Bayesian Hierarchical MPT Modeling

Theory

Daniel W. Heck

Philipps Universität Marburg

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Bayesian Hierarchical MPT Modeling

1) MPT models & heterogeneity
2) Hierarchical MPT models
3) Bayesian estimation with MCMC sampling
4) Advantages of MCMC

MPT models & heterogeneity

Standard MPT models

Standard MPT models assume that . . .

- . . . people behave identically
- . . . items are similarly difficult
- Technical assumption
  - Fixed-effects model: Observations are "independent and identically" (i.i.d.) distributed
  - The likelihood of all observations $i = 1, \dots, n$ is the product of the likelihood of a single observation x_i

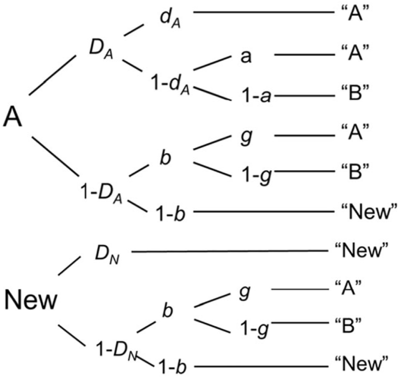
$$p(x_1, \dots, x_n \mid \theta) = \prod_{i=1}^{n} p(x_i \mid \theta)$$

What about real data?

Source-Monitoring Model

Source-Monitoring

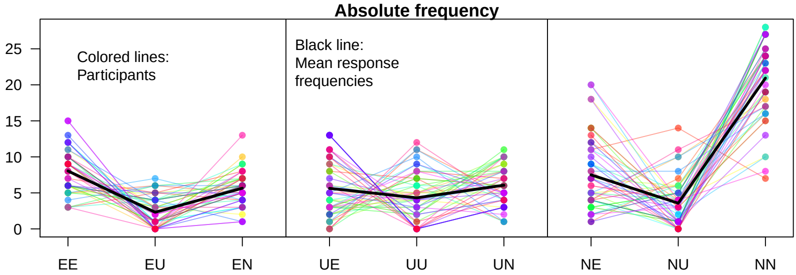
1) Study phase: List of words from Source A and B.
2) Test phase: Is the presented item from Source A/B/New?



People Behave Differently

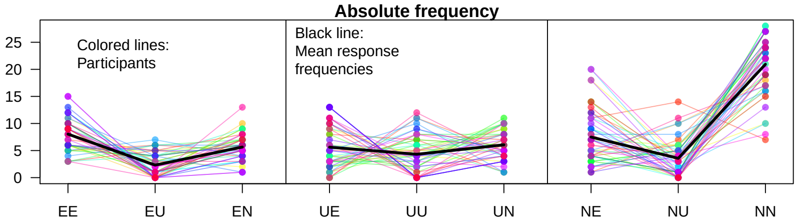
Distribution of individual response frequencies (Arnold et al., 2013)

1) Study phase: Words from Source E (= expected) and U (= unexpected)
2) Test phase: Is the presented item from Source E/U/New?



- Substantial variance in the choice patterns of participants
  - Differences in memory? Response bias?
- If we fit a standard MPT model to the aggregated data, these differences are ignored (treated as random, unsystematic noise)

People Behave Differently



Heterogeneity of participants

- Response frequencies are often aggregated across subjects
  - Dependent variable: Summed individual frequencies
- However, responses are likely not i.i.d.
  - Assumption can be tested statistically (Smith & Batchelder, 2008)
- Heterogeneity may result in biased statistical inference
  - Biased point estimates if parameter are correlated
  - Over-/underestimation of confidence intervals
  - Inflated model-fit statistics

How to Handle Heterogeneity?

a. **Complete pooling**: Analysis of aggregated frequencies
    - Ignores differences between persons
    - High power, but possibly biased statistical inference
b. **No pooling**: A separate MPT model per person
    - Low power, parameter estimates will have a large variance
    - Often, not enough data per participant
    - Problem: How to aggregate results across models?
c. **Partial pooling**: Hierarchical model
    - Account for differences AND similarities between persons jointly
    - Higher efficiency than separate analysis
    - Individual and group-level parameters inform each other

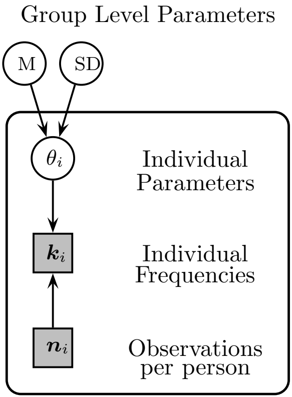
Note: This classification is very general and not limited to MPT models.

