Seminarul 12

① Pt. fc. $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + 3xy^2 - 15x - 12y$ și pct. $a = (-2,-1)$, precizați:

a) $\nabla f(a)$, $H(f)(a)$, $d^2 f(a)$

* $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) = (3x^2 + 3y^2 - 15,\ 6xy - 12)$

$\nabla f(a) = (0,0)$

* $H(f)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & 6y \\ 6y & 6x \end{pmatrix}$ (= matricea hessiană)

$H(f)(a) = \begin{pmatrix} -12 & -6 \\ -6 & -12 \end{pmatrix}$

* $d^2 f(x,y)(u_1,u_2) = \frac{\partial^2 f}{\partial x^2}(x,y) \cdot u_1^2 + \frac{\partial^2 f}{\partial y^2}(x,y) \cdot u_2^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x,y) u_1 u_2$

$= 6x \cdot u_1^2 + 6x \cdot u_2^2 + 12y \cdot u_1 u_2$ (= diferențiala de ordinul 2)

$d_2 f(a)(u_1,u_2) = -12u_1^2 - 12u_2^2 - 12u_1u_2$

$= -12(u_1^2 + u_2^2 + u_1u_2)$

b) natura pct. a

a [pct. minim / pct. maxim / pct. șa

$\Delta_1 = h_{11}$ $\quad \Delta_2 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}$ ... $\Delta_m = \det(H(f)(a))$

↳ determinanții matricei hessiene

* Dacă $d^2 f(a)$ este
  - pozitiv definită ⇒ a pct. de minim local
  - negativ definită ⇒ a pct. de maxim local
  - indefinită ⇒ a pct. șa

* $d^2 f(a)$
  - poz. def. ⇔ $\Delta_k > 0$, $\forall k = \overline{1,m}$ ($d^2 f(a)(u) > 0$, $\forall u \in \mathbb{R}^m \setminus \{0_m\}$)
  - neg. def. ⇔ $(-1)^k \cdot \Delta_k > 0$, $\forall k = \overline{1,m}$ ($d^2 f(a)(u) < 0$, $\forall u \in \mathbb{R}^m \setminus \{0$
  - indef. dacă $\exists u, v \in \mathbb{R}^m$ a.î. $d^2 f(a)(u) < 0 < d^2 f(a)(v)$

* a e punct critic ($\nabla f(a) = (0,0)$).

$$\left.\begin{array}{l}\Delta_1 = -12 < 0 \\ \Delta_2 = \begin{vmatrix} -12 & -6 \\ -6 & -12 \end{vmatrix} = 144-36 > 0\end{array}\right\} \Rightarrow d^2 f(a) \text{ negativ definită} \Rightarrow$$

a pct. de maxim local.

② Det. punctele critice și punctele de extrem local (specificându-le tipul) pt. următoarele funcții.

a) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = 2x^2 - xy + 2xz - y + y^3 + z^2$.

$\boxed{P_1}$ Căutăm punctele critice. Ele sunt soluțiile ecuației

$$\nabla f(x,y,z) = (0,0,0)$$

$$\Rightarrow (4x-y+2z,\ -x-1+3y^2,\ 2x+2z) = (0,0,0)$$

$$\begin{cases} 4x-y+2z = 0 \\ -x+3y^2 = 1 \Rightarrow x = 3y^2-1 \\ x+z = 0 \Rightarrow -x = z \end{cases}$$

(I ec.) $\Rightarrow$

$$4(3y^2-1) - y - 2(3y^2-1) = 0$$

$$6y^2 - y - 2 = 0$$

$\Delta = 49$

$$y_1 = \frac{1+7}{12} = \frac{8}{12} = \frac{2}{3} \Rightarrow x_1 = 3 \cdot \frac{4}{9} - 1 = \frac{1}{3} \Rightarrow z_1 = -\frac{1}{3}$$

$$y_2 = \frac{1-7}{12} = -\frac{1}{2} \Rightarrow x_2 = -\frac{1}{4},\ z_2 = \frac{1}{4}$$

Deci punctele critice sunt $P_1\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ și $P_2\left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right)$

