Exerciții suplimentare

① Studiați natura următoarelor S.T.P.:

a) $S = \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$.

Am arătat sem. trecut că $\lim_{n\to\infty} \sqrt[n]{n} = 1$, deci $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \neq 0$

$\Rightarrow S$ divergentă.

b) $S = \sum_{n=0}^{\infty} \frac{2^n}{n+3^n}$. $\lim_{n\to\infty} \frac{2^n}{n+3^n} = \lim_{n\to\infty} \frac{2^n}{3^n\left(\underbrace{\frac{n}{3^n}}_{\to 0}+1\right)} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

$\frac{2^n}{n+3^n} < \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$. Cum $\sum \left(\frac{2}{3}\right)^n$ e seria geometrică cu

$q = \frac{2}{3} \in (-1, 1) \Rightarrow \sum \left(\frac{2}{3}\right)^n$ C $\underset{\text{Crit. comp.}}{\Rightarrow}$ S convergentă.

c) $S = \sum_{n=1}^{\infty} \sin^3 \frac{1}{n}$. $\lim_{n\to\infty} \sin^3 \frac{1}{n} = 0$

$l = \lim_{n\to\infty} \frac{\sin^3 \frac{1}{n}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\left[\sin\left(\frac{1}{n}\right)\right]^3}{\frac{1}{n^3}} = \lim_{n\to\infty} \left(\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right)^3 = 1^3 = 1 \in (0, \infty)$.

Deci seriile au aceeași natură. Cum $\sum \frac{1}{n^3}$ e convergentă

$\Rightarrow S$ e convergentă.

d) $S = \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2n+1}$

$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\cancel{(2n-1)!!}}{\cancel{(2n)}!!} \cdot \frac{1}{2n+1} \cdot \frac{\overset{2n+2}{\cancel{(2n+2)!!}}}{\underset{2n+1}{\cancel{(2n+1)}!!}} \cdot \frac{2n+3}{1}$

$= \lim_{n\to\infty} \frac{(2n+2)(2n+3)}{(2n+1)^2} = \lim_{n\to\infty} \frac{4n^2+10n+6}{4n^2+4n+1} = 1$

$R = \lim_{n\to\infty} n \cdot (D-1) = \lim_{n\to\infty} n \cdot \left(\frac{\cancel{4n^2}+10n+6-\cancel{4n^2}-4n-1}{4n^2+4n+1}\right) = \lim_{n\to\infty} \frac{6n^2+5n}{4n^2+4n+1}$

$= \frac{6}{4} > 1 \Rightarrow S$ convergentă.

e) $S=\sum_{n=1}^{\infty} a^{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}$, $a>0$.

Cum $\sum_{n=1}^{\infty} \frac{1}{n}$ e D $\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = +\infty$

Dacă $a>1$, atunci $a^{1+\frac{1}{2}+\dots+\frac{1}{n}} \to +\infty$, deci S e divergentă.

Dacă $a=1$, atunci $S=\sum_{n=1}^{\infty} 1 = \infty \Rightarrow$ seria e divergentă.

Dacă $a\in(0,1)$, atunci $a^{1+\frac{1}{2}+\dots+\frac{1}{n}} \to 0$.

$$D=\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{a^{1+\frac{1}{2}+\dots+\frac{1}{n}}}{a^{1+\frac{1}{2}+\dots+\frac{1}{n}+\frac{1}{n+1}}} = \lim_{n\to\infty} a^{-\frac{1}{n+1}} = a^0 = 1.$$

$$R=\lim_{n\to\infty} n\cdot(D-1) = \lim_{n\to\infty} n\cdot\left(a^{-\frac{1}{n+1}}-1\right) = \lim_{n\to\infty} n\cdot \underbrace{\frac{\left(\frac{1}{a}\right)^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}}_{\ln\left(\frac{1}{a}\right)} \cdot \frac{1}{n+1}$$

$$=\lim_{n\to\infty} \ln\left(\frac{1}{a}\right)\cdot\frac{n}{n+1} = \underbrace{\ln\left(\frac{1}{a}\right)}_{=-\ln a} > 0 \qquad \frac{1}{a}>1$$

1) $a\in\left(0,\frac{1}{e}\right) \Rightarrow \ln\frac{1}{a} > 1 \Rightarrow \sum x_n$ C.

2) $a\in\left(\frac{1}{e},1\right) \Rightarrow \ln\frac{1}{a} < 1 \Rightarrow \sum x_n$ D.

