Seminar 6

① Calc. derivata de ordinul $n \in \mathbb{N}$ a fc. de mai jos și precizați mulțimea pe care aceste fc. sunt indefinit derivabile.

a) $f(x) = \sin x$

$f'(x) = \cos x \quad f''(x) = -\sin x \quad f'''(x) = -\cos x \quad f^{(4)}(x) = \sin x = f(x)$

$$f^{(n)}(x) = \begin{cases} \sin x, & n = 4k \\ \cos x, & n = 4k+1 \\ -\sin x, & n = 4k+2 \\ -\cos x, & n = 4k+3 \end{cases}, k \in \mathbb{N}. \quad \text{sau} \quad f^{(n)}(x) = \begin{cases} (-1)^k \cdot \sin x, & n = 2k \\ (-1)^k \cdot \cos x, & n = 2k+1 \end{cases}$$

f indefinit derivabilă pe $\mathbb{R}$.

! Relațiile obținute pt. derivata de ordin n se dem. prin inducție. !

b) $f(x) = \ln(x+1) \qquad x+1>0 \Rightarrow x \in (-1, \infty)$

$f'(x) = \dfrac{1}{x+1} = (x+1)^{-1}$

$f''(x) = -1 \cdot (x+1)^{-2}$

$f'''(x) = (-1) \cdot (-2) \cdot (x+1)^{-3}$

$f^{IV}(x) = (-1) \cdot (-2) \cdot (-3) \cdot (x+1)^{-4}$

$f^{(n)}(x) = (-1)^{n+1} \cdot (n-1)! \cdot (x+1)^{-n}$ → inducție pt. dem.

f indef. deriv. pe $(-1, \infty)$, $\forall n \geq 1$.

c) $f(x) = (x^2 - x) \cdot e^x$

f indef. der. pe $\mathbb{R}$.

Vom determina derivata de ordinul n folosind Formula lui Leibniz:

$$(u \cdot v)^{(n)} = \sum_{k=0}^{n} C_n^k \cdot u^{(k)} \cdot v^{(n-k)}$$

unde $u = u(x)$, $v = v(x)$, iar $u^{(k)}$ = derivata de ordin k a fc. u.

Luăm u fc. care devine după câtiva pași, prin derivare, 0.
În cazul nostru $u(x) = x^2 - x$

$v(x) = e^x$ (are aceeași derivată pt. orice ordin)

$$\left((x^2-x)\cdot e^x\right)^{(n)} = C_n^0 \cdot (x^2-x)\cdot (e^x)^{(n)} + C_n^1 (x^2-x)' \cdot (e^x)^{(n-1)} +$$

$$+ C_n^2 (x^2-x)'' \cdot (e^x)^{(n-2)} + C_n^3 (x^2-x)''' \cdot (e^x)^{(n-3)} + \dots$$

$$= (x^2-x)\cdot e^x + \frac{n!}{1!\cdot(n-1)!}\cdot(2x-1)\cdot e^x + \frac{n!}{2!\cdot(n-2)!}\cdot 2\cdot e^x +$$

$$+ \frac{n!}{3!(n-3)!}\cdot 0 \cdot e^x + \underbrace{---}$$

de aici toți termenii vor fi 0, pt. că $(x^2-2x)^{(k)} = 0, \ \forall k \geq 3$

$$= (x^2-x)\cdot e^x + n(2x-1)e^x + n(n-1)e^x$$

$$= e^x\left(x^2 + (2n-1)x + n^2 - 2n\right) \quad \forall n \in \mathbb{N},$$

$n=0$: $f^{(0)}(x) = e^x(x^2 + (-1)\cdot x) = e^x(x^2 - x) = f(x)$ ✓

$n=1$: $f'(x) = e^x(x^2 + 1\cdot x + 1^2 - 2\cdot 1) = e^x(x^2 + x - 1)$

$[f(x)]' = (x^2-x)'\cdot e^x + (x^2-x)\cdot(e^x)' = e^x(2x-1+x^2-x)$

$= e^x(x^2+x-1)$ ✓

d) $f(x) = \sqrt{1-x}$, $x \leq 1$

$f(x) = (1-x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(1-x)^{-1/2}\cdot(-1)$

