Subiect 212

1. Studiați convergența și calculați suma. $\sum_{n=0}^{\infty} \frac{3+(-1)^n}{2^{n+2}}$

$= \sum_{n=0}^{\infty} \frac{3}{2^{n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+2}}$

serie alternantă, $\lim_{n\to\infty} \frac{1}{2^{n+2}} = 0 \Rightarrow$ e convergentă ②

$3\sum_{n=0}^{\infty} \frac{1}{2^{n+2}}$ - convergentă ①

pt. că $\lim_{n\to\infty} \Sigma = 0$

①, ② $\Rightarrow \sum_{n=0}^{\infty} \frac{3+(-1)^n}{2^{n+2}}$ - convergentă.

$$\sum_{n=0}^{\infty} \frac{3+(-1)^n}{2^{n+2}} = 3\sum_{n=0}^{\infty} \frac{1}{2^{n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+2}} = \frac{3}{4}\sum_{n=0}^{\infty} \frac{1}{2^n} + \frac{1}{4}\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \dots + \frac{1}{2^n} = 1 \cdot \frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} = \left(\left(\frac{1}{2}\right)^n - 1\right) \cdot (-2) = 2 \ (n \to \infty).$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots \left(-\frac{1}{2}\right)^n = 1 \cdot \frac{\left(-\frac{1}{2}\right)^n - 1}{-\frac{1}{2} - 1} = -\frac{2}{3}\left(\left(-\frac{1}{2}\right)^n - 1\right) = \frac{2}{3} \ (n \to \infty)$$

$$\frac{3}{4}\sum_{n=0}^{\infty} \frac{1}{2^{n+2}} + \frac{1}{4}\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{3}{\cancel{4}_2} \cdot \cancel{2} + \frac{1}{\cancel{4}_2} \frac{\cancel{2}}{3} = \frac{3}{2} + \frac{1}{6} = \frac{10}{6} = \frac{5}{3}.$$

2. Calculați derivatele parțiale $f(x,y) = \arcsin \frac{x^2}{y+x}$ în $(1,1)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{\sqrt{1 - \left(\frac{x^2}{y+x}\right)^2}} \cdot \frac{2x(y+x) - x^2}{(y+x)^2} =$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{\sqrt{1 - \left(\frac{x^2}{x+y}\right)^2}} \cdot \frac{-x^2}{(y+x)^2}$$

$$\frac{\partial f}{\partial x}(1,1) = \frac{1}{\sqrt{1 - \left(\frac{1}{2}\right)^2}} \cdot \frac{2 \cdot 2 - 1}{2^2} = \frac{1}{\sqrt{1 - \frac{1}{4}}} \cdot \frac{3}{4} = \frac{\cancel{2}}{\sqrt{3}} \cdot \frac{3}{\cancel{4}_2} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}.$$

$$\frac{\partial f}{\partial y}(1,1) = \frac{\cancel{2}}{\sqrt{3}} \cdot \left(\frac{-1}{2^{\cancel{2}}}\right) = -\frac{1}{2\sqrt{3}}.$$

S.t.p: $\sum_{n=0}^{\infty} a_n$ - convergentă $\Rightarrow \lim_{n\to\infty} a_n = 0$.

dacă nu e 0 $\Rightarrow$ e divergentă

3. Calculați seria Taylor și mulțimea de convergență.

$$f(x) = \frac{x+3}{1+3x} = \frac{x+3}{3\left(x+\frac{1}{3}\right)} = \frac{1}{3}\cdot\frac{x+\frac{1}{3}-\frac{1}{3}+3}{x+\frac{1}{3}} = \frac{1}{3}\left(\frac{\cancel{x+\frac{1}{3}}}{\cancel{x+\frac{1}{3}}} + \frac{\frac{8}{3}}{x+\frac{1}{3}}\right)$$

$$f(x) = \frac{1}{3} + \frac{8}{9}\cdot\frac{1}{x+\frac{1}{3}} = \frac{1}{3} + \frac{8}{9}\underbrace{\left(x+\frac{1}{3}\right)^{-1}}$$ – notăm $(x+c)^{-1}$

calculăm derivata de ordin n.

$f'(x) = \left((x+c)^{-1}\right)' = -1(x+c)^{-2}$

$f''(x) = 2(x+c)^{-3}$ ... $f^{(n)}(x) = (-1)^n\cdot n!\,(x+c)^{-n-1} = P(n)\ \forall n \geq 1$

I Verificăm $P(1)$

II Presupunem $P(k)$

III Demonstrăm $P(k+1)$

$$f^n(x) = \begin{cases} \frac{1}{3} + \frac{8}{9}\left(x+\frac{1}{3}\right)^{-1}, & n=0 \\ \frac{8}{9}\left[(-1)^n\cdot n!\left(x+\frac{1}{3}\right)^{-n-1}\right], & n \neq 0. \end{cases}$$

$$f^n(0) = \begin{cases} \frac{1}{3} + \frac{8}{\cancel{9}_3}\cdot\cancel{3} = 3, & n=0 \\ \frac{8}{9}\left((-1)^n\cdot n!\cdot 3^{n+1}\right), & n\neq 0 \end{cases}$$

seria Taylor: $\displaystyle\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}\cdot(x-x_0)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot x^n$

$$3+\sum_{n=1}^{\infty}\frac{\frac{8}{9}(-1)^n\cancel{n!}\cdot 3^{n+1}}{\cancel{n!}}\cdot x^n = 3+\frac{8}{9}\sum_{n=1}^{\infty}(-1)^n\cdot 3^{n+1}\cdot x^n$$

$$= 3+8\sum_{n=1}^{\infty}(-1)^n\cdot 3^{n-1}\cdot x^n$$

aflăm raza de convergență.

