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Grade Selection under Uncertainty: Least Cost Last and Other Anomalies

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Grade-selection rules for heterogeneous resource deposits with increasing marginal extraction costs are derived. It is shown that, contrary to the conventional wisdom, it can be optimal to extract the least-cost deposits last. There are a number of rules that are compatible with dynamic profit maximization under certainty. Nevertheless, none explains empirical regularities in actual extraction patterns. When uncertainty about future price is introduced, however, the theoretical model conforms closely not only to observed grade-selection patterns but also to observed mineral-industry price paths.

I. INTRODUCTION

This paper investigates a seeming contradiction between theoretical conclusions concerning the extraction of exhaustible resources of varying quality and observed mineral-industry extraction patterns.

The idea that the least-cost deposits will be extracted first is so firmly embedded in our minds that it is an often-made but rarely tested assumption underlying the construction of theoretical exhaustible-resource models.[1] Those who attempt to endogenize the sequencing of extraction of heterogeneous deposits, however, usually make very restrictive assumptions about costs—that marginal cost is constant for a given deposit.[2]

When marginal extraction cost rises with the rate of extraction from a given deposit, it is well known that it is optimal to extract from several deposits simultaneously. And in general, no simple rule determines when the least-cost deposits are extracted. In Section III, families of cost functions are constructed that give rise to a least-cost-first rule, to a least-cost-in-periods-of-high-present-value-price rule, and to other more complex sequencing patterns.

Unfortunately, none of the theoretically derived rules is consistent with empirical regularities in mineral-industry extraction profiles. For example, in the 20th century most mineral industries experienced a secular decline in both present-value price and average grade of ores mined. This pattern is compatible with both least-cost-last and least-cost-in-periods-of-high-price rules. The anomaly is that a *nominal* price increase (decrease) is observed to be accompanied by a decline (increase) in the

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[1] See, for example, Gordon [8], Schultze [23], Levhari and Leviatan [16], Fisher [7], Hanson [9], Pindyck [21, 22], Slade [25], and Devarajan and Fisher [4].

[2] See, for example, Herfindahl [11], Solow and Wan [30], Hartwick [10], Ulph [31], Kemp and Long [13], Lewis [17], and Cairns and Lasserre [3]. An exception is Krautkraemer [14].

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average grade of ores mined.[3] This pattern implies a negative correlation between grade and price over the business cycle that is inconsistent with both rules.

A long-run positive grade–price correlation coupled with a short-run negative correlation is a puzzling occurrence. In some instances, the inverse grade–price relationship is the result of conscious national policy concerning a mining industry. For example, the South African government constrains companies to mine to their average grade of gold reserves. Because reserves are defined as material that is profitable under current price and cost conditions, when price increases reserves increase and higher cost ores are mined. The South African policy is often thought to be at variance with private profit-maximizing decisions on the part of firms [18]. It is shown here, however, that similar behavior results when all decision makers are private companies (in the U.S. copper industry, for example).[4]

The seeming contradiction between theoretically derived and empirically observed profit-maximizing decisions can be resolved if uncertainty about future price is introduced. A model in which price is generated by a stochastic differential equation is postulated. Testable parameter restrictions on the differential equation result in the price path being a martingale. It is shown that not only are observed price paths consistent with those implied by the restrictions but, in addition, the theoretical grade-selection profiles that result when the restrictions are imposed conform closely to profiles observed in practice. The seeming inconsistency is therefore resolved.

The organization of the paper is as follows. In the next section, standard methods of modeling grade heterogeneity are presented and their implications for grade-selection profiles are analyzed. In Section III, grade selection under more general cost conditions is discussed and classes of cost functions that lead to specific grade-selection rules are constructed. Section IV discusses empirical regularities in grade-selection patterns. The model of price uncertainty is introduced in Section V and its implications are derived. Section VI assesses the stochastic behavior of mineral-commodity prices. In particular, it is shown that many commodity-price series are reasonably well approximated by martingales. Finally, in Section VII, results are summarized and conclusions are drawn.

II. MODELING GRADE HETEROGENEITY

Consider the case of a competitive industry where firms own ore deposits of different grades. The word "ore" is used to denote a mineral in the ground. "Ore" can contain metal (iron or copper, for example) or it can be a fuel such as petroleum or coal. And the word "grade" is used as a proxy for cost conditions. That is, high "grade" means low cost whether the low cost results from high metal content or from locational or other advantages. All quantities are measured in standardized units such as contained metal. Finally, the industry is analyzed in isolation; that is, the analysis is partial equilibrium.

[3] The word nominal is taken to mean deflated but not discounted. Nominal price is contrasted with present-value price.

[4] The model developed here is for a price-taking firm. South Africa accounts for approximately 80% of free-world gold production and thus might be a price setter. Current gold production, however, is only a small fraction of supply and therefore the price-taking assumption seems reasonable.

When marginal extraction cost is constant within a deposit but differs across deposits, the firm's problem is not well defined. That is, at any instant in time the firm will extract the entire stock of a given deposit, will not extract at all, or will be indifferent about when it extracts.

