## Exerciții suplimentare

① Evaluați integralele iterate

$$I_1=\int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}\,dx\right)dy \quad \text{și} \quad I_2=\int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}\,dy\right)dx$$

Contrazice acest lucru teorema lui Fubini?

$$I_1:\ \int_0^1 \frac{x-y}{(x+y)^3}\,dx=\int_0^1 \frac{x+y-y}{(x+y)^3}\,dx - y\cdot\int_0^1 \frac{1}{(x+y)^3}\,dx=$$

$$=\int_0^1 \frac{x+y}{(x+y)^3}\,dx-2y\int_0^1\frac{1}{(x+y)^3}\,dx=\int_0^1 (x+y)^{-2}\,dx-2y\int_0^1 (x+y)^{-3}\,dx$$

$$=-\frac{1}{x+y}\Big|_0^1-2y\cdot\frac{1}{(x+y)^2\cdot(-2)}\Big|_0^1=-\frac{1}{1+y}+\cancel{\frac{1}{y}}+\frac{y}{(1+y)^2}-\cancel{\frac{1}{y}}$$

$$I_1=\int_0^1\left(-\frac{1}{1+y}+\frac{y}{(1+y)^2}\right)dy=\int_0^1\left(-\frac{1}{1+y}+\frac{y+1}{(1+y)^2}-\frac{1}{(1+y)^2}\right)dy$$

$$=-\int_0^1(1+y)^{-2}\,dy=\frac{1}{1+y}\Big|_0^1=\frac{1}{2}-1=-\frac{1}{2}$$

$$I_2:\ \int_0^1\frac{x-y}{(x+y)^3}\,dy=\int_0^1\frac{2x-x-y}{(x+y)^3}\,dy=2x\int_0^1(x+y)^{-3}\,dy-\int_0^1(x+y)^{-2}\,dy$$

$$=\cancel{2}x\cdot\frac{(x+y)^{-2}}{-\cancel{2}}\Big|_0^1-\frac{(x+y)^{-1}}{-1}\Big|_0^1=-\frac{x}{(x+y)^2}\Big|_0^1+\frac{1}{x+y}\Big|_0^1=$$

$$=\frac{-x}{(x+1)^2}+\cancel{\frac{1}{x}}+\frac{1}{x+1}-\cancel{\frac{1}{x}}$$

$$I_2=\int_0^1\left(\frac{-x}{(x+1)^2}+\frac{1}{x+1}\right)dx=\int_0^1\frac{-\cancel{x}+\cancel{x}+1}{(x+1)^2}\,dx=\frac{(x+1)^{-1}}{-1}\Big|_0^1=$$

$$=-\frac{1}{x+1}\Big|_0^1=-\frac{1}{2}+1=\frac{1}{2}$$

Nu contrazice T. Fubini, deoarece funcția trebuie să fie <u>continuă</u> pe dreptunghiul $[0,1]\times[0,1]$, iar în acest caz nu e cont. în $(0,0)$ (limitele iterate sunt diferite):

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x-y}{(x+y)^3}\right)=\lim_{x\to 0}\frac{x}{x^3}=\lim_{x\to 0}\frac{1}{x^2}=+\infty$$

$$\neq$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x-y}{(x+y)^3}\right)=\lim_{y\to 0}\frac{-y}{y^3}=\lim_{y\to 0}\frac{-1}{y^2}=-\infty$$

② Evaluați integralele duble pe mulțimile specificate:

a) $I=\iint_A \frac{1+x^2}{1+y^2}\,dx\,dy$, $A=[0,1]^2$

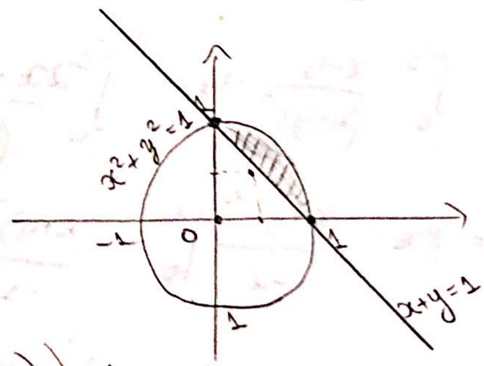
$$I=\int_0^1\left(\int_0^1\frac{1+x^2}{1+y^2}\,dx\right)dy=\int_0^1\frac{1}{1+y^2}\cdot\left(x+\frac{x^3}{3}\right)\Big|_0^1\,dy$$

$$=\int_0^1\frac{1}{1+y^2}\cdot\frac{4}{3}\,dy=\frac{4}{3}\cdot\operatorname{arctg} y\Big|_0^1=\frac{4}{3}\cdot\frac{\pi}{4}=\frac{\pi}{3}$$

b) $I=\iint_A dx\,dy$, $A\subseteq\mathbb{R}^2$ e regiunea mărginită de curba $x^2+y^2=1$ și situată deasupra dreptei $x+y=1$.

