JOURNAL OF ENVIRONMENTAL ECONOMICS AND MANAGEMENT 13 212–224 (1986)

Set-Up Costs and Theory of Exhaustible Resources

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Received July 11, 1983; revised June 19, 1985

Let there be several identical deposits of an exhaustible, non-reproducible resource, the working of a deposit entailing a set-up cost but no other costs. The deposits must be extracted in strict sequence with jump discontinuities of marginal benefit at transition points. Moreover, the average rate of increase of marginal benefit is less than the rate of interest. © 1986 Academic Press, Inc.

1. INTRODUCTION

The opening of a mine may involve once-and-for-all "set-up" costs of exploration, clearing, tunneling, training, equipping; and its closure once-and-for-all costs of dismantling and cleaning-up. Indeed such non-recurrent costs may dominate the fixed and variable flow costs of resource extraction. It is puzzling then that in the formal economic analysis of exhaustible resources, attention has been focussed on the latter to the almost complete neglect of the former. It is our purpose in the present note to repair this deficiency and, in particular, to set forth a generalized form of Hotelling's "r per cent rule" for the behaviour over time of the expected marginal current net benefit (MNB) derived from extraction. Our most important finding is that if set-up costs are present then, along an optimal path, MNB must increase not continuously and exponentially at the current rate of interest but in discontinuous saw-tooth fashion at an average rate less than the rate of interest. In addition, it is shown that, in general, the optimal path of extraction cannot be reproduced by competitive markets and that, in any case, set-up costs render the assumption of competitive spot markets highly implausible. Finally, it is shown that under monopoly the rate of extraction is suboptimal, and that this is so even when the demand for the resource is of constant elasticity (too little being extracted).

Of course it is well known that the conditions under which Hotelling's rule is valid are quite strict. In particular, it is known that the rule must be modified (or, at least, rephrased) if current extraction exerts a direct influence on expected benefit or cost of extraction. Thus if costs of extraction increase with cumulative extraction, then any increase in the current rate of extraction adds to the expected cost of any

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planned future extraction (Levhari and Leviatan [11]). If the resource-stock is of unknown extent, then any addition to the current rate of extraction increases the probability that the stock will be exhausted before any assigned future point of time and therefore reduces the expected benefit associated with the extraction planned for that time. In both sets of circumstances, it is optimal for MNB to rise at a rate less than the current rate of interest. (For costs of extraction which depend on cumulative output, see Herfindahl [5], Levhari and Liviatan [11], Hartwick [4], and Kemp and Long [9]; for stock uncertainty, see Kemp [8].) However, in these circumstances it is optimal for marginal benefit to increase continuously; this is so even when the average cost of extraction jumps discontinuously from one deposit to another and even when the distribution function (defined on alternative stock sizes) contains jump discontinuities. Moreover, in these circumstances the optimal path of extraction can be reproduced by laisser-faire competitive markets if only there are enough of them. (See Kemp and Long [9].)

2. ANALYSIS OF THE SIMPLEST CASE

We begin with the simplest case. There are n identical deposits of some resource. Each deposit is of known initial extent R. The extracted resource is consumable but perishable; the rates of extraction and consumption therefore are always equal.

There is a second perishable commodity, leisure, available in a steady *flow*. Like the extracted resource, leisure may be consumed; but, as we shall see, it has other uses too.

Associated with each deposit is a set-up cost K, expressed in leisure. Since leisure comes as a finite flow, it is not possible for the community to incur finite set-up costs in a moment of time; they must be spread over a non-degenerate interval of time. We therefore suppose that, to prepare a deposit for extraction, leisure must be spent at some fixed rate (not greater than the rate at which it becomes available) over some fixed interval of time. The single number K is then the value of the expenditure flow compounded to the end of the interval. There are no other costs of extraction.

Social utility is a strictly concave function u of resource-consumption and a linear function of leisure. There is a positive and constant rate of time preference.