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{3^{n-1}}{3^n} = \lim_{n\to\infty}\frac{1}{3}\cdot\frac{\cancel{3^n}}{\cancel{3^n}} = \frac{1}{3}.$$

$(x_0-r,\ x_0+r) \subseteq J \subseteq [x_0-r,\ x_0+r].$

$\left(-\frac{1}{3},\frac{1}{3}\right) \subseteq J \subseteq \left[-\frac{1}{3},\frac{1}{3}\right]$, e converg. pe $\left(-\frac{1}{3},\frac{1}{3}\right)$, verificăm în $\frac{1}{3}$ și $-\frac{1}{3}$

pt $x=\frac{1}{3} \Rightarrow \displaystyle\sum_{n=1}^{\infty}(-1)^n 3^{n-1}\left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty}(-1)^n\cdot 3^{n-1-n} = \sum_{n=1}^{\infty}(-1)^n\cdot 3^{-1} = \frac{1}{3}\sum_{n=1}^{\infty}(-1)^n$

pt n = par $\lim_{n\to\infty} 1-1+1-1\ldots 1 = 0$

pt n – impar $\lim_{n\to\infty} 1-1+1-1+\ldots-1 = 1$

} avem două subșiruri ale lui S_n care nu au aceeași limită

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Pt $x = -\frac{1}{3}$ la fel $\Rightarrow$ seria Taylor e divergentă.

4. Studiați convergența lui $I(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ și $I\left(\frac{1}{2}, \frac{1}{2}\right)$

$I(p,q)$ nu e definită în 1 dacă $q-1<0$.
nu e definită în 0 dacă $p-1<0$

luăm $p,q \in (0,1)$ pentru a acoperi toate cazurile.

$$\int_0^1 x^{p-1}(1-x)^{q-1}dx = \underbrace{\int_0^{\frac{1}{2}} x^{p-1}(1-x)^{q-1}dx}_{I_1} + \underbrace{\int_{\frac{1}{2}}^1 x^{p-1}(1-x)^{q-1}dx}_{I_2}$$

$$I_1 = \int_0^{\frac{1}{2}} x^{p-1}(1-x)^{q-1}dx$$

I, $p-1 \geq 0 \Rightarrow I_1$ - definită $\Rightarrow$ convergentă.

II $p-1<0$, proprietățile cu p și λ

P3: $\left(0, \frac{1}{2}\right]$

$\lim\limits_{x \searrow 0} x^{p'} \cdot x^{p-1}(1-x)^{q-1}$ găsim p' pt care $\lim \in (0,\infty)$

alegem $p' = -p+1 \Rightarrow \lim\limits_{x \searrow 0} x^{\cancel{-p+1}} \cdot x^{\cancel{p-1}}(1-x)^{q-1} = 1. \in (0,\infty)$.

$1-p<1 \Rightarrow p>0$ „A" $\Rightarrow I_1$ - convergentă $\forall p>0$.

$$I_2 = \int_{\frac{1}{2}}^1 x^{p-1}(1-x)^{q-1}dx$$

I $q-1 \geq 0$ - definită $\Rightarrow$ e convergentă

II $q-1<0$, propr. cu p și λ

P1. $\left[\frac{1}{2}, 1\right)$

$\lim\limits_{x \nearrow 1} (1-x)^{p'} \cdot x^{p-1} \overset{=1}{\cdot} (1-x)^{q-1} =$

alegem $p' = 1-q \Rightarrow \lim\limits_{x \nearrow 1} (1-x)^{1-q+q-1} \cdot x^{p-1} = \lim\limits_{x \nearrow 1} x^{p-1} = 1 \in (0,\infty)$

$1-q<1 \Rightarrow q>0 \Rightarrow I_2$ - convergentă $\forall q>0$.

$I_1 + I_2 \Rightarrow$ convergentă $\forall q, p>0$.

$I\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx = \int_0^1 \frac{1}{\sqrt{x(1-x)}}dx =$ $\quad t = \sqrt{x}$, $dt = \frac{1}{2\sqrt{x}}dx$

$= 2\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = 2\lim\limits_{u \searrow 0} \arcsin\sqrt{x}\Big|_u^{\frac{1}{2}} + \lim\limits_{u \nearrow 1} \arcsin\sqrt{x}\Big|_{\frac{1}{2}}^{u} = -\frac{\pi}{4}$