Seminar 11

Aplicații

$\boxed{3.2.40}$ Se consideră în $\mathbb{R}^3$ lista de vectori $v = [v_1, v_2, v_3]^t$. Folosind două metode (definiția bazei, respectiv lema substituției) să se găsească $a \in \mathbb{R}$ a.î. v este o bază a lui $\mathbb{R}^3$, unde:

(1) $v_1 = (1, -2, 0)$; $v_2 = (2, 1, 1)$, $v_3 = (0, a, 1)$

(2) $v_1 = (2, 1, -1)$; $v_2 = (0, 3, -1)$, $v_3 = (1, a, 1)$

Soluție: (1) • Definiția:

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ $\to$ liniar independența este suficientă

$$\alpha_1 \cdot (1,-2,0) + \alpha_2 (2,1,1) + \alpha_3 (0,a,1) = (0,0,0)$$

$$(\alpha_1, -2\alpha_1, 0) + (2\alpha_2, \alpha_2, \alpha_2) + (0, a\alpha_3, \alpha_3) = (0,0,0)$$

$$(\alpha_1 + 2\alpha_2 + 0, -2\alpha_1 + \alpha_2 + a\alpha_3, \alpha_2 + \alpha_3) = (0,0,0)$$

$$\begin{cases} \alpha_1 + 2\alpha_2 + 0 \cdot \alpha_3 = 0 \\ -2\alpha_1 + \alpha_2 + a\alpha_3 = 0 \\ \alpha_1 \cdot 0 + \alpha_2 + \alpha_3 = 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & a \\ 0 & 1 & 1 \end{pmatrix}$$

Trebuie să obținem $\alpha_1 = \alpha_2 = \alpha_3 = 0$,

$\Leftrightarrow \det A \neq 0$

$$\Leftrightarrow \begin{vmatrix} 1 & 2 & 0 \\ -2 & 1 & a \\ 0 & 1 & 1 \end{vmatrix} \neq 0$$

$\Longleftrightarrow 1 + (-2)\cdot 1\cdot 0 + 0\cdot 2\cdot a - (0 + a - 4) \neq 0$

$1 + 4 - a \neq 0$

$5 - a \neq 0 \Rightarrow a \neq 5$

$\Rightarrow a \in \mathbb{R} \setminus \{5\}$

• Lema substituției

Fie $b = [b_1, b_2, \dots, b_n]^t$ o bază a K - spațiului vectorial V și $v \in V$ cu coordonatele $[d_1, d_2, \dots, d_n]$ în raport cu baza b. ($v = d_1 \cdot b_1 + d_2 \cdot b_2 + \dots + d_n b_n$)

Considerăm o listă de vectori $b' = [b_1, \dots, v, \dots, b_n]$ care rezultă din v prin înlocuirea vectorului b_i cu v.

Atunci:

(a) b' este bază dacă $\boxed{d_i \neq 0}$

(b) Dacă b' este bază și $x \in V$ are coordonatele $[x_1, x_2, \ldots, x_n]$ în raport cu b și $[x_1', x_2', \ldots, x_n']$ în raport cu b', atunci:

$$\boxed{\begin{cases} x_i' = \alpha_i^{-1} \cdot x_i \\ x_j' = \alpha_i^{-1} (\alpha_i x_j - \alpha_j x_i), \quad j \neq i \end{cases}}$$

$v_1 = (1, -2, 0)$; $v_2 = (\underset{x_1}{2}, \underset{x_2}{1}, \underset{x_3}{1})$

$$v_1 = \underbrace{1}_{\alpha_1} \cdot e_1 + \underbrace{(-2)}_{\alpha_2} \cdot e_2 + \underbrace{0}_{\alpha_3} \cdot e_3$$

$\underline{\alpha_1 = 1 \neq 0} \Rightarrow u_1 = (v_1, e_2, e_3)$ este o bază în $\mathbb{R}^3$

