Bayesian Hierarchical MPT Modeling
Theory

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STATISTICAL MODELING in PSYCHOLOGY
FREIBURG HEIDELBERG LANDAU MANNHEIM TÜBINGEN

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1) MPT models & heterogeneity
2) Hierarchical MPT models
3) Bayesian estimation with MCMC sampling
4) Advantages of MCMC
5) Application: Linking covariates to MPT parameters

MPT models & heterogeneity

Standard MPT models assume that . . .

- ... people behave identically
- ... items are similarly difficult
- Technical assumption
  - Fixed-effects model: Observations are "independent and identically" (i.i.d.) distributed
  - The likelihood of all observations $i = 1, \ldots, n$ is the product of the likelihood of a single observation x_i

$$p(x_1, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} p(x_i \mid \theta)$$
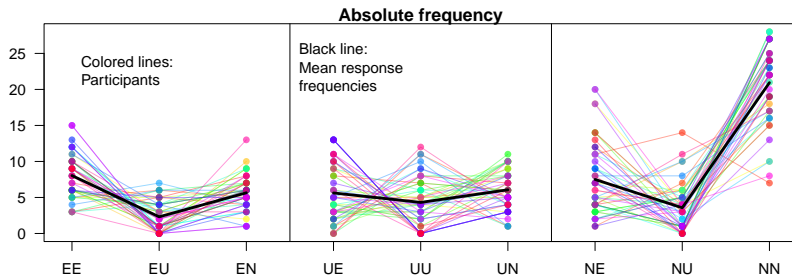
What about real data?

People Behave Differently

Source-monitoring task

1. Study phase: List of words from Source A and B.
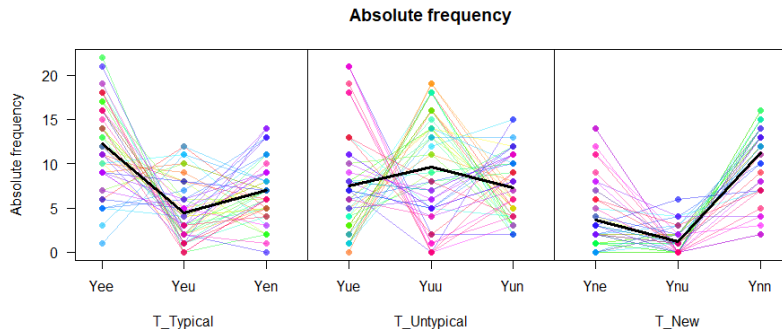2. Test phase: Is the presented item from Source A/B/New?

Distribution of individual response frequencies

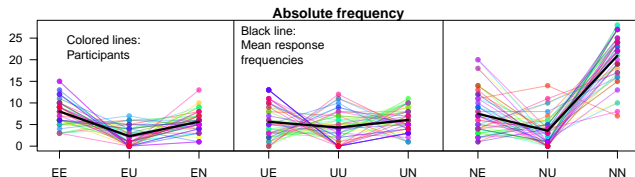- Example: Experiment on schema activation (Arnold et al., 2013)

Data from a different experiment (Bayen, 2011)

- ■ Substantial variance in the choice patterns of participants
  - ■ Differences in memory? Response bias?
- ■ If we fit a standard MPT model to the aggregated data, these differences are ignored (treated as random, unsystematic noise)

Absolute frequency

Heterogeneity of participants

- Response frequencies are often aggregated across subjects
    - Dependent variable: Summed individual frequencies
- However, responses are likely not i.i.d.
    - Assumption can be tested statistically (Smith & Batchelder, 2008)
- Heterogeneity may result in biased statistical inference
    - Biased point estimates if parameter are correlated
    - Over-/underestimation of confidence intervals
    - Inflated model-fit statistics

How to Handle Heterogeneity?

- **A** **Complete pooling**: Analysis of aggregated frequencies
  - Ignores differences between persons
  - High power, but possibly biased statistical inference
- **B** **No pooling**: A separate MPT model per person
  - Low power, parameter estimates will have a large variance
  - Often, not enough data per participant
  - Problem: How to aggregate results across models?
- **C** **Partial pooling**: Hierarchical model
  - Account for differences AND similarities between persons jointly
  - Higher efficiency than separate analysis
  - Individual and group-level parameters inform each other

Note: This classification is very general and not limited to MPT models.

