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Foundations of Statistical Modeling II

SMiP Core Course, Spring 2019

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STATISTICAL MODELING in PSYCHOLOGY

FREIBURG HEIDELBERG LANDAU MANNHEIM TÜBINGEN

Foundations of Statistical Modeling II

Multinomial Processing Tree (MPT) Modeling: Basic Methods and Recent Advances, Block 1

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1) Basics

- 1.1) Introduction to standard MPT models
- 1.2) Examples
- 1.3) Model development
- 1.4) Formal model structure
- 1.5) Identifiability
- 1.6) Parameter estimation
- 1.7) Model assessment
- 1.8) Selected literature

1.1) Introduction to standard MPT models

- **Required type of data:**
- Standard multinomial models are tailored to discrete (i.e., categorical) data.
- Psychological data are typically discrete in nature (e.g., yes/no responses, correct/incorrect judgments, ratings, choices, ...).
- If not, they can be transformed into discrete data
  - Response times: Categorization into bins
  - Numerical judgments: Rank-orders of judgments
- Hence, many psychological paradigms generate frequency data that are appropriate for MPT modeling.

1.1) Introduction to standard MPT models

- **Distributional assumptions:**
- Standard MPT models assume that observations are sampled independently from
  - one multinomial distribution (simple multinomial model)
  - several multinomial distributions (joint multinomial model)
- This includes simple and joint binomial models as special cases.
- The frequency data structure can be univariate or multivariate.

MPT models ...

- … provide explanations of observed frequency data in terms of basic parameters with clear-cut psychological interpretations;
- … these parameters represent probabilities of latent psychological processes (or latent psychological states) underlying human behavior;
- … in other words, these models disentangle and measure the contributions of different psychological processes to frequencies of observable behaviors.
- In this sense, multinomial models allow for a “measurement of cognitive processes” (Riefer & Batchelder, 1988)

1.2) Examples

A very simple example:

- *Paradigm:*
  - Yes-No recognition test
- *Two Conditions:*
  - Old Items
  - New Items
- *Categorical (dichotomous) dependent variable:*
  - „Old“ vs. „New“ Judgment

A) One-High Threshold Model (Blackwell, 1963)

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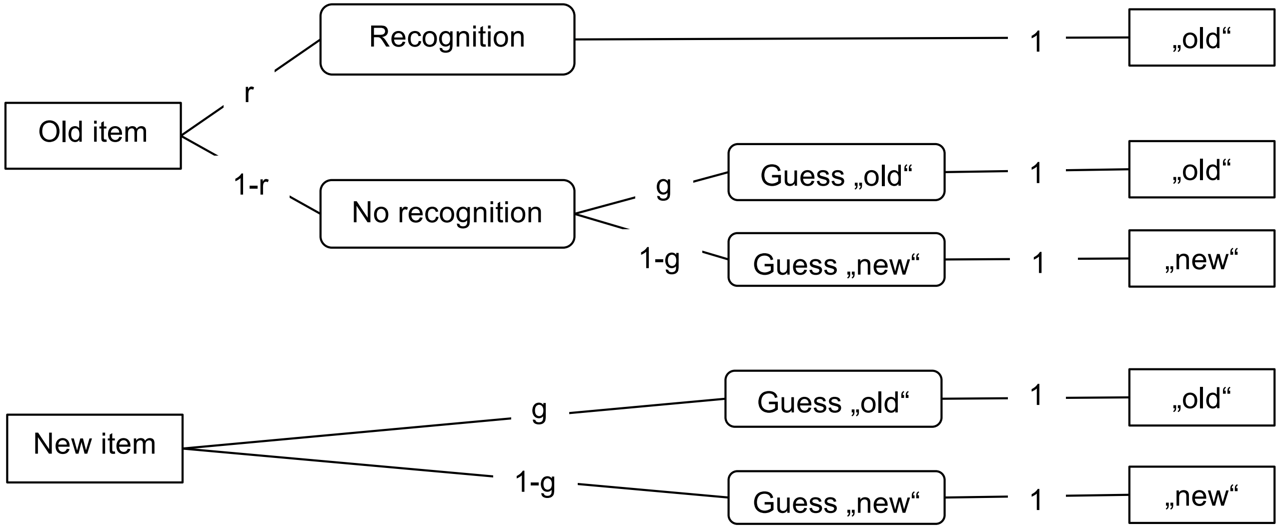
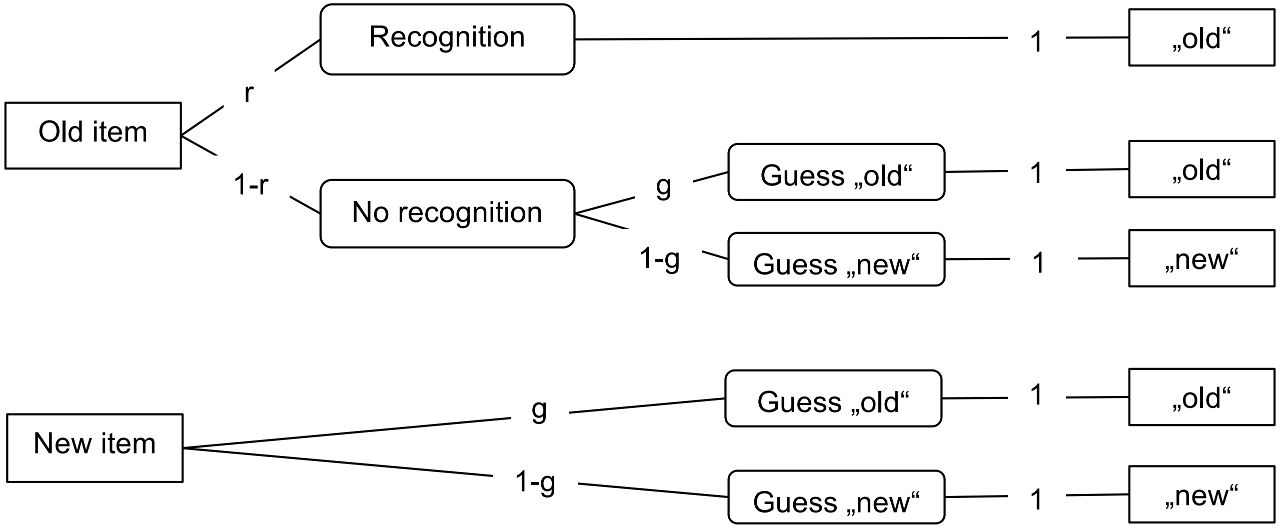