$f''(x) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot(1-x)^{-3/2}\cdot(-1)^2 = \frac{1}{2^2}\cdot(-1)^3\cdot(1-x)^{-3/2}$

$f'''(x) = \frac{1}{2^2}\cdot(-1)^3\cdot\left(-\frac{3}{2}\right)\cdot(1-x)^{-5/2}\cdot(-1) = \frac{1\cdot 3}{2^3}\cdot(-1)^5\cdot(1-x)^{-5/2}$

$f^{IV}(x) = \frac{1\cdot 3}{2^3}\cdot(-1)^5\cdot\left(-\frac{5}{2}\right)\cdot(1-x)^{-7/2}\cdot(-1) = \frac{1\cdot 3\cdot 5}{2^4}\cdot(-1)^7\cdot(1-x)^{-7/2}$

$f^{(n)}(x) = \frac{1\cdot 3\cdot 5\cdot \dots \cdot(2n-3)}{2^n}\cdot(-1)^{2n-3}\cdot(1-x)^{-\frac{2n-1}{2}}$

$$f^{(n)}(x) = \frac{(2n-3)!!}{2^n} \cdot \underbrace{(1-x)^{-\frac{2n-1}{2}}}_{}, n \geq 2$$

inducție.. f indef. der. pe $(-\infty, 1)$

② Pt. fc. de la ex. anterior, pct. $x_0 = 0$ și nr. $n \in \mathbb{N}$ det.

a) Polinomul lui Taylor de grad n asociat funcției f în pct. x_0.

Pol. lui Taylor: $(T_n f)(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} \cdot f^{(k)}(x_0)$

a) $f(x) = \sin x$

$$f^{(n)}(x) = \begin{cases} (-1)^k \cdot \sin x, & n = 2k \\ (-1)^k \cdot \cos x, & n = 2k+1 \end{cases} \rightarrow f^{(n)}(0) = \begin{cases} 0, & n = 2k \\ (-1)^k, & n = 2k+1 \end{cases}$$

$$(T_n f)(x) = \frac{(x-0)^0}{0!} \cdot \underbrace{f(0)}_{=0} + \frac{(x-0)^1}{1!} \cdot f'(0) + \frac{(x-0)^2}{2!} \cdot \underbrace{f''(0)}_{=0} +$$

$$\frac{(x-0)^3}{3!} \cdot f'''(0) + \frac{(x-0)^4}{4!} \cdot \underbrace{f^{(4)}(0)}_{=0} + \frac{(x-0)^5}{5!} \cdot f^{(5)}(0) + \dots + \frac{(x-0)^n}{n!} \cdot f^{(n)}(0)$$

$$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots + f^{(n)}(0) \cdot \frac{x^n}{n!}$$

b) $f(x) = \ln(x+1)$

$$f^{(n)}(x) = (-1)^{n+1} \cdot (n-1)! \cdot (x+1)^{-n}, n \geq 1$$

$$f^{(n)}(0) = (-1)^{n+1} \cdot (n-1)! \cdot \underbrace{1^{-n}}_{=1}, n \geq 1$$

$$(T_n f)(x) = \frac{x^0}{0!} \cdot \ln 1 + \frac{x}{1!} \cdot (-1)^2 \cdot 0! + \frac{x^2}{2!} \cdot (-1)^3 \cdot 1! + \frac{x^3}{3!} \cdot (-1)^4 \cdot 2! +$$

$$+ \dots + (-1)^{n+1} \cdot (n-1)! \cdot \frac{x^n}{n!} =$$

$$= \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n+1} \cdot \frac{x^n}{n}$$

c) $f(x) = (x^2 - x) \cdot e^x$,

$f^{(m)}(x) = e^x\left(x^2 + (2m-1)x + m^2 - 2m\right)$, $m \in \mathbb{N}$

$f^{(m)}(0) = e^0\,(0 + 0 + m^2 - 2m) = m^2 - 2m$

$$(T_m f)(x) = \frac{x^0}{0!} \cdot \underbrace{f(0)}_{=0} + \frac{x^1}{1!} \cdot \underbrace{f'(0)}_{=-1} + \frac{x^2}{2!} \cdot \underbrace{f''(0)}_{=0} + \frac{x^3}{3!} \cdot \underbrace{f'''(0)}_{3} + \dots + \frac{x^m}{m!} \cdot f^{(m)}(0)$$