Hierarchical MPT models

Hierarchical MPT Models

Bayesian hierarchical MPT
(Klauer, 2010; Smith & Batchelder, 2010)

- Explicit
  model for participant heterogeneity
- Assumption:
  MPT structure holds for each
  person, but with different parameters!
- One parameter
  vector $\theta_i = (D_i, d_i, g_i, \dots)$ per person
- On the group
  level, the θ_i have a specific distribution
  - [A] Beta-MPT: Beta distribution
  - [B] Latent-trait MPT:
    multivariate normal distribution
    for the probit-transformed parameters

Group Level Parameters

Individual
Parameters

Individual
Frequencies

Observations
per person
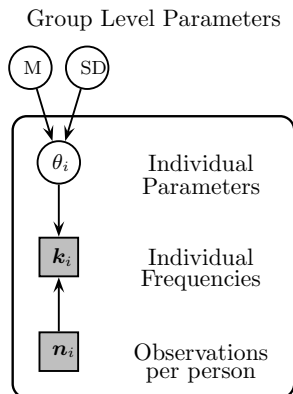
Beta-MPT

Beta distribution

- Ideally suited to model the distribution of an MPT parameter:
    - Allows values between 0 and 1
    - Two shape parameters: α and β
- On the group level, the mean for the MPT parameter equals: $\alpha/(\alpha + \beta)$
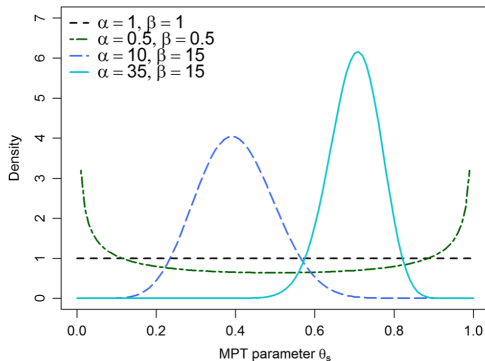
Beta-MPT
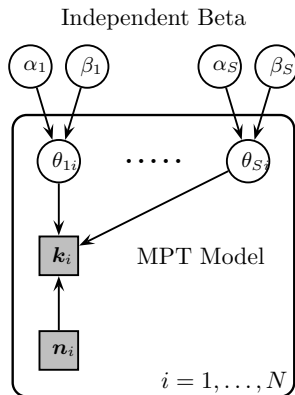
Beta-MPT (Smith & Batchelder, 2010)

Parameters:

- Level-1: MPT parameters θ_{si} of person i
- Level-2: Shape parameters
  α_s and β_s of beta distributions

Data:

- k_i: Individual choice frequencies
- n_i: Number of responses per person

Priors:

- Uniform or gamma on α_s and β_s
- Truncation to $\alpha_s \geq 1$ and $\beta_s \geq 1$:
  Unimodal group-level distribution

Independent Beta

Latent-Trait MPT

Latent-trait MPT (Klauer, 2010)

Parameters:

- Level-1: Person
  parameters are probit-transformed
  - $\theta_{si} = \Phi(\theta'_{si})$
  - Φ = cumulative density function of the standard normal
- Level-2: Probit-transformed parameters
  have a multivariate normal distribution
  - Mean μ and
   covariance matrix Σ (on probit scale)

Prior distributions:

- Standard normal distributions for μ
- Scaled inverse-Wishart prior for Σ
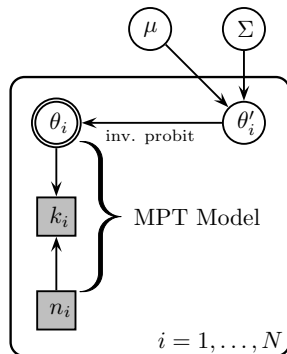
# The Probit-Transformation