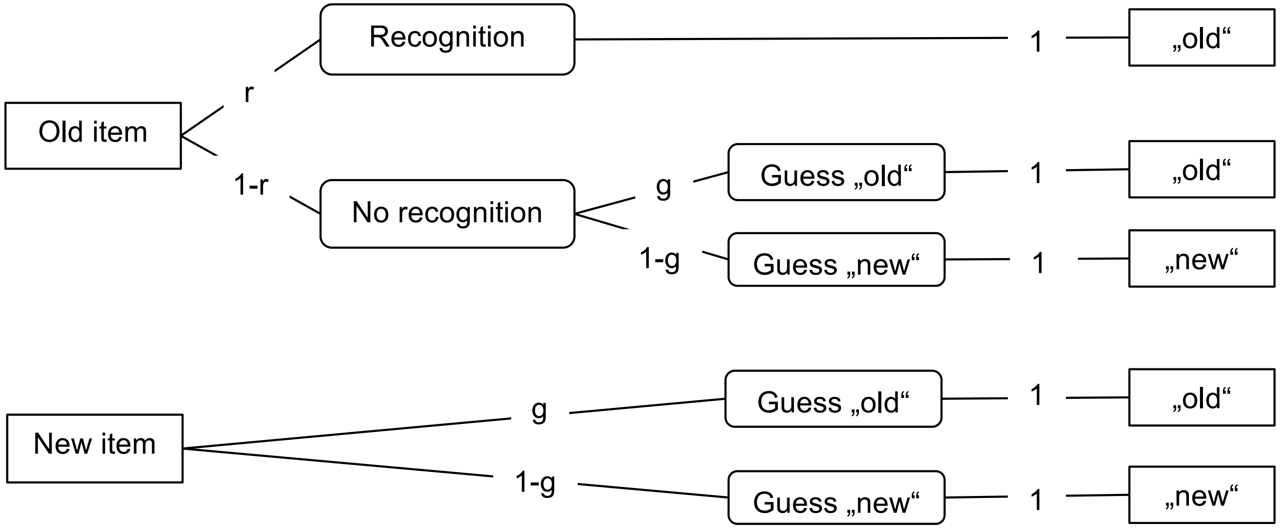
A) One-High Threshold Model (Blackwell, 1963)



Model equations:

p(„old“ | old item) = $r + (1\text{-}r) \cdot g$

A) One-High Threshold Model (Blackwell, 1963)



Model equations:

p(„old“ | old item) $= r + (1\text{-}r) \cdot g$

p(„old“ | new item) $= g$

B) Measuring storage and retrieval in long-term memory

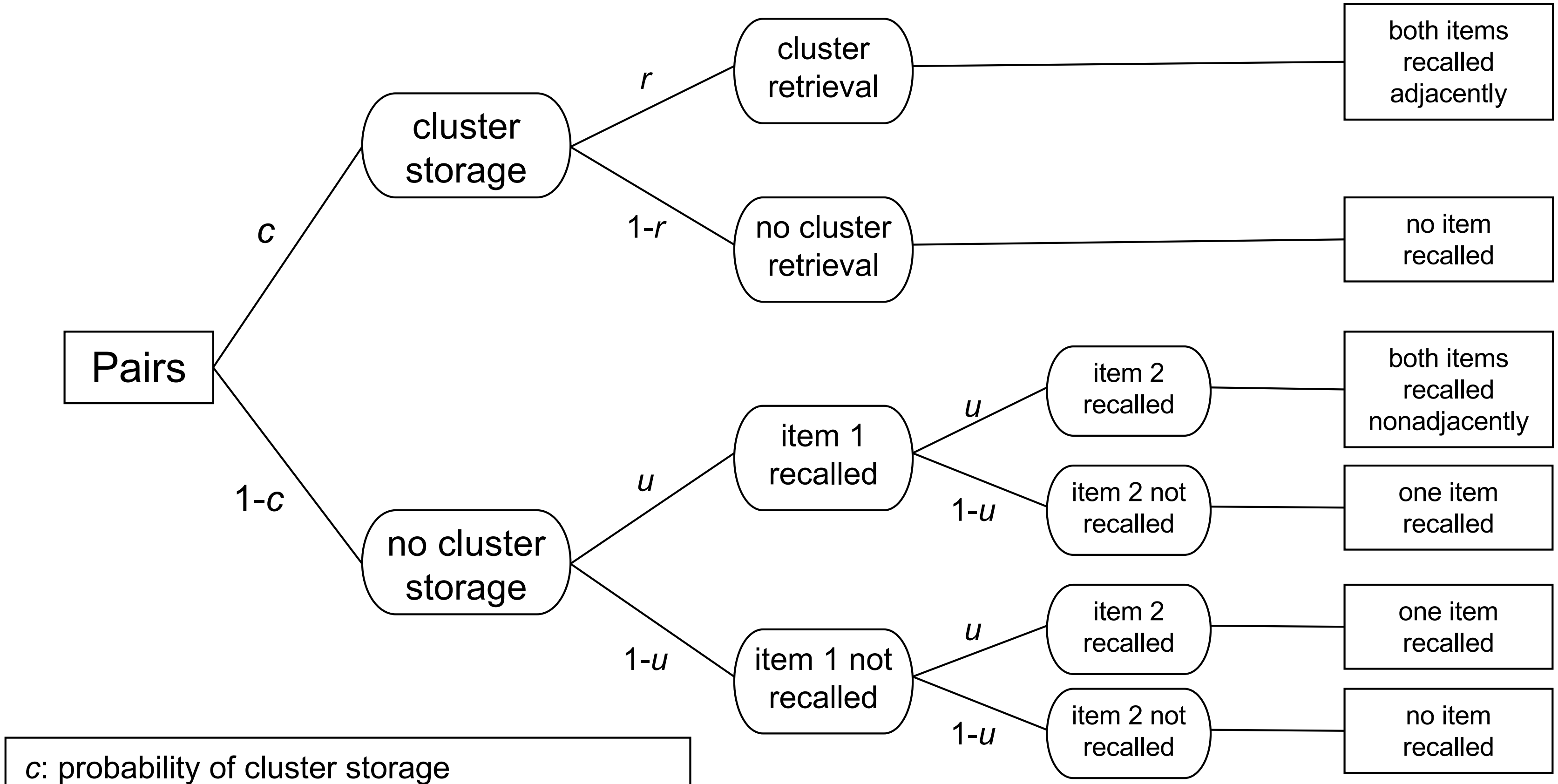
- *Empirical Paradigm*:
  - Free recall of word list consisting of
    - Word pairs (e.g. „chair“ und „table“)
    - Singletons (e.g., „rose“ and no other flower).
    - „Primacy–“ and „Recency–Buffer“.
- *Two distributions of responses*:
  - pairs
  - singletons

