Exercitii suplimentare

① Calc. derivata de ordinul $n \in \mathbb{N}$ a fc. de mai jos și precizați mulțimea pe care aceste fc. sînt indef. deriv.

a) $f(x) = \cos x$

$f'(x) = -\sin x \quad f''(x) = -\cos x \quad f'''(x) = \sin x \quad f^{IV}(x) = \cos x$

$$f^{(n)}(x) = \begin{cases} \cos x, & n = 4k \\ -\sin x, & n = 4k+1 \\ -\cos x, & n = 4k+2 \\ \sin x, & n = 4k+3 \end{cases}$$

Inducție..

$$f^{(n)}(x) = \begin{cases} (-1)^k \cdot \cos x, & n = 2k \\ (-1)^{k+1} \cdot \sin x, & n = 2k+1 \end{cases}$$

$D = \mathbb{R}$.

b) $f(x) = x\sqrt{x+1}$

$f(x) = x \cdot (x+1)^{\frac{1}{2}}$

$$\left(x \cdot \sqrt{x+1}\right)^{(n)} = \sum_{k=0}^{n} C_n^k \cdot x^{(k)} \cdot \left(\sqrt{x+1}\right)^{(n-k)} =$$

$$= C_n^0 \cdot x \cdot \left(\sqrt{x+1}\right)^{(n)} + C_n^1 \cdot x' \cdot \left(\sqrt{x+1}\right)^{(n-1)} + C_n^2 \cdot \underbrace{x''}_{=0} \cdot \left(\sqrt{x+1}\right)^{(n-2)}$$

0 toți termenii

$$= x \cdot \left(\sqrt{x+1}\right)^{(n)} + n \cdot \left(\sqrt{x+1}\right)^{(n-1)}$$

$g(x) = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$

$g'(x) = \frac{1}{2}(x+1)^{-1/2}$

$g''(x) = (-1)^1 \cdot \frac{1}{2^2} \cdot (x+1)^{-3/2}$

$g'''(x) = (-1)^2 \cdot \frac{1}{2^2} \cdot \frac{3}{2} \cdot (x+1)^{-5/2} = (-1)^2 \cdot \frac{3!!}{2^3} \cdot (x+1)^{-5/2}$

$g^{(IV)}(x) = (-1)^3 \cdot \frac{5!!}{2^4} (x+1)^{-7/2}$

$g^{(n)}(x) = (-1)^{n+1} \cdot \frac{(2n-3)!!}{2^n} \cdot (x+1)^{-\frac{(2n-1)}{2}} \quad \forall n \geq 2$

inducție.

$\boxed{x > -1}$

$$f^{(m)}(x) = (-1)^{m+1} \cdot \frac{(2m-3)!!}{2^m} \cdot (x+1)^{-\frac{(2m-1)}{2}} \cdot x +$$

$$+ (-1)^{m}, \frac{(2m-5)!!}{2^{m-1}} \cdot (x+1)^{-\frac{(2m-3)}{2}} \cdot m, \; m \geq 3$$

$n=1$: $f'(x) = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}}$, $m=2$: $f''(x) = \frac{1}{2\sqrt{x+1}} + \frac{x+2}{4(x+1)^{3/2}}$

c) $f(x) = \ln(1-x^2)$

$$f(x) = \ln[(1-x)(1+x)] = \ln(1-x) + \ln(1+x)$$

$$f'(x) = \frac{-1}{1-x} + \frac{1}{1+x} = (1+x)^{-1} - (1-x)^{-1}$$

$$f''(x) = (-1)(1+x)^{-2} - (-1)\cdot(-1)\cdot(1-x)^{-2}$$

$$= (-1)(1+x)^{-2} - (1-x)^{-2}$$

$$f'''(x) = (-1)(-2)(1+x)^{-3} - (1-x)^{-3} \cdot 1 \cdot 2$$

$$f^{IV}(x) = (-1)^3 \cdot 1 \cdot 2 \cdot 3 \, (1+x)^{-4} - 1 \cdot 2 \cdot 3 \cdot (1-x)^{-4}$$

$$f^{(m)}(x) = (-1)^{m-1} \cdot (m-1)! \, (1+x)^{-m} - (m-1)! \, (1-x)^{-m}, \; m \geq 1$$

inducție

$x \in (-1, 1)$

d) $f(x) = \frac{1}{1-x^2}$

$$f(x) = \frac{1}{(1-x)(1+x)} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)$$