Industry extraction and price profiles, however, are well defined. The price trajectory is an envelope of curves of the form $c_i + \lambda_i e^{rt}$, where c_i is the constant marginal extraction cost, r is the firms' discount rate, and $\lambda_i e^{rt}$ is the user cost or rent associated with the ith deposit. That is, λ_i is the Lagrange multiplier associated with the constraint that total extraction from the ith deposit cannot exceed its initial stock.

In this case, reserves are extracted in order of increasing cost and the entire stock of ore of a given grade is exhausted before extraction from the deposit with the next highest grade begins. Because price increases at a rate strictly less than the rate of interest, the grade-selection pattern is consistent with both a least-cost-first and a high-PV-price/high-grade rule.

When marginal extraction cost is an increasing function of the rate of extraction from a deposit, the firm's problem is well defined. In this case, price and marginal cost are related by

$$P_t = c_i\left(q_t^i\right) + \lambda_i e^{rt}, \tag{1}$$

where $c_i(q^i)$ is the marginal-cost function for the ith deposit. Given an exogenous price path, each firm varies its extraction rate until (1) holds and deposits of different grades are mined simultaneously.

For a given cost function, the larger the initial stock S_0^i, the lower λ_i. And for a given size, the higher the cost function, the lower λ_i. Extraction from a given deposit therefore depends on the size of the stock as well as on the shape of the marginal-cost function. All else being equal, extraction from larger deposits is faster and marginal cost at the optimal extraction rate is higher.

For the industry, aggregate extraction depends on the individual cost functions and on the size distribution of deposits. Grade-selection patterns under these cost conditions are discussed in the next section.

It is often assumed that extraction cost depends on cumulative extraction as well as the rate of extraction (Gordon [8], Levhari and Leviatan [16], and Pindyck [22], for example). There are two reasons for making this assumption. First, for a given deposit extraction itself may cause the increase in cost. This would be the case for an oil pool of homogeneous quality where cost rises as pressure is depleted. In this case, it is trivially true that the "least-cost ore" is extracted first. No choice is involved, however, and the grade-selection pattern is not an economic problem.

Second, there may be heterogeneous deposits each characterized by different cost conditions. If the low-cost ores are extracted first, then by assumption aggregate extraction cost rises with cumulative extraction. The grade-selection profile here, however, is an economic problem and should be endogenous. If, for example, the low-cost deposits are not used first cost could fall with cumulative extraction.

The emphasis in this paper is on the economic problem of grade selection. For this reason, neither stock approach to modeling depletion is adopted. The first is rejected because no choice is involved and the second is rejected because it assumes the answer. Instead, each deposit is assumed to have a cost function that depends only on the rate of extraction. Whether or not aggregate extraction cost rises with cumulative extraction depends on the optimizing decisions of mine managers, where mines can consist of many shafts containing ores of different grades.[5]

III. GRADE SELECTION WITH INCREASING MARGINAL EXTRACTION COSTS

With increasing marginal costs, even though it is optimal to extract from several deposits simultaneously, the ratio of extraction rates from different deposits may vary over time or with present-value price. In this section, the question of systematic tendencies in these ratios is explored.

A simple two-period two-deposit example that abstracts from all but cost differences suffices to illustrate endogenous grade-selection patterns. Assume that the two deposits are of equal size with initial stock equal S. Deposit i is said to be of lower cost than deposit j if $c_i(q) < c_j(q)$ and $c_i'(q) \leqslant c_j'(q)$ for all positive q. Let deposit one be the low-cost deposit.

Necessary conditions for an interior solution to the firms' maximization problems are

$$P_1 = c_i(q^i) + \lambda_i, \qquad i = 1,2, \tag{2}$$

and

$$\delta P_2 = \delta c_i(S - q^i) + \lambda_i. \tag{3}$$

Subtracting (2) from (3) yields

$$\Theta \equiv \delta P_2 - P_1 = \delta c_i(S - q^i) - c_i(q^i), \tag{4}$$

where Θ is the difference in present-value prices.

Equation (4) shows that extraction from both deposits tends to be tilted toward the period of higher present-value price (because $0 < \delta < 1$ and $c_i' > 0$). The switch from more extraction in period j to more in period i, however, does not occur when $\delta P_2 = P_1$ but occurs at prices where $\delta P_2 < P_1$.

To investigate relative extraction rates q^1/q^2, consider the linear example

$$c_i(q^i) = \alpha_i q^i + \beta_i, \tag{5}$$

with $\alpha_1 \leqslant \alpha_2$ and $\beta_1 \leqslant \beta_2$. Equation (4) shows that

$$\delta\{\alpha_i(S - q^i) + \beta_i\} - \alpha_i q^i - \beta_i = \Theta \tag{6}$$

and

$$q^i = \delta S/(1 + \delta) - \{(1 - \delta)\beta_i + \Theta\}/\{\alpha_i(1 + \delta)\}. \tag{7}$$

If $\alpha_1 = \alpha_2$ (i.e., if the cost curves are parallel), then $q^1 > q^2$ and $S - q^1 < S - q^2$. In this case, extraction follows a least-cost-first pattern. If, in contrast, $\beta_1 = \beta_2 = 0$

[5]An additional complication with stock effects relates to exploration. There is no guarantee that the least-cost deposits will be found first.