Scoring of responses

- Observation categories
  - For word pairs:
    - E1 — both words recalled adjacently
    - E2 — both words recalled nonadjacently
    - E3 — one word recalled
    - E4 — no word recalled
  - For singletons:
    - F1 — Recalled
    - F2 — Not recalled

Storage-Retrieval Model

(Batchelder & Riefer, 1980, 1986)



Pairs

c

cluster storage

r

cluster retrieval

both items recalled adjacently

1-*r*

no cluster retrieval

no item recalled

1-*c*

no cluster storage

u

item 1 recalled

u

item 2 recalled

both items recalled nonadjacently

1-*u*

item 2 not recalled

one item recalled

1-*u*

item 1 not recalled

u

item 2 recalled

one item recalled

1-*u*

item 2 not recalled

no item recalled

c: probability of cluster storage

r: probability of cluster retrieval

u: probability of singleton storage and retrieval

Model equations

- *Word pairs*:
- $p(\mathrm{E}_1) = c \cdot r$
- $p(\mathrm{E}_2) = (1 - c) \cdot u^2$
- $p(\mathrm{E}_3) = (1 - c) \cdot 2 \cdot u \cdot (1 - u)$
- $p(\mathrm{E}_4) = c \cdot (1 - r) + (1 - c) \cdot (1 - u)^2$
- *Singletons*:
- $p(\mathrm{F}_1) = u$
- $p(\mathrm{F}_2) = 1 - u$

1.3) Model development

- Preliminary summary:
- Select a paradigm (e.g., a task)
- Define the conditions of the paradigm
- Define the category system for each condition
- List relevant processes/parameters
- Construct theoretically reasonable processing branches („trees“) for each condition
- Derive corresponding model equations.
- General rules:
  - As simple as possible!!
  - Ignore unlikely events

1.4) Formal model structure

- Multinomial models
- Parameterized multinomial models
- Multinomial processing tree (MPT) models
- Processing-tree representation of MPT models

Multinomial Models

a) One condition (simple multinomial model)

- One variable with J categories and sample frequencies $n_1, n_2, ..., n_j, ..., n_J$.

- $\boldsymbol{\pi} = (p_1, p_2, ..., p_J)$ is the vector of category probabilities.

- Given independent sampling, the sample frequencies follow a multinomial distribution:

$$p_{N,\pi}(n_1, n_2, \ldots, n_J) = \frac{N}{n_1! n_2! \ldots n_J!} p_1^{n_1} p_2^{n_2} \ldots p_J^{n_J}$$

b) Several conditions (joint multinomial model)

- In each condition *k* (*k* = 1, ..., *K*), one categorical variable with *J*(*k*) categories is observed.
- For each of the *K* conditions a simple multinomial model holds.
- Given independence between conditions, the overall probability of the sample frequencies across conditions is

$$p = \prod_{k=1}^{K} p_{N(k),\pi(k)}(n_{1(k)},\ n_{2(k)}\ ,\ \ldots,\ n_{J(k)})$$

Parameterized Multinomial Models

- The category probabilities p_1, p_2 etc. are rewritten as functions of “latent parameters” $\theta_1, \theta_2, ..., \theta_S$
- Thus, in case of a simple multinomial model we have
  - $p_1 = f_1(\theta_1, \theta_2, ..., \theta_S)$
  - $p_2 = f_2(\theta_1, \theta_2, ..., \theta_S)$
  - ....
  - $p_J = f_J(\theta_1, \theta_2, ..., \theta_S)$
- These equations are called model equations.
- The set of possibles values of S latent parameters is called “parameter space“ Ω of the model.

Multinomial Processing Tree (MPT) Models

- MPT models form a subclass of parameterized multinomial models.
- Additional assumptions:

1) Each θ_s is in [0, 1]

Multinomial Processing Tree (MPT) Models

- MPT models form a subclass of parameterized multinomial models.
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2) Structure of the model equations (Hu & Batchelder, 1994):

$$p_j = \sum_{i=1}^{I(j)} c_{ij} \prod_{s=1}^{S} \boldsymbol{\theta}_s^{\,a_{ijs}} \cdot \left(1 - \boldsymbol{\theta}_s\right)^{b_{ijs}}, \qquad \sum_{j=1}^{J} p_j = 1, \qquad \boldsymbol{\theta}_s \in [0,\, 1]$$

where s : Parameter index

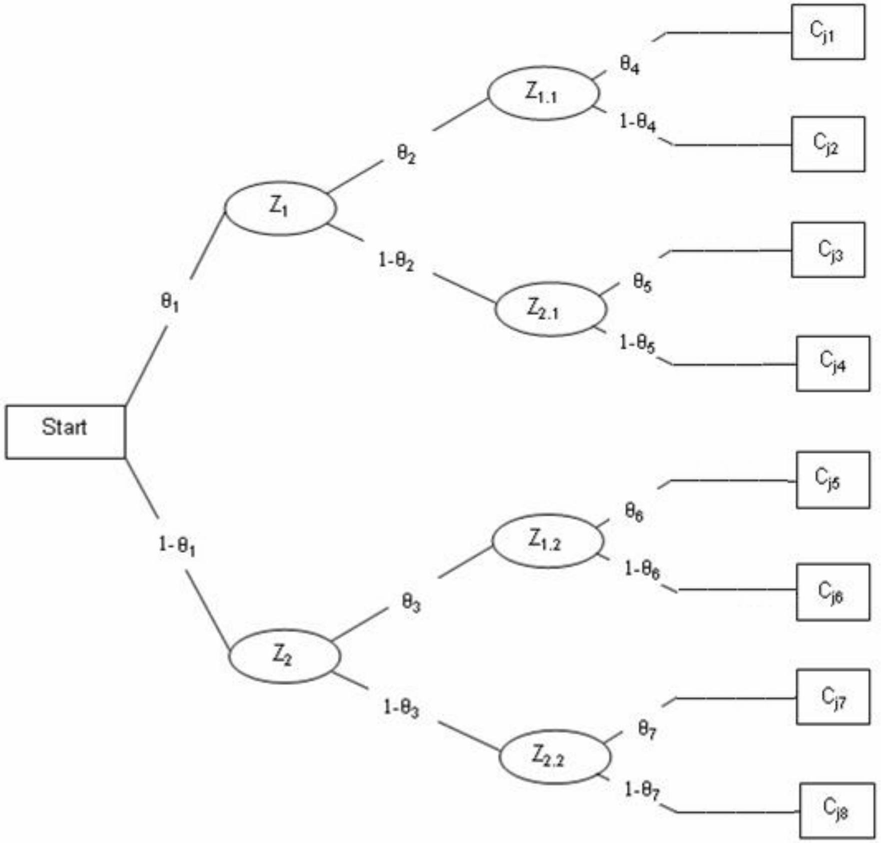
j : Category index

i : *B*ranch index

c_{ij} : positive real number

a_{ijs}, b_{ijs} : nonnegative integer number (often 0 or 1)

