Exerciții suplimentare

① Calculați der. parțiale de ordinul 1, gradientul ∇f și diferențiala df pt. funcțiile:

a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sin^2(x^2+y)$

$$\frac{\partial f}{\partial x}(x,y) = 2\sin(x^2+y)\cdot\cos(x^2+y)\cdot 2x = 2x\sin(2x^2+2y)$$

$$\frac{\partial f}{\partial y}(x,y) = 2\sin(x^2+y)\cdot\cos(x^2+y)\cdot 1 = \sin(2x^2+2y)$$

$$\nabla f(x,y) = \left(2x\sin(2x^2+2y),\ \sin(2x^2+2y)\right)$$

$$df(x,y)(u_1,u_2) = 2x\sin(2x^2+2y)\cdot u_1 + \sin(2x^2+2y)u_2$$

b) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = (x+y+z)\cdot e^{x^2+y^2+z^2}$

$$\frac{\partial f}{\partial x}(x,y,z) = e^{x^2+y^2+z^2} + (x+y+z)\cdot e^{x^2+y^2+z^2}\cdot 2x =$$

$$= e^{x^2+y^2+z^2}\left(1+2x^2+2xy+2xz\right)$$

$$\frac{\partial f}{\partial y}(x,y,z) = e^{x^2+y^2+z^2}\left(1+2y^2+2xy+2yz\right)$$

$$\frac{\partial f}{\partial z}(x,y,z) = e^{x^2+y^2+z^2}\left(1+2z^2+2xz+2yz\right)$$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \dots$$

$$df(x,y,z)(u_1,u_2,u_3) = \frac{\partial f}{\partial x}(x,y,z)\cdot u_1 + \frac{\partial f}{\partial y}(x,y,z)u_2 + \frac{\partial f}{\partial z}(x,y,z)\cdot u_3 = \dots$$

② Calculați matricea Jacobi $J(f)$ în pct. $(1,1)$ pt. următoarele fc. vectoriale:

a) $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (x^2 - y,\ 3x - 2y,\ 2xy + y^2)$

$$J(f)(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \\ \frac{\partial f_3}{\partial x}(x,y) & \frac{\partial f_3}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 2x & -1 \\ 3 & -2 \\ 2y & 2x+2y \end{pmatrix}$$

$$J(f)(1,1) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \\ 2 & 4 \end{pmatrix}$$

b) $f: (0,\infty)^2 \to \mathbb{R}^2$, $f(x,y) = \left(\frac{1}{xy},\ \operatorname{arctg}\frac{y}{x}\right)$

$$\frac{\partial f_1}{\partial x}(x,y) = \frac{1}{y}\cdot(-x^{-2}) = -\frac{1}{x^2 y}$$

$$\frac{\partial f_1}{\partial y}(x,y) = \frac{1}{x}\cdot(-y^{-2}) = -\frac{1}{xy^2}$$

$$\frac{\partial f_2}{\partial x}(x,y) = \frac{1}{\left(\frac{y}{x}\right)^2+1}\cdot\left(\frac{y}{x}\right)'_x = \frac{x^2}{y^2+x^2}\cdot\frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial f_2}{\partial y}(x,y) = \frac{1}{\left(\frac{y}{x}\right)^2+1}\cdot\left(\frac{y}{x}\right)'_y = \frac{x^2}{y^2+x^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$J(f)(x,y) = \begin{pmatrix} -\frac{1}{x^2 y} & -\frac{1}{xy^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

$$J(f)(1,1) = \begin{pmatrix} -1 & -1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Funcții omogene

Fie $p \in \mathbb{R}$. O fc. $f: (0,\infty)^m \to \mathbb{R}$ s.m. omogenă (de grad p) dacă $f(tx) = t^p \cdot f(x)$, $\forall x \in (0,\infty)^m$ și $\forall t > 0$.