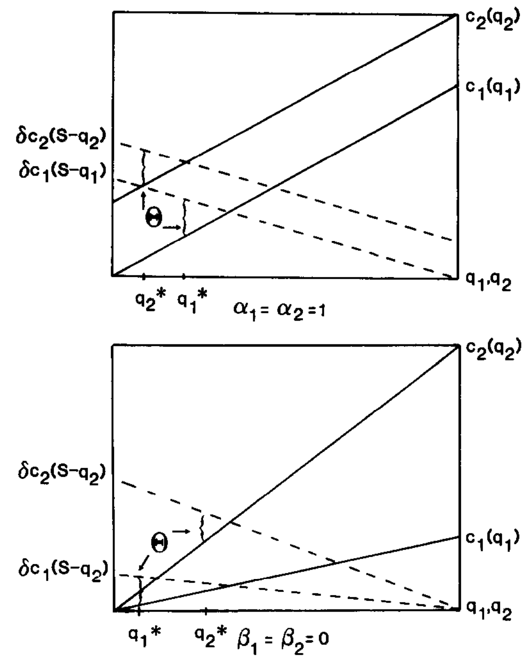

FIG. 1. Possible grade-selection patterns.

(i.e., the cost curves differ by a scalar multiple), then $q^2 > q^1$ and $S - q^2 < S - q^1$ for positive Θ. In this case, extraction follows a least-cost-in-periods-of-high-PV-price pattern. By varying the parameters α_i and β_i, it is possible to obtain many patterns including $q^1 = q^2$ when $(1 - \delta)(\alpha_2\beta_1 - \alpha_1\beta_2) = \Theta(\alpha_1 - \alpha_2)$.

Figure 1, where $c_i(q^i)$ and $\delta c_i(S - q^i)$ are graphed as functions of q^i, illustrates extraction patterns for the two cases $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 = 0$. With positive Θ, case one exhibits a least-cost-first pattern and case two a least-cost-last pattern.[6]

The existence of the second class of cost functions, coupled with endogenous grade selection, implies that the standard practice of including stock effects in aggregate cost functions as proxies for depletion may be suspect. Whereas it is true that extraction cost changes with cumulative extraction, the direction of the change cannot be predicted a priori.

It has been shown that there are many grade-selection patterns that are consistent with dynamic profit maximization under conditions of certainty. Whether these theoretical patterns are consistent with empirically observed extraction patterns is discussed next.

IV. OBSERVED SECULAR AND CYCLICAL GRADE VARIATIONS

In the 20th century, most mineral industries experienced a secular decline in both average grade and present-value price. This pattern is consistent with both a

[6] Many mineral commodities have experienced periods of over a decade when prices were rising faster than the real rate of interest. This is true, for example, of iron ore for the period 1944–1958 and of copper and silver for the period 1958–974.

least-cost-first and a high-PV-price/high-grade rule. Anyone familiar with the mineral industries, however, knows that when nominal price increases (falls) lower (higher) grade ores are mined.

To demonstrate this point, it would be desirable to obtain time-series data on average grade for a large number of mineral commodities. Unfortunately, this is not possible. There is, however, one commodity for which consistent statistics have been recorded over a number of decades—copper in the United States. For this reason, copper data are examined here.

Define average yield to be equal to metal produced divided by ore mined. Yield is not equivalent to grade (metal content of ore in the ground) but it is a proxy for it. Yield is bounded above by grade but depends on recovery efficiency as well.[7]

For U.S. copper in the last few decades, yield is a reasonable proxy for cost. While it is true that high-grade stratabound deposits such as those found in Africa may be higher cost than low-grade porphyries, most copper mined in the United States in the last few decades has come from porphyry deposits. Within this class, there is a strong positive relationship between yield and grade and a strong negative relationship between yield and cost.

Let "yield_t" be U.S. primary copper production divided by U.S. copper ore mined in period t for the years 1925–1977.[8] And let P_t be the U.S. producer price of copper in cents per pound divided by the U.S. wholesale-price index (1967 = 1) in period t.[9] P_t is thus deflated but not discounted.

The price used here is the price of copper, not the price of copper ore. This substitution is made necessary by the fact that copper ore was not sold in a market and thus did not have a market price. The complications arising from this fact are discussed in the Appendix, where it is shown that the shadow price of ore is positively related to the market price of metal. A negative metal-price/yield relationship thus implies a negative ore-shadow-price/yield relationship and vice versa.

The first thing to test is the secular behavior of yield. Yield can never be negative; therefore, an exponential decline seems more reasonable than a linear trend. Accordingly, an equation of the form $\text{yield}_t = ae^{bt}$ was estimated. The result is

$$\ln\left(\widehat{\text{yield}}_t\right) = 1.04 - \underset{(24)}{0.024}t, \qquad R^2 = 0.92 \tag{8}$$

or

$$\widehat{\text{yield}}_t = 2.8e^{-0.024t},$$

where the t statistic is shown in the parenthesis. Equation (8) indicates a significant secular decline in yield at the rate of 2.4% per year.

