① Integrarea funcțiilor raționale

* **Descompunerea în fracții simple**

1) $\dfrac{\alpha x+\beta}{(x-a)(x-b)} = \dfrac{A}{x-a} + \dfrac{B}{x-b}$, $a \neq b$

2) $\dfrac{\alpha x+\beta}{(x-a)^2} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2}$

3) $\dfrac{\alpha x^2+\beta x+\gamma}{(x-a)(x^2+px+q)} = \dfrac{A}{x-a} + \dfrac{Bx+C}{x^2+px+q}$, cu $\Delta = p^2-4q<0$

4) $\dfrac{\alpha x^3+\beta x^2+\gamma x+\delta}{(x^2+a^2)(x^2+b^2)} = \dfrac{Ax+B}{x^2+a^2} + \dfrac{Cx+D}{x^2+b^2}$

5) $\dfrac{P(x)}{(x^2-a^2)(x^2+b^2)^2} = \dfrac{A}{x-a} + \dfrac{B}{x+a} + \dfrac{Cx+D}{x^2+b^2} + \dfrac{Ex+F}{(x^2+b^2)^2}$

a) $\displaystyle\int \frac{3x-4}{x^2+x-6}\,dx$, $x \in J \subset \mathbb{R} \setminus \{2,-3\}$

$x^2+x-6 = (x-2)(x+3)$

$\dfrac{3x-4}{x^2+x-6} = \dfrac{3x-4}{(x-2)(x+3)} = \dfrac{^{x+3)}A}{x-2} + \dfrac{^{x-2)}B}{x+3}$.

Aducem la același numitor.

$3x-4 = Ax+3A+Bx-2B$

$3x-4 = (A+B)x + (3A-2B)$

Egalăm coeficienții

$\begin{cases} A+B=3 \quad |\cdot 2 \\ 3A-2B=-4 \end{cases}$ ⊕

$5A = 2 \Rightarrow A = \frac{2}{5} \Rightarrow B = 3-\frac{3}{5} = \frac{13}{5}$ ①

$\Rightarrow y = \displaystyle\int\left(\frac{\frac{2}{5}}{x-2} + \frac{\frac{13}{5}}{x+3}\right)dx =$

$= \frac{2}{5}\displaystyle\int \frac{1}{x-2}\,dx + \frac{13}{5}\int\frac{1}{x+3}\,dx =$

$= \frac{2}{5}\ln|x-2| + \frac{13}{5}\ln|x+3| + \mathcal{C}$

b) $\int \frac{4x+3}{4x^2-4x+1}\,dx,\quad x\in J\subset \mathbb{R}\setminus\{\frac{1}{2}\}$

$$\frac{4x+3}{4x^2-4x+1}=\frac{4x+3}{(2x-1)^2}=\frac{\overset{2x-1}{A}}{2x-1}+\frac{B}{(2x-1)^2}.$$

$$4x+3=A(2x-1)+B$$

$$\underline{4}x+\underline{3}=\underline{2A}x+\underline{B-A}$$

$2A=4 \Rightarrow A=2$

$B-A=3 \Rightarrow B=5$

$$\mathcal{I}=\int\frac{2}{2x-1}\,dx+\int\frac{5}{(2x-1)^2}\,dx=\ln|2x-1|-\frac{5}{2}\cdot\frac{1}{2x-1}+\mathcal{C}$$

!c) $\int \frac{1}{(x^2+3)^2}\,dx$

$$\frac{1}{(x^2+3)^2}=\frac{\overset{x^2+3}{Ax+B}}{x^2+3}+\frac{Cx+D}{(x^2+3)^2}$$

$$1=Ax^3+Bx^2+3Ax+3B+Cx+D$$

$$\underline{0}\cdot x^3+\underline{0}\cdot x^2+\underline{0}\cdot x+\underline{1}=\underline{A}x^3+\underline{B}x^2+(\underline{3A+C})x+\underline{\underline{3B+D}}$$

$A=0,\ B=0,\ C=0,\ D=1$, deci obținem aceeași fracție.