$$= \frac{1}{2}\left((1-x)^{-1} + (1+x)^{-1}\right)$$

$$f'(x) = \frac{1}{2}\left[1(1-x)^{-2} - 1\cdot(1+x)^{-2}\right]$$

$$f''(x) = \frac{1}{2}\left[1\cdot 2(1-x)^{-3} + 1\cdot 2(1+x)^{-3}\right]$$

$$f'''(x) = \frac{1}{2}\left[1\cdot 2\cdot 3(1-x)^{-4} - 1\cdot 2\cdot 3(1+x)^{-4}\right]$$

$$f^{(m)}(x) = \frac{1}{2}\left[(-1)^m \cdot m! \, (1+x)^{-(m+1)} + m! \, (1-x)^{-(m+1)}\right], \; m \in \mathbb{N}$$

inducție

② Pentru fc. de la ex. ①, pct. $x_0=0$ și nr. $n\in\mathbb{N}$, det.

a) Polinomul Taylor de grad n asociat fc. f in pct. x_0

b) Mulțimea de convergență a seriei Taylor corespunzătoare

* $f(x)=\cos x$

$$f^{(n)}(0)=\begin{cases}1, & n=4k\\ 0, & n=4k+1\\ -1, & n=4k+2\\ 0, & n=4k+3\end{cases}$$

$$(T_n f)(x)=\frac{x^0}{0!}\cdot f(0)+\frac{x^1}{1!}\cdot f'(0)+\frac{x^2}{2!}\cdot f''(0)+\frac{x^3}{3!}\cdot f'''(0)+\dots$$

$$+\frac{x^n}{n!}\cdot f^{(n)}(0)=$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}+\dots+\frac{x^n}{n!}\cdot f^{(n)}(0)$$

Fie $s_n=\sum\limits_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!}$ $\Rightarrow a_n=\frac{(-1)^n}{(2n)!}$

$$r=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{\overset{(2n+1)(2n+2)}{\cancel{(2n+2)!}}}{\underset{1}{\cancel{(2n)!}}}=\infty$$

$\Rightarrow I=\mathbb{R}$

* $f(x)=x\sqrt{x+1}$

$$f^{(n)}(0)=(-1)^{n-2}\cdot\frac{(2n-5)!!}{2^{n-1}}\cdot n\quad,\ n\geq 3$$

$f(0)=0$

$$f'(x)=\sqrt{x+1}+\frac{x}{2\sqrt{x+1}}\Rightarrow f'(0)=1$$

$$f''(x)=\frac{1}{2\sqrt{x+1}}+\frac{x+2}{4(x+1)^{3/2}}\qquad\Rightarrow f''(0)=\frac{1}{2}+\frac{2}{4}=1$$

$$\Rightarrow (T_n f)(x) = \frac{x^0}{0!}\cdot 0 + \frac{x^1}{1!}\cdot 1 + \frac{x^2}{2!}\cdot 1 + \sum_{k=3}^{n} (-1)^{k-2}\cdot \frac{(2k-5)!!}{2^{k-1}}\cdot k \cdot \frac{x^k}{k!}$$

$$= 1 + x + x^2 + \sum_{k=3}^{n} (-1)^{k-2}\cdot k\cdot \frac{(2k-5)!!}{2^{k-1}}\cdot \frac{x^k}{k!}$$

$$* \; S_n = \sum_{n=3}^{\infty} (-1)^n \cdot \cancel{n}\cdot \frac{(2n-5)!!}{2^{n-1}}\cdot \frac{x^n}{\underbrace{\cancel{n!}}_{(n-1)!}} + 1 + x + x^2$$

$$a_n = \frac{(2n-5)!!}{(n-1)!\cdot 2^{n-1}}\cdot (-1)^n$$

$$\lambda = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\cancel{(2n-5)!!}}{\cancel{(n-1)!}\cdot \cancel{2^{n-1}}}\cdot \frac{\overset{2}{\cancel{2^n}}\cdot \overset{n}{\cancel{n!}}}{\underbrace{\cancel{(2n-3)!!}}_{2n-3}} = \lim_{n\to\infty}\frac{2n}{2n-3} = 1$$

$$(-1,1) \subset J$$

$$x=1. \qquad 3 + \sum_{n=3}^{\infty} \frac{(-1)^n\cdot (2n-5)!!}{(n-1)!\cdot 2^{n-1}}$$

