Exerciții suplimentare

① Determinați mulțimea pct. de acumulare A':

a) $A = \left\{ \frac{n!}{3^n}, n \in \mathbb{N} \right\}$

$(x_n)_{n\in\mathbb{N}}$, $x_n = \frac{n!}{3^n}$ șir cu termeni pozitivi

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{n!}{3^n} \cdot \frac{3^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{3}{n+1} < 1, \ n>2,$$

(Vezi Ex. 3, Sem. 4)

deci conform Crit. Raportului pt. șiruri

$\lim_{n\to\infty} x_n = +\infty$.

Deci $A' = \{+\infty\}$.

b) $A = (0,1) \setminus \mathbb{Q}$. → nr. iraționale din $(0,1)$.

Fie $y \in [0,1]$. Cum orice nr. real e limita unui șir de nr. iraționale (Vezi Ex 7, Sem. 1) ⇒

$\exists (s_n) \in A \setminus \{y\}$ a.î. $\lim_{n\to\infty} s_n = y$, deci $y \in A'$.

y arbitrar ⇒ $A' = [0,1]$.

③ Det. pct. de extrem local și valorile extreme ale fc:

a) $f: [-1,1] \to \mathbb{R}$, $f(x) = |x|(1-x)$

$$f(x) = \begin{cases} -x(1-x) = x^2 - x, & x<0 \\ x(1-x) = -x^2+x, & x \geq 0 \end{cases}$$

f nu e derivabilă în 0!

$$l_s'(0) = \lim_{\substack{x\to 0\\ x<0}} \frac{f(x)-f(0)}{x} = \lim_{\substack{x\to 0\\ x<0}} \frac{\overbrace{x^2-x}^{x(x-1)}}{x} = -1$$

$$l_d'(0) = \lim_{\substack{x\to 0\\ x>0}} \frac{f(x)-f(0)}{x} = \lim_{\substack{x\to 0\\ x>0}} \frac{\overbrace{-x^2+x}^{x(-x+1)}}{x} = 1$$

$$f'(x) = \begin{cases} 2x-1, & x<0 \\ -2x+1, & x>0 \end{cases}$$

x	-1		0		$\frac{1}{2}$		1
$f'(x)$		$-$		$+$	0	$-$	
$f(x)$	2	$\searrow$	0	$\nearrow$	$\frac{1}{4}$	$\searrow$	0

$2x-1=0 \Rightarrow x=\frac{1}{2} > 0$, nu convine

$-2x+1=0 \Rightarrow x=\frac{1}{2}$ convine.

$f(-1)=2$

$f(0)=0$

$f\left(\frac{1}{2}\right)=\frac{1}{4}$

$f(1)=0$

$-1, \frac{1}{2}$ pct. de maxim local

$0, 1$ pct. de minim local.

$\inf f(A) = 0 = f(0) = f(1)$

$\sup f(A) = 2 = f(-1)$

$\downarrow$

valorile extreme.

d) $f:(0,\infty) \to \mathbb{R}$, $f(x) = \dfrac{|\ln x|}{\sqrt{x}}$

$\ln x < 0$ pt. $x \in (0,1)$

$\ln x \geq 0$ pt. $x \geq 1$.

$$f(x) = \begin{cases} -\dfrac{\ln x}{\sqrt{x}}, & x \in (0,1). \\ \dfrac{\ln x}{\sqrt{x}}, & x \geq 1. \end{cases}$$

f nu e derivabilă în 1!

$$f'(x) = \begin{cases} \dfrac{\ln x - 2}{2x\sqrt{x}}, & x \in (0,1) \\ \dfrac{2-\ln x}{2x\sqrt{x}}, & x \geq 1 \end{cases}$$

