1.571 Modeling and Analysis of Structures Fall 2014

Lecture 11

Displacement Method (3)

Stiffness Matrix of a Frame Element

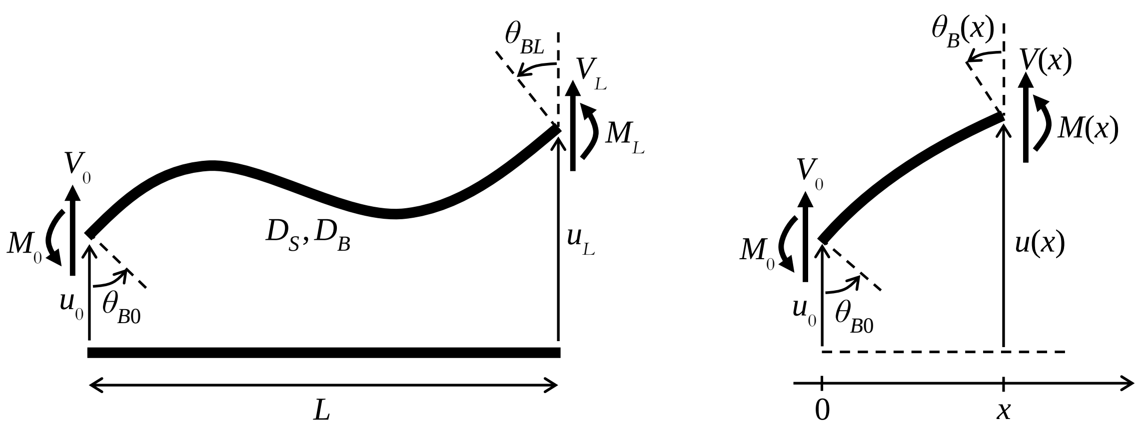
The matrix formulation of the displacement method was introduced in lecture 10 on a truss example. The method is however general, and all we need to apply it to other structures is to construct the stiffness matrices of the corresponding elements. In the following, we construct the stiffness matrix of a general frame element.

Beam Element

A frame element is the superposition of a truss element (axial behavior) and a beam element (shear and bending behavior). To obtain the stiffness matrix of the frame element, we propose to construct the matrix of the beam element and combine it with the matrix of the truss element, which we already know.

We consider a beam element in its local coordinate system, with the following assumptions:

- Small deflections u and rotations θ (linear problem).
- No axial force, i.e. $D_A = 0$ (beam element, not frame element).
- No span load (such loads would have to be turned into equivalent loads applied to the nodes).



The transverse forces V_0 and V_L and the moments M_0 and M_L are the local end actions of the element. They are the total forces and moments acting onto the element and may include loads, reactions and/or forces and moments applied by adjacent elements. The rotations of interest are the bending rotations, which must be continuous across adjacent elements.

The local stiffness matrix of the element must relate the local end actions (V_0, V_L, M_0, M_L) and the local displacements (u_0, u_L, θ_{B0}, θ_{BL}). The global stiffness matrix can then be obtained by applying a rotation to the local displacements and end actions.

We start by integrating the beam equations, from the shear force $V(x)$ to the deflection $u(x)$:

$$\frac{dV}{dx} = -w = 0 \qquad V(0) = -V_0 \qquad \rightarrow \quad V(x) = -V_0$$

$$\frac{dM}{dx} = -V - n = -V \qquad M(0) = -M_0 \qquad \rightarrow \quad M(x) = V_0 x - M_0$$

$$\frac{d\theta_B}{dx} = \frac{M}{D_B} \qquad \theta_B(0) = \theta_{B0} \qquad \rightarrow \quad \theta_B(x) = \frac{1}{D_B}\left(V_0 \frac{x^2}{2} - M_0 x\right) + \theta_{B0}$$

$$\frac{du}{dx} = \theta_B + \frac{V}{D_S} \qquad u(0) = u_0 \qquad \rightarrow \quad u(x) = \frac{1}{D_B}\left(V_0 \frac{x^3}{6} - M_0 \frac{x^2}{2}\right) + \left(\theta_{B0} - \frac{V_0}{D_S}\right)x + u_0$$

We obtain 4 equations between the displacements and the end actions by evaluating the above quantities at $x = L$:

$$V_L = V(L) = -V_0$$

$$M_L = M(L) = V_0 L - M_0$$

$$\theta_{BL} = \theta_B(L) = \frac{1}{D_B}\left(V_0 \frac{L^2}{2} - M_0 L\right) + \theta_{B0}$$

$$u_L = u(L) = \frac{1}{D_B}\left(V_0 \frac{L^3}{6} - M_0 \frac{L^2}{2}\right) + \left(\theta_{B0} - \frac{V_0}{D_S}\right)L + u_0$$

We can solve for the end actions in terms of the displacements:

$$V_0 = \frac{D_B D_S}{L^3 D_S + 12 D_B L}\left(12u_0 - 12u_L + 6L\theta_{B0} + 6L\theta_{BL}\right)$$

$$V_L = \frac{D_B D_S}{L^3 D_S + 12 D_B L}\left(-12u_0 + 12u_L - 6L\theta_{B0} - 6L\theta_{BL}\right)$$

$$M_0 = \frac{D_B D_S}{L^3 D_S + 12 D_B L}\left(6Lu_0 - 6Lu_L + \left(\frac{4L^2 D_S + 12 D_B}{D_S}\right)\theta_{B0} + \left(\frac{2L^2 D_S - 12 D_B}{D_S}\right)\theta_{BL}\right)$$

$$M_L = \frac{D_B D_S}{L^3 D_S + 12 D_B L}\left(6Lu_0 - 6Lu_L + \left(\frac{2L^2 D_S - 12 D_B}{D_S}\right)\theta_{B0} + \left(\frac{4L^2 D_S + 12 D_B}{D_S}\right)\theta_{BL}\right)$$

