Analytical-Numerical Method of Analysis and Functional Method of Building of Non-linear Dynamic Systems Models

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***Abstract*— An analytical-numerical method for solving non-linear integro-differential equation systems, that is based on the application of generalized functions, whose regular components are described by the Taylor series, is proposed. The procedure for determing the parameters of oscillatory regimes in deterministic, non-linear, non-autonomous mathematical models with lumped parameters is described. Within the framework of the "black box" principle on establishing the input-output relationship of an object the functional models of non-linear compensators for a power amplifier are constructed and the results of the models comparison are represented.**

Keywords— non-linear model; dynamic system; analytical-numerical method; neural network; non-linear compensator

I. Introduction

Designing the constructive model of a studied object or phenomenon is extremely complicated and many-sided task. The analytical-numerical method of the mutual mathematical modelling and calculation of dynamic systems [1], as well as the methods of complex devices modelling, based on establishing input-output mapping [1]–[5], significantly expand the mathematical modelling possibilities of complex objects.

The analytical-numerical method is intended for the analysis and parametric synthesis of deterministic, non-linear, non-autonomous models with lumped parameters. According to this method, we vary the Taylor polynomial degree and adapt the calculation step on analyzing the model, as well as manage the limits of local and total absolute calculation errors [1]. The proposed method replys to the following main questions:

- What necessary and sufficient complication of dynamic model mathematical description, the research character and mathematical means chosen for these researches admit.
- What form of the mathematical description of required solutions is expedient and preferable.
- What way to choose the existence and uniqueness of the required solutions of the model dynamics equation should be examined in a given time-interval; also the possibility of the solution obtainment with the help of chosen mathematical means should be cleared up.

One of advanced methods of complicated device modelling is based on mathematical description connecting only the input and output variables of a device. This approach is very useful, for instance, in the following situations. Firstly, taking into account of device component structure the mathematical model of complicated device is described by the equation system with high order. The problem of bad conditionality is emerged on solving this equation system. Secondly, for lack of sufficient information about the device components it is impossible to design the model as the equation system. In the mentioned cases to apply the model describing the input-output relation is the effective way of the complicated device modelling [1]–[5].

II. Matrix Form of Model in Analitical-Numerical Method

Let us describe the mathematical models of deterministic dynamic systems taking into account the non-autonomy and nonlinearity of the objects being modeled, the discontinuities of the first kind for exciters, the discontinuities of the first and second kinds for the phase space coordinates and the derivatives of them. Random changes in the model parameters and external exciters, as well as the distributed model parameters are excluded from consideration.

The dynamics of the distinguished class of models is described by ordinary, non-linear, non-autonomous, integro-differential equations with non-stationary coefficients and deterministic right sides. The equation system is reduced to the following constructive matrix form, in which all the phase space coordinates of the model have a certain meaning and are accessible to observation [1]:

$$\mathbf{A}(D, D^{-1})\,\mathbf{X}(t) = \mathbf{G}(D, D^{-1})\,\mathbf{F}(t) + \mathbf{H}(\mathbf{X}, \mathbf{F}, t), \qquad (1)$$

where D is the operator of generalized differentiation with respect to independent variable t; D^{-1} is the operator of integration with respect to t up to variable higher limit of t, lower limit is pre-initial instant in every integration interval; $\mathbf{A}(D, D^{-1})$ is the quadratic matrix of size $L_x \times L_x$ with polynomial elements $a_{l,k}(D)$; $\mathbf{G}(D, D^{-1})$ is the matrix of size $L_x \times L_f$ with polynomial elements $g_{l,k}(D)$; $\mathbf{X}(t)$ and $\mathbf{F}(t)$ are the column matrices of the phase space coordinates of the model and external exciters, correspondingly; $\mathbf{H}(\mathbf{X}, \mathbf{F}, t)$ is the column matrix with elements being the sum of the products of the following factors: time, non-stationary coefficients, the classical derivatives of arbitrary order and the integrals of arbitrary multiplicity, starting from zero, with respect to the phase space coordinates (required solutions) and external exciters (it should be mentioned that all the multipliers have arbitrary fractional-rational powers).

Polynomials $a_{l,k}(D)$, $g_{l,k}(D)$ are formed according to expressions:

$$a_{l,k}(D) = \sum_{m=-M}^{M} a_{l,k}^{[m]}(t) D^m ,$$

$$g_{l,k}(D) = \sum_{m=-M}^{M} g_{l,k}^{[m]}(t)\, D^m ,$$

where m is the derivative order (under $m > 0$) or the integral multiplicity (under $m < 0$) with respect to variable *t*. Some or all coefficients of the represented polynomials can be zero.

By definition [1], the row str_u with number u, $u \in [1,\ L_x]$ in matrix $\mathbf{H}(\mathbf{X},\ \mathbf{F},\ t)$ is written as

$$\mathrm{str}_u\ \mathbf{H}(\mathbf{X}, \mathbf{F}, t) = \sum_{v=1}^{V_u} t^{k_{u,v}} h_{u,v}(t) \times$$
$$\times \prod_{l=1}^{L} \prod_{n=-N_{u,v}}^{N_{u,v}} \left(D^{\,n} x_l(t) \right)^{k_{u,v}^{\,l,n}} , \qquad (2)$$

where $L = L_x + L_f$; $k_{u,v}, k_{u,v}^{l,n} \in Q$, Q is the set of rational numbers; $N_{u,v} \in Z$, Z is the set of integers.