[7] There is some empirical evidence for a positive relationship between price and processing intensity [27]. If a negative yield/price correlation is found, therefore, this evidence implies a still stronger negative grade/price correlation.

[8] Earlier years were eliminated because production from porphyries was much less significant in the first quarter of this century.

[9] These data can be found in the U.S. Bureau of Mines "Minerals Yearbook" [32].

The second thing to test is the price-yield relationship. The result of regressing ln(yield) on price is

$$\ln\left(\widehat{\text{yield}_t}\right) = 1.05 - \underset{(8.2)}{0.036P_t}, \qquad R^2 = 0.57. \tag{9}$$

Equation (9) shows a significant negative nominal-price/yield relationship.[10]

Finally, it is shown that this negative relationship is not observed when nominal price is replaced by PV-price. Let $\text{PV}P_t = e^{-rt}P_t$. Using a real discount rate of 3% ($r = 0.03$) the following relationship was obtained:

$$\ln\left(\widehat{\text{yield}_t}\right) = -7.2 + \underset{(6.9)}{0.094\text{PV}P_t}, \qquad R^2 = 0.48. \tag{10}$$

A positive relationship between present-value price and grade coupled with a negative nominal-price/grade relationship seems incompatible with any theoretically derived grade-selection pattern. This is especially true when we consider that both nominal and present-value prices have been very unstable, exhibiting both upward and downward swings over the period 1925–1977 but that the overall trend for both has been negative.

An equation which summarizes both secular and cyclical grade variations is

$$\ln\left(\widehat{\text{yield}_t}\right) = 1.11 - \underset{(15)}{0.022t} - \underset{(1.8)}{0.005P_t}, \qquad R^2 = 0.92. \tag{11}$$

This equation shows an overall secular decline in grade (least-cost-first) with a countercyclical grade cycle superimposed.

V. A MODEL WITH EXOGENOUS AND UNCERTAIN PRICES

The models discussed in Section III were derived under the assumption that the firm knows the entire future price path with certainty. In practice, however, there are many reasons why uncertainty over future prices may exist. Uncertainties on the demand side may stem from technical change in the using sectors, cyclical changes in GNP, changes in prices of substitutes and complements, and tariffs imposed by importing countries. Supply-side uncertainties include the discovery of new deposits, strikes in the mines, and political disruptions or cartel actions on the part of producing countries. Finally, policies that affect price directly include price controls and royalties. Given the numerous sorts of uncertainty, it seems reasonable to treat price as a random variable.

In the model of price uncertainty, the firm is a price taker and makes decisions on the basis of expected future prices. Expectations are rational and are thus based on

[10] It could be claimed that this result is due to a positive growth rate for price that is less than r coupled with an overall decline in yield. The trend in undiscounted prices however is not positive [25]. In addition, an examination of the data reveals a negative year-to-year correlation in these series. For example, in 1972 when price was 42.5 cents per pound (in 1967 cents) yield was 0.55%. By 1974, when deflated price had risen to 47.8 cents per pound, yield had fallen to 0.49%. And in 1977, when the deflated price had fallen to 33.9 cents per pound, yield climbed back up to 0.52%.

a knowledge of the model that generates prices. Before deriving the implications of price uncertainty for the grade-selection profile, it is necessary to make an assumption about the stochastic behavior of price.

Suppose that price is generated by a stochastic differential equation

$$\begin{aligned} dP &= \alpha P\,dt + \sigma\,dz \\ &= \alpha P\,dt + \sigma\epsilon\sqrt{dt}\,, \end{aligned} \tag{12}$$

where z is a Wiener process (ϵ is a standard normal variable). Equation (12) nests two models of theoretical interest. If $\sigma = 0$, price grows at a deterministic rate α, as would be the case in the simplest Hotelling [12] model. In this case, α should equal r. If, in contrast, $\alpha = 0$, the price path is a martingale—the sum or integral of independent random increments of zero mean—as would be the case if speculation were the principal factor determining price and if the commodity were sold in an efficient market. In addition to the Hotelling and efficient-market models, when both α and σ are nonzero, the rate of growth of price is a random variable with mean α.

If producers are risk neutral, the firm's problem is to

$$\max_{q^1,\dots,q^n} E_0\left\{\int_0^{\infty} e^{-rt}\left(P_t\left(\sum q_t^i\right) - \sum C_i\left(q_t^i\right)\right) dt\right\} = E_0\int_0^{\infty}\Pi_t\,dt \tag{13}$$

subject to

$$\int_0^{\infty} q_t^i\,dt \leqslant S_0^i \qquad \text{and} \qquad q_t^i \geqslant 0, \tag{14}$$

where E is the expectation operator, its subscript denotes the time at which expectations are formed, and $C_i(q^i)$ is the total-cost function for the ith deposit. Marginal costs are assumed to be increasing.