Așadar, nu putem aplica descompunerea în acest caz.!

$$\mathcal{I}=\int\frac{x^2+3-x^2}{3(x^2+3)^2}\,dx=\frac{1}{3}\int\frac{1}{x^2+3}\,dx-\frac{1}{3}\int\frac{x^2}{(x^2+3)^2}\,dx$$

↓ cu formula

↓ prin părți cu

$f=x \Rightarrow f'=1$

$g'=\frac{x}{(x^2+3)^2}\Rightarrow g=-\frac{1}{2}\cdot\frac{1}{x^2+3}$

d) $\int \frac{2x^2+x+3}{(x^2+3)(x-1)}\,dx$

$$\frac{2x^2+x+3}{(x^2+3)(x-1)} = {}^{x^2+3/}\frac{A}{x-1} + {}^{x-1/}\frac{Bx+C}{x^2+3}$$

$$2x^2+x+3 = Ax^2+3A+Bx^2+Cx-Bx-C$$

$$= (A+B)x^2+(C-B)x+3A-C$$

$$\begin{cases} A+B=2 \\ C-B=1 \\ 3A-C=3 \end{cases} \Rightarrow \begin{cases} A=\frac{3}{2} \\ B=\frac{1}{2} \\ C=\frac{3}{2} \end{cases}$$

$$y = \frac{3}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{x+3}{x^2+3}\,dx =$$

$$= \frac{3}{2}\ln|x-1| + \frac{1}{2}\int \frac{x}{x^2+3}\,dx + \frac{1}{2}\int \frac{3}{x^2+3}\,dx$$

$$= \frac{3}{2}\ln|x-1| + \frac{1}{4}\ln(x^2+3) + \frac{\sqrt{3}}{2}\operatorname{arctg}\frac{x}{\sqrt{3}} + C$$

e) $\int \frac{x^2+x+1}{(x+2)^2(x-3)}\,dx$

$$\frac{x^2+x+1}{(x+2)^2(x-3)} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

f) $\int \frac{x^3+x+2}{(x-2)^2(x^2+1)}\,dx$

$$\frac{x^3+x+2}{(x-2)^2(x^2+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+1}$$

g) $\int \frac{2x^3-x^2-x+2}{(x^2+1)(x^2+4)}\,dx$

$$\frac{2x^3-x^2-x+2}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

h) $\int \frac{3x^3-x+4}{(x+1)(x^2+x+1)^2}\,dx$

$$\frac{3x^3-x+4}{(x+1)(x^2+x+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

② Substituții trigonometrice pentru integrale algebrice

i) $\int R(x, \sqrt{a^2-x^2})\,dx$ $\qquad x = a\sin t$ sau $x = a\cos t$

1) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$

$a=1.$ $\quad x = \sin t$ $\qquad x=0 \Rightarrow t = \arcsin 0 = 0$

$dx = \cos t\,dt$ $\qquad x=1 \Rightarrow t = \arcsin 1 = \frac{\pi}{2}.$

$$I = \int_0^{\pi/2} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}}\cdot \cos t\,dt = \int_0^{\pi/2} \frac{\sin^2 t}{\underbrace{\sqrt{\cos^2 t}}_{=|\cos t| = \cos t,\ t\in[0,\pi/2]}}\cdot \cos t\,dt =$$

$$= \int_0^{\pi/2} \sin^2 t\,dt = \int_0^{\pi/2} \frac{1-\cos 2t}{2}\,dt = \frac{1}{2}t\Big|_0^{\pi/2} - \frac{1}{2}\cdot\frac{\sin 2t}{2}\Big|_0^{\pi/2} =$$

$\cos 2t = 1 - 2\sin^2 t$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$$= \frac{\pi}{4} - \frac{1}{4}(\underbrace{\sin\pi - \sin 0}_{=0}) = \frac{\pi}{4}.$$

2) $\displaystyle\int_0^{1/2} \frac{(1+x)}{\sqrt{1-x^2}}\,dx$

ii) $\int R(\sqrt{a^2+x^2}, x)\,dx$ $\qquad x = a\,tg\,t$ sau $x = a\,ctg\,t$

1) $\displaystyle I = \int_1^{\sqrt3} \frac{\sqrt{1+x^2}}{x}\,dx$

$a=1,$ $\quad x = tg\,t$ $\qquad x=1 \Rightarrow t = arctg\,1 = \pi/4$

$\qquad\qquad\qquad\qquad x=\sqrt3 \Rightarrow t = arctg\sqrt3 = \frac{\pi}{3}$

$dx = \frac{1}{\cos^2 t}\,dt$

$$\sqrt{1+x^2} = \sqrt{1+tg^2 t} = \sqrt{1+\frac{\sin^2 t}{\cos^2 t}} = \sqrt{\frac{1}{\cos^2 t}} = \frac{1}{\cos t}$$