Coordonatele lui v_2 în baza u_2:

$$x_1' = \alpha_1^{-1} \cdot x_1 = \frac{1}{\alpha_1} \cdot x_1 = \frac{1}{1} \cdot 2 = $$
$$= 2$$

$$x_2' = \frac{1}{\alpha_1} \cdot (\alpha_1 \cdot x_2 - \alpha_2 x_1) =$$

$$= \frac{1}{1} \cdot (1 \cdot 1 - (-2) \cdot 2) = 1 + 4 = 5$$

$$x_3' = \frac{1}{\alpha_1} (\alpha_1 \cdot x_3 - \alpha_3 x_1) =$$

$$= \frac{1}{1} (1 \cdot 1 - 0 \cdot 2) = 1 - 0 = 1$$

$$v_2 = (2, \boxed{5}, 1) = 2 \cdot v_1 + 5 \cdot e_2 + e_3$$

$\alpha_2 = 5 \neq 0 \quad \Rightarrow \quad u_2 = (v_1, v_2, e_3)$ bază în $\mathbb{R}^3$

$$v_3 = (0, a, 1) = \underbrace{0}_{x_1} \cdot e_1 + \underbrace{a}_{x_2} \cdot e_2 + \underbrace{1}_{x_3} \cdot e_3$$

- în baza u_1

$$x_1' = \frac{1}{\alpha_1} \cdot x_1 = \frac{1}{1} \cdot 0 = 0$$

$$x_2' = \frac{1}{\alpha_1} \cdot (\alpha_1 x_2 - \alpha_2 x_1) =$$

$$= \frac{1}{1} \cdot (1 \cdot a - (-2) \cdot 0) = a$$

$$x_3' = \frac{1}{\alpha_1}(\alpha_1 x_3 - \alpha_3 x_1) =$$

$$= \frac{1}{1}(1 \cdot 1 - 0 \cdot 0) = 1$$

$$v_3 = \underbrace{0}_{x_1} \cdot v_1 + \underbrace{a}_{x_2} \cdot e_2 + \underbrace{1}_{x_3} \cdot e_3$$

- în baza u_2:

$$x_2' = \frac{1}{\alpha_2} \cdot x_2 = \frac{1}{5} \cdot a = \frac{a}{5}$$

$$x_1' = \frac{1}{\alpha_2}(\alpha_2 x_1 - \alpha_1 x_2) =$$

$$= \frac{1}{5}(5 \cdot 0 - 2 \cdot a) = -\frac{2a}{5}$$

$$x_3' = \frac{1}{\alpha_2}(\alpha_2 x_3 - \alpha_3 x_2) =$$

$$= \frac{1}{5}(5 \cdot 1 - 1 \cdot a) = \frac{5-a}{5}$$

$$v_3 = -\frac{2a}{5} v_1 + \frac{a}{5} v_2 + \frac{5-a}{5} e_3$$

$$\frac{5-a}{5} \neq 0 \Rightarrow 5 - a \neq 0$$

$$a \neq 5$$

$$\Rightarrow a \in \mathbb{R} \setminus \{5\}$$

(2) $v_1 = (2, 1, -1)$

$v_2 = (0, 3, -1)$

$v_3 = (1, a, 1)$

• Definiția:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 =$$

$$= \alpha_1 (2, 1, -1) + \alpha_2 (0, 3, -1) + \alpha_3 (1, a, 1) =$$

$$= (2\alpha_1, \alpha_1, -\alpha_1) + (0 \cdot \alpha_2, 3\alpha_2, -\alpha_2) + (\alpha_3, a\alpha_3, \alpha_3)$$

$$= (2\alpha_1 + 0 \cdot \alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + a\alpha_3, -\alpha_1 - \alpha_2 + \alpha_3)$$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow \begin{cases} 2\alpha_1 + 0 \cdot \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + 3\alpha_2 + a\alpha_3 = 0 \\ -\alpha_1 - \alpha_2 + \alpha_3 = 0 \end{cases}$$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & a \\ -1 & -1 & 1 \end{pmatrix}$$