Verificăm dacă e abs. C.

$$D = \lim_{n\to\infty} \frac{2n}{2n-3} = 1$$

$$R = \lim_{n\to\infty} n\left(\frac{2n}{2n-3} - 1\right) = \lim_{n\to\infty} n\cdot \frac{\cancel{2n}-\cancel{2n}+3}{2n-3} = \frac{3}{2} > 1$$

$\Rightarrow$ seria e abs. C $\Rightarrow$ seria e $\boxed{C.}$

$$x=-1 \qquad 1 + \sum_{n=3}^{\infty} \frac{\overset{=1}{(-1)^{2n}}\cdot (2n-5)!!}{(n-1)!\cdot 2^{n-1}} \qquad \text{S.T.P.}$$

D și R sunt ca la $x=1$, deci seria e $\boxed{C}$

$\Rightarrow J = [-1, 1]$.

* $f(x) = \ln(1-x^2)$

$f(0)=0$

$f^{(n)}(0) = (-1)^{n-1}\cdot(n-1)! - (n-1)!\ , n \geq 1.$

$$(T_n f)(x) = \sum_{k=1}^{n}\left((-1)^{k-1}\cdot(k-1)! - (k-1)!\right)\cdot\frac{x^k}{k!}$$

$$= 0 + \frac{x^2}{2!}(-1!-1!) + \frac{x^4}{4!}(-3!-3!) + \dots + \frac{x^n}{n!}\left((-1)^{n-1}\cdot(n-1)! - (n-1)!\right)$$

$$= -\left(\frac{x^2}{2!}\cdot 2\cdot 1! + \frac{x^4}{4!}\cdot 2\cdot 3! + \frac{x^6}{6!}\cdot 2\cdot 5! + \dots + \frac{x^n}{n!}\cdot(n-1)!\left((-1)^{n-1}-1\right)\right)$$

$$= -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \dots + \frac{1}{2}\cdot\frac{x^n}{n!}(n-1)!\left((-1)^{n-1}-1\right)\right)$$

Fie $s_n = -2\sum_{n=1}^{\infty}\frac{x^{2n}}{2n}$

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{2n+2}{2n} = 1$$

$\Rightarrow (-1,1) \subset J$

* $x=1.$

$$s_n = -2\sum_{n=1}^{\infty}\frac{1}{2n} = -\sum_{n=1}^{\infty}\frac{1}{n}\quad \boxed{D}$$

* $x=-1$

$$s_n = -2\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{2n} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

$\boxed{C}$ din criteriul lui Leibniz

şir $\searrow \left(\frac{1}{n}\right)$ cu lim $=0$

$\Rightarrow J = [-1,1)$

* $f(x) = \frac{1}{1-x^2}$

$f^{(m)}(0) = \frac{1}{2}\left[(-1)^m \cdot m! + m!\right] = \frac{1}{2} \cdot m! \left((-1)^m + 1\right)$

$(T_m f)(x) = 1 + \frac{x^2}{2!} \cdot \frac{1}{2} \cdot 2! \cdot 2 + \frac{x^4}{4!} \cdot \frac{1}{2} \cdot 4! \cdot 2 + \frac{x^6}{6!} \cdot \frac{1}{2} \cdot 6! \cdot 2 +$

$\dots + \frac{x^m}{m!} \cdot \frac{1}{2} \cdot m! \cdot \left((-1)^m + 1\right)$

$= 1 + x^2 + x^4 + \dots + \frac{x^m}{2}\left(1 + (-1)^{m+1}\right)$

Fie $s_m = \sum_{m=0}^{\infty} x^{2m}$

$\Rightarrow a_m = 1$

$R = \lim_{m \to \infty} \left| \frac{a_m}{a_{m+1}} \right| = 1$

$(-1, 1) \subset J$

* $x = 1 \Rightarrow s_m = \sum_{m=0}^{\infty} 1^{2m} = \sum_{m=0}^{\infty} 1$, $\boxed{D}$

* $x = -1 \Rightarrow s_m = \sum_{m=0}^{\infty} (-1)^{2m} = \sum_{m=0}^{\infty} 1$ $\boxed{D}$

$\Rightarrow J = (-1, 1)$

③ Determinați mulțimea de convergență a seriilor de puteri:

a) $\sum\limits_{n=0}^{\infty} \frac{2^n}{(n+1)^3} \cdot (x+1)^n$

$x_0=-1$

$a_n=\frac{2^n}{(n+1)^3}$

$r=\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} \frac{2^n}{(n+1)^3} \cdot \frac{(n+2)^3}{2^{n+1}} = \lim\limits_{n\to\infty} \frac{n^3+\dots}{2n^3+\dots} = \frac{1}{2}$.