Hierarchical MPT models

Hierarchical MPT Models

Bayesian hierarchical MPT
(Klauer, 2010; Smith & Batchelder, 2010)

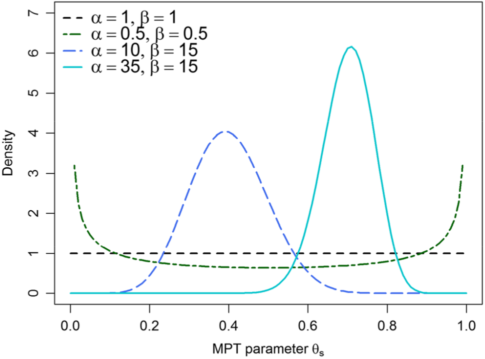
- Explicit model for participant heterogeneity
- Assumption: MPT structure holds for each person, but with different parameters!
- One parameter vector $\theta_i = (D_i, d_i, g_i, \dots)$ per person
- On the group level, the θ_i have a specific distribution
  - A) Beta-MPT: Beta distribution
  - B) Latent-trait MPT: multivariate normal distribution for the probit-transformed parameters



Beta-MPT

Beta distribution

- Ideally suited to model the distribution of an MPT parameter:
  - Allows values between 0 and 1
  - Two shape parameters: α and β
- On the group level, the mean for the MPT parameter equals: $\alpha/(\alpha + \beta)$



Beta-MPT

Beta-MPT (Smith & Batchelder, 2010)

Parameters:

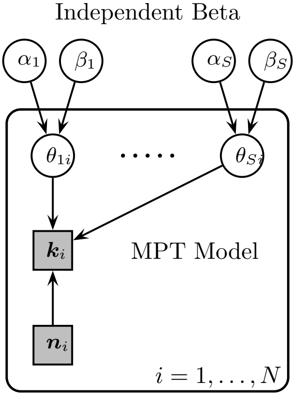
- Level-1: MPT parameters θ_{si} of person i
- Level-2: Shape parameters α_s and β_s of beta distributions

Data:

- k_i: Individual choice frequencies
- n_i: Number of responses per person

Priors:

- Uniform or gamma on α_s and β_s
- Truncation to $\alpha_s \geq 1$ and $\beta_s \geq 1$: Unimodal group-level distribution



Latent-Trait MPT

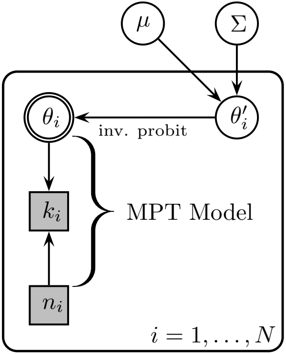
Latent-trait MPT (Klauer, 2010)

Parameters:

- Level-1: Person parameters are probit-transformed
  - $\theta_{si} = \Phi(\theta'_{si})$
  - Φ = cumulative density function of the standard normal
- Level-2: Probit-transformed parameters have a multivariate normal distribution
  - Mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ (on probit scale)

Prior distributions:

- Standard normal distributions for $\boldsymbol{\mu}$
- Scaled inverse-Wishart prior for $\boldsymbol{\Sigma}$



The Probit-Transformation

Transformation of MPT parameters

- We need to transform the probability parameters (d, g, $\ldots$)
- We want parameters between $(-\infty, +\infty)$ (to work with normal distributions)
- Solution: Transform parameters using the cumulative density function Φ of the standard-normal distribution (similar as in logistic regression)