$\boxed{P_2}$ Natura punctelor critice

$$* \ H(f)(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 6y & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

* Pt. P_1:

$$H(f)\left(\frac{1}{3},\frac{2}{3},-\frac{1}{3}\right)=\begin{pmatrix}4 & -1 & 2\\ -1 & 4 & 0\\ 2 & 0 & 2\end{pmatrix}$$

$\Delta_1 = 4 > 0$

$\Delta_2 = \begin{vmatrix}4 & -1\\ -1 & 4\end{vmatrix} = 16-1 = 15 > 0$

$\Delta_3 = \begin{vmatrix}4 & -1 & 2\\ -1 & 4 & 0\\ 2 & 0 & 2\end{vmatrix} = 14 > 0$

$\Rightarrow d^2f\left(\frac{1}{3},\frac{2}{3},-\frac{1}{3}\right)$ pozitiv definită

$\Rightarrow P_1$ pct. de minim local

* Pt. P_2:

$$H(f)\left(-\frac{1}{4},-\frac{1}{2},\frac{1}{4}\right)=\begin{pmatrix}4 & -1 & 2\\ -1 & -3 & 0\\ 2 & 0 & 2\end{pmatrix}$$

$\Delta_1 = 4 > 0$

$\Delta_2 = \begin{vmatrix}4 & -1\\ -1 & -3\end{vmatrix} = -12-1 = -13 < 0$

$\Delta_3 = \begin{vmatrix}4 & -1 & 2\\ -1 & -3 & 0\\ 2 & 0 & 2\end{vmatrix} = -14 < 0$

$\Rightarrow$ nu putem trage o concluzie cu această metodă, calculăm $d^2f(P_2)$

$$d^2f(x,y,z)(u_1,u_2,u_3)=\frac{\partial^2 f}{\partial x^2}(x,y,z)\cdot u_1^2+\frac{\partial^2 f}{\partial y^2}(\dots)\cdot u_2^2+\frac{\partial^2 f}{\partial z^2}(\dots)\cdot u_3^2+$$

$$+2\frac{\partial^2 f}{\partial x\partial y}(\dots)u_1u_2+2\frac{\partial^2 f}{\partial x\partial z}(\dots)u_1u_3+2\frac{\partial^2 f}{\partial y\partial z}(\dots)u_2u_3$$

$$\Rightarrow d^2f\left(-\frac{1}{4},-\frac{1}{2},\frac{1}{4}\right)(u_1,u_2,u_3)=4u_1^2-3u_2^2+2u_3^2-2u_1u_2+4u_1u_3$$

$d^2f\left(-\frac{1}{4},-\frac{1}{2},\frac{1}{4}\right)(1,0,0)=4>0$

$d^2f\left(-\frac{1}{4},-\frac{1}{2},\frac{1}{4}\right)(0,1,0)=-3<0$

$\Rightarrow d^2f$ indefinită $\Rightarrow$ P_2 punct șa.

b) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^4 + y^4 - 2x^2$

I. $\nabla f(x,y) = (0,0) \Rightarrow (4x^3 - 4x, 4y^3) = (0,0) \Rightarrow$

$$\begin{cases} 4x^3 - 4x = 0 \Rightarrow 4x(x^2-1) = 0 \Rightarrow x_1 = 0,\ x_2 = 1,\ x_3 = -1 \\ 4y^3 = 0 \Rightarrow y = 0 \end{cases}$$

Pct. critice sunt: $P_1(0,0)$, $P_2(1,0)$, $P_3(-1,0)$

II. $H(f)(x,y) = \begin{pmatrix} 12x^2-4 & 0 \\ 0 & 12y^2 \end{pmatrix}$

* $P_1(0,0)$

$H(f)(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}$

$\Delta_1 = -4 < 0$

$\Delta_2 = \begin{vmatrix} -4 & 0 \\ 0 & 0 \end{vmatrix} = 0$ $\Big\}$ $\to$ nu putem stabili astfel.