I. $f'(x)=0 \Rightarrow \ln x - 2 = 0 \Rightarrow \ln x = 2 \Rightarrow x = e^2 \notin (0,1)$.

II. $f'(x)=0 \Rightarrow 2-\ln x = 0 \Rightarrow \ln x = 2 \Rightarrow x = e^2 \in [1,\infty)$

x	0		1		e^2		∞
$f'(x)$		$- -$		$+$	0	$- -$	
$f(x)$	$-\infty$	$\searrow$	0	$\nearrow$	$\frac{2}{e}$	$\searrow$	0

$$\lim_{x\to 0} f(x) = \lim_{\substack{x\to 0 \\ x>0}} \frac{-\ln x}{\sqrt{x}} \overset{l'H}{=} \lim_{\substack{x\to 0 \\ x>0}} \frac{-\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{\substack{x\to 0 \\ x>0}} -\frac{2}{\sqrt{x}} = -\infty$$

$\inf f(A) = -\infty$ (nu se atinge)

$\sup f(A) = \frac{2}{e} = f(e^2)$.

$f(1)=0$

$f(e^2)=\frac{2}{e}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

$\Rightarrow$ 1 pct. de minim local, e^2 pct. maxim local

① Folosind regula lui l'Hopital, calculați:

a) $\lim\limits_{\substack{x\to 0\\ x>0}} (\cos\sqrt{x})^{\frac{1}{x}} = l$

Fie $\ln l = \lim\limits_{\substack{x\to 0\\ x>0}} \ln(\cos\sqrt{x})^{\frac{1}{x}} = \lim\limits_{\substack{x\to 0\\ x>0}} \frac{\ln(\cos\sqrt{x})}{x} \overset{L'H}{=}$

$= \lim\limits_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{\cos\sqrt{x}} \cdot (-\sin\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}}{1} = \lim\limits_{\substack{x\to 0\\ x>0}} \frac{-\frac{\sin\sqrt{x}}{\cos\sqrt{x}}}{2\sqrt{x}} =$

$= \lim\limits_{\substack{x\to 0\\ x>0}} \underbrace{\frac{-\operatorname{tg}\sqrt{x}}{2\sqrt{x}}}_{\to 1} = -\frac{1}{2}$

$\ln l = -\frac{1}{2} \Rightarrow l = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$.

b) $\lim\limits_{x\searrow 0} x^{\alpha} \cdot \ln(\sin x)$, $\alpha > 0$.

$\lim\limits_{x\searrow 0} \frac{\ln(\sin x)}{x^{-\alpha}} \overset{\boxed{\frac{\infty}{\infty}}}{\underset{l'H}{=}} \lim\limits_{x\searrow 0} \frac{\frac{\cos x}{\sin x}}{-\alpha \cdot x^{-\alpha-1}} = \lim\limits_{x\searrow 0} \underbrace{\operatorname{ctg} x}_{\to\infty} \cdot \underbrace{x^{\alpha+1}}_{\to 0} \cdot \frac{1}{-\alpha}$

$= \lim\limits_{x\searrow 0} \frac{\operatorname{ctg} x}{-\alpha \cdot x^{-\alpha-1}} \overset{\boxed{\frac{\infty}{\infty}}}{\underset{l'H}{=}} \lim\limits_{x\searrow 0} \frac{-\frac{1}{\sin^2 x}}{-\alpha(-\alpha-1) x^{-\alpha-2}} =$

$= \lim\limits_{x\searrow 0} -\frac{1}{\sin^2 x} \cdot \frac{1}{\alpha(\alpha+1) \cdot x^{-(\alpha+2)}} = \lim\limits_{x\searrow 0} -\frac{x^{\alpha} \cdot x^2}{\alpha(\alpha+1) \cdot \sin^2 x} =$ ($\frac{x^2}{\sin^2 x} \to 1$)

$= \lim\limits_{x\searrow 0} -\frac{x^{\alpha}}{\alpha(\alpha+1)} = 0.$

Sau $= \lim\limits_{x\searrow 0} \operatorname{ctg} x \cdot x^{\alpha+1} \cdot \frac{1}{-\alpha} = \lim\limits_{x\searrow 0} \frac{\cos x}{\sin x} \cdot x^{\alpha} \cdot x \cdot \frac{1}{-\alpha} =$