The stiffness matrix of a beam element is therefore:

$$\begin{pmatrix} V_0 \\ M_0 \\ V_L \\ M_L \end{pmatrix} = \frac{D_B D_S}{L^3 D_S + 12 D_B L}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & \frac{4L^2 D_S + 12 D_B}{D_S} & -6L & \frac{2L^2 D_S - 12 D_B}{D_S} \\ -12 & -6L & 12 & -6L \\ 6L & \frac{2L^2 D_S - 12 D_B}{D_S} & -6L & \frac{4L^2 D_S + 12 D_B}{D_S} \end{pmatrix}\begin{pmatrix} u_0 \\ \theta_{B0} \\ u_L \\ \theta_{BL} \end{pmatrix} \qquad (1)$$

Bending Beam Element

A bending beam is a beam with infinite shear rigidity. The stiffness matrix of a bending beam element is obtained by setting D_S to infinity in (1):

e.g. mult. factor: $$\lim_{D_S\to\infty}\left(\frac{D_B D_S}{L^3 D_S + 12 D_B L}\right) = \frac{D_B D_S}{L^3 D_S} = \frac{D_B}{L^3}$$

e.g. inner term 2,2: $$\lim_{D_S\to\infty}\left(\frac{4L^2 D_S + 12 D_B}{D_S}\right) = \frac{4L^2 D_S}{D_S} = 4L^2$$

$$\to \quad \begin{pmatrix} V_0 \\ M_0 \\ V_L \\ M_L \end{pmatrix} = \frac{D_B}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}\begin{pmatrix} u_0 \\ \theta_{B0} \\ u_L \\ \theta_{BL} \end{pmatrix} \qquad (2)$$

Shear Beam Element

A shear beam is a beam with infinite bending rigidity. The stiffness matrix of a shear beam element is obtained by setting D_B to infinity in (1):

e.g. mult factor: $$\lim_{D_B\to\infty}\left(\frac{D_B D_S}{L^3 D_S + 12 D_B L}\right) = \frac{D_B D_S}{12 D_B L} = \frac{D_S}{12L}$$

e.g. inner term 2,2 $$\lim_{D_B\to\infty}\left(\frac{4L^2 D_S + 12 D_B}{D_S}\right) = \frac{12 D_B}{D_S} = \infty$$

$$\to \quad \begin{pmatrix} V_0 \\ M_0 \\ V_L \\ M_L \end{pmatrix} = \frac{1}{12L}\begin{pmatrix} 12D_S & 6LD_S & -12D_S & 6LD_S \\ 6LD_S & \infty & -6LD_S & \infty \\ -12D_S & -6LD_S & 12D_S & -6LD_S \\ 6LD_S & \infty & -6LD_S & \infty \end{pmatrix}\begin{pmatrix} u_0 \\ \theta_{B0} \\ u_L \\ \theta_{BL} \end{pmatrix}$$

The end moments M_0 and M_L cannot be infinite, so the rotations θ_{B0} and θ_{BL} must be zero. This is consistent with the shear beam model, which assumes the sections of the beam do not rotate. With $\theta_{B0} = \theta_{BL} = 0$, the two columns of the stiffness matrix corresponding to θ_{B0} and θ_{BL} can be removed. The stiffness matrix of the shear beam is then:

$$\begin{pmatrix} V_0 \\ M_0 \\ V_L \\ M_L \end{pmatrix} = \frac{D_S}{2L}\begin{pmatrix} 2 & -2 \\ L & -L \\ -2 & 2 \\ L & -L \end{pmatrix}\begin{pmatrix} u_0 \\ u_L \end{pmatrix} \qquad (3)$$

Rotational Spring Element

Suppose the beam has zero shear rigidity, so it cannot carry any shear force. The bending moment must be uniform, and the beam acts as some sort of rotational spring between 2 nodes that are at a distance from each other. The stiffness matrix is obtained by setting D_S to zero in (1):

e.g. full term 1,1: $$\left(\frac{D_B D_S}{L^3 D_S + 12 D_B L}\right)(12) \xrightarrow[D_S=0]{} \frac{12 D_B D_S}{12 D_B L} = 0$$

e.g. full term 2,2: $$\left(\frac{D_B D_S}{L^3 D_S + 12 D_B L}\right)\left(\frac{4L^2 D_S + 12 D_B}{D_S}\right) \xrightarrow[D_S=0]{} \frac{12 D_B{}^2}{12 D_B L} = \frac{D_B}{L}$$