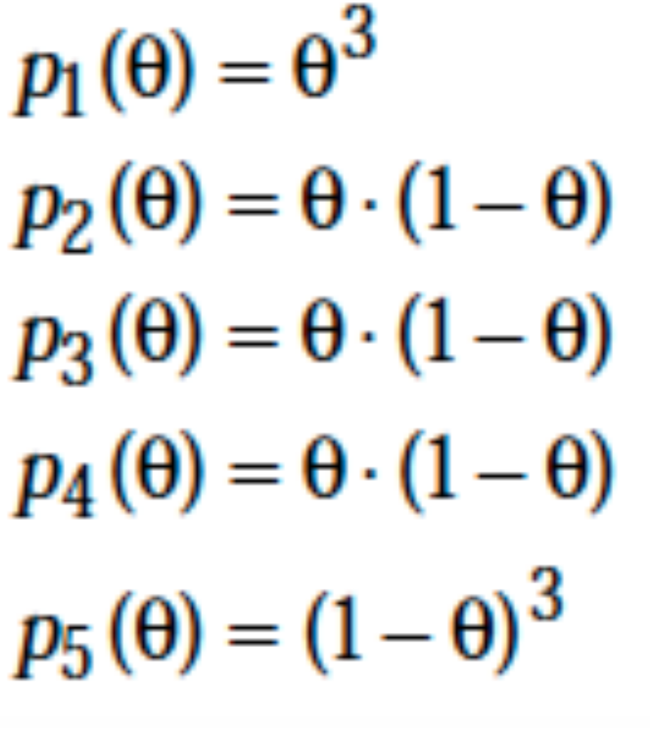
Binary probabilistic event trees can always be translated in MPT-model equations:



... as an aside ...

- Although any binary processing tree diagram uniquely determines a system of MPT model equations ...
- ... it is not true that any system of MPT model equations uniquely determines a specific processing tree diagram.
- Counter examples:
  - Level switching in independence models
  - Category switching given identical branch probabilities

Another counter example is this MPT model:



$$p_1(\theta) = \theta^3$$

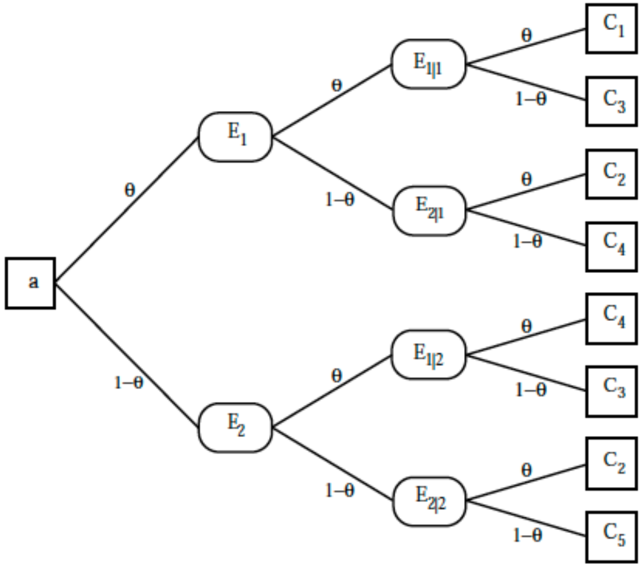
$$p_2(\theta) = \theta \cdot (1 - \theta)$$

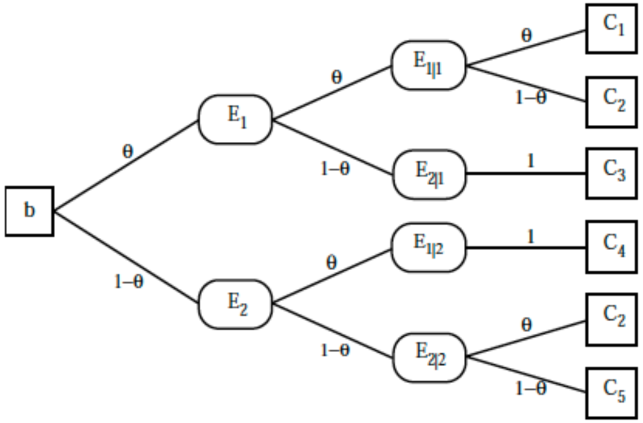
$$p_3(\theta) = \theta \cdot (1 - \theta)$$

$$p_4(\theta) = \theta \cdot (1 - \theta)$$

$$p_5(\theta) = (1 - \theta)^3$$

Two different processing trees compatible with the model on the previous slide:





1.5) Identifiability

Any MPT model equation system

$$p_j = \sum_{i=1}^{I(j)} c_{ij} \prod_{s=1}^{S} \theta_s^{\,a_{ijs}} \cdot (1-\theta_s)^{b_{ijs}}, \qquad \sum_{j=1}^{J} p_j = 1, \qquad \theta_s \in [0,\ 1]$$

where s: Parameter index

j: Category index

i: Branch index

c_{ij}: positive real number

a_{ijs}, b_{ijs}: nonnegative integer number (often 0 or 1)

defines a mapping $f\colon \Omega \rightarrow P$

Parameter Space and Data Space

Ω is called „Parameter Space“:

= Set of all possible parameter vectors

P is called „Data Space“ (more precisely: space of category probabilities)

= Set of all possible category probability vectors

Definition (global identifiability):

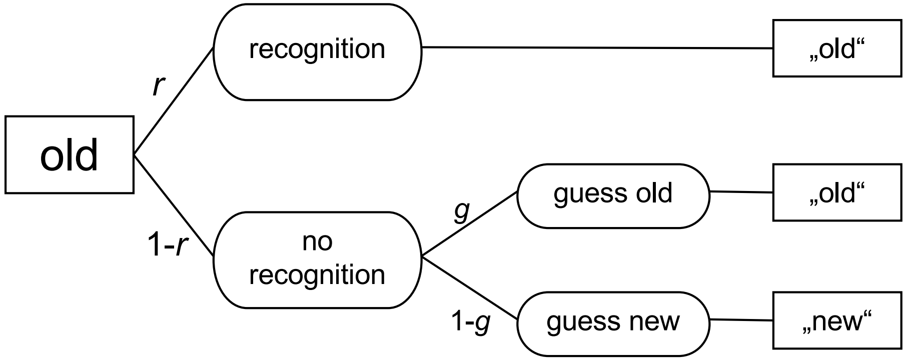
A MPT model is globally identified if *f* is one-to-one.

