Exerciții suplimentare

① Fie $x = (-1,2,3)$ și $y = (-2,1,-3) \in \mathbb{R}^3$.

a) Det. $r > 0$ a.î. $y \notin B(x,r)$

$B(x,r) = \{z \in \mathbb{R}^3 : d(x,z) < r\}$

$y \notin B(x,r) \Rightarrow d(x,y) \geq r \Rightarrow \|x-y\| \geq r \Rightarrow \sqrt{(-1+2)^2+(2-1)^2+(3+3)^2} \geq r \Rightarrow$

$\sqrt{38} \geq r \quad \Rightarrow r \in (0, \sqrt{38}]$

b) Det. $t \in \mathbb{R}$ a.î. $\underbrace{(1,-1,t)}_{v} \in \overline{B}(x,5)$

$\Rightarrow d(x,v) \leq 5$

$\sqrt{(-1-1)^2+(2+1)^2+(3-t)^2} \leq 5$

$\sqrt{13+(3-t)^2} \leq 5 \Rightarrow 13+(3-t)^2 \leq 25$

$(3-t)^2 \leq 12 \Rightarrow |3-t| \leq 2\sqrt{3}$

$-2\sqrt{3} \leq 3-t \leq 2\sqrt{3} \quad |-3$

$-3-2\sqrt{3} \leq -t \leq 2\sqrt{3}-3 \quad |\cdot(-1)$

$t \leq 3+2\sqrt{3}$ și $t \geq 3-2\sqrt{3}$

$t \in [3-2\sqrt{3}, 3+2\sqrt{3}]$

② Fie $x,y \in \mathbb{R}^m$. Dem. că:

a) $x \cdot y = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$

$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}\left((x+y)(x+y) - (x-y)(x-y)\right) =$

$= \frac{1}{4}(x\cdot x + 2xy + y\cdot y - x\cdot x + 2xy - y\cdot y) =$

$= \frac{1}{4}\cdot 4\cdot x\cdot y = x\cdot y$

b) $\left|\, \|x\| - \|y\| \,\right| \leq \|x-y\|$

Din inegalitatea Δ avem că:

$$\|x+y\| \le \|x\| + \|y\|$$

$$\|x\| = \|x+y-y\| = \|(x-y)+y\| \overset{i.\Delta}{\le} \|x-y\| + \|y\| \Rightarrow$$

$$\|x\| - \|y\| \le \|x-y\|$$

$$\|y\| = \|(y-x)+x\| \le \|y-x\| + \|x\| \Rightarrow$$

$$\|y\| - \|x\| \le \|y-x\| = \|x-y\| \quad | \cdot (-1)$$

$$\|x\| - \|y\| \ge -\|x-y\|$$

$$\Rightarrow \quad -\|x-y\| \le \|x\| - \|y\| \le \|x-y\| \Rightarrow$$

$$\big| \|x\| - \|y\| \big| \le \|x-y\|$$

3 Doi vectori $x, y \in \mathbb{R}^m$ s.m. <u>ortogonali</u> dacă $x \cdot y = 0$. Justificați că:

$x, y \in \mathbb{R}^m$ ortogonali $\Leftrightarrow \|x-y\|^2 = \|x\|^2 + \|y\|^2$

$$\|x-y\|^2 = (x-y)\cdot(x-y) = x\cdot x - 2xy + y\cdot y =$$

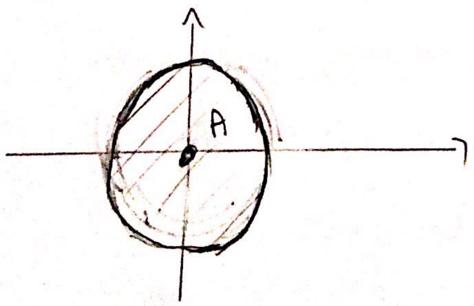
$$= \|x\|^2 + \|y\|^2 - 2\cdot x\cdot y$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 \quad \Leftrightarrow$$

$\Leftrightarrow x\cdot y = 0 \Leftrightarrow x, y$ ortogonali

4 Det. $\text{int} A$, $\text{Fr} A$, precum și dacă A e închisă sau deschisă.

a) $A = \overline{B}(0_2, 1) \setminus \{0_2\} \subseteq \mathbb{R}^2$



$\text{int} A = B(0_2, 1) \setminus \{0_2\}$

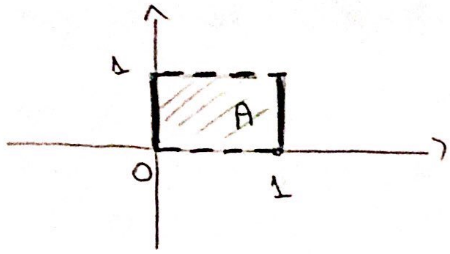
$\text{Fr} A = \{x \in \mathbb{R}^2 \mid \|x\| = 1\} \cup \{0_2\}$

$A \cap \text{Fr} A \ne \emptyset$

2 $\text{Fr} A \not\subseteq A$

$\Rightarrow$ A nici deschisă, nici închisă

b) $A = [0,1] \times (0,1) \subseteq \mathbb{R}^2$



$\text{int}\, A = (0,1) \times (0,1)$

$\text{Fr}\, A = (\{0\} \times [0,1]) \cup ([0,1] \times \{1\}) \cup (\{1\} \times [0,1]) \cup [0,1] \times \{0\})$

$\left.\begin{array}{l} A \cap \text{Fr}\, A \neq \emptyset \\ \text{Fr}\, A \not\subseteq A \end{array}\right| \Rightarrow$ A nu e nici deschisă, nici închisă.

c) $A = \mathbb{Q} \times \mathbb{Q} \subseteq \mathbb{R}^2$

$\text{int}\, A = \emptyset$ $\quad A \cap \text{Fr}\, A \neq \emptyset$ $\Rightarrow$ A nu e nici deschisă, nici închisă.

