Exercitii suplimentare

1. Studiați existența limitelor de fc:

a) $\lim\limits_{(x,y)\to(0,2)} \frac{\sin(xy)}{x}$

$$\lim\limits_{(x,y)\to(0,2)} \frac{\sin(xy)}{xy}\cdot \overset{\nearrow 2}{y} = 2 \cdot \overset{\nearrow 1}{\lim\limits_{t\to 0}\frac{\sin t}{t}} = 2$$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

$$\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right| = \frac{|xy|}{\sqrt{x^2+y^2}}$$

$$x^2 \le x^2+y^2 \Rightarrow \sqrt{x^2} \le \sqrt{x^2+y^2} \Rightarrow |x| \le \sqrt{x^2+y^2}$$

analog $|y| \le \sqrt{x^2+y^2}$

$$\Rightarrow \frac{|xy|}{\sqrt{x^2+y^2}} = \frac{|x|\cdot|y|}{\sqrt{x^2+y^2}} \le \frac{\sqrt{x^2+y^2}\cdot\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2}}} = \sqrt{x^2+y^2} \to 0$$

deci limita e 0.

c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2 y^2}$

Fie $f(x,y) = \frac{x^4-y^4}{x^2y^2}$ și $a^n = \left(\frac{1}{n}, \frac{2}{n}\right) \to (0,0)$, $n\to\infty$; $b^n = \left(\frac{2}{n}, \frac{1}{n}\right) \to (0,0)$, $n\to\infty$

$$f(a^n) = \frac{\frac{1}{n^4}-\frac{16}{n^4}}{\frac{1}{n^2}\cdot\frac{4}{n^2}} = \frac{-15}{n^4}\cdot\frac{n^4}{4} \to -\frac{15}{4}, \ n\to\infty$$

$$f(b^n) = \frac{\frac{16}{n^4}-\frac{1}{n^4}}{\frac{4}{n^2}\cdot\frac{1}{n^2}} \to \frac{15}{4}, \ n\to\infty$$

$$f(a^n) \ne f(b^n) \Rightarrow \nexists \lim.$$

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2y)}{x^2+y^2}$

$l = \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2y)}{x^2y} \cdot \dfrac{x^2y}{x^2+y^2} = \overbrace{\left(\lim\limits_{t\to0} \dfrac{\sin t}{t}\right)}^{\to 1} \cdot \underbrace{\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}}_{l'}$

$\left|\dfrac{x^2y}{x^2+y^2} - 0\right| = \left|\dfrac{x^2y}{x^2\left(1+\left(\frac{y}{x}\right)^2\right)}\right| = \left|\dfrac{y}{\underbrace{1+\left(\frac{y}{x}\right)^2}_{\geq 1}}\right| \leq |y| \to 0$

Deci $l' = 0 \Rightarrow l = 0$

e) $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2}$

$l = \lim\limits_{(x,y)\to(0,0)} e^{\ln(x^2+y^2)^{x^2y^2}} = e^{\lim\limits_{(x,y)\to(0,0)} x^2y^2 \cdot \ln(x^2+y^2)}$

$l' = \lim\limits_{(x,y)\to(0,0)} x^2 y^2 \cdot \ln(x^2+y^2)$: $|x^2y^2 \ln(x^2+y^2)| \overset{*}{\leq} |(x^2+y^2)^2 \cdot \ln(x^2+y^2)|$

$\underset{t = x^2+y^2}{=} |t^2 \ln t|$, dar $\lim\limits_{t\to0} t^2 \ln t = \lim\limits_{t\to0} \dfrac{\ln t}{\frac{1}{t^2}} \overset{L'H}{=} \lim\limits_{t\to0} \dfrac{\frac{1}{t}}{-\frac{2}{t^3}} =$

$(x,y)\to(0,0) \Rightarrow t\to 0_+$

$= \lim\limits_{t\to0} \dfrac{t^{\cancel{3}2}}{-2\cancel{t}} = 0 \Rightarrow l' = 0 \Rightarrow \boxed{l = e^0 = 1.}$