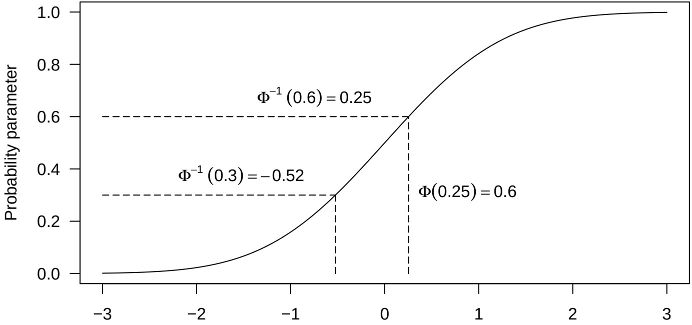
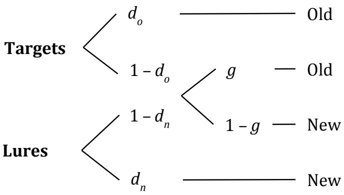


Illustration: Separate MPT Structure for each Person

Example: 2HTM for two persons

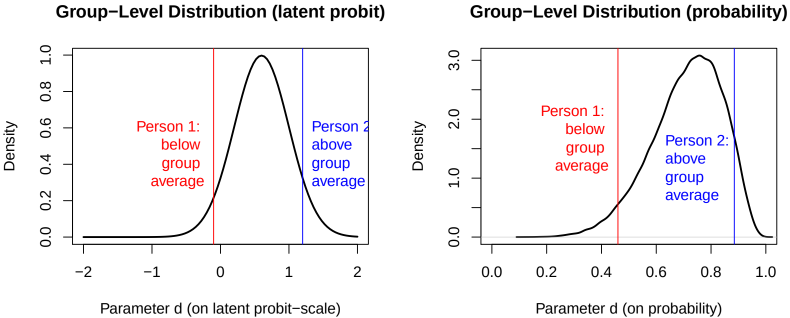
- Probit scores for memory parameter d are: $-.10$ and 1.20
- What is the predicted probability of correct OLD responses (hits)?
- We assume symmetric and identical guessing for everybody ($g = .50$)
- **Person 1**:
  1) Transform: $d = \Phi(-.10) = .46$
  2) MPT: $P(hit) = d + (1-d)g = .46 + (1-.46).50 = .73$
- **Person 1**:
  1) Transform: $d = \Phi(1.20) = .88$
  2) MPT: $P(hit) = d + (1-d)g = .88 + (1-.88).50 = .94$



Group Level: Normal Distribution

Assumption: Normal distribution of probit parameters

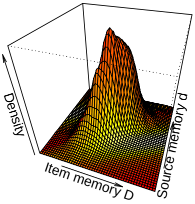
- Illustration: Normal distribution with mean $\mu_d = .80$ and standard deviation $\sigma_d = .3$
- For interpretation, it matters whether parameters are on the probit or the probability scale

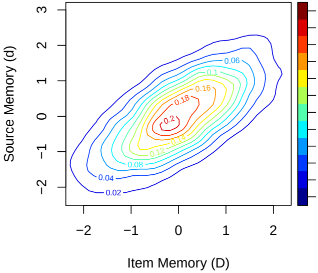


Comparison of Groups

Parameter correlations

- Item and source memory might be correlated (parameters g and d)
- "The more likely I remember the item, the more likely I also remember the source."
- Solution: Assumption that the vector θ_i' with probit-transformed MPT parameters follows a *multivariate* normal distribution
- Caveat: Correlation estimates are often very unprecise and require both large number of responses and large number of participants

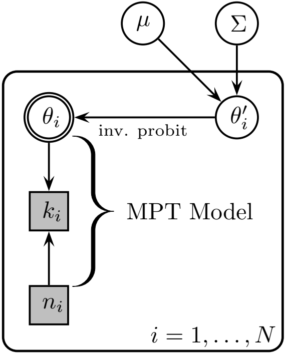




Summary: Hierarchical Models

Core ideas of hierarchical models

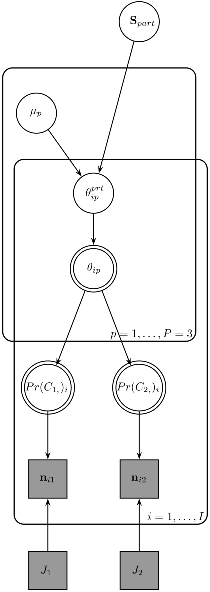
- Assume an MPT model with separate MPT parameters θ_i per person
- On the group-level, the parameters have a specific distribution
  1) Beta-MPT: Beta distribution
  2) Latent-trait MPT: multivariate normal distribution of probit-parameters with mean μ and covariance matrix Σ
  3) Other option (not discussed here): Discrete latent classes (Klauer, 2006)