Definition (local identifiability):

A MPT model is locally identified if *f* is one-to-one in the neighborhood of θ_0 in Ω.

One-High-Threshold recognition model

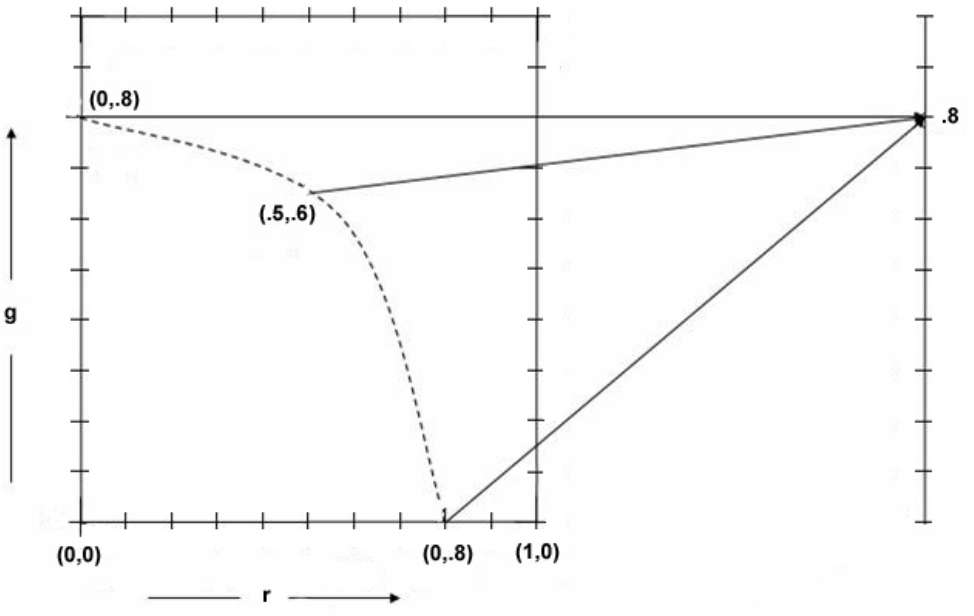
(Blackwell, 1953)



r: probability of recognition

g: probability of guessing old given recognition

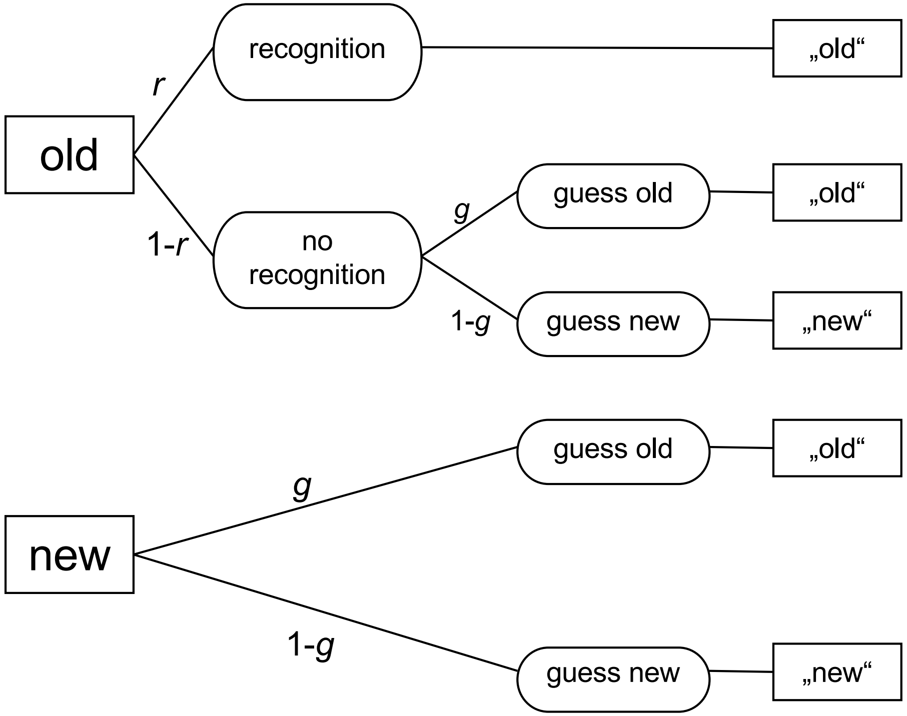
Example 1 (nonidentifiability)



- One-high-Threshold-Modell limited to old items:
  - p("alt" | old item) = $r + (1\text{-}r) \cdot g$
- Model is not identified.