③ Ar. că fc. $f: (0,\infty)^3 \to \mathbb{R}$, $f(x,y,z) = \frac{1}{x+y+z} \cdot \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$ este omogenă (de un anumit grad) și justificați:

$$x \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} = -2f, \ \forall (x,y,z) \in (0,\infty)^3$$

$$f(tx,ty,tz) = \frac{1}{tx+ty+tz}\left(\frac{1}{tx} + \frac{1}{ty} + \frac{1}{tz}\right) =$$

$$= \frac{1}{t(x+y+z)} \cdot \frac{1}{t}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) =$$

$$= \frac{1}{t^2}\left[\frac{1}{x+y+z} \cdot \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)\right] =$$

$$= t^{-2} \cdot f(x,y,z) \Rightarrow \text{omogenă (de grad } -2)$$

$$f(x,y,z) = (x+y+z)^{-1} \cdot \left(x^{-1} + y^{-1} + z^{-1}\right)$$

$$\frac{\partial f}{\partial x}(x,y,z) = -(x+y+z)^{-2}\left(x^{-1} + y^{-1} + z^{-1}\right) + (x+y+z)^{-1} \cdot (-1) \cdot x^{-2}$$

$$= -\frac{1}{(x+y+z)^2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{x+y+z}{x^2}\right)$$

$$\frac{\partial f}{\partial y}(x,y,z) = -\frac{1}{(x+y+z)^2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{x+y+z}{y^2}\right)$$

$$\frac{\partial f}{\partial z}(x,y,z) = -\frac{1}{(x+y+z)^2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{x+y+z}{z^2}\right)$$

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$$x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}+z\cdot\frac{\partial f}{\partial z}=$$

$$=-\frac{1}{(x+y+z)^2}\left(1+\frac{x}{y}+\frac{x}{z}+\frac{x+y+z}{x}+\frac{y}{x}+1+\frac{y}{z}+\frac{x+y+z}{y}+\frac{z}{x}+\frac{z}{y}+1\right.$$

$$\left.+\frac{x+y+z}{z}\right)$$

$$=-\frac{1}{(x+y+z)^2}\left(6+2\frac{x}{y}+2\frac{x}{z}+2\frac{y}{x}+2\frac{z}{x}+2\frac{z}{y}+2\frac{y}{z}\right)$$

$$=\frac{-2}{(x+y+z)^2}\left(\underbrace{3}_{\frac{x}{x}+\frac{y}{y}+\frac{z}{z}}+\frac{x}{y}+\frac{y}{x}+\frac{x}{z}+\frac{z}{x}+\frac{y}{z}+\frac{z}{y}\right)$$

$$=-\frac{2}{(x+y+z)^2}\left[\left(\frac{x}{x}+\frac{x}{y}+\frac{x}{z}\right)+\left(\frac{y}{x}+\frac{y}{y}+\frac{y}{z}\right)+\left(\frac{z}{x}+\frac{z}{y}+\frac{z}{z}\right)\right]$$

$$=-\frac{2}{(x+y+z)^2}\left[x\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+y\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+z\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\right]$$

$$=-\frac{2}{(x+y+z)^{\cancel{2}}}\left[\cancel{(x+y+z)}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\right]$$

$$=-2\cdot\frac{1}{x+y+z}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=-2f(x,y,z)$$

4) Arătați că fc. de mai jos nu e continuă în $(0,0)$, dar admite derivate după orice direcție în acest punct:

$$f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=\begin{cases}\dfrac{xy}{x+y}, & x+y\neq 0\\ 0, & x+y=0\end{cases}$$

- Fie $a^n=\left(\frac{1}{n},\frac{3}{n}\right)\to(0,0),\ n\to\infty$

  $b^n=\left(\frac{1}{n},\frac{1}{n^2}-\frac{1}{n}\right)\to(0,0),\ n\to\infty$

$$f(a^n) = \frac{\frac{3}{n^2}}{\frac{4}{n}} = \frac{3}{n^2} \cdot \frac{n}{4} = \frac{3}{4n} \to 0,\ n \to \infty$$

$$f(b^n) = \frac{\frac{1}{n}\left(\frac{1}{n^2} - \frac{1}{n}\right)}{\frac{1}{n} + \frac{1}{n^2} - \frac{1}{n}} = \frac{\frac{1}{n^3} - \frac{1}{n^2}}{\frac{1}{n^2}} = \frac{n^2}{n^3} - \frac{n^2}{n^2} = \frac{1}{n} - 1$$

$$\to -1,\ n \to \infty$$

$\Rightarrow \nexists$ lim. în $(0,0)$, deci f nu e cont. în $(0,0)$.

