Exerciții suplimentare

① Evaluați integralele improprii:

a) $J = \int_0^1 \frac{\sqrt{x} + \ln x}{x} dx = \int_0^1 \frac{\sqrt{x}}{x} dx + \int_0^1 \frac{\ln x}{x} dx =$

$$= \underbrace{\int_0^1 \frac{1}{\sqrt{x}} dx}_{J_1} + \underbrace{\int_0^1 \frac{\ln x}{x}}_{J_2}$$

$$J_1 = \int_{0+0}^1 \frac{1}{\sqrt{x}} dx = \frac{\sqrt{x}}{2} \Big|_{0+0}^1 = \lim_{\substack{t \to 0 \\ t>0}} \frac{\sqrt{x}}{2} \Big|_t^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

J_2: $\ln x = t$

$\frac{1}{x} dx = dt$

$x = 0+0 \Rightarrow t = -\infty$

$x = 1 \Rightarrow t = 0$

$$J_2 = \int_{-\infty}^0 t\, dt = \frac{t^2}{2} \Big|_{-\infty}^0 = -\lim_{t \to -\infty} \frac{t^2}{2} = -\infty$$

$$J = \frac{1}{2} - \infty = -\infty.$$

b) $J = \int_0^1 \sqrt{\frac{1+x}{1-x}} dx$

$$J' = \int \sqrt{\frac{1+x}{1-x}} dx = \int \sqrt{\frac{(1+x)(1+x)}{(1-x)(1+x)}} dx = \int \frac{1+x}{\sqrt{1-x^2}} dx =$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx = \arcsin x - \sqrt{1-x^2} + C$$

$$J = \int_0^{1-0} \sqrt{\frac{1+x}{1-x}} = \lim_{u \nearrow 1} \left(\arcsin x - \sqrt{1-x^2}\right) \Big|_0^u = \arcsin 1 - 0 - \arcsin 0 + 1 = 1 + \pi/2$$

c) $J = \int_0^\infty e^{-x} \cdot \cos x\, dx$

$$J' = \int e^{-x} \cos x\, dx$$

$f' = e^{-x}$ $\qquad f = -e^{-x}$

$g = \cos x$ $\qquad g' = -\sin x$

$$J' = -e^{-x}\cos x - \underbrace{\int e^{-x}\sin x \, dx}_{J''}$$

J'': $f' = e^{-x}$ $\qquad f = -e^{-x}$ $\qquad J'' = -e^{-x}\sin x + \underbrace{\int e^{-x}\cos x \, dx}_{J'}$

$g = \sin x$ $\qquad g' = \cos x$

$$J' = -e^{-x}\cos x + e^{-x}\sin x - J'$$

$$J' = \frac{e^{-x}(\sin x - \cos x)}{2}$$

$$J = \lim_{u\to\infty} \frac{(\sin x - \cos x)}{2e^{x}}\Big|_0^u = \lim_{u\to\infty}\Big(\underbrace{\frac{\sin u}{2e^u}}_{\downarrow \atop 0} - \underbrace{\frac{\cos u}{2e^u}}_{\downarrow \atop 0}\Big) - \frac{\overbrace{\sin 0}^{0}}{2e^0} + \frac{\overbrace{\cos 0}^{1}}{2e^0}$$

(criteriul majorării)

$$= \frac{1}{2}.$$

d) $J = \int_0^1 \frac{\ln x}{\sqrt{1-x}} \, dx$

$$J' = \int \frac{\ln x}{\sqrt{1-x}} \, dx$$

$f' = \frac{1}{\sqrt{1-x}}$ $\qquad f = -2\sqrt{1-x}$

$g = \ln x$ $\qquad g' = \frac{1}{x}$

$$J' = -2\ln x\sqrt{1-x} + 2\underbrace{\int \frac{\sqrt{1-x}}{x} \, dx}_{J''}$$

J'': $\sqrt{1-x} = t$

$1 - x = t^2$

$x = 1 - t^2$

$dx = -2t \, dt$

$$J'' = -2\cdot\int \frac{t}{1-t^2}\cdot t \, dt = 2\int \frac{t^2}{t^2-1} \, dt = 2\left(\int \frac{t^2-1}{t^2-1} \, dt + \int \frac{1}{t^2-1} \, dt\right)$$

