SUBIECT MIRU

1. Studiați ∃-ța derivatelor după direcție ale funcției $f(x,y)=\sqrt[3]{x+y^2}$ în pct. $(0,0)$
Este funcția derivabilă parțial în acest punct? Justificați.

$$\lim_{t\to 0}\frac{f(x+t\cdot v)-f(x)}{t}$$, unde x - orice punct (usually 0)

$$\lim_{t\to 0}\frac{f((0,0)+t(v_1,v_2))-f(0,0)}{t}=\lim_{t\to 0}\frac{f(tv_1,tv_2)}{t}=\lim_{t\to 0}\frac{\sqrt[3]{tv_1+t^2v_2^2}}{t}=$$

$$=\sqrt[3]{\lim_{t\to 0}\frac{tv_1+t^2v_2^2}{t^3}}\overset{\frac{0}{0}}{\underset{l'H}{=}}\sqrt[3]{\lim_{t\to 0}\frac{v_1+2tv_2^2}{3t^2}}=\begin{cases}\text{I } v_1>0\Rightarrow \lim\limits_{t\to 0}=+\infty\\ \text{II } v_1<0\Rightarrow \lim\limits_{t\to 0}=-\infty\\ \text{III } v_1=0\Rightarrow \sqrt[3]{\lim\limits_{t\to 0}\frac{2\cancel{t}v_2^2}{3t^{\cancel{2}}}} \text{ nu are limită (lim. laterale} \neq)\end{cases}$$

funcția e deriv. parțial în (x,y) dacă ∃ derivatele parțiale și sunt finite
Derivatele parțiale sunt cazuri particulare de derivate după direcție!

$$\frac{\partial f}{\partial x}=f'_{(1,0)}(0,0)=+\infty \Rightarrow \text{nu e deriv. parțial}$$

$$\frac{\partial f}{\partial y}=f'_{(0,1)}(0,0)=\text{nu are limită}\Rightarrow \text{nu e deriv. parțial}$$

2. Calculați integrala improprie $\int_3^\infty \frac{1}{x^2-x-2}dx$

$$\int_3^\infty\frac{1}{x^2-x-2}dx=\lim_{v\to\infty}\int_3^v\frac{1}{x^2-x-2}dx=\lim_{v\to\infty}\int_3^v\frac{1}{x^2-\frac{1}{2}\cdot 2\cdot x-2}dx=\lim_{v\to\infty}\int_3^v\frac{1}{\left(x-\frac{1}{2}\right)^2-\frac{9}{4}}dx=$$

$$=\lim_{v\to\infty}\int_{\frac{5}{2}}^{v-\frac{1}{2}}\frac{1}{t^2-\frac{9}{4}}dt=\lim_{v\to\infty}\frac{2}{3}\cdot\operatorname{arctg}\frac{2t}{3}\Big|_{\frac{5}{2}}^{v-\frac{1}{2}}=\lim_{v\to\infty}\frac{2}{3}\left(\operatorname{arctg}\frac{2(v-\frac{1}{2})}{3}-\operatorname{arctg}\frac{2\cdot\frac{5}{2}}{3}\right)=$$

not $t=x-\frac{1}{2}$

$x=3\Rightarrow t=\frac{2}{\cancel{2}}3-\frac{1}{2}=\frac{5}{2}$

$x=v\Rightarrow t=v-\frac{1}{2}$

$$=\lim_{v\to\infty}\frac{2}{3}\left(\operatorname{arctg}\frac{2v-1}{3}-\operatorname{arctg}\frac{5}{3}\right)=$$

$$=\frac{2}{3}\left(\frac{\pi}{2}-\operatorname{arctg}\frac{5}{3}\right)$$

3. Fie funcția $f:(0,\infty)^2 \to \mathbb{R}$ $f(x,y) = x\sqrt{y} + \frac{y}{\sqrt{x}}$. Determinați $\alpha \in \mathbb{R}$ a.î.

$$\alpha \frac{x^2}{y^2} \cdot \frac{\partial^2 f}{\partial x^2}(x,y) + 2\frac{\partial^2 f}{\partial y^2}(x,y) + \frac{x}{y} \cdot \frac{\partial^2 f}{\partial x \partial y}(x,y) = 0 \quad \forall x,y \in (0,\infty)^2$$

$$\frac{\partial f}{\partial x}(x,y) = \left(x\sqrt{y} + y \cdot x^{-\frac{1}{2}}\right)'_x = \sqrt{y} + y \cdot \left(-\frac{1}{2}\right) \cdot x^{-\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \left(\sqrt{y} + y \cdot \left(-\frac{1}{2}\right) \cdot x^{-\frac{3}{2}}\right)'_x = -\frac{1}{2}y \cdot \left(-\frac{3}{2}\right) \cdot x^{-\frac{5}{2}} = \frac{3}{4}y \cdot x^{-\frac{5}{2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \left(xy^{\frac{1}{2}} + y \cdot x^{-\frac{1}{2}}\right)'_y = x \cdot \frac{1}{2}y^{-\frac{1}{2}} + x^{-\frac{1}{2}}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \left(\frac{x}{2}y^{-\frac{1}{2}} + x^{-\frac{1}{2}}\right)'_y = \frac{x}{2} \cdot \left(-\frac{1}{2}\right) \cdot y^{-\frac{3}{2}} = -\frac{1}{4}x \cdot y^{-\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \left(\frac{1}{2}xy^{-\frac{1}{2}} + x^{-\frac{1}{2}}\right)'_x = \frac{1}{2}y^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$

