Seminarul 11

① Calculați derivatele parțiale de ordinul I, gradientul ∇f, diferențiala df pt. funcțiile:

a) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2y^3 + y\sin x - 2z$

* Derivatele parțiale de ordinul I: $\frac{\partial f}{\partial x}(x,y,z)$, $\frac{\partial f}{\partial y}(x,y,z)$, $\frac{\partial f}{\partial z}(x,y,z)$

$$\frac{\partial f}{\partial x}(x,y,z) = 2xy^3 + y\cos x$$

$$\frac{\partial f}{\partial y}(x,y,z) = 3x^2y^2 + \sin x$$

$$\frac{\partial f}{\partial z}(x,y,z) = -2$$

! Obs Calculul derivatei parțiale în raport cu o variabilă se poate efectua utilizând regulile de derivare cunoscute, păstrând celelalte variabile ale funcției ca parametri constanți.

* Gradientul : $\nabla f(x^0) = \left(\frac{\partial f}{\partial x_1}(x^0), \dots, \frac{\partial f}{\partial x_m}(x^0)\right) \in \mathbb{R}^m$

$$\nabla f(x,y,z) = \left(2xy^3 + y\cos x,\ 3x^2y^2 + \sin x,\ -2\right)$$

* Diferențiala $df(x^0): \mathbb{R}^m \to \mathbb{R}$

$$df(x^0)(u) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(x^0) \cdot u_i \quad , \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m$$

$df(x,y,z): \mathbb{R}^3 \to \mathbb{R}$

$$df(x,y,z)(u_1,u_2,u_3) = \frac{\partial f}{\partial x}(x,y,z) \cdot u_1 + \frac{\partial f}{\partial y}(x,y,z)u_2 + \frac{\partial f}{\partial z}(x,y,z) \cdot u_3$$

$$= (2xy^3 + y\cos x)u_1 + (3x^2y^2 + \sin x)u_2 - 2u_3$$

b) $f: (0,\infty)^2 \to \mathbb{R}$, $f(x,y) = \operatorname{arctg} \frac{x-y}{x+y}$

- $$\frac{\partial f}{\partial x}(x,y) = \frac{1}{\left(\frac{x-y}{x+y}\right)^2+1} \cdot \left(\frac{x-y}{x+y}\right)'_x = \frac{1}{\frac{(x-y)^2+(x+y)^2}{(x+y)^2}} \cdot \frac{x+y-x+y}{\cancel{(x+y)^2}} =$$

$$= \frac{2y}{(x^2+y^2)\cdot 2} = \frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{\left(\frac{x-y}{x+y}\right)^2+1} \cdot \left(\frac{x-y}{x+y}\right)'_y = \frac{1}{\frac{2(x^2+y^2)}{\cancel{(x+y)^2}}} \cdot \frac{-x-y-x+\cancel{y}}{\cancel{(x+y)^2}} =$$

$$= \frac{-2x}{2(x^2+y^2)} = \frac{-x}{x^2+y^2}$$

- $$\nabla f(x,y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$$

- $$df(x,y)(u_1,u_2) = \frac{y}{x^2+y^2}\cdot u_1 + \left(\frac{-x}{x^2+y^2}\cdot u_2\right) =$$

$$= \frac{y}{x^2+y^2}\cdot u_1 - \frac{x}{x^2+y^2}\cdot u_2$$

c) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x\sqrt{x^2+y^2}$

- $$\frac{\partial f}{\partial x}(x,y) = \sqrt{x^2+y^2} + x\cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = x\cdot \frac{2y}{2\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}} \qquad \forall (x,y)\in \mathbb{R}^2\setminus\{0_2\}$$

- $$\nabla f(x,y) = \left(\frac{2x^2+y^2}{\sqrt{x^2+y^2}}, \frac{xy}{\sqrt{x^2+y^2}}\right)$$

- $$df(x,y)(u_1,u_2) = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}\, u_1 + \frac{xy}{\sqrt{x^2+y^2}}\, u_2 \qquad \forall (x,y)\in\mathbb{R}^2\setminus\{0_2\}$$