$\cos t > 0$

$$I = \int_{\pi/4}^{\pi/3} \frac{1}{\cos t\cdot \underbrace{tg\,t}_{=\frac{\sin t}{\cos t}}}\cdot\frac{1}{\cos^2 t}\,dt = \int_{\pi/4}^{\pi/3}\frac{1}{\cos^2 t\cdot\sin t}\,dt = \int_{\pi/4}^{\pi/3}\frac{\sin^2 t+\cos^2 t}{\cos^2 t\cdot \sin t} =$$

$$= \int_{\pi/4}^{\pi/3}\frac{\sin t}{\cos^2 t}\,dt + \int_{\pi/4}^{\pi/3}\frac{1}{\sin t}\,dt = \frac{1}{\cos t}\Big|_{\pi/4}^{\pi/3} + \ln\left(tg\frac{t}{2}\right)\Big|_{\pi/4}^{\pi/3} = \ldots$$

$\boxed{2)}$ $J=\int_1^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x^2}\,dx$

iii) $\int R(x, \sqrt{x^2-a^2})\,dx$ $x=\frac{a}{\sin t}$ sau $x=\frac{a}{\cos t}$

$\boxed{T}$ $J=\int_{2/5}^{4/5} \frac{\sqrt{25x^2-4}}{x}\,dx$

3) Substituțiile lui Euler pt. integrale algebrice $R(\sqrt{ax^2+bx+c}, x)$

1) $\sqrt{ax^2+bx+c}=t(x-r)$, dacă r e rădăcină a ec. $ax^2+bx+c=0$

$$J=\int_0^{1/2} \frac{1}{(1+x)\sqrt{1-x^2}}\,dx$$

$\sqrt{1-x^2}=t(x+1)$

$1-x^2=t^2(x+1)^2$

~~$1-x^2=t^2(x^2+2x+1)$~~

$(1+x)(1-x)=t^2(x+1)^2 \Rightarrow 1-x=t^2x+t^2$

$t^2x+x=-t^2+1$

$x=\frac{-t^2+1}{t^2+1}$

$x=0 \Rightarrow t=+1$

$x=\frac{1}{2} \Rightarrow t=\frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}}=+\frac{\sqrt{3}}{3}$

$dx=\frac{-2t(t^2+1)-2t(-t^2+1)}{(t^2+1)^2}=\frac{-4t}{(t^2+1)^2}$

$$J=\int_1^{\frac{\sqrt{3}}{3}} \frac{1}{\left(1+\frac{1-t^2}{1+t^2}\right)\cdot t\left(\frac{1-t^2}{1+t^2}+1\right)}\cdot\frac{-4t}{(t^2+1)^2}\,dt=\int_1^{\frac{\sqrt{3}}{3}} \frac{1}{\cancel{t}\cdot\left(\frac{2}{1+t^2}\right)^2}\cdot\frac{-4\cancel{t}}{(t^2+1)^2}\,dt$$

$$=-\int_1^{\frac{\sqrt{3}}{3}} 1\,dt=-\frac{\sqrt{3}}{3}+1.$$

$\boxed{2)}$ $J=\int \frac{1}{(1+x^2)\cdot\sqrt{1-x^2}}$

ii) $\sqrt{ax^2+bx+c} = x\sqrt{a} + t$, dacă $a>0$

1) $J = \int \frac{1}{(1+x)\sqrt{1+x+x^2}}\, dx$

$\sqrt{x^2+x+1} = x\cdot\sqrt{1} + t$
$= x + t$

$\cancel{x^2} + x + 1 = \cancel{x^2} + 2xt + t^2$

$x(1-2t) = t^2 - 1$

$x = \frac{t^2-1}{1-2t}$

$dx = \frac{2t(1-2t) + 2(t^2-1)}{(1-2t)^2}\, dt$

$dx = \frac{-2t^2+2t-2}{(1-2t)^2}$

$J = \int \frac{1}{\left(1+\frac{t^2-1}{1-2t}\right)\cdot\left(\frac{t^2-1}{1-2t}+t\right)} \cdot \frac{-2t^2+2t-2}{(1-2t)^2}\, dt$

