Seminar 7

① Evaluați integralele:

a) $I=\int_0^1 \frac{e^x}{\sqrt{e^{2x}+1}}\,dx$

$e^x=t$

$e^x dx = dt$

$x=0 \Rightarrow t=1$

$x=1 \Rightarrow t=e$

$$I=\int_1^e \frac{1}{\sqrt{t^2+1}}\,dt = \ln\left(t+\sqrt{t^2+1}\right)\Big|_1^e = \ln\left(e+\sqrt{e^2+1}\right)-\ln\left(1+\sqrt{2}\right)$$

$$=\ln\left(\frac{e+\sqrt{e^2+1}}{1+\sqrt{2}}\right)$$

b) $I=\int_0^2 \max\{x, x^2\}\,dx$

Fie $f:[0,2]\to\mathbb{R}$, $f(x)=x-x^2$

$f'(x)=1-2x$

$f'(x)=0 \Leftrightarrow x=\frac{1}{2}$

x	0	$1/2$	1	2
$f'(x)$	$+\ +$	0	$-\ -$	
$f(x)$	$0\ \nearrow$	$1/4$	$\searrow\ 0\ \searrow$	-2

Pe $[0,1]$: $x-x^2\geq 0 \Rightarrow x\geq x^2$

Pe $[1,2]$: $x-x^2\leq 0 \Rightarrow x\leq x^2$

$$I=\int_0^1 x\,dx+\int_1^2 x^2\,dx = \frac{x^2}{2}\Big|_0^1+\frac{x^3}{3}\Big|_1^2 = \frac{1}{2}-0+\frac{8}{3}-\frac{1}{3}= {}^{3)}\frac{1}{2}+{}^{2)}\frac{7}{3}=\frac{17}{6}$$

c) $I=\int_1^{\sqrt{3}} \frac{\operatorname{arctg} x}{x^2}\,dx$

$f'=\frac{1}{x^2} \Rightarrow f=-\frac{1}{x}$

$g=\operatorname{arctg} x \Rightarrow g'=\frac{1}{x^2+1}$

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$$I=\left(-\frac{1}{x}\cdot \operatorname{arctg} x\right)\Big|_1^{\sqrt{3}} + \underbrace{\int_1^{\sqrt{3}} \frac{1}{x(x^2+1)}\,dx}_{J} = -\frac{1}{\sqrt{3}}\operatorname{arctg}\sqrt{3} + \operatorname{arctg} 1 + J =$$

$$= -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4} + J$$

$$J=\int_1^{\sqrt{3}} \frac{1}{x(x^2+1)}\,dx$$

Descompunem în fracții simple:

$$\frac{1}{x(x^2+1)} = \frac{^{x^2+1/}A}{x} + \frac{^{x/}Bx+C}{x^2+1}$$

$$1 = Ax^2+A+Bx^2+Cx$$

$$(A+B)\,x^2 + Cx + A = 0\cdot x^2 + 0\cdot x + 1$$

$$\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \Rightarrow B=-1$$

$$\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$$

$$J=\int_1^{\sqrt{3}}\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \ln x\Big|_1^{\sqrt{3}} - \frac{1}{2}\ln(x^2+1)\Big|_1^{\sqrt{3}} =$$

$$= \ln\sqrt{3} - \ln 1 - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 2$$

$$= \ln\sqrt{3} - \ln 2 + \ln\sqrt{2} = \ln\frac{\sqrt{6}}{2}$$

$$\Rightarrow I = \frac{-\pi}{3\sqrt{3}} + \frac{\pi}{4} + \ln\frac{\sqrt{6}}{2}$$

d) $I=\int_{-1}^{1}\sqrt{1-x^2}\,dx$

$$I' = \int\sqrt{1-x^2}\,dx$$

$$f'=1 \qquad f=x$$

$$g=\sqrt{1-x^2} \quad g' = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

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$$J' = x\sqrt{1-x^2} \quad \int \frac{-x^2}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$$

$$J' = x\sqrt{1-x^2} - \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx =$$

$$J' = x\sqrt{1-x^2} - \underbrace{\int \sqrt{1-x^2}\,dx}_{J'} + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$2J' = x\sqrt{1-x^2} + \arcsin x + C$$

$$J' = \frac{x\sqrt{1-x^2} + \arcsin x}{2} + C$$

$$J = \frac{x\sqrt{1-x^2} + \arcsin x}{2}\Bigg|_{-1}^{1} = \frac{0+\arcsin 1}{2} - \frac{0+\arcsin(-1)}{2}$$

$$= \frac{\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)}{2} = \frac{\pi}{2}$$

$$* \int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & f \text{ impară } (f(-x) = -f(x)) \\ 2\cdot\int_0^a f(x)\,dx, & f \text{ pară } (f(-x) = f(x)) \end{cases}$$

e) $J = \int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$

Substituție trigonometrică:

$$\boxed{\int R(x, \sqrt{x^2-a^2}) \qquad x = \frac{a}{\sin t}, \quad x = \frac{a}{\cos t}}$$