În $(0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{x\sqrt{x^2}-0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y-0} = 0$$

$$\nabla f(0,0) = (0,0)$$

$$df(0,0)(u_1,u_2)=0$$

(2) Arătați că fc. $f(x,y)=y\ln(x^2-y^2)$ verifică relația:

$$\frac{1}{x}\frac{\partial f}{\partial x}+\frac{1}{y}\cdot\frac{\partial f}{\partial y}=\frac{f}{y^2},\ \forall\ x>y>0.$$

$$\frac{\partial f}{\partial x}(x,y)=y\cdot\frac{1}{x^2-y^2}\cdot 2x$$

$$\frac{\partial f}{\partial y}(x,y)=\ln(x^2-y^2)-y\cdot\frac{2y}{x^2-y^2}$$

$$\frac{1}{x}\cdot\frac{\partial f}{\partial x}+\frac{1}{y}\cdot\frac{\partial f}{\partial y}=\cancel{\frac{2y}{x^2-y^2}}+\frac{\ln(x^2-y^2)}{y}-\cancel{\frac{2y}{x^2-y^2}}=\frac{\ln(x^2-y^2)}{y}$$

$$=\frac{f}{y^2}$$

(3) Studiați existența derivatelor parțiale în origine și a derivatelor după direcție în origine:

$$f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{x}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{0-0}{y-0}=0$$

$\Rightarrow \exists$ deriv. parțiale în O_2

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* Fie $f: A \to \mathbb{R}$ o fc., $x^0 \in A$ și $v \in \mathbb{R}^m$. Dacă $\exists$ limita
$A \subseteq \mathbb{R}^m$

$$\lim_{t \to 0} \frac{f(x^0 + tv) - f(x^0)}{t}$$

ea s.n. derivata lui f în x^0 după direcția vectorului v și se notează cu $f'_v(x^0)$. Dacă lim. e finită, atunci f este derivabilă în x^0 după direcția vectorului v.

$$\lim_{t \to 0} \frac{f((0,0) + t(v_1, v_2)) - f(0,0)}{t} = \lim_{t \to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{t^3 v_1^2 v_2}{t^4 v_1^4 + t^2 v_2^2} \cdot \frac{1}{t} = \lim_{t \to 0} \frac{\cancel{t^2} v_1^2 v_2}{\cancel{t^2}(t^2 v_1^4 + v_2^2)} = \lim_{t \to 0} \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2}$$

$$\frac{v_1^2 \cdot v_2}{v_2^2} = \begin{cases} 0, & v_2 = 0 \\ \dfrac{v_1^2}{v_2}, & v_2 \neq 0 \end{cases}$$

(4) Calculați derivatele parțiale ale fc. compuse $g \circ f$, unde $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (xe^y + xe^{-y}, xe^y - xe^{-y})$ și $g = g(u,v): \mathbb{R}^2 \to \mathbb{R}$ e o fc. oarecare de clasă C^1 pe $\mathbb{R}^2$.

O fc. $g: A \to \mathbb{R}$ s.n. de clasă C^1 în pct. $x^0 \in A$ dacă:

(1°) $\exists r > 0$ a.î. f derivabilă parțial în orice pct. al mulțimii $B(x^0, r) \cap A$.

(2°) Fc. $\frac{\partial f}{\partial x_i}: B(x^0, r) \cap A \to \mathbb{R}$ sunt cont. în x^0, $\forall i = \overline{1,m}$.

$$\nabla(g \circ f)(x^0) = \nabla g(f(x^0)) \cdot J(f)(x^0)$$

$$J(f)(x^0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x^0) & \dots & \frac{\partial f_1}{\partial x_m}(x^0) \\ \vdots & & \\ \frac{\partial f_p}{\partial x_1}(x^0) & \dots & \frac{\partial f_p}{\partial x_m}(x^0) \end{pmatrix}$$

↳ matricea Jacobi

$f = (f_1, \dots, f_p): A \to \mathbb{R}^p$, $A \subseteq \mathbb{R}^m$ deschisă; $g: B \to \mathbb{R}$ (4), $B \subseteq \mathbb{R}^p$ deschisă.

$$\frac{\partial}{\partial x_i}(g\circ\varphi)(x^0)=\sum_{k=1}^{p}\frac{\partial g}{\partial u_k}\cdot(\varphi(x^0))\cdot\frac{\partial f_k}{\partial x_i}(x^0)\qquad i=\overline{1,m}$$