$$= 2t + \frac{2}{2}\ln\left|\frac{t-1}{t+1}\right| + C = 2t + \ln\left|\frac{t-1}{t+1}\right| = 2\sqrt{1-x} + \ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| + C$$

$$J' = -2\ln x\sqrt{1-x} + 4\sqrt{1-x} + 2\ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| + C$$

$$\mathcal{I} = \int_{0+0}^{1/2} \frac{\ln x}{\sqrt{1-x}}\,dx + \int_{1/2}^{1-0} \frac{\ln x}{\sqrt{1-x}}\,dx$$

$$= \lim_{u \searrow 0} \left(-2\ln x \sqrt{1-x} + 4\sqrt{1-x} + 2\ln \left| \frac{\sqrt{1-x}-1}{\sqrt{1-x}+1} \right| \right) \Bigg|_{u}^{1/2} +$$

$$+ \lim_{v \nearrow 1} \left(-2\ln x \sqrt{1-x} + 4\sqrt{1-x} + 2\ln \left| \frac{\sqrt{1-x}-1}{\sqrt{1-x}+1} \right| \Bigg|_{1/2}^{v} \right) =$$

$$= -2\ln\tfrac{1}{2}\cdot\sqrt{1/2} \cancel{+4\sqrt{1/2}} + 2\ln\left|\frac{\sqrt{1/2}-1}{\sqrt{1/2}+1}\right| -$$

$$- \lim_{u \searrow 0} \left(-2\ln u \sqrt{1-u} + 4\sqrt{1-u} + 2\ln\left|\frac{\sqrt{1-u}-1}{\sqrt{1-u}+1}\right| \right) +$$

$$+ \lim_{v \nearrow 1} \left(-2\ln v \sqrt{1-v} + 4\sqrt{1-v} + 2\ln\left|\frac{\sqrt{1-v}-1}{\sqrt{1-v}+1}\right| \right. -$$

$$- \left(-2\ln 1/2 \cdot \sqrt{1/2} \cancel{+4\sqrt{1/2}} + 2\ln\left|\frac{\sqrt{1/2}-1}{\sqrt{1/2}+1}\right| \right)$$

$$= -\lim_{u \searrow 0} \left(-2\ln u \sqrt{1-u} + 4 + 2\ln\left|\frac{\sqrt{1-u}-1}{\sqrt{1-u}+1}\right| \right)$$

$$+ \lim_{v \nearrow 1} \left(-2\ln v \underbrace{\sqrt{1-v}}_{=0} + 4\cdot 0 + \underbrace{2\ln\left|\frac{\sqrt{1-v}-1}{\sqrt{1-v}+1}\right|}_{2\ln 1 = 0} \right) =$$

$$= -4 - 2\lim_{u \searrow 0} \left(-\ln u \cdot \underbrace{\sqrt{1-u}}_{=1} + \ln(\sqrt{1-u}-1) - \underbrace{\ln(\sqrt{1-u}+1)}_{\ln 2} \right)$$

$$= -4 + 2\ln 2 + 2\lim_{u \searrow 0} \left(\ln u - \ln(\sqrt{1-u}-1) \right)$$

$$= -4 + 2\ln 2 + 2\lim_{u \searrow 0} \ln\left|\frac{u}{(\sqrt{1-u}-1)}\right| = -4 + 2\ln 2 + 2\ln|-2| = -4 + 2\ln 2 + 2\ln 2$$

$$= -4 + 4\ln 2$$

$$\lim_{u \searrow 0} \frac{u}{\sqrt{1-u}-1} \overset{l'H}{=} \lim_{u \searrow 0} \frac{1}{\frac{-1}{2\sqrt{1-u}}} = -2$$

e) $\int_1^{\infty} \frac{1}{(x^2+1)(\sqrt{x^2-1})} dx$

$$J' = \int \frac{1}{(x^2+1)\sqrt{x^2-1}} dx$$

Cum 1 e rădăcină pt. x^2-1, folosim substituția lui Euler.

