Seminar 9

① Să se rezolve sistemul unde a, b, c, d sunt nr. diferite între ele, două câte două:

$$\begin{cases} x+y+z+t=1 \\ ax+by+cz+dt=m \\ a^2x+b^2y+c^2z+d^2t=m^2 \\ a^3x+b^3y+c^3z+d^3t=m^3 \end{cases}$$

Soluție: $A=\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} =$$

$$= (b-a)(c-a)(c-b)(d-c)(d-b)(d-a) \neq 0$$

(a, b, c, d dif. 2 câte 2)

$\Rightarrow$ sist. compatibil determinat

$$d_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ m & b & c & d \\ m^2 & b^2 & c^2 & d^2 \\ m^3 & b^3 & c^3 & d^3 \end{vmatrix} =$$

$$= (b-m)(c-m)(c-b)(d-c)(d-b)(d-m)$$

$$x = \frac{d_1}{\det A} = \frac{(b-m)(c-m)(d-m)}{(b-a)(c-a)(d-a)}$$

$$y = \frac{d_2}{\det A}$$

$$d_2 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & m & c & d \\ a^2 & m^2 & c^2 & d^2 \\ a^3 & m^3 & c^3 & d^3 \end{vmatrix} =$$

$$= (m-a)(c-m)(c-a)(d-c)(d-m)(d-a)$$

$$y = \frac{(m-a)(c-m)(d-m)}{(b-a)(c-b)(d-b)}$$

$$z = \frac{d_3}{\det A}$$

$$d_3 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & m & d \\ a^2 & b^2 & m^2 & d^2 \\ a^3 & b^3 & m^3 & d^3 \end{vmatrix} =$$

$$= (b-a)(m-b)(m-a)(d-m)$$
$$(d-b)(d-a)$$

$$z = \frac{(m-b)(m-a)(d-m)}{(c-b)(c-a)(d-c)}$$

$$d_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & m \\ a^2 & b^2 & c^2 & m^2 \\ a^3 & b^3 & c^3 & m^3 \end{vmatrix} =$$

$$= (b-a)(c-b)(c-a)(m-c)$$
$$(m-b)(m-a)$$

$$t = \frac{d_4}{\det A} = \frac{(m-c)(m-b)(m-a)}{(d-c)(d-b)(d-a)}$$

② Să se determine rangul matricelor:

a) $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 4 & 5 & 3 & -1 \\ 2 & 1 & -3 & 1 \end{pmatrix} = A$

$a_{11} = 1 \neq 0$

$d_1 = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$

$d_2 = \begin{vmatrix} 1 & 2 & -1 \\ 4 & 5 & 3 \\ 2 & 1 & -3 \end{vmatrix} = 1(-15-3) -$

$-2(-12-6) - 1(4-10) = -18 + 36$

$+6 = 18 + 6 = 24 \neq 0$

$\Longrightarrow \text{rang } A = 3$

b.) $A = \begin{pmatrix} 6 & 2 & 3 & -2 \\ 5 & 3 & 7 & -6 \\ 8 & 0 & -5 & 6 \\ 4 & -2 & -7 & 5 \end{pmatrix} \begin{matrix} L_2 - \frac{5}{6}L_1 \\ \sim \\ L_3 - \frac{8}{6}L_1 \\ L_4 - \frac{4}{6}L_1 \end{matrix}$

$\begin{pmatrix} 6 & 2 & 3 & -2 \\ 0 & \frac{4}{3} & \frac{9}{2} & -\frac{13}{3} \\ 0 & -\frac{8}{3} & -9 & \frac{26}{3} \\ 0 & -\frac{10}{3} & -9 & \frac{19}{3} \end{pmatrix} \begin{matrix} L_3 + 2L_2 \\ \sim \\ L_4 + \frac{10}{4}L_2 \end{matrix} \begin{pmatrix} 6 & 2 & 3 & -2 \\ 0 & \frac{4}{3} & \frac{9}{2} & -\frac{13}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{63}{2} & \frac{84}{3} \end{pmatrix}$

$$\sim \begin{pmatrix} 6 & 2 & 3 & -2 \\ 0 & \frac{4}{3} & \frac{9}{2} & -\frac{13}{2} \\ 0 & 0 & -\frac{63}{2} & \frac{84}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ rang $A = 3$ (pe o diagonală avem 3 valori $\neq 0$)

c.) $\begin{pmatrix} 2 & \alpha & -5 \\ \beta & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix} = A$, $\alpha, \beta \in \mathbb{R}$