$d^2 f(0,0)(u_1,u_2) = -4u_1^2 \le 0$ (Ex: pt $(0,1)$, $(0,2)$ e $= 0$, iar $(0,1)$, $(0,2) \neq 0_2$)

$\Rightarrow$ nu e neg. definită $\Rightarrow$ nu putem stabili astfel

$f(x,0) = x^4 - 2x^2 = x^2(x^2-2) =$
$= x^2(x-\sqrt{2})(x+\sqrt{2}) < 0$ pt. $x \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\overset{\shortparallel}{f(0,0)}$ $\Rightarrow$ $(0,0)$ nu poate fi pct. de minim local

$f(0,y) = y^4 > 0$, $\forall y \neq 0$

$\quad\quad\quad\quad\quad\quad\quad$ $f(0,0)$ $\Rightarrow$ $(0,0)$ nu poate fi nici de max. local

$\Rightarrow (0,0)$ pct. șa (avem valori diferite în vecinătatea sa)

* $P_2(1,0)$

$H(f)(1,0) = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad$ $\Delta_1 = 8 > 0$, $\Delta_2 = 0$ $\Rightarrow$ nu putem stabili astfel

$d^2 f(1,0)(u_1,u_2) = 8u_1^2 \ge 0 \Rightarrow$ nu e poz. def. $\Rightarrow$ nu putem stabili astfel.

$f(1,0) = -1$

$f(x,y) = x^4 + y^4 - 2x^2 = x^4 - 2x^2 + 1 + y^4 - 1 = \underbrace{(x^2-1)^2}_{\geq 0} + \underbrace{y^4}_{\geq 0} - 1 \geq -1$
$\forall (x,y) \in \mathbb{R}^2$

$\Rightarrow (1,0)$ pct. de minim.

Analog și $(-1,0)$ pct. de minim. (se obțin aceleași valori)

③ Determinați punctele de extrem condiționat (specificând tipul lor) și valorile extreme ale fc. f relativ la mulțimea S indicată (știind că e compactă):

$f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = (1-x)(1-y), \quad S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

$\left.\begin{array}{l} f \text{ cont.} \\ S \text{ compactă} \end{array}\right\} \overset{T.W}{\Rightarrow}$ f mărginită și își atinge marginile

$\Rightarrow f$ are minim și maxim condiționat relativ la S.

Vom folosi Metoda Multiplicatorilor lui Lagrange.

1) Fie $F(x,y) = x^2 + y^2 - 1$

Atunci $S = \{(x,y) \in \mathbb{R}^2 \mid F(x,y) = 0\}$.

2) Considerăm fc. $L(x,y,\lambda) = f(x,y) + \lambda \cdot F(x,y)$

$L(x,y) = f(x,y) + \lambda F(x,y) = (1-x)(1-y) + \lambda(x^2 + y^2 - 1)$

3) Rezolvăm sistemul $\nabla L(x,y,\lambda) = (0,0,0)$ pt. a determina punctele critice.

$\nabla L(x,y,\lambda) = (-1 + y + 2\lambda x, -1 + x + 2\lambda y, x^2 + y^2 - 1) = (0,0,0)$

$$\Rightarrow \begin{cases} y-1+2\lambda x=0 \\ x-1+2\lambda y=0 \\ x^2+y^2-1=0 \end{cases} \xrightarrow{(2)-(1)} \Rightarrow x-y+2\lambda y-2\lambda x=0$$

$$(x-y)-2\lambda(x-y)=0$$

$$(x-y)(1-2\lambda)=0$$

$$\Rightarrow x=y \quad \text{sau} \quad \lambda=\frac{1}{2}.$$

1) $x=y$

$\overset{(3)}{\Rightarrow} 2x^2=1 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\pm\frac{\sqrt{2}}{2} \Rightarrow y=\pm\frac{\sqrt{2}}{2}$