$$\sqrt{x^2-1} = t(x-1) \quad |()^2$$

$$x^2-1 = t^2(x-1)^2$$

$$\cancel{(x-1)}(x+1) = t^2(x-1)^{\cancel{2}}$$

$$x+1 = t^2(x-1)$$

$$t^2 = \frac{x+1}{x-1} = \frac{x-1+2}{x-1} = 1 + \frac{2}{x-1}$$

$$\frac{2}{x-1} = t^2 - 1 \Rightarrow x-1 = \frac{2}{t^2-1} \Rightarrow x = \frac{2+t^2-1}{t^2-1} = \frac{1+t^2}{t^2-1}.$$

$$dx = \frac{2t(t^2-1) - 2t(1+t^2)}{(t^2-1)^2} dt \Rightarrow \frac{-4t}{(t^2-1)^2} dt = dx$$

$$x^2+1 = \left(\frac{1+t^2}{t^2-1}\right)^2 + 1 = \frac{1+t^4+\cancel{2t^2}+t^4-\cancel{2t^2}+1}{(t^2-1)^2} = \frac{2(t^4+1)}{(t^2-1)^2}$$

$$\sqrt{x^2-1} = t(x-1) = t \cdot \frac{2}{t^2-1} = \frac{2t}{t^2-1}$$

$$J' = \int \frac{\cancel{(t^2-1)^2}}{\cancel{2}(t^4+1)} \cdot \frac{t^2-1}{\cancel{2t}} \cdot \frac{-\cancel{4t}}{\cancel{(t^2-1)^2}} dt = -\int \frac{t^2-1}{t^4+1} dt =$$

$$= -\int \frac{\cancel{t^2}\left(1-\frac{1}{t^2}\right)}{\cancel{t^2}\left(t^2+\frac{1}{t^2}\right)} dt = -\int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2} dt$$

$$\left(t+\frac{1}{t}\right)^2 = t^2+2+\frac{1}{t^2} \Rightarrow t^2+\frac{1}{t^2} = \left(t+\frac{1}{t}\right)^2 - 2$$

$$u = t + \frac{1}{t}$$

$$du = 1 - \frac{1}{t^2} dt$$

$$J' = -\int \frac{du}{u^2-2} = -\frac{1}{2\sqrt{2}} \ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| = -\frac{1}{2\sqrt{2}} \ln\left|\frac{{}^{t)}t+\frac{1}{t}-\sqrt{2}^{\,t)}}{{}^{t)}t+\frac{1}{t}+\sqrt{2}^{\,t)}}\right| + C =$$

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$$-\frac{1}{2\sqrt{2}}\ln\left|\frac{t^2+1-\sqrt{2}t}{t^2+1+\sqrt{2}t}\right|+C \underset{\substack{t^2=\frac{x+1}{x-1}\\ t=\sqrt{\frac{x+1}{x-1}}}}{=} -\frac{1}{2\sqrt{2}}\ln\left|\frac{\frac{x+1}{x-1}+\overset{\frac{x-1}{x-1}}{1}-\overset{\frac{\sqrt{x-1}}{\sqrt{x-1}}}{\sqrt{2}}\cdot\sqrt{\frac{x+1}{x-1}}}{\frac{x+1}{x-1}+\overset{\frac{x-1}{x-1}}{1}+\overset{\frac{\sqrt{x-1}}{\sqrt{x-1}}}{\sqrt{2}}\cdot\sqrt{\frac{x+1}{x-1}}}\right|+C$$

$$=-\frac{1}{2\sqrt{2}}\ln\left|\frac{x+1+x-1-\sqrt{2}\cdot\sqrt{x^2-1}}{x+1+x-1+\sqrt{2}\cdot\sqrt{x^2-1}}\right|+C=-\frac{1}{2\sqrt{2}}\ln\left|\frac{2x-\sqrt{2}\sqrt{x^2-1}}{2x+\sqrt{2}\sqrt{x^2-1}}\right|+C$$

$$I=\int_{1\leftarrow 0}^{2}\frac{1}{(x^2+1)(\sqrt{x^2-1})}\,dx+\int_{2}^{\infty}\frac{1}{(x^2+1)\sqrt{x^2-1}}\,dx=$$

$$=\lim_{u\searrow 1}\left(\cancel{-\frac{1}{2\sqrt{2}}\ln\left|\frac{4-\sqrt{2}\cdot\sqrt{3}}{4+\sqrt{2}\cdot\sqrt{3}}\right|}+\frac{1}{2\sqrt{2}}\ln\left|\frac{2u-\sqrt{2}\cdot\sqrt{u^2-1}}{2u+\sqrt{2}\cdot\sqrt{u^2-1}}\right|\right)+$$