From the point of view of the community, if it is worth exploiting the deposits at all then it is optimal to exploit them in strict sequence, completely exhausting each deposit before moving on the next; indeed, since the rate of time preference is positive, it is suboptimal to do otherwise. Of course it is a matter of indifference which of the n identical deposits is first exploited.

It is a necessary condition of optimality that, from the moment at which a deposit is first worked to the moment of exhaustion, MNB rise exponentially at the rate of interest; for, otherwise, it would be possible to increase the value, at any arbitrarily chosen point of time, of the flow of net benefits or rents derived by the community from the deposit. A more difficult question concerns the behaviour of MNB at the moment of transition from one deposit to another. We proceed to show that at all points of transition, MNB drops discontinuously.

It suffices to consider the case $n = 2$. Let $q_i(t)$ be the rate of extraction from the ith deposit at time t, so that total extraction is $q(t) \equiv q_1(t) + q_2(t)$; let t_1 be the moment of transition from one deposit to the other; and let ρ be the rate of time preference. Then the social problem is to find

$$\max_{t_1, \{q_1, q_2\}} \left\{ \int_0^\infty \exp(-\rho t) \cdot u(q_1(t) + q_2(t))\, dt - K - K \cdot \exp(-\rho t_1) \right. \qquad \text{(P.1)}$$

$$\text{s.t.} \int_0^\infty q_i(t)\, dt \leqq R \qquad i = 1, 2$$

$$q_1(t) \geqq 0 \text{ if } t \leqq t_1, \qquad q_1(t) = 0 \text{ if } t > t_1$$

$$q_2(t) \geqq 0 \text{ if } t \geqq t_1, \qquad q_2(t) = 0 \text{ if } t < t_1.$$

Now to solve (P.1), it is necessary to solve the subproblem

$$\max_{\{q_2\}} \left\{ \exp(\rho t_1) \int_{t_1}^\infty \exp(-\rho t) \cdot u(q_2(t))\, dt - K \right\} \equiv V(t_1) - K \qquad \text{(P.2)}$$

$$\text{s.t.} \int_{t_1}^\infty q_2(t)\, dt \leqq R$$

$$q_2(t) \geqq 0.$$

Hence the maximand of (P.1) can be re-written as

$$\int_0^{t_1} \exp(-\rho t) \cdot u(q_1(t))\, dt - K - K \cdot \exp(-\rho t_1) + V(t_1)\exp(-\rho t_1). \qquad (1)$$

Throughout our further calculations it will be assumed that $V(t_1)$ is greater than K, so that it will be optimal to exhaust the second deposit and suboptimal to fail to do so.

The Hamiltonian for (P.1), with the revised maximand (1), is

$$H(t) = u(q_1(t)) - \psi_1(t) q_1(t).$$

As necessary conditions of a maximum, we then have

$$u'(q_1(t)) - \psi_1(t) \leqq 0 \qquad (= \text{if } q_1(t) > 0)$$
$$\dot{\psi}_1 = \rho \psi_1 \qquad (2)$$

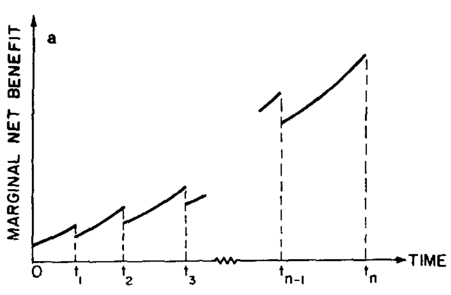
and the transversality condition associated with the choice of t_1,

$$H(t_1^-) \cdot \exp(-\rho t_1^-) + \rho K \cdot \exp(-\rho t_1^+) = -\frac{d}{dt_1} \left[V(t_1^+) \cdot \exp(-\rho t_1^+) \right]. \qquad (3)$$

(See Hestenes [6, Theorem 11.1] and Long and Vousden [12, Theorem 1].) The left side of (3) may be interpreted as the marginal gain from delaying the exhaustion of the first deposit, and the right side may be interpreted as the marginal cost of doing so.