Techniques of stochastic control can be used to solve the maximization (13). Let S_t^i be the stock of ore remaining in deposit i at time t. A stochastic version of the Euler equation in the calculus of variations is (see Pindyck [22] for a derivation)

$$\partial\mathring{\Pi}/\partial q^i = -\partial\Pi/\partial S^i = 0, \tag{15}$$

where the symbol $^\circ$ denotes the average or expected time rate of change of a function of a stochastic process.[11] The second equality in (15) results because S^i does not enter Π. Equation (15) can be rewritten as

$$(P \mathbin{\mathring{-}} c_i)e^{-rt} = 0, \tag{16}$$

or

$$r(P - c_i) = \mathring{P} - \mathring{c}_i. \tag{17}$$

[11] If $f(P, t)$ is a function of the stochastic process P, then

$$f(P\mathring{,}\, t) = \lim_{h\to 0} E_t\{(f(P_{t+h}, t+h) - f(P_t, t))/h\}.$$

For a theoretical treatment of the differential generator and Itô's lemma, see Kushner [15] and for a discussion of applications in economics, see Merton [20]

$\mathring{P} = \alpha P$ by assumption. Because q^i is a function of P along an optimal path, Itô's lemma can be used to find $\mathring{c}_i$:

$$\mathring{c}_i = c_i'\{\alpha\, \partial q^i/\partial P + \partial q^i/\partial t + \tfrac{1}{2}\partial^2 q^i/\partial P^2\} + \tfrac{1}{2}\sigma^2 c_i''\, \partial q^i/\partial P^2 \tag{18}$$

$$= c_i'\mathring{q}^i + \tfrac{1}{2}\sigma^2 c_i''\, \partial q^i/\partial P^2, \tag{19}$$

where Itô's lemma was used a second time in substituting $\mathring{q}^i$ for the expression in braces in (18). Substituting (19) into (17) results in

$$\mathring{q}^i = -1/c_i'\{r(P - c_i) - \alpha P + \tfrac{1}{2}\sigma^2 c_i''\, \partial q^i/\partial P^2\}. \tag{20}$$

To see how (20) compares to the nonstochastic case where $\dot{P}/P = \alpha$ with certainty ($\sigma = 0$), Eq. (1) can be differentiated with respect to time and rearranged to obtain

$$\dot{q}^i = -1/c_i'\{r(P - c_i) - \alpha P\}, \tag{21}$$

where a dot over a variable denotes its (deterministic) time rate of change.

If marginal-cost functions are straight lines ($c_i'' = 0$), output is the same in the two cases (certainty equivalence holds). If however marginal-cost functions are strictly convex ($c_i'' > 0$), then extraction is more rapid than under certainty and the more convex the marginal-cost function, the greater the bias toward the present.

A similar result is obtained by Pindyck [22] who gives the intuition as follows. When marginal cost is strictly convex, the effect of incremental output changes is asymmetric. The expected net gain in present value from a future price increase is therefore less than the net-present-value loss from an equal price decline. The producer therefore has less incentive to withhold production in anticipation of higher prices than under certainty, and extraction is more rapid.

Grade-selection profiles are first examined under the hypotheses that $\alpha = 0$ (the price path is a martingale) and $c_i'' = 0$ (certainty equivalence holds). The reason for emphasizing the martingale case is that it is consistent with the empirical evidence presented in the next section. That is, it is shown that the stochastic behavior of the prices of many major mineral commodities is well approximated by martingales, at least over a period of several decades. For longer periods, there may be a trend in $\dot{P}$ or $\dot{P}/P$ [25]. Nevertheless, all that is required for the grade-selection behavior implied by the martingale model to occur in practice is that mine managers make decisions on the basis of optimal ARIMA forecasts of price.[12]

The distinguishing feature of a martingale is that the optimal forecast, ${}_tP_{t'}^*$, for the price at time t' made at time t (with $t \leqslant t'$) is P_t. That is,

$${}_tP_{t'}^* = E_t\{P_{t'} | \text{the history of price up to time } t\} = P_t. \tag{22}$$

When the price path is a martingale, the firm expects the current price to prevail in all future time periods. It should be noted that this is an optimal forecast and is not the same as the firm having static expectations (which usually means that the firm ignores the process that generates price).

[12]An ARIMA model is a discrete time-series model whereas Eq. (12) is continuous. In making forecasts, however, it is always necessary to approximate continuous processes by discrete.

Beginning at $t_0 \geqslant 0$, the firm wishes to

$$\max_{q^1,\ldots,q^n} E_{t_0}\left\{\int_{t_0}^{\infty} e^{-rt}\left(P_t\left(\Sigma q_t^i\right) - \Sigma C_i\left(q_t^i\right)\right) dt\right\} \tag{23}$$

subject to

$$\int_{t_0}^{\infty} q_t^i \, dt \leqslant S_0^i - \int_0^{t_0} q_t^i \, dt = S_{t_0}^i \quad \text{and} \quad q_t^i \geqslant 0. \tag{24}$$

Under certainty equivalence, in all periods when extraction is positive it must be true that

$$E_{t_0}\{P_t\} = P_{t_0} = c_i\left(E_{t_0}\left(q_t^i\right)\right) + \lambda_i e^{rt}. \tag{25}$$

All deposits for which $P_{t_0} > c_i(0)$ are exploited.[13] Because price is falling rapidly in present value, extraction from all economic deposits is tilted toward the present. In particular, extraction falls over time in such a way that $P_{t_0} - c_i(E_{t_0}(q_t^i))$ grows at the discount rate r.