One-High-Threshold recognition model

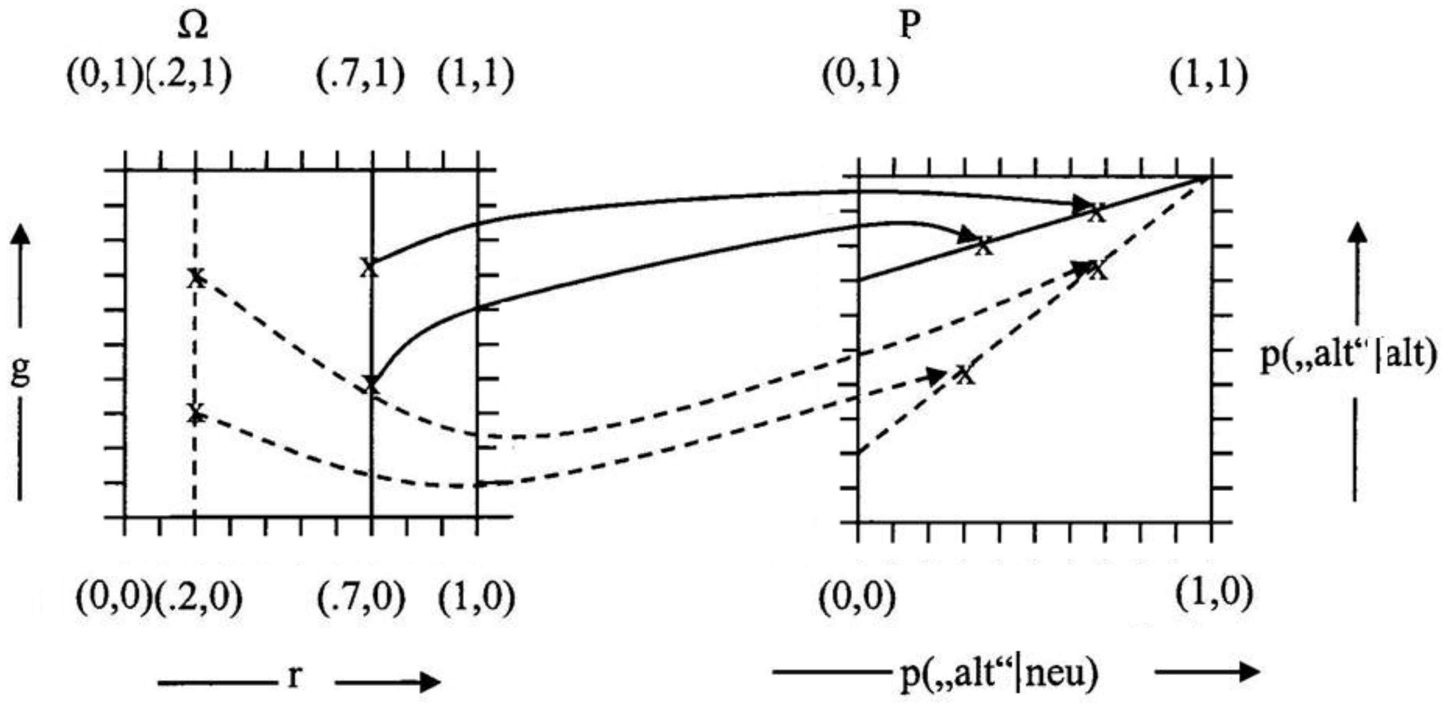
(Blackwell, 1953)



r: probability of recognition

g: probability of guessing old given recognition

Example 2 (identifiability)



- One-high-Threshold-Model (old and new items)
  - p(“alt” | old item) $= r + (1\text{-}r) \cdot g$
  - p(„alt“ | new item) = g
- Model ist globally identified.

Two important theorems:

- „*Observable branches*“:

  A model is always globally identified if each of its branches terminates in a unique empirical category (Hu & Batchelder, 1994).

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- „*Observable branches*“:

  A model is always globally identified if each of its branches results in a new empirical category (Hu & Batchelder, 1994).

- „*No more parameters than degrees of freedom in the data*“:

  A necessary but not sufficient condition of identifiability is

$$S \le \sum_{k=1}^{K} (J(k)-1).$$

Jacobian Matrix

- Jacobian: Matrix of the 1st partial derivates of the model equations with respect to parameters θ_s , $s = 1, \dots S$.
- r: maximum rank of the Jacobian across Ω
- If $r < S$, then the model is neither locally nor globally identified.
- If $r = S$, then the model is locally identified (but not necessarily globally).

Remedies for nonidentifiable models

- Parameter constraints
  - Parameter fixations ($\theta_s = c$, with c = constant)
  - Equality constraints ($\theta_s = \theta_{s'}$)
- Increase the number of empirical categories
  - Additional conditions, no (or few) additional parameters
  - Selective manipulations of parameters

1.6) Parameter Estimation

Find a parameter vector $\boldsymbol{\theta} = (\theta_1, ..., \theta_s, ..., \theta_S)$, $\boldsymbol{\theta} \in \Omega$, such that the distance between the sample frequencies $n_1, n_2, ..., n_J$ and the expected category frequencies under the model, $N \cdot p_1(\boldsymbol{\theta}), N \cdot p_2(\boldsymbol{\theta}), ..., N \cdot p_J(\boldsymbol{\theta})$, becomes a minimum.

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Which distance measure?

1.6) Parameter Estimation

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Which distance measure?

The likelihood ratio statistic G^2 and Pearson‘s χ^2 are used most often:

$$G^2(\boldsymbol{\theta}) = 2\sum_{j=1}^{J} n_j \ln\left(\frac{n_j}{N \cdot p_j(\boldsymbol{\theta})}\right) \qquad \chi^2(\boldsymbol{\theta}) = \sum_{j=1}^{J} \frac{[n_j - N \cdot p_j(\boldsymbol{\theta})]^2}{N \cdot p_j(\boldsymbol{\theta})}$$

Both distance measures are special cases of the Power-Divergence-family (PD$_\lambda$-Statistics) (Read & Cressie, 1988):

$$s_{\lambda} = \frac{2}{\lambda(\lambda+1)} \sum_{k=1}^{k} \sum_{i=1}^{J(k)} n_{j(k)} \cdot \left[\left(\frac{n_{j(k)}}{e_{j(k)}} \right)^{\lambda} - 1 \right]$$

Note that:

Pearson-χ^2 $= S_{\lambda = 1}$
Likelihood-ratio-G^2 $= \lim_{\lambda \to 0} \; S_\lambda$.

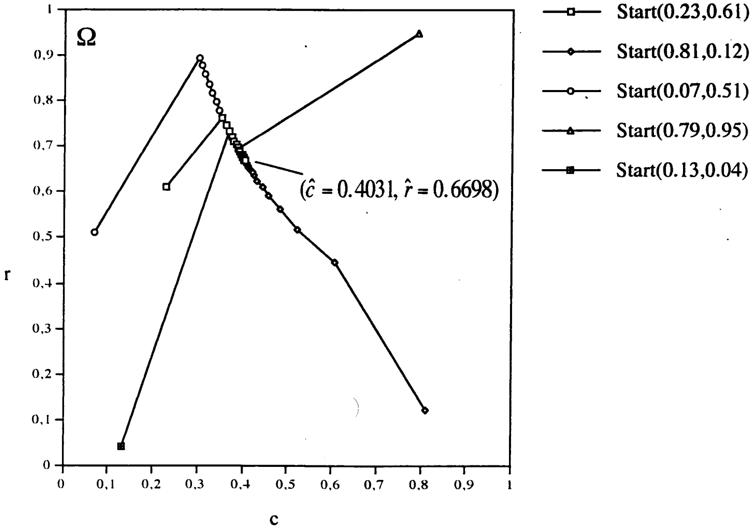
What is the best estimation method?