$$= -x + \frac{x^3}{2} + \frac{x^4}{4!} \cdot 8 + \dots + \frac{x^m}{m!} \cdot (m^2 - 2m)$$

$$= -x + \frac{x^3}{2} + \frac{x^4}{3!} \cdot 2 + \dots + \frac{x^m}{(m-1)!} \cdot (m-2)$$

d) $f(x) = \sqrt{1-x}$ $\qquad f'(0) = -\frac{1}{2}$

$f^{(m)}(x) = -\frac{(2m-3)!!}{2^m} \cdot (-1-x)^{-\frac{2m-1}{2}}$, $m \geq 2$ $\qquad f(0) = 1$

$$f^{(m)}(0) = -\frac{(2m-3)!!}{2^m} \cdot 1$$

$$(T_m f)(x) = \frac{x^0}{0!} \cdot \underbrace{f(0)}_{1} + \frac{x^1}{1!} \cdot f'(0) + \frac{x^2}{2!} \cdot f''(0) + \frac{x^3}{3!} \cdot f'''(0) + \dots + \frac{x^m}{m!} \cdot f^{(m)}(0)$$

$$= 1 - \frac{x}{2} - \frac{x^2}{2!} \cdot \frac{1}{2^2} - \frac{x^3}{3!} \cdot \frac{1 \cdot 3}{2^3} - \frac{x^4}{4!} \cdot \frac{1 \cdot 3 \cdot 5}{2^4} - \dots - \frac{x^m}{m!} \cdot \frac{(2m-3)!!}{2^m}$$

b) Mulțimea de convergență a seriei Taylor corespunzătoare

a) $f(x) = \sin x$

$$T_m f(x) = \sum_{k=1}^{m} (-1)^{k+1} \cdot \frac{1}{(2k-1)!} \cdot x^{2k-1} = \sum_{k=0}^{m} (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!}$$

Considerăm s.t.p $\sum \left|(-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!}\right| = \sum_{k \geq 0} \frac{|x|^{2k+1}}{(2k+1)!}$ pt. seria $\sum_{k \geq 0} \frac{x^{2k+1}}{(2k+1)!} \cdot (-1)^k$

$$D = \lim_{m \to \infty} \frac{|x|^{2m+1}}{(2m+1)!} \cdot \frac{(2m+3)!}{|x|^{2m+3}} = \lim_{m \to \infty} \frac{(2m+3) \cdot (2m+2)}{|x|^2} = \infty > 1$$

Deci seria e abs. conv. $\Rightarrow$ seria conv. $\forall x \in \mathbb{R}$, deci seria Taylor coresp. fc. $\sin x$ are mulț. de conv. $\mathbb{R}$.

$$\sin x = \sum_{m=0}^{\infty} (-1)^m \cdot \frac{x^{2m+1}}{(2m+1)!} , \ \forall x \in \mathbb{R}.$$

b) $f(x) = \ln(x+1)$

$$(T_n f)(x) = \sum_{k=1}^{n} (-1)^{k+1} \cdot \frac{x^k}{k}$$

(M1) Seria Taylor $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$ e serie de puteri. Îi studiem convergența. Considerăm s.t.p $\sum\limits_{n=1}^{\infty} \left| (-1)^{n+1} \cdot \frac{x^n}{n} \right| = \sum\limits_{n=1}^{\infty} \frac{|x|^n}{n}$

$$D = \lim_{n \to \infty} \frac{|x|^n}{n} \cdot \frac{n+1}{|x|^{n+1}} = \lim_{n \to \infty} \frac{n+1}{|x| \cdot n}$$

* $D > 1 \Leftrightarrow |x| < 1 \Leftrightarrow x \in (-1, 1)$

$D = 1 \Leftrightarrow |x| = 1.$

1) $x = 1$

Seria e: $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$ care e seria armonică alternant

și e convergentă (conform crit. Leibniz)

2) $x = -1$

Seria e: $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(-1)^n}{n} = \sum \underbrace{(-1)^{2n+1}}_{=-1} \cdot \frac{1}{n} = -\sum \frac{1}{n}$

care e divergentă, deoarece e seria armonică.

