Seminarul 10

① Studiați existența limitelor de funcții:

a) $l = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{1+xy}-1}$

$$l = \lim_{\substack{u\to 0}} \frac{u}{\sqrt{u+1}-1} = \lim_{u\to 0} \frac{u(\sqrt{u+1}+1)}{u+1-1} = \lim_{u\to 0} \frac{\cancel{u}(\sqrt{u+1}+1)}{\cancel{u}} = 2$$

$x\cdot y = u$

$\Rightarrow$ limita există.

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

Considerăm 2 șiruri a^n și b^n, $a^n, b^n \subseteq \mathbb{R}^2$ și arătăm că $\lim\limits_{n\to\infty} a^n = (0,0) = \lim\limits_{n\to\infty} b^n$, dar $\lim\limits_{n\to\infty} f(a^n) \neq \lim\limits_{n\to\infty} f(b^n)$,

unde $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$.

Fie $a^n = \left(\frac{1}{n}, 0\right) \to (0,0)$

$b^n = \left(0, \frac{1}{n}\right) \to (0,0)$ $\quad n\to\infty$

$$\lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} \frac{\frac{1}{n^2}-0}{\frac{1}{n^2}+0} = 1$$

$$\lim_{n\to\infty} f(b^n) = \lim_{n\to\infty} \frac{0-\frac{1}{n^2}}{0+\frac{1}{n^2}} = -1$$

$\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$.

c) $\lim\limits_{(x,y)\to(\infty,\infty)} \dfrac{x^2+y^2}{x^4+y^4}$

$\lim\limits_{x\to x^0} f(x) = l \Leftrightarrow \lim\limits_{x\to x^0} |f(x)-l| = 0$, $x, x^0 \in \mathbb{R}^m$, $l \in \overline{\mathbb{R}}$.

Vom arăta că $l=0$.

$$\left|\frac{x^2+y^2}{x^4+y^4} - 0\right| = \left|\frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4}\right| \le \left|\frac{x^2}{x^4+y^4}\right| + \underbrace{\left|\frac{y^2}{x^4+y^4}\right|}_{\text{pozitive}} = \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4},$$

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$$\le \frac{x^2}{x^4} + \frac{y^2}{y^4} = \frac{1}{x^2} + \frac{1}{y^2} \xrightarrow[(x,y)\to(\infty,\infty)]{} 0 \quad \Rightarrow \lim_{(x,y)\to(\infty,\infty)} \frac{x^2+y^2}{x^4+y^4} = 0.$$

d) $\lim\limits_{(x,y)\to(0,0)} \frac{x \cdot \sin(x^2-y^2)}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2} \cdot \frac{\sin(x^2-y^2)}{x^2-y^2} \cdot (x^2-y^2) \overset{t=x^2-y^2}{=}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x^2-y^2)}{x^2+y^2} \cdot \underbrace{\lim_{t\to 0} \frac{\sin t}{t}}_{1}$$

Calculăm $\lim\limits_{(x,y)\to(0,0)} \frac{x(x^2-y^2)}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2}$

$$\left| \frac{x^3-xy^2}{x^2+y^2} - 0 \right| = \frac{|x^3+(-xy^2)|}{x^2+y^2} \le \frac{|x^3|}{x^2+y^2} + \frac{|-xy^2|}{x^2+y^2} =$$

$$= \frac{|x|\cdot x^2}{x^2+y^2} + \frac{|x|\cdot y^2}{x^2+y^2} = \frac{|x|}{\cancel{x^2+y^2}}\left(\cancel{x^2+y^2}\right) \xrightarrow[x\to 0]{} 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x \cdot \sin(x^2-y^2)}{x^2+y^2} = 0 \cdot 1 = 0$$

e) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy}$

Fie $a^n = \left(\frac{1}{n^2}, \frac{1}{n^2}\right) \to (0,0)$, $f(x,y) = \frac{x^3+y^3}{xy}$

$$\lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} \frac{\frac{1}{n^6}+\frac{1}{n^6}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{2n^4}{n^6} = 0$$

Fie $b^n = \left(\frac{1}{n}, \frac{1}{n^2}\right) \to (0,0)$

$$\lim_{n\to\infty} f(b^n) = \lim_{n\to\infty} \frac{\frac{1}{n^3}+\frac{1}{n^6}}{\frac{1}{n^3}} = \lim_{n\to\infty}\left(\frac{n^3}{n^3} + \frac{n^3}{n^6}\right) = 1+0 = 1$$

