Exerciții suplimentare

① Calculați limita șirurilor:

a) $y_n = \dfrac{1+\sqrt{2}+\dots+\sqrt{n}}{n\sqrt{n}}$

$a_n = 1+\sqrt{2}+\dots+\sqrt{n}$

$b_n = n\sqrt{n} \nearrow \infty$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{(n+1)\sqrt{n+1}-n\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{(n+1)^3}-\sqrt{n^3}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}\left(\sqrt{(n+1)^3}+\sqrt{n^3}\right)}{(n+1)^3-n^3} = \lim_{n\to\infty} \frac{(n+1)^2+\sqrt{n^4+n^3}}{3n^2+3n+1} =$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1+n^2\sqrt{1+\frac{1}{n}}}{3n^2+3n+1} = \frac{2}{3}, \exists$$

$$\overset{S-C}{\Rightarrow} \lim_{n\to\infty} y_n = \frac{2}{3}.$$

b) $y_n = \dfrac{1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}}{\ln n}$ $\qquad a_n = 1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}$

$b_n = \ln n \nearrow \infty$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\ln\frac{n+1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{(n+1)\ln\left(1+\frac{1}{n}\right)} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}}{\ln\underbrace{\left(1+\frac{1}{n}\right)^{n+1}}_{\to e}} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{1} = \infty$$

$$\overset{S-C}{\Rightarrow} \lim_{n\to\infty} y_n = \infty.$$

c) $y_n = \dfrac{\sqrt[n]{(n+1)(n+2)\cdot\ldots\cdot(n+n)}}{n}$

$$y_n = \sqrt[n]{\frac{(n+1)\cdot\ldots\cdot(n+n)}{n^n}} . \quad \text{Fie } x_n = \frac{(n+1)\cdot\ldots\cdot(n+n)}{n^n}, \ x_n > 0.$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{(n+1)\cdot\ldots\cdot(n+n)}\cdot(n+n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{(n+1)\cdot\ldots\cdot(n+n)}}$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n}{n+1}\right)^n}_{\to \frac{1}{e}} \cdot \frac{(2n+\overset{2}{1})}{n+1} = \frac{2}{e}$$

Deci $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = \lim\limits_{n\to\infty} y_n = \dfrac{2}{e}$.

d) $y_n = \dfrac{\ln n!}{n \ln n}$

$$y_n = \frac{\ln n!}{\ln n^n} \qquad a_n = \ln n! \qquad b_n = \ln n^n \nearrow \infty$$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\ln \frac{(n+1)!}{n!}}{\ln \frac{(n+1)^{n+1}}{n^n}} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln\left[\underbrace{\left(\frac{n+1}{n}\right)^n}_{e}\cdot(n+1)\right]} =$$

$$= \lim_{n\to\infty} \frac{\ln(n+1)}{\ln e + \ln(n+1)} = 1.$$

$\overset{S-C}{\Rightarrow} \lim\limits_{n\to\infty} y_n = 1.$

③ Fie $(x_n)_{n\in\mathbb{N}}$ un şir cu termeni strict pozitivi.
Dacă $\exists \lim\limits_{n\to\infty} x_n = l$, atunci

$$\lim_{n\to\infty} \frac{x_1+x_2+\ldots+x_n}{n} = \lim_{n\to\infty} \sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n} = l.$$

* Fie $a_n = x_1+x_2+\ldots+x_n$, $b_n = n \uparrow \infty$.

* $\lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim\limits_{n\to\infty} \dfrac{x_{n+1}}{1} = l$

$\overset{S-C}{\Rightarrow} \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = l$, deci $\lim\limits_{n\to\infty} \dfrac{x_1+\ldots+x_n}{n} = l$.

* Fie $c_n = x_1\cdot\ldots\cdot x_n$

$\lim\limits_{n\to\infty} \dfrac{c_{n+1}}{c_n} = \lim\limits_{n\to\infty} x_{n+1} = l$

$\overset{Ex.1}{\Rightarrow} \lim\limits_{n\to\infty} \sqrt[n]{c_n} = l$, deci $\lim\limits_{n\to\infty} \sqrt[n]{x_1\cdot\ldots\cdot x_n} = l$.