Transformation of MPT parameters

- We need to transform the probability parameters (d, g, . . .)
- We want parameters between $(-\infty, +\infty)$ (to work with normal distributions)
- Solution: Transform parameters using the cumulative density function Φ of the standard-normal distribution (similar as in logistic regression)
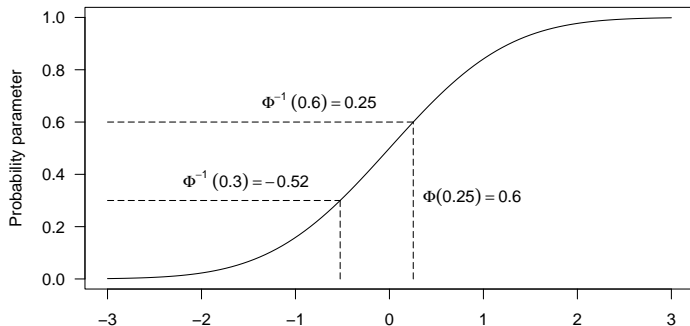
Illustration: Separate MPT Structure for each Person

Example: 2HTM for two persons

- Probit scores for memory parameter d are: $-.10$ and 1.20
- What is the predicted probability of correct OLD responses (hits)?
- We assume symmetric and identical guessing for everybody ($g = .50$)
- **Person 1**:
  1. Transform: $d = \Phi(-.10) = .46$
  2. MPT: $P(hit) = d + (1 - d)g = .46 + (1 - .46).50 = .73$
- **Person 1**:
  1. Transform: $d = \Phi(1.20) = .88$
 2. MPT: $P(hit) = d + (1 - d)g = .88 + (1 - .88).50 = .94$
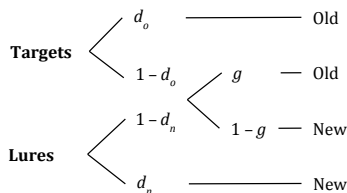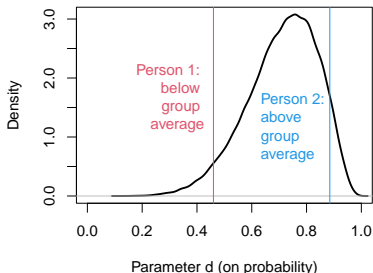
Group Level: Normal Distribution

Assumption: Normal distribution of probit parameters

- Illustration: Normal distribution with mean $\mu_d = .80$ and standard deviation $\sigma_d = .3$
- For interpretation, it matters whether parameters are on the probit or the probability scale

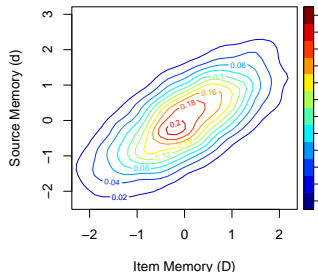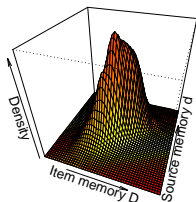**Group–Level Distribution (latent probit)**

Person 1: below group average

Person 2: above group average

Parameter d (on latent probit–scale)
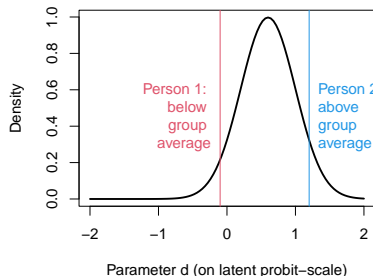
Group–Level Distribution (probability)

Person 1: below group average

Person 2: above group average

Parameter d (on probability)

Parameter correlations

- Item and source memory might be correlated (parameters g and d)
- "The more likely I remember the item, the more likely I also remember the source."
- Solution: Assumption that the vector θ_i' with probit-transformed MPT parameters follows a *multivariate* normal distribution
- Caveat: Correlation estimates are often very unprecise and require both large number of responses and large number of participants

# Summary: Hierarchical Models

Core ideas of hierarchical models

- Assume an MPT model with separate MPT parameters θ_i per person
- On the group-level, the parameters have a specific distribution
  1) Beta-MPT: Beta distribution
  2) Latent-trait MPT: multivariate normal distribution of probit-parameters with mean μ and covariance matrix Σ
  3) Other option (not discussed here): Discrete latent classes (Klauer, 2006)
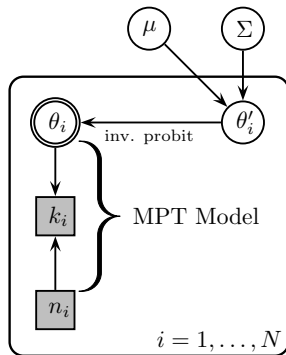
Excursion: Graphical Models