3) $a=\frac{1}{e} \Rightarrow S=\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^{1+\frac{1}{2}+\dots+\frac{1}{n}}$.

$$B=\lim_{n\to\infty} \ln n\cdot\left(\frac{n}{n+1}-1\right) = \lim_{n\to\infty} \frac{-\ln n}{n+1} = 0 < 1 \Rightarrow \sum x_n \text{ D}.$$

Deci S este $\begin{cases} \text{C, dacă } a\in\left(0,\frac{1}{e}\right) \\ \text{D, dacă } a\geq\frac{1}{e} \end{cases}$.

g) $S=\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)^{n^2}$

$$c=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n+1}{n+2}\right)^{n^2}}=\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^{\frac{n^2}{n}}=\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^{n}\overset{[1^\infty]}{=}e^{\lim_{n\to\infty}\left(\frac{n+1}{n+2}-1\right)\cdot n}$$

$$=e^{\lim_{n\to\infty}\frac{-n}{n+2}}=e^{-1}=\frac{1}{e}<1\ \underset{\text{Radicalului}}{\overset{\text{Crit.}}{\Longrightarrow}}\ S\text{ convergentă.}$$

h) $S=\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

$$l=\lim_{n\to\infty}\frac{\frac{\ln n}{n^2}}{\frac{1}{n^{3/2}}}=\lim_{n\to\infty}\frac{\ln n}{n^2}\cdot n^{3/2}=\lim_{n\to\infty}\frac{\ln n}{n^{1/2}}<+\infty,$$

iar $\sum\frac{1}{n^{3/2}}$ C, deoarece $3/2>1$ (seria armonică). Atunci, $\Rightarrow S$ convergentă conform crit. comparației sub formă de limită.

② Studiați convergența și absolut convergența seriilor:

a) $S=\sum_{n=1}^{\infty}(-1)^n\cdot n$

$S'=\sum_{n=1}^{\infty}|(-1)^n\cdot n|=\sum_{n=1}^{\infty}n$ care e divergentă, deci seria nu e absolut convergentă.

$$S_n=\underbrace{-1+2}_{1}\underbrace{-3+4}_{1}\underbrace{-5+6}_{1}-\dots\underbrace{-(n-1)+n}_{1}\begin{cases}\frac{n}{2}, & n \text{ par}\\ -\frac{n+1}{2}, & n\text{ impar.}\end{cases}$$

~~Deci~~ Evident, suma e divergentă.

b) $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{\sqrt{n}}{n+\sqrt{2}}$

$$S=\sum_{n=1}^{\infty}\left|(-1)^{n+1}\cdot\frac{\sqrt{n}}{n+\sqrt{2}}\right|=\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+\sqrt{2}}$$

$$\frac{\sqrt{n}}{n+\sqrt{2}} > \frac{\sqrt{n}}{n+\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n}(\sqrt{n}+1)} = \frac{1}{\sqrt{n}+1} > \frac{1}{n}$$

$\sum \frac{1}{n}$ e divergentă, fiind seria armonică

Deci $\sum \frac{\sqrt{n}}{n+\sqrt{2}}$ e D, nu e abs. convergentă.

Folosim Crit. Leibniz: dacă $(a_n)_{n\in\mathbb{N}}$ e un şir descrescator de nr. poz. cu $\lim\limits_{n\to\infty} a_n = 0$, atunci $\sum\limits_{n=0}^{\infty} (-1)^n \cdot a_n$ e convergentă.

$$a_n - a_{n+1} > 0 \iff \frac{\sqrt{n}}{n+\sqrt{2}} > \frac{\sqrt{n+1}}{n+1+\sqrt{2}} \iff$$

$$\frac{\sqrt{n}}{n+\sqrt{2}} - \frac{\sqrt{n+1}}{n+1+\sqrt{2}} > 0 \iff \sqrt{n}(n+1+\sqrt{2}) - (n+\sqrt{2})\cdot\sqrt{n+1} > 0$$

$$\iff \sqrt{n}(n+1+\sqrt{2}) > (n+\sqrt{2})\cdot\sqrt{n+1} \quad \|()^2 \iff$$

$$n(n^2+1+2+2n+2\sqrt{2}n+2\sqrt{2}) > (n^2+2\sqrt{2}n+2)(n+1) \iff$$

$$\cancel{n^3} + 3n + 2n^2 + \cancel{2\sqrt{2}n^2} + \cancel{2\sqrt{2}n} - \cancel{n^3} - n^2 - \cancel{2\sqrt{2}n^2} - \cancel{2\sqrt{2}n} - 2n - 2 > 0 \iff$$

$n^2 + n - 2 > 0$, adev. pt. $n \geq 2$.