$$+\lim_{v\to\infty}\left(-\frac{1}{2\sqrt{2}}\ln\left|\frac{2v-\sqrt{2}\sqrt{v^2-1}}{2v+\sqrt{2}\sqrt{v^2-1}}\right|+\cancel{\frac{1}{2\sqrt{2}}\ln\left|\frac{4-\sqrt{2}\cdot\sqrt{3}}{4+\sqrt{2}\cdot\sqrt{3}}\right|}\right)$$

$$=\frac{1}{2\sqrt{2}}\underbrace{\ln\frac{2-\sqrt{2}\cdot 0}{2+\sqrt{2}\cdot 0}}_{\ln 1=0}-\frac{1}{2\sqrt{2}}\cdot\lim_{v\to\infty}\ln\left|\frac{2v-\sqrt{2}\sqrt{v^2-1}}{2v+\sqrt{2}\sqrt{v^2-1}}\right|=$$

$$=-\frac{1}{2\sqrt{2}}\lim_{v\to\infty}\ln\frac{(2v-\sqrt{2}\cdot\sqrt{v^2-1})^2}{\underbrace{4v^2-2(v^2-1)}}$$

$$\lim_{v\to\infty}\frac{4v^2-4\sqrt{2}v\sqrt{v^2-1}+2v^2-2}{2v^2+2}=\lim_{v\to\infty}\frac{6v^2-4\sqrt{2}v\overbrace{\sqrt{v^2-1}}^{\sim v^2}-2}{2v^2+2}$$

$$=\frac{6-4\sqrt{2}}{2}=3-2\sqrt{2}$$

$$I=-\frac{1}{2\sqrt{2}}\cdot\ln\left(3-2\sqrt{2}\right)$$

② Studiați convergența integralelor improprii:

a) $\int_0^1 \frac{1}{\sqrt[4]{1-x^4}}\,dx.$

Avem pb. în $x=1$. Fie $f:[0,1)\to[0,\infty)$, $f(x)=\frac{1}{\sqrt[4]{1-x^4}}$.

$b=1$, în $\boxed{P1}$

$$\lambda=\lim_{x\nearrow 1}(1-x)^p\cdot\frac{1}{\sqrt[4]{(1-x^4)}}=\lim_{x\nearrow 1}(1-x)^p\frac{1}{\sqrt[4]{(1-x^2)(1+x^2)}}=$$

$$=\lim_{x\nearrow 1}(1-x)^p\cdot\frac{1}{\sqrt[4]{(1-x)}\cdot\sqrt[4]{(1+x)(1+x^2)}}=\lim_{x\nearrow 1}(1-x)^{p-1/4}\cdot\frac{1}{\sqrt[4]{(1+x)(1+x^2)}}$$

Pt. $p=1/4$ avem $\lambda=\lim_{x\nearrow 1}(1-x)^{1/4-1/4}\cdot\frac{1}{\sqrt[4]{(1+x)(1+x^2)}}=$

$$=1\cdot\frac{1}{\sqrt[4]{4}}<\infty$$

$\left.\begin{array}{l}p=\frac{1}{4}<1\\ \lambda<\infty\end{array}\right\}\overset{\boxed{P1}}{\Rightarrow}$ Integrala e convergentă.

b) $\int_0^1 \frac{1}{\sqrt{x(e^x-e^{-x})}}\,dx$

Avem pb. în $x=0$. Fie $f:(0,1]\to[0,\infty)$, $f(x)=\frac{1}{\sqrt{x}\cdot\sqrt{e^x-e^{-x}}}$

$a=0$ în $\boxed{P3}$.