$= \int \frac{\cancel{(1-2t)^2}}{(t^2-2t)\cdot(-t^2+t-1)} \cdot \frac{-2t^2+2t-2}{\cancel{(1-2t)^2}}\, dt$

$= 2\int \frac{1}{t(t-2)}\, dt = \frac{2}{2}\int\left(\frac{1}{t-2} - \frac{1}{t}\right) dt$

$= \ln|t-2| - \ln|t| + \mathcal{C}$

$= \ln\left|\sqrt{x^2+x+1} - x - 2\right| - \ln\left|\sqrt{x^2+x+1} - x\right| + \mathcal{C}$

$\boxed{2}$ $J = \int \frac{x^2}{\sqrt{1+x^2}}\, dx$

iii) $\sqrt{ax^2+bx+c} = \sqrt{c} + tx$, dacă $c>0$

1) $J = \int \frac{dx}{(x+1)\sqrt{-x^2+x+1}}$, $c=1$

$\sqrt{-x^2+x+1} = 1 + tx$

$-x^2 + x + \cancel{1} = \cancel{1} + 2tx + t^2x^2$

$x^2(t^2+1) + (2t-1)x = 0$

$x(t^2+1) + (2t-1) = 0$

$x = \frac{1-2t}{1+t^2}$ $\quad dx = \frac{-2(1+t^2) - 2t(1-2t)}{(1+t^2)^2}\, dt$

$dx = \frac{2t^2-2t-2}{(t^2+1)^2}\, dt$

$x + 1 = \frac{1-2t}{1+t^2} + 1 = \frac{t^2-2t+2}{t^2+1}$

$\sqrt{-x^2+x+1} = 1 + tx = 1 + \frac{t-2t^2}{1+t^2} = \frac{-t^2+t+1}{1+t^2}$

$J = \int \frac{1}{\frac{t^2-2t+2}{\cancel{t^2+1}} \cdot \frac{-t^2+t+1}{\cancel{t^2+1}}} \cdot \frac{\overset{-2}{\cancel{2t^2-2t-2}}}{\cancel{(t^2+1)^2}}\, dt$

$= \int \frac{-2}{t^2-2t+2}\, dt = \int \frac{-2}{(t-1)^2+1}\, dt =$

$= -2\cdot \operatorname{arctg}(t-1) + \mathcal{C}$
$t = \dots$
$= \dots$

$\boxed{2}$ $J = \int \frac{1}{1+\sqrt{1-2x-x^2}}\, dx$

④ Substituțiile lui Cebâșev pt. Integrale binome

$\int x^m \cdot (ax^n+b)^p\,dx \qquad m,n,p\in\mathbb{Q},\ a,b\in\mathbb{R}$

i) $x=t^r$, dacă $p\in\mathbb{Z}$ și r e multiplu comun al numitorilor lui m și n

1) $J=\int_1^{64}\frac{\left(1+\sqrt[3]{x^2}\right)^2}{\sqrt{x^3}}\,dx = \int_1^{64} x^{-3/2}\cdot\left(x^{2/3}+1\right)^2\,dx$

$p=2\in\mathbb{Z}$

$m=-3/2$, $n=2/3$ $\Rightarrow$ multiplu comun al numitorilor $=6$

$x=t^6$ $\qquad x=1\Rightarrow t=1$

$dx=6t^5$ $\qquad x=64\Rightarrow t=2$

$x^{-3/2}=\left(t^6\right)^{-3/2}=t^{-9}$

$x^{2/3}=\left(t^6\right)^{2/3}=t^4$

$J=\int_1^2 t^{-9}\cdot\left(t^4+1\right)^2\cdot 6t^5\,dt=$

$=6\int_1^2 t^{-4}\left(t^8+2t^4+1\right)dt=$

$=6\int_1^2\left(t^4+2+t^{-4}\right)dt=\dots$

T ② $J=\int\frac{1}{\sqrt[3]{x^2}\cdot\left(1+\sqrt[3]{x}\right)}\,dx$

ii) $ax^n+b=t^r$, dacă $\frac{m+1}{n}\in\mathbb{Z}$ și r e numitorul lui p.