Let

$$J(t_1) \equiv \exp(-\rho t_1) V(t_1) \equiv \int_{t_1}^\infty \exp(-\rho t) u(q_2^*(t))\, dt.$$



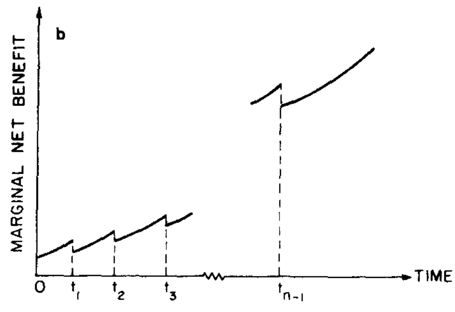


FIGURE 1

Then, following Hadley and Kemp [3, pp. 117–120],

$$\frac{d}{dt_1}[J(t_1)]\bigg|_{t_1^+} = -\left[\exp(-\rho t_1^+)u(q_2^*(t_1^+)) - \psi_2(t_1^+)\exp(-\rho t_1^+)q_2^*(t_1^+)\right], \quad (4)$$

where $\psi_2(t_1^+) = u'(q_2^*(t_1^+))$. Hence (3) can be re-written as

$$\begin{aligned} u(q_1(t_1^-)) - \psi_1(t_1^-)q_1(t_1^-) &\equiv H(t_1^-) + \rho K \\ &= H(t_1^+) \equiv u(q_2(t_1^+)) - \psi_2(t_1^+)q_2(t_1^+), \end{aligned} \quad (5)$$

implying that the Hamiltonian values $H(t_1^-)$ and $H(t_1^+)$ differ by the amount ρK. Recalling (2), we can interpret (5) as requiring that at t_1 the consumers' surplus $u - qu'$ jumps up to compensate for the interest on the new set-up cost.

With (5) in our hands, it is easy to show that $q_1(t_1^-) < q_2(t_1^+)$, that is, that at t_1 the rate of extraction jumps up and, since u is concave, MNB jumps down. Recalling (2) and substituting for $\psi_1(t_1^-)$ and $\psi_2(t_1^+)$ in (5), we obtain

$$\begin{aligned} &\left[u(q_1(t_1^-)) - u'(q_1(t_1^-)) \cdot q_1(t_1^-)\right] + \rho K \\ &\quad = u(q_2(t_1^+)) - u'(q_2(t_1^+)) \cdot q_2(t_1^+). \end{aligned} \quad (5')$$

The required conclusion then follows from the fact that $u(q) - u'(q) \cdot q$ is an increasing function of q.[1]

That completes our demonstration that optimal MNB follows a saw-tooth path with an average rate of growth less than the rate of interest. If there is some natural upper bound on MNB (stemming, for example, from the availability of a standby technology for the production of a resource-substitute or from the possibility of extinguishing demand at finite prices), then the n deposits must be exhausted in finite time, as in Fig. 1a; otherwise, the last deposit to be worked will be exhausted only asymptotically, as in Fig. 1b. Given the rate of time preference and the size of the deposits, the extent of the jumps at t_i ($i = 1, \ldots, n-1$) is determined by the amount of the set-up costs K. In the extreme case studied by Hotelling, the costs are zero and the jumps disappear.

[1]See Fig. 1.

Gathering together our findings to this point, we obtain

PROPOSITION 1. *Under the assumptions of this section, deposits must be extracted in strict sequence. Marginal net benefit follows the saw-tooth path of Fig.* 1, *with an average rate of increase less than the market rate of interest. The optimal rate of extraction follows a complementary path, generally declining but jumping up at points of transition from one deposit to another.*

3. RELATED CASES

Nothing of any importance changes if, in addition to or instead of set-up costs, there are set-down costs, incurred at the end of the working life of a deposit and the same for each deposit. It remains true that the optimal MNB follows the saw-tooth path of Fig. 1, that the optimal working life of a deposit is greater the longer its exploitation is delayed, and that the order in which the deposits are worked is a matter of indifference.

