Seminar 10

Aplicații

Un K-spațiu vectorial este format dintr-un grup abelian $(V,+)$ și o operație externă $\cdot : K \times V \to V$ care respectă următoarele axiome:

1. $\alpha \cdot (x+y) = \alpha x + \alpha y$
2. $(\alpha+\beta)x = \alpha x + \beta x$
3. $\alpha(\beta x) = (\alpha\beta)x$
4. $1 \cdot x = x$

pt. orice $x, y \in V$ și $\alpha, \beta \in K$

1. Să se arate că $\mathbb{R}_+^* = (0, \infty)$ este un $\mathbb{R}$-spațiu vectorial în raport cu adunarea vectorilor:

$\boxplus : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$

$x \boxplus y = x \cdot y, \forall x, y \in \mathbb{R}_+^*$

și cu înmulțirea cu scalari:

$\boxdot : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}_+^*$,

$\alpha \boxdot x = x^\alpha$, $\forall x \in \mathbb{R}_+^*$, $\alpha \in \mathbb{R}$

<u>Soluție</u>: I. $(\mathbb{R}_+^*, \boxplus)$ grup abelian?

1. p.s.:

$\forall x, y \in \mathbb{R}_+^* \Longrightarrow x \boxplus y \in \mathbb{R}_+^*$

$x \boxplus y = \underbrace{x}_{\in \mathbb{R}_+^*} \cdot \underbrace{y}_{\in \mathbb{R}_+^*} \in \mathbb{R}_+^*$

2. asociativitate:

$\forall x, y, z \in \mathbb{R}_+^*, (x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$

$(x \boxplus y) \boxplus z = (xy) \boxplus z = xyz = x(yz) = x \boxplus (yz) = x \boxplus (y \boxplus z)$

3. comutativitate:

$\forall x, y \in \mathbb{R}_+^*, x \boxplus y = y \boxplus x$

$$x \boxplus y = x \cdot y = y \cdot x = y \boxplus x$$

4. element neutru:

$$\exists\, e \in \mathbb{R}_+^*,\ \forall\, x \in \mathbb{R}_+^*$$

$$\underbrace{x \boxplus e = e \boxplus x}_{\text{com.}} = x$$

$$x \boxplus e = x \implies x \cdot e = x \ \Big|\, : x \neq 0$$

$$e = 1 \in \mathbb{R}_+^*$$

5. elemente simetrizabile

$$\forall\, x \in \mathbb{R}_+^*,\ \exists\, x' \in \mathbb{R}_+^* \text{ a.î.}$$

$$\underbrace{x \boxplus x' = x' \boxplus x}_{\text{com.}} = e$$

$$\implies x \boxplus x' = 1 \iff x \cdot x' = 1$$

$$x' = \frac{1}{x},$$

$$\forall x \in \mathbb{R}_+^*$$

1., ..., 5. $\implies (\mathbb{R}_+^*, \boxplus)$ grup abelian

$\boxed{\text{II}}$ 1.) $\alpha \boxdot (x \boxplus y) = (\alpha \boxdot x) \boxplus (\alpha \boxdot y)$

$$\alpha \boxdot (x \boxplus y) = \alpha \boxdot (xy) = (xy)^\alpha =$$

$$= x^\alpha y^\alpha = x^\alpha \boxplus y^\alpha =$$

$$= (\alpha \boxdot x) \boxplus (\alpha \boxdot y),$$

$\forall x, y \in \mathbb{R}_+^*, \alpha \in \mathbb{R}$

2.) $(\alpha+\beta) \boxdot x = (\alpha \boxdot x) \boxplus (\beta \boxdot x)$

$(\alpha+\beta) \boxdot x = x^{\alpha+\beta} = x^{\alpha} \cdot x^{\beta}$

$= (\alpha \boxdot x) \boxplus (\beta \boxdot x), \forall x \in \mathbb{R}_+^*, \alpha, \beta \in \mathbb{R}$