- All S_λ-estimates have the same asymptotic properties:
  - Consistent, efficient, asympotically unbiased
  - (multivariate) normal sampling distributions
- Choosing λ=0 (i.e., minimizing G^2) provides Maximum Likelihood (ML) estimates:
  - Sample frequencies have highest likelihood
  - Standard method in typical applications
- All S_λ-estimates are easily implemented using the robust Expectation-Maximization (EM) estimation algorithm.

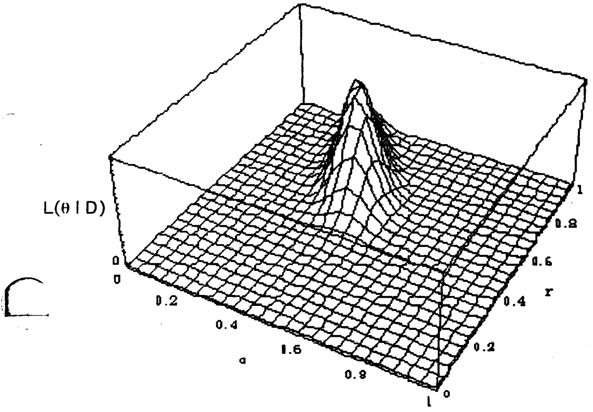
***E**xpectation-**M**aximization-(EM) Algorithm*

- 1) Choose a start vector $\boldsymbol{\theta}_i$.
- 2) **E**(xpectation)-Step:
  - Estimate the expected frequencies of the branches given $\boldsymbol{\theta}_i$ and the observed category frequencies $n_{j(k)}$
- 3) **M**(aximization)-Step:
  - Let i = i + 1
  - Compute new S_λ estimates $\boldsymbol{\theta}_i$ given the expected frequencies from step 2)
- 4) Convergence ?
  - If Abs($\boldsymbol{\theta}_i$- $\boldsymbol{\theta}_{i-1}$) > Criterion go back to step 2).
- 5) Otherwise accept $\boldsymbol{\theta}_i$ as final parameter estimates $\hat{\boldsymbol{\theta}}$

Graphical Illustration of the EM-Algorithm applied to *c* and *r* in storage-retrieval model:



EM paths in Ω for λ = 0, given five different starting values (constructed data).



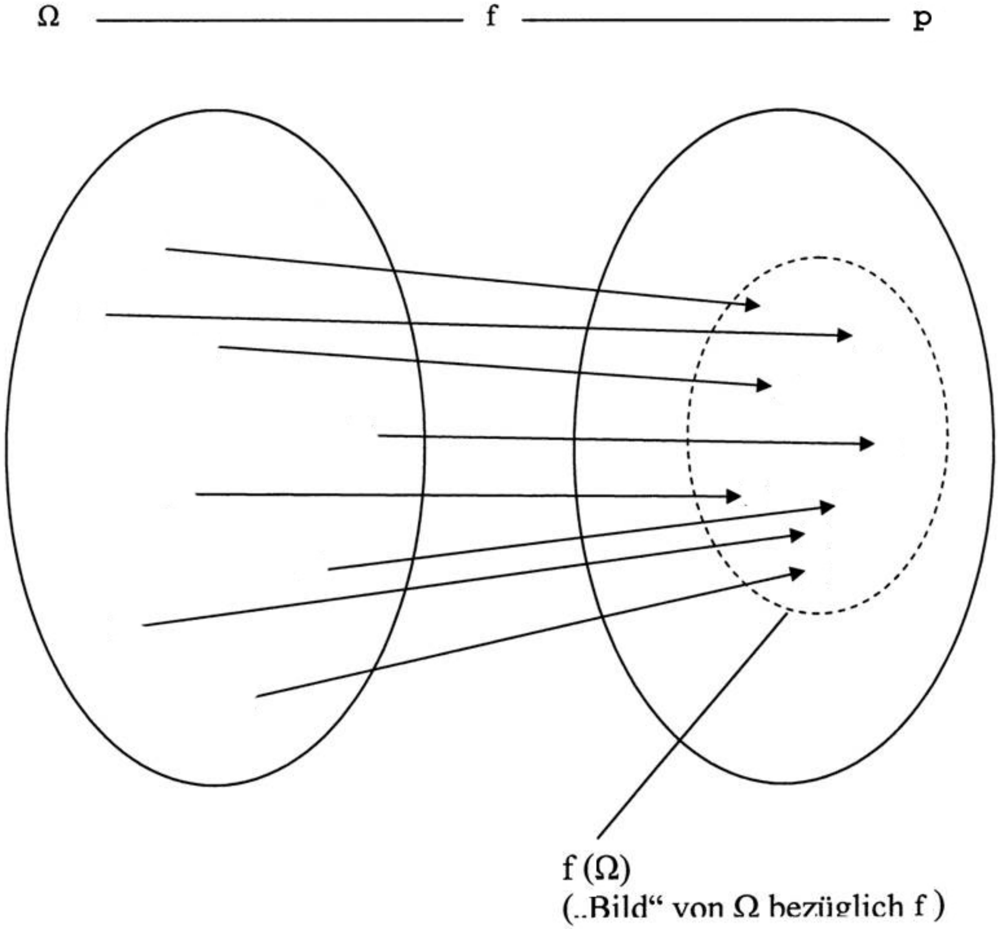
Likelihood (y-axis) as a function of *c* und *r* given the same data.

711

Bottom line

- Use G^2-minimization (= likelihood maximization) as the default estimation method.
- Repeat the estimation process several times using random start values to make sure that you really found a global maximum of the likelihood function.

1.7) Model assessment



How to test H_0: $\pi \in f(\Omega)$?