$\det A \neq 0$

$$\iff \begin{vmatrix} 2 & 0 & 1 \\ 1 & 3 & a \\ -1 & -1 & 1 \end{vmatrix} \neq 0$$

$$6 - 1 + 0 + 3 + 2a \neq 0$$

$$2a + 8 \neq 0$$

$$2a \neq -8$$

$$a \neq -4 \implies a \in \mathbb{R} \setminus \{-4\}$$

- Lema substituției

$v_1 = (2, 1, -1)$, $v_2 = (\underset{x_1}{0}, \underset{x_2}{3}, \underset{x_3}{-1})$

$$v_1 = \underset{\alpha_1}{2} \cdot e_1 + \underset{\alpha_2}{1} \cdot e_2 + \underset{\alpha_3}{(-1)} \cdot e_3$$

$\alpha_1 \neq 0 \implies \mathcal{U}_1 = \{v_1, e_2, e_3\}$ este bază în $\mathbb{R}^3$

Coord. lui v_2 în baza $\mathcal{U}_1$:

$$x_1' = \frac{1}{\alpha_1} \cdot x_1 = \frac{1}{2} \cdot 0 = 0$$

$$x_2' = \frac{1}{\alpha_1}(\alpha_1 x_2 - \alpha_2 x_1) =$$

$$= \frac{1}{2}(2 \cdot 3 - 1 \cdot 0) = 3$$

$$x_3' = \frac{1}{\alpha_1}(\alpha_1 x_3 - \alpha_3 x_1) =$$

$$= \frac{1}{2}(2 \cdot (-1) - (-1) \cdot 0) = -1$$

$$v_2 = (0, 3, -1) = \underset{\alpha_1}{0} \cdot v_1 + \underset{\alpha_2}{3} \cdot e_2 + \underset{\alpha_3}{(-1)} \cdot e_3$$

$\underline{\underline{\alpha_2 = 3 \neq 0}} \Rightarrow \mathcal{U}_2 = (v_1, v_2, e_3)$ este bază în $\mathbb{R}^3$

$$v_3 = (1, a, 1)$$

- bază în $\mathcal{U}_1$:

$$x_1' = \frac{1}{\alpha_1} \cdot x_1 = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$x_2' = \frac{1}{\alpha_1}(\alpha_1 x_2 - \alpha_2 x_1) = \frac{1}{2}(2a - 1 \cdot 1)$$

$$= \frac{2a-1}{2}$$

$$x_3' = \frac{1}{\alpha_1}(\alpha_1 x_3 - \alpha_3 x_1) = \frac{1}{2}\Big(2\cdot 1 -$$
$$-(-1)\cdot 1\Big) = \frac{3}{2}$$

$$v_3 = \left(\frac{1}{2}, \frac{2a-1}{2}, \frac{3}{2}\right) =$$
$$= \frac{1}{2} v_1 + \frac{2a-1}{2} e_2 + \frac{3}{2} e_3$$

- baza u_2:

$$x_1' = \frac{1}{\alpha_2}\cdot(\alpha_2 x_1 - \alpha_1 x_2) =$$
$$= \frac{1}{3}\cdot\left(3\cdot\frac{1}{2} - 0\cdot\frac{2a-1}{2}\right)$$
$$= \frac{1}{2}$$

$$x_2' = \frac{1}{\alpha_2}\cdot x_2 = \frac{1}{3}\cdot\frac{2a-1}{2} = \frac{2a-1}{6}$$

$$x_3' = \frac{1}{\alpha_2}(\alpha_2 x_3 - \alpha_3 x_2) =$$
$$= \frac{1}{3}\left(3\cdot\frac{3}{2} - (-1)\cdot\frac{2a-1}{2}\right) =$$
$$= \frac{a+4}{3}$$

$$v_3 = \frac{1}{2} v_1 + \frac{2a-1}{6} v_2 + \frac{a+4}{3} e_3$$

$$\frac{a+4}{3} \neq 0 \Rightarrow a+4 \neq 0$$

$$a \neq -4$$

$$\Rightarrow a \in \mathbb{R} \setminus \{-4\}$$

$\boxed{3.2.41}$ Să se arate că $b = [b_1, b_2, b_3, b_4]^t$ unde $b_1 = [1, 2, -1, 2]$, $b_2 = [1, 2, 1, 4]$, $b_3 = [2, 3, 0, -1]$, $b_4 = [1, 3, -1, 0]$ este o bază a lui $\mathbb{R}^4$ și să se determine coordonatele lui $X = [2, 3, 2, 10]$ în raport cu acea bază.