$x\in[1-y,\sqrt{1-y^2}]$

$y\in[0,1]$




$$I=\int_0^1\left(\int_{1-y}^{\sqrt{1-y^2}}dx\right)dy=\int_0^1\Big(\underbrace{\sqrt{1-y^2}}_{I^*}-(1-y)\Big)\,dy=$$

$$=\frac{\arcsin(y)+y\sqrt{1-y^2}}{2}\Big|_0^1-y\Big|_0^1+\frac{y^2}{2}\Big|_0^1=\frac{\arcsin 1}{2}-1+\frac{1}{2}=\frac{\pi}{4}-\frac{1}{2}$$

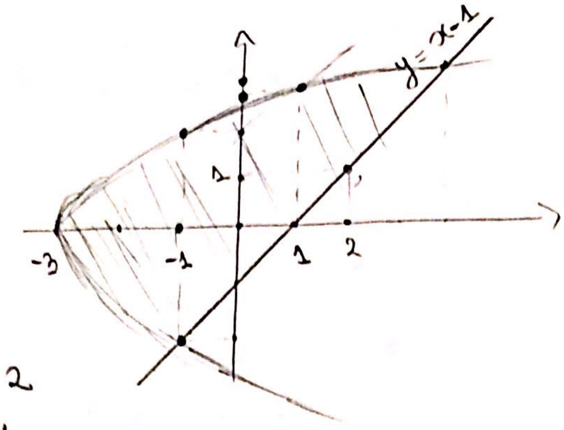
$$I^*=\int\sqrt{1-y^2}\,dy=y\sqrt{1-y^2}-\int\frac{-y^2}{\sqrt{1-y^2}}\,dy=y\sqrt{1-y^2}-\int\frac{1-y^2-1}{\sqrt{1-y^2}}\,dy=y\sqrt{1-y^2}-\underbrace{\int\sqrt{1-y^2}\,dy}_{I^*}+\int\frac{dy}{\sqrt{1-y^2}}$$

$f=\sqrt{1-y^2}\Rightarrow f'=\frac{-y}{\sqrt{1-y^2}}$

$g'=1\Rightarrow g=y$

$I^*=y\sqrt{1-y^2}-I^*+\arcsin y \Leftrightarrow I^*=\frac{\arcsin y+y\sqrt{1-y^2}}{2}$

c) $I=\iint_A xy\,dx\,dy$, $A\subseteq\mathbb{R}^2$ e regiunea mărginită de dreapta $y=x-1$ și parabola $y^2=2x+6$




$x=y+1$

$x=\frac{y^2-6}{2}$

$\Rightarrow y+1=\frac{y^2-6}{2} \Rightarrow 2y+2=y^2-6$

$y^2-2y-8=0 \Rightarrow y_1=-2$, $y_2=4$

$y\in[-2,4]$, $x\in\left[\frac{y^2-6}{2}, y+1\right]$

$$I=\int_{-2}^{4}\left(\int_{\frac{y^2-6}{2}}^{y+1} xy\,dx\right)dy=\int_{-2}^{4} y\cdot\frac{x^2}{2}\Big|_{\frac{y^2-6}{2}}^{y+1}dy=$$

$$=\int_{-2}^{4}\frac{y}{2}\cdot\left((y+1)^2-\frac{(y^2-6)^2}{4}\right)dy=$$

$$=\int_{-2}^{4}\frac{y}{2}\left(\frac{4(y^2+2y+1)-y^4+12y^2-36}{4}\right)dy=$$

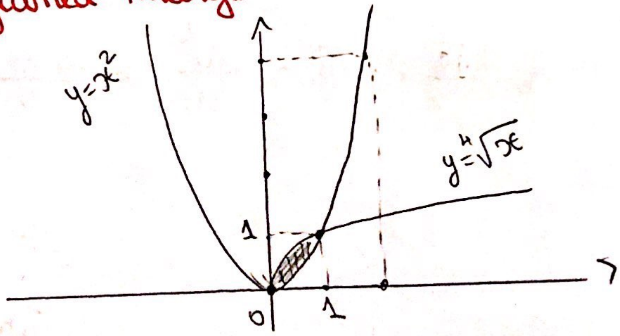
$$=\frac{1}{8}\int_{-2}^{4}\left(-y^5+16y^3+8y^2-32y\right)dy=\frac{1}{8}\left(-\frac{y^6}{6}+16\frac{y^4}{4}+\frac{8y^3}{3}-32\frac{y^2}{2}\right)\Big|_{-2}^{4}=$$