Bayesian graphical models

- In publications, graphical models look more difficult
- Example: Matzke et al. (2015)
- However, most models use exactly the same ingredients
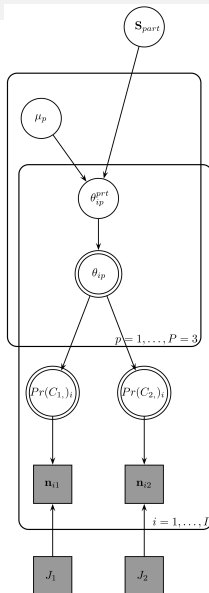
$\mathbf{S}_{part} \sim \text{Scaled} - \text{Inverse} - \text{Wishart}\left(\mathbf{W}, df = P + 1, \boldsymbol{\xi}_{part}\right)$

$\xi_{part_p} \sim \text{Uniform}(0, 100)$

$\mu_p \sim \text{Normal}(0, 1)$

$\boldsymbol{\theta}_i^{prt} \sim \text{Multivariate} - \text{Normal}\left((\mu_1, \ldots, \mu_P), \mathbf{S}_{part}^{-1}\right)$

$\theta_{ip} = \phi\left(\theta_{ip}^{prt}\right)$

$Pr(C_{11})_i = \theta_{i1} \times \theta_{i2}$

$Pr(C_{12})_i = (1 - \theta_{i1}) \times \theta_{i3}^2$

$Pr(C_{13})_i = (1 - \theta_{i1}) \times 2 \times \theta_{i3} \times (1 - \theta_{i3})$

$Pr(C_{14})_i = \theta_{i1} \times (1 - \theta_{i2}) + (1 - \theta_{i1}) \times (1 - \theta_{i3})^2$

$Pr(C_{21})_i = \theta_{i3}$

$Pr(C_{22})_i = (1 - \theta_{i3})$

$\mathbf{n}_{i1} \sim \text{Multinomial}\left(Pr(C_{1\cdot})_i, J_1\right)$

$\mathbf{n}_{i2} \sim \text{Multinomial}\left(Pr(C_{2\cdot})_i, J_2\right)$

Some Advantages

Benefits of hierarchical MPT models

- Avoid aggregation biases
- "Shrinkage" of parameter estimates
    - Parameter estimates for each person are closer together compared to fitting each person separately
    - Hence, extreme estimates are less likely
    - Overall, this ensures that parameter estimates are closer to the true values on average
- The basic idea of hierarchical models can easily applied to any other model
    1) Assume that model holds for each person
    2) Specificy group-level distribution of parameters across persons

Effect of Shrinkage

Bayesian estimation with MCMC
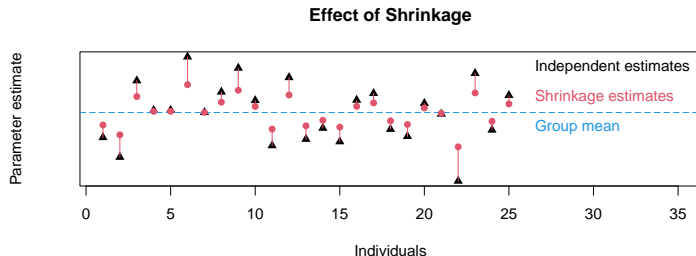
Fitting Hierarchical MPT Models

Parameter estimation

- How can we actually fit such models?
- Which are the "best" parameters given the data?
    - Standard MPT models: Maximum likelihood estimation
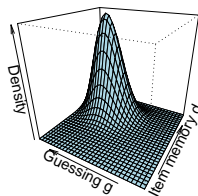   - Not an option for hierarchical models (intractable likelihood function due to high-dimensional integrals)