To see what happens when price changes, suppose that in some time period $t_1 > t_0$ price increases. The firm will re-solve the maximization (23) with t_0 replaced by t_1. Because $P_{t_1} > P_{t_0}$, some deposits that were previously uneconomical may now be viable and therefore exploited.

At time t_1, user costs for the low-grade deposits are apt to be small for three reasons. First, all else being equal, there is a positive relationship between λ_i and grade. Second, high-grade deposits are economical at lower prices than low-grade deposits. Extraction in previous time periods thus depletes the stock of high-grade deposits and raises their associated multipliers at time t_1. And finally, there is some geostatistical evidence for an inverse relationship between grade and size of deposits [24]. High-cost deposits may therefore start out larger (at $t = 0$) in addition to being exploited in fewer time periods, and both factors are associated with low multipliers at time t_1.

If the user cost associated with two deposits is the same, extraction from the deposit with the lower marginal extraction cost is greater. If, however, the user cost associated with the high-cost deposit is relatively small, extraction from this deposit may exceed extraction from the low-cost deposit. When price increases therefore if extraction from previously uneconomical deposits begins at high rates, there will be a significant decline in the average grade of ores mined.

The predictions from the certainty-equivalence model are thus twofold. First, because low-cost deposits are viable in all time periods and because extraction from all deposits is tilted toward the present, there will be a general secular decline in the average grade of ores mined. And second, because high-cost deposits become (cease to be) viable when price increases (falls) there will be a negative relationship between nominal price and average grade.

[13] Under price uncertainty, the option value of the resource creates an incentive to slow down production (see, for example, Pindyck [22] or Brennan and Schwartz [2]). There seems no reason to believe, however, that this tendency would systematically bias ratios of extraction rates from deposits of different costs.

If marginal extraction costs are strictly convex ($c_i'' > 0$), these conclusions can be reinforced. Suppose, for example, that deposit j is of higher cost than deposit i and that marginal-cost functions are quadratic,

$$c_i(q) = d_i + e_i q + f_i/2q^2. \tag{26}$$

The assumption that $c_i'(q) = e_i + f_i q \leqslant e_j + f_j q = c_j'(q)$ for all positive q implies that $c_i'' = f_i \leqslant c_j'' = f_j$. That is, the certainty-equivalent bias toward more rapid extraction can be greater for high-cost deposits implying that the previous predictions are reinforced.

With more general marginal-cost functions, unless there is a systematic correlation between grade and degree of convexity of marginal-cost functions, ratios of extraction rates from different deposits should not be systematically affected by certainty-equivalent bias and the previous predictions remain unchanged.

The case where $\alpha \neq 0$ is not analyzed because it is systematically rejected by the empirical evidence that follows. If mine managers were to make forecasts on the basis of longer run autoregressive models, different grade-selection patterns might result. There seems to be little evidence of longer run forecasting, however, and this subject is therefore not explored.

The model could be complicated by adding technical progress. However, unless extremely restrictive assumptions were made about the nature of technical progress, no definitive predictions about changes in grade-selection patterns could be made.

The next section discusses whether the grade-selection tendencies predicted by the theoretical model with price uncertainty are observed in practice.

VI. THE STOCHASTIC BEHAVIOR OF MINERAL-COMMODITY PRICES

The prices of major mineral commodities produced in North America are examined. Time-series data on prices are much easier to obtain than data on cost or grade. Accordingly, prices of several major North American mineral commodities were collected for the years 1906–1973.[14] The commodities are copper, pig iron, lead, bauxite, silver, petroleum, and coal.[15] All prices were deflated by the U.S. wholesale-price index.

To investigate the hypothesis that the price series are martingales ($\alpha = 0$), a discrete-time approximation to Eq. (12) was estimated:

$$\Delta P_t = \alpha P_t + u_t, \qquad u \sim N(0, \sigma^2), \tag{27}$$

[14] With the exception of silver, these price data can be found in Manthy [19]. Silver price was obtained from the U.S. Bureau of Mines "Minerals Yearbook" [32].

[15] Bauxite is the only commodity considered for which the price-taking assumption for North American producers is highly suspect (because the aluminum industry is highly concentrated worldwide). For other commodities, cartels have been formed (CIPEC for copper and OPEC for petroleum, for example) but the principal actors have not been North American countries or companies. Commodities such as nickle and molybdenum where the dominant firm is North American (INCO and AMAX, respectively) were eliminated from consideration. Also eliminated were commodities such as tin that are not produced in North America. Finally, gold was not considered because its price was linked to the value of the dollar over a large portion of the sample period.