③ Calculaţi suma seriilor:

a) $\sum\limits_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n = \left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)^2 + \ldots + \left(-\frac{2}{3}\right)^n + \ldots =$

$$\Big(\underbrace{1+\left(-\tfrac{2}{3}\right)+\left(-\tfrac{2}{3}\right)^2+\ldots+\left(-\tfrac{2}{3}\right)^n+\ldots}_{\text{serie geom. cu } q=-\frac{2}{3}\in(-1,1)}\Big) - 1 = \frac{1}{1-\left(-\frac{2}{3}\right)} - 1 =$$

$= \frac{3}{5} - 1 = -\frac{2}{5}.$

b) $\sum\limits_{n=2}^{\infty} \dfrac{1}{C_n^2}$

$S_k = \sum\limits_{k=2}^{n} \dfrac{1}{C_k^2} = \sum\limits_{k=2}^{n} \dfrac{1}{\frac{k!}{(k-2)!\cdot 2!}} = \sum\limits_{k=2}^{n} \dfrac{2}{(k-1)\cdot k} = 2\sum\limits_{k=2}^{n}\left(\dfrac{1}{k-1} - \dfrac{1}{k}\right) =$

$= 2 \cdot \left(1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{n-1}} - \frac{1}{n}\right) = 2 \cdot \left(1 - \frac{1}{n}\right)$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{C_n^2} = \lim_{n \to \infty} 2\left(1 - \frac{1}{n}\right) = 2.$

c) $\sum_{n=1}^{\infty} \operatorname{arctg} \frac{1}{n^2+n+1}$

$$\boxed{\operatorname{arctg} x - \operatorname{arctg} y = \operatorname{arctg} \frac{x-y}{1+x \cdot y}}$$

Luăm $x = n+1$, $y = n$

$\operatorname{arctg} \frac{(n+1)-n}{1+(n+1) \cdot n} = \operatorname{arctg}(n+1) - \operatorname{arctg} n \Leftrightarrow \operatorname{arctg} \frac{1}{n^2+n+1} = \operatorname{arctg}(n+1) - \operatorname{arctg} n$

$S_k = \sum_{k=1}^{n} \operatorname{arctg} \frac{1}{k^2+k+1} = \operatorname{arctg} 2 - \operatorname{arctg} 1 + \operatorname{arctg} 3 - \operatorname{arctg} 2 + \dots +$
$\operatorname{arctg}(n+1) - \operatorname{arctg} n = \operatorname{arctg}(n+1) - \operatorname{arctg} 1$

$\sum_{n=1}^{\infty} \operatorname{arctg} \frac{1}{n^2+n+1} = \lim_{n \to \infty} \left[\operatorname{arctg}(n+1) - \operatorname{arctg} 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$

d) $\sum_{n=0}^{\infty} \frac{1+a^n}{(1+a)^n}$, $a > 0$

$\sum_{n=0}^{\infty} \frac{1+a^n}{(1+a)^n} = \sum_{n=0}^{\infty} \left[\left(\frac{1}{1+a}\right)^n + \left(\frac{a}{a+1}\right)^n\right] = \sum_{n=0}^{\infty} \left(\frac{1}{1+a}\right)^n + \sum_{n=0}^{\infty} \left(\frac{a}{1+a}\right)^n$

$= \underbrace{\left(1 + \frac{1}{1+a} + \left(\frac{1}{1+a}\right)^2 + \dots\right) + \left(1 + \frac{a}{1+a} + \left(\frac{a}{1+a}\right)^2 + \dots\right)}_{\text{serii geometrice}} =$

$\frac{1}{1+a} \in (-1,1), a > 0$ $\qquad$ $\frac{a}{1+a} \in (-1,1), a > 0.$

$= \frac{1}{1 - \frac{1}{1+a}} + \frac{1}{1 - \frac{a}{1+a}} = \frac{1+a}{a} + \frac{1+a}{1} =$

$= \frac{1+a}{a} + \frac{a+a^2}{a} = \frac{a^2+2a+1}{a} = \frac{(a+1)^2}{a}.$