$f(a^n) \neq f(b^n) \Rightarrow \nexists \lim$

f) $\lim\limits_{(x,y)\to(1,1)} \dfrac{(x-1)(y-1)}{xy-1}$

Fie $x-1=u,\ y-1=v \Rightarrow \begin{matrix} x=u+1 \\ y=v+1 \end{matrix}$

$$\lim_{(u,v)\to(0,0)} \frac{u\cdot v}{(u+1)(v+1)-1} = \lim_{(u,v)\to(0,0)} \frac{u\cdot v}{uv+u+v+1-1} = \lim_{(u,v)\to(0,0)} \frac{uv}{uv+u+v}$$

Fie $f(u,v)=\dfrac{uv}{uv+u+v}$ și $a^n=\left(\frac{1}{n},-\frac{1}{n}\right)$, $b^n=\left(\frac{1}{n},\frac{1}{n}\right)$

$$\lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} \frac{\frac{1}{n}\cdot\left(-\frac{1}{n}\right)}{\frac{1}{n}\cdot\left(-\frac{1}{n}\right)+\frac{1}{n}-\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{1}{n^2}}{-\frac{1}{n^2}} = 1$$

$$\lim_{n\to\infty} f(b^n) = \lim_{n\to\infty} \frac{\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n}\cdot\frac{1}{n}+\frac{1}{n}+\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}+\frac{2}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1+2n}{n^2}} =$$

$$= \lim_{n\to\infty} \frac{1}{1+2n} = 0$$

$$\left.\begin{matrix} \lim\limits_{n\to\infty} f(a^n) \neq \lim\limits_{n\to\infty} f(b^n) \\ \lim\limits_{n\to\infty} a^n = \lim\limits_{n\to\infty} b^n = 0_2 \end{matrix}\right\} \Rightarrow \nexists \lim.$$

g) $\lim\limits_{(x,y,z)\to 0_3} \dfrac{(x+y+z)^2}{x^2+y^2+z^2}$

Fie $f(x,y,z)=\dfrac{(x+y+z)^2}{x^2+y^2+z^2}$

$a^n=\left(\frac{1}{n},\frac{1}{n},\frac{1}{n}\right) \to 0_3,\ n\to\infty$

$b^n=\left(\frac{1}{n},0,0\right) \to 0_3,\ n\to\infty$

$$\lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} \frac{\left(\frac{1}{n}+\frac{1}{n}+\frac{1}{n}\right)^2}{\frac{1}{n^2}+\frac{1}{n^2}+\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{9}{n^2}}{\frac{3}{n^2}} = \frac{9}{3} = 3.$$

$$\lim_{n\to\infty} f(b^n) = \lim_{n\to\infty} \frac{\left(\frac{1}{n}+0+0\right)^2}{\frac{1}{n^2}+0+0} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1$$

$\lim\limits_{n\to\infty} f(a^n) \neq \lim\limits_{n\to\infty} f(b^n)$

$\Rightarrow \nexists \lim.$

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② Se dă fc. $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} x \cdot \cos\frac{1}{y^2} + y \cdot \cos\frac{1}{x^2}, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

Este fc. continuă în $(0,0)$? Dar în $(1,0)$?

* f continuă în $(0,0) \Leftrightarrow \exists \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$f(0,0) = 0$.

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$, deoarece:

$$\left|x \cdot \cos\frac{1}{y^2} + y \cdot \cos\frac{1}{x^2} - 0\right| \leq \left|x \cdot \cos\frac{1}{y^2}\right| + \left|y \cdot \cos\frac{1}{x^2}\right| =$$

$$= |x| \cdot \underbrace{\left|\cos\frac{1}{y^2}\right|}_{\leq 1} + |y| \cdot \underbrace{\left|\cos\frac{1}{x^2}\right|}_{\leq 1} \leq |x| + |y| \xrightarrow[(x,y)\to(0,0)]{} 0$$

Deci f cont. în $(0,0)$.

* f cont. în $(1,0) \Leftrightarrow \exists \lim\limits_{(x,y)\to(1,0)} f(x,y) = f(1,0)$

$f(1,0) = 0$.