$*$. $a,b \geq 0 \Rightarrow ab \leq (a+b)^2$
$\Leftrightarrow a^2+2ab+b^2-ab \geq 0$
$a^2+ab+b^2 \geq 0$

$(a+b)^2 + a^2 + b^2 \geq 0$, evident
$\Rightarrow 2(a^2+ab+b^2) \geq 0 \Rightarrow$
$a^2+ab+b^2 \geq 0$. ✓

pt. $a = x^2$, $b = y^2$, avem
$x^2y^2 \leq (x^2+y^2)^2$.

f) $\lim\limits_{(x,y)\to(\infty,\infty)} \dfrac{x+y}{x^2+xy+y^2}$

$\left|\dfrac{x+y}{x^2+xy+y^2} - 0\right| \leq \dfrac{|x|}{|x^2+xy+y^2|} + \dfrac{|y|}{|x^2+xy+y^2|} = \dfrac{x}{x^2+xy+y^2} + \dfrac{y}{y^2+xy+x^2} \leq$

$\leq \dfrac{x}{x^2} + \dfrac{y}{y^2} = \dfrac{1}{x} + \dfrac{1}{y} \to 0,\ (x,y)\to(\infty,\infty) \Rightarrow l = 0$

g) $\lim\limits_{(x,y)\to(1,1)} \dfrac{\ln(1+x^2)-\ln(1+y^2)}{x^2-y^2}$

$$\frac{1}{x^2-y^2}\cdot \ln\frac{1+x^2}{1+y^2} = \ln\left(\frac{1+x^2}{1+y^2}\right)^{\frac{1}{x^2-y^2}}$$

$$\lim_{(x,y)\to(1,1)}\left(\frac{1+x^2}{1+y^2}\right)^{\frac{1}{x^2-y^2}} = \lim_{(x,y)\to(1,1)}\left(1+\frac{x^2-y^2}{1+y^2}\right)^{\frac{1+y^2}{x^2-y^2}\cdot\frac{1}{1+y^2}}$$

$$= e^{\lim\limits_{(x,y)\to(1,1)}\frac{1}{1+y^2}} = e^{\frac{1}{2}}$$

$\Rightarrow$ limita inițială e $\ln e^{\frac{1}{2}} = \frac{1}{2}$.

h) $\lim\limits_{(x,y,z)\to 0_3} \dfrac{xyz}{x^2+y^2+z^2}$

$$\left|\frac{xyz}{x^2+y^2+z^2}-0\right| = \frac{|x|\cdot|y|\cdot|z|}{x^2+y^2+z^2}$$

$$m_g \le m_p \Rightarrow \sqrt[3]{|x|\cdot|y|\cdot|z|} \le \sqrt{\frac{|x|^2+|y|^2+|z|^2}{3}} \quad \overset{()^3}{\Rightarrow}$$

$$|x|\cdot|y|\cdot|z| \le \frac{(x^2+y^2+z^2)^{\frac{3}{2}}}{3^{\frac{3}{2}}}$$

$$\Rightarrow \frac{|x|\cdot|y|\cdot|z|}{x^2+y^2+z^2} \le \frac{(x^2+y^2+z^2)^{\frac{3}{2}}}{3^{\frac{3}{2}}\cdot(x^2+y^2+z^2)} = \frac{(x^2+y^2+z^2)^{\frac{1}{2}}}{3^{3/2}} \to 0$$

$\Rightarrow l=0$

i) $\lim\limits_{(x,y,z)\to O_3} (xy+yz+zx)^2 \ln(x^2+y^2+z^2)$

Cauchy - Schwarz: $(xy+yz+zx)^2 \le (x^2+y^2+z^2)\cdot(x^2+y^2+z^2)$

$= (x^2+y^2+z^2)^2$

$\left|(xy+yz+zx)^2 \cdot \ln(x^2+y^2+z^2) - 0\right| \le (x^2+y^2+z^2)^2 \cdot |\ln(x^2+y^2+z^2)|$