Solution

- Hierarchical models are often fitted using Bayesian statistics

Maximum Likelihood

- Logic of parameter estimation with maximum-likelihood
    1) Define likelihood function $p(x \mid \theta)$
    2) Find parameters θ that maximize f
- Interpretation: "The estimator $\hat{\theta}$ has the highest likelihood."
- Computational solution: search for the "top of the mountain"
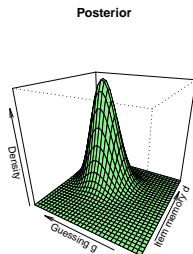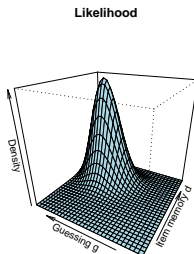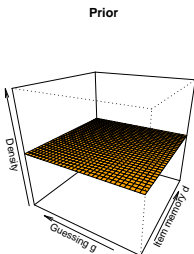
Likelihood

Bayesian Estimation

- Logic of Bayesian parameter estimation
  1) Define likelihood $p(x \mid \theta)$ and prior distribution $p(\theta)$
  2) Derive the posterior distribution of the parameters via Bayes' theorem:

$$p(\theta \mid x) = \frac{p(x \mid \theta)p(\theta)}{p(x)}$$

- Interpretation: "What have we learned about the parameters θ given the data x?"

Bayesian Estimation

- Problem: We need to work with the posterior function $p(\theta \mid x)$
  - What is the mean/mode/95% credibility interval of θ?
  - Often, this is analytically not tractable
- Solution: We draw random samples from the posterior distribution
  - Logic: It is easier to draw conclusions from these random samples than deriving solutions for the analytical posterior (which is a function!)
  - Example: Computing the mean of a normal distribution requires to solve:

Bayesian Estimation
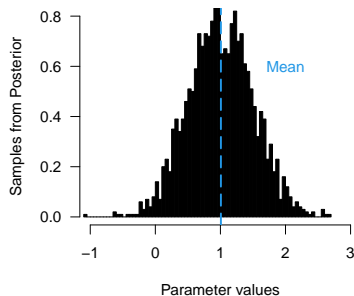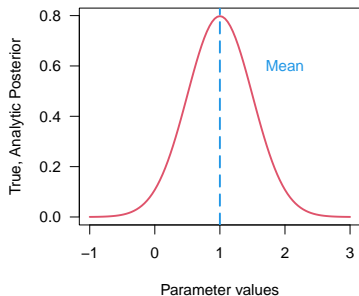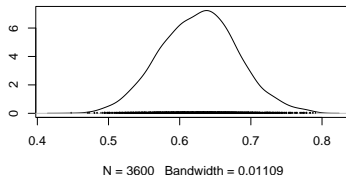
Markov Chain Monte Carlo (MCMC) Sampling

1. Draw random samples of the posterior distribution for *all* parameters (individual and group level)
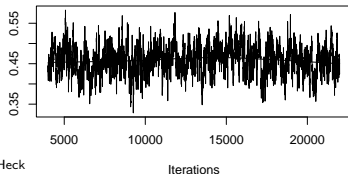2. Summarize parameter samples (e.g., mean, SD, density, . . .)
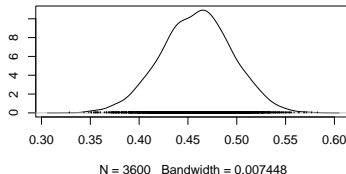
Trace of mean[dn]

Density of mean[dn]

N = 3600   Bandwidth = 0.01109
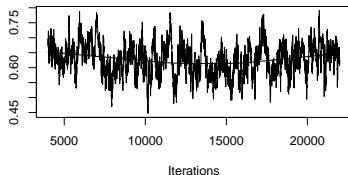
Trace of mean[g]

Density of mean[g]

Iterations

N = 3600   Bandwidth = 0.007448

Bayesian Estimation

Markov chain Monte Carlo (MCMC)

- General method to draw posterior samples
- In a hierarchical model, there are many (!) parameters
  - Group-level means and covariances, person parameters, . . .
  - Intuitively, this method moves around and searches for parameter values with high posterior density
- There are software packages that draw random samples for many models of interest
  - JAGS, WinBUGS, OpenBUGS, Stan, . . .