$0 \cdot \cos 1 = 0$

$$\lim_{(x,y)\to(1,0)} f(x,y) = \lim_{(x,y)\to(1,0)} \left(x \cdot \cos\frac{1}{y^2} + y \cdot \cos\frac{1}{x^2}\right) = \lim_{(x,y)\to(1,0)} x \cdot \cos\frac{1}{y^2}.$$

Fie $a^n = \left(1, \frac{1}{\sqrt{2n\pi}}\right)$ și $b^n = \left(1, \frac{1}{\sqrt{(2n+1)\pi}}\right)$ cu $a^n \to (1,0)$, $b^n \to (1,0)$, $n \to \infty$

și $f_1(x,y) = x \cdot \cos\frac{1}{y^2}$

$$\lim_{n\to\infty} f_1(a^n) = \lim_{n\to\infty} 1 \cdot \cos\frac{1}{\frac{1}{2n\pi}} = \lim_{n\to\infty} \cos(2n\pi) = 1$$

$$\lim_{n\to\infty} f_1(b^n) = \lim_{n\to\infty} 1 \cdot \cos\frac{1}{\frac{1}{(2n+1)\pi}} = \lim_{n\to\infty} \cos((2n+1)\pi) = -1$$

$\lim\limits_{n\to\infty} f_1(a^n) \neq \lim\limits_{n\to\infty} f_1(b^n) \Rightarrow \nexists \lim$, deci f nu e cont. în $(1,0)$

③ Verificați dacă fc. următoare își ating valorile extreme și determinați aceste valori.

a) $f:(0,\infty)^2 \to \mathbb{R}$, $f(x,y)=\frac{x}{y}+\frac{y}{x}$.

! T. Weierstrass Fie $A \subseteq \mathbb{R}^m$ mulțime compactă (= mărginită și închisă) și $f: A \to \mathbb{R}$ o fc. continuă. Atunci f mărginită și își atinge extremele pe A.

$A=(0,\infty)^2$ nu e mărginită, deci nu e compactă, deci nu putem aplica T.W.

Fie $f:(0,\infty)\to\mathbb{R}$, $f(a)=a+\frac{1}{a}$

$f'(a)=0 \Rightarrow 1-\frac{1}{a^2}=0 \Rightarrow a=1$

a	0		1		∞
$f'(a)$		$-\,-$	0	$+\,+$	
$f(a)$		$\searrow$		$\nearrow$	

1 - pct. de minim $\Rightarrow f(a)\geq f(1)$

$f(1)=2$.

Vom arăta că $f(x,y)\geq 2$.

$$\frac{x}{y}+\frac{y}{x}\geq 2 \Leftrightarrow \frac{x^2+y^2}{xy}\geq 2 \Leftrightarrow \frac{x^2+y^2}{xy}-2\geq 0 \Leftrightarrow$$

$$\frac{x^2+y^2-2xy}{xy}\geq 0 \Leftrightarrow \frac{(x-y)^2}{xy}\geq 0, \text{ „A"}, \text{ deoarece } (x-y)^2\geq 0,\ x,y>0$$

$f(1,1)=2 \Rightarrow \inf f(A)=2$ și se atinge. (de fapt se atinge pt. orice pct de forma (x,x))

$\sup f(A)=\infty$, nu se atinge

(Fie $(x,y)=(n,1) \Rightarrow f(n,1)=n+\frac{1}{n}\to\infty,\ n\to\infty$)

b) $f: B(0_2,1) \to \mathbb{R}$, $f(x,y) = \frac{1}{1+x^2+y^2}$.

$B(0_2,1)$ nu e închisă, deci mulț. nu e compactă $\Rightarrow$ nu putem aplica T.W.

$(x,y) \in B(0_2,1) \Rightarrow \|(x,y)\| < 1 \Rightarrow \sqrt{x^2+y^2} < 1 \Rightarrow x^2+y^2 < 1$

$0 \leq x^2+y^2 < 1 \quad | +1$

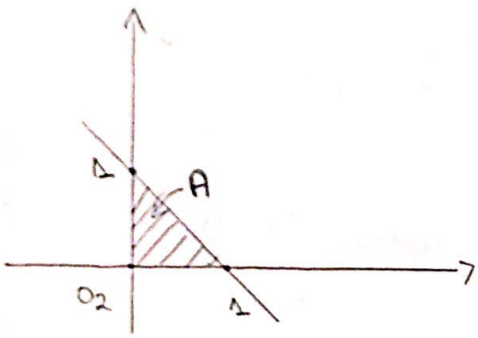
$1 \leq x^2+y^2+1 < 2 \quad \overset{()^{-1}}{\Rightarrow}$

$1 \geq \frac{1}{x^2+y^2+1} > \frac{1}{2}$

Deci $\inf f(A) = \frac{1}{2}$, nu se atinge

$\sup f(A) = 1$ se atinge: $f(0,0) = 1$.

c) $f: A \to \mathbb{R}$, $f(x,y) = xy(1-x-y)$, $A = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$.



$\left.\begin{array}{l} A \text{ închisă} \\ A \text{ mărginită} \end{array}\right\} \Rightarrow A$ compactă $\overset{T.W.}{\Longrightarrow}$

f mărg. și își atinge marginile.