3.) $(\alpha \cdot \beta) \boxdot x = \alpha \boxdot (\beta \boxdot x)$

$(\alpha \cdot \beta) \boxdot x = x^{\alpha \cdot \beta} = (x^{\beta})^{\alpha} =$

$= (\beta \boxdot x)^{\alpha} = \alpha \boxdot (\beta \boxdot x)$

4.) $1 \boxdot x = x, \forall x \in \mathbb{R}_+^*$

$1 \boxdot x = x^1 = x$

$\underline{\underline{I, II}} \Rightarrow \mathbb{R}_+^*$ este un $\mathbb{R}$-spațiu vectorial împreună cu $\boxplus$ și op. externă $\boxdot$

(2.) Să se verifice că operațiile

$\boxplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$x \boxplus y = \sqrt[5]{x^5 + y^5}, \forall x, y \in \mathbb{R}$

$\boxdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \alpha \boxdot x = \sqrt[5]{\alpha} \cdot x$

$\forall \alpha, x \in \mathbb{R}$

definesc o structură de $\mathbb{R}$ – spațiu vectorial.

<u>Soluție</u>: $\boxed{I}$ $(\mathbb{R}, \boxplus)$ gr. abelian

1.) p.s:

$\forall x, y \in \mathbb{R} \Rightarrow x \boxplus y \in \mathbb{R}$

$$x \boxplus y = \sqrt[5]{\underbrace{x^5}_{\in \mathbb{R}} + \underbrace{y^5}_{\in \mathbb{R}}} \in \mathbb{R}$$

2.) asociativitatea:

$\forall x, y, z \in \mathbb{R} \Rightarrow (x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$

$$(x \boxplus y) \boxplus z = \sqrt[5]{x^5 + y^5} \boxplus z =$$

$$= \sqrt[5]{\left(\sqrt[5]{x^5+y^5}\right)^5 + z^5} = \sqrt[5]{x^5 + y^5 + z^5}$$

$$x \boxplus (y \boxplus z) = x \boxplus \sqrt[5]{y^5 + z^5} =$$

$$= \sqrt[5]{x^5 + y^5 + z^5}$$

3.) comutativitatea:

$\forall x, y \in \mathbb{R}, \quad x \boxplus y = y \boxplus x$

$x \boxplus y = \sqrt[5]{x^5+y^5} = \sqrt[5]{y^5+x^5}$
$= y \boxplus x$

4.) element neutru

$\exists\, e \in \mathbb{R},\ \forall\, x \in \mathbb{R}$ a.î. $x \boxplus e \overset{\text{com}}{=}$
$= e \boxplus x = x$

$x \boxplus e = x \iff \sqrt[5]{x^5+e^5} = x \mid^5$

$x^5+e^5 = x^5$

$e^5 = 0 \Rightarrow e = 0 \in \mathbb{R}$

5.) elemente simetrizabile

$\forall\, x \in \mathbb{R},\ \exists\, x' \in \mathbb{R}$ a.î. $x \boxplus x' \overset{\text{com.}}{=}$
$= x' \boxplus x = e$

$x \boxplus x' = 0 \iff \sqrt[5]{x^5+x'^5} = 0 \mid^5$

$x^5 + x'^5 = 0$

$x'^5 = -x^5$

$x' = -x,\ \forall\, x \in \mathbb{R}$

Din 1.), ..., 5.) $\Rightarrow$ $(\mathbb{R}, \boxplus)$ este grup abelian

II. 1.) $\alpha \boxdot (x \boxplus y) = (\alpha \boxdot x) \boxplus (\alpha \boxdot y)$
$\forall\, x, y, \alpha \in \mathbb{R}$

$$\alpha \boxdot x = \sqrt[5]{\alpha} \cdot x$$

$$x \boxplus y = \sqrt[5]{x^5 + y^5}$$

$$\alpha \boxdot \sqrt[5]{x^5 + y^5} = \sqrt[5]{\alpha} \cdot \sqrt[5]{x^5 + y^5}$$