$\Rightarrow$ Seria e convergentă pe $\left(-1-\frac{1}{2}, -1+\frac{1}{2}\right) = \left(-\frac{3}{2}, -\frac{1}{2}\right)$.

Verificăm în $-\frac{3}{2}$ și $-\frac{1}{2}$.

* $x=-\frac{3}{2}$

$S=\sum\limits_{n=0}^{\infty} \frac{2^n}{(n+1)^3} \cdot \left(-\frac{1}{2}\right)^n = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3}$

Seria e abs. convergentă ($\sum \frac{1}{(n+1)^3}$ e seria armonica gen. cu $p=3>1$) deci convergentă.

* $x=-\frac{1}{2}$

$S=\sum\limits_{n=0}^{\infty} \frac{2^n}{(n+1)^3} \cdot \frac{1}{2^n} = \sum\limits_{n=0}^{\infty} \frac{1}{(n+1)^3} \rightarrow$ convergentă (seria armonică gen. cu $p=3>1$)

$\Rightarrow J=[-3/2, -1/2]$

b) $\sum\limits_{n=0}^{\infty} (\pi/2 - \text{arctg}\, n)\cdot x^n$

$x_0=0$

$a_n=\pi/2-\text{arctg}\, n = \text{arcctg}\, n$

$r=\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty}\left|\frac{\text{arcctg}\, n}{\text{arcctg}(n+1)}\right| = 1.$

$(-1,1) \subset J$

$\boxed{\text{arctg}\, n + \text{arctg}\frac{1}{n} = \pi/2}$

* $x=1$

~~$S=\sum\limits_{n=0}^{\infty} \text{arcctg}\, n$~~ $S=\sum\limits_{n=0}^{\infty}(\pi/2-\text{arctg}\, n) = \sum\limits_{n=0}^{\infty} \text{arctg}\frac{1}{n}$.

$\lim\limits_{n\to\infty} \frac{\text{arctg}\frac{1}{n}}{\frac{1}{n}} = \lim\limits_{x_n\to 0} \frac{\text{arctg}\, x_n}{x_n} = 1$ (limită remarcabilă)

$1\in(0,\infty) \Rightarrow$ seriile $\sum \operatorname{arctg}\frac{1}{n}$ şi $\sum \frac{1}{n}$ au aceeaşi natură $\Rightarrow$

seria e divergentă.

* $x=-1$.

$$S=\sum_{n=0}^{\infty} \operatorname{arctg}\frac{1}{n}\cdot(-1)^n$$

Seria nu e absolut conv. (se obţine cazul precedent).

$$\left.\begin{aligned} &n<n+1 \\ &\frac{1}{n}>\frac{1}{n+1} \\ &\operatorname{arctg} \text{ fc. cresc.} \end{aligned}\right| \Rightarrow \operatorname{arctg}\left(\frac{1}{n}\right)>\operatorname{arctg}\left(\frac{1}{n+1}\right) \Rightarrow$$

şir descrescator

$$\lim_{n\to\infty} \operatorname{arctg}\frac{1}{n} = \operatorname{arctg} 0 = 0$$

$\overset{\text{Leibniz}}{\Rightarrow}$ seria e C.

$J=[-1,1)$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}\cdot x^n$ $\qquad x_0=0$

$$a_n=\frac{n^n}{n!}$$

$$r=\lim_{n\to\infty}\left|\frac{n^n}{\cancel{n!}}\cdot\frac{\overset{n+1}{\cancel{(n+1)!}}}{(n+1)^{n+1}}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$$

$$\left(-\frac{1}{e},\frac{1}{e}\right) \subset J$$

* $x=\frac{1}{e}$: $\displaystyle s_n=\sum_{n=1}^{\infty}\left(\frac{n}{e}\right)^n\cdot\frac{1}{n!}$