$\left.\begin{array}{l} x+y \leq 1 \Rightarrow 1-x-y \geq 0 \\ x \geq 0 \\ y \geq 0 \end{array}\right\} \Rightarrow f(x,y) = (1-x-y) \cdot xy \geq 0 \Rightarrow$

$\inf f(A) = 0$ (se atinge de ex. în $(1,0)$ sau $(0,1)$, etc)

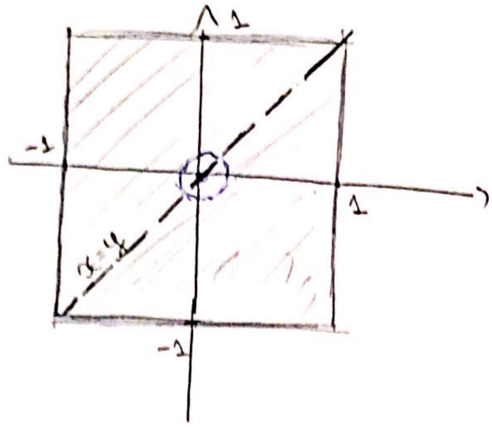
$m_a \geq m_g \Rightarrow \frac{x+y+(1-x-y)}{3} \geq \sqrt[3]{xy(1-x-y)} \Rightarrow$

$\frac{1}{3} \geq \sqrt[3]{f(x,y)} \Rightarrow \frac{1}{27} \geq f(x,y) \Rightarrow \sup f(A) = \frac{1}{27}$

(se atinge de ex. în $(\frac{1}{3}, \frac{1}{3})$)

④ Se dă mulțimea $A = \{(x,y) \in [-1,1]^2 \mid x \neq y\}$ și

$f: A \to \mathbb{R},\ f(x,y) = \dfrac{x^2+y^2}{(x-y)^2}$.



a) Există limita fc. f în origine?

Fie $a^n = \left(\frac{1}{n}, 0\right) \to (0,0),\ n\to\infty$

$b^n = \left(\frac{1}{n}, \frac{2}{n}\right) \to (0,0),\ n\to\infty$

$$\lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1$$

$$\lim_{n\to\infty} f(b^n) = \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{4}{n^2}}{\left(\frac{1}{n}-\frac{2}{n}\right)^2} = \lim_{n\to\infty} \frac{\frac{5}{n^2}}{\frac{1}{n^2}} = 5$$

$\Rightarrow \nexists$ lim.

b) Este A compactă?

A mărginită.

$O_2 \in \operatorname{fr} A,\ O_2 \notin A \Rightarrow \operatorname{fr} A \not\subset A \Rightarrow$ A nu e compactă, pt. că nu e închisă.

c) Det. valorile extreme ale lui f pe mulțimea A. Atinge f aceste valori?

$$\frac{x^2+y^2}{(x-y)^2} \geq \frac{1}{2} \Leftrightarrow \frac{x^2+y^2}{(x-y)^2} - \frac{1}{2} \geq 0 \Leftrightarrow \frac{2x^2+2y^2-x^2+2xy-y^2}{2(x-y)^2} \geq 0 \Leftrightarrow$$

$$\frac{(x+y)^2}{2(x-y)^2} \geq 0 \text{ „A"}.$$ Deci $\inf f(A) = \frac{1}{2} = f(1,-1)$

Fie $x = 1-\frac{1}{n},\ y = 1-\frac{2}{n}$, $x,y \in [-1,1],\ x \neq y$

$$f(x,y) = \frac{\left(1-\frac{1}{n}\right)^2+\left(1-\frac{2}{n}\right)^2}{\left(1-\frac{1}{n}-1+\frac{2}{n}\right)^2} = \frac{1-\frac{2}{n}+\frac{1}{n^2}+1-\frac{4}{n}+\frac{4}{n^2}}{\frac{1}{n^2}} = n^2\left(2-\frac{6}{n}+\frac{5}{n^2}\right) = 2n^2-6n+5 \to \infty,\ n\to\infty$$

deci $\sup f(A) = +\infty$, nu se atinge.

* Inegalitatea mediilor

$n \in \mathbb{N}^*;\ x_1, \dots, x_n \in [0, \infty) \Rightarrow G \leq A \Rightarrow$

$$\sqrt[n]{x_1 \cdot \ldots \cdot x_n} \leq \frac{x_1 + \ldots + x_n}{n}$$

* Inegalitatea Cauchy-Schwarz

$n \in \mathbb{N}^*;\ x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R} \Rightarrow$

$$(x_1 y_1 + \ldots + x_n y_n)^2 \leq (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2)$$