$$= \sqrt[5]{\alpha (x^5 + y^5)}$$

$$(\alpha \boxdot x) \boxplus (\alpha \boxdot y) = (\sqrt[5]{\alpha} \cdot x) \boxplus (\sqrt[5]{\alpha} \cdot y) = \sqrt[5]{\alpha \cdot x^5 + \alpha \cdot y^5}$$

$$= \sqrt[5]{\alpha (x^5 + y^5)}$$

2.) $$(\alpha + \beta) \boxdot x = (\alpha \boxdot x) \boxplus (\beta \boxdot x)$$

$$(\alpha + \beta) \boxdot x = \sqrt[5]{\alpha + \beta} \cdot x$$

$$(\alpha \boxdot x) \boxplus (\beta \boxdot x) =$$

$$= (\sqrt[5]{\alpha} \cdot x) \boxplus (\sqrt[5]{\beta} \cdot x)$$

$$= \sqrt[5]{(\sqrt[5]{\alpha} \cdot x)^5 + (\sqrt[5]{\beta} \cdot x)^5}$$

$$= \sqrt[5]{\alpha \cdot x^5 + \beta \cdot x^5} = x \cdot \sqrt[5]{\alpha + \beta}$$

3.) $$\alpha \boxdot (\beta \boxdot x) = (\alpha \cdot \beta) \boxdot x$$

$\alpha \boxdot (\beta \boxdot x) = \alpha \boxdot (\sqrt[5]{\beta} \cdot x) =$

$= \sqrt[5]{\alpha} \cdot \sqrt[5]{\beta} \cdot x$

$(\alpha \cdot \beta) \boxdot x = \sqrt[5]{\alpha \cdot \beta} \cdot x = \sqrt[5]{\alpha} \cdot \sqrt[5]{\beta} \cdot x$

4.) $1 \boxdot x = x$

$1 \boxdot x = \sqrt[5]{1} \cdot x = x$

$\xRightarrow{I, II}$ $\mathbb{R}$ este un $\mathbb{R}$-spațiu vectorial împreună cu op. internă $\boxplus$ și cea externă $\boxdot$

Fie V un K-sp. vectorial, $U \subseteq V$. U.A.S.E:

(1): $U \leq_K V$

(2): a.) $0 \in U$

b.) $x, y \in U \Rightarrow x + y \in U$

c.) $x \in U, \alpha \in K \Rightarrow \alpha x \in U$

(3): a.) $0 \in U$

b.) $\left.\begin{array}{l} x, y \in U \\ \alpha, \beta \in K \end{array}\right\} \Rightarrow \alpha x + \beta y \in U$

③ Care dintre următoarele submulțimi ale mulțimii $\mathbb{R}^3$ sunt $\mathbb{R}$ - subspații:

a) $A = \{ x = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 0 \}$

Soluție: $0 = [0,0,0] \in \mathbb{R}^3$

$2 \cdot 0 + 0 - 0 = 0 \Rightarrow 0 \in A$

$x, y \in A$

$x = [x_1, x_2, x_3], \quad 2x_1 + x_2 - x_3 = 0$

$y = [y_1, y_2, y_3], \quad 2y_1 + y_2 - y_3 = 0$

$x + y = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$

$2(x_1 + y_1) + x_2 + y_2 - x_3 - y_3 =$

$= 2x_1 + 2y_1 + x_2 + y_2 - x_3 - y_3$

$= 2x_1 + x_2 - x_3 + 2y_1 + y_2 - y_3$

$= 0 + 0 = 0 \Rightarrow x + y \in A$

$\alpha \in \mathbb{R}$

$x = [x_1, x_2, x_3] \in \mathbb{R}^3, \quad x \in A$

$\alpha x = [\alpha x_1, \alpha x_2, \alpha x_3]$

$$2\alpha x_1 + \alpha x_2 - \alpha x_3 = \alpha(2x_1 + x_2 - x_3)$$
$$= \alpha \cdot 0 = 0$$
$$\Longrightarrow \quad \alpha x \in A$$
$$\Longrightarrow \quad A \leq_{\mathbb{R}} \mathbb{R}^3$$