Summary of Bayesian estimation

1. Develop a model ($=>$ psychological theory, multiTree)
2. Get posterior (MCMC) samples (JAGS, TreeBUGS)
3. Summarize these samples (e.g., mean of group-level parameters μ_D, μ_g, . . .)

Advantages of MCMC

Advantages of MCMC sampling

- Theoretical:
  - No asymptotic assumptions
  - Maximum likelihood: requires a sufficient number of observations
- Practical: It is easy to quantify uncertainty
  - Bayesian credibility interval (BCI): What are the 2.5%- and 97.5%-quantiles of the parameter values?
  - Highest posterior density interval (HPD or HDI): What are the 95% most plausible parameter values?
  - For probability parameters, these intervals will always be in the interval $[0, 1]$
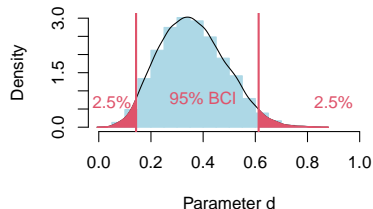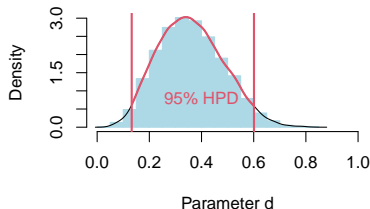
Advantages of MCMC: Transformed Parameters

- Often, we are interested in parameter/group comparisons
    - Example: Do healthy controls vs. schizophrenics differ in memory?
    - Test: Does the group-mean parameter μ_D differ?
- Based on MCMC samples, we can directly estimate functions of the parameters

MCMC estimation of transformed parameters

1. Draw MCMC samples
2. Compute transformed parameters for all samples
    - Example: $\delta^{(t)} = \theta_1^{(t)} - \theta_2^{(t)}$
3. Summarize the new values
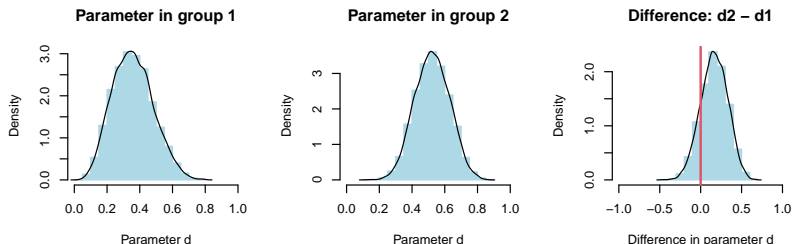
Advantages of MCMC: Model Fit

Does the model fit the data?

- Graphical comparison: observed vs. predicted frequencies
    - Use posterior samples of the MPT parameters to sample new data (= posterior predictive)
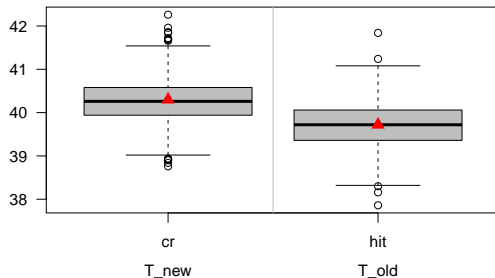   - Compare whether these predicted data (boxplot) are in line with the observations (red points)

Observed (red) and predicted (boxplot) mean frequencies

Advantages of MCMC: Model Fit

How to quantify model fit for MPT models?

- Test statistic similar to Pearson's X^2 statistic (Klauer, 2010)
  - T1 statistic: Mean structure of frequences
  - T2 statistic: Covariance matrix of frequences
- Posterior predictive p-value (PPP) measures model fit:
  1) Compute T1 for the observed data
  2) Compute T1 for the posterior predicted data
  3) PPP = probability that T1(predicted) is larger than T1(observed)
- Ideally, PPP should be around .50

Application

Linking Covariates to MPT Parameters

**Interindividual differences**

- Personality
  as a predictor for MPT parameters
- Statistical approach in latent-trait
  MPT: Similar to logistic regression

$$\theta_i = \Phi(\mu + \boxed{\beta \cdot x_i} + \delta_i)$$

MPT (y-axis label)

Predictor: Personality trait

Cognitive Psychometrics (Riefer et al., 2002)