Similarly the analysis changes only in minor detail if there are constant average costs of extraction, the same for each deposit. One need only impose the additional assumption that something will be demanded at a price equal to the average costs of extraction, and recall that MNB is net of extraction costs.

Our conclusions change if the deposits are of different size. Two polar cases may be distinguished. (a) If the same set-up costs are incurred whatever the size of a deposit, then it is optimal to work the deposits in descending order of size, for then the average delay in incurring costs is greatest. This finding has an interesting corollary. Suppose that the deposits are of equal size but with set-up costs payable not just once for each deposit, but recurrently, once for each R' (or part thereof) of resource extracted. Then each deposit can be viewed as a collection of $n' + 1$ subdeposits, all but one of size R', the remaining subdeposit of size $R - n'R'$. The initial endowment can then be viewed as consisting of $n'n$ subdeposits each of size R' and n subdeposits each of size $R - n'R' \leqq R'$. Applying (a), it is optimal to extract the $n'n$ larger subdeposits before turning to the n smaller subdeposits.

We next note the possibility that the deposits are identical in size and set-up costs but are subject to different constant average variable costs of extraction. It is easy to see that in this case it is optimal to work the deposits in ascending order of average variable cost, a conclusion which generalizes a well-known result of Herfindahl [5]. (See also Hartwick [4].)

Throughout this and the preceding section we have imagined that the resource-good can be obtained only from resource-stocks. Finally, we broaden our analysis by allowing for the possibility that there is available a relatively high-cost standby technology for the production of a flow substitute for the extracted resource. Specifically, we assume that the community possesses a single homogeneous resource-stock with no set-up costs and no variable costs of extraction; but that after incurring set-up costs it can produce a perfect substitute for the extracted resource at constant average variable cost of extraction v. (In effect, the community has two deposits, one of infinite size.) Without a space-consuming re-working of Section 2, it can be accepted, perhaps, that the optimal path of marginal benefit is as depicted in Fig. 2.[2] Depending on the size of the resource-deposit, the initial marginal benefit

[2] Kemp and Long [10] have studied the bearing of (flow) fixed costs of extraction on the optimal path of extraction and have found that if two deposits are alike in all respects but fixed cost, then (a) it is

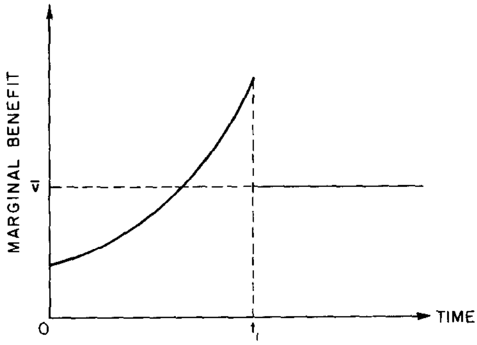


FIGURE 2

may be greater or less than v. Of course the standby technology may improve over time; in that case, marginal benefit declines after t_1.

4. MARKET OUTCOMES

Having described the socially optimal path of extraction, we now consider whether the path can be supported by a system of competitive markets. Since the answer to that question will be NO, the argument can proceed in terms of a counter example.

EXAMPLE. Let $n = 2$ and let the social utility function take the special constant-elasticity form

$$u(q) = q^{\alpha} \qquad (0 < \alpha < 1). \tag{6}$$

The solution to (P.2) is then

$$q_2^*(t) = \frac{\rho R}{1-\alpha}\exp\left(-\frac{\rho}{1-\alpha}(t-t_1)\right)$$

so that

$$V(t_1) = \frac{1-\alpha}{\rho}\cdot\left(\frac{\rho R}{1-\alpha}\right)^{\alpha}$$

optimal to work one deposit at a time, with MNB increasing at the rate of time preference while any particular deposit is being worked and (b) at points of transition the rate of extraction jumps up or down according as the new deposit is associated with a higher or lower fixed cost. By interpreting the interest on earlier set-up expenditures as the fixed cost of current extraction, it can be seen that our present result is closely related to that of Kemp and Long.