$$\Rightarrow \alpha \cdot \frac{x^2}{y^2} \cdot \frac{3}{4}y \cdot x^{-\frac{5}{2}} + 2 \cdot \left(-\frac{1}{4}\right)x \cdot y^{-\frac{3}{2}} + \frac{x}{y} \cdot \left(\frac{1}{2}y^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}\right) = 0$$

$$\Leftrightarrow \frac{3}{4}\alpha\, x^{-\frac{1}{2}} \cdot y^{-1} + \cancel{\left(-\frac{1}{2}\right)x \cdot y^{-\frac{3}{2}}} + \cancel{\frac{1}{2}x \cdot y^{-\frac{3}{2}}} - \frac{1}{2} \cdot x^{-\frac{1}{2}} \cdot y^{-1} = 0$$

$$x^{-\frac{1}{2}} \cdot y^{-1}\left(\frac{3}{4}\alpha - \frac{1}{2}\right) = 0 \quad \forall x,y \in (0,\infty)$$

$$\Rightarrow \frac{3}{4}\alpha - \frac{1}{2} = 0 \Rightarrow \boxed{\alpha = \frac{2}{3}}$$

4. a) $$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\ln n} \overset{S-C}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln(n+1) - \ln n} = \lim_{n\to\infty} \frac{1}{\ln\left(\frac{n+1}{n}\right)^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{1}{\ln\left[\left(\underbrace{\frac{n+1}{n} - 1}_{} + 1\right)^{n}\right]^{\frac{1}{n}(n+1)}} = \frac{1}{\ln e^{\lim \frac{n+1}{n}}} = \frac{1}{\ln e} = 1$$

b) Studiați convergența s.t.p $\sum_{n=1}^{\infty} \left(\frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{n}\right)^a$ în funcție de valorile lui $a > 0$

când avem $1+\frac{1}{2}+\ldots+\frac{1}{n}$ încercăm să îl scriem $/\ln n$ => e mai ușor de calculat convergența

notăm $a_n = \left(\frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{n}\right)^a$ $\quad b_n = \left(\frac{\ln n}{n}\right)^a$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\underbrace{\left(\frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{\ln n}\right)^a}_{\text{datorită nr. } \gamma} = 1^a = 1 \Rightarrow \sum a_n \sim \sum b_n$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^a$$

* D'Alembert: $\lim_{n\to\infty}\frac{b_n}{b_{n+1}} = \lim_{n\to\infty}\left(\frac{\ln n}{n}\right)^a\cdot\left(\frac{n+1}{\ln(n+1)}\right)^a = 1 \Rightarrow$ NU DECIDE

* Raabe-Duhamel: $\lim_{n\to\infty} n\left(\frac{b_n}{b_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\left(\frac{\ln n\,(n+1)}{n\ln(n+1)}\right)^a-1\right] =$

$\frac{\ln n\,(n+1)}{n\ln(n+1)} \to 1$

$$\lim_{x\to 0}\frac{(x+1)^a-1}{x} = a \Rightarrow \lim_{x\to 1}\frac{x^a-1}{x-1} = a$$

$$= \lim_{n\to\infty} n\cdot\underbrace{\frac{\left(\frac{\ln n\,(n+1)}{n\ln(n+1)}\right)^a-1}{\frac{\ln n\,(n+1)}{n\ln(n+1)}-1}}_{\to a}\cdot\left(\frac{(n+1)\ln n}{n\ln(n+1)}\cancel{-n\ln(n+1)}-1\right) =$$

$$= a\lim_{n\to\infty}\cancel{n}\cdot\frac{(n+1)\ln n - n\ln(n+1)}{\cancel{n}\ln(n+1)} = a\lim\frac{n\ln n - n\ln(n+1)+\ln n}{\ln(n+1)} =$$

$$= a\lim_{n\to\infty}\frac{n(\ln n-\ln(n+1))+\ln n}{\ln(n+1)} = a\left[1+\lim_{n\to\infty}\frac{n\ln\frac{n}{n+1}}{\ln(n+1)}\right] =$$

$\frac{\ln n}{\ln(n+1)} \to 1$

$$= a\left[1+\lim_{n\to\infty}\frac{\ln\left[\frac{n}{n+1}\right]^n}{\ln(n+1)}\right] = a\left[1+\lim_{n\to\infty}\frac{\ln\left(-\frac{1}{n+1}+1\right)^{\frac{n+1}{-1}\cdot\frac{-1}{n+1}\cdot n}}{\ln(n+1)}\right] =$$

$$= a\left[1+\underbrace{\frac{-1}{\infty}}_{\to 0}\right] = a$$

I $a>0 \Rightarrow \Sigma$ - convergentă

II $a=0 \Rightarrow \sum\frac{\ln n}{n} > \sum\frac{1}{n} \Rightarrow$ divergentă

III $a<0 \Rightarrow \Sigma$ - divergentă