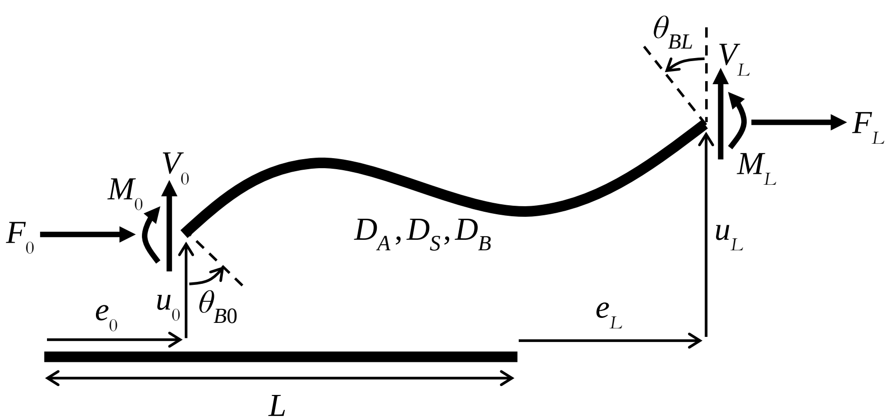
$$\rightarrow \begin{pmatrix} V_0 \\ M_0 \\ V_L \\ M_L \end{pmatrix} = \frac{D_B}{L}\begin{pmatrix} & & & \\ & 1 & & -1 \\ & & & \\ & -1 & & 1 \end{pmatrix}\begin{pmatrix} u_0 \\ \theta_{B0} \\ u_L \\ \theta_{BL} \end{pmatrix}$$

The empty rows and columns can be removed:

$$\begin{pmatrix} M_0 \\ M_L \end{pmatrix} = \frac{D_B}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \theta_{B0} \\ \theta_{BL} \end{pmatrix} \qquad D_B/L \text{ is a rotational stiffness} \tag{4}$$

Frame Element

Back to the original problem of constructing the stiffness matrix of a frame element, we superpose the axial behavior of a truss element to the shear and bending behaviors of a beam element:



The axial end actions F_0 and F_L only depend on the axial displacements e_0 and e_L, and these quantities are related through the stiffness matrix of a truss element:

$$\begin{pmatrix} F_0 \\ F_L \end{pmatrix} = \frac{D_A}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e_0 \\ e_L \end{pmatrix} \tag{5}$$

The stiffness matrix of the frame element must relate the 6 displacements and 6 end actions. With a transverse rigidity D_T defined below to simplify the notation, the stiffness matrix is obtained by combining (1) and (5):

$$\begin{pmatrix} F_0 \\ V_0 \\ M_0 \\ F_L \\ V_L \\ M_L \end{pmatrix} = \frac{1}{L} \begin{pmatrix} D_A & & & -D_A & & \\ & 12D_T D_S & 6LD_T D_S & & -12D_T D_S & 6LD_T D_S \\ & 6LD_T D_S & D_T\left(4L^2 D_S + 12D_B\right) & & -6LD_T D_S & D_T\left(2L^2 - 12D_B\right) \\ -D_A & & & D_A & & \\ & -12D_T D_S & -6LD_T D_S & & 12D_T D_S & -6LD_T D_S \\ & 6LD_T D_S & D_T\left(2L^2 - 12D_B\right) & & -6LD_T D_S & D_T\left(4L^2 + 12D_B\right) \end{pmatrix} \begin{pmatrix} e_0 \\ u_0 \\ \theta_{B0} \\ e_L \\ u_L \\ \theta_{BL} \end{pmatrix} \qquad (6)$$

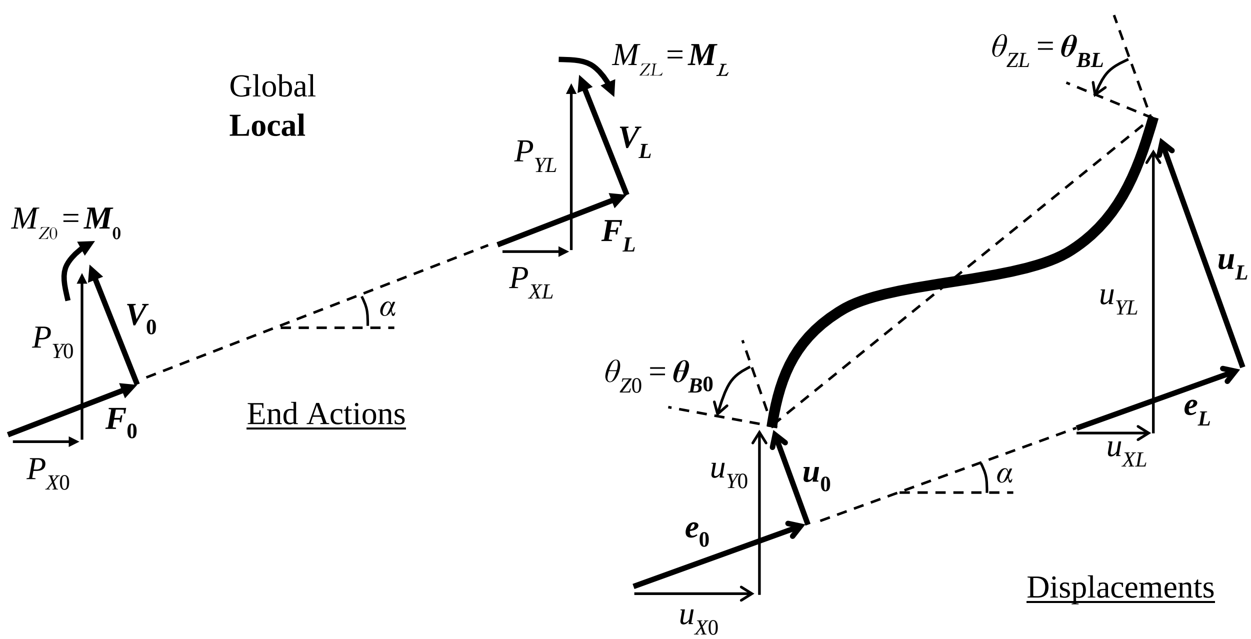
with $$D_T = \frac{D_B}{L^2 D_S + 12D_B}$$