Aşadar, şirul e $\searrow$, deci S convergentă.

c) $S = \sum\limits_{n=1}^{\infty} \sin(\pi \cdot \sqrt{n^2+1})$

$$\sin(\pi\cdot\sqrt{n^2+1}) = \sin[\pi(\sqrt{n^2+1}-n) + n\pi] = \sin[\pi(\sqrt{n^2+1}-n)]\cdot\cos(n\pi) + \cos[\pi(\sqrt{n^2+1}-n)]\cdot\underbrace{\sin(n\pi)}_{=0}$$

$$= \sin[\pi(\sqrt{n^2+1}-n)]\cdot\underbrace{\cos(n\pi)}_{(-1)^n}$$

$$= (-1)^n\cdot\sin[\pi(\sqrt{n^2+1}-n)] = (-1)^n\cdot\sin\left(\pi\cdot\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}\right) =$$

$= (-1)^n \cdot \sin \frac{\pi}{\sqrt{n^2+1}+n}$

Șirul $\sin \frac{\pi}{\sqrt{n^2+1}+n}$ e descrescător spre 0 (are termeni pozitivi deoarece $\frac{\pi}{\sqrt{n^2+1}+n}$ e în cadranul 1), deci conform Leibniz $\Rightarrow S$ e convergentă.

* Verificăm absolut convergența : $\sum \left|(-1)^n \cdot \sin \frac{\pi}{\sqrt{n^2+1}+n}\right| = \sum \sin \frac{\pi}{\sqrt{n^2+1}+n}$

$$\lim_{n\to\infty} \frac{\sin \frac{\pi}{\sqrt{n^2+1}+n}}{\frac{1}{n}} = \lim_{n\to\infty} \underbrace{\frac{\sin \frac{\pi}{\sqrt{n^2+1}+n}}{\frac{\pi}{\sqrt{n^2+1}+n}}}_{1} \cdot n \cdot \frac{\pi}{\sqrt{n^2+1}+n} =$$

$$= \lim_{n\to\infty} \frac{\pi \cdot \cancel{n}}{\cancel{n}\left(1+\sqrt{1+\underbrace{\frac{1}{n^2}}_{0}}\right)} = \frac{\pi}{2} \in (0,\infty) \Rightarrow \text{ seriile } \sum \sin \frac{\pi}{\sqrt{n^2+1}+n} \text{ și } \sum \frac{1}{n}$$

au aceeași natură. $\sum \frac{1}{n}$ D $\Rightarrow$ seria nu e abs. conv.

③ Calc. limita șirului $x_n = \frac{3^n \cdot n!}{n^n}$.

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\cancel{3^n} \cdot \cancel{n!}}{n^n} \cdot \frac{(n+1)^{n+1}}{\underset{3}{\cancel{3^{n+1}}} \cdot \underset{(n+1)}{\cancel{(n+1)!}}} = \frac{1}{3} \cdot \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \frac{e}{3} < 1$$

$\Rightarrow x_n \to \infty,\ n \to \infty.$

④ Fie $\sum_{n=1}^{\infty} x_n$ o S.T.P. convergentă. Care din următoarele afirmații sunt întotdeauna adevărate?

i) Seria $\sum_{n=1}^{\infty} x_n^2$ e convergentă.

ii) Seria $\sum_{n=1}^{\infty} \sqrt{x_n}$ e convergentă.

i) ~~F~~ Fals. Ex: $\sum x_n = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}}$ e convergentă (conform Leibniz, cu $\frac{1}{\sqrt{n}} \downarrow 0$), dar $\sum_{n=1}^{\infty} x_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ e divergentă

ii) Fals. $\sum_{n=1}^{\infty} x_n = \sum \frac{1}{n^2}$, convergentă, fiind seria armonică cu $p=2>1$.

Dar $\sum_{n=1}^{\infty} \sqrt{x_n} = \sum \frac{1}{n}$ care e divergentă.

5. Fie $(x_n)_{n\in\mathbb{N}}$ un șir cu termeni pozitivi. Care din următ. implicații sunt adevărate?

a) Dacă $\sum_{n=1}^{\infty} x_n$ C $\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ C. „A"

b) Dacă $\sum_{n=1}^{\infty} x_n$ D $\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ D. „F".

a) $m_a \geq m_g \rightarrow \frac{x_n + \frac{1}{n^2}}{2} \geq \sqrt{x_n \cdot \frac{1}{n^2}}$

$$\frac{1}{2}\left(x_n + \frac{1}{n^2}\right) \geq \frac{\sqrt{x_n}}{n}.$$

Cum $\sum x_n$ C, $\sum \frac{1}{n^2}$ C $\Rightarrow \frac{1}{2}\sum\left(x_n + \frac{1}{n^2}\right)$ C $\xRightarrow[\text{comparației}]{\text{Crit.}}$

$\sum \frac{\sqrt{x_n}}{n}$ C.

b) Fie $x_n = \frac{1}{n}$. $\sum \frac{1}{n}$ D.

$\sum \frac{\sqrt{x_n}}{n} = \sum \frac{1}{n\sqrt{n}} = \sum \frac{1}{n^{3/2}}$ C, deoarece e seria armonică cu $p = 3/2 > 1$.