$\det A = 0$ $\quad (L_3 = 0)$

$a_{11} = 2 \neq 0$

$d_2 = \begin{vmatrix} 2 & \alpha \\ \beta & 3 \end{vmatrix} = 6 - \alpha \cdot \beta$

Dacă $\boxed{\text{I.}}$ $\boxed{\alpha \cdot \beta = 6} \Rightarrow d_2 = 0$

$d_3 = \begin{vmatrix} \alpha & -5 \\ 3 & -1 \end{vmatrix} = -\alpha + 15 = 15 - \alpha$

C_1: $\alpha = 15 \Rightarrow d_3 = 0$

$\Rightarrow$ rang $A = 1$

C_2. $\alpha \neq 15 \Rightarrow d_3 \neq 0 \Rightarrow$ rang $A = 2$

$\boxed{\text{II}}$ $\left.\begin{array}{l} \boxed{\alpha \cdot \beta \neq 6} \\ \text{Cum } \det A = 0 \end{array}\right\} \Rightarrow$ rang $A = 2$

③ Să se rezolve sistemele:

a) $\begin{cases} x+2y+3z=1 \\ 2x+3y+6z=2 \end{cases}$

$$\bar{A}=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 6 & 2 \end{array}\right)$$

$$d_1=\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}=3-4=-1\neq 0$$

$$d_2=\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}=6-6=0$$

$$d_3=\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}=0$$

$\Rightarrow$ sist. compatibil simplu nedeterminat

$$\begin{cases} x+2y=1-3z \quad \Big| \cdot(-2) \\ 2x+3y=2-6z \end{cases}$$

$$\Rightarrow \begin{array}{r} -2x-4y=-2+6z \\ 2x+3y=2-6z \\ \hline \cancel{} \quad -y=0 \end{array} \quad \oplus$$

$$y=0$$

$$x=1-3z \Rightarrow S=\{(1-3z,0,z)|$$

$z \in \mathbb{R}\}$

b.) $\begin{cases} x+y+z-2t=5 \\ 2x+y-2z+t=1 \\ 2x-3y+z+2t=3 \end{cases}$

$$\bar{A}=\left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 3 \end{array}\right)$$

$a_{11}=1\neq 0$

$d_1=\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}=1-2=-1\neq 0$

$d_2=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix}=1(1-6)-1(2+4)+$
$+1(-6-2)$

$=-5-6-8=-19\neq 0$

$\nexists$ min. carac.

$\Rightarrow$ sist comp simplu nedeterminat

$\begin{cases} x+y+z=5+2t \\ 2x+y-2z=1-t \\ 2x-3y+z=3-2t \end{cases}$

$$d_x = \begin{vmatrix} 5+2t & 1 & 1 \\ 1-t & 1 & -2 \\ 3-2t & -3 & 1 \end{vmatrix}$$

$$= (5+2t)(1-6) - (1-t+6-4t)$$
$$+ (-3+3t-3+2t)$$

$$= 5-30\cancel{+2t}\cancel{-12t}-1\cancel{+t}-6\cancel{+4t}-3$$
$$\cancel{+3t}-3\cancel{+2t}$$

$$= -38$$

$$\boxed{x} = -\frac{38}{-19} = \boxed{2}$$

$$d_y = \begin{vmatrix} 1 & 5+2t & 1 \\ 2 & 1-t & -2 \\ 2 & 3-2t & 1 \end{vmatrix} =$$

$$= 1-t+6-4t-(5+2t)(2+4)$$
$$+(6-4t-2+2t)$$

$$= 7-5t-30-12t+4-2t$$

$$= -19t-19$$

$$\boxed{y} = \frac{d_y}{d_z} = \frac{-19t-19}{-19} = \boxed{t+1}$$

$$d_z = \begin{vmatrix} 1 & 1 & 5+2t \\ 2 & 1 & 1-t \\ 2 & -3 & 3-2t \end{vmatrix} =$$