$= t^2 \cdot |\ln t|$

Cum $(x,y,z) \to O_3 \Rightarrow t\to 0, t>0$

$$\lim_{t\to 0} t^2 \cdot |\ln t| = \lim_{t\to 0} \frac{|\ln t|}{\frac{1}{t^2}} \overset{L'H}{=} \lim_{\substack{t\to 0\\ t>0}} \frac{\frac{1}{t}}{\frac{-2}{t^3}} = \lim_{\substack{t\to 0\\ t>0}} -\frac{t^2}{2} = 0$$

$\Rightarrow$ limita e 0.

② Studiați continuitatea următ. fc. în origine:

a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x,y)\ne(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

f cont. în $(0,0) \Leftrightarrow \exists \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

$$\left|\frac{x^3+y^3}{x^2+y^2} - 0\right| \le \frac{|x|^3+|y|^3}{x^2+y^2} \le \frac{|x|^3+|x|^2\cdot|y|+|x|\cdot|y|^2+|y|^3}{x^2+y^2} =$$

$$= \frac{|x|^2(|x|+|y|)+|y|^2(|x|+|y|)}{x^2+y^2} = \frac{(|x|+|y|)\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} = |x|+|y|$$

Cum $|x|+|y| \to 0 \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$, deci f cont. în 0.

b) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} (1+x^2y^2)^{\frac{1}{x^2+y^2}}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$

f cont. în $(0,0) \Leftrightarrow \exists \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 1$.

$$l = \lim_{(x,y)\to(0,0)} (1+x^2y^2)^{\frac{1}{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \left[(1+x^2y^2)^{\frac{1}{x^2y^2}}\right]^{\frac{x^2y^2}{x^2+y^2}} =$$

$$= e^{\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}}$$

$$y^2 \leq x^2+y^2 \Rightarrow \left|\frac{x^2y^2}{x^2+y^2} - 0\right| \leq \frac{x^2(x^2+y^2)}{x^2+y^2} = x^2 \to 0$$

$\Rightarrow l = e^0 = 1 = f(0,0) \Rightarrow f$ cont. în $(0,0)$.

③ Verificați dacă funcțiile următ. își ating valorile extreme și det. aceste valori.

a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \frac{xy}{x^2+y^2+1}$

$\mathbb{R}^2$ nu e mărginită $\Rightarrow$ $\mathbb{R}^2$ nu e compactă $\Rightarrow$ nu putem aplica T.W.

$$\frac{xy}{x^2+y^2} \leq \frac{1}{2} \Leftrightarrow \frac{2xy}{x^2+y^2} - \frac{1}{2} \leq 0 \Leftrightarrow \frac{2xy - x^2 - y^2}{2(x^2+y^2)} \leq 0 \Leftrightarrow \frac{-(x-y)^2}{2(x^2+y^2)} \leq 0,$$
$\forall A^4$,

cu eg. $\Leftrightarrow x = y$.

$x^2+y^2 < x^2+y^2+1 \Rightarrow$

$\frac{xy}{x^2+y^2+1} < \frac{xy}{x^2+y^2} \leq \frac{1}{2}$

dacă $xy > 0$.

⑤ (dacă $xy < 0 \Rightarrow f(x,y) < 0$)

Deci $\sup f(A) = \frac{1}{2}$, dar nu se atinge.

$f(n,n) = \frac{n^2}{2n^2+1} \to \frac{1}{2}$, $n \to \infty$

Dacă $xy < 0$, putem pp. că $x < 0$. $\Rightarrow$ $y > 0$

$-x > 0 \Rightarrow \frac{-xy}{x^2+y^2+1} < \frac{1}{2} \overset{\cdot(-1)}{\Rightarrow} \frac{xy}{x^2+y^2+1} > -\frac{1}{2}$.