It may be noted that Eq. (5) can be re-derived by introducing cumulative discounted set-up costs as a state variable subject to jumps and then applying the results of Vind [13].

and

$$H(t_1^+) = (1-\alpha)\left(\frac{\rho R}{1-\alpha}\right)^{\alpha}. \tag{7}$$

Now consider (P.1), with maximand (1). If $(t_1^*, \{q_1^*(t)\})$ is the solution to (P.1), then $\{q_1^*(t)\}$ must solve

$$\max_{\{q_1\}} \int_0^{t_1^*} \exp(-\rho t) u(q_1(t))\, dt \qquad (t_1^* \text{ fixed}) \tag{P.3}$$

$$\text{s.t.} \int_0^{t_1^*} q_1(t)\, dt \leqq R$$

$$q_1(t) \geqq 0.$$

Given (6), the solution to (P.3) is

$$q_1^*(t) = \frac{\rho R}{1-\alpha} \cdot \frac{\exp\left(-\dfrac{\rho t}{1-\alpha}\right)}{1-\exp\left(-\dfrac{\rho t_1^*}{1-\alpha}\right)}. \tag{8}$$

Substituting (7) and (8) into (5′), we obtain

$$(1-\alpha)\left(\frac{\rho R}{1-\alpha}\right)^{\alpha}\left[\frac{\exp\left(-\dfrac{\rho t_1}{1-\alpha}\right)}{1-\exp\left(-\dfrac{\rho t_1}{1-\alpha}\right)}\right]^{\alpha} = (1-\alpha)\left(\frac{\rho R}{1-\alpha}\right)^{\alpha} - \rho K, \tag{5″}$$

from which the optimal t_1, if it exists, may be determined. It is easy to show that the optimal t_1 exists and is both unique and finite. Thus, taking logarithms in (5″), and defining $X = (1-\alpha)[\rho R/(1-\alpha)]^{\alpha}$, we obtain

$$\alpha \ln\left[1-\exp\left(-\frac{\rho t_1}{1-\alpha}\right)\right] + \left(\frac{\rho \alpha}{1-\alpha}\right)t_1 - \ln\left(\frac{X}{X-\rho K}\right) \equiv g(t_1; \rho, \alpha, K, R) = 0. \tag{9}$$

(Notice that, since $X/\rho = H(t_1^+)/\rho = V(t_1) > K$, $X/(X-\rho K) > 1$.) The desired properties then follow from the easily verified facts that $g < 0$ if $t_1 = 0$, $g = \infty$ if $t_1 = \infty$, and $\partial g/\partial t_1 > 0$.

With the existence and uniqueness of an optimal program assured, we can sensibly ask whether the program could be reproduced by competitive markets.[3] For such an outcome it is necessary that the resource sell at a current utility- or leisure-price of $u'(q(t))$. But it is also necessary that each deposit earn the same present value of profits or rents; for, otherwise, not all owners would be content with the optimal sequence of exploitation of the deposits. We now show that these two requirements

[3] The special case in which $u(q)$ is of constant elasticity is examined in the Appendix.

may be incompatible, that if the prices $u'(q(t))$ prevail then the two deposits may differ in profitability; and we conclude that set-up costs may destroy the Pareto-optimality of competitive outcomes. Thus if the prices $u'(q(t))$ prevail, then the total profit derived from the first deposit, referred to time t_1, is

$$\begin{aligned}\pi_1(t_1) &\equiv R \cdot u'\big(q_1^*(t_1^-)\big) - K \cdot \exp(\rho t_1)\\ &= \alpha R\left(\frac{\rho R}{1-\alpha}\right)^{\alpha-1}\left[\frac{\exp\left(-\dfrac{\rho t_1}{1-\alpha}\right)}{1-\exp\left(-\dfrac{\rho t_1}{1-\alpha}\right)}\right]^{\alpha-1} - K \cdot \exp(\rho t_1) \qquad (10)\end{aligned}$$

and the total profit from the second deposit, also referred to time t_1, is

$$\begin{aligned}\pi_2(t_1) &\equiv Ru'\big(q_2^*(t_1^+)\big) - K\\ &= \alpha R\left(\frac{\rho R}{1-\alpha}\right)^{\alpha-1} - K. \qquad (11)\end{aligned}$$