$$= 3-2t+3-3t-(6-4t-2+2t)$$
$$+(5+2t)(-6-2)$$
$$= 6-5t-4+2t-40-16t$$
$$= -19t-38$$

$$\boxed{z} = \frac{d_z}{d_1} = \frac{-19t-38}{-19} = \boxed{t+2}$$

$$S = \{(2, t+1, t+2, t) \mid t \in \mathbb{R}\}$$

c)
$$\begin{cases} x+y+z=2 \\ 2x-y-2z=-2 \\ x+4y+5z=8 \\ 2x-5y+6z=10 \end{cases}$$

$$\tilde{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & -2 & -2 \\ 1 & 4 & 5 & 8 \\ 2 & -5 & 6 & 10 \end{array}\right)$$

$L_2 - 2L_1$

$\sim$ $L_3 - L_1$; $L_4 - 2L_1$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -4 & -6 \\ 0 & 3 & 4 & 6 \\ 0 & -7 & 4 & 6 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -4 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{40}{3} & 20 \end{array}\right)$$

$L_3 + L_2$

$L_4 - \frac{7}{3}L_2$

$$\widetilde{L_4 \longleftrightarrow L_3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -4 & -6 \\ 0 & 0 & \frac{40}{3} & 20 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Longrightarrow \quad \frac{40}{3} z = 20$$

$$z = \frac{\cancel{20} \cdot 3}{\cancel{40}_2} \qquad \boxed{z = \frac{3}{2}}$$

$$-3y - \overset{2}{\cancel{4}} \cdot \frac{3}{\cancel{2}} = -6$$

$$-3y-6=-6 \implies \boxed{y=0}$$

$$x = 2 - \frac{3}{2}$$

$$\boxed{x = \frac{1}{2}} \implies S = \left\{ \left(\frac{1}{2}, 0, \frac{3}{2} \right) \right\}$$

c.) $\begin{cases} (1+\lambda)x + y + z = 1 \\ x + (1+\lambda)y + z = \lambda \\ x + y + (1+\lambda)z = \lambda^2 \end{cases}$

$$\bar{A} = \left(\begin{array}{ccc|c} 1+\lambda & 1 & 1 & 1 \\ 1 & (1+\lambda) & 1 & \lambda \\ 1 & 1 & (1+\lambda) & \lambda^2 \end{array} \right)$$

$$\det A = \begin{vmatrix} 1+\lambda & 1 & 1 \\ 1 & 1+\lambda & 1 \\ 1 & 1 & 1+\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} 3+\lambda & 1 & 1 \\ 3+\lambda & 1+\lambda & 1 \\ 3+\lambda & 1 & 1+\lambda \end{vmatrix}$$

$$= (3+x)\cdot\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+x \end{vmatrix}$$

$$= (3+x)\cdot\left[(1+x)^2 - 1 - \left((\cancel{1}+x) - \cancel{1}\right) + (\cancel{1}-\cancel{1}-x)\right]$$

$$= (3+x)\left[(1+x)^2 - 1 - x - x\right]$$

$$= (3+x)\left(\cancel{1} + \cancel{2x} - \cancel{2x} + x^2 - \cancel{1}\right)$$

$$= x^2(3+x)$$

I. $\det A \neq 0$

$\Rightarrow$ sist. comp. det.

II. $\det A = 0$

$\Rightarrow x^2(3+x) = 0$

$\Rightarrow x = 0$ sau $x = -3$

C1: $x = 0$

$$\left.\begin{aligned} x+y+z&=1 \\ x+y+z&=0 \end{aligned}\right\} \quad S_1 = \emptyset$$

C_2: $\lambda = -3$

$$\begin{cases} -2x + y + z = 1 \\ x - 2y + z = -3 \\ x + y - 2z = 9 \end{cases}$$

$a_{11} = -2 \neq 0$

$d_1 = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 = 3 \neq 0$

$$m_{car_1} = \begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & -3 \\ 1 & 1 & 9 \end{vmatrix}$$

$$= -2(-18+3) - 1(9+3) + 1(1+2)$$

$$= -2 \cdot (-15) - 12 + 3$$

$$= 30 - 9 = 21 \neq 0$$

$\Rightarrow$ sist. incompatibil