↳ fc. rațională

$$x = \frac{2}{\sin t} \qquad dx = \frac{-2\cos t}{\sin^2 t}$$

$$\frac{1}{x} = \frac{\sin t}{2}$$

$$\sin t = \frac{2}{x}$$

$$t = \arcsin\left(\frac{2}{x}\right) \qquad \begin{aligned} x = 2 &\Rightarrow t = \arcsin(1) = \pi/2 \\ x = 4 &\Rightarrow t = \arcsin(1/2) = \pi/6 \end{aligned}$$

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$$J=\int_{\pi/2}^{\pi/6}\frac{\sqrt{\left(\frac{2}{\sin t}\right)^2-4}}{\frac{2}{\sin t}}\cdot\frac{-2\cos t}{\sin^2 t}\,dt=$$

$$=\int_{\pi/6}^{\pi/2}\sqrt{\frac{4}{\sin^2 t}-4}\cdot\frac{\cos t}{\sin t}\,dt=\int_{\pi/6}^{\pi/2}\sqrt{\frac{4-4\sin^2 t}{\sin^2 t}}\cdot\frac{\cos t}{\sin t}\,dt=$$

$$=\int_{\pi/6}^{\pi/2}\sqrt{\frac{4(1-\sin^2 t)}{\sin^2 t}}\cdot\frac{\cos t}{\sin t}\,dt=\int_{\pi/6}^{\pi/2}2\sqrt{\frac{\cos^2 t}{\sin^2 t}}\cdot\frac{\cos t}{\sin t}\,dt=$$

$$=2\int_{\pi/6}^{\pi/2}\left|\frac{\cos t}{\sin t}\right|\cdot\frac{\cos t}{\sin t}\,dt\underset{\downarrow}{=}2\int_{\pi/6}^{\pi/2}\frac{\cos^2 t}{\sin^2 t}\,dt=$$

$\left|\frac{\cos t}{\sin t}\right|=\frac{\cos t}{\sin t}$ pt. că $t\in[\pi/6,\pi/2]$,

deci $\sin,\cos>0$

$$=2\int_{\pi/6}^{\pi/2}\frac{1-\sin^2 t}{\sin^2 t}\,dt=2\left(\int_{\pi/6}^{\pi/2}\frac{1}{\sin^2 t}\,dt-\int_{\pi/6}^{\pi/2}1\,dt\right)=$$

$$2\left(-\mathrm{ctg}\,t\Big|_{\pi/6}^{\pi/2}-t\Big|_{\pi/6}^{\pi/2}\right)=2\left(0+\sqrt{3}-\pi/2+\pi/6\right)=\frac{-2\pi+6\sqrt{3}}{6}\cdot 2$$

$$=\frac{-2\pi+6\sqrt{3}}{3}$$

$\boxed{M2}$ Substituție trigonometrică pt. d)

$$J=\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

$$\boxed{\int R(x,\sqrt{a^2-x^2})\qquad x=a\sin t,\ a\cos t}\ !$$

$x=\sin t$ $\quad x=-1\Rightarrow t=-\pi/2$

$t=\arcsin x$ $\quad x=1\Rightarrow t=\pi/2$

$dx=\cos t\,dt$

$$J=\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2 t}\cdot\cos t\,dt=\int_{-\pi/2}^{\pi/2}|\cos t|\cdot\cos t\,dt=$$

$$=\int_{-\pi/2}^{\pi/2}\cos^2 t\,dt=\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2t}{2}\,dt=$$

$$=\int_{-\pi/2}^{\pi/2}\frac{1}{2}\,dt+\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos(2t)\,dt$$

$$\frac{1}{2}t\Big|_{-\pi/2}^{\pi/2}+\frac{1}{2}\cdot\frac{\sin(2t)}{2}\Big|_{-\pi/2}^{\pi/2}$$

$$=\frac{1}{2}(\pi/2+\pi/2)+\frac{1}{4}(\underbrace{\sin\pi}_{0}-\underbrace{\sin(-\pi)}_{0})$$

$$=\frac{\pi}{2}$$

$$\cos^2 t=\frac{1+\cos 2t}{2}$$

$\boxed{\text{Obs}}$. Altă substituție trig.

$$\boxed{\int R(x,\sqrt{a^2+x^2}):\ x=a\,\mathrm{ctg}\,t,\ x=a\,\mathrm{tg}\,t}\ !$$

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