- $$\lim_{t \to 0} \frac{f(0+tv_1,\ 0+tv_2) - f(0,0)}{t} = \lim_{t \to 0} \frac{f(tv_1, tv_2)}{t} =$$

$$\lim_{t \to 0} \frac{t^2 v_1 v_2}{t(v_1+v_2)} \cdot \frac{1}{t} = \frac{v_1 v_2}{v_1+v_2} = \begin{cases} 0, & v_1 = 0 \text{ sau } v_2 = 0 \text{ sau } v_1 = -v_2 \ * \\ \frac{v_1 v_2}{v_1+v_2}, & \text{dacă } v_1 \neq 0 \text{ și } v_2 \neq 0 \end{cases}$$

* Dacă $v_1 = -v_2 \Rightarrow \lim\limits_{t \to 0} \frac{\overbrace{f(tv_1, -tv_1)}^{=0} - f(0,0)}{t} = \lim\limits_{t \to 0} \frac{0-0}{t} = 0$

$\Rightarrow f$ derivabilă după orice direcție în $(0,0)$.

(4) Arătați că fc. $f(x,y) = (x^2+y^2) \cdot \operatorname{arctg} \frac{y}{x}$ verifică relația

$$x^2 \cdot \frac{\partial^2 f}{\partial x^2} + 2xy \cdot \frac{\partial^2 f}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 f}{\partial y^2} = 2f,\ \forall (x,y) \in (0,\infty)^2$$

$$\frac{\partial f}{\partial x} = 2x \operatorname{arctg} \frac{y}{x} + (x^2+y^2) \cdot \frac{-y}{x^2+y^2} = 2x \operatorname{arctg} \frac{y}{x} - y$$

$$\frac{\partial f}{\partial y} = 2y \operatorname{arctg} \frac{y}{x} + (x^2+y^2) \cdot \frac{x}{x^2+y^2} = 2y \operatorname{arctg} \frac{y}{x} + x$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \operatorname{arctg} \frac{y}{x} + 2x \cdot \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = 2\,\mathrm{arctg}\,\frac{y}{x} + 2y\cdot\frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y\cdot\frac{-y}{x^2+y^2} + 1$$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} =$$

$$= 2x^2\,\mathrm{arctg}\,\frac{y}{x} - \frac{2x^3y}{x^2+y^2} - \frac{4xy^3}{x^2+y^2} + 2xy + 2y^2\,\mathrm{arctg}\,\frac{y}{x} + \frac{2xy^3}{x^2+y^2}$$

$$= 2(x^2+y^2)\,\mathrm{arctg}\,yx + \frac{2xy(x^2+y^2) - 2x^3y - 2xy^3}{x^2+y^2} =$$

$$= 2f(x,y) + \frac{2xy(\overbrace{x^2+y^2-x^2-y^2}^{0})}{x^2+y^2} = 2f(x,y) + 0 = 2f(x,y)$$

⑤ Calculați derivatele parțiale ale f.c. compuse $g\circ f$, unde $f:\mathbb{R}^2\to\mathbb{R}^3$, $f(x,y) = (x^2-y,\ 3x-2y,\ 2xy+y^2)$ și $g = g(u,v,w):\mathbb{R}^3\to\mathbb{R}$ este o f.c. oarecare de clasă C^1 pe $\mathbb{R}^3$.