Rotation into Global Coordinates

The local displacements and end actions are related through (6), for which a compact form is:

$$\overline{F} = \underline{K}^{loc}\,\overline{E} \qquad \text{with} \qquad \begin{array}{ll} \overline{F} & \text{local end actions} \\ \overline{E} & \text{local displacements} \\ \underline{K}^{loc} & \text{local stiffness matrix} \end{array} \qquad (7)$$

The global end forces are noted P_{X0}, P_{Y0}, P_{XL} and P_{YL}. In 2D, the global end moments M_{Z0} and M_{ZL} match the local end moments M_0 and M_L. The global translations are noted u_{X0}, u_{Y0}, u_{XL} and u_{YL}. In 2D, the global rotations θ_{Z0} and θ_{ZL} match the local rotations θ_{B0} and θ_{BL}:



We express the local displacements in terms of the global ones:

$$\begin{pmatrix} e_0 \\ u_0 \\ \theta_{B0} \\ e_L \\ u_L \\ \theta_{BL} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & & & & \\ -\sin\alpha & \cos\alpha & & & & \\ & & 1 & & & \\ & & & \cos\alpha & \sin\alpha & \\ & & & -\sin\alpha & \cos\alpha & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} u_{X0} \\ u_{Y0} \\ \theta_{Z0} \\ u_{XL} \\ u_{YL} \\ \theta_{ZL} \end{pmatrix}$$

$\overline{E} = \underline{R}\overline{U}$ with $\overline{U}$ global displacements (8)

$\underline{R}$ rotation matrix

We express the global end actions in terms of the local ones:

$$\begin{pmatrix} P_{X0} \\ P_{Y0} \\ M_{Z0} \\ P_{XL} \\ P_{YL} \\ M_{ZL} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & & & & \\ \sin\alpha & \cos\alpha & & & & \\ & & 1 & & & \\ & & & \cos\alpha & -\sin\alpha & \\ & & & \sin\alpha & \cos\alpha & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} F_0 \\ V_0 \\ M_0 \\ F_L \\ V_L \\ M_L \end{pmatrix}$$

$\overline{P} = \underline{R}^T \overline{F}$ with $\overline{P}$ global end actions (9)

Combining (7), (8) and (9):

$$\overline{P} = \underline{R}^T \underline{K}^{loc} \underline{R}\overline{U}$$

$\overline{P} = \underline{K}\overline{U}$ with $\underline{K} = \underline{R}^T \underline{K}^{loc} \underline{R}$ global stiffness matrix (10)

The rotation matrix depends on the type of the element. For the other line elements (truss, bending beam, shear beam), the local stiffness matrix relates some of the local end actions of the frame element to some of its local displacements, so the rotation matrix is obtained by extracting the corresponding rows and columns from the rotation matrix of the frame element. If, like for the bending beam element, the number of local and global DOFs do not match, the matrix is no longer a rotation matrix, but a mere one-way mapping between local and global DOFs. And if, like for the shear beam element, the local end actions and displacements do not cover the same set of DOFs, the two mapping matrices are no longer transpose of each other. The global stiffness matrix of the shear beam element is constructed below.

The local stiffness matrix of the shear beam element in (3) relates the local transverse forces and moments V_0, M_0, V_L and M_L to the local deflections u_0 and u_L.

We express the global end actions in terms of the local ones by removing from (9) the columns corresponding to the local axial forces, which are not involved in the behavior of the shear beam element:

$$\begin{pmatrix} P_{X0} \\ P_{Y0} \\ M_{Z0} \\ P_{XL} \\ P_{YL} \\ M_{ZL} \end{pmatrix} = \begin{pmatrix} -\sin\alpha & & & \\ \cos\alpha & & & \\ & 1 & & \\ & & -\sin\alpha & \\ & & \cos\alpha & \\ & & & 1 \end{pmatrix} \begin{pmatrix} V_0 \\ M_0 \\ V_L \\ M_L \end{pmatrix} \tag{11}$$

We express the local displacements in terms of the global ones by removing from (8) the rows corresponding to the local elongations and rotations, which are not involved in the behavior of the shear beam element. The global rotations have no effect of the remaining displacements, so the corresponding columns can also be removed.

$$\begin{pmatrix} u_0 \\ u_L \end{pmatrix} = \begin{pmatrix} -\sin\alpha & \cos\alpha & & & & \\ & & & -\sin\alpha & \cos\alpha & \end{pmatrix} \begin{pmatrix} u_{X0} \\ u_{Y0} \\ \theta_{Z0} \\ u_{XL} \\ u_{YL} \\ \theta_{ZL} \end{pmatrix} = \begin{pmatrix} -\sin\alpha & \cos\alpha & & \\ & & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u_{X0} \\ u_{Y0} \\ u_{XL} \\ u_{YL} \end{pmatrix} \tag{12}$$