$\varphi = (x_1, \dots, x_m)$; $g = g(u_1, \dots, u_p)$

$$\left(\frac{\partial}{\partial x}(g\circ\varphi)(x,y),\ \frac{\partial}{\partial y}(g\circ\varphi)(x,y)\right)=$$

$$=\left(\frac{\partial g}{\partial u}(\varphi(x,y)),\ \frac{\partial g}{\partial v}(\varphi(x,y))\right)\cdot\underbrace{\begin{pmatrix} e^y+e^{-y} & xe^y-xe^{-y} \\ e^y-e^{-y} & xe^y+xe^{-y}\end{pmatrix}}_{J(\varphi)(x,y)}$$

$$J(\varphi)(x,y)=\begin{pmatrix}\frac{\partial \varphi_1}{\partial x}(x,y) & \frac{\partial \varphi_1}{\partial y}(x,y)\\ \frac{\partial \varphi_2}{\partial x}(x,y) & \frac{\partial \varphi_2}{\partial y}(x,y)\end{pmatrix}=\begin{pmatrix} e^y+e^{-y} & xe^y-xe^{-y} \\ e^y-e^{-y} & xe^y+xe^{-y}\end{pmatrix}$$

$$\Rightarrow \frac{\partial}{\partial x}(g\circ\varphi)(x,y)=\frac{\partial g}{\partial u}(\varphi(x,y))\left(e^y+e^{-y}\right)+\frac{\partial g}{\partial v}(\varphi(x,y))\cdot\left(e^y-e^{-y}\right)$$

$$\frac{\partial}{\partial y}(g\circ\varphi)(x,y)=\frac{\partial g}{\partial u}(\varphi(x,y))\cdot\left(xe^y-xe^{-y}\right)+\frac{\partial g}{\partial v}(\varphi(x,y))\cdot\left(xe^y+xe^{-y}\right)$$

⑤ Exprimați ecuația

$$u\frac{\partial g}{\partial u}(u,v)+v\frac{\partial g}{\partial v}(u,v)=\sqrt{u^2+v^2}\ ,\ \forall (u,v)\in(0,\infty)^2$$

în variabilele $(x,y)\in(0,\infty)\times(0,\frac{\pi}{2})$, efectuând transformarea $u = x\cos y$, $v = x\sin y$. Det. apoi o fc. de clasă C^1 ce verifică relația respectivă.

$$(*)\quad x\cdot\cos y\cdot\frac{\partial g}{\partial u}(x\cos y, x\sin y)+x\sin y\,\frac{\partial g}{\partial v}(x\cos y, x\sin y)=\sqrt{x^2(\cos^2 y+\sin^2 y)}$$

$$=\sqrt{x^2\cdot 1}=|x|=x$$

Fie $\varphi(x,y) = (x\cos y, x\sin y)$

$G = g\circ\varphi$

$$\frac{\partial G}{\partial x}(x,y) = \frac{\partial g}{\partial u}(\varphi(x,y))\cdot\frac{\partial \varphi_1}{\partial x}(x,y) + \frac{\partial g}{\partial v}(\varphi(x,y))\cdot\frac{\partial \varphi_2}{\partial x}(x,y)$$

$$= \frac{\partial g}{\partial u}(x\cos y, x\sin y)\cdot \cos y + \frac{\partial g}{\partial v}(x\cos y, x\sin y)\cdot \sin y$$

Din (*) avem $x\cos y\,\frac{\partial g}{\partial u}(x\cos y, x\sin y) + x\sin y\,\frac{\partial g}{\partial v}(x\cos y, x\sin y) = x \quad |:x$

$$\Rightarrow \cos y\,\frac{\partial g}{\partial u}(x\cos y, x\sin y) + \sin y\,\frac{\partial g}{\partial v}(x\cos y, x\sin y) = \frac{\partial G}{\partial x}(x,y) = 1$$

$$\Rightarrow G(x,y) = \int 1\,dx = x + C(y)$$

$$g(\varphi(x,y)) = x + C(y)$$

$$g(x\cos y, x\sin y) = x + C(y)$$

$u = x\cos y$

$v = x\sin y$ $\Rightarrow g(u,v) = \sqrt{u^2+v^2} + C$

$C = ?$

$$\frac{u}{v} = \operatorname{ctg} y \Rightarrow \frac{v}{u} = \operatorname{tg} y \Rightarrow y = \operatorname{arctg}\left(\frac{v}{u}\right)$$

$$\Rightarrow g(u,v) = \sqrt{u^2+v^2} + C\left(\operatorname{arctg}\left(\frac{v}{u}\right)\right)$$

