Mixture Models for Continuous Data

Generalized Processing Tree Models

Daniel W. Heck

STATISTICAL MODELING in PSYCHOLOGY
FREIBURG HEIDELBERG LANDAU MANNHEIM TÜBINGEN

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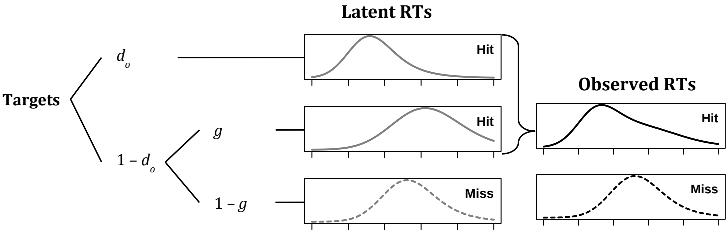
MPT Modeling with Continuous Data

1. MPT-RT: Modeling response times with histograms
   - Heck & Erdfelder (2016)
2. GPT (generalized processing tree): Parametric modeling
   - Heck, Erdfelder, & Kieslich (in press)
3. RT-MPT: Serial-process model for response times
   - Klauer & Kellen (2018)

MPT Models and Continuous Variables

Mixture distribution

- All of the MPT extensions assume mixture distributions for discrete and continuous observations
  1. Latent RTs: Different processing branches of the MPT model result in different latent distributions $g_j(t)$
  2. Observed RTs: A mixture distribution, defined as $f(t) = \sum_j p_j g_j(t)$
  3. The mixture weights p_j are determined by the MPT structure ($=$ branch probabilities)

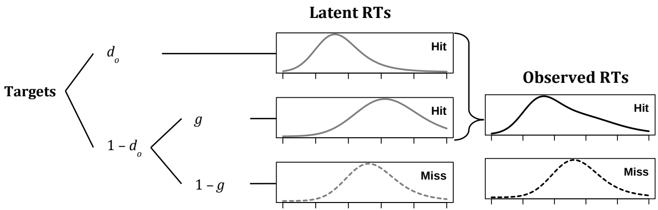


Generalized Processing Tree Models

- Heck, D. W., Erdfelder, E., & Kieslich, P. J. (in press). Generalized processing tree models: Jointly modeling discrete and continuous variables. *Psychometrika.*

Generalized processing tree (GPT) models

- Main difference: Parametric assumptions for component distributions
- The type of distribution depends on continuous variable
    - RTs: log-normal, ex-Gaussian, ...
    - Mouse-tracking measures (see below): Normal distribution
    - Neuro-psychological measures: ...
- The distributions are described by parameters $\boldsymbol{\eta}$
    - Normal distribution: mean and SD
    - ex-Gaussian: mean, SD, and mean of exponential



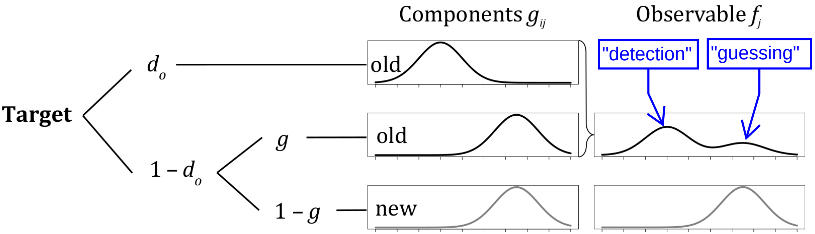
Benefits of the GPT Framework

1. Increased precision in estimating MPT parameters $\boldsymbol{\theta}$
2. Unidentifiable MPT models can become identifiable
3. Flexibility and simplicity

GPTs: Increased Precision

Higher precision of MPT-parameter estimates in GPTs

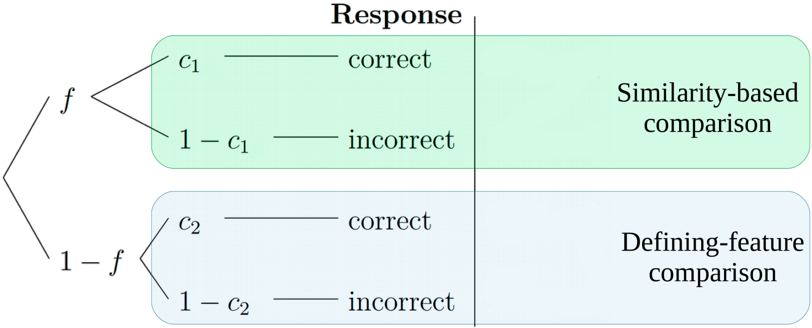
- The more distinct the latent distributions, the smaller the standard error
- Intuition: Continuous variables improve the “classification” which trials belong to which latent processing states
- 2HTM: “Fast RTs are due to detection, slow RTs are due to guessing”