$$D=\lim_{n\to\infty} \frac{n^n}{e^n}\cdot\frac{1}{\cancel{n!}}\cdot\frac{\overset{n+1}{\cancel{(n+1)!}}\cdot e^{n+1}}{(n+1)^{n+1}} = \lim_{n\to\infty} e\cdot\left(\frac{n}{n+1}\right)^n = e\cdot\frac{1}{e}=1.$$

$$R=\lim_{n\to\infty} n\left(e\cdot\left(\frac{n}{n+1}\right)^n-1\right)=\lim_{n\to\infty}\frac{e\left(\left(\frac{n}{n+1}\right)^n-\frac{1}{e}\right)}{\frac{1}{n}}=$$

$$=\lim_{n\to\infty}\frac{\left(\frac{n}{n+1}\right)^n-\frac{1}{e}}{\frac{1}{ne}}$$

Considerăm limita

$$\lim_{x\to\infty} \frac{\left(\frac{x}{x+1}\right)^x - \frac{1}{e}}{\frac{1}{x\cdot e}} \underset{L'H}{\overset{\left[\frac{0}{0}\right]}{=}} \lim_{x\to\infty} \frac{\left(\left(\frac{x}{x+1}\right)^x\right)'}{\frac{1}{e}\cdot\frac{-1}{x^2}}$$

$$\overset{(*)}{=} \lim_{x\to\infty} \frac{\overset{\to\frac{1}{e}}{\left(1-\frac{1}{x+1}\right)^x}\cdot\left(\ln\frac{x}{x+1}+\frac{1}{x+1}\right)}{\frac{1}{e}\cdot\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\ln\frac{x}{x+1}+\frac{1}{x+1}}{-\frac{1}{x^2}} \underset{\left[\frac{0}{0}\right]}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}}{\frac{2}{x^3}} =$$

$$= \lim_{x\to\infty} \frac{(x+1)^2 - x(x+1) - x}{x(x+1)^2}\cdot\frac{x^3}{2} =$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+2\cancel{x}+1-\cancel{x^2}-\cancel{x}-\cancel{x}}{x^3+2x^2+x}\cdot\frac{x^3}{\cancel{2}} = \lim_{x\to\infty}\frac{x^3}{2x^3+\dots} = \frac{1}{2}$$

$R < 1 \Rightarrow \boxed{D}$

$$(*)\ \lim_{x\to\infty}\left(\left(\frac{x}{x+1}\right)^x\right)' = \lim_{x\to\infty}\left(\left(1-\frac{1}{x+1}\right)^x\right)' \overset{(**)}{=}$$

$$= \lim_{x\to\infty}\left(\left(1-\frac{1}{x+1}\right)^x\cdot\left(1\cdot\ln\left(1-\frac{1}{x+1}\right) + \frac{x}{1-\frac{1}{x+1}}\cdot\frac{1}{(x+1)^2}\right)\right) =$$

$$= \frac{x}{\frac{x}{x+1}}\cdot\frac{1}{(x+1)^2} = \frac{1}{x+1}$$

$$(**)\ (f^g)' = \left(e^{\ln f^g}\right)' = \left(e^{g\cdot\ln f}\right)' = \underbrace{e^{g\cdot\ln f}}_{= f^g}\cdot\left[g'\cdot\ln f + g\cdot(\ln f)'\right] =$$

$$= f^g\cdot\left[g'\cdot\ln f + \frac{g}{f}\cdot f'\right]$$

* $x = -\frac{1}{e}$

$$s_n = \sum_{n=1}^{\infty} \frac{n^n}{n!} \cdot \frac{(-1)^n}{e^n}$$

Fie $a_n = \frac{n^n}{e^n \cdot n!}$

$$a_{n+1} - a_n = \frac{(n+1)^{n+1}}{e^{n+1} \cdot (n+1)!} - \frac{n^n}{e^n \cdot n!} = \frac{(n+1)^{n+1} - (n+1) \cdot e \cdot n^n}{e^{n+1} (n+1)!} =$$

$$= \frac{(n+1) \cdot n^n \left((n+1)^n \cdot \frac{1}{n^n} - e \right)}{e^{n+1} (n+1)!} =$$

$$= \frac{(n+1) \cdot n^n \left(\left(1+\frac{1}{n}\right)^n - e \right)}{e^{n+1} (n+1)!} < 0,$$ deoarece $\left(1+\frac{1}{n}\right)^n < e$.

(se poate arăta cu inducție)

$\Rightarrow$ șir descrescător.