Sau, folosind Caracterizarea mulț. de convergență a unei serii de puteri.

* Dacă $I \subseteq \mathbb{R}$ notează mulț. de convergență a unei serii de puteri centrată în x_0, atunci $\exists!\, r \in [0, \infty]$ a.î. seria e abs. conv. $\forall x \in (x_0 - r, x_0 + r)$ și seria e divergentă $\forall$ $x \in (-\infty, x_0 - r) \cup (x_0 + r, \infty)$.

Teorema nu precizează natura seriei în $x = x_0 - r$ și ~~x = x~~ $x = x_0 + r$.

Calculul razei de conv.: $\sum\limits_{n=0}^{\infty} a_n \cdot (x - x_0)^n$ serie de puteri cu raza de conv. $r \in [0, \infty]$. Atunci $r = \lim\limits_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$.

(M2)

Pt. $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{x^n}{n}$ avem $a_n=(-1)^{n+1}\cdot\frac{1}{n}$

$$r=\lim_{n\to\infty}\left|(-1)^{n+1}\cdot\frac{1}{n}\cdot\frac{n+1}{1}\cdot(-1)^{n+2}\right|=\frac{n+1}{n}=1.$$

$\Rightarrow$ Serie conv. pe $(-1,1)$.

* Cazurile $x=1$, $x=-1$ se tratează analog.

Deci mulț. de conv. e $J=(-1,1]$.

c) $f(x)=(x^2-x)\cdot e^x$

$-x+\cancel{\sum_{n=3}^{\infty} x^{2n+1}}$ $\qquad \underbrace{-x+\sum_{n=3}^{\infty}\frac{x^n}{(n-1)!}\cdot(n-2)}_{\text{mulț. de conv ?}}$ $\qquad a_n=\frac{n-2}{(n-1)!}$

$$r=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{n-2}{(n-1)!}\cdot\frac{n!^{\,n}}{n-1}=\lim_{n\to\infty}\frac{n^2-2n}{n-1}=\infty$$

$\Rightarrow$ $J=\mathbb{R}$.

d) $f(x)=\sqrt{1-x}$

$1-\frac{x}{2}-\underbrace{\sum_{n=2}^{\infty}\frac{x^n}{n!}\cdot\frac{(2n-3)!!}{2^n}}_{\text{mulț. de conv.}}$ $\qquad a_n=\frac{(2n-3)!!}{n!\cdot 2^n}$

$$r=\lim_{n\to\infty}\frac{(2n-3)!!}{n!\cdot 2^n}\cdot\frac{(n+1)!\cdot 2^{n+1}}{(2n-1)!!}=\frac{2n+2}{2n-1}=1$$

$\Rightarrow$ $J=(-1,1)\subset[-1,1]$

* $x=+1$ $\qquad \frac{1}{2}-\sum_{n=2}^{\infty}\frac{(2n-3)!!}{n!\cdot 2^n}$

$D=\lim_{n\to\infty}\frac{2n+2}{2n-1}=1$

$$R=\lim_{n\to\infty}n\left(\frac{2n+2}{2n-1}-1\right)=\lim_{n\to\infty}\left(\frac{2n+2-2n+1}{2n-1}\right)\cdot n=\lim_{n\to\infty}\frac{3n}{2n-1}=\frac{3}{2}>1$$

$\Rightarrow$ serie convergentă.

* $x=-1$ $\quad \frac{3}{2}-\sum_{n=2}^{\infty}(-1)^n\cdot\frac{(2n-3)!!}{2^n\cdot n!}$

Seria e abs. conv. (se obține ca la $x=1$), deci conv.

$\Rightarrow I=[-1,1]$.