Identifiability of GPT Models

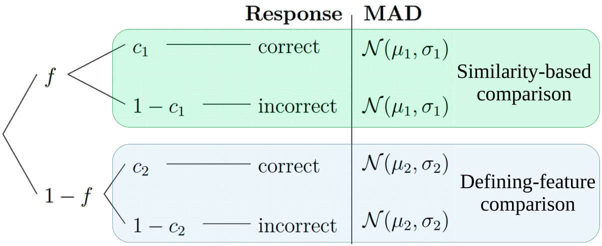
Example: The feature comparison model of semantic categorization

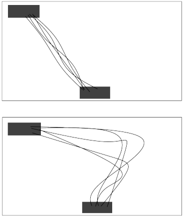
- The theory assumes two different processes
- f = probability of Process 1 (similarity-based comparison)
- c_1 = accuracy of similarity-based comparison
- c_2 = accuracy of defining-feature comparison
- With discrete responses only, the (MPT) model is not identifiable
    - only 1 free category for 3 free parameters



Identifiability of the feature comparison model

- Solution: Assume Gaussian component distributions for continuous variable
  - MAD = maximum absolute deviation in mouse-tracking
- Order constraint for mean parameters: $\mu_c < \mu_d$
  - Interpretation: More direct trajectories/small MADs for similarity-based comparison
- With both discrete *and* continuous data, model is identifiable
  - The different component distribution allow to disentangle the two processes





Flexibility: GPT Model Specification

GPTs as general-purpose measurement models

- GPTs can be specified as easily as MPTs
  - Implemented in the R package gpt
  - Currently under development: https://github.com/danheck/gpt
  - Define model in a text file similar to EQN
  - Type of latent distribution(s) defined within R (e.g., `latent="normal"`)

GPT version of 2-high-threshold model

```
# Tree ; Categ. ; MPT equation ; mean, SD (normal distr.)
target ; hit    ;  d           ; m_d,  sig
target ; hit    ; (1-d)*g      ; m_g,  sig
target ; miss   ; (1-d)*(1-g)  ; m_g,  sig

lure   ; cr     ;  d           ; m_d,  sig
lure   ; fa     ; (1-d)*g      ; m_g,  sig
lure   ; cr     ; (1-d)*(1-g)  ; m_g,  sig
```

Illustration of the gpt Package

```r
library("gpt")
# data from 2(response bias) x 2(memory strength) design:
# labels: "o30s_cr" = 30% old items / strong memory / correct rejection
head(heck2016, 3)
```

```
##       cat   rt
## 1 o30s_cr 1123
## 2 o30s_cr  671
## 3 o30s_cr  728
```

```r
modelfile <- "models/2htm_exgauss_2x2.txt"
# first lines:
```

```
## # 30% old / strong memory
## lure_s30;     o30s_cr   ;  (1-dn_s)*(1-g30) ; mu,sig,lambda_g_new30
## lure_s30;     o30s_cr   ;  dn_s             ; mu,sig,lambda_dn_s
## lure_s30;     o30s_fa   ;  (1-dn_s)*g30     ; mu,sig,lambda_g_old30
##
## target_s30;   o30s_hit  ;   do_s            ; mu,sig,lambda_do_s
## target_s30;   o30s_hit  ;  (1-do_s)*g30     ; mu,sig,lambda_g_old30
## target_s30;   o30s_miss ;  (1-do_s)*(1-g30) ; mu,sig,lambda_g_new30
##
## # 30% old / weak memory
## lure_w30;     o30w_cr   ;  (1-dn_w)*(1-g30)  ; mu,sig,lambda_g_new30
## lure_w30;     o30w_cr   ;  dn_w              ; mu,sig,lambda_dn_w
## lure_w30;     o30w_fa   ;  (1-dn_w)*g30      ; mu,sig,lambda_g_old30
```

gpt Package: Model Fitting

```r
fit <- gpt_fit(x = "cat",          # MPT category
               y = "rt",           # name of continuous variable(s)
               data = heck2016,    # example data for 1 person
               file = modelfile,   # GPT model file
               latent="exgauss",   # family of latent RT distributions
               restrictions=list("dn_s=do_s", "dn_w=do_w"))
fit
```