1) $J=\int_1^2 x^3\cdot\frac{1}{\sqrt{x^2+1}}\,dx=\int_1^2 x^3\cdot\left(x^2+1\right)^{-1/2}dx$

$m=3$, $n=2$ $\qquad \frac{m+1}{n}=2\in\mathbb{Z}$ $\qquad$ numitorul lui $p=\frac{-1}{2}$ este 2.

$x^2+1=t^2$

$2x\,dx=2t\,dt$

$x\,dx=t\,dt$

$x^2=t^2-1$

$x=1\Rightarrow t=\sqrt2$

$x=2\Rightarrow t=\sqrt5$

$J=\int_{\sqrt2}^{\sqrt5}\left(t^2-1\right)\cdot\left(t^2\right)^{-1/2}\cdot t\,dt=\int_{\sqrt2}^{\sqrt5}\left(t^2-1\right)\cdot\frac{1}{t}\cdot t\,dt$

$=\int_{\sqrt2}^{\sqrt5}\left(t^2-1\right)dt=$

$=\frac{t^3}{3}\Big|_{\sqrt2}^{\sqrt5}-t\Big|_{\sqrt2}^{\sqrt5}=\frac{5\sqrt5}{3}-\frac{2\sqrt2}{3}-\frac{3\sqrt5}{3}+\frac{3\sqrt2}{3}=\frac{2\sqrt5+\sqrt2}{3}$

T ② $J=\int x^3\sqrt[4]{1+x^4}\,dx$

iii) $a+b\cdot x^{-n}=t^{s}$, dacă $\frac{m+1}{n}+p\in\mathbb{Z}$ și s e numitorul lui p.

1) $I=\int_2^3 \frac{1}{x^4\sqrt{x^2+1}}\,dx=\int_2^3 x^{-4}\left(x^2+1\right)^{-1/2}\,dt$

$m=-4$ $\quad \frac{m+1}{n}=\frac{-3}{2}\in\mathbb{Z}$ $\quad \frac{m+1}{n}+p=\frac{-4}{2}=-2\in\mathbb{Z}$

$n=2$

$p=-1/2$ $\quad r=2$

$1+x^{-2}=t^2$

$x^{-2}=t^2-1$

$\frac{1}{x^2}=t^2-1 \Rightarrow x^2=\frac{1}{t^2-1}$ $\quad x^2+1=\frac{t^2-1+1}{t^2-1}=\frac{t^2}{t^2-1}$

$x=\sqrt{\frac{1}{t^2-1}}$

$dx=\left[\left(t^2-1\right)^{-1/2}\right]'=-\frac{1}{2}\cdot\left(t^2-1\right)^{-3/2}\cdot 2t=-t\left(t^2-1\right)^{-3/2}\,dt$

$x=2\Rightarrow t^2=1+\frac{1}{4}=\frac{5}{4}\Rightarrow t=\frac{\sqrt{5}}{2}$

$x=3\Rightarrow t^2=1+\frac{1}{9}=\frac{10}{9}\Rightarrow t=\frac{\sqrt{10}}{3}$

$I=\int_{\frac{\sqrt{5}}{2}}^{\frac{\sqrt{10}}{3}}\left(t^2-1\right)^{\cancel{2}}\cdot\left(\frac{t^2}{\cancel{t^2-1}}\right)^{-1/2}\cdot\frac{-t}{\cancel{(t^2-1)}\cdot(t^2-1)^{1/2}}=\int_{\frac{\sqrt{5}}{\sqrt{2}}}^{\frac{\sqrt{10}}{3}}\frac{-t\left(t^2-1\right)}{t}=$

$=\int_{\frac{\sqrt{5}}{\sqrt{2}}}^{\frac{\sqrt{10}}{3}}\left(1-t^2\right)dt=$

$t\Big|_{\frac{\sqrt{5}}{\sqrt{2}}}^{\frac{\sqrt{10}}{\sqrt{3}}}-\frac{t^3}{3}\Big|_{\frac{\sqrt{5}}{\sqrt{2}}}^{\frac{\sqrt{10}}{\sqrt{3}}}=\ldots$

T

$\boxed{2}$ $\quad I=\int\frac{1}{\sqrt[3]{1+x^3}}\,dx$