Seminarul 3

Criteriul Stolz-Cesaro

Fie $(a_n)_{n\in\mathbb{N}}$ un șir oarecare de nr. reale și $(b_n)_{n\in\mathbb{N}}$ un șir strict monoton și divergent.

Dacă $\exists \lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} = l \in \overline{\mathbb{R}}$ atunci $\exists \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = l$.

① Fie $(x_n)_{n\in\mathbb{N}}$ un șir cu termeni strict pozitivi. Dacă $\exists \lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = l$, atunci $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = l$. Reciproca e adevărată?

* În Stolz-Cesaro luăm $a_n = \ln x_n$ ($\exists$, deoarece (x_n) are termeni strict pozitivi)

$b_n = n$, crescător și divergent.

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\ln(x_{n+1}) - \ln x_n}{n+1-n} = \lim_{n\to\infty} \ln \frac{x_{n+1}}{x_n} =$$

$$= \ln\left(\lim_{n\to\infty} \frac{x_{n+1}}{x_n}\right) = \ln l.$$

Deci $\exists \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \ln l \Rightarrow \lim\limits_{n\to\infty} \dfrac{\ln x_n}{n} = \ln l \Rightarrow$

$$\lim_{n\to\infty} \ln x_n^{\frac{1}{n}} = \ln l \Rightarrow \ln\left(\lim_{n\to\infty} \sqrt[n]{x_n}\right) = \ln l \Rightarrow$$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = l,\ l \in [0, +\infty]$$

Convenție: $\ln 0 = -\infty$, $\ln \infty = \infty$.

* Reciproca NU e adevărată.

Fie $x_n = e^{(-1)^n}$. Evident are termeni >0.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{e^{(-1)^{n+1}}}{e^{(-1)^n}} = \lim_{n\to\infty} e^{(-1)^{n+1}-(-1)^n} = \begin{cases} e^{-2}, & n \text{ par} \\ e^{2}, & n \text{ impar} \end{cases}, \text{ deci } \nexists$$

Totuşi, $\lim_{n\to\infty} \sqrt[n]{e^{(-1)^n}} = \lim_{n\to\infty} e^{(-1)^n\cdot\frac{1}{n}\to 0} = e^0 = 1$, care ∃.

② Calculaţi limita şirurilor:

a) $y_n = \dfrac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n}$

$a_n = 1+\frac{1}{2}+\dots+\frac{1}{n}$

$b_n = \ln n \nearrow$ cu $\lim_{n\to\infty} b_n = +\infty$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln\frac{n+1}{n}} = \lim_{n\to\infty} \frac{1}{\ln\left(1+\frac{1}{n}\right)^{n+1}} =$$

$$= \cancel{\lim_{n\to\infty}} \frac{1}{\ln e} = 1.$$

Deci, conform S-C $\lim_{n\to\infty} y_n = 1$.

b) $y_n = \sqrt[n]{n!}$

Folosim Ex.1. cu $x_n = n!$, $x_n > 0$.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = +\infty. (=l)$$

Deci $\lim_{n\to\infty} \sqrt[n]{x_n} = l \Rightarrow \lim_{n\to\infty} \sqrt[n]{n!} = +\infty$.

c) $y_n = \dfrac{\sqrt[n]{n!}}{n}$

$y_n = \sqrt[n]{\dfrac{n!}{n^n}}$.

Folosim Ex.1 cu $x_n = \dfrac{n!}{n^n}$, $x_n > 0$.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{(n+1)}!}{(n+1)^{\cancel{n+1}\,n}} \cdot \frac{n^n}{\cancel{n!}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n =$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}.$$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = \frac{1}{e}$, deci $\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e}$.

a) $y_n = \sqrt[n]{n}$ (din Sem. 2)

$x_n = n$. $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \frac{n+1}{n} = 1 \overset{Ex.1}{\Rightarrow} \lim\limits_{n\to\infty} \sqrt[n]{x_n} = \lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

(3) Pt. șirurile de mai jos

$a_n = \sum\limits_{k=1}^{n} \frac{1+(-1)^k}{2}$, $b_n = n$, $\forall n \in \mathbb{N}^*$

calculați valoarea limitelor $\lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ și $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$.