⑥ Calculați derivatele parțiale de ordinul 2 ale fc:

a) $f:(1,\infty)\times\mathbb{R}\to\mathbb{R}$ $f(x,y)=\ln(x+y^2-1)$

$$\boxed{\frac{\partial^2 f}{\partial x^2}(x,y)\,;\ \frac{\partial^2 f}{\partial x\partial y}(x,y)\,;\ \frac{\partial^2 f}{\partial y\partial x}(x,y)\,;\ \frac{\partial^2 f}{\partial y^2}(x,y)}$$

$$\frac{\partial f}{\partial x}(x,y)=\frac{1}{x+y^2-1}=(x+y^2-1)^{-1}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y)=(-1)(x+y^2-1)^{-2}$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)=-(x+y^2-1)^{-2}\cdot 2y$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{2y}{x+y^2-1}=2y(x+y^2-1)^{-1}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=2(x+y^2-1)^{-1}-2y(x+y^2-1)^{-2}\cdot 2y$$

$$=(x+y^2-1)^{-2}(2x+2y^2-2-4y^2)$$

$$=(x+y^2-1)^{-2}(2x-2y^2-2)$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y)=-2y(x+y^2-1)^{-2}$$

b) $f:\mathbb{R}\times(0,\infty)\to\mathbb{R},\ f(x,y)=xye^{\frac{x}{y}}$

$$\frac{\partial f}{\partial x}(x,y)=ye^{\frac{x}{y}}+xy\cdot e^{\frac{x}{y}}\cdot\left(\frac{x}{y}\right)'_x=ye^{\frac{x}{y}}+xy\cdot e^{\frac{x}{y}}\cdot\frac{1}{y}=$$

$$=(x+y)e^{\frac{x}{y}}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = e^{\frac{x}{y}} + (x+y)\cdot e^{\frac{x}{y}}\cdot\frac{1}{y} = e^{\frac{x}{y}}\left(2+\frac{x}{y}\right)$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y) = e^{\frac{x}{y}} + (x+y)e^{\frac{x}{y}}\cdot\left(-\frac{x}{y^2}\right)$$

$$= e^{\frac{x}{y}}\left(1-\frac{x^2}{y^2}-\frac{x}{y}\right)$$

$$\frac{\partial f}{\partial y}(x,y) = xe^{\frac{x}{y}} + e^{\frac{x}{y}}\cdot xy\cdot\frac{-x}{y^2} =$$

$$= e^{\frac{x}{y}}\left(x-\frac{x^2}{y}\right)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = e^{\frac{x}{y}}\cdot\left(\frac{-x}{y^2}\right)\left(x-\frac{x^2}{y}\right) + e^{\frac{x}{y}}\cdot\frac{x^2}{y^2} =$$

$$= e^{\frac{x}{y}}\cdot\frac{x}{y^2}\left(-x+\frac{x^2}{y^2}+x\right) =$$

$$= e^{\frac{x}{y}}\cdot\frac{x^3}{y^4}$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y) = e^{\frac{x}{y}}\cdot\frac{1}{y}\left(x-\frac{x^2}{y}\right) + e^{\frac{x}{y}}\left(1-\frac{2x}{y}\right) =$$

$$= e^{\frac{x}{y}}\left(\frac{x}{y}-\frac{x^2}{y^2}+1-\frac{2x}{y}\right) = e^{\frac{x}{y}}\left(1-\frac{x^2}{y^2}-\frac{x}{y}\right)$$

! __Criteriul lui Schwarz__: Dacă $f: A \to \mathbb{R}$ e o fc. de clasă C^2 în $x^0 \in A$, atunci $\frac{\partial^2 f}{\partial x_j \partial x_i}(x^0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x^0)$ $\forall\, \overline{i,j} = \overline{1,m}$, $i \neq j$

(vezi curs pt. def. fc. de clasă C^2)

În ex. anterior: $\frac{\partial^2 f}{\partial x\partial y}(x,y) = \frac{\partial^2 f}{\partial y\partial x}(x,y)$