- Using cognitive (MPT) models to learn about interindividual differences

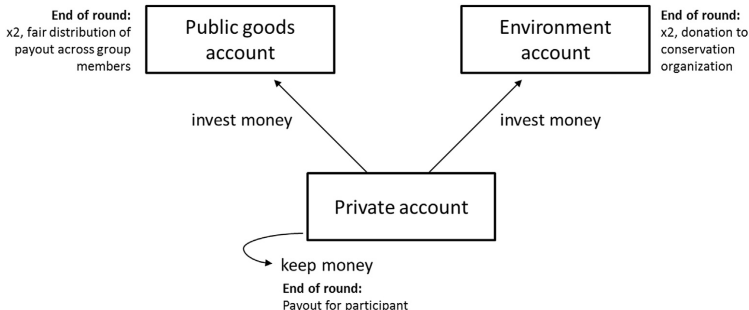**Example: Linking personality to MPT parameters**
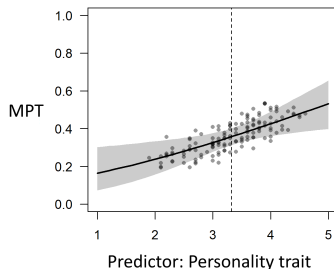
- "Which is the greater good? A social dilemma paradigm disentangling environmentalism and cooperation"
  - Klein, Hilbig, & Heck (2017). *Journal of Environmental Psychology*)
- Research question: How can we distinguish between 3 types of behavior?
  - Pro-environmental behavior
  - Pro-social behavior
  - Selfish behavior

Application: The Greater Good Game

- Greater Good Game
    - Participants decide whether to keep the money for themselves or contribute it to either a public goods or an environment account.
    - Important: Participants are forced to decide between the group and the environment!
    - The game is a variant of a nested public goods game

End of round:
x2, fair distribution of payout across group members

Public goods account

Environment account

End of round:
x2, donation to conservation organization

invest money

invest money

Private account

keep money
End of round:
Payout for participant

Application: MPT Model

MPT model for the Greater Good Game

- $s =$ probability of selfish behavior
- $e =$ probability of pro-environmental behavior
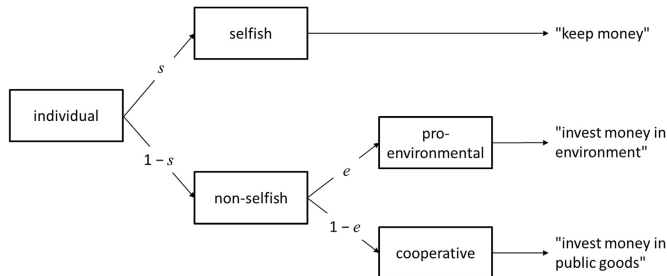
Results

- Honesty Humility ($=$ sincerity, fairness) is associated with less selfish behavior
- Selfish behavior decreases from 33.4% to 13.9% for participants $-1/+1$ SD on Honesty Humility

Summary

Hierarchical MPT Models

- Individual level
 - Assume a separate MPT model for each person
- Group level
  - Beta-MPT: Beta distribution of person parameters
  - Latent-trait MPT: Normal distribution of probit-transformed parameters
- Bayesian model fitting: Drawing posterior samples

Appendix & References

Source-Monitoring

1) Study phase: List of words from Source A and B.
2) Test phase: Is the presented item from Source A/B/New?
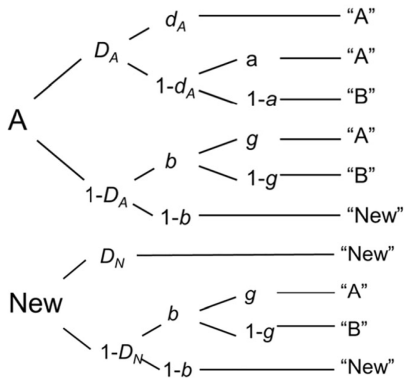
Appendix B: Meta-Analysis of Raw Data

- Linking process and measurement models of recognition-based decisions (Heck & Erdfelder, 2017, PsychReview)
- Reanalysis of about 400,000 decisions
  - 3-level hierarchical latent-trait MPT:

$$\theta_{sij} = \Phi(\mu_s + \boxed{\xi_{sj}} + \delta_{si})$$

  - Overall mean of MPT parameters (μ_s)
  - Participants nested in studies (random effect: ξ_{sj})
  - Responses nested in participants (random effect: δ_{si})