b) $B = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 1\}$

$$0 = [0, 0, 0] \in \mathbb{R}^3$$
$$2 \cdot 0 + 0 - 0 = 0 \neq 1$$
$$\Longrightarrow \quad B \not\leq_{\mathbb{R}} \mathbb{R}^3$$

c) $C = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$

$$[0, 0, 0] \in \mathbb{R}^3$$
$$0 = 0 = 0 \quad \Longrightarrow \quad 0 \in C$$
$$x, y \in C \overset{?}{\Longrightarrow} x + y \in C$$
$$x = [x_1, x_2, x_3], \quad \underline{x_1 = x_2 = x_3}$$
$$y = [y_1, y_2, y_3], \quad \underline{y_1 = y_2 = y_3}$$
$$x + y = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$$
$$x_1 + y_1 = x_2 + y_2 = x_3 + y_3$$
$$\Longrightarrow \quad x + y \in C$$

$x \in C \quad , \quad \alpha \in \mathbb{R} \quad$ a.î. $\quad \alpha x \overset{?}{\in} C$

$x = [x_1, x_2, x_3] \quad , \quad x_1 = x_2 = x_3$

$\alpha x = [\alpha x_1, \alpha x_2, \alpha x_3] \quad \Downarrow$

$\alpha x_1 = \alpha x_2 = \alpha x_3$

$\Rightarrow \quad \alpha x \in C$

$\Rightarrow \quad C \leq_{\mathbb{R}} \mathbb{R}^3$

d.) $D = \{ [x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1^2 + x_2 = 0 \}$

$[0,0,0] \in \mathbb{R}^3$

$0^2 + 0 = 0 \quad \Rightarrow \quad 0 \in D$

$x \in D \quad , \quad x = [x_1, x_2, x_3] \in \mathbb{R}^3$

cu $\quad \underline{x_1^2 + x_2 = 0}$

$y \in D \quad , \quad y = [y_1, y_2, y_3] \in \mathbb{R}^3$

cu $\quad \underline{y_1^2 + y_2 = 0}$

$x + y = [x_1 + y_1, \ x_2 + y_2, \ x_3 + y_3]$

$(x_1 + y_1)^2 + (x_2 + y_2) =$

$= x_1^2 + 2x_1 y_1 + y_1^2 + x_2 + y_2$

$= x_1^2 + x_2 + y_1^2 + y_2 + 2x_1 y_1$

$= 0 + 0 + 2x_1 y_1 = 2x_1 y_1 ,$

$\forall \quad x_1, y_1 \in \mathbb{R}$

$\Rightarrow \quad \Delta \not\leq_{\mathbb{R}} \mathbb{R}^3$

e) $E = \mathbb{R}^3 \setminus A$

$A = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 0\}$

$0 \in A$

$\Rightarrow \quad 0 \notin \mathbb{R}^3 \setminus A$

$\Rightarrow \quad 0 \notin E$

$\Rightarrow E \not\leq_{\mathbb{R}} \mathbb{R}^3$

f.) <u>(temă)</u> $\quad F = (\mathbb{R}^3 \setminus A) \cup \{0\}$

$F = E \cup \{0\}$

V un K - sp. vectorial

O aplicație f este liniară dacă $f(\alpha \underline{x} + \beta \underline{y}) = \underline{\alpha f(x)} + \underline{\beta f(y)}$

$x, y \in V, \quad \alpha, \beta \in K$

④ Care dintre următoarele aplicații sunt liniare:

a) $f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f[x_1, x_2, x_3] = [x_1 - x_2, x_2 - x_3, x_3 - x_1]$