$\text{Fr}\, A = \mathbb{R}^2$ $\quad \text{Fr}\, A \not\subseteq A$

⑤ $\forall A \subseteq \mathbb{R}^m$ mulțime nevidă, au loc afirmațiile:

a) $A' \subseteq A \cup \text{Fr}\, A$

Fie $x \in A' \Rightarrow \forall r>0: B(x,r) \cap (A \setminus \{x\}) \neq \emptyset$, și mai mult

$\Rightarrow \forall r>0: B(x,r) \cap A \neq \emptyset$

Pp. că $x \notin A$. Vom arăta că $x \in \text{Fr}\, A$.

Avem din faptul că $x \in A'$ că $\forall r>0: B(x,r) \cap A \neq \emptyset$.

Cum $x \notin A \Rightarrow \forall r>0, B(x,r) \not\subseteq A$, deci $\forall r>0\ B(x,r) \cap (\mathbb{R}^m \setminus A) \neq \emptyset$

$x \in \text{Fr}\, A$. ☑

b) $\mathrm{int}\, A \cap \mathrm{int}(\mathbb{R}^m \setminus A) = \emptyset$

Pp. prin absurd că $\exists x \in \mathrm{int}\, A \cap \mathrm{int}(\mathbb{R}^m \setminus A)$. Atunci $x \in \mathrm{int} A$ și $x \in \mathrm{int}(\mathbb{R}^m \setminus A)$.

$x \in \mathrm{int}\, A \Rightarrow \exists r > 0$ a.î. $B(x,r) \subseteq A \Rightarrow B(x,r) \cap (\mathbb{R}^m \setminus A) = \emptyset$,

contradicție cu $x \in \mathrm{int}(\mathbb{R}^m \setminus A)$, deci $B(x,r) \cap (\mathbb{R}^m \setminus A) \neq \emptyset$.

Așadar $\mathrm{int}\, A \cap \mathrm{int}(\mathbb{R}^m \setminus A) = \emptyset$.

c) $\mathrm{Fr}\, A = \mathrm{Fr}(\mathbb{R}^m \setminus A)$

$$\left.\begin{array}{l} \mathrm{Fr}\, A = \mathbb{R}^m \setminus (\mathrm{int}\, A \cup \mathrm{int}(\mathbb{R}^m \setminus A)) \\ \mathrm{Fr}(\mathbb{R}^m \setminus A) = \mathbb{R}^m \setminus [\mathrm{int}(\mathbb{R}^m \setminus A) \cup \underbrace{\mathrm{int}(\mathbb{R}^m \setminus (\mathbb{R}^m \setminus A))}_{= \mathrm{int}\, A}] \end{array}\right\} \Rightarrow \text{„="}$$

d) $\mathrm{int}\, A = A \setminus \mathrm{Fr}\, A$

Fie $x \in \mathrm{int}\, A \Rightarrow \exists r > 0$ a.î. $B(x,r) \subseteq A$, deci $B(x,r) \cap (\mathbb{R}^m \setminus A) = \emptyset$.

$\Downarrow$

$x \notin \mathrm{Fr}\, A$

Cum $\mathrm{int}\, A \subseteq A \Rightarrow x \in A$.

Deci $x \in A \setminus \mathrm{Fr}\, A$.

Fie $x \in A \setminus \mathrm{Fr}\, A$. $\Rightarrow \exists r > 0$ a.î. $B(x,r) \cap A = \emptyset$ sau $B(x,r) \cap (\mathbb{R}^m \setminus A) = \emptyset$.

$$\left.\begin{array}{l} \text{și } x \in A. \\ x \in B(x,r) \end{array}\right\} \Rightarrow B(x,r) \cap A \neq \emptyset \Rightarrow$$

$\Rightarrow B(x,r) \cap (\mathbb{R}^m \setminus A) = \emptyset \Rightarrow B(x,r) \subseteq A \Rightarrow x \in \mathrm{int}\, A$.

⑦ Fie $x = (x_1, \dots, x_m) \in \mathbb{R}^m$. nr. real pozitiv

$$\|x\|_m \overset{not}{=} |x_1| + |x_2| + \dots + |x_m|$$

s.m. <u>norma Minkowski</u> a vectorului x. Ar. că verifică propr. normei euclidiene.

1°. $\|x\|_m = 0 \iff x = 0_m$

$\|x\|_m = |x_1| + |x_2| + \dots + |x_m|$

Cum $|a| \geq 0$, avem că $\|x\|_m = 0 \iff |x_i| = 0, \forall i = \overline{1,m}$,

deci $x_i = 0, \forall i = \overline{1,m}$, deci $x = 0_m$.

2°. $\|\alpha x\|_m = |\alpha| \cdot \|x\|_m$, $\alpha \in \mathbb{R}$.

$\|\alpha x\|_m = |\alpha x_1| + |\alpha x_2| + \dots + |\alpha x_m| =$

$= |\alpha| \cdot |x_1| + |\alpha| \cdot |x_2| + \dots + |\alpha| \cdot |x_m| =$

$= |\alpha| \cdot (|x_1| + \dots + |x_m|)$

$= |\alpha| \cdot \|x\|_m$

3°. $\|x+y\|_m \leq \|x\|_m + \|y\|_m$

$\|x+y\|_m = |x_1+y_1| + |x_2+y_2| + \dots + |x_m+y_m| \leq$

$\leq |x_1| + |y_1| + \dots + |x_m| + |y_m| =$

$= (|x_1| + \dots + |x_m|) + (|y_1| + \dots + |y_m|)$

$= \|x\|_m + \|y\|_m$