③ Utilizând operații cu serii de puteri, justificați egalitățile:

a) $\sum_{n=0}^{\infty}(-1)^n\cdot(n+1)\cdot x^n=\frac{1}{(1+x)^2}$, $\forall x\in(-1,1)$

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}, \quad \forall x\in(-1,1)$$

$$\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n\cdot x^n=\frac{1}{1+x}, \quad \forall x\in(-1,1) \qquad \Big|\text{ derivăm}$$

$$\sum_{n=0}^{\infty}(-1)^n\cdot n\cdot x^{n-1}=\frac{-1}{(1+x)^2} \qquad \Big|\cdot(-1)$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot n\cdot x^{n-1}=\frac{1}{(1+x)^2}$$

Luăm $n=n-1$

$$\sum_{n=0}^{\infty}(-1)^n\cdot(n+1)\cdot x^n=\frac{1}{(1+x)^2}$$

* Avem aceeași rază de convergență. $r=1$.

* $x=\pm1$ va fi divergentă. $\sum_{n=0}^{\infty}(-1)^n\cdot\underline{\underline{(n+1)}}\cdot(\pm1)^n$

b) $1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot x^n=\frac{1}{\sqrt{1-x}}$, $\forall x\in[-1,1)$

$$\sum_{n=2}^{\infty}x^n\cdot\frac{(2n-3)!!}{n!\cdot 2^n}=1-\frac{x}{2}-\sqrt{1-x} \qquad \forall x\in[-1,1] \qquad \Big|\text{ derivăm}$$

$$\sum_{n=2}^{\infty}n\cdot x^{n-1}\cdot\frac{(2n-3)!!}{n!\cdot 2^n}=-\frac{1}{2}-\frac{-1}{2\sqrt{1-x}} \qquad \Big|\cdot 2$$

$$\sum_{n=2}^{\infty} \frac{(2n-3)!!}{(n-1)!\cdot 2^{n-1}} \cdot x^{n-1} = -1 + \frac{1}{\sqrt{1-x}} \quad |+1$$

$$1+\sum_{n=2}^{\infty} \frac{(2n-3)!!}{(n-1)!\cdot 2^{n-1}} \cdot x^{n-1} = \frac{1}{\sqrt{1-x}}$$

$$(n-1)!\cdot 2^{n-1} = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot$$
$$2\cdot 2\cdot 2\cdot \ldots \cdot 2$$

$$= 2\cdot 1 \cdot 2\cdot 2 \cdot 2\cdot 3 \cdot \ldots \cdot 2(n-1) = (2n-2)!!$$

$$1+\sum_{n=2}^{\infty} \frac{(2n-3)!!}{(2n-2)!!} \cdot x^{n-1} = \frac{1}{\sqrt{1-x}}$$

$n = n-1$

$$1+\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot x^{n} = \frac{1}{\sqrt{1-x}}$$

$(-1,1) \subset J$

* $\boxed{x=1}$ $1+\sum \frac{(2n-1)!!}{(2n)!!}$

$$D = \lim_{n\to\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{(2n+2)!!}{(2n+1)!!} = \lim_{n\to\infty} \frac{2n+2}{2n+1} = 1$$

$$R = \lim_{n\to\infty} \left(\frac{2n+2}{2n+1} - 1\right)\cdot n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1 \Rightarrow \text{divergentă}$$

* $\boxed{x=-1}$:

$$\frac{a_n}{a_{n+1}} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{(2n+2)!!}{(2n+1)!!} = \frac{2n+2}{2n+1} > 1 \Rightarrow \text{șir descrescător}$$

$\lim = 0$

$\overset{\text{Leibniz}}{\Rightarrow}$ serie convergentă

$J = [-1,1)$

4. Determinați mulțimea de convergență a seriei de puteri $\sum_{n=1}^{\infty} \frac{1}{n^2}(x-1)^n$

$a_n = \frac{1}{n^2}$ $\quad x_0 = 1.$

$$r = \lim_{n\to\infty} \left| \frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}} \right| = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} = 1$$

$\Rightarrow (1-1, 1+1) \subset J \quad \Leftrightarrow \quad (0,2) \subset J$

* $x = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^2} \cdot (-1)^n$

Serie absolut convergentă deoarece $\left(\sum \left| \frac{1}{n^2} \cdot (-1)^n \right| = \sum \frac{1}{n^2} \right)$ convergentă: serie armonică generalizată cu $p = 2 > 1$.

* $x = 2$

$\sum_{n=1}^{\infty} \frac{1}{n^2} \cdot 1^n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ C (serie armonică gen. $p = 2 > 1$)

$J = [0, 2]$