Excursion: Graphical Models

Bayesian graphical models

- In publications, graphical models look more difficult
- Example: Matzke et al. (2015)
- However, most models use exactly the same ingredients



$\mathbf{S}_{part} \sim \text{Scaled} - \text{Inverse} - \text{Wishart}(\mathbf{W}, df = P + 1, \boldsymbol{\xi}_{part})$

$\xi_{part_p} \sim \text{Uniform}(0, 100)$

$\mu_p \sim \text{Normal}(0, 1)$

$\boldsymbol{\theta}_i^{prt} \sim \text{Multivariate} - \text{Normal}\left((\mu_1, \ldots, \mu_P), \mathbf{S}_{part}^{-1}\right)$

$\theta_{ip} = \phi\left(\theta_{ip}^{prt}\right)$

$Pr(C_{11})_i = \theta_{i1} \times \theta_{i2}$

$Pr(C_{12})_i = (1 - \theta_{i1}) \times \theta_{i3}^2$

$Pr(C_{13})_i = (1 - \theta_{i1}) \times 2 \times \theta_{i3} \times (1 - \theta_{i3})$

$Pr(C_{14})_i = \theta_{i1} \times (1 - \theta_{i2}) + (1 - \theta_{i1}) \times (1 - \theta_{i3})^2$

$Pr(C_{21})_i = \theta_{i3}$

$Pr(C_{22})_i = (1 - \theta_{i3})$

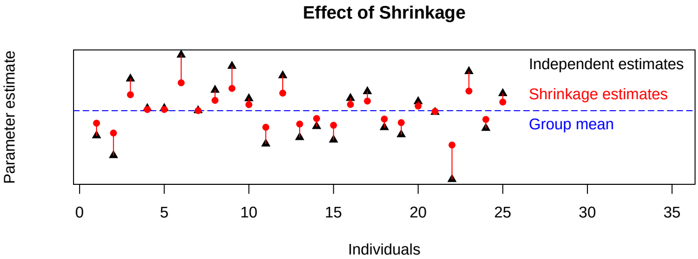
$\mathbf{n}_{i1} \sim \text{Multinomial}\left(Pr(C_{1,})_i, J_1\right)$

$\mathbf{n}_{i2} \sim \text{Multinomial}\left(Pr(C_{2,})_i, J_2\right)$

Some Advantages

Benefits of hierarchical MPT models

- Avoid aggregation biases
- "Shrinkage" of parameter estimates
  - Parameter estimates for each person are closer together compared to fitting each person separately
  - Hence, extreme estimates are less likely
  - Overall, this ensures that parameter estimates are closer to the true values on average
- The basic idea of hierarchical models can easily applied to any other model
  1) Assume that model holds for each person
  2) Specificy group-level distribution of parameters across persons



Bayesian estimation with MCMC

Fitting Hierarchical MPT Models

Parameter estimation

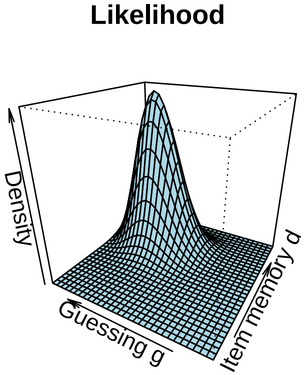
- How can we actually fit such models?
- Which are the “best” parameters given the data?
  - Standard MPT models: Maximum likelihood estimation
  - Not an option for hierarchical models (intractable likelihood function due to high-dimensional integrals)

Solution

- Hierarchical models are often fitted using Bayesian statistics

Maximum Likelihood

- Logic of parameter estimation with maximum-likelihood
  1) Define likelihood function $p(x \mid \theta)$
  2) Find parameters θ that maximize f
- Interpretation: “The estimator $\hat{\theta}$ has the highest likelihood.”
- Computational solution: Algorithm searches for the “top of the mountain”