TABLE I
Statistics for Hypothesis Tests

Commodity	Test: t for $\alpha = 0$	g for $P = 1$	t for $\overline{\Delta P} = 0$
Copper	0.06	0.033	0.04
Pig iron	0.25	0.012	0.06
Lead	0.15	0.022	0.01
Bauxite	0.10	0.016	0.07
Silver	0.93	0.024	0.05
Petroleum	0.60	0.018	0.06
Coal	0.59	0.007	0.14
Average	0.38	0.019	0.06

where the symbol Δ stands for the first difference. Testing for $\alpha = 0$ in Eq. (27) is equivalent to testing for a unit root of a first-order autoregressive process. That is, Eq. (27) is equivalent to

$$P_t = \rho P_{t-1} + v_t = 1/(1 - \alpha) P_{t-1} + u_t/(1 - \alpha). \tag{28}$$

It is well known that there are no completely satisfactory methods of testing this hypothesis (for a discussion of the issues, see Evans and Savin [6]). Two admittedly imperfect tests are therefore reported.

The second column of Table I shows ordinary t statistics for the test of the hypothesis that $\alpha = 0$ in Eq. (27). A t value of 1.67 or larger would be grounds for rejecting the null hypothesis at the 95% confidence level. It is clear that the hypothesis of zero rate of growth for price is never rejected.

The second test, a test for $\rho = 1$ in Eq. (28), makes use of the g statistic developed by Berenbluth and Webb [1]. The g statistic is closely related to the Durbin–Watson statistic [5] and provides the locally most powerful invariant test in the region of $\rho = 1$. The third column of Table I shows the g statistic calculated for the series P_t. If g is less than 1.58, the hypothesis that $\rho = 1$ cannot be rejected

TABLE II
Descriptive Statistics for $\dot{P}/P$

Commodity	Mean (%) $= \hat{\alpha} \cdot 100$	Standard error (%) $= \hat{\sigma}/\overline{P} \cdot 100$
Copper	−0.13	17.4
Iron	0.48	16.2
Lead	0.32	17.6
Bauxite	−0.18	14.5
Silver	1.75	17.6
Petroleum	1.34	18.6
Coal	1.29	18.2
Average	0.70	17.2

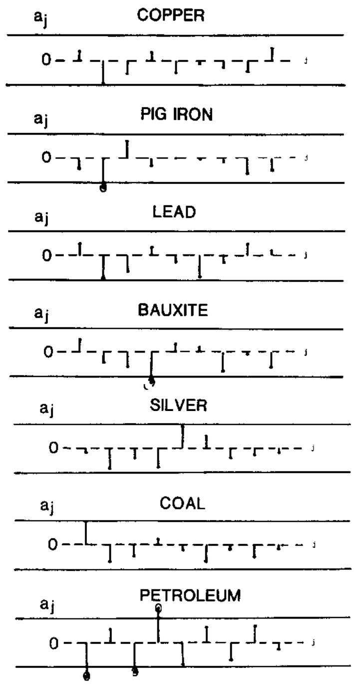

FIG. 2. Autocorrelation functions.

(at the 95% confidence level with 67 degrees of freedom). Once again, the hypothesis of zero rate of growth for price is never rejected.

It is possible to calculate descriptive statistics for $\dot{P}/P$ from Eq. (27). $\dot{P}/P$ has mean α and standard error σ/P. Table II shows estimates of these numbers. Because the standard error of $\dot{P}/P$ is nonconstant, it is evaluated at the mean of each price series. Both the mean and standard error are given in percentage terms.

Table II presents a uniform picture of the behavior of $\dot{P}/P$ (the rate of growth of price) for all commodities analyzed. Each series has a small or zero mean combined with a large standard error. Price uncertainty is seen to be a dominant factor whereas deterministic trends are small or negligible.

The martingale assumption is investigated further. Sufficient conditions for a discrete-time series to be a martingale are (1) first differences must have zero mean and (2) successive changes must be independent. To test these hypotheses, first differences were calculated and the means and autocorrelation functions for the differenced series, ΔP_t, were estimated.

Table I and Fig. 2 show the results of these estimations. The fourth column of Table I lists t statistics for the test of zero mean. In Fig. 2, a_j is the correlation coefficient between ΔP_t and ΔP_{t-j}. The dashed lines show the zero axes and the solid lines are the 95% confidence intervals. If all a_j lie within the solid lines, successive price changes are uncorrelated.

Examination of Fig. 2 reveals that the hypothesis that petroleum prices conform to a martingale must be rejected. One third of the estimated autocorrelation coefficients lie outside the 95% confidence bands (those circled). In addition, the a_j follow a regular pattern, alternating in sign.

In contrast to petroleum prices, the hypothesis that the stochastic processes that generate the remaining price series are martingales cannot be rejected. All easily pass the test of zero mean, and of the 54 estimated autocorrelation coefficients, only two are marginally significant. A priori, one would expect three coefficients to be significant at the 95% level of confidence. In addition, no regular pattern can be detected in the a_j's, such as cycling or alternating in sign. It can therefore be concluded that a martingale provides a reasonably close approximation to the stochastic process that generates these price series.[16]

In particular, the hypothesis that mineral-commodity prices are martingales conforms much more closely to the observed data than the hypothesis that these prices are generated by Hotelling-style models with no uncertainty. For example, Smith [29] considers the following equation (that can be derived by differentiating Eq. (3) with respect to time):

$$\Delta P_t/P_t = (P_t - c_t)r + (c_t/P_t)(\Delta c_t/c_t). \tag{29}$$

He fits variants of this equation to many of the price series used here and meets with little success.