$$=\frac{1}{8}\left(-\frac{4^6-2^6}{6}+4\left(4^4-2^4\right)+\frac{8}{3}\left(4^3+2^3\right)-16(16-4)\right)=$$

$$\frac{1}{8}\left(-672+960+192-192\right)=36$$

d) $\iint_A(\sqrt{x}-y^2)\,dx\,dy$, $A\subseteq\mathbb{R}^2$ e regiunea mărginită de curbele $y=x^2$ și $x=y^4$,

$y=\sqrt[4]{x}$




$y\in[x^2,\sqrt[4]{x}]$

$x\in[0,1]$

$$I=\int_0^1\left(\int_{x^2}^{\sqrt[4]{x}}(\sqrt{x}-y^2)\,dy\right)dx=\int_0^1\left(\sqrt{x}\cdot y\Big|_{x^2}^{\sqrt[4]{x}}-\frac{y^3}{3}\Big|_{x^2}^{\sqrt[4]{x}}\right)dx=$$

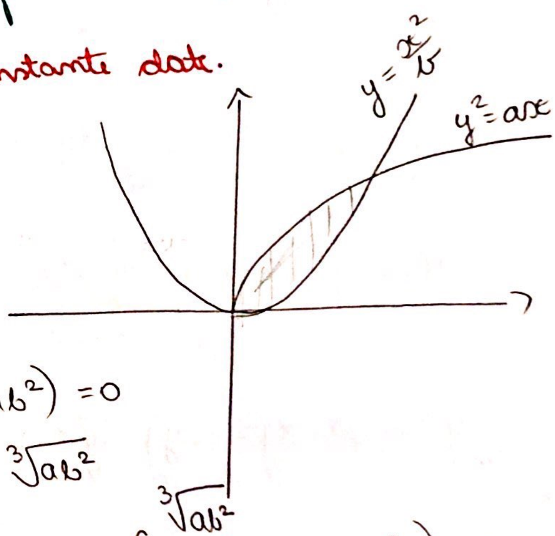
$$=\int_0^1\left[x^{1/2}\left(x^{1/4}-x^2\right)-\frac{1}{3}\cdot\left(x^{\frac{3}{4}}-x^6\right)\right]dx=$$

$$=\frac{x^{1/2+1/4+1}}{1/2+1/4+1}\Big|_0^1-\frac{x^{2+1/2+1}}{2+1/2+1}\Big|_0^1-\frac{1}{3}\cdot\frac{x^{3/4+1}}{3/4+1}\Big|_0^1+\frac{x^7}{3\cdot 7}\Big|_0^1=$$

$$=\frac{4}{7}-\frac{2}{7}-\frac{1}{3}\cdot\frac{4}{7}+\frac{1}{21}=\frac{6-4+1}{21}=\frac{3}{21}=\frac{1}{7}.$$

3 Determinați aria mulțimii plane mărginită de curbele

a) $y^2=ax$, $x^2=by$, $a,b>0$ constante date.




$y=\sqrt{ax}$ $\qquad y=\frac{x^2}{b}$

$$\frac{x^2}{b}=\sqrt{ax}\quad |()^2$$

$$\frac{x^4}{b^2}=ax\Rightarrow x^4=ab^2x\Rightarrow x(x^3-ab^2)=0$$

$$x=0,\ x=\sqrt[3]{ab^2}$$

$$\mathcal{A}=\iint_A dx\,dy=\int_0^{\sqrt[3]{ab^2}}\left(\int_{\frac{x^2}{b}}^{\sqrt{ax}}dy\right)dx=\int_0^{\sqrt[3]{ab^2}}\left(\sqrt{a}\cdot\sqrt{x}-\frac{x^2}{b}\right)dx$$

$$=\left(\sqrt{a}\cdot\frac{x^{3/2}}{3/2}-\frac{x^3}{3b}\right)\Big|_0^{\sqrt[3]{ab^2}}=\frac{2\sqrt{a}}{3}\cdot\left(\sqrt[3]{ab^2}\right)^{3/2}-\frac{ab^2}{3b}$$

$$=\frac{2\sqrt{a}}{3}\cdot\sqrt{ab^2}-\frac{ab^2}{3b}=\frac{2a\cdot b}{3}-\frac{ab^{\cancel{2}}}{3\cancel{b}}\ \frac{2ab-ab}{3}=\frac{ab}{3}$$



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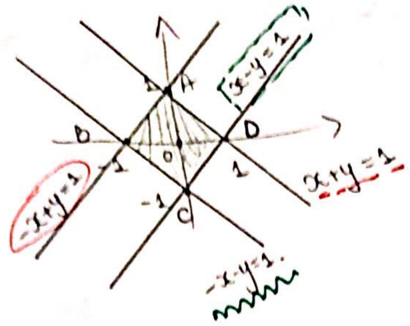
b) $|x|+|y|=1$.