$= \lim\limits_{x\searrow 0} \cos x \cdot x^{\alpha} \cdot \underbrace{\frac{x}{\sin x}}_{\to 1} \cdot \frac{1}{-\alpha} = \lim\limits_{x\searrow 0} \left(-\frac{1}{\alpha} \cdot x^{\alpha} \cdot \cos x\right) =$

$= -\frac{1}{\alpha} \cdot 0 \cdot 1 = 0$

② Fie $A \subseteq \mathbb{R}$ și $f: A \to A$ o fc. cu proprietățile:

i) f derivabilă pe A

ii) $\exists q < 1$ a.î. $|f'(x)| \leq q$, $\forall x \in A$.

iii) $\exists a \in A$ a.î. $f(a) = a$.

Definim recursiv șirul $x_{n+1} = f(x_n)$, $\forall n \in \mathbb{N}$ și $x_0 \in A$. Justificați:

a) $|x_{n+1} - a| \leq q \cdot |x_n - a|$, $\forall n \in \mathbb{N}$

b) (x_n) e convergent și are limita a.

Construiți o fc. și un șir neconstant cu proprietățile de mai sus.

a) Aplicăm T. Lagrange pe int. $[a, x_n]$ (sau $[x_n, a]$)

$\exists c \in (a, x_n)$ (sau (x_n, a)) a.î.

$$f'(c) = \frac{f(x_n) - f(a)}{x_n - a} \overset{x_{n+1} = f(x_n)}{\underset{iii)\ f(a) = a}{=}} \frac{x_{n+1} - a}{x_n - a}$$

$\Rightarrow x_{n+1} - a = f'(c)(x_n - a) \Rightarrow$

$$|x_{n+1} - a| = |f'(c)| \cdot |x_n - a| \underset{\substack{ii)\ |f'(c)| \leq q,\ \forall c \in A \\ q < 1}}{\leq} q \cdot |x_n - a| \quad ✓$$

b) $|x_n - a| \overset{a)}{\leq} q|x_{n-1} - a| \overset{a)}{\leq} q \cdot q \cdot |x_{n-2} - a| \leq q^3 |x_{n-3} - a| \leq \dots$

$\leq q^n |x_0 - a| \to 0$, deoarece $q < 1$.

Deci $|x_n - a| \to 0 \Rightarrow (x_n)$ convergent cu limita a.

Fie $f:[1,\infty)\to[1,\infty)$, $f(x)=\frac{1}{2}\left(x+\frac{1}{x}\right)$

* f derivabilă pe $[1,\infty)$
* $D=[1,\infty)$, $\operatorname{Im} f=[1,\infty)$

$$f'(x)=\frac{1}{2}\left(1-\frac{1}{x^2}\right)=\frac{1}{2}\cdot\frac{x^2-1}{x^2}$$

$f'(x)=0 \iff x=1.$

x	1		∞
$f'(x)$	0	$+\ +\ +$	
$f(x)$	$f(1)=1$	$\nearrow$	$\lim\limits_{x\to\infty} f(x)=\infty$

* $|f'(x)|=\frac{1}{2}\cdot\left|\underbrace{\frac{x^2-1}{x^2}}_{<1}\right|<\frac{1}{2}$

Deci $\exists q=\frac{1}{2}<1$.

* $f(1)=1$, $1\in[1,\infty)$

Şirul va fi: $x_0=2$

$$x_1=\frac{1}{2}\left(x_0+\frac{1}{x_0}\right)=\frac{1}{2}\left(2+\frac{1}{2}\right)=\frac{5}{4}$$

$$x_2=\frac{1}{2}\left(x_1+\frac{1}{x_1}\right)=\frac{1}{2}\left(\frac{5}{4}+\frac{4}{5}\right)=\frac{41}{40}$$

...

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{1}{x_n}\right)$$

Şirul va avea limita 1.