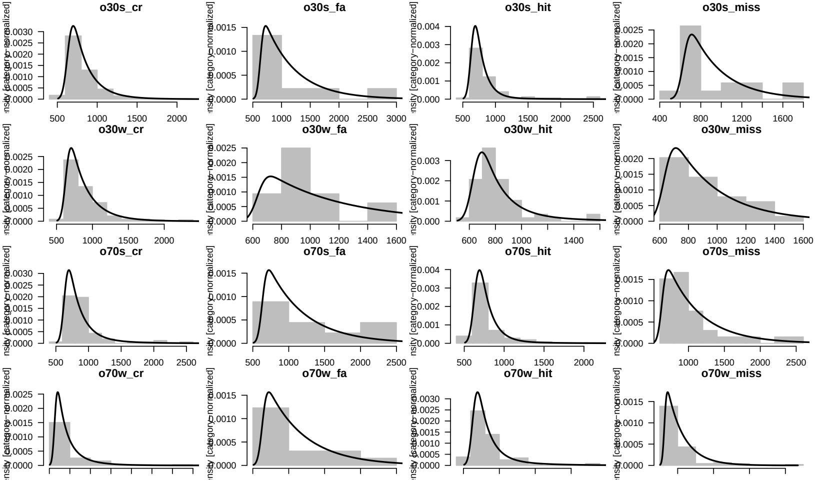
```
##                 Estimate     SE CI.lower CI.upper
## dn_s               0.741  0.027    0.687    0.794
## dn_w               0.477  0.033    0.411    0.542
## g30                0.189  0.031    0.128    0.250
## g70                0.261  0.038    0.186    0.335
## lambda_dn_s      172.252 18.192  136.597  207.907
## lambda_dn_w      192.571 35.080  123.815  261.327
## lambda_do_s      128.099 14.723   99.244  156.955
## lambda_do_w      151.588 18.455  115.417  187.760
## lambda_g_new30   311.057 30.895  250.504  371.610
## lambda_g_new70   460.131 36.179  389.221  531.042
## lambda_g_old30   531.286 86.541  361.669  700.903
## lambda_g_old70   516.769 74.457  370.837  662.702
## mu               633.896  5.376  623.360  644.432
## sig               49.279  4.047   41.347   57.212
```

```r
test_fit(fit, bins = 4)$test     # Dzhaparidze-Nikulin statistic
```

```
##   statistic df      p.value
## 1  83.46215 42 0.0001469729
```

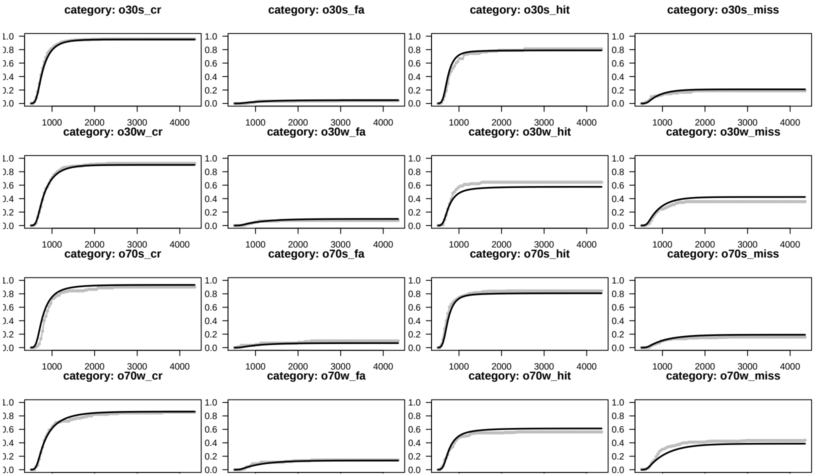
gpt Package: Model Fit

```r
hist(fit)
```



gpt Package: Model Fit

```
plot(fit)   # cumulative densities
```

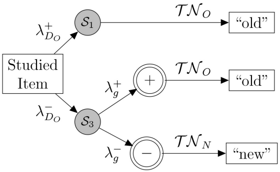


RT-MPT: Serial-process modeling of RTs

- Klauer, K. C., & Kellen, D. (2018). RT-MPTs: Process models for response-time distributions based on multinomial processing trees with applications to recognition memory. *Journal of Mathematical Psychology, 82*, 111–130.