Options:

$$\text{Pearson}\,\chi^2 = \sum_{k=1}^{K}\sum_{j=1}^{J(k)} (n_{j(k)} - e_{j(k)})^2 / e_{j(k)},$$

or

$$\text{Likelihood - ratio - G}^2 = 2 \cdot \sum_{k=1}^{K}\sum_{j=1}^{J(k)} n_{j(k)} \cdot ln(n_{j(k)} / e_{j(k)});$$

with $e_{j(k)} = p_{j(k)}(\hat{\theta}) \cdot N(k)$

Both statistics are special cases of Read und Cressies (1988) „Power Divergence“ – Statistics:

$$S_\lambda = \frac{2}{\lambda(\lambda+1)} \sum_{k=1}^{k} \sum_{l=1}^{J(k)} n_{j(k)} * \left[\left(\frac{n_{j(k)}}{e_{j(k)}} \right)^{\lambda} - 1 \right]$$

Again:
Pearson-$\chi^2 = S_{\lambda=1}$
Likelihood-ratio-$G^2 = \lim_{\lambda \to 0} S_\lambda$.

All S_λ – Statistics are asymptotically (i.e., for N → oo) chi-square distributed under H_0 (i.e., the model holds) with

df = $\sum_{k=1}^{K} (J(k)-1) - S.$

What is the best goodness-of-fit statistic?

- In case of small sample sizes, $S_{\lambda=1}$ and $S_{\lambda=2/3}$ outperform other S_{λ}-statistics in terms of accuracy of Chi-square approxination (cf. Read & Cressie, 1988).
- However, small samples are typically less of a problem in MPT model applications (unless models are tested for single participants).
- Given the fact that G^2 is a by-product of ML-parameter estimation, G^2 (bzw. $S_{\lambda=0}$) can be recommended für moderate to large sample sizes.
- However, you cannot choose G^2 in case of samples with zero cells.
- Remedies: 1) Ignore 0 cells, 2) Add constant ε to all counts

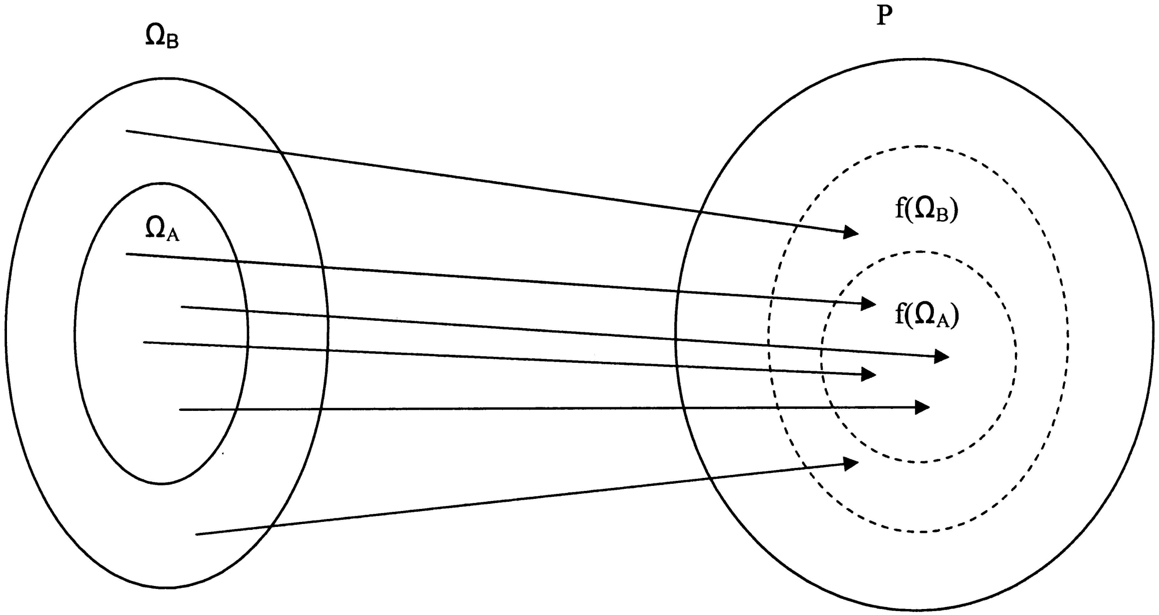
Model comparisons

- Assume that model B (M_B) holds for your data: $G^2(M_B)$ is not significant.
- Then often M_B is compared to an alternative model A (M_A)
- M_A typically is a special case of M_B:

  M_A is obtained by applying a parameter restriction to M_B.
- How to decide whether M_A fits the data worse than M_B?

Model M_A as a special case of M_B:

Illustration of model-specific mapping of Ω_A and Ω_B in P.



This is called a „hierarchical model family“.

Model comparisons in hierarchical model families

- If both M_A and M_B hold then

  G^2_A is chi-square distributed with df_A

  G^2_B is chi-square distributed with df_B

  $\Delta G^2_{A\text{-}B}: = G^2_A – G^2_B$ is chi-square distributed with $df_{A\text{-}B} = df_A - df_B$.
- In other words, G^2 is additive (irrespective of *N*):

  $G^2_A = G^2_B + \Delta G^2_{A\text{-}B}$
- This additivity property does not hold for other PD_λ-stats.
- Using $\Delta G^2_{A\text{-}B,}$ it is easy to compare different models in a hierarchical model family using chi-square tests
- This is perhaps the strongest argument for relying on G^2 for purposes of model fitting and testing.

Model comparisons in nonhierarchical model families

- Unfortunately, $\Delta G^2_{A\text{-}B} := G^2_A - G^2_B$ cannot be used in nonhierarchical model families.
- How to proceed then?

Information-theoretic measures of goodness-of-fit

- Akaike Information Criterion (AIC):
  - $\text{AIC}(\text{M}_0) = -2 \cdot \ln(\text{L}(\boldsymbol{\theta};\mathbf{y})) + 2 \cdot S$
  - $\Delta\text{AIC}(\text{M}_0) = \text{AIC}(\text{M}_0)\text{-AIC(sat.)} = \text{G}^2(\text{M}_0) - 2 \cdot \text{df}(\text{M}_0)$
- Bayesian Information Criterion (BIC):
  - $\text{BIC}(\text{M}_0) = -2 \cdot \ln(\text{L}(\boldsymbol{\theta};\mathbf{y})) + S \cdot \ln(N)$
  - $\Delta\text{BIC}(\text{M}_0) = \text{BIC}(\text{M}_0)\text{-BIC(sat.)} = \text{G}^2(\text{M}_0)\text{-df}(\text{M}_0) \cdot \ln(N)$
- Rule:
  - Choose the model with the smaller AIC / BIC
- Recommendation:
  - Fit is excellent if ΔAIC or ΔBIC < 0.

1.8) Selected literature

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