$$\lambda=\lim_{x\searrow 0}(x-a)^p\cdot\frac{1}{\sqrt{x}\cdot\sqrt{e^x-e^{-x}}}=\lim_{x\searrow 0}x^p\cdot\frac{1}{\sqrt{e^x-e^{-x}}\cdot\sqrt{x}}=$$

$$=\lim_{x\searrow 0}\frac{x^{p-1/2}}{\sqrt{e^x-e^{-x}}}$$

$p=1.$

$$\lambda=\lim_{x\searrow 0}\frac{\sqrt{x}}{\sqrt{e^x-e^{-x}}}=\sqrt{\lim_{x\searrow 0}\frac{x}{e^x-e^{-x}}}\ \underset{\left[\frac{0}{0}\right]}{\overset{L'H}{=}}\ \sqrt{\lim_{x\searrow 0}\frac{1}{e^x+e^{-x}}}=\sqrt{\frac{1}{2}}.$$

$\left.\begin{array}{l}p=1\\ \lambda=\sqrt{\frac{1}{2}}>0\end{array}\right\}\overset{\boxed{P3}}{\Rightarrow}$ Integrala e divergentă.

c) $\int_0^{\pi} \left(1-\frac{\sin x}{x}\right)^{-1} dx$

$J=\int_0^{\pi}\left(\frac{x-\sin x}{x}\right)^{-1} dx=\int_0^{\pi}\frac{x}{x-\sin x}\,dx$

Avem pb. în $x=0$. Fie $f:(0,\pi]\to[0,\infty)$, $f(x)=\frac{x}{x-\sin x}$

$a=0$, $\boxed{P_3}$

$\lambda=\lim_{x\searrow 0} x^p\cdot\frac{x}{x-\sin x}$

Fie $p=1$

$\lambda=\lim_{x\searrow 0}\frac{x^2}{x-\sin x}\overset{\left[\frac{0}{0}\right]}{\underset{L'H}{=}}\lim_{x\searrow 0}\frac{2x}{1-\cos x}\overset{\left[\frac{0}{0}\right]}{\underset{L'H}{=}}\lim_{x\searrow 0}\frac{2}{1+\sin x}=2$

$\left.\begin{array}{l} p=1\\ \lambda=2>0\end{array}\right\}\overset{\boxed{P_3}}{\Rightarrow}$ Integrala e divergentă.

③ Determinați $\alpha>0$ a.î. integrala

$$J(\alpha)=\int_1^{\infty}\frac{x-1}{x^{\alpha}-1}\,dx$$

să fie convergentă. Calculați $J(3)$.

$J=\underbrace{\int_1^2\frac{x-1}{x^{\alpha}-1}\,dx}_{J_1}+\underbrace{\int_2^{\infty}\frac{x-1}{x^{\alpha}-1}\,dx}_{J_2}$

J_2: $f:[2,\infty)\to[0,\infty)$, $f(x)=\frac{x-1}{x^{\alpha}-1}$.

$\boxed{P_2}$ $\lambda=\lim_{x\to\infty} x^p\cdot\frac{x-1}{x^{\alpha}-1}=\lim_{x\to\infty}\frac{x^{p+1}-x^p}{x^{\alpha}-1}=1$ dacă $p+1=\alpha$.

Cum $0<1<\infty$, avem: $\begin{cases} p>1 \text{ și } \lambda=1<\infty \Rightarrow J_2 \text{ convergentă}\\ \quad p>1\Rightarrow \alpha=p+1>2\\ p\le 1 \text{ și } \lambda=1>0\Rightarrow J_2 \text{ divergentă}\\ \quad p\le 1\Rightarrow \alpha=p+1\le 2\end{cases}$

Rezolvare.

$J_1: f:(1,2] \to [0,\infty), \ f(x) = \frac{x-1}{x^\alpha - 1}$.

$\boxed{P_3}$ $l := \lim\limits_{x\to 1} \frac{(x-1)^p \cdot (x-1)}{x^\alpha - 1} = \cancel{\lim\limits_{x\to 1} \frac{(x-1)^{p+1}}{x^\alpha - 1}}$

$= \lim\limits_{x\to 1} (x-1)^p \cdot \frac{x-1}{x^\alpha - 1} = \lim\limits_{x\to 1} \frac{1}{\alpha} \cdot (x-1)^p$

$\text{l'H: } \lim\limits_{x\to 1} \frac{1}{\alpha \cdot x^{\alpha-1}} = \frac{1}{\alpha}$

Luăm $p=0$ și obținem $l = \frac{1}{\alpha}$

$\left.\begin{array}{l} p = 0 < 1 \\ l = \frac{1}{\alpha} < \infty \text{ pt. că } \alpha > 0 \end{array}\right\} \xRightarrow{\boxed{P_3}} J_1$ convergentă.