$\overset{(1)}{\Rightarrow} 2\lambda x=1-y$

$$\lambda=\frac{1-y}{2x} \Rightarrow \begin{cases} \lambda=\dfrac{1-\frac{\sqrt{2}}{2}}{\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}}=\dfrac{2-\sqrt{2}}{2\sqrt{2}}=\dfrac{\sqrt{2}(\sqrt{2}-1)}{2\sqrt{2}}=\dfrac{\sqrt{2}-1}{2} \\ \lambda=\dfrac{1+\frac{\sqrt{2}}{2}}{-\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}}=-\dfrac{2+\sqrt{2}}{2\sqrt{2}}=-\dfrac{1+\sqrt{2}}{2} \end{cases}$$

* valorile lui λ nu contează pt. determinarea min f, max f, doar în rezolvarea sistemului

2) $\lambda=\frac{1}{2} \Rightarrow \begin{cases} y-1+x=0 \\ x-1+y=0 \\ x^2+y^2-1=0 \end{cases}$

$$\left.\begin{array}{l} x+y=1 \\ x^2+y^2=1 \end{array}\right| \Rightarrow (x+y)^2-2xy=x^2+y^2=1 \Rightarrow$$

$$1-2xy=1 \Rightarrow 2xy=0 \Rightarrow xy=0$$

$\Rightarrow x=0 \Rightarrow y=1$

sau $y=0 \Rightarrow x=1$

Deci pct. critice sunt: $P_1\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}-1}{2}\right)$, $P_2\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{1+\sqrt{2}}{2}\right)$

$P_3\left(0, 1, \frac{1}{2}\right)$, $P_4\left(1, 0, \frac{1}{2}\right)$

$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)= \left(1-\frac{\sqrt{2}}{2}\right)\left(1-\frac{\sqrt{2}}{2}\right)=1-\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}+\frac{1}{2}=\frac{3}{2}-\sqrt{2}$

$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)=\left(1+\frac{\sqrt{2}}{2}\right)^2=1+\sqrt{2}+\frac{1}{2}=\frac{3}{2}+\sqrt{2}=\max f|_S$

$f(0,1)=0=\min f|_S$

$f(1,0)=0=\min f|_S$

$\Rightarrow (1,0), (0,1)$ pct. min. condiționat; $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ max cond.

4. Determinați valorile extreme ale următoarelor funcții relative la mulțimea S indicată:

a) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x+2y+3z$, $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 1\}$

f cont., S compactă $\overset{T.W}{\Rightarrow}$ f mărg. și își atinge marginile (pct. de min. și max. condiționat, relativ la S)

S închisă $\Rightarrow$ $\boxed{S = \operatorname{int} S \cup \operatorname{Fr} S}$

1) în $\operatorname{int} S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 < 1\}$

$\nabla f(x,y,z) = (0,0,0) \Leftrightarrow (1,2,3) = (0,0,0)$, imposibil $\Rightarrow$

$\nexists$ pct. critice $\Rightarrow \nexists$ min, max în $\operatorname{int} S$.

2) pe $\operatorname{Fr} S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

Folosim metoda multipl. lui Lagrange.

Fie $F(x,y,z) = x^2+y^2+z^2-1$.

$\Rightarrow \operatorname{Fr} S = \{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = 0\}$

Fie $L(x,y,z,\lambda) = f(x,y,z) + \lambda \cdot F(x,y,z) =$
$= x+2y+3z+\lambda(x^2+y^2+z^2-1)$

$\nabla L(x,y,z,\lambda) = (0,0,0,0) \Rightarrow \begin{cases} 1+2\lambda x = 0 \\ 2+2\lambda y = 0 \\ 3+2\lambda z = 0 \\ x^2+y^2+z^2-1 = 0 \end{cases}$ $\Rightarrow x = -\frac{1}{2\lambda}$, $y = -\frac{2}{2\lambda}$, $z = -\frac{3}{2\lambda}$ $\quad \lambda \ne 0$

$\overset{(4)}{\Rightarrow} \frac{1+4+9}{4\lambda^2} - 1 = 0 \Rightarrow \frac{14}{4\lambda^2} = 1 \Rightarrow 4\lambda^2 = 14 \Rightarrow$

$\lambda^2 = \frac{7}{2} \Rightarrow \lambda = \pm\frac{\sqrt{7}}{\sqrt{2}} = \pm\frac{\sqrt{14}}{2}$