RT-MPT Models (Klauer & Kellen, 2018)

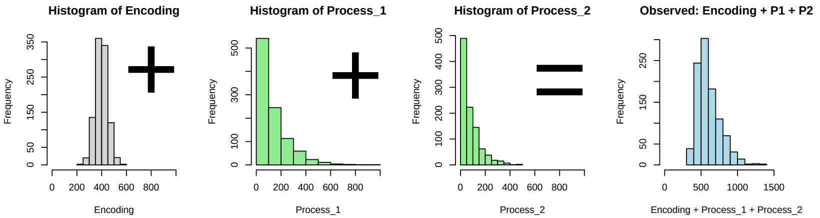
- Serial processing assumption: Observed RTs within each MPT branch are the result of a sequence of underlying processes
    1. Time for encoding and response execution
    2. Completion time for each state in the MPT model
    3. Observed RTs in a branch are the sum of encoding and all relevant processing times
- 2HTM components:
    - "Detection RTs": Encoding + Detection
    - "Guessing RTs": Encoding + unsuccessful detection + guessing



Parametric Assumptions

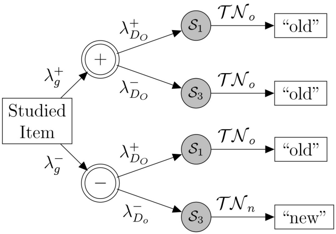
Illustration of the parametric assumptions

- Encoding and response execution (gray): truncated normal distribution (with mean μ and SD σ)
- Completion times (green): exponential distribution (with rate parameters λ)
- Observed time (blue): sum of encoding and processing times
- Note: Encoding and all completion times are independent



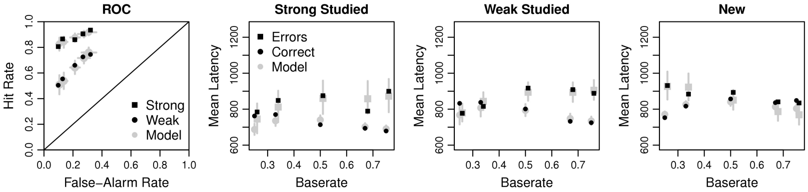
The ordering of the latent MPT states matters

- The 2HTM permits two possible orders:
  1. "Detect-Guess Model": First detection, then guessing (previous slides)
  2. "Default-Interventionist Model": First guessing (by default) and the detection process can intervent (see below)
- With the RT-MPT extension, both versions can be tested against each other



Empirical test of different 2HTM versions

- Bayesian hierarchical model (person random effects)
- The Default-Interventionist model fits 5 data sets better
- Effect of manipulations
  - No effect of memory strength on completion time of detection
  - Faster completion times for guessing if response matches the response-bias-condition
- The model fit is satisfactory:



Summary

Summary & Conclusion

RT-Extended MPT Models (Heck & Erdfelder, 2016)

1. No assumptions about shape of latent distributions
2. RT-extended MPT models are also MPT models
3. Testing relative speed of processes (stochastic dominance)

GPT Models (Heck, Erdfelder, & Kieslich; 2016)

1. General approach for modeling discrete and continuous data
2. New tests of psychological theories (e.g., mouse-tracking)
3. MPT parameters estimated more precisely
4. User-friendly software (R package gpt)

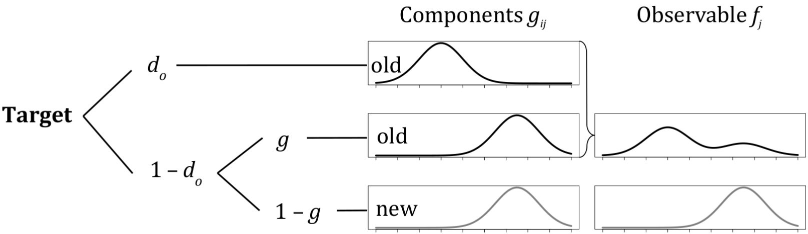
RT-MPT (Klauer & Kellen, 2018)

1. Assumption of serial processing (sum of encoding and completion times)
2. Hierarchical Bayesian

Appendix

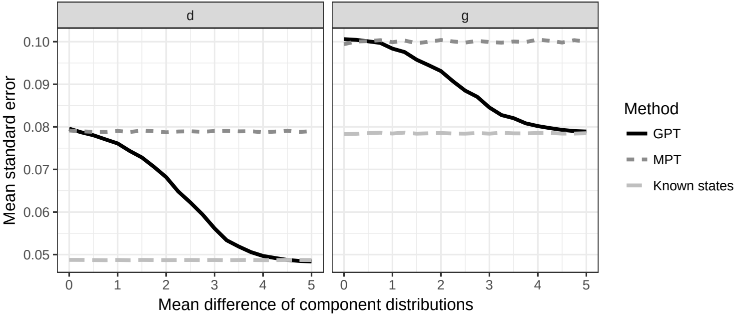
Appendix: GPT Estimates are More Precise