$\alpha, \beta \in \mathbb{R}, \quad x, y \in \mathbb{R}^3$

$$x = [x_1, x_2, x_3], \quad y = [y_1, y_2, y_3]$$

$$\alpha x + \beta y = [\underline{\alpha x_1 + \beta y_1}, \underline{\alpha x_2 + \beta y_2}, \underline{\alpha x_3 + \beta y_3}]$$

$$\underline{f(\alpha x + \beta y)} = [\alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2, \alpha x_2 + \beta y_2 - \alpha x_3 - \beta y_3, \alpha x_3 + \beta y_3 - \alpha x_1 - \beta y_1] =$$

$$= [\alpha(x_1 - x_2), \alpha(x_2 - x_3), \alpha(x_3 - x_1)] + [\beta(y_1 - y_2), \beta(y_2 - y_3), \beta(y_3 - y_1)]$$

$$= \alpha[x_1 - x_2, x_2 - x_3, x_3 - x_1] + \beta[y_1 - y_2, y_2 - y_3, y_3 - y_1]$$

$$= \underline{\alpha f(x) + \beta f(y)}$$ $\Rightarrow$ f este aplicație liniară

$$\operatorname{Ker} f = \{x = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid f(x) = 0\}$$

$$f(x) = 0 \Rightarrow [x_1 - x_2, x_2 - x_3, x_3 - x_1] = [0, 0, 0]$$

$$\Rightarrow \begin{cases} x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \\ x_3 - x_1 = 0 \end{cases} \Leftrightarrow x_1 = x_2 = x_3$$

$\Rightarrow \quad \operatorname{Ker} f = \{ x = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 = x_2 = x_3 \}$

b) $f: \mathbb{R}^3 \to \mathbb{R}^3, \quad f[x_1, x_2, x_3] =$
$= [x_1 - 1, \; x_2 + 2, \; x_3 + 1]$

$x = [x_1, x_2, x_3] \in \mathbb{R}^3$
$y = [y_1, y_2, y_3] \in \mathbb{R}^3$

$\alpha, \beta \in \mathbb{R}$

$\alpha x + \beta y = [\alpha x_1 + \beta y_1, \; \alpha x_2 + \beta y_2, \; \alpha x_3 + \beta y_3]$

$f(\alpha x + \beta y) = [\alpha x_1 + \beta y_1 - 1, \; \alpha x_2 + \beta y_2 + 2, \; \alpha x_3 + \beta y_3 + 1]$

$= [\alpha x_1, \alpha x_2, \alpha x_3] + [\beta y_1, \beta y_2, \beta y_3] + [-1, 2, 1]$

$= [\alpha x_1 - \alpha, \; \alpha x_2 + 2\alpha, \; \alpha x_3 + \alpha] +$
$+ [\beta y_1 - \beta, \; \beta y_2 + 2\beta, \; \beta y_3 + \beta]$
$+ [\alpha, -2\alpha, -\alpha] + [\beta, -2\beta, -\beta]$

$+ [-1, 2, 1]$

$= \alpha f(x) + \beta f(y) + [\alpha+\beta-1,$
$-2(\alpha+\beta-1), -\alpha-\beta+1]$
$\neq \alpha f(x) + \beta f(y)$
$\Rightarrow$ f nu este aplicație liniară

c) $f: \mathbb{R}^2 \to \mathbb{R}^3$

$f[x_1, x_2] = [x_1 + x_2, x_1 - x_2, 2x_1 + x_2]$

$x, y \in \mathbb{R}^2$

$x = [x_1, x_2), y = [y_1, y_2]$

$\alpha x + \beta y = [\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2]$

$f(\alpha x + \beta y) = [\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2,$
$\alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2, 2\alpha x_1 + 2\beta y_1$
$+ \alpha x_2 + \beta y_2]$

$= [\alpha(x_1 + x_2), \alpha x_1 - \alpha x_2, 2\alpha x_1 + \alpha x_2]$
$+ [\beta y_1 + \beta y_2, \beta y_1 - \beta y_2, 2\beta y_1 + \beta y_2]$
$= \alpha[x_1 + x_2, x_1 - x_2, 2x_1 + x_2]$
$+ \beta[y_1 + y_2, y_1 - y_2, 2y_1 + y_2]$

$= \alpha f(x) + \beta f(y) \Rightarrow f$ apl. liniară