Contrazice acest lucru criteriul Stolz-Cesaro?

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\frac{1+(-1)}{2} + \dots + \frac{1+(-1)^n}{2} + \frac{1+(-1)^{n+1}}{2} - \frac{1+(-1)}{2} - \dots - \frac{1+(-1)^n}{2}}{n+1-n}$$

$$= \lim_{n\to\infty} \frac{1+(-1)^{n+1}}{2} = \begin{cases} 0, & n \text{ par} \\ 1, & n \text{ impar} \end{cases}$$

deci $\nexists$ limita.

$$\cancel{\lim} \quad a_n = \begin{cases} \overbrace{0+1+\dots+0+1}^{n/2 \text{ ori}}, & n \text{ par} \\ \underbrace{0+1+\dots+0}_{n-1/2 \text{ ori}}, & n \text{ impar} \end{cases} \qquad \frac{1+(-1)^k}{2} = \begin{cases} 0, & k \text{ impar} \\ 1, & k \text{ par} \end{cases}$$

$$\lim_{k\to\infty} \frac{a_{2k}}{b_{2k}} = \lim_{k\to\infty} \frac{\frac{2k}{2}}{2k} = \frac{1}{2}$$

$$\lim_{k\to\infty} \frac{a_{2k+1}}{b_{2k+1}} = \lim_{k\to\infty} \frac{\frac{2k+1-1}{2}}{2k+1} = \frac{1}{2}$$

deci $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$ și $\exists$.

Așadar $\exists \lim \frac{a_n}{b_n} \not\Rightarrow \exists \lim \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$, în

consecință Reciproca T. Stolz-Cesaro NU e adevărată.

④ Scrieți următoarele serii cu ajutorul simbolului suma:

a) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$

$$S=\sum_{k=0}^{\infty}\frac{1}{2k+1}$$

b) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$

$$S=\sum_{k=0}^{\infty}\frac{1}{2^k}$$

c) $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\ldots$

$$S=\sum_{k=1}^{\infty}(-1)^{k+1}\cdot\frac{1}{k^2}$$

⑤ Calculați suma următoarelor serii:

a) $\sum_{n=0}^{\infty}\frac{1}{n!}=e$ (Seminar 2)

b) $\sum_{n=1}^{\infty}\frac{1}{5^n}=\frac{1}{5}+\frac{1}{5^2}+\ldots=\underbrace{1+\frac{1}{5}+\frac{1}{5^2}+\ldots}_{\text{seria geom. de rație } \frac{1}{5}}-1\overset{*}{=}\frac{1}{1-\frac{1}{5}}-1=$

$$=\frac{5}{4}-1=\frac{1}{4}$$

(*) $\sum_{n=0}^{\infty}a^n=1+a^1+a^2+\ldots=\frac{1}{1-a}$, dacă $a\in(-1,1)$

c) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n-1}}$

$$S_n=\sum_{k=1}^{n}\frac{1}{\sqrt{k}+\sqrt{k-1}}=\frac{1}{\sqrt{1}+\sqrt{0}}^{\sqrt{1}-\sqrt{0}}+\frac{1}{\sqrt{2}+\sqrt{1}}+\ldots+\frac{1}{\sqrt{n-1}+\sqrt{n-2}}+\frac{1}{\sqrt{n}+\sqrt{n-1}}=$$

$$=\frac{\cancel{\sqrt{1}}-\sqrt{0}}{1-0}+\frac{\cancel{\sqrt{2}}-\cancel{\sqrt{1}}}{2-1}+\ldots+\frac{\cancel{\sqrt{n-1}}-\cancel{\sqrt{n-2}}}{n-1-n+2}+\frac{\sqrt{n}-\cancel{\sqrt{n-1}}}{n-n+1}=\sqrt{n}$$