It follows from (10) and (11) that

$$\left[\frac{\exp\left(-\dfrac{\rho t_1}{1-\alpha}\right)}{1-\exp\left(-\dfrac{\rho t_1}{1-\alpha}\right)}\right]^{\alpha-1} = \frac{\pi_1(t_1) + K \cdot \exp(\rho t_1)}{\pi_2(t_1) + K}. \qquad (12)$$

Choosing K large enough to make $\pi_2(t_1) = 0$, and setting $\pi_1(t_1) = \pi_2(t_1)$, (12) reduces to

$$\left[1 - \exp\left(-\frac{\rho t_1}{1-\alpha}\right)\right]^{1-\alpha} = 1.$$

But this is not possible because, as we have seen, t_1 is finite.

Thus, in general, $\pi_1(t_1) \neq \pi_2(t_1)$. This may be found surprising. However, it must be remembered that the future seen from time 0 is not at all the same as the future seen from time t_1; for at time 0 there are two deposits to exploit, at time t_1 there is just one. But, surprising or not, the finding carries an important implication: Laissez-faire competitive markets cannot be relied on to reproduce the optimal program of extraction. To make this implication quite clear, let the set-up costs be such that $\pi_2(t_1) = 0$. Then, as will be verified, $\pi_1(t_1)$ must be negative, implying that, for the competitive outcome to be optimal, it is necessary that the owner of one deposit be subsidized, with payment of the subsidy conditional upon his extracting first. The verification is achieved by setting $\pi_2(t_1) = 0$ in (12), obtaining

$$1 = \left[\frac{\pi_1 + K \cdot \exp(\rho t_1)}{K \cdot \exp(\rho t_1)}\right]\left[\frac{1}{1-\exp\left(-\dfrac{\rho t_1}{1-\alpha}\right)}\right]^{1-\alpha},$$

and then noting that, since the second square-bracketed term is greater than one, the first must be less than one, which is possible if and only if π_1 is negative.

That completes our discussion of the counter example. The outcome of that discussion is

PROPOSITION 2. *In general, the socially optimal path of extraction cannot be reproduced by laissez-faire competitive markets.*

We now note an additional implication of the sequential nature of optimal extraction. Suppose that there is a complete set of spot markets and that one has calculated the set of taxes and subsidies which, if markets were competitive, would support the optimal program of extraction. Since that program is strictly sequential, the number of unexhausted deposits steadily declines until, after a finite interval of time, there remains only one. Why should the owner of the surviving deposit not exercise his new-found monopoly power? Indeed, why should competition not break down when only two or three deposits remain? Whether or not owners recognize at time zero the advantages of being last, the assumption of competition emerges as highly implausible.[4]

We are led therefore to consider markets containing elements of monopoly. Suppose that a single monopolist controls each of two identical deposits; and, to avail ourselves of the calculations of our Example, suppose again that the utility function is of the constant-elasticity type (6), so that the leisure-price of the resource is $u'(q) = \alpha q^{\alpha-1}$ and total revenue from sales is αq^{α}. If the monopolist seeks to maximize the present value of revenue less the present value of set-up costs, we obtain, instead of (5″),

$$\alpha(1-\alpha)\left(\frac{\rho R}{1-\alpha}\right)^{\alpha}\left[\frac{\exp\left(-\frac{\rho t_1}{1-\alpha}\right)}{1-\exp\left(-\frac{\rho t_1}{1-\alpha}\right)}\right]^{\alpha} = \alpha(1-\alpha)\left(\frac{\rho R}{1-\alpha}\right)^{\alpha} - \rho K. \quad (5''')$$