Așadar $J = J_1 + J_2$ e convergentă când J_2 e C, deci $\boxed{\alpha > 2}$.

$$J(3) = \int_1^\infty \frac{x-1}{x^3-1}\,dx = \int_1^\infty \frac{x-1}{(x-1)(x^2+x+1)}\,dx = \int_1^\infty \frac{1}{x^2+x+1}\,dx =$$

$$= \int_1^\infty \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx = \frac{1}{\frac{\sqrt{3}}{2}} \cdot \operatorname{arctg}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\Bigg|_1^{v\to\infty} =$$

$$= \lim_{v\to\infty} \frac{2}{\sqrt{3}} \cdot \left[\operatorname{arctg} \frac{2v+1}{\sqrt{3}} - \operatorname{arctg} \frac{3}{\sqrt{3}}\right] = \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{3\sqrt{3}}$$

$\Rightarrow \boxed{J(\alpha)\ C, \ \forall \alpha > 0}$.

④ Fie $\alpha > 0$. Studiați convergența integralei

$$J(\alpha) = \int_1^\infty \left[\frac{1}{x^\alpha} - \frac{1}{(x+1)^\alpha}\right] dx$$

și calculați $J\left(\frac{1}{2}\right)$.

1) $\boxed{\alpha = 1}$

$J(1) = \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) dx =$

$= \lim\limits_{v\to\infty} \ln\left|\frac{x}{x+1}\right|\Big|_1^\infty =$

$= \lim\limits_{v\to\infty} \ln \underbrace{\frac{v}{v+1}}_{\to 1} - \ln\frac{1}{2}$

$= -\ln\frac{1}{2} \Rightarrow J(1) \boxed{C}$

2) $\alpha > 1$

$J(\alpha) = \int_1^\infty x^{-\alpha} - (x+1)^{-\alpha}\,dx =$

$\lim\limits_{v\to\infty} \left(\frac{x^{-\alpha+1}}{-\alpha+1} - \frac{(x+1)^{-\alpha+1}}{-\alpha+1}\right)\Big|_1^v =$

$= \frac{1}{1-\alpha} \lim\limits_{v\to\infty} \left(\frac{1}{x^{\alpha-1}} - \frac{1}{(x+1)^{\alpha-1}}\right)\Big|_1^v =$

$= \frac{1}{1-\alpha}\left(0 \ 8 - \frac{1}{1} + \frac{1}{2^{\alpha-1}}\right) \Rightarrow \boxed{C}$

3) $\alpha < 1$.

$J(\alpha) = \lim\limits_{v\to\infty} \frac{1}{1-\alpha}\Big(\underbrace{v^{1-\alpha} - (v+1)^{1-\alpha}}_{[\infty - \infty]} - 1 + 2^{1-\alpha}\Big)$

$\lim\limits_{v\to\infty} \left(v^{1-\alpha} - (v+1)^{1-\alpha}\right) \overset{?}{=} 0$

se amplifică cu conjugata și la numărător se obține -1, iar la numitor o sumă de $v^\beta \cdot (v+1)^\beta$ ce tinde la ∞, deci $l = 0$ și $J(\alpha) \boxed{C}$

$$J(1/2) = \int_1^{\infty}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}\right) dx = \left(2\sqrt{x} - 2\sqrt{x+1}\right)\Big|_1^{v\to\infty}$$

$$= \lim_{v\to\infty}\left(2\sqrt{v} - 2\sqrt{v+1}\right) - 2 + 2\sqrt{2} =$$

$$= -2+2\sqrt{2} + 2\underbrace{\lim_{v\to\infty}\frac{v-v-1}{\sqrt{v}+\sqrt{v+1}}}_{=0} = -2+2\sqrt{2}$$

5 Fie $f: [1,\infty) \to [0,\infty)$ o fc. continuă, pozitivă și descrescătoare. Arătați că:

i) $f(n+1) \le \int_n^{n+1} f(x)\,dx \le f(n)$, $\forall n \in \mathbb{N}^*$.