Soluție:

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 1 & 0 & -1 \\ 2 & 4 & -1 & 0 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 1 & 0 & -1 \\ 2 & 4 & -1 & 0 \end{vmatrix} =$$

$$\begin{array}{c} C_1+C_2 \\ = \\ C_4+C_2 \end{array} \begin{vmatrix} 2 & 1 & 2 & 2 \\ 4 & 2 & 3 & 5 \\ 0 & 1 & 0 & 0 \\ 6 & 4 & -1 & 4 \end{vmatrix} =$$

$$= 1 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 2 & 2 & 2 \\ 4 & 3 & 5 \\ 6 & -1 & 4 \end{vmatrix} =$$

$$\begin{array}{c} C_1-C_2 \\ = \\ C_3-C_2 \end{array} (-1) \cdot \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & 2 \\ 7 & -1 & 5 \end{vmatrix} = (-1) \cdot 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & 2 \\ 7 & 5 \end{vmatrix}$$

$$= 2 \cdot (5 - 14) = -18 \neq 0$$

$\Rightarrow$ rang $A = 4$ (rang maximal)

$\dim \mathbb{R}^4 = 4$

$\Rightarrow$ $\mathcal{B}$ este bază în $\mathbb{R}^4$

$$X = (2, 3, 2, 10)$$

$$(2, 3, 2, 10) = \underline{\alpha_1}\, b_1 + \underline{\alpha_2}\, b_2 + \underline{\alpha_3}\, b_3 + \underline{\alpha_4}\, b_4$$

$$(2,3,2,10) = \alpha_1(1,2,-1,2) + \alpha_2(1,2,1,4) + \alpha_3(2,3,0,-1) + \alpha_4(1,3,-1,0)$$

$$(2,3,2,10) = (\alpha_1, 2\alpha_1, -\alpha_1, 2\alpha_1) + (\alpha_2, 2\alpha_2, \alpha_2, 4\alpha_2) + (2\alpha_3, 3\alpha_3, 0, -\alpha_3) + (\alpha_4, 3\alpha_4, -\alpha_4, 0)$$

$$(2,3,2,10) = (\alpha_1+\alpha_2+2\alpha_3+\alpha_4,\ 2\alpha_1+2\alpha_2+3\alpha_3+3\alpha_4,\ -\alpha_1+\alpha_2+0-\alpha_4,\ 2\alpha_1+4\alpha_2-\alpha_3+0)$$

$$\begin{cases} \alpha_1+\alpha_2+2\alpha_3+\alpha_4 = 2 \\ 2\alpha_1+2\alpha_2+3\alpha_3+3\alpha_4 = 3 \\ -\alpha_1+\alpha_2+0\cdot\alpha_3-\alpha_4 = 2 \\ 2\alpha_1+4\alpha_2-\alpha_3+0\cdot\alpha_4 = 10 \end{cases}$$

$$\bar{A} = \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 2 & 2 & 3 & 3 & 3 \\ -1 & 1 & 0 & -1 & 2 \\ 2 & 4 & -1 & 0 & 10 \end{array}\right)$$

$$\det A = -18 \neq 0$$

$\Rightarrow$ sistem compatibil determinat (sistem Cramer)

$$\alpha_i = \frac{d_{\alpha_i}}{\det A}$$

$$\alpha_1 = \frac{d_{\alpha_1}}{-18}$$

$$d_{\alpha_1} = \begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 2 & 3 & 3 \\ 2 & 1 & 0 & -1 \\ 10 & 4 & -1 & 0 \end{vmatrix} \begin{matrix} C_1-2C_2 \\ = \\ C_4+C_2 \end{matrix}$$

$$= \begin{vmatrix} 0 & 1 & 2 & 2 \\ -1 & 2 & 3 & 5 \\ 0 & 1 & 0 & 0 \\ 2 & 4 & -1 & 4 \end{vmatrix} =$$

$$= 1 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 0 & 2 & 2 \\ -1 & 3 & 5 \\ 2 & -1 & 4 \end{vmatrix}$$