$\Rightarrow 2\lambda = \pm\sqrt{14}$

$\Rightarrow x = -\frac{1}{\sqrt{14}},\ y = -\frac{2}{\sqrt{14}},\ z = -\frac{3}{\sqrt{14}} \Rightarrow P_1\left(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{\sqrt{14}}{2}\right)$

$x = \frac{1}{\sqrt{14}},\ y = \frac{2}{\sqrt{14}},\ z = \frac{3}{\sqrt{14}} \Rightarrow P_2\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{\sqrt{14}}{2}\right)$

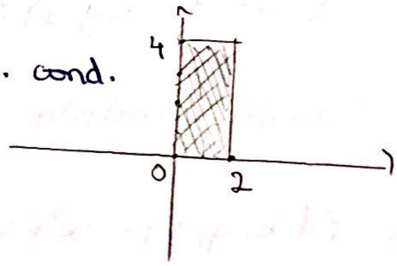
sunt pct. critice ale lui L.

$$f\left(-\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right) = \frac{-1-4-9}{\sqrt{14}} = -\sqrt{14} = \min f\big|_S$$

$$f\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) = \frac{1+4+9}{\sqrt{14}} = \sqrt{14} = \max f\big|_S$$

Temă

b) $f: \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = x^2 - 2xy + 2y,\quad S = [0,2] \times [0,4]$



f cont., S compactă $\overset{T.W}{\Longrightarrow}$ f are min. și max. cond.

S închisă $\Rightarrow S = \operatorname{int} S \cup \operatorname{Fr} S$.

$\operatorname{int} S = (0,2) \times (0,4)$

$\operatorname{Fr} S = (\{0\} \times [0,4]) \cup (\{2\} \times [0,4]) \cup ([0,2] \times \{0\}) \cup ([0,2] \times \{4\})$

1) int S:

$$\nabla f(x,y) = (0,0) \Rightarrow \begin{cases} 2x - 2y = 0 \Rightarrow x = y \\ 2 - 2x = 0 \Rightarrow x = 1 \end{cases} \Rightarrow P(1,1)$$

$H(f)(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}$ $\quad H(f)(1,1) = \begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}$ $\quad \Delta_1 = 2 > 0$, $\Delta_2 = -4 < 0$

$d^2 f(1,1)(u_1,u_2) = 2u_1^2 - 4u_1u_2$

$\left.\begin{array}{l} d^2 f(1,1)(1,0) = 2 > 0 \\ d^2 f(1,1)(1,2) = 2 - 8 < 0 \end{array}\right\} \Rightarrow d^2 f$ indef. $\Rightarrow (1,1)$ pct. șa

2) $f|_S$:

* $\{0\} \times [0,4] \Rightarrow x=0$

$$\min_{y\in[0,4]} f(0,y) = \min_{y\in[0,4]} 2y = 0 \quad (y=0)$$

$$\max_{y\in[0,4]} f(0,y) = \max_{y\in[0,4]} 2y = \underset{\sim}{8} \quad (y=4)$$

* $\{2\} \times [0,4] \Rightarrow x=2$

$$\min_{y\in[0,4]} f(2,y) = \min_{y\in[0,4]} 4-2y = \underline{-4} \quad (y=4)$$

$$\max_{y\in[0,4]} f(2,y) = 4 \quad (y=0)$$

* $[0,2] \times \{0\} \Rightarrow y=0$

$$\min_{x\in[0,2]} f(x,0) = \min_{x\in[0,2]} x^2 = 0 \quad (x=0)$$

$$\max_{x\in[0,2]} f(x,0) = 4 \quad (x=2)$$

* $[0,2] \times \{4\} \Rightarrow y=4$

$$\min_{x\in[0,2]} f(x,4) = \min_{x\in[0,2]} x^2 - 8x + 8 = \underline{-4} \quad (x=2)$$

$$\max_{x\in[0,2]} f(x,4) = \underset{\sim}{8} \quad (x=0)$$

$$\min f|_S = f(2,-4) = -4$$

$$\max f|_S = f(0,4) = 8$$

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