Avem 4 cazuri:

* $x,y\ge 0$ : $x+y=1$
* $x\ge 0,\ y\le 0$: $x-y=1$
* $x,y\le 0$ : $-x-y=1$
* $x\le 0,\ y\ge 0$: $-x+y=1$




Obs. Figura obținută e un pătrat, deci $A=l^2$, iar $l=\sqrt{2}$, deci $A=2$.

Cu integrale duble: $A_{ABCD}=A_{ABD}+A_{BCD}$.

ABD: $y\in[0,1]$, $x\in[y-1, 1-y]$

$$A_{ABD}=\int_0^1\left(\int_{y-1}^{1-y}dx\right)dy=\int_0^1 x\Big|_{y-1}^{1-y}dx=\int_0^1(1-y-y+1)\,dy$$

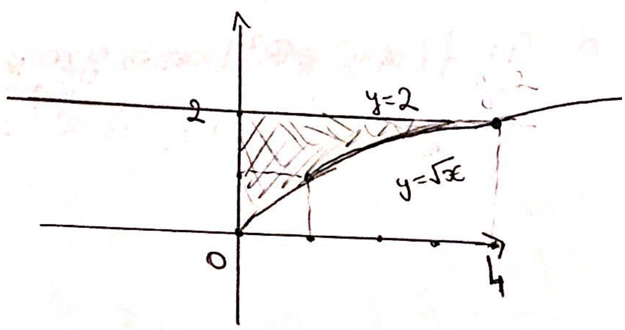
$$=\int_0^1(2-2y)\,dy=2y\Big|_0^1-y^2\Big|_0^1=2-1=1.$$

Analog $A_{BCD}=1$ ( se ia $y\in[-1,0]$, $x\in[-y-1, \boxed{1+y}]$! )

$\Rightarrow A_{ABCD}=1+1=2$.

4) Evaluați integralele iterate, schimbând în prealabil ordinea de integrare:

a) $I=\int_0^4\left(\int_{\sqrt{x}}^2\frac{1}{y^3+1}dy\right)dx$



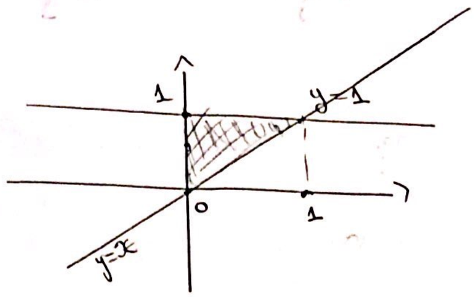

$y=\sqrt{x}\Rightarrow x=y^2$

$x\in[0,y^2]$, $y\in[0,2]$

$$I = \int_0^2 \left( \int_0^{y^2} \frac{1}{y^3+1}\, dx \right) dy = \int_0^2 \frac{x}{y^3+1} \Big|_0^{y^2} dy = \int_0^2 \frac{y^2}{y^3+1}\, dy =$$

$$= \int_0^2 \frac{1}{3} \cdot \frac{(y^3+1)'}{y^3+1}\, dy = \frac{1}{3} \ln (y^3+1) \Big|_0^2 = \frac{1}{3} \ln 9 = \frac{2}{3} \ln 3$$

b) $J = \int_0^1 \left( \int_x^1 e^{x/y}\, dy \right) dx$




$y \in [0,1], \quad x \in [0,y]$

$$J = \int_0^1 \left( \int_0^y e^{x/y}\, dx \right) dy =$$

$$= \int_0^1 \left( e^{x/y} \cdot y \Big|_0^y \right) dy = \int_0^1 (e \cdot y - y)\, dy = (e-1) \cdot \frac{y^2}{2} \Big|_0^1 = \frac{e-1}{2}$$

5) Fie un obiect plan, subțire și omogen, a cărui imagine este mulțimea compactă $A \subseteq \mathbb{R}^2$. Coordonatele centrului de greutate al acestui obiect sunt