Evidently,

$$\begin{aligned}(1-\alpha)\left(\frac{\rho R}{1-\alpha}\right)^{\alpha}\left[\frac{\exp\left(-\frac{\rho t_1}{1-\alpha}\right)}{1-\exp\left(-\frac{\rho t_1}{1-\alpha}\right)}\right]^{\alpha} &= (1-\alpha)\left(\frac{\rho R}{1-\alpha}\right)^{\alpha} - (\rho K/\alpha) \\ &\leqq (1-\alpha)\left(\frac{\rho R}{1-\alpha}\right)^{\alpha} - \rho K.\end{aligned}$$

Thus, if $K = 0$, the extraction path under monopoly is socially optimal; otherwise, the monopolist extracts the first deposit too slowly. The price paths under monopoly and perfect planning, with K positive, are displayed in Fig. 3. (The superscripts m and s stand for monopoly and social optimum, respectively, and $p(t)$ = price of resource in terms of leisure.)

[4] Of course the foregoing argument is inapplicable if there is a complete set of futures markets, so that all contracts are concluded at time zero.

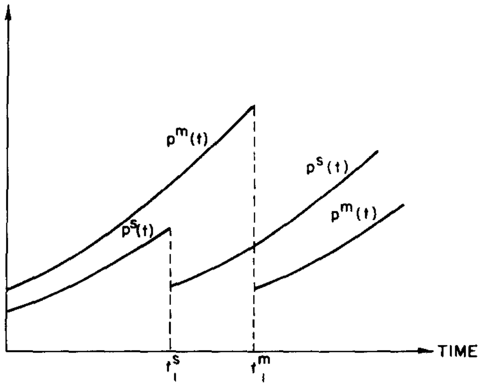


FIGURE 3

The same conclusion emerges if, following the final paragraph of Section 3, it is supposed that the second deposit is infinite. (A proof may be found in the Appendix.) Thus we have

PROPOSITION 3. *In general, the extraction path under monopoly is suboptimal. This is so even if the utility function is of constant elasticity; in that case, with K positive, the rate of extraction is too slow. However, if $K = 0$ and if the utility function is of constant elasticity, then the extraction path under monopoly is optimal.*[5]

APPENDIX

Let there be two deposits, the first finite and the second infinite. For the first deposit there are neither set-up nor variable costs of extraction, and for the second deposit there is a set-up cost K and a constant average variable cost of extraction v. The utility function is of constant elasticity: $u(q) = q^{\alpha}$. If exploitation of the second deposit begins at t_1, society solves the problem

$$\max_{\{q_2\}} \int_{t_1}^{\infty} \exp(-\rho(t - t_1))(q_2^{\alpha} - vq_2)\, dt \equiv V(t_1).$$

The first order condition

$$\alpha q_2^{\alpha-1} = v$$

[5] Eswaran *et al.* [2] have presented an elegant case of non-convexities and market failure which appears to have been in part inspired by our contribution here. (They refer to the mimeographed version of our paper (Hartwick, Kemp, and Long, Queen's University Discussion Paper #412).) In Eswaran *et al.* [2], many identical price-taking extraction firms have average extraction costs U-shaped. The candidate market solution has simultaneous extraction by all firms to stock exhaustion at the point average extraction costs equals marginal extraction costs. But this exhaustion point implies quantity extracted by each firm at the end of the program is positive. At the next instant price will jump up to the price at which the backstop comes on stream. Hence, each firm will choose to depart from the candidate path in order to reap the gain from the price jump. Hence, the candidate path is not sustainable as a market solution. Weitzman [14] considered the case of many deposits, declining average extraction costs and an exogenously given constant quantity of output demanded. He focused attention on the planning problem of optimal sequencing of different sized deposits with different declining average extraction costs.