Cum șirul e descrescător $\Rightarrow \frac{a_n}{a_{n+1}} > 1$.

Cum are termenii strict pozitivi

$\Bigg\}$ Crit. rap. $\Rightarrow$ pt. șiruri

$\lim_{n \to \infty} a_n = 0$

C. Leibniz $\Rightarrow$ seria e $\boxed{C}$

$I = \left[-\frac{1}{e}, \frac{1}{e}\right)$

④ Utilizând operații cu serii de puteri, justificați egalitățile:

a) $\sum_{n=1}^{\infty} n \cdot x^n = \frac{x}{(x-1)^2}$, $\forall x \in (-1,1)$

$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$, $|x|<1$ | derivăm

$n\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{(1-x)^2}$ $\forall x \in (-1,1)$ $|\cdot x$

$\sum_{n=1}^{\infty} n \cdot x^n = \frac{x}{(x-1)^2}$, $\forall x \in (-1,1)$

$(-1,1) \subset J$

$\forall x = 1, x = -1$

seria e divergentă.

$\Rightarrow J = (-1,1)$

b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{1}{2}(e^x + e^{-x})$, $\forall x \in \mathbb{R}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{n!}$

$e^x + e^{-x} = \sum_{n=0}^{\infty} \frac{x^n + (-1)^n \cdot x^n}{n!} \underset{\downarrow}{=} \sum_{n=0}^{\infty} \frac{x^{2n} + x^{2n}}{(2n)!} = 2\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$,

termenii impari se reduc, rămân doar cei pari

$\Rightarrow \frac{1}{2}(e^x + e^{-x}) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $R = \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} = +\infty$ $\Rightarrow J = \mathbb{R}$

c) $x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!! \cdot (2n+1)} \cdot x^{2n+1} = \arcsin x$, $\forall x \in [-1,1]$

Considerăm fc. $\varphi(x) = (1+x)^r$

$\varphi'(x) = r(1+x)^{r-1}$

$\varphi''(x) = r(r-1)(1+x)^{r-2}$

...

$\varphi^{(n)}(x) = r(r-1)\cdot\ldots\cdot(r-(n-1))\cdot(1+x)^{r-n}$

Polinomul Taylor asociat ei va fi, în jurul lui $x_0=0$:

$$(T_n f)(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \cdot f^{(k)}(0)$$

$$= \sum_{k=0}^{n} \frac{x^k}{k!} \cdot r(r-1) \cdot \ldots \cdot (r-k+1)$$

$$(1+x)^{r} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot r(r-1) \cdot \ldots \cdot (r-n+1) \quad \text{(convergentă pe } [-1,1])$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = (1+(-x^2))^{-1/2} = \cancel{\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \cdot r(r-1) \cdot \ldots \cdot (r-n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \cdot (-1/2)(-1/2-1) \cdot \ldots \cdot \left(-\frac{1}{2}-n+1\right) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!} \cdot (-1)^n \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{2n-1}{2} =$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \cdot \frac{(2n-1)!!}{2^n} = \sum_{n=0}^{\infty} \frac{x^{2n} \cdot (2n-1)!!}{\underbrace{1 \cdot 2 \cdot \ldots \cdot n \cdot \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{n \text{ ori}}}} = \sum_{n=0}^{\infty} \frac{x^{2n} (2n-1)!!}{(2n)!!}$$

Integrăm

$$\int_0^t \frac{1}{\sqrt{1-x^2}}\, dx = \int_0^t \sum_{n=0}^{\infty} \frac{x^{2n}(2n-1)!!}{(2n)!!}\, dx$$

$$\arcsin x \Big|_0^t = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{x^{2n+1}}{2n+1} \Big|_0^t$$

$$\arcsin t = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{t^{2n+1}}{2n+1} = t + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{t^{2n}}{2n+1}$$

* $t=1$: $1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2n+1}$ $\quad R = \lim_{n\to\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{2n+3}{2n+1} \cdot \frac{(2n+2)!!}{(2n+1)!!}$

$$R = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1} - 1\right) = \lim_{n\to\infty} \frac{6n^2+5n}{4n^2+\ldots} = \frac{6}{4} > 1 \Rightarrow \boxed{C}$$

* $t=-1$: $-1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{(-1)^{2n}}{2n+1}$, $(-1)^{2n}=1$ $\to$ ca anterior $\Rightarrow \boxed{C}$

$I=[-1,1]$