Deci $\inf f(A) = -\frac{1}{2}$, nu se atinge. ($f(-n,n) \to -\frac{1}{2}$)

b) $f: \overline{B}(0_2,1) \to \mathbb{R}$, $f(x,y) = x^2 + xy + y^2$

$\overline{B}(0_2,1)$ mărginită și închisă, f cont. $\overset{T.W}{\Longrightarrow}$ f mărg. și își atinge extremele.

Fie $x \neq 0$.

$$f(x,y) = x^2\left(1 + \frac{xy}{x^2} + \frac{y^2}{x^2}\right) = x^2\left(1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2\right)$$

Fie $\frac{y}{x} = t$. Atunci $t^2 + t + 1 > 0$, deoarece $\Delta < 0$ și $a = 1 > 0$.

Deci $\left(1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2\right) > 0$ și $x^2 > 0$ pt. $x \neq 0$ $\Rightarrow f(x,y) > 0$, $x \neq 0$

Dacă $(x,y) = (0,0) \Rightarrow f(0,0) = 0$

$\Rightarrow \inf f(A) = 0 = f(0,0)$

Punctul maxim se va afla pe frontieră $(x^2 + y^2 = 1)$ $\to$ vom arăta în seminariile viitoare că punctul critic din interiorul lui $\overline{B}(0_2,1)$ este $(0,0)$ care e pct. de minim.

$$x^2 + y^2 = 1 \Rightarrow y = \pm\sqrt{1-x^2}$$

Fie $y = \sqrt{1-x^2}$.

$$\Rightarrow f(x) = 1 + x\sqrt{1-x^2}$$

$$f'(x) = x \cdot \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} + \sqrt{1-x^2} = \frac{-x^2+1-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}, \quad x \neq \pm 1.$$

$$f'(x) = 0 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

x	-1		$-1/\sqrt{2}$		$1/\sqrt{2}$		1
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$		$\searrow$		$\nearrow$		$\searrow$	

$$\sup f(A) = f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

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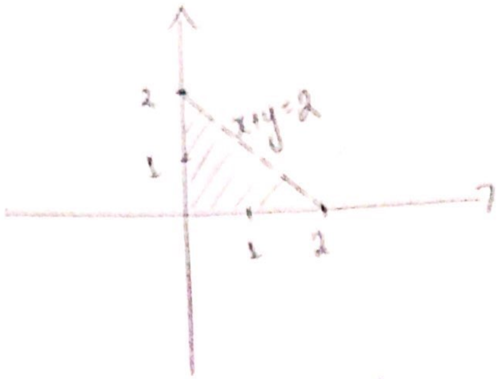
c) $f: A \to \mathbb{R}$, $f(x,y) = xy + \frac{1}{xy}$, $A = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0, x+y < 2\}$



A nu e închisă, deci nici compactă $\Rightarrow$ nu putem aplica T.W.

Fie $x = \frac{1}{n+3}$, $y = \frac{1}{n+3}$, $x > 0, y > 0$, $x+y = \frac{2}{n+3} < 2$, $n \geq 0$.

$f\left(\frac{1}{n+3}, \frac{1}{n+3}\right) = \frac{1}{(n+3)^2} + (n+3)^2 \to \infty$ când $n \to \infty$

$\Rightarrow \sup f(A) = +\infty$, nu se atinge

Fie $\varphi: (0,\infty) \to \mathbb{R}$, $\varphi(t) = t + \frac{1}{t}$

t	0	1	∞
$\varphi'(t)$	$-$	0	$+$
$\varphi(t)$	$\searrow$		$\nearrow$

$\varphi'(t) = 1 - \frac{1}{t^2} = 0 \Leftrightarrow t = 1$.

1 - minim $\Rightarrow \varphi(t) \geq \varphi(1) = 2$.

$\Rightarrow f(x,y) \geq 2$.

Cum $\varphi(t) = 2 \Leftrightarrow t = 1 \Rightarrow xy = 1 \Rightarrow x = \frac{1}{y}$.