In equation (2), the enumeration of the phase space coordinates and exciters is ordered so that

$$x_l(t) = \begin{cases} x_l(t) \text{ at } \forall\ l \in [1,\ L_x], \\ f_l(t) \text{ at } \forall\ l \in [L_x + 1,\ L]. \end{cases}$$

The lower limit of the integration operators D^{-n}, $n \in \left[-N_{u,v},\ N_{u,v} \right]$ in expression (2) (in contrast to operators also indicated in matrices $\mathbf{A}(D, D^{-1})$, $\mathbf{G}(D, D^{-1})$) is the initial time instant in every integration interval.

III. Computational Procedure of Analitical-Numerical Method

Dynamic processes proceed in the phase space of coordinates accessible to observation and registration. In general case, these processes characterize the following [1]: the discontinuities of the first kind for the differentiable phase coordinates of the model and external exciters; the discontinuities of the second kind for the phase space coordinates; the time intervals alternation of the fast and slow changes of the phase space coordinates; the instability of the phase space coordinates in finite or semi-infinite time intervals.

In order to take correctly into account the nonsmoothness of processes in dynamic systems, for example, on differentiating discontinuities, the solutions of equation (1) must be sought in the class of generalized functions, in the form of the sums of singular and regular components [1]:

$$x_l(t) = x_l^-(t) + x_l^+(t) = \sum_{j=0}^{-J_l} S_{l,j}\ \delta_j(t) + \sum_{i=0}^{\infty} R_{l,i}\ t^i / i!\ , \qquad (3)$$

where $x_l^-(t)$, $x_l^+(t)$ are the singular and regular components of solution $x_l(t)$, accordingly; $\delta_j(t)$ is an impulse function; $S_{l,j}$ are the weight coefficients of impulse functions, defined at the initial point with abscissa $t = 0^+$ of the considered calculation interval; $R_{l,i}$ are the coefficients of a power series with the expansion center, existing at the same point.

The analytical-numerical method has been developed to find the solution of equation (1) in the class of generalized functions (3) with regular components, described by a power series. The computational procedure of the method includes analytical and numerical parts.

The essence of the analytical-numerical method consists in replacing dynamic systems with deterministic, non-autonomous, non-linear models with lumped non-stationary parameters and then searching for equations solutions of their dynamics in the form of generalized functions, whose regular components are described by the Taylor series.

The method procedure with analytical and numerical parts consists in the step-by-step (with estimation) construction of dynamic processes in mathematical models.

The higher bound for absolute total error $|\Delta x_l^+(t_k, I_l)|$ on calculating the approximate solution value $x_l^+(t_k)$ is formed in the following form

$$|\Delta x_l^+(t_k, I_l)| = (1 + h_k \chi_{l,k})\, |\Delta x_l^+(0^+, I_l)| + |\Delta x_l^+(h_k, I_l)|\ , \qquad (4)$$

where $\chi_{l,k}$ is a coefficient, that takes into account the inaccuracy of initial conditions in the calculation step t_k, $k \geq 2$.

Using the approximate solution value and computed estimation (4), the region, which contains the unknown exact value of the regular solution component of equation (1), is constructed.

IV. Operator Approximation Based on Input-Output Relationship

Complicated device can be generally represented as non-linear dynamic system, whose input-output relationship is described by non-linear operator F_S. On modelling non-linear dynamic system, it is required to approximate the non-linear operator F_S by the non-linear operator F_ε, which maps the input set X in the output set Y^o on producing error ε, $\varepsilon > 0$, so that $y(n) = F_\varepsilon[x(n)]$ under condition $\left\| y^o(n) - y(n) \right\| \le \varepsilon$ for all $x(n) \in X$, $y^o(n) \in Y^o$. The parameters of the non-linear operator F_ε (mathematical model) are resulted from soluting the approximation problem, as a rule, in the uniform norm or root-mean-square one [1], [4], [5].

The way of solving the approximation problem (1) depends on the form of mathematical model. The universal forms of non-linear model can be distinguished into two classes. The first class includes polynomial models, such as the Volterra series and the Volterra polynomial, multidimensional polynomials of split signals, regression models [1], [4]–[6]. The second class of non-linear models is composed of different types of neural networks [1]–[3]. Neural networks are useful when increasing the polynomial degree causes slowly reducing the error of the operator approximation.

Non-linear models based on the operator approach is built for different devices, for instance, filters, compensators, detectors, equalizers, etc. [1]–[6], as well as the compensators of non-linear signal distortions in power amplifiers included in communication channels (CC) [1], [7]. The power amplifier is linear in narrow-band and low-power applications. However, the power amplifier operates with width-band signals and as close as possible to saturation to achieve maximum power efficiency and output power. The power amplifier nonlinearity leads to the signal distortion and appearance of the intermodulation components in adjacent channel that cannot be filtered out. The process of out-of-band emission is known as spectral regrowth [7].

The bandpass filter and power amplifier form the Wiener structure (the cascade connection of a linear dynamic circuit and an memoryless nonlinearity) describing the communication channel. The methods of noiseless coding do not provide the required level of information transfer reliability. It is expedient to apply the compensators synthesized according to the operational approach on sets of input and output signals to cancel non-linear signal distortions in CC [7].

Let's execute the compensation of the non-linear signal distortions in