$$\overset{C_3-C_2}{=} - \begin{vmatrix} 0 & 2 & 0 \\ -1 & 3 & 2 \\ 2 & -1 & 5 \end{vmatrix} =$$

$$= -2 \cdot (-1)^{1+2} \begin{vmatrix} -1 & 2 \\ 2 & 5 \end{vmatrix} =$$

$$= 2 \cdot (-5-4) = -18$$

$$\boxed{\alpha_1} = \frac{-18}{-18} = \boxed{1}$$

$$\alpha_2 = \frac{d_{\alpha_2}}{\det A} = \frac{d_{\alpha_2}}{-18}$$

$$d_{\alpha_2} = \begin{vmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 3 \\ -1 & 2 & 0 & -1 \\ 2 & 10 & -1 & 0 \end{vmatrix} =$$

$$\overset{C_2+2C_1}{\underset{C_4-C_1}{=}} \begin{vmatrix} 1 & 4 & 2 & 0 \\ 2 & 7 & 3 & 1 \\ -1 & 0 & 0 & 0 \\ 2 & 14 & -1 & -2 \end{vmatrix} =$$

$$= (-1) \cdot (-1)^4 \cdot \begin{vmatrix} 4 & 2 & 0 \\ 7 & 3 & 1 \\ 14 & -1 & -2 \end{vmatrix} =$$

$$\overset{C_1-2C_2}{=} - \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & 1 \\ 16 & -1 & -2 \end{vmatrix} = -2 \cdot (-1)^3 \begin{vmatrix} 1 & 1 \\ 16 & -2 \end{vmatrix}$$

$$= 2(-2-16) = -36$$

$$\boxed{\alpha_2} = \frac{-36}{-18} = \boxed{2}$$

$$\alpha_3 = \frac{d_{\alpha_3}}{-18}$$

$$d_{\alpha_3} = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 1 & 2 & -1 \\ 2 & 4 & 10 & 0 \end{vmatrix} \begin{matrix} C_1 - C_2 \\ = \\ C_3 - 2C_2 \\ C_4 - C_2 \end{matrix}$$

$$= \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 \\ -2 & 1 & 0 & -2 \\ -2 & 4 & 2 & -4 \end{vmatrix} = 1 \cdot (-1)^3 \cdot \begin{vmatrix} 0 & -1 & 1 \\ -2 & 0 & -2 \\ -2 & 2 & -4 \end{vmatrix}$$

$$\underset{=}{C_2+C_3} - \begin{vmatrix} 0 & 0 & 1 \\ -2 & -2 & -2 \\ -2 & -2 & -4 \end{vmatrix} = -(-1)^4 \cdot \begin{vmatrix} -2 & -2 \\ -2 & -2 \end{vmatrix}$$

$$= 0$$

$$\boxed{\alpha_3} = \frac{0}{-18} = \boxed{0}$$

$$\alpha_4 = \frac{d_{\alpha_4}}{-18}$$

$$d_{\alpha_4} = \begin{vmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ -1 & 1 & 0 & 2 \\ 2 & 4 & -1 & 10 \end{vmatrix} \begin{matrix} C_1 + C_2 \\ = \\ C_4 - 2C_2 \end{matrix}$$

$$= \begin{vmatrix} 2 & 1 & 2 & 0 \\ 4 & 2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 6 & 4 & -1 & 2 \end{vmatrix} = 1 \cdot (-1)^5 \cdot \begin{vmatrix} 2 & 2 & 0 \\ 4 & 3 & -1 \\ 6 & -1 & 2 \end{vmatrix}$$

$C_1 - C_2$

$$= - \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & -1 \\ 7 & -1 & 2 \end{vmatrix} = -2 \cdot (-1)^3 \begin{vmatrix} 1 & -1 \\ 7 & 2 \end{vmatrix}$$

$$= 2(2+7) = 18$$

$$\boxed{\alpha_4} = \frac{18}{-18} = \boxed{-1}$$

$$X = (\underset{\alpha_1}{\underset{\downarrow}{1}}, \underset{\alpha_2}{\underset{\downarrow}{2}}, \underset{\alpha_3}{\underset{\downarrow}{0}}, \underset{\alpha_4}{\underset{\downarrow}{-1}})$$ (în baza b)