yields

$$q_2^* = (v/\alpha)^{1/(\alpha-1)} \tag{13}$$

whence

$$V(t_1) = \frac{1-\alpha}{\rho}(q_2^*)^\alpha.$$

(It is assumed, of course, that $V(t_1) > K$.) On the other hand, (8) continues to yield the optimal value of $q_1(t)$ for $t < t_1$. Hence

$$\begin{aligned} H(t_1^-) &= u(q_1(t_1^-)) - \psi_1(t_1^-)q_1(t_1^-) \\ &= u(q_1(t_1^-)) - u'(q_1(t_1^-))q_1(t_1^-) \\ &= (1-\alpha)(q_1^*)^\alpha \end{aligned} \tag{14}$$

and

$$\begin{aligned} H(t_1^+) &= u(q_2^*) - vq_2^* \\ &= (1-\alpha)(q_2^*)^\alpha. \end{aligned} \tag{15}$$

The transversality condition at t_1 is

$$H(t_1^-) = H(t_1^+) - \rho K. \tag{16}$$

Substituting from (8) and (13) into (14) and (15) and thence into (16), we obtain

$$\begin{aligned} (1-\alpha)\left(\frac{\rho R}{1-\alpha}\right)^\alpha\left[\frac{\exp(-\rho t_1/(1-\alpha))}{1-\exp(-\rho t_1/(1-\alpha))}\right]^\alpha &= (1-\alpha)(q_2^*)^\alpha - \rho K \\ &= (1-\alpha)\left(\frac{v}{\alpha}\right)^{\alpha/(\alpha-1)} - \rho K, \end{aligned} \tag{17}$$

which can be solved uniquely for t_1. Let the solution be t_1^s.

Consider now the monopolist's problem. If exploitation of the second deposit begins at t_1, the monopolist solves the problem

$$\max_{\{q_2\}} \int_{t_1}^{\infty} \exp(-\rho(t - t_1^m))(\alpha q_2^\alpha - vq_2)\, dt \equiv V_1^m(t_1).$$

The first order condition

$$\alpha^2 q_2^{\alpha-1} = v$$

yields

$$q_2^* = (v/\alpha^2)^{1/(\alpha-1)}$$

whence

$$V^m(t_1) = \frac{\alpha(1-\alpha)}{\rho}(q_2^*)^\alpha.$$

On the other hand, the monopolist's allocation of the first deposit, over the interval $[0, t_1]$, is again given by (8). Hence,

$$H(t_1^-) = \alpha(1-\alpha)(q_1^*)^\alpha$$

and

$$H(t_1^+) = \alpha(1-\alpha)(q_2^*)^\alpha.$$

Substituting into the transversality condition (16), we obtain

$$(1-\alpha)\left(\frac{\rho R}{1-\alpha}\right)^\alpha \left[\frac{\exp(-\rho t_1/(1-\alpha))}{1-\exp(-\rho t_1/(1-\alpha))}\right]^\alpha = (1-\alpha)(q_2^*)^\alpha - \rho K/\alpha, \quad (18)$$

which can be solved uniquely for t_1. Let the solution be t_1^m. Now, if $K > 0$, the right hand side of (17) is greater than that of (18); for

$$(v/\alpha^2)^{\alpha/(\alpha-1)} = (v/\alpha)^{\alpha/(\alpha-1)} \cdot \alpha^{\alpha/(1-\alpha)} < (v/\alpha)^{\alpha/(\alpha-1)}$$

and $-\rho K/\alpha < -\rho K$. Hence, $t_1^m > t_1^s$, if $K > 0$; that is, if there are set-up costs, the monopolist works the first deposit at a slower rate and therefore switches to the second deposit at a later date than is socially desirable. Indeed it is possible that the monopolist will choose to withhold the second deposit forever; for

$$V^s(t_1^s) = \frac{1-\alpha}{\rho}(q_2^*)^\alpha > V^m(t_1^m),$$

implying that

$$V^s(t_1^s) > K > V^m(t_1^m)$$

is possible. This generalizes a finding of Dasgupta and Stiglitz [1] who assumed that $K = 0$.

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