VII. SUMMARY AND CONCLUSIONS

Grade heterogeneity is a ubiquitous feature of the mineral industries. When cost conditions are particularly simple (marginal costs are constant for a given deposit) it is well known that it is optimal to extract the least-cost ores first.[17] When, however, marginal extraction costs are increasing, many deposits may be extracted simultaneously and more complicated grade-selection rules arise. In particular, it is shown here that there are circumstances in which it is optimal to extract the least-cost deposits last. This possibility is seen to have implications for the way that we model depletion effects in aggregate extraction-cost functions.

Several grade-selection rules are derived that are consistent with dynamic profit maximization under conditions of certainty. None of these, however, provides a good approximation to the grade-selection patterns observed in practice. In particular, none can explain the coexistence of a long-run positive and short-run negative grade–price relationship.

A model of price uncertainty is introduced in an attempt to reconcile theory and practice. Price, which is exogenous to the firm, is generated by a stochastic differential equation. Testable parameter restrictions on this differential equation result in the price path being a martingale. The implications of the stochastic model are derived and the theoretical grade-selection patterns that result when the restrictions are imposed are seen to be consistent with observed regularities in the data.

The stochastic differential equation is used to assess price patterns. This equation nests two models of theoretical interest: a simple Hotelling model and an efficient-

[16] These results are not inconsistent with those obtained in Slade [25, 26]. The lag lengths used in the test of the martingale model are too short to detect the long-run regularities in price behavior that are discussed there.

[17] Even with the simplest cost conditions, the least-cost-first rule may not hold in a general-equilibrium context [13].

market model. Empirical support is given to the latter and it is shown that in the medium run (several decades) price uncertainty and volatility overwhelm any deterministic trends. The idea that the optimal forecast for future price is the current price cannot be rejected.

The notion that the optimal forecast for future mineral-commodity prices is the current price could explain other observed and puzzling behavior in the mineral industries. For example, overinvestment is a common phenomenon, as witnessed by current problems in the nickel and molybdenum industries.[18] One explanation for overinvestment is myopic behavior on the part of firms. Myopic behavior however is not rational unless static expectations are optimal (as is the case with martingales).

The model presented here is very simple and neglects many real-world complications. It is therefore especially surprising that its predictions are so accurate.

APPENDIX

In this Appendix, it is assumed that ore does not have a market price. Ore is processed by a vertically integrated firm to obtain metal, which is then sold in a market. These assumptions correspond to conditions in many mineral industries. Their implications for the model set out in this paper are explored here.

Let R^i be the rate of ore extraction from the ith deposit. Suppose that ore must be combined with a vector of variable inputs X to produce metal q. Define the ith processing-production function by

$$q = f^i(R^i, X).^{19} \tag{A1}$$

If variable inputs X sell at prices V, the firm's variable-profit function is defined by

$$\Pi(P, V, R) = \max_{X^1,\dots,X^m} \left\{ P\left(\Sigma f^i(R^i, X^i)\right) - V^T(\Sigma X^i) \right\} - \Sigma C_i(R^i, V), \tag{A2}$$

where P is metal price, R is a vector of ore-extraction rates (R^i), and X^i is a vector of variable inputs (X_j^i) used in processing ore from the ith deposit.

By the envelope theorem, the implicit or shadow price of ore from the ith deposit, w^i, is[20]

$$w^i = \partial\Pi/\partial R^i = Pf^i_{R^i}(R^i, X^{i*}) - c_i(R^i, V), \tag{A3}$$

where X^{i*} denotes the optimal value of X^i and $c_i = C_i'$. Equation (A3) implies that

$$\partial w^i/\partial P = f^i_{R^i}(R^i, X^{i*}) > 0. \tag{A4}$$

In general therefore there will be many shadow prices, one for each grade of ore.

[18] These industries however do not fit the model of Section V because the price-taking assumption is violated (see footnote 15).

[19] Including technical change in the production function will not alter the flavor of the results.

[20] The implicit or shadow price is the implied value of the extracted ore, not the value of ore in the ground ($\lambda_i e^{rt}$).

If we assume that deposits are numbered in order of increasing cost and that $f_{R^i}^i > f_{R^j}^j$ for $j > i$ (that more metal can be obtained from higher grade ore) then $w^1 > w^2 > \cdots w^m > 0$ and $\partial w^i/\partial P > \partial w^2/\partial P > \cdots > \partial w^m/\partial P > 0$. High-grade ores have higher shadow prices and an increase in P increases the shadow price for high-grade ores more than that for low-grade ores.

Because P and w^i are positively correlated, the empirical finding of a negative metal-price/yield relationship implies a negative ore-shadow-price/yield relationship and vice versa.

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