Combining (3), (11) and (12) give the global stiffness matrix of the shear beam element:

$$\begin{pmatrix} P_{X0} \\ P_{Y0} \\ M_{Z0} \\ P_{XL} \\ P_{YL} \\ M_{ZL} \end{pmatrix} = \frac{D_S}{2L} \begin{pmatrix} -\sin\alpha & & & \\ \cos\alpha & & & \\ & 1 & & \\ & & -\sin\alpha & \\ & & \cos\alpha & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ L & -L \\ -2 & 2 \\ L & -L \end{pmatrix} \begin{pmatrix} -\sin\alpha & \cos\alpha & & \\ & & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u_{X0} \\ u_{Y0} \\ u_{XL} \\ u_{YL} \end{pmatrix}$$

$$\begin{pmatrix} P_{X0} \\ P_{Y0} \\ M_{Z0} \\ P_{XL} \\ P_{YL} \\ M_{ZL} \end{pmatrix} = \frac{D_S}{2L} \begin{pmatrix} 2\sin^2\alpha & -2\sin\alpha\cos\alpha & -2\sin^2\alpha & 2\sin\alpha\cos\alpha \\ -2\sin\alpha\cos\alpha & 2\cos^2\alpha & 2\sin\alpha\cos\alpha & -2\cos^2\alpha \\ -L\sin\alpha & L\cos\alpha & L\sin\alpha & -L\cos\alpha \\ -2\sin^2\alpha & 2\sin\alpha\cos\alpha & 2\sin^2\alpha & -2\sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & -2\cos^2\alpha & -2\sin\alpha\cos\alpha & 2\cos^2\alpha \\ -L\sin\alpha & L\cos\alpha & L\sin\alpha & -L\cos\alpha \end{pmatrix} \begin{pmatrix} u_{X0} \\ u_{Y0} \\ u_{XL} \\ u_{YL} \end{pmatrix}$$

Slope-Deflection Method

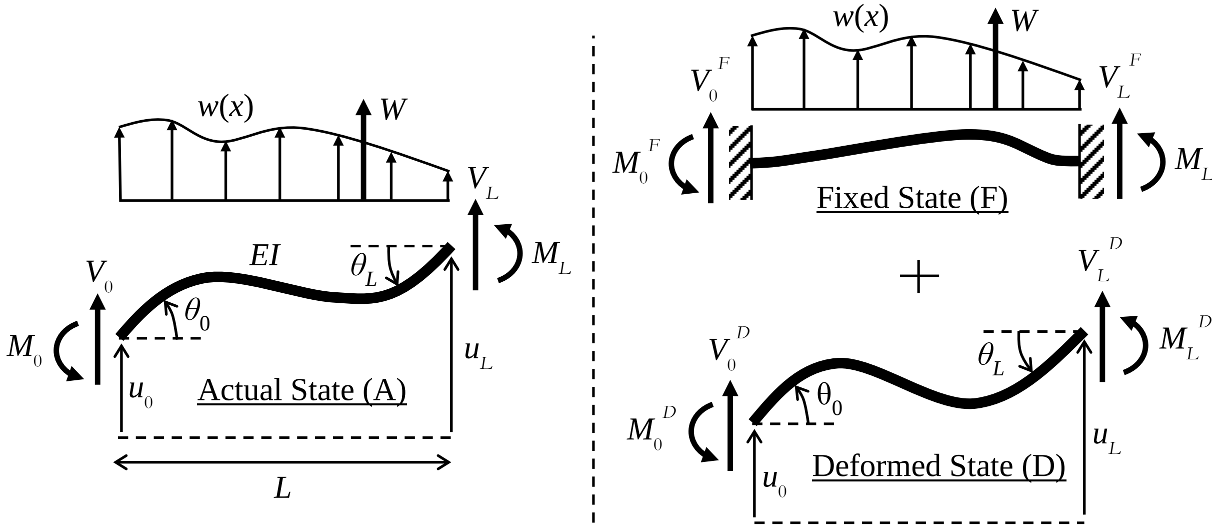
In the matrix formulation of the displacement method, the loads must be concentrated loads applied to the nodes of the model. If concentrated or distributed loads are applied directly to an element, a computer implementation of the displacement method (i.e. finite element analysis) would replace them with equivalent nodes acting on the nodes. In the following, we introduce another method to take into account element loads that is geared toward hand calculations: the Slope-Deflection Method. We only consider structures made of bending beam elements, where axial and shear deformation is neglected. Only for such structures does the Slope-Deflection Method remain (somewhat) manageable by hand.

Slope-Deflection Equations

In the actual state (A), the beam shown below is in equilibrium under its end actions (V_0, M_0, V_L and M_L) and some span loads (concentrated load W and distributed load $w(x)$). As shear deformation is neglected, the end rotations are the total rotations and are noted θ_0 and θ_L. Because of the span loads, the displacements and end actions cannot be directly related using the stiffness matrix of the element. We consider a pair of virtual states to work around this problem:

- Fixed state (F): the beam is in equilibrium under the span loads (W and w) and the <u>fixed end actions</u> (V_0^F, M_0^F, V_L^F and M_L^F). The fixed end actions are the end actions required to prevent displacement and rotation at both ends of the beam. These end actions are identical to the reactions that would develop if the beam was fully fixed at both ends.
- Deformed state (D): the deformed beam is in equilibrium under the <u>displaced end actions</u> (V_0^D, M_0^D, V_L^D and M_L^D). The displaced end actions are the end actions required to obtain the actual displacements without the span loads and are therefore related to the displacements through the stiffness matrix of the element, obtained in (2):

$$(2) \;\rightarrow\; \begin{pmatrix} V_0^D \\ M_0^D \\ V_L^D \\ M_L^D \end{pmatrix} = \frac{D_B}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}\begin{pmatrix} u_0 \\ \theta_0 \\ u_L \\ \theta_L \end{pmatrix} \qquad (13)$$



The displacements in state A are the superposition of the displacements in states F and D, so the loads and end actions in state A must also be the superposition of the loads and end action in states F and D. This is immediately satisfied for the loads, and for the end actions we must have:

$$V_0 = V_0^D + V_0^F$$

$$M_0 = M_0^D + M_0^F$$

$$V_L = V_L^D + V_L^F$$

$$M_L = M_L^D + M_L^F$$

Expressing the displaced end actions in terms of the displacements using (13) gives the <u>Slope-Deflection Equations</u>:

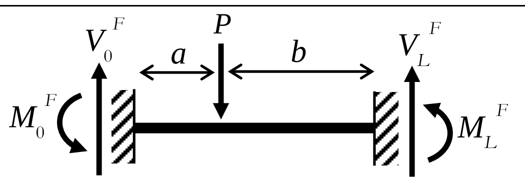
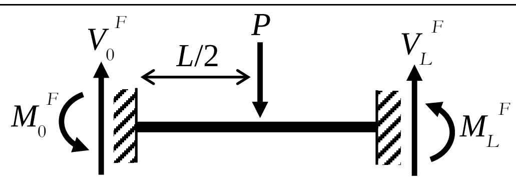
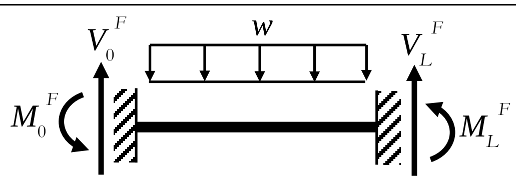
$$V_0 = \frac{6EI}{L^3}\left(2u_0 + L\theta_0 - 2u_L + L\theta_L\right) + V_0^F$$

$$M_0 = \frac{2EI}{L^3}\left(3Lu_0 + 2L^2\theta_0 - 3Lu_L + L^2\theta_L\right) + M_0^F$$

$$V_L = \frac{6EI}{L^3}\left(-2u_0 - L\theta_0 + 2u_L - L\theta_L\right) + V_L^F$$

$$M_L = \frac{2EI}{L^3}\left(3Lu_0 + L^2\theta_0 - 3Lu_L + 2L^2\theta_L\right) + M_L^F$$

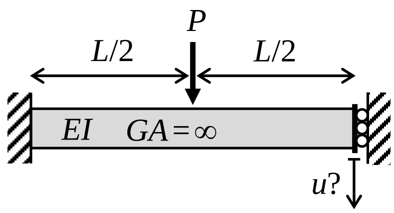
The fixed end actions can be tabulated for generic loading conditions:

	V_0^F	M_0^F	V_L^F	M_L^F
	$\frac{Pb^2(3a+b)}{L^3}$	$\frac{Pab^2}{L^2}$	$\frac{Pab^2}{L^2}$	$-\frac{Pba^2}{L^2}$
	$\frac{P}{2}$	$\frac{PL}{8}$	$\frac{P}{2}$	$-\frac{PL}{8}$
	$\frac{wL}{2}$	$\frac{wL^2}{12}$	$\frac{wL}{2}$	$-\frac{wL^2}{12}$

The examples below show how the slope-deflection equations can be used to determine displacements in indeterminate beams and frames with span loads.

Slope-Deflection Method: Example 1

We seek to determine the displacement u in the following beam:



The slope-deflection equation for the shear force at $x = L$ can conveniently solve this particular problem, because that shear force is zero due to the sliding support and the only non-zero displacement is the one to be determined:

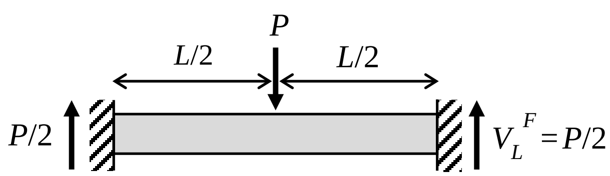
$$V_L = \frac{6EI}{L^3}\left(-2u_0 + 2u_L - L\theta_0 - L\theta_L\right) + V_L^F = 0$$

We replace the displacements:

$$\theta_0 = 0$$
$$\theta_L = 0$$
$$u_0 = 0$$
$$u_L = -u \quad (u_L > 0 \text{ upwards})$$

$$\rightarrow \quad u = \frac{V_L^F L^3}{12EI}$$

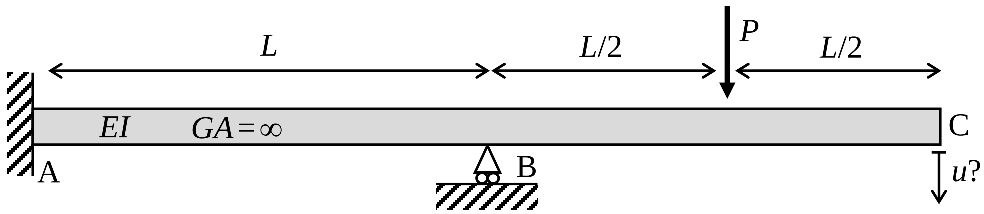
We need to determine the fixed end action V_L^F. Because the fixed state is symmetric, the vertical reaction at each end is half of the applied load:



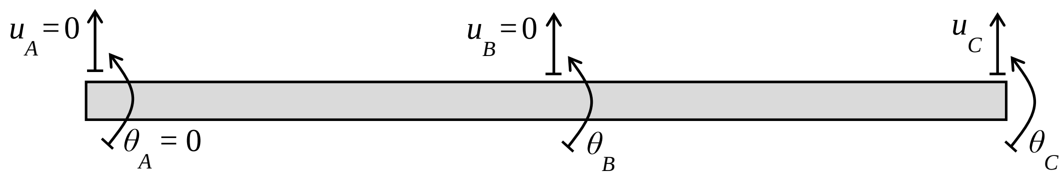
$$\rightarrow \quad u = \frac{PL^3}{24EI}$$