$$J(f)(x,y) = \begin{pmatrix} 2x & -1 \\ 3 & -2 \\ 2y & 2x+2y \end{pmatrix}$$

$$\left(\frac{\partial}{\partial x}(g\circ f)(x,y),\ \frac{\partial}{\partial y}(g\circ f)(x,y)\right) =$$

$$= \left(\frac{\partial g}{\partial u}(f(x,y)),\ \frac{\partial g}{\partial v}(f(x,y)),\ \frac{\partial g}{\partial w}(f(x,y))\right)\cdot\underbrace{\begin{pmatrix} 2x & -1 \\ 3 & -2 \\ 2y & 2x+2y \end{pmatrix}}_{J(f)(x,y)}$$

$$\Rightarrow \frac{\partial}{\partial x}((g\circ f)(x,y)) = 2x\frac{\partial g}{\partial u}(f(x,y)) + 3\frac{\partial g}{\partial v}(f(x,y)) + 2y\frac{\partial g}{\partial w}(f(x,y))$$

$$\frac{\partial}{\partial y}((g\circ f)(x,y)) = -\frac{\partial g}{\partial u}(f(x,y)) - 2\frac{\partial g}{\partial v}(f(x,y)) + (2x+2y)\frac{\partial g}{\partial w}(f(x,y))$$

⑥ Exprimați ecuația

$$v \cdot \frac{\partial g}{\partial u}(u,v) - u \cdot \frac{\partial g}{\partial v}(u,v) = 1, \quad \forall (u,v) \in (0,\infty)^2$$

în variabilele $(x,y) \in (0,\infty) \times (0,\pi/2)$, efectuând transformarea $u = x\cos y$, $v = x\sin y$. Determinați apoi o fc. de clasă C^1 ce verifică relația respectivă.

$$(*)\quad x\sin y \frac{\partial g}{\partial u}(x\cos y, x\sin y) - x\cos y \frac{\partial g}{\partial v}(x\cos y, x\sin y) = 1.$$

Fie $\varphi(x,y) = (x\cos y, x\sin y)$

$G = g \circ \varphi$

$$\frac{\partial G}{\partial x}(x,y) = \frac{\partial g}{\partial u}(\varphi(x,y)) \cdot \frac{\partial \varphi_1}{\partial x}(x,y) + \frac{\partial g}{\partial v}(\varphi(x,y)) \cdot \frac{\partial \varphi_2}{\partial x}(x,y)$$

$$= \frac{\partial g}{\partial u}(x\cos y, x\sin y) \cdot \cos y + \frac{\partial g}{\partial v}(x\cos y, x\sin y) \cdot \sin y$$

↳ nu mă ajută.

Încercăm cu $\frac{\partial G}{\partial y}$

$$\frac{\partial G}{\partial y}(x,y) = \frac{\partial g}{\partial u}(\varphi(x,y)) \cdot \frac{\partial \varphi_1}{\partial y}(x,y) + \frac{\partial g}{\partial v}(\varphi(x,y)) \cdot \frac{\partial \varphi_2}{\partial y}(x,y)$$

$$= \frac{\partial g}{\partial u}(x\cos y, x\sin y) \cdot (-x\sin y) + \frac{\partial g}{\partial v}(x\cos y, x\sin y) \cdot x\cos y$$

$$= -\left(x\sin y \frac{\partial g}{\partial u}(x\cos y, x\sin y) - x\cos y \frac{\partial g}{\partial v}(x\cos y, x\sin y)\right)$$

$$\overset{*}{=} -1$$

$$\Rightarrow \frac{\partial G}{\partial y}(x,y) = -1 \Rightarrow \int -1\, dy = G(x,y) \Rightarrow G(x,y) = -y + C(x)$$

$$g(\varphi(x,y)) = -y + C(x) \Rightarrow \boxed{g(u,v) = -\operatorname{arctg}\left(\frac{v}{u}\right) + C\left(\sqrt{u^2+v^2}\right)}$$

(†) $\frac{v}{u} = \operatorname{tg} y \Rightarrow y = \operatorname{arctg}\left(\frac{v}{u}\right)$

$u^2 + v^2 = x^2 \Rightarrow x = \sqrt{u^2+v^2}$