Relative speed of latent processes (N=1074)

Appendix C: Large-Scale Reanalysis
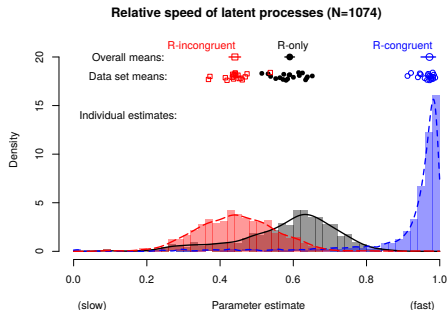
Open questions:

- How much do results actually differ between different MPT versions?
- Which MPT version should be used in practice?

Large-scale reanalysis project

- Network of MPT researchers (organized by Beatrice Kuhlmann & Julia Groß)
- Reanalysis of existing data sets to compare:
    - Fixed-effects vs. hierarchical
    - Maximum-likelihood vs. Bayes
    - Different hierarchical level-2 structures
      (beta, multiv. normal, independent univ. normal)
- Software: "A multiverse pipeline for MPT models"
    - Maximum likelihood: `MPTinR` (Henrik Singmann)
    - Bayes: `TreeBUGS`
    - Available at: https://github.com/mpt-network/MPTmultiverse

Appendix C: Reanalysis with Different Models

- Source-monitoring model (data by Bayen & Kuhlmann, 2011)
- Plot: Difference in parameters across two groups

References
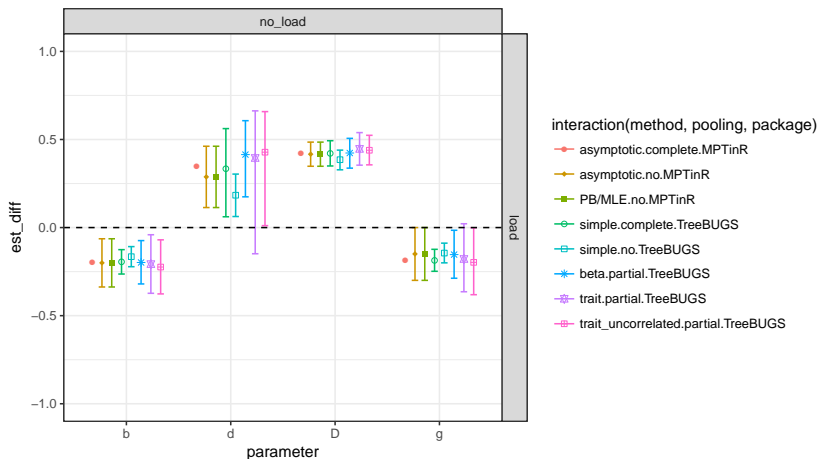
- TreeBUGS and a simple introduction to hierarchical MPT models
  - Heck, D. W., Arnold, N. R., & Arnold, D. (in press). TreeBUGS: An R package for hierarchical multinomial-processing-tree modeling. Behavior Research Methods. https://doi.org/10.3758/s13428-017-0869-7
- The latent-trait model (very technical)
  - Klauer, K. C. (2010). Hierarchical multinomial processing tree models: A latent-trait approach. Psychometrika, 75, 70–98. https://doi.org/10.1007/s11336-009-9141-0
- The latent-trait model with crossed-random effects and a JAGS implementation
  - Matzke, D., Dolan, C. V., Batchelder, W. H., & Wagenmakers, E.-J. (2015). Bayesian estimation of multinomial processing tree models with heterogeneity in participants and items. Psychometrika, 80, 205–235. https://doi.org/10.1007/s11336-013-9374-9

References

- Alternative hierarchical group structure of parameters
  - Smith, J. B., & Batchelder, W. H. (2010). Beta-MPT: Multinomial processing tree models for addressing individual differences. Journal of Mathematical Psychology, 54, 167–183. https://doi.org/10.1016/j.jmp.2009.06.007
- Benefits of hierarchical cognitive models
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