- Higher precision of parameter estimates $\hat{\boldsymbol{\theta}}$
  - The more distinct the latent distributions, the smaller the standard error of $\hat{\boldsymbol{\theta}}$
  - Intuition: Continuous variables improve the classification of trials to latent processing states
  - Upper bound: Precision of MPT model
  - Lower bound: Latent states known



Appendix: GPT Estimates are More Precise

- Simulation of the 2HTM with Gaussian component distributions
  - $\mu^{\text{detect}} = 0$ vs. $\mu^{\text{guess}} = 0, \ldots, 5$
- Results
  - More distinct distributions: smaller standard error of $\hat{\boldsymbol{\theta}}$
  - Upper bound: Precision of MPT model
  - Lower bound: Latent states known



Appendix: Formal Definition of GPT Models

For a vector of discrete responses $\boldsymbol{x}$, a matrix of continuous variables $\boldsymbol{Y}$:

- The **joint distribution** f is a finite mixture
  - For a discrete response x and continuous response(s) $\boldsymbol{y}$:

$$f(x, \boldsymbol{y} \mid \boldsymbol{\theta}, \boldsymbol{\eta}) = \sum_{j=1}^{J} \delta_{C_j}(\{x\}) \sum_{i=1}^{I_j} p_{ij}(\boldsymbol{\theta}) g_{ij}(\boldsymbol{y} \mid \boldsymbol{\eta})$$

- **Mixture weights** $p_{ij}(\boldsymbol{\theta})$
  - Identical to MPT-branch probabilities (probability parameters $\boldsymbol{\theta}$)

$$p_{ij}(\boldsymbol{\theta}) = c_{ij} \prod_{s=1}^{S} \theta_s^{a_{ijs}} (1 - \theta_s)^{b_{ijs}}$$

- **Basis distributions** $g_{ij}(\boldsymbol{y} \mid \boldsymbol{\eta})$ for latent states
  - E.g., normal, exGaussian, exWald,... distributions
  - Product-distributions for multivariate continuous data

Appendix: Formal Definition of GPT Models

- GPT models are a set of parameterized distributions:

$$\mathcal{M}^{\mathsf{GPT}}(\Theta = [0,1]^{S_1}, \Lambda \subset \mathbb{R}^{S_2}) = \{f(x, \boldsymbol{y} \mid \boldsymbol{\theta}, \boldsymbol{\eta}) \mid \boldsymbol{\theta} \in \Theta, \boldsymbol{\eta} \in \Lambda\}$$

- Likelihood where trials $k = 1, \ldots, K$ fall into disjoint sets M_t that are modeled by $t = 1, \ldots, T$ processing trees

$$L(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \boldsymbol{x}, \boldsymbol{Y}) = \prod_{t=1}^{T} \prod_{k \in M_t} f_t(x_k, \boldsymbol{y}_k \mid \boldsymbol{\theta}, \boldsymbol{\eta})$$

Appendix: GPT Parameter Estimation

Expectation-Maximization (EM) algorithm

- E: Compute expected probabilities $\boldsymbol{z}$ of being in the cognitive states
  - Continuous variables inform the state-vector $\boldsymbol{z}$
- M: Maximize likelihood of continuous parameters given the latent-states $\boldsymbol{z}$

Illustration

- E-step estimates the probability to be in state i in trial k:

$$P(z_k = i \mid \boldsymbol{\theta}, \boldsymbol{\eta}, x_k, \boldsymbol{y}_k) = \frac{P(z_k = i \mid \boldsymbol{\theta}, x_k) g_{ij}(\boldsymbol{y}_k \mid \boldsymbol{\eta})}{\sum_i P(z_k = i \mid \boldsymbol{\theta}, x_k) g_{ij}(\boldsymbol{y}_k \mid \boldsymbol{\eta})}$$

Cat.	RT [ms]	Conf. [1-10]	ERP [mV]	$z_k = 1$	...	$z_k = I$
c_1	551	3	1.324	0.43	...	0.09
c_1	502	1	0.921	0.19	...	0.56
c_2	470	6	2.231	0.30	...	0.00
c_1	733	4	1.010	0.14	...	0.47

Appendix: Identifiability

Identifiability: GPT models with identifiable MPT structure are identifiable if (cf. distribution-free approach):

- Component distributions are observable (Yantis et al., 1991)
- Stepwise deletion of identifiable component distributions (Heck & Erdfelder, 2016)
- Specific matrix has full rank (Heck & Erdfelder, 2016)

Alternative strategy

- Using order constraints to identify GPT models with a nonidentifiable MPT structure
- Label switching of processing paths with component distributions