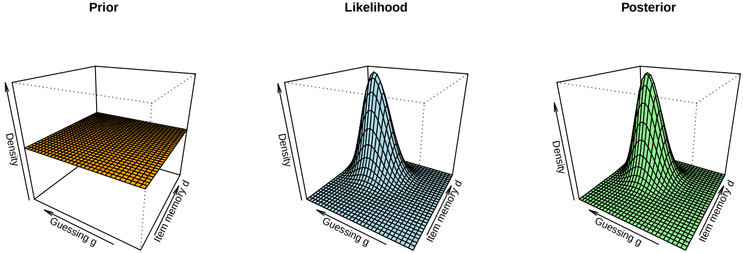


Bayesian Estimation

- Logic of Bayesian parameter estimation
  1) Define likelihood $p(x \mid \theta)$ and prior distribution $p(\theta)$
  2) Derive the posterior distribution of the parameters via Bayes' theorem:

$$p(\theta \mid x) = \frac{p(x \mid \theta)p(\theta)}{p(x)}$$

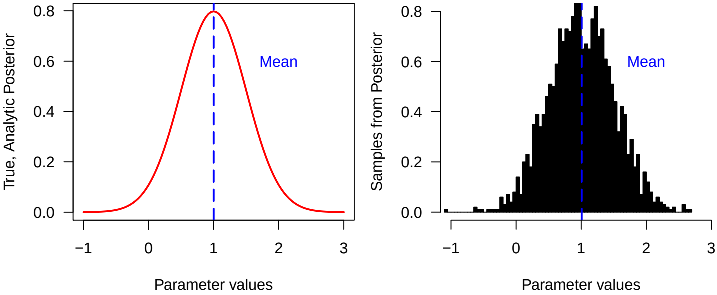
- Interpretation: "What have we learned about the parameters θ given the data x?"



Prior **Likelihood** **Posterior**

Bayesian Estimation

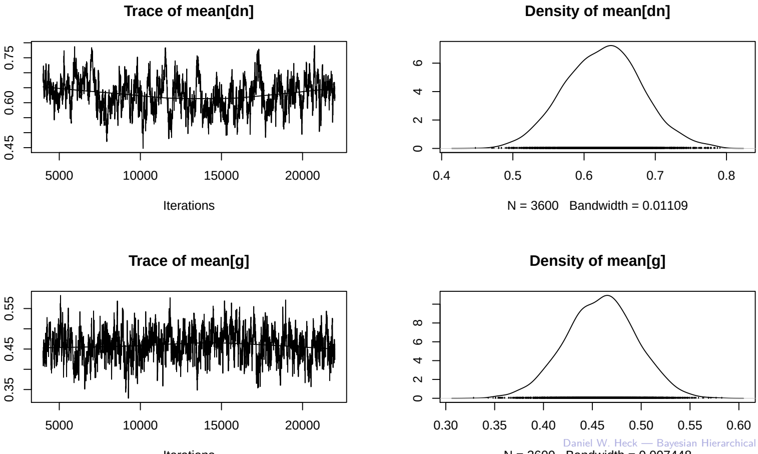
- Problem: We need to work with the posterior function $p(\theta \mid x)$
  - What is the mean/mode/95% credibility interval of θ?
  - Often, this is analytically not tractable
- Solution: We draw random samples from the posterior distribution
  - Logic: It is easier to draw conclusions from these random samples than deriving solutions for the analytical posterior (which is a function!)
  - Example: Computing the mean of a normal distribution requires to solve:



Bayesian Estimation

Markov Chain Monte Carlo (MCMC) Sampling

1. Draw random samples of the posterior distribution for *all* parameters (individual and group level)
2. Summarize parameter samples (e.g., mean, SD, density, . . .)



Bayesian Estimation

Markov chain Monte Carlo (MCMC)

- General method to draw posterior samples
- In a hierarchical model, there are many (!) parameters
  - Group-level means and covariances, person parameters, . . .
  - Intuitively, this method moves around and searches for parameter values with high posterior density
- There are software packages that draw random samples for many models of interest
  - JAGS, WinBUGS, OpenBUGS, Stan, . . .