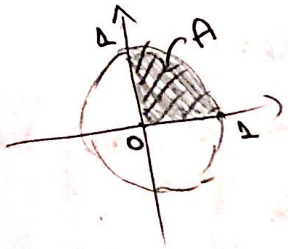
$$x_G = \frac{1}{\sigma} \iint_A x\, dx\, dy, \quad y_G = \frac{1}{\sigma} \iint_A y\, dx\, dy,$$

$$\sigma = \iint_A dx\, dy = \text{aria mulțimii } A.$$

Determinați coordonatele c.g. al obiectului plan, subțire și omogen, a cărui imagine este mulțimea A de mai jos:

a) $A = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1\}$

$x \in [0,1], \quad y \in [0, \sqrt{1-x^2}]$



$$\nabla = \iint_A dx\,dy = \int_0^1 \left( \int_0^{\sqrt{1-x^2}} dy \right) dx = \int_0^1 \sqrt{1-x^2}\, dx = \frac{\arcsin x + x\sqrt{1-x^2}}{2} \Big|_0^1$$

$$= \frac{\pi}{4}$$

$$x_G = \frac{4}{\pi} \iint_A x\, dx\, dy = \frac{4}{\pi} \int_0^1 \left( \int_0^{\sqrt{1-x^2}} x\, dy \right) dx =$$

$$= \frac{4}{\pi} \int_0^1 x\sqrt{1-x^2}\, dx = \frac{4}{\pi} \cdot \frac{1}{2} \int_0^1 \sqrt{t}\, dt = \frac{2}{\pi} \cdot \frac{t^{3/2}}{\frac{3}{2}} \Big|_0^1 = \frac{4}{3\pi}$$

$1 - x^2 = t$ $\quad x = 0 \Rightarrow t = 1$

$-2x\, dx = dt$ $\quad x = 1 \Rightarrow t = 0$

$x\, dx = -\frac{1}{2} dt$

$$y_G = \frac{4}{\pi} \iint_A y\, dx\, dy = \frac{4}{\pi} \int_0^1 \left( \int_0^{\sqrt{1-x^2}} y\, dy \right) dx = \frac{4}{\pi} \int_0^1 \frac{y^2}{2} \Big|_0^{\sqrt{1-x^2}} dx$$

$$= \frac{2}{\pi} \int_0^1 (1-x^2)\, dx = \frac{2}{\pi} \left( x - \frac{x^3}{3} \right) \Big|_0^1 = \frac{2}{\pi} \left( \overset{3}{1} - \frac{1}{3} \right) = \frac{4}{3\pi}$$

$$G\left( \frac{4}{3\pi}, \frac{4}{3\pi} \right)$$

b) $A = \{ (x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 ,\ 0 \le y \le 1 - x^2 \}$

$$\nabla = \iint_A dx\, dy = \int_0^1 \left( \int_0^{1-x^2} dy \right) dx = \int_0^1 y \Big|_0^{1-x^2} dx = \int_0^1 (1-x^2)\, dx$$

$$= \left( x - \frac{x^3}{3} \right) \Big|_0^1 = \frac{2}{3}$$

$$x_G = \frac{3}{2} \iint_A x\, dx\, dy = \frac{3}{2} \int_0^1 \left( \int_0^{1-x^2} x\, dy \right) dx = \frac{3}{2} \int_0^1 \left( x \cdot y \Big|_0^{1-x^2} \right) dx$$

$$= \frac{3}{2} \int_0^1 (x - x^3)\, dx = \frac{3}{2} \left( \frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 = \frac{3}{2} \left( \overset{2}{\frac{1}{2}} - \frac{1}{4} \right) = \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}$$

$$y_G = \frac{3}{2} \iint_A y\, dx\, dy = \frac{3}{2} \int_0^1 \left( \int_0^{1-x^2} y\, dy \right) dx = \frac{3}{2} \int_0^1 \left( \frac{y^2}{2} \Big|_0^{1-x^2} \right) dx =$$

$$= \frac{3}{4} \int_0^1 (1 - 2x^2 + x^4)\, dx = \frac{3}{4} \left( x - \frac{2x^3}{3} + \frac{x^5}{5} \right) \Big|_0^1 = \frac{3}{4} \left( \overset{15}{1} - \overset{5}{\frac{2}{3}} + \overset{3}{\frac{1}{5}} \right) = \frac{3}{4} \cdot \frac{8}{15} = \frac{2}{5}$$

$$\boxed{G\left( \frac{3}{8}, \frac{2}{5} \right)}$$