Fie $x \in [n, n+1]$. $\Rightarrow$ $n \le x \le n+1 \overset{f \searrow}{\Rightarrow} f(n+1) \le f(x) \le f(n)$ $\forall x \in [n, n+1]$

$$\Rightarrow \int_n^{n+1} \underbrace{f(n+1)}_{\text{nu depinde de } x} dx \le \int_n^{n+1} f(x)\,dx \le \int_n^{n+1} \underbrace{f(n)}_{\text{nu dep. de } x} dx$$

$$\Rightarrow f(n+1)\cdot x\Big|_n^{n+1} \le \int_n^{n+1} f(x)\,dx \le f(n)\cdot x\Big|_n^{n+1}$$

$$f(n+1) \le \int_n^{n+1} f(x)\,dx \le f(n).$$

ii) $f(2)+f(3)+\dots+f(n) \le \int_1^n f(x)\,dx \le f(1)+f(2)+\dots+f(n)$, $\forall n \in \mathbb{N}, n \ge 2$

Folosim i): $f(2) \le \int_1^2 f(x)\,dx \le f(1)$

$f(3) \le \int_2^3 f(x)\,dx \le f(2)$

$\vdots$

$f(n) \le \int_{n-1}^n f(x)\,dx \le f(n-1)$

$$f(2)+f(3)+\dots+f(n) \le \underbrace{\int_1^2 f(x)\,dx + \int_2^3 f(x) + \dots + \int_{n-1}^n f(x)\,dx}_{=\int_1^n f(x)\,dx} \le f(1)+f(2)+\dots+f(n-1)$$

$$\Rightarrow \sum_{i=2}^n f(i) \le \int_1^n f(x)\,dx \le \sum_{i=1}^{n-1} f(i) \underset{f \text{ pozitivă}}{\le} \sum_{i=1}^n f(i)$$

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iii) Criteriul integral

Seria $\sum_{n=1}^{\infty} f(n)$ este convergentă $\Leftrightarrow$ integrala $\int_1^{\infty} f(x)\,dx$ este convergentă

Deoarece $\int_1^n f(x)\,dx \le f(1)+\ldots+f(n) \overset{n\to\infty}{\Rightarrow}$

$$\int_1^{\infty} f(x)\,dx = \left(\sum_{n=1}^{\infty} \int_n^{n+1} f(x)\,dx\right) \le \sum_{n=1}^{\infty} f(n)$$

Așadar, conform Criteriului Comparației,

dacă $\sum_{n=1}^{\infty} f(n)$ C $\Rightarrow \int_1^{\infty} f(x)\,dx$ C. (sunt STP)

Deoarece $f(2)+f(3)+\ldots+f(n) \le \int_1^n f(x)\,dx \overset{n\to\infty}{\Rightarrow}$

$$-f(1)+\sum_{n=1}^{\infty} f(n) \le \int_1^{\infty} f(x)\,dx = \sum_{n=1}^{\infty} \int_n^{n+1} f(x)\,dx$$

Așadar, tot din C. Comp., dacă $\int_1^{\infty} f(x)\,dx$ C $\Rightarrow$

$\sum_{n=1}^{\infty} f(n)$ C (sunt STP)

Așadar $\int_1^{\infty} f(x)\,dx$ C $\Leftrightarrow \sum_{n=1}^{\infty} f(n)$ C.

iv) Șirul $c_n = f(1)+f(2)+\ldots+f(n) - \int_1^n f(x)\,dx$ e convergent.

$$c_{n+1}-c_n = f(n+1) - \int_1^{n+1} f(x)\,dx + \int_1^n f(x)\,dx =$$

$$= f(n+1) + \int_{n+1}^{1} f(x)\,dx + \int_1^n f(x)\,dx = f(n+1) + \int_{n+1}^{n} f(x)\,dx$$

$$= f(n+1) - \int_n^{n+1} f(x)\,dx \le 0$$ (din prima ineg. dem. la i))

$\Rightarrow (c_n) \searrow$ mărg. superior de c_1.

$f(1)+\ldots+f(n) - \int_1^n f(x)\,dx \ge 0$, deoarece din ii) avem

$\int_1^n f(x)\,dx \le f(1)+\ldots+f(n)$ $\Rightarrow (c_n)$ mărg. inf. de 0

$\Rightarrow (c_n)$ convergent