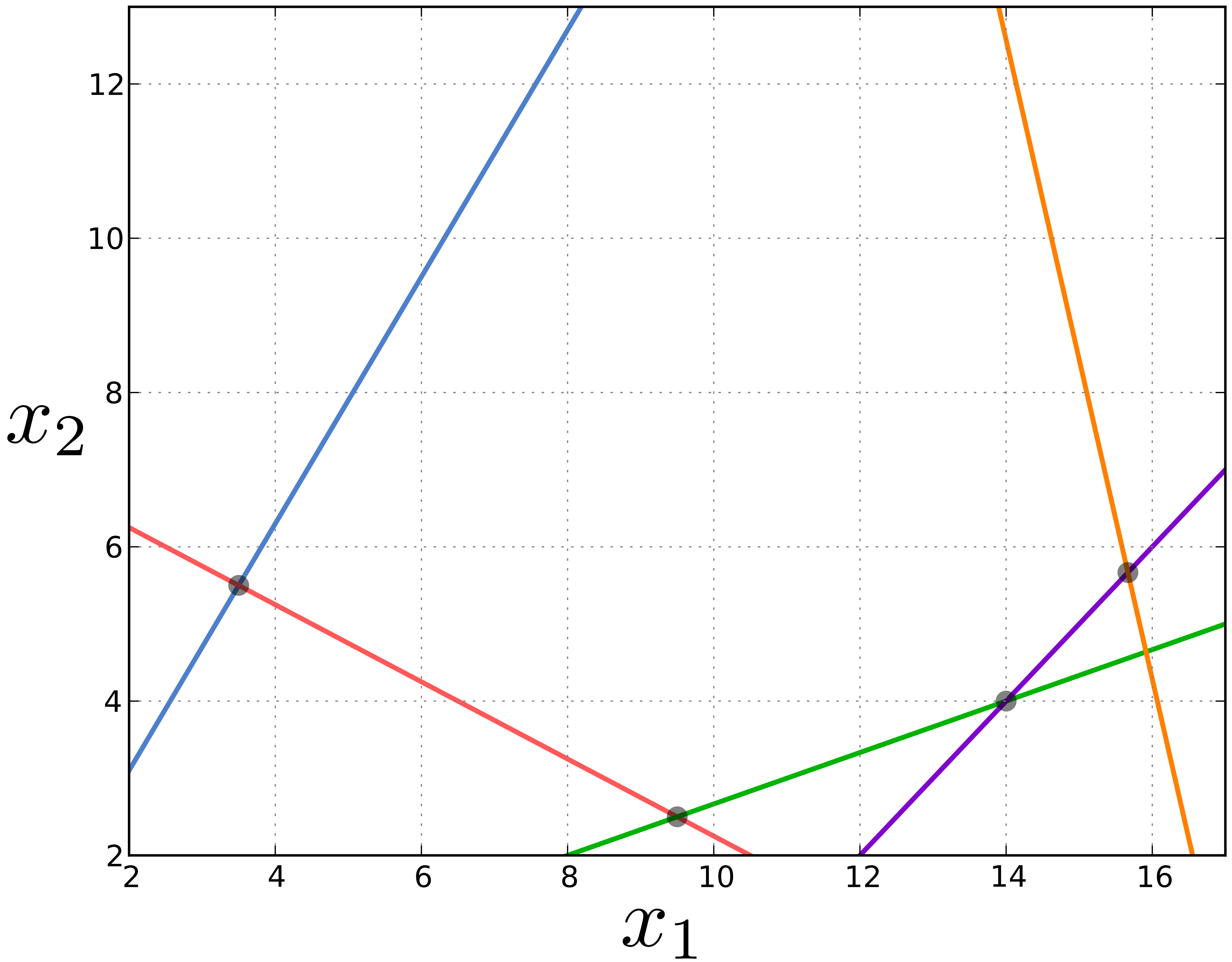
$$1.5x_1 \quad - \quad 4.5x_2 \quad \leq \quad 3.00$$

Hyperplanes



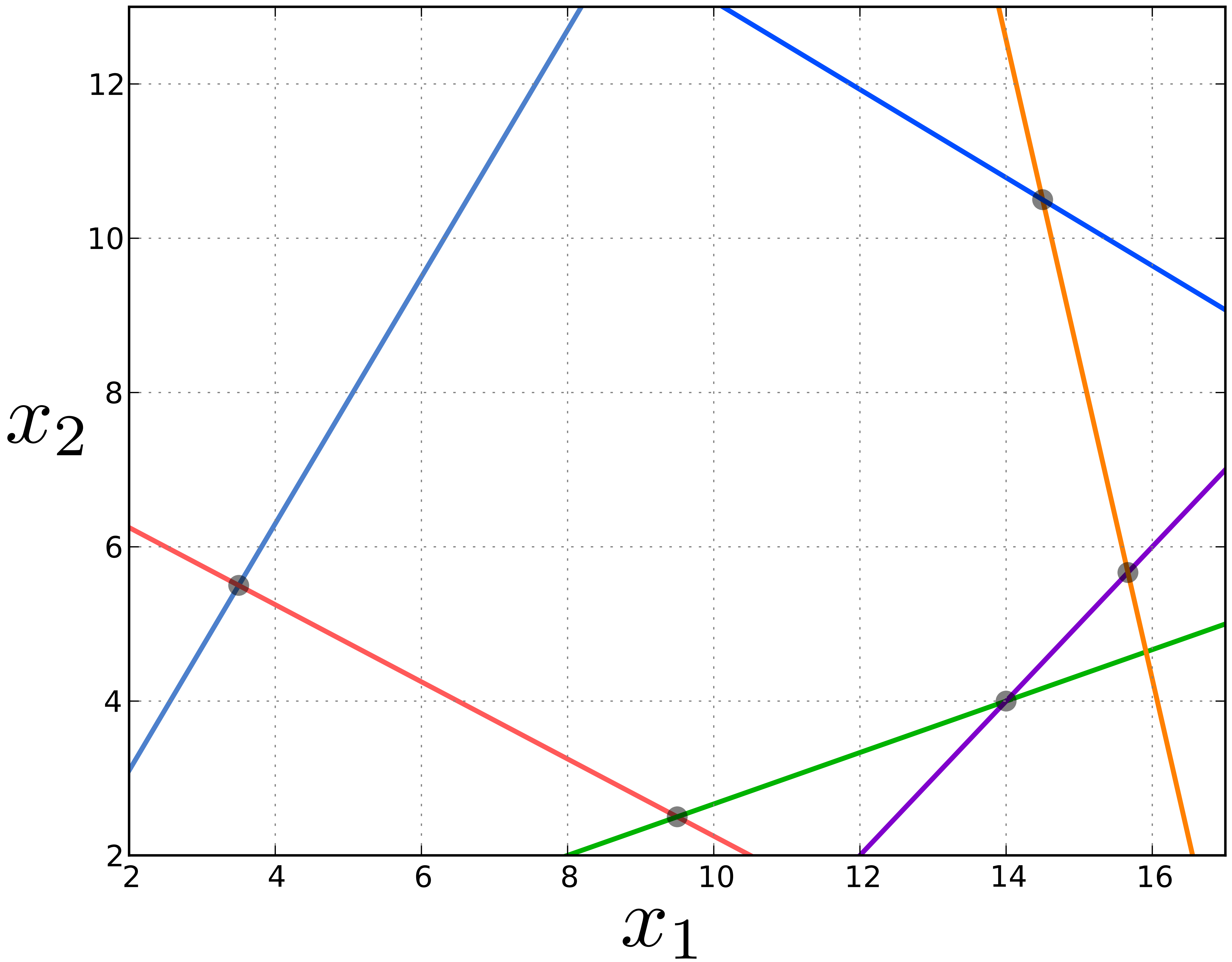
$$1.7x_1 \quad - \quad 1.7x_2 \quad \leq \quad 16.67$$

Hyperplanes



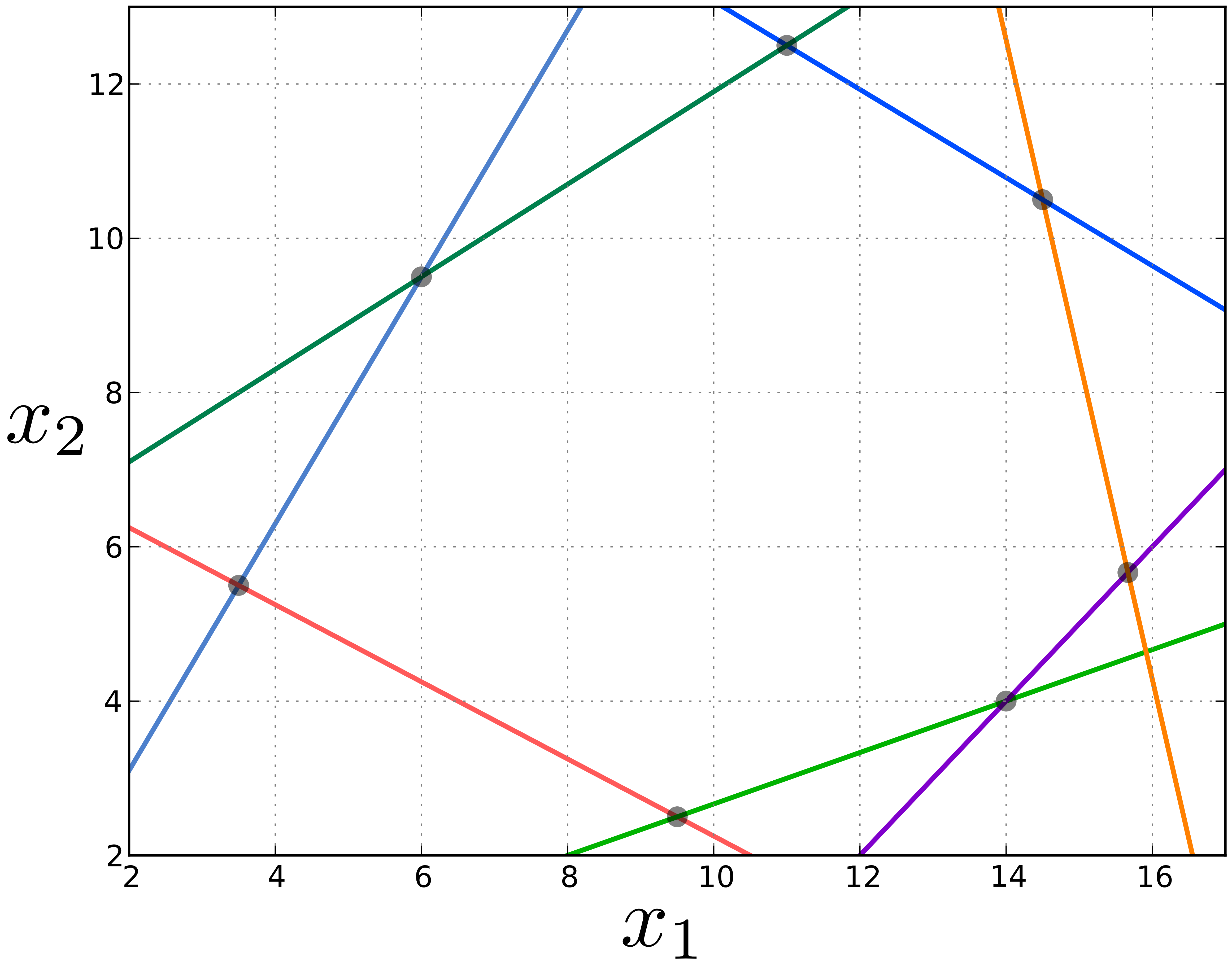
$$4.8x_1 \quad + \quad 1.2x_2 \quad \leq \quad 82.33$$

Hyperplanes



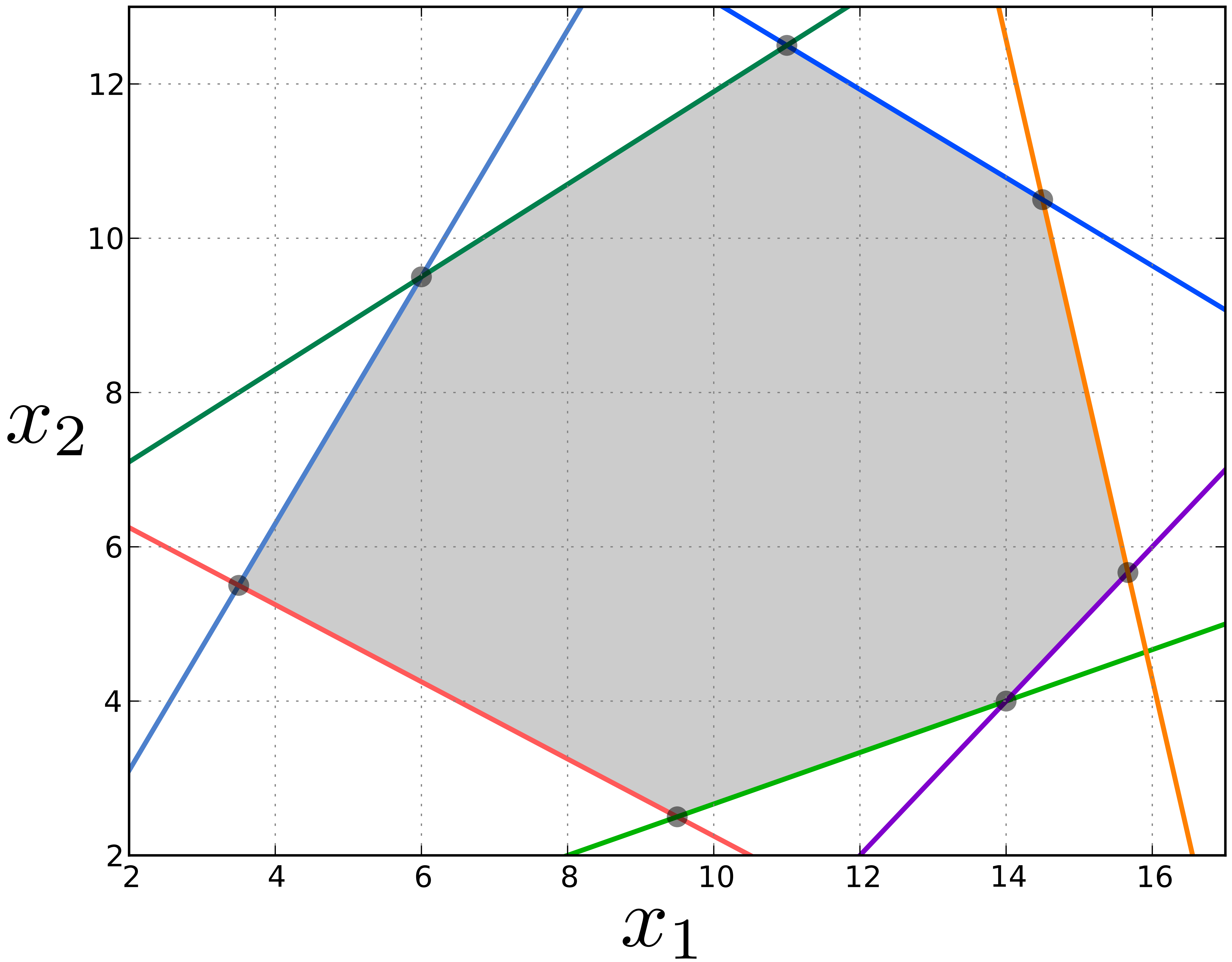
$$2.0x_1 \quad + \quad 3.5x_2 \quad \leq \quad 65.75$$

Hyperplanes



$$-3.0x_1 \quad + \quad 5.0x_2 \quad \leq \quad 29.50$$

Hyperplanes



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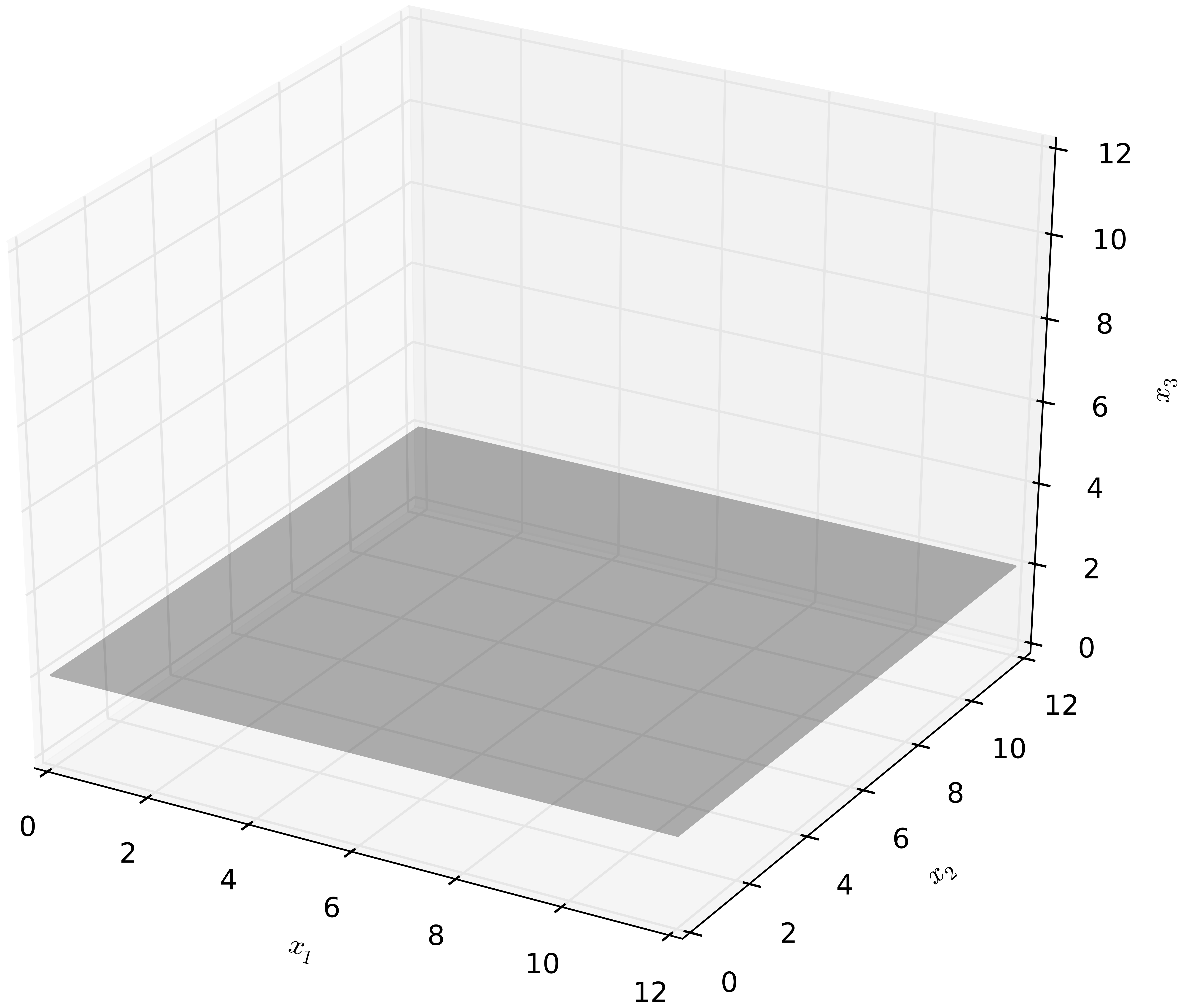
- For n variables
  - dimension n-1 (one hyperplane): facet
  - dimension 0 (n hyperplane): vertex

3D Hyperplanes



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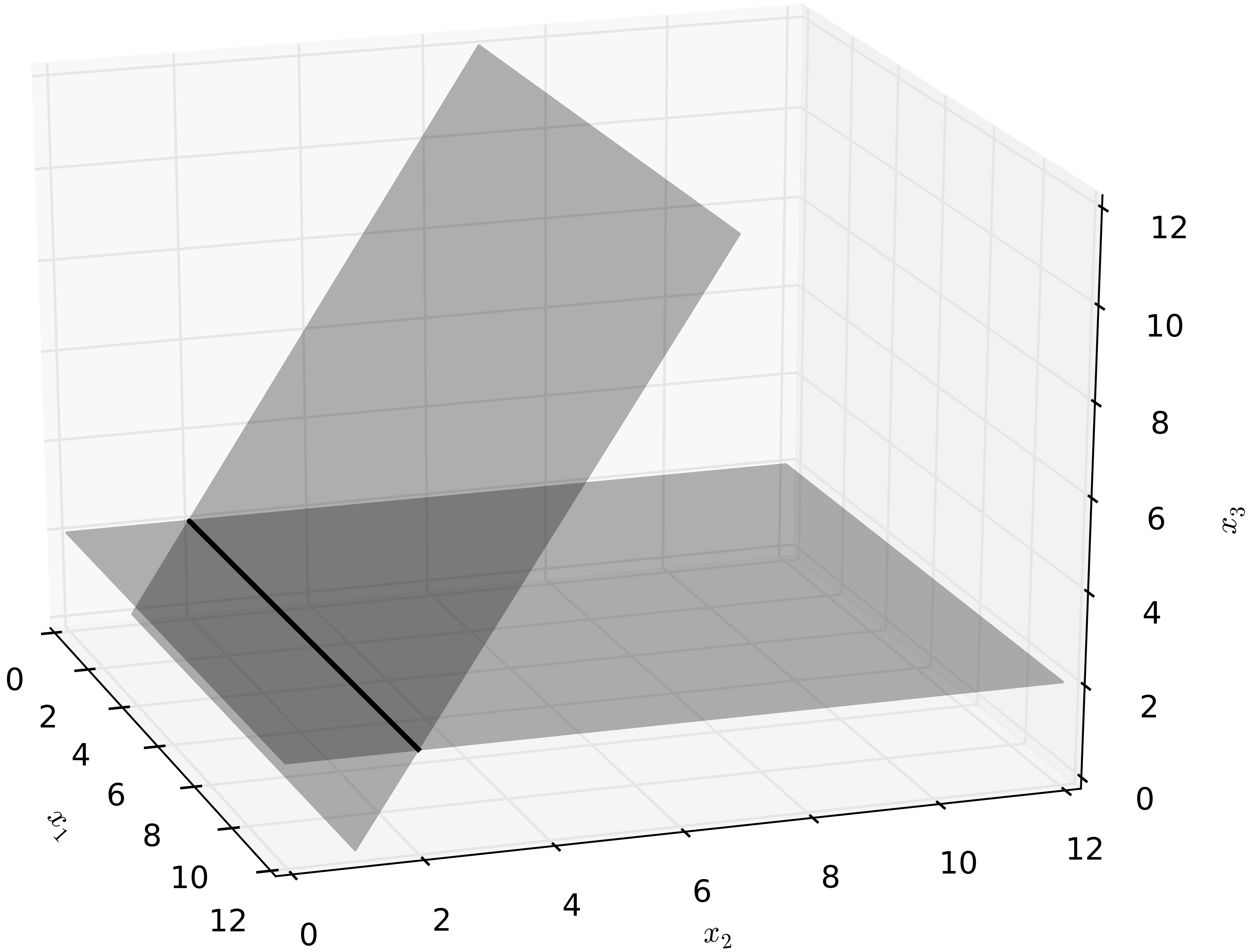

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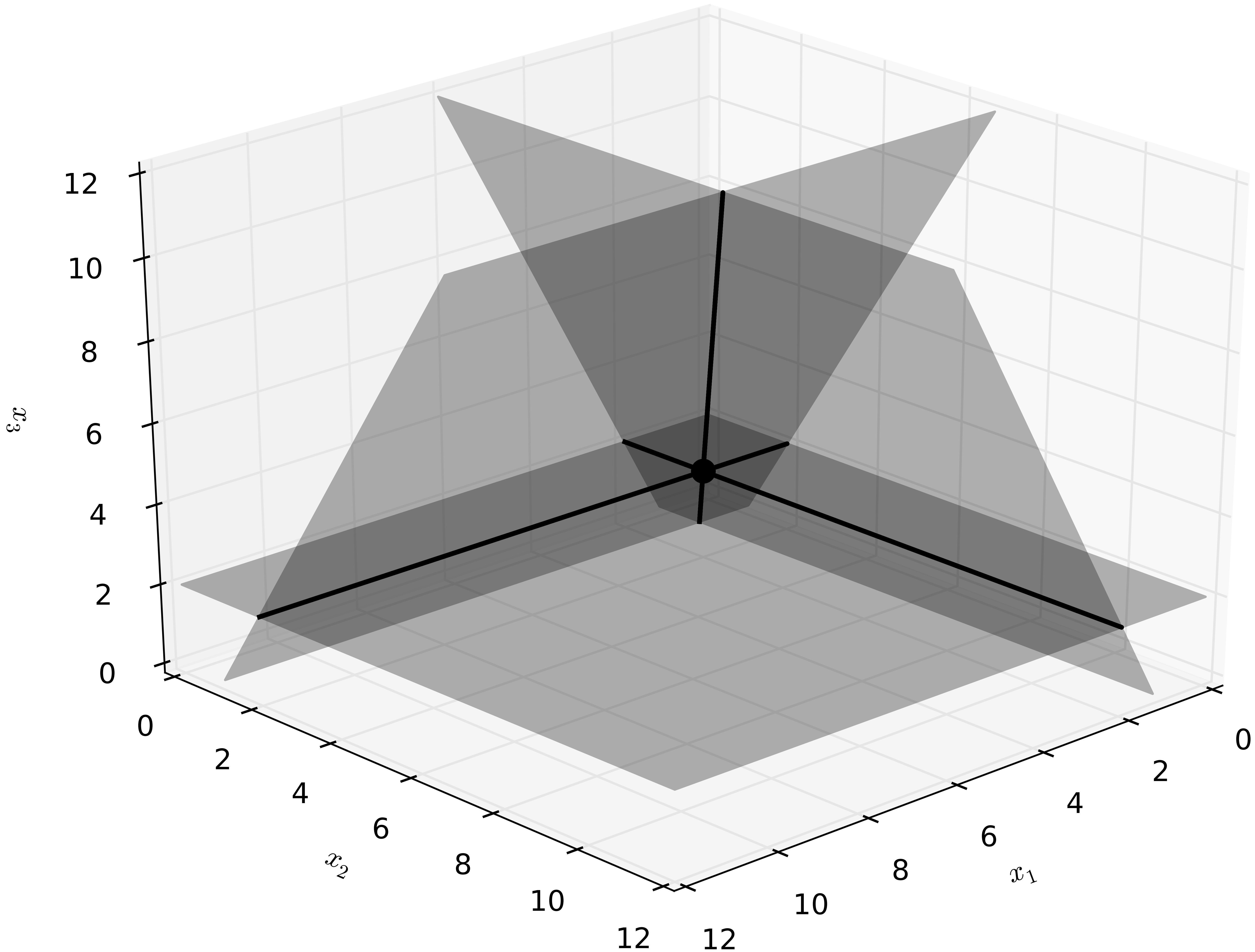
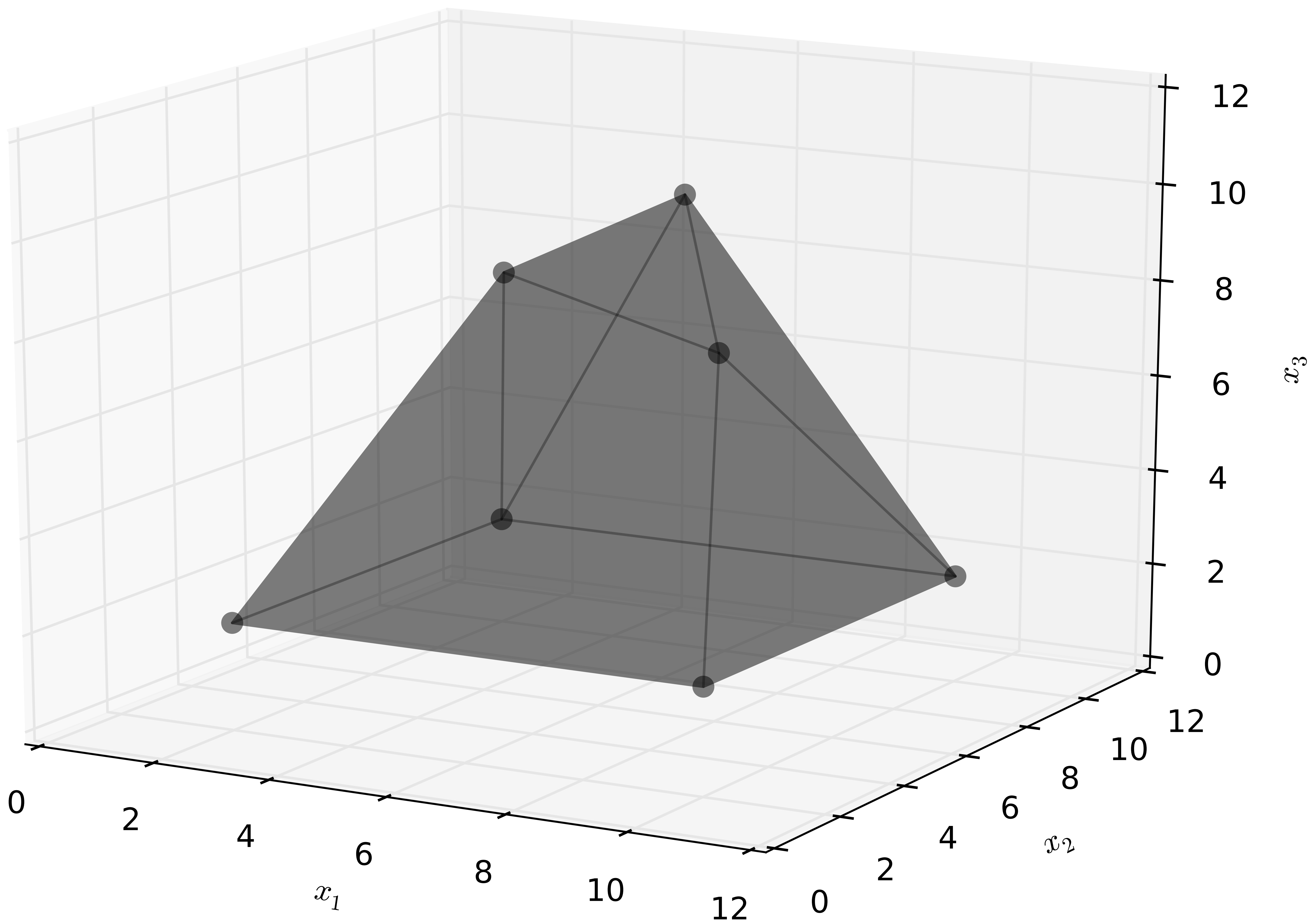


Tuesday, 11 June 13

Hyperplane, Facets, and Vertices



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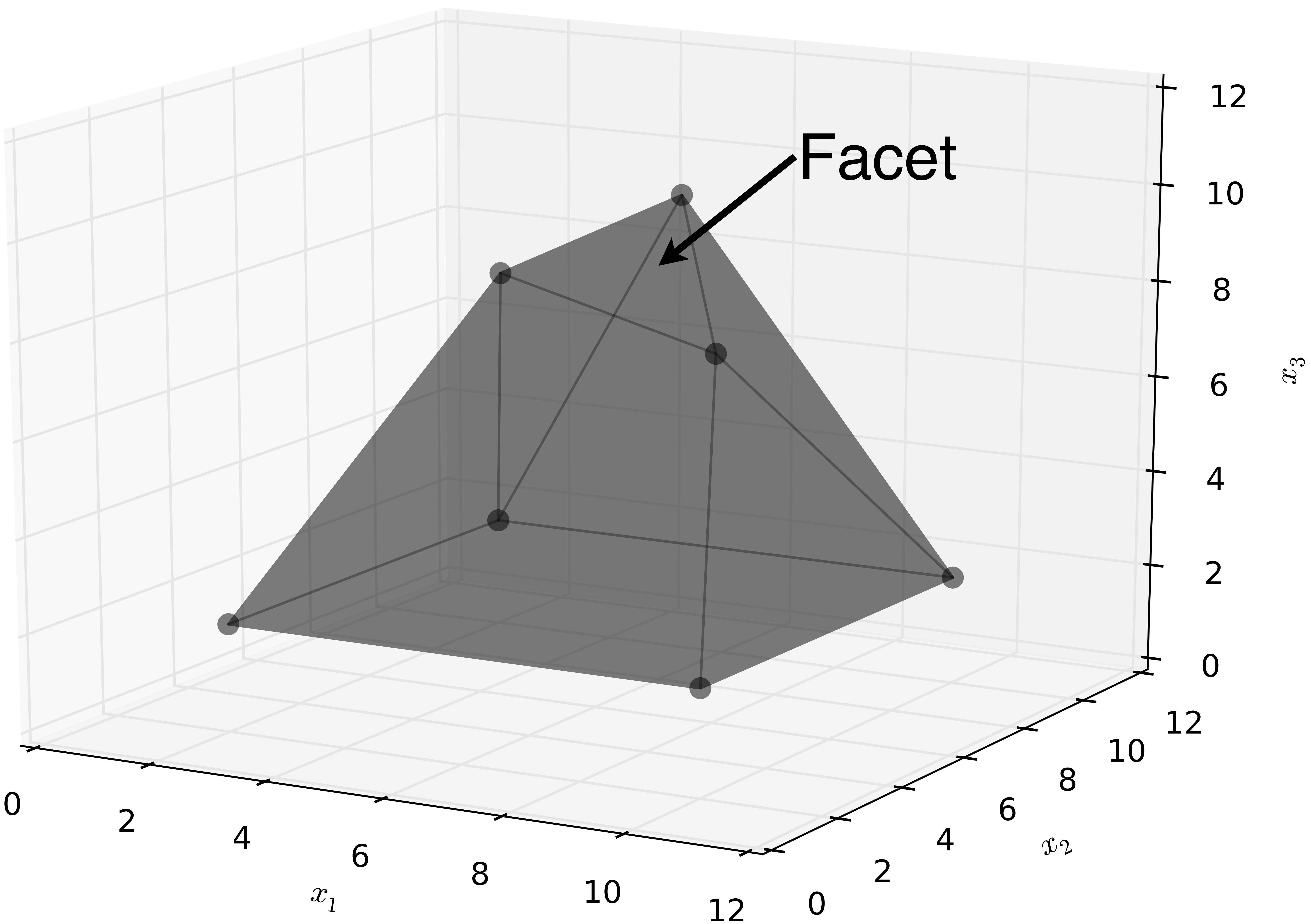

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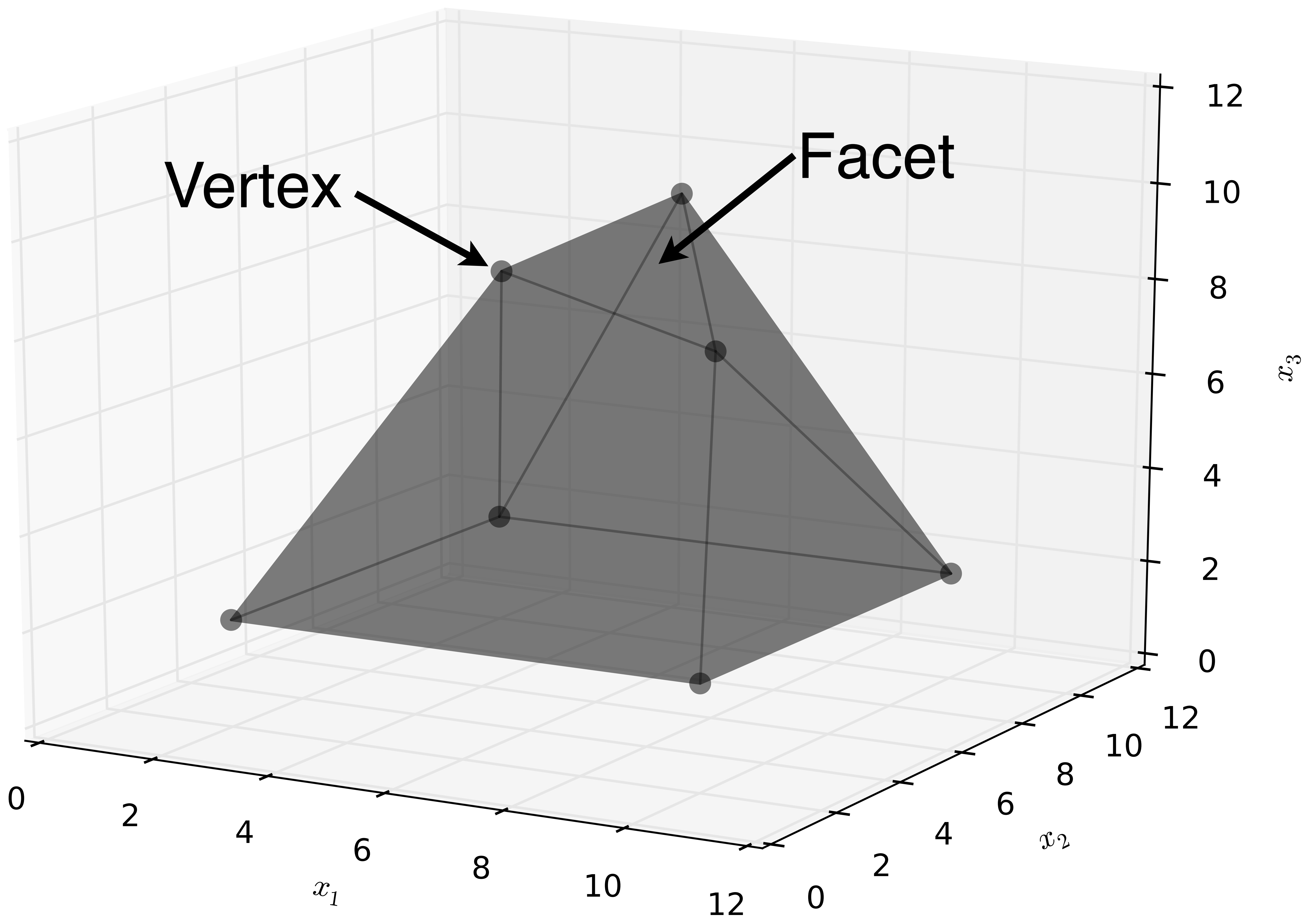

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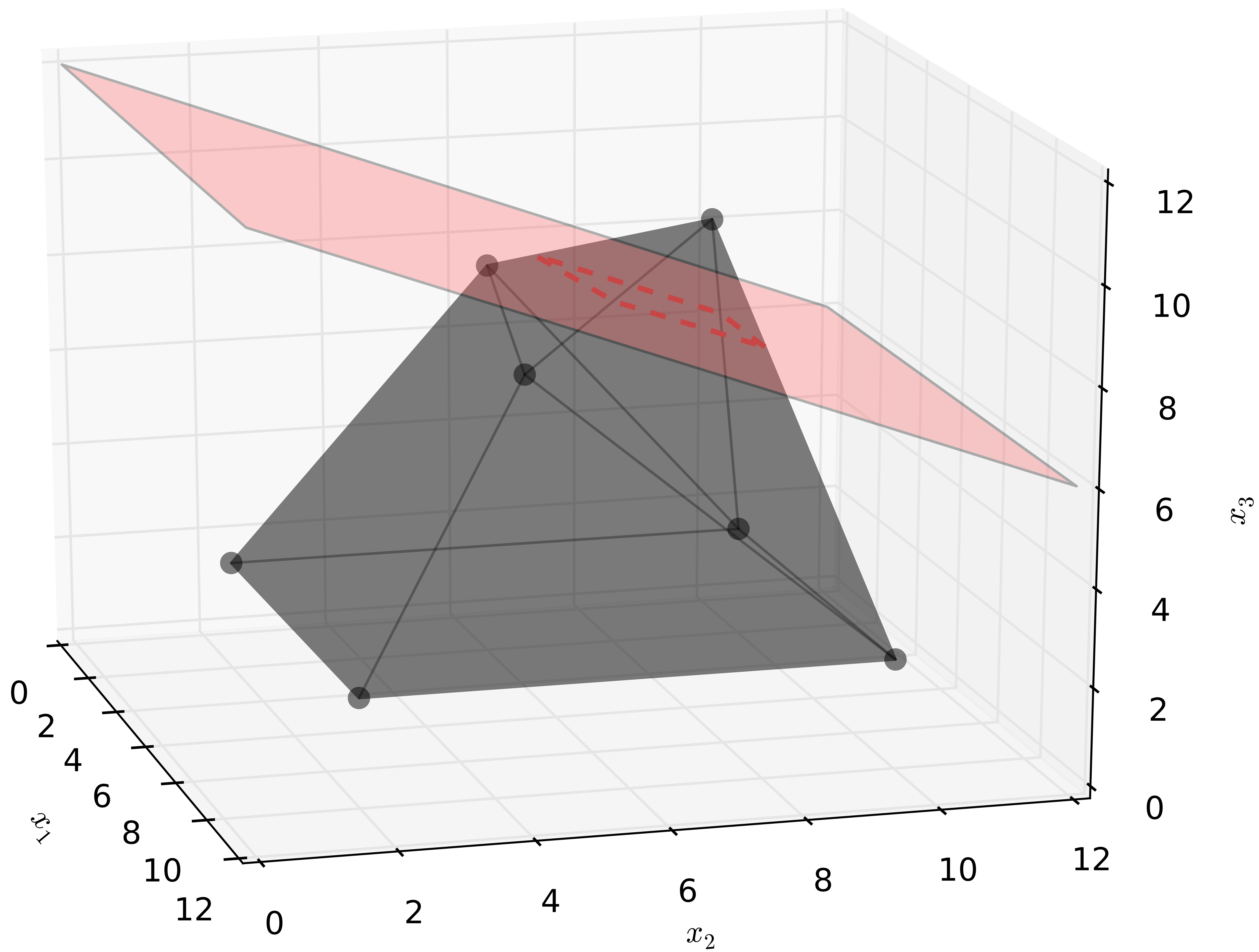

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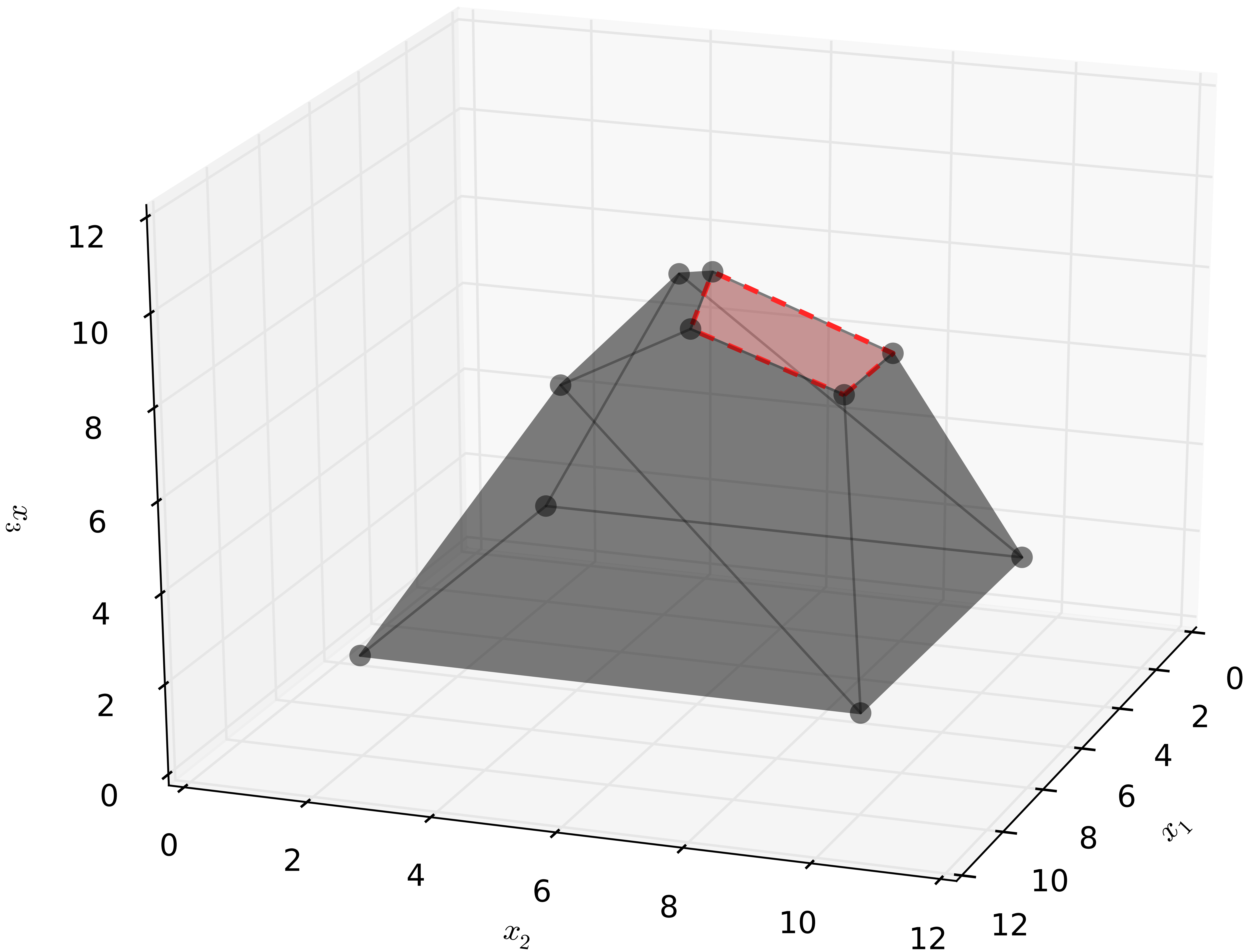

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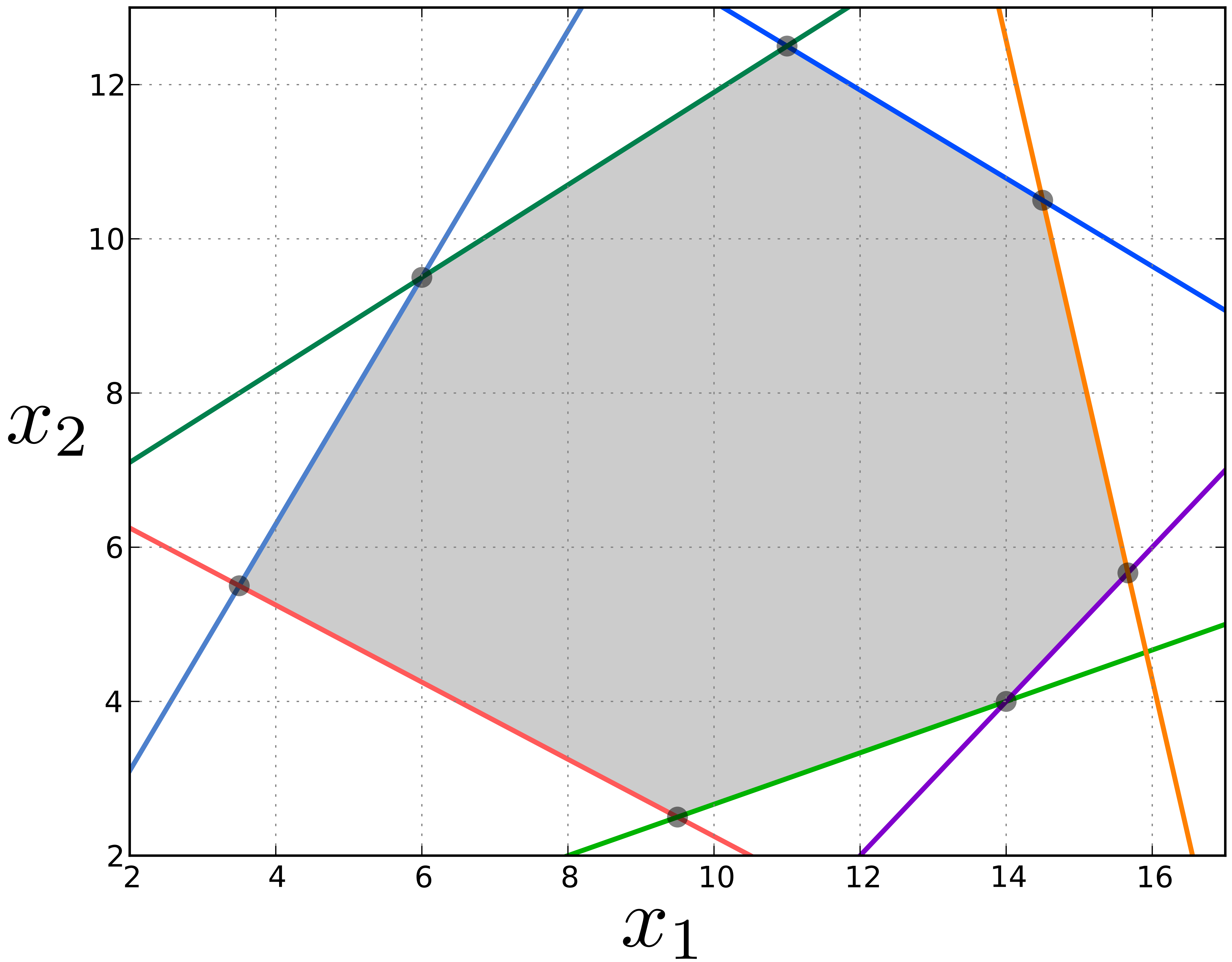

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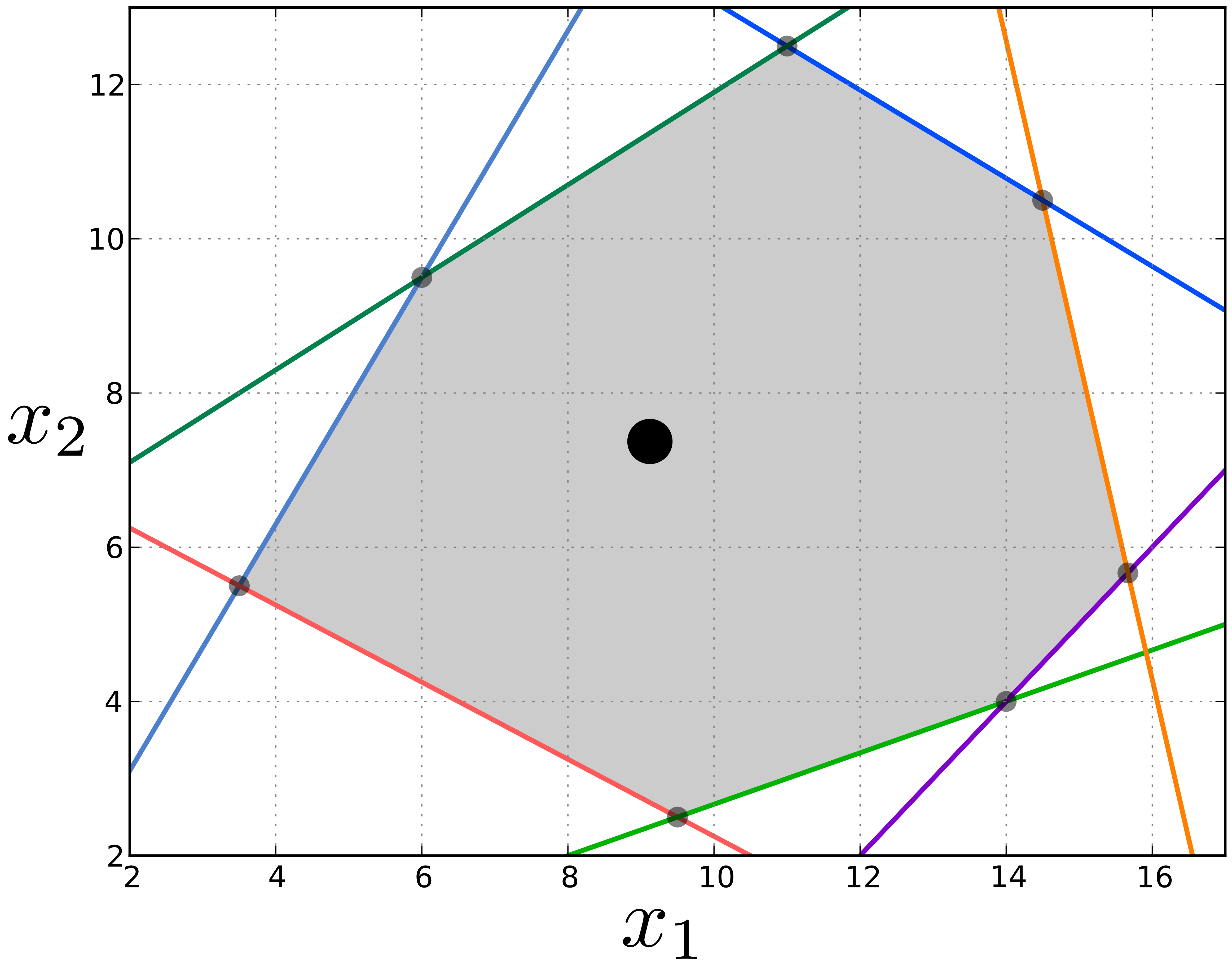

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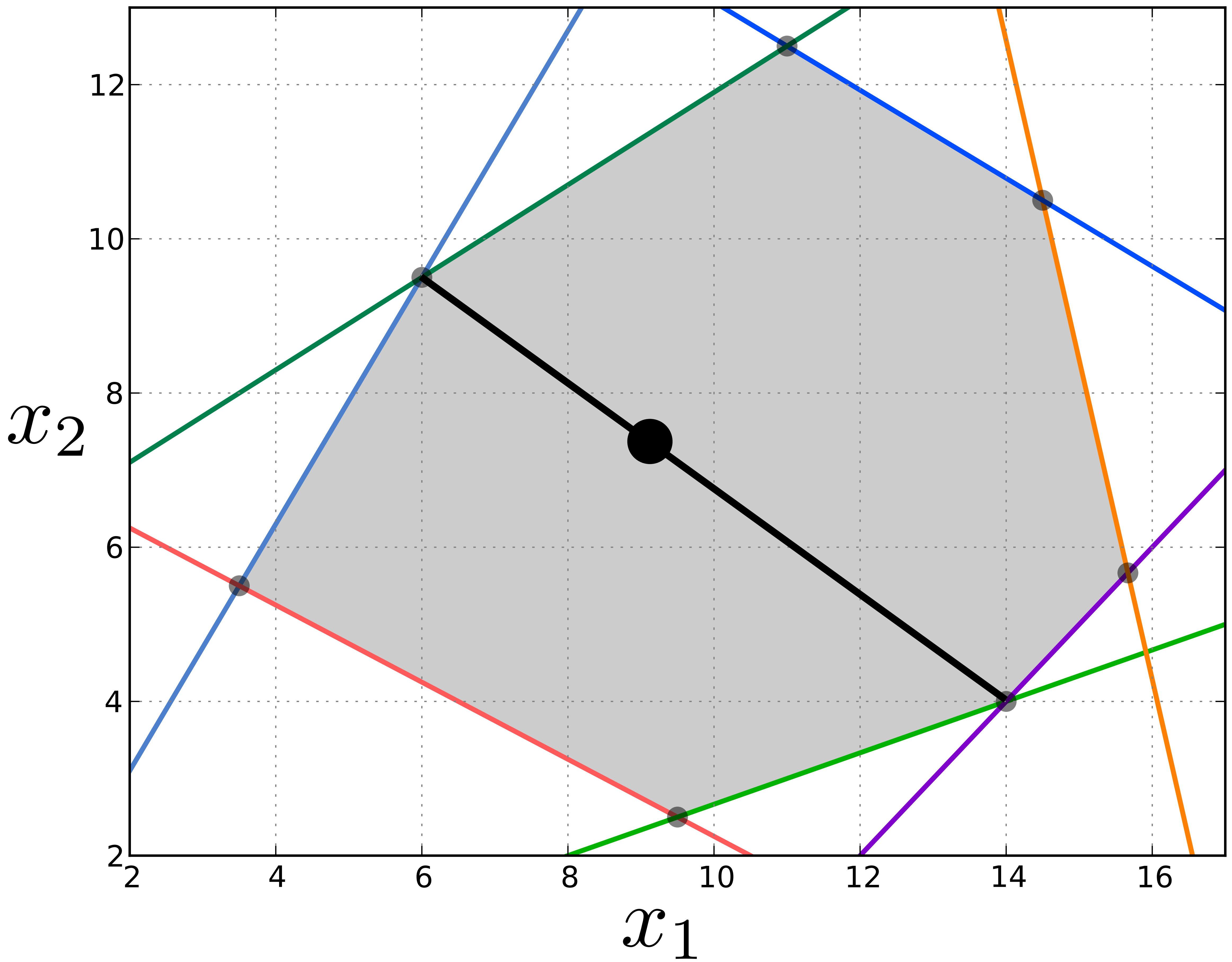

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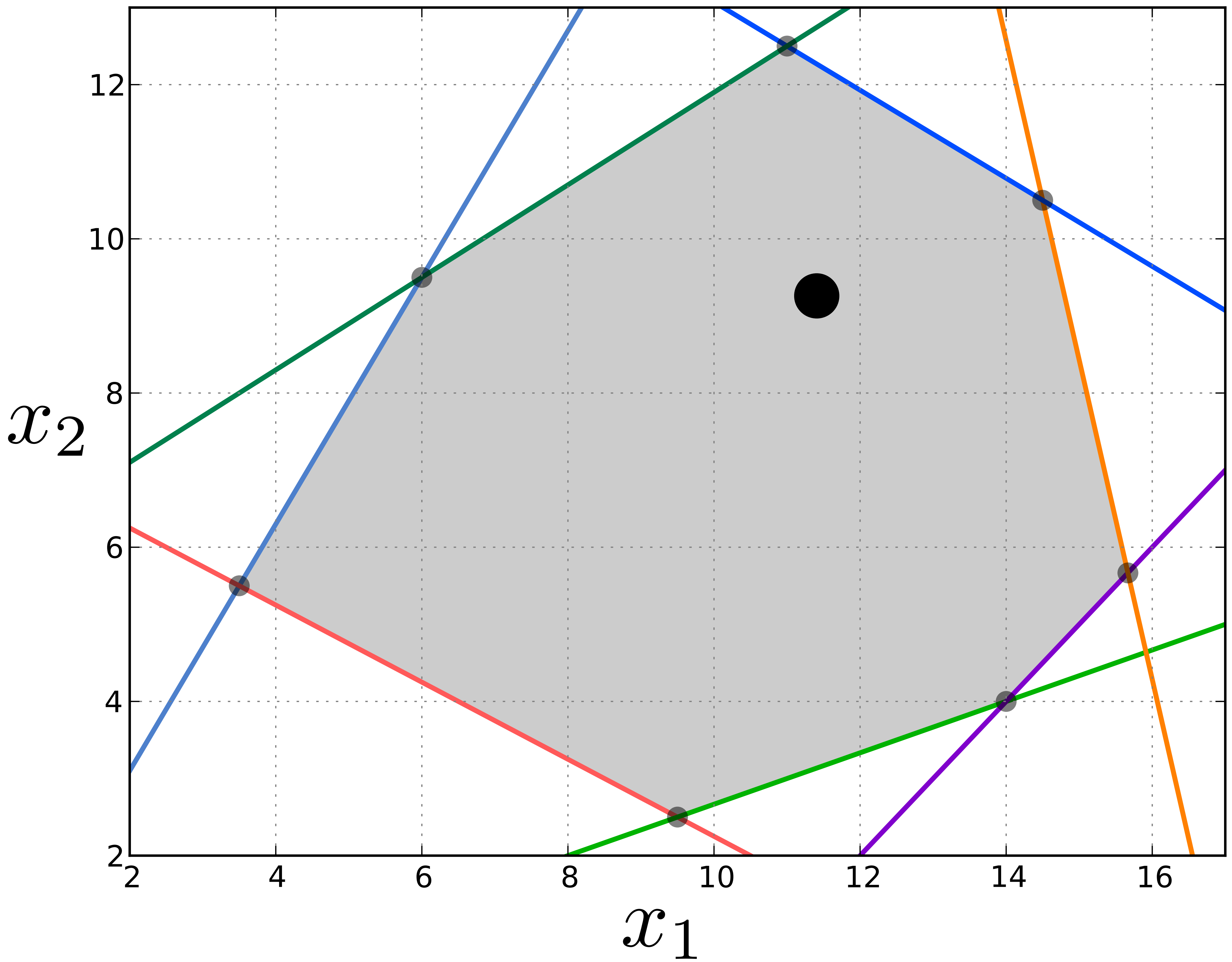

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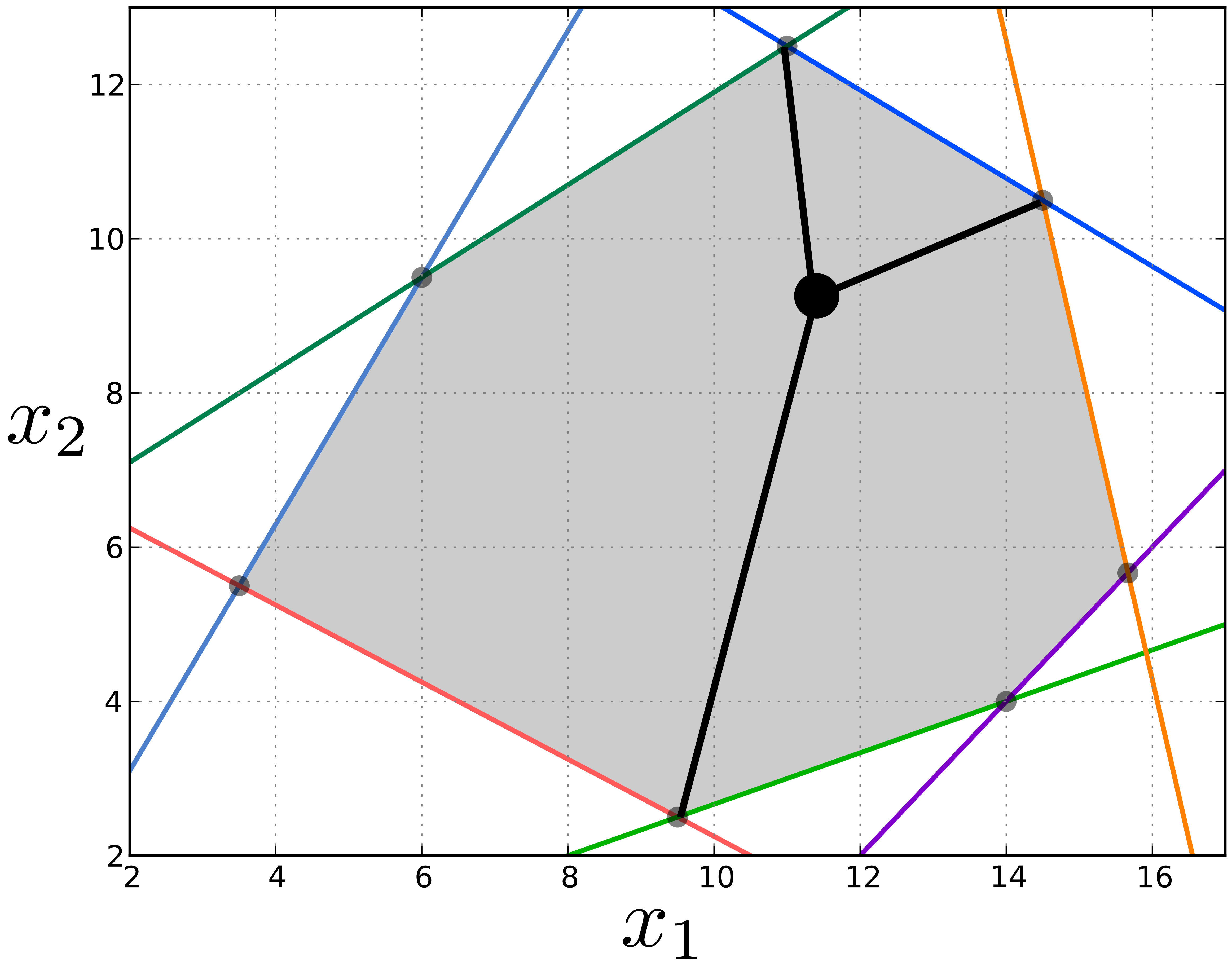

Why I Love These Vertices

$$\begin{aligned}
&\min c_1x_1 + \ldots + c_nx_n \\
&\text{subject to} \\
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&\quad \ldots \\
&\quad a_{m1}x_1 + \ldots + a_{mn}x_n \le b_m \\
&\\
&\quad x_i \ge 0 \quad (1 \le i \le n)
\end{aligned}$$

- Theorem: At least one of the points where the objective value is minimal is a vertex.

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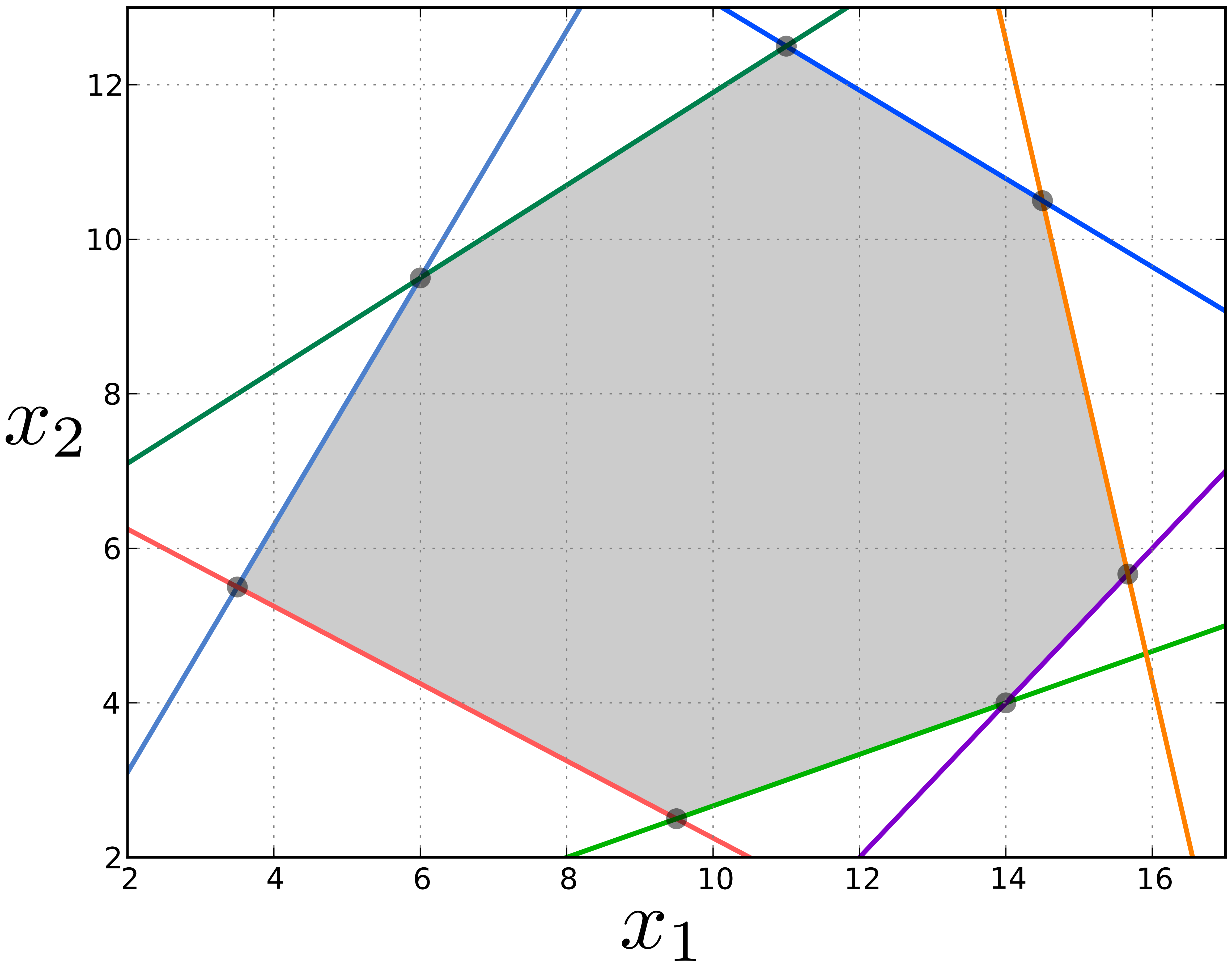

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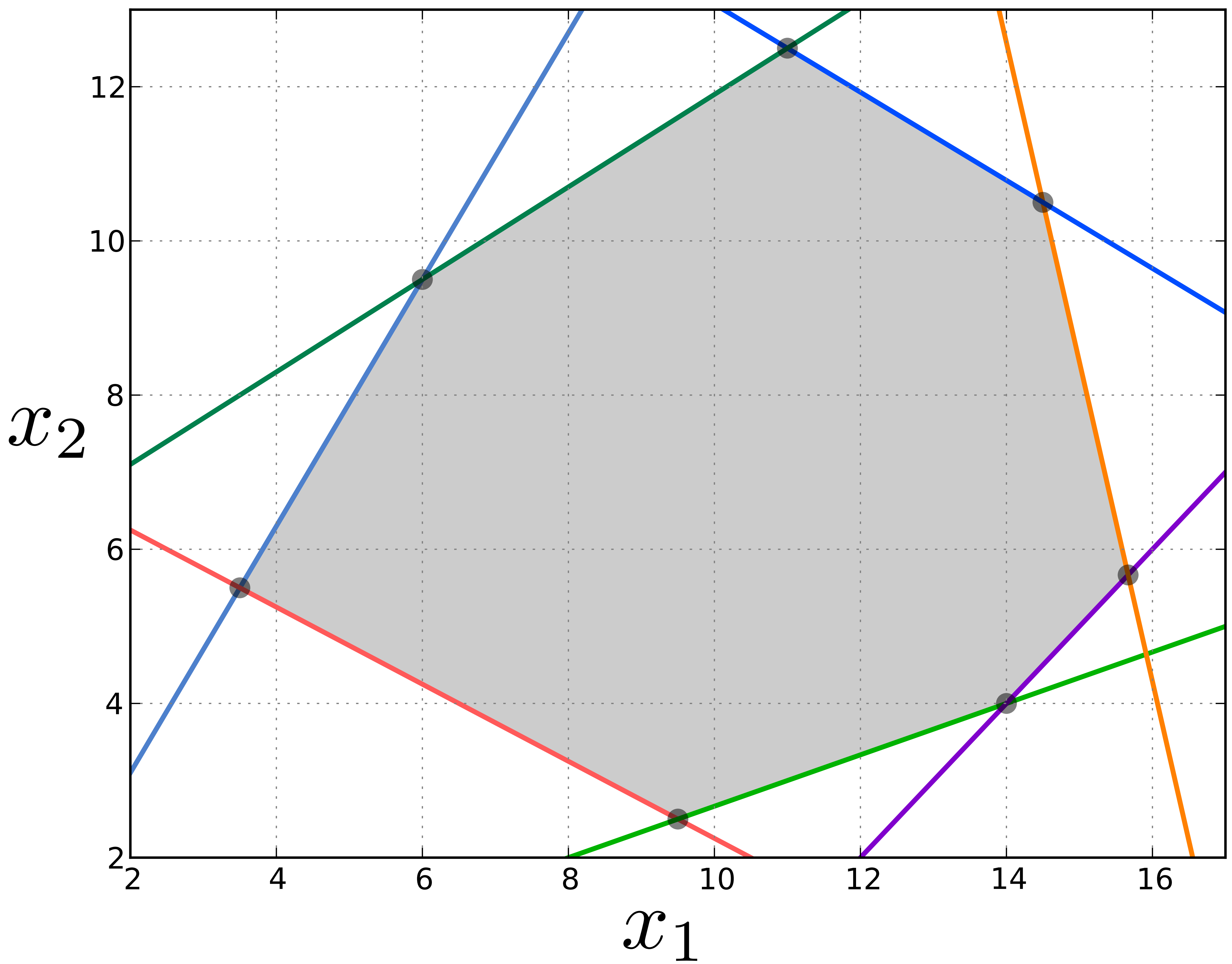

$$c_1 x_1 + \ldots + c_n x_n = b$$

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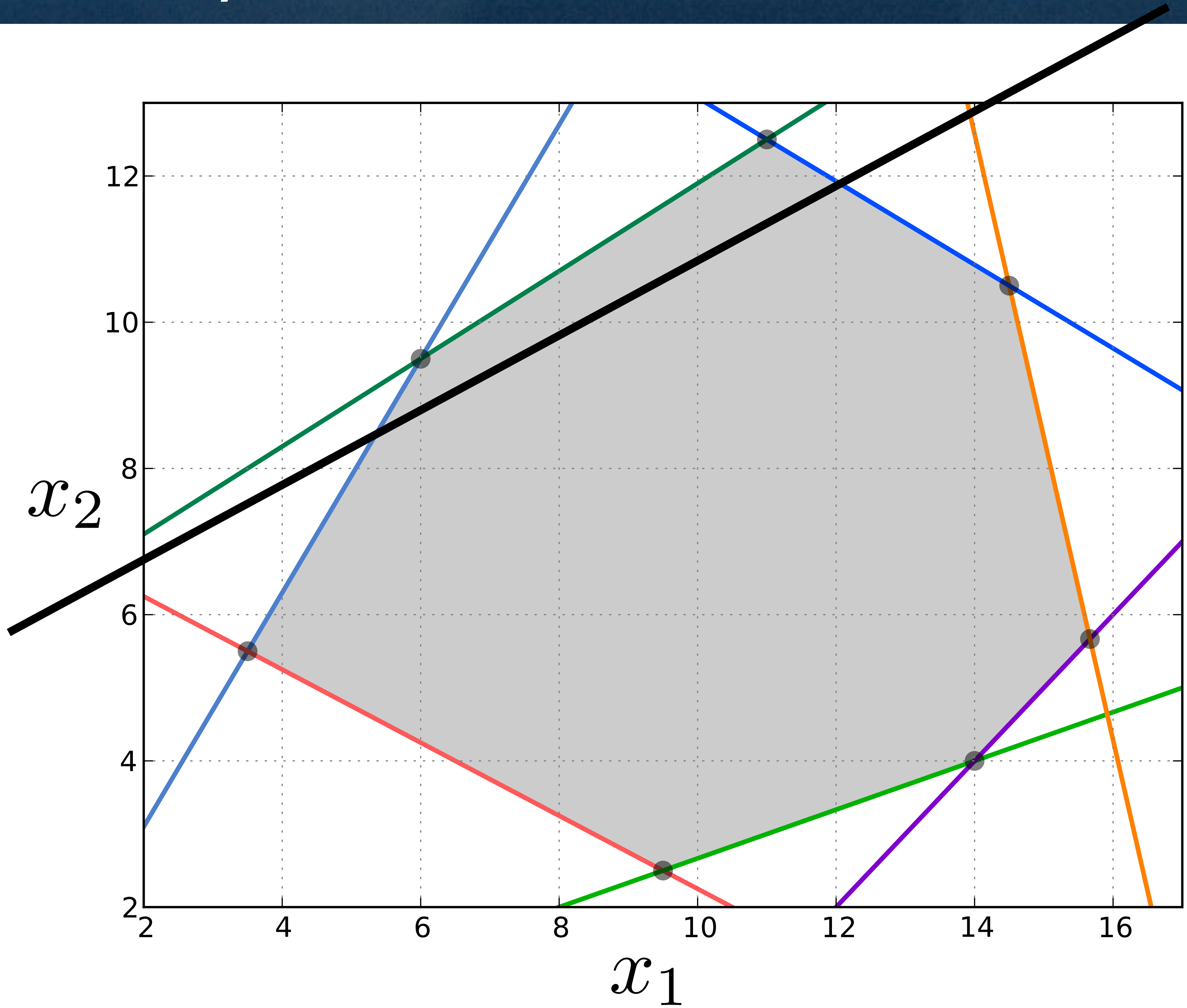

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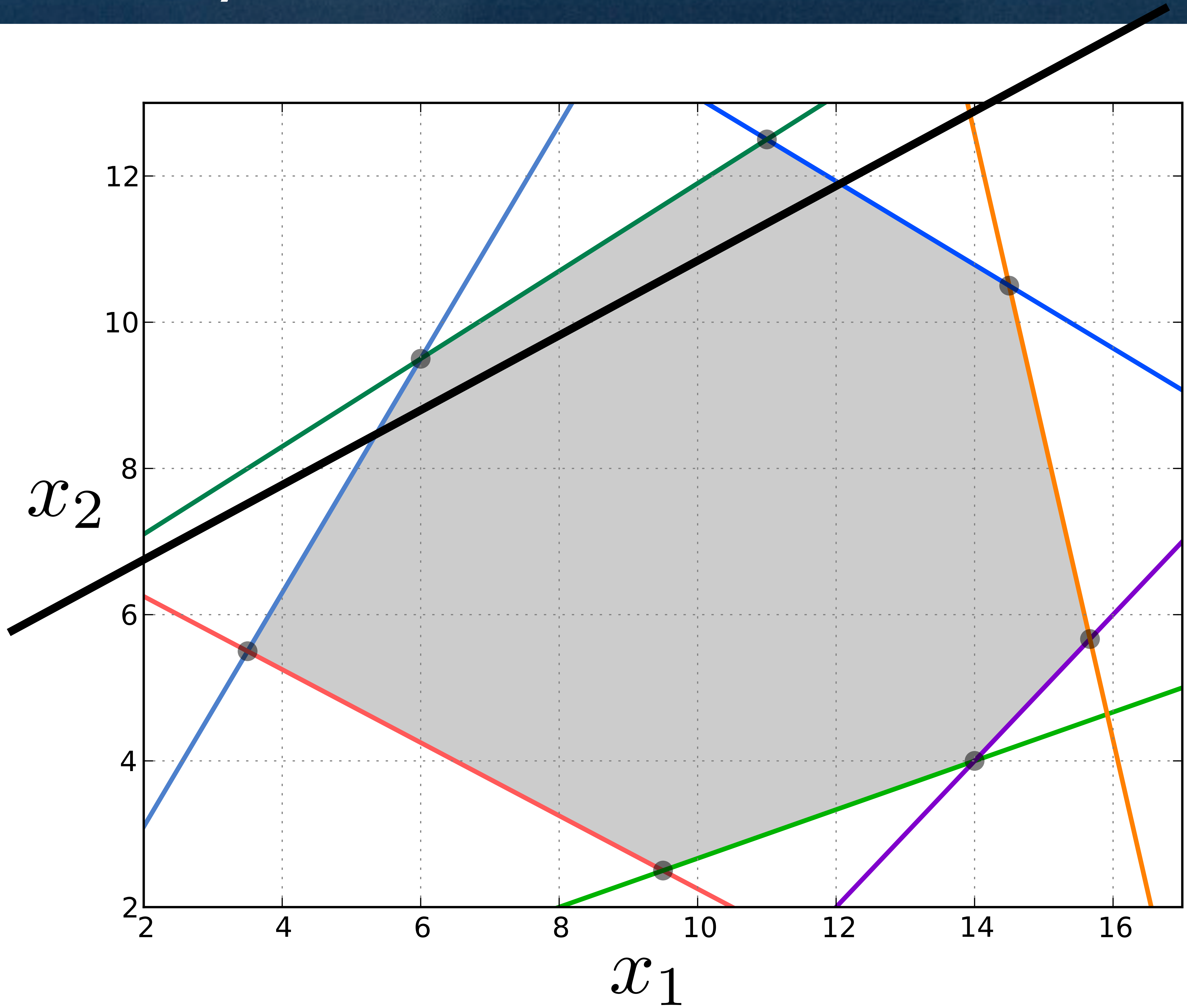
Why I Love These Vertices



$$\min \; c_1 x_1 + \ldots + c_n x_n = b$$

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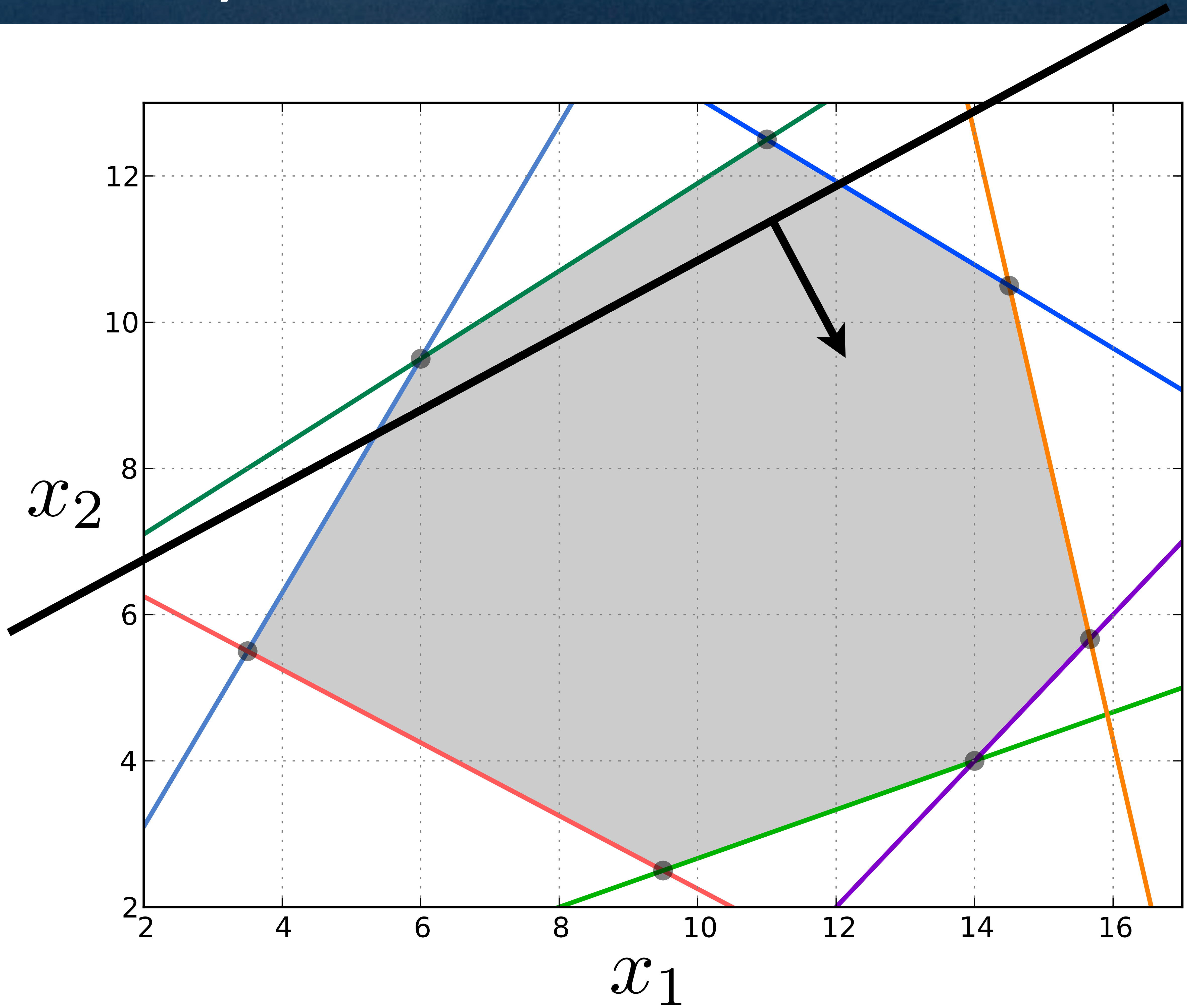

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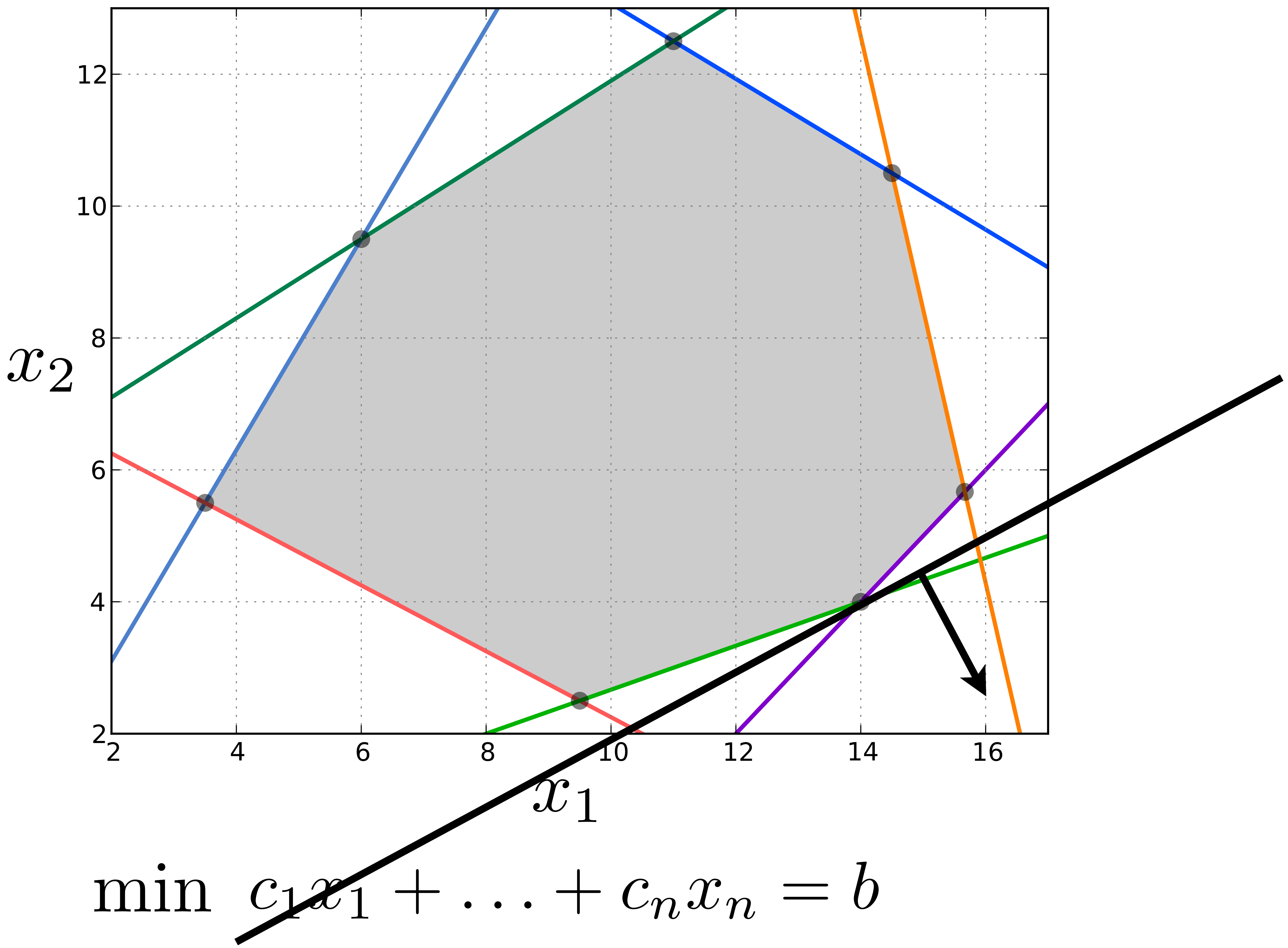

$$\min\ c_1x_1 + \ldots + c_nx_n = b$$

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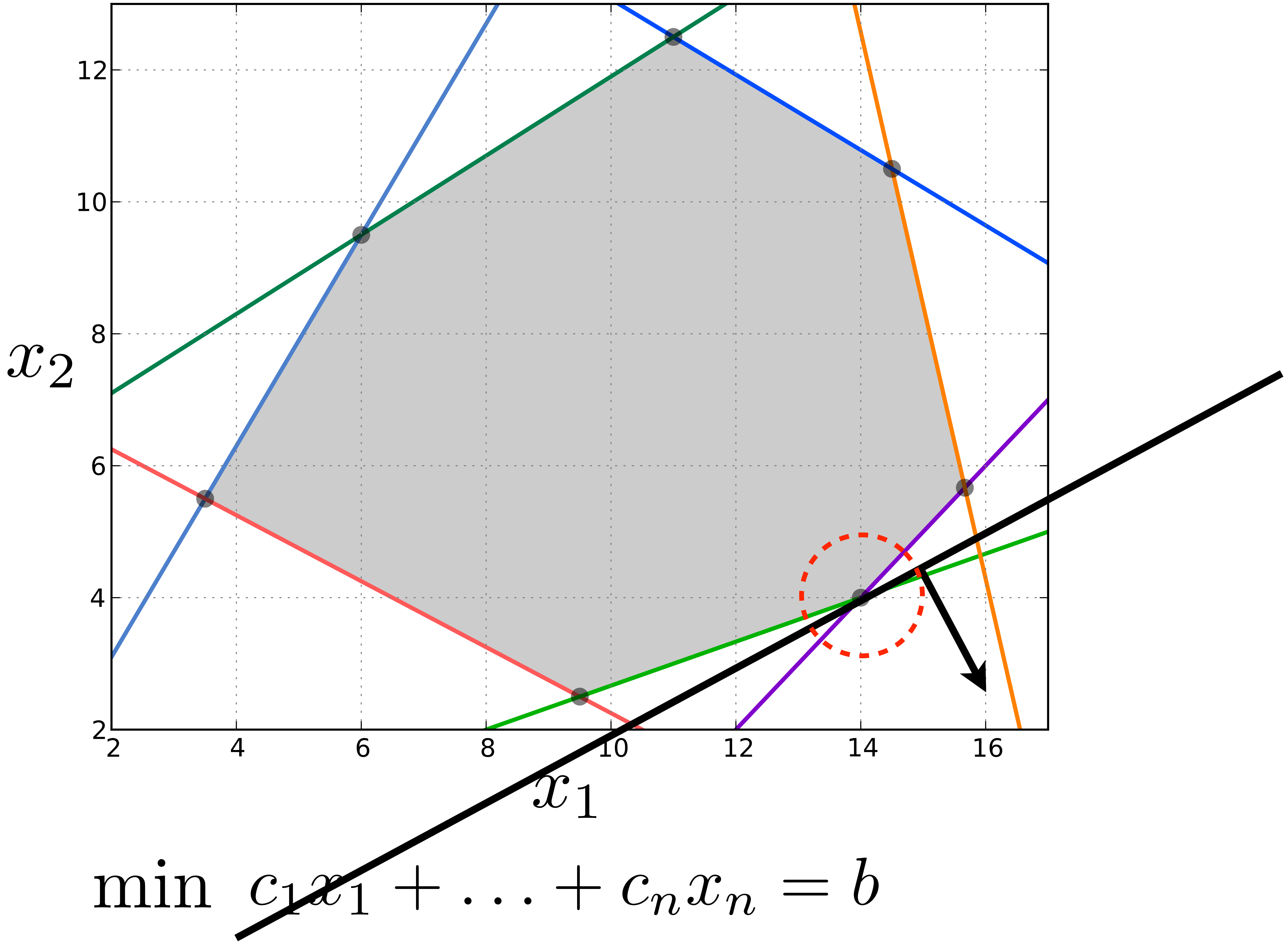

$$\min\ c_1x_1 + \ldots + c_nx_n = b$$

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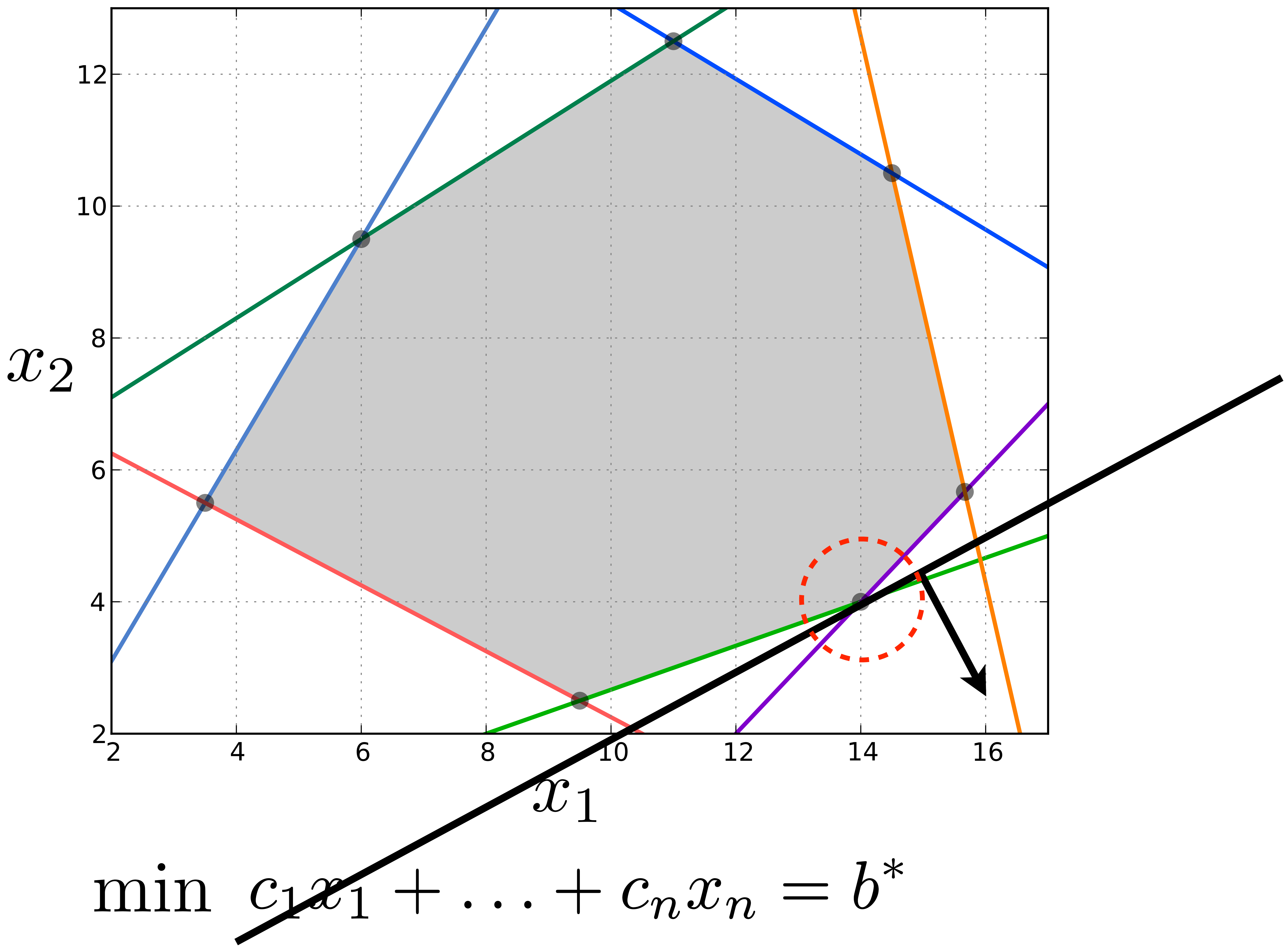
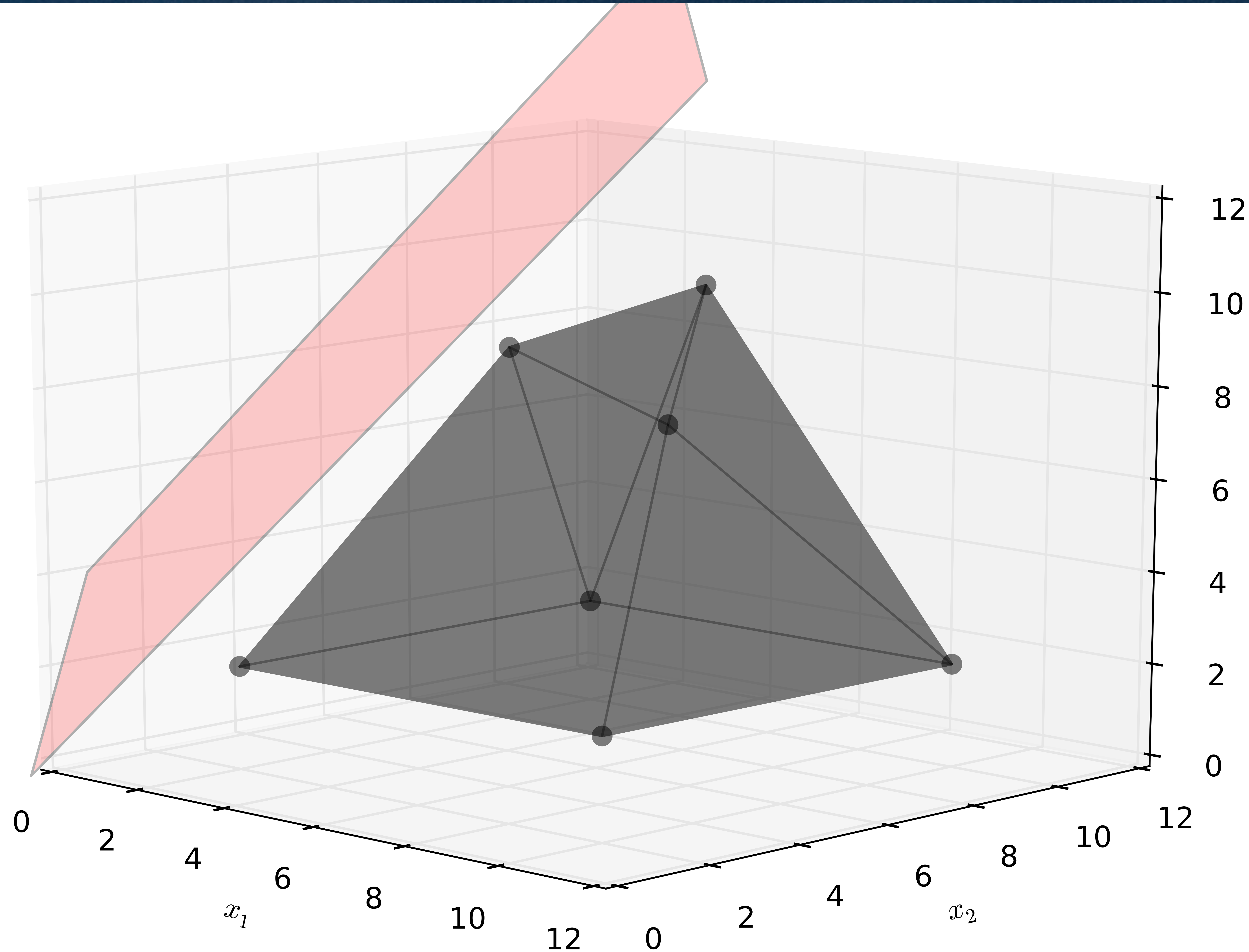
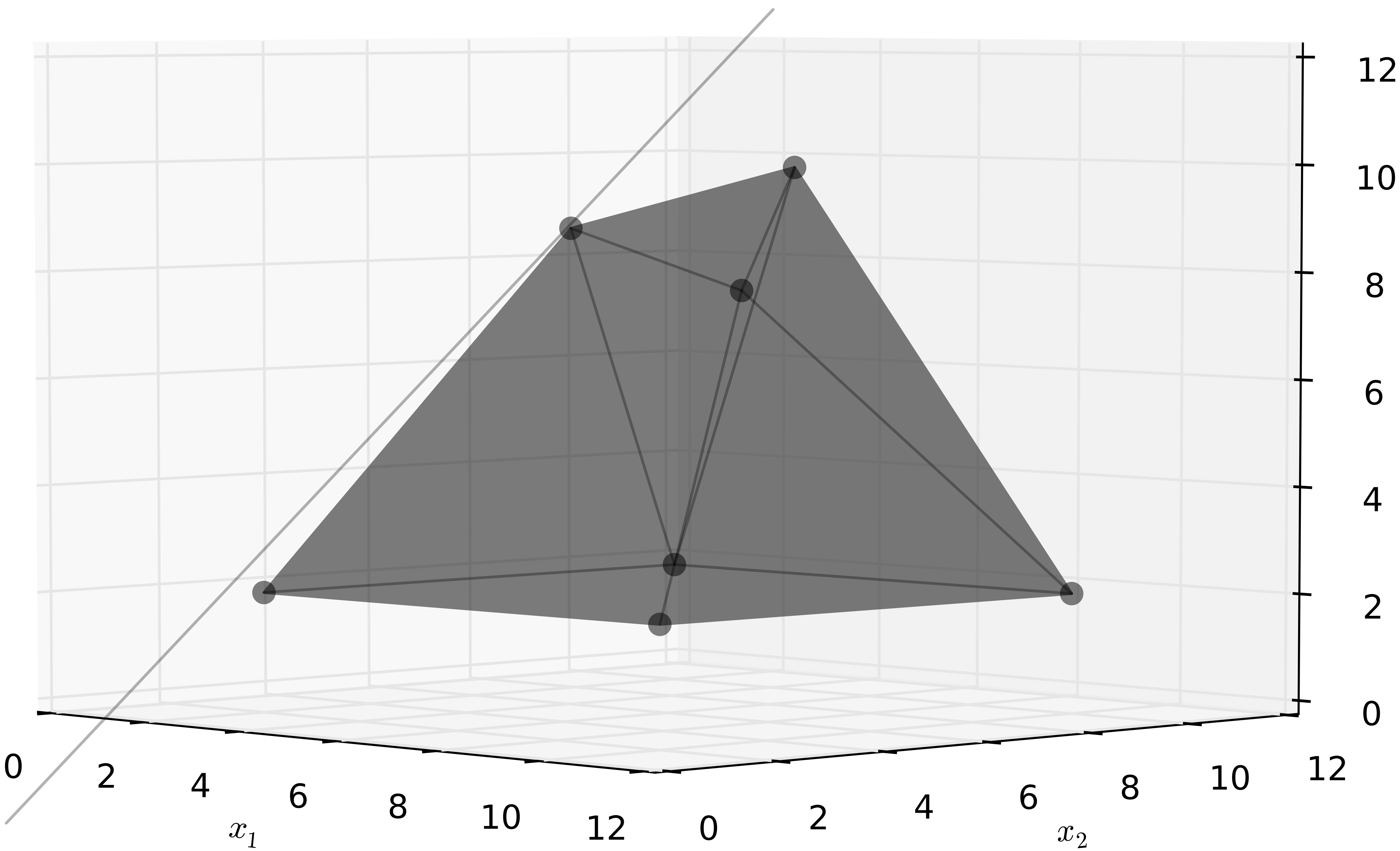


$$\min \; c_1 x_1 + \ldots + c_n x_n = b^*$$

Why I Love These Vertices Now in 3D!



Why I Love These Vertices Now in 3D!



Why I Love These Vertices

- Theorem: At least one of the points where the objective value is minimal is a vertex.

Why I Love These Vertices

Why I Love These Vertices

Let x^* be the minimum. Since each point in a polytope is a convex combination of the vertices $v_1, \ldots, v_t$, we have

$$x^* = \lambda_1 v_1 + \ldots + \lambda_t v_t$$

and the objective value at optimality can be expressed as

$$cx^* = \lambda_1 * (cv_1) + \ldots + \lambda_t (cv_t).$$

Assume that the minimum is not at a vertex, i.e.,

$$cx^* < cv_i \quad \forall i : 1 \leq i \leq t.$$

It follows that

$$\begin{array}{rcl} cx^* & = & \lambda_1 * (cv_1) + \ldots + \lambda_t (cv_t) \\ & > & \lambda_1 * (cx^*) + \ldots + \lambda_t (cx^*) \\ & > & (\lambda_1 + \ldots + \lambda_t)(cx^*) \\ & > & cx^*. \end{array}$$

Hence, it must be the case that $x^* = v_i$ for some $1 \leq i \leq t$.

Geometry of Linear Programming

Geometry of Linear Programming

- How to solve a linear program "geometrically"?
  - enumerate all the vertices
  - select the one with the smallest objective value

Until Next Time