$x + y < 2 \Leftrightarrow \frac{1}{y} + y - 2 < 0 \Leftrightarrow \frac{1 + y^2 - 2y}{y} < 0 \Leftrightarrow \frac{(y-1)^2}{y} < 0$,

imposibil.

Deci $f(x,y) > 2$.

Fie $x = \frac{n-1}{n}$ și $y = \frac{n+2}{n+1}$. $\begin{cases} x, y > 0 \\ \text{pt. } n > 1 \end{cases}$; $x + y = \frac{n-1}{n} + \frac{n+2}{n+1} =$

$= \frac{n^2 - 1 + n^2 + 2n}{n^2 + n} = \frac{2n^2 + 2n - 1}{n^2 + n} < 2$

$f(x,y) = \frac{(n-1)(n+2)}{n(n+1)} + \frac{n(n+1)}{(n-1)(n+2)} = \frac{n^2 + n - 2}{n^2 + n} + \frac{n^2 + n}{n^2 + n - 2} \to 1 + 1 = 2$, $n \to \infty$

Deci $\inf f(A) = 2$, nu se atinge

3) Fie $A = \overline{B}(0_3, 1) \setminus \{0_3\}$ și $f: A \to \mathbb{R}$, $f(x,y,z) = \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}$

a) Există limita fc. f în origine?

Fie $a^n = \left(\frac{1}{n}, 0, 0\right)$ și $b^n = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right)$, $a^n, b^n \to 0_3$, $n \to \infty$

$$f(a^n) = \frac{\frac{1}{n}}{\sqrt{\frac{1}{n^2}}} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1, \quad n \to \infty$$

$$\neq \qquad \Rightarrow \nexists \text{ limită.}$$

$$f(b^n) = \frac{\frac{3}{n}}{\sqrt{\frac{3}{n^2}}} = \frac{\frac{3}{n}}{\frac{\sqrt{3}}{n}} = \sqrt{3}, \quad n \to \infty$$

b) Este A compactă?

$\forall r > 0$, $B(0_3, r) \cap A \neq \emptyset$ și $0_3 \in \mathbb{R}^n \setminus A$, deci $\forall r>0$ $\underbrace{B(0_3, r)}_{B_3} \cap (\mathbb{R}^n \setminus A) \neq \emptyset \Rightarrow$

$\left.\begin{array}{l} \Rightarrow 0_3 \in \text{fr}\, A \\ 0_3 \notin A \end{array}\right\} \Rightarrow \text{fr}\, A \not\subset A \Rightarrow A$ nu e închisă, deci nici compactă.

c) Determinați valorile extreme ale lui f pe mulțimea A. Atinge f aceste valori?

$x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$ atunci, din Cauchy-Schwarz ~~at x~~ avem:

$$(x_1y_1 + x_2y_2 + x_3y_3)^2 \leq (x_1^2 + x_2^2 + x_3^2) \cdot (y_1^2 + y_2^2 + y_3^2)$$

Fie $x_1 = x$, $x_2 = y$, $x_3 = z$; $y_1 = y_2 = y_3 = 1 \Rightarrow$

$$(x+y+z)^2 \leq (x^2+y^2+z^2) \cdot 3 \Rightarrow |x+y+z| \leq \sqrt{3} \cdot \sqrt{x^2+y^2+z^2} \Rightarrow$$

$$\Rightarrow \frac{|x+y+z|}{\sqrt{x^2+y^2+z^2}} \leq \sqrt{3} \Rightarrow -\sqrt{3} \leq \frac{x+y+z}{\sqrt{x^2+y^2+z^2}} \leq \sqrt{3}.$$

$\sup f(A) = \sqrt{3}$, se atinge pt. $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, iar $\inf f(A) = -\sqrt{3}$,

$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \in A$

$\left\|\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{3}{4}} < 1$, $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \in A$. se atinge pt. $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

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