Deci $S=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n-1}}=\lim_{n\to\infty}\sqrt{n}=\infty.$

d) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$.

$$S_k = \sum_{k=1}^{n} \frac{1}{4k^2-1}$$

$$\frac{1}{4k^2-1} = \frac{1}{(2k-1)(2k+1)} = \frac{(2k+1)-(2k-1)}{(2k-1)(2k+1)} \cdot \frac{1}{2} =$$

$$= \frac{1}{2} \cdot \frac{1}{2k-1} - \frac{1}{2} \cdot \frac{1}{2k+1}$$

$$S_k = \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \dots + \cancel{\frac{1}{2n-3}} - \cancel{\frac{1}{2n-1}} + \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)$$

$$= \frac{1}{2} \cdot \left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}.$$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}.$$

e) $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$

$$S_k = \sum_{k=2}^{n} \ln\left(1-\frac{1}{k^2}\right) = \ln\left(1-\frac{1}{4}\right) + \ln\left(1-\frac{1}{9}\right) + \dots + \ln\left(1-\frac{1}{n^2}\right) =$$

$$= \ln\left[\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{9}\right)\cdot \dots \cdot \left(1-\frac{1}{n^2}\right)\right] =$$

$$= \ln\left(\frac{3}{4} \cdot \frac{8}{9} \cdot \dots \cdot \frac{n^2-1}{n^2}\right) = \ln\left(\frac{(2-1)(2+1)}{2^2} \cdot \frac{(3-1)(3+1)}{3^2} \cdot \dots \cdot \frac{(n-2)n}{(n-1)^2} \cdot \frac{(n-1)(n+1)}{n^2}\right)$$

$$= \ln\left[\frac{1\cdot\cancel{3}}{2^2} \cdot \frac{\cancel{2}\cdot\cancel{4}}{\cancel{3^2}} \cdot \frac{\cancel{3}\cdot\cancel{5}}{\cancel{4^2}} \cdot \frac{\cancel{4}\cdot 6}{\cancel{5^2}} \cdot \dots \cdot \frac{(n-2)\cdot\cancel{n}}{\cancel{(n-1)^2}} \cdot \frac{\cancel{(n-1)}(n+1)}{n^{\cancel{2}}}\right] =$$

$$= \ln\left(\frac{n+1}{2n}\right)$$

$$\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right) = \lim_{n\to\infty} \ln\left(\frac{n+1}{2n}\right) = \ln\frac{1}{2}.$$

f) $\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{(n+2)!}$

$$\frac{n \cdot 2^n}{(n+2)!} = \frac{(n+2-2) \cdot 2^n}{(n+2)!} = \frac{n+2}{(n+2)!} \cdot 2^n - \frac{2^{n+1}}{(n+2)!} = \frac{2^n}{(n+1)!} - \frac{2^{n+1}}{(n+2)!}$$

$$S_n = \sum_{k=1}^{n} \frac{k \cdot 2^k}{(k+2)!} = \sum_{k=1}^{n} \left[\frac{2^k}{(k+1)!} - \frac{2^{k+1}}{(k+2)!} \right] =$$

$$= \frac{2}{2!} - \cancel{\frac{2^2}{3!}} + \cancel{\frac{2^2}{3!}} - \cancel{\frac{2^3}{4!}} + \cancel{\frac{2^3}{4!}} - \cancel{\frac{2^4}{5!}} + \ldots + \cancel{\frac{2^{n-1}}{n!}} - \cancel{\frac{2^n}{(n+1)!}} + \cancel{\frac{2^n}{(n+1)!}} - \frac{2^{n+1}}{(n+2)!}$$

$$= 1 - \frac{2^{n+1}}{(n+2)!}$$

Am arătat în Sem. trecut că $\lim_{n \to \infty} \frac{2^n}{n!} = 0$

Deci $\lim_{n \to \infty} \frac{2^{n+1}}{(n+2)!} = 0$.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n \cdot 2^n}{(n+2)!} = 1.$$