Summary of Bayesian estimation

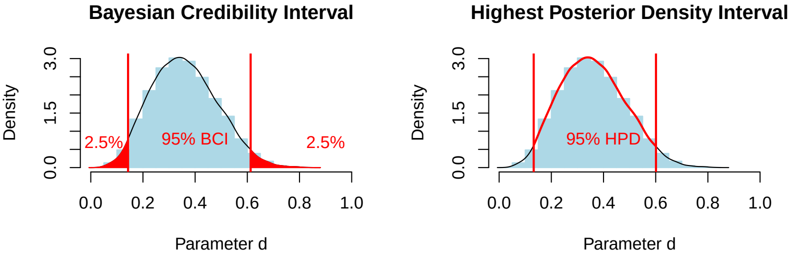
1) Develop a model (=> psychological theory, multiTree)
2) Get posterior (MCMC) samples (JAGS, TreeBUGS)
3) Summarize these samples (e.g., mean of group-level parameters μ_D, μ_g,. . .)

Advantages of MCMC

Advantages of MCMC: Uncertainty

Advantages of MCMC sampling

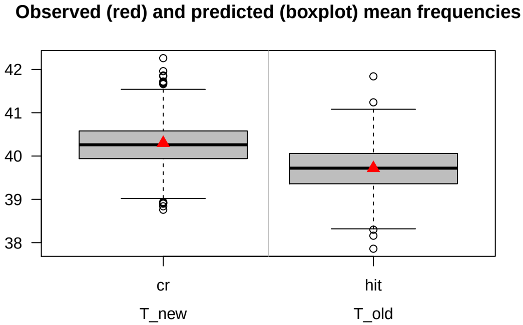
- Theoretical: No asymptotic assumptions about minimal sample size
- Practical: It is easy to quantify uncertainty
  - Bayesian credibility interval (BCI): What are the 2.5%- and 97.5%-quantiles of the parameter values?
  - Highest posterior density interval (HPD or HDI): What are the 95% most plausible parameter values?
  - For probability parameters, these intervals will always be in the interval $[0, 1]$



Advantages of MCMC: Model Fit

Does the model fit the data?

- Graphical comparison: observed vs. predicted frequencies
- Use posterior samples of the MPT parameters to sample new data (= posterior predictive)
- Compare whether these predicted data (boxplot) are in line with the observations (red points)



Summary

Hierarchical MPT Models

- Individual level: Assume separate MPT parameters for each person
- Group level
  - Beta-MPT: Beta distribution of person parameters
  - Latent-trait MPT: Normal distribution of probit-transformed parameters
- Bayesian model fitting: Draw posterior samples via MCMC

Appendix

Appendix: Standard vs. Hierarchical MPT Modeling

Currently open questions:

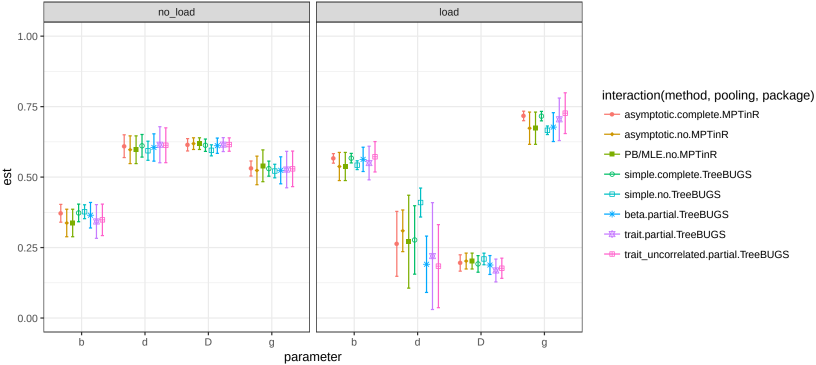
- How much do results actually differ when using different MPT model versions (standard, hierarchical, beta, latent-trait, . . .)?
- Which MPT model version should be used in practice?

Large-scale reanalysis project

- Network of MPT researchers (organized by Beatrice Kuhlmann & Julia Groß)
- Reanalysis of existing data sets to compare:
  - Fixed-effects vs. hierarchical
  - Maximum-likelihood vs. Bayes
  - Different hierarchical level-2 structures (beta, multiv. normal, independent univ. normal)
- Software: "A multiverse pipeline for MPT models"
  - Maximum likelihood: `MPTinR` (Henrik Singmann)
  - Bayes: `TreeBUGS`
  - Available at: https://github.com/mpt-network/MPTmultiverse

Appendix: Standard vs. Hierarchical MPT Modeling: Reanalysis

- Source-monitoring model (data by Bayen & Kuhlmann, 2011)
- Plot: Difference in parameters across two groups



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