Slope-Deflection Method: Example 2

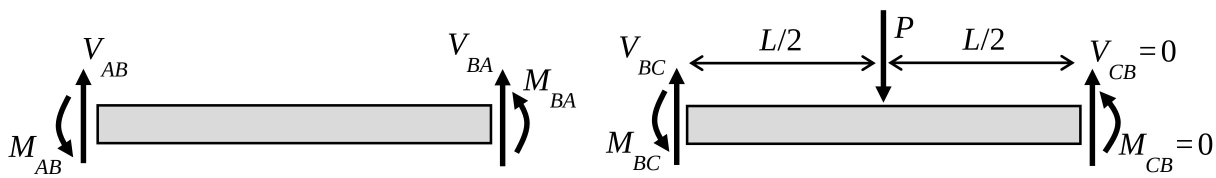
We seek to determine the displacement u in the following beam:



The displacements and rotations are as follows:



As an example of the notation used in the following, M_{AB} designates the moment applied by node A to element AB. The end actions of the elements are as follows:



We can determine the fixed end actions:

$$V_{AB}^F = V_{BA}^F = 0 \qquad M_{AB}^F = M_{BA}^F = 0 \qquad V_{BC}^F = V_{CB}^F = \frac{P}{2} \qquad M_{BC}^F = -M_{CB}^F = \frac{PL}{8}$$

We need 3 equations between the 3 unknown displacements. We start by using the slope-deflection equations corresponding to the zero end actions. This way, we generate equations without introducing new variables for the end actions.

$$V_{CB} = 0 \qquad \rightarrow \quad \frac{6EI}{L^3}\left(-L\theta_B + 2u_C - L\theta_C\right) + \frac{P}{2} = 0 \tag{14}$$

$$M_{CB} = 0 \qquad \rightarrow \quad \frac{2EI}{L^3}\left(L^2\theta_B - 3Lu_C + 2L^2\theta_C\right) - \frac{PL}{8} = 0 \tag{15}$$

To obtain a 3rd equation, we use the absence of moment reaction at node B. We can write the moment equilibrium of that node without introducing a new variable for the reaction moment. The moments in equilibrium at node B are the end moments of the elements connected to node B, which can be expressed in terms of the displacements using the slope-deflection equations:

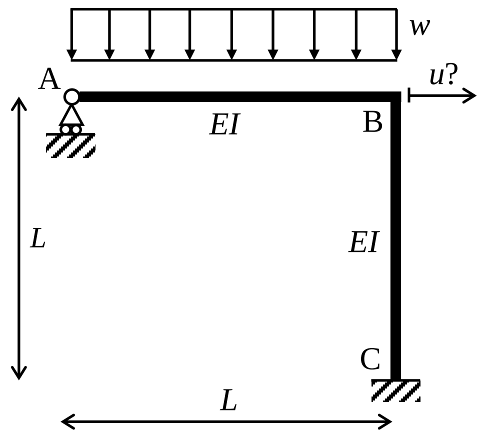
$$M_{BA} + M_{BC} = 0 \quad \rightarrow \quad \frac{2EI}{L^3}\left(4L^2\theta_B - 3Lu_C + L^2\theta_C\right) + \frac{PL}{8} = 0 \tag{16}$$

(14), (15) and (16) can be solved θ_B, θ_C and u_C:

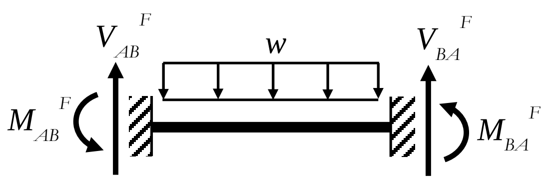
$$\theta_B = -\frac{5PL^2}{32EI} \qquad \theta_C = -\frac{11PL^2}{32EI} \qquad u_C = -\frac{7PL^3}{24EI} \qquad \rightarrow \quad u = \frac{7PL^3}{24EI}$$

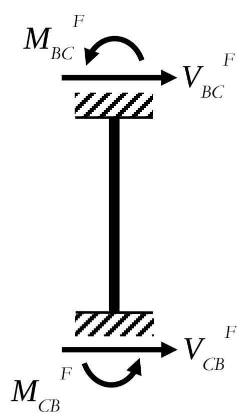
Slope-Deflection Method: Example 3

We seek to determine the displacement *u* in the following frame:



We first determine the fixed end actions of each element:

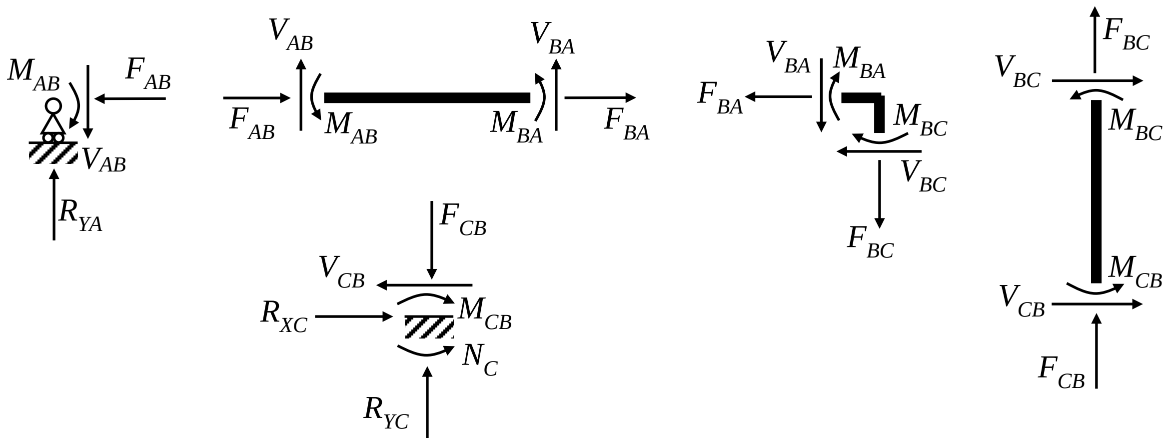




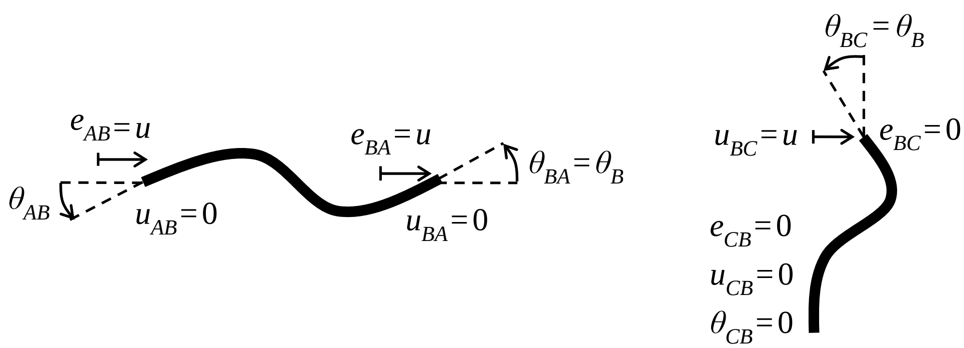
$$V_{AB}^{F} = \frac{wL}{2} \qquad V_{BA}^{F} = \frac{wL}{2} \qquad V_{BC}^{F} = 0 \qquad V_{CB}^{F} = 0$$

$$M_{AB}^{F} = \frac{wL^2}{12} \qquad M_{BA}^{F} = -\frac{wL^2}{12} \qquad M_{BC}^{F} = 0 \qquad M_{CB}^{F} = 0$$

We now remove the load and consider the equilibrium of each node and element under the end actions and the support reactions. Expressing the equilibrium of only some nodes or elements is often sufficient to solve for a particular displacement.



The end displacements of the elements are as follows:



We now express the slope-deflection equations for each element. The first equation is:

$$V_0 = \frac{6EI}{L^3}\left(2u_0 + L\theta_0 - 2u_L + L\theta_L\right)$$

For element AB:

$$V_{AB} = \frac{6EI}{L^3}\left(L\theta_{AB} + L\theta_{BA} + 2u_{AB} - 2u_{BA}\right) + V_{AB}^F$$

With the known displacements and fixed end action:

$$V_{AB} = \frac{6EI}{L^2}\left(\theta_A + \theta_B\right) + \frac{wL}{2}$$

Similarly, the other 3 slope-deflection equations for element AB are:

$$V_{BA} = -\frac{6EI}{L^2}\left(\theta_A + \theta_B\right) + \frac{wL}{2}$$

$$M_{AB} = \frac{2EI}{L}\left(2\theta_A + \theta_B\right) + \frac{wL^2}{12} \qquad (17)$$

$$M_{BA} = \frac{2EI}{L}\left(\theta_A + 2\theta_B\right) - \frac{wL^2}{12} \qquad (18)$$

For completeness, we can add an equation expressing axial force equilibrium:

$$F_{AB} + F_{BA} = 0 \qquad (19)$$

A similar set of equations is obtained for element BC:

$$V_{BC} = \frac{6EI}{L^3}\left(L\theta_B + 2u\right) \qquad (20)$$

$$V_{CB} = -\frac{6EI}{L^3}\left(L\theta_B + 2u\right)$$

$$M_{BC} = \frac{2EI}{L^2}\left(2L\theta_B + 3u\right) \qquad (21)$$

$$M_{CB} = \frac{2EI}{L^2}\left(L\theta_B + 3u\right)$$

$$F_{BC} + F_{CB} = 0$$

We now express the force and moment equilibrium of the 3 nodes:

Node A:

$$-F_{AB} = 0 \qquad (22)$$

$$R_{YA} - V_{AB} = 0$$

$$-M_{AB} = 0 \qquad (23)$$

Node B:

$$-F_{BA} - V_{BC} = 0 \qquad (24)$$

$$-V_{BA} - F_{BC} = 0$$

$$-M_{BA} - M_{BC} = 0 \qquad (25)$$

Node C:

$$R_{XC} - V_{CB} = 0$$

$$R_{YC} - F_{CB} = 0$$

$$N_C - M_{CB} = 0$$

We have a set of 19 equations, and we could solve for any of the 19 unknown forces and displacements. In the following, we use a subset of equations to solve for the displacement u.

(19), (22) and (24) require:

$$V_{BC} = 0 \tag{26}$$

This could have been concluded directly from the support condition at point A, which prevents any horizontal reaction. Using (20) to replace V_{BC} in (26) gives:

$$L\theta_B + 2u = 0 \tag{27}$$

Combining (17) and (23) gives:

$$2\theta_A + \theta_B + \frac{wL^3}{24EI} = 0 \tag{28}$$

Combining (18), (21) and (25) gives:

$$L^2\theta_A + 2L^2\theta_B - \frac{wL^5}{12EI} + 2L^2\theta_B + 3Lu = 0 \tag{29}$$

(27), (28) and (29) are 3 equations between u, θ_A and θ_B. We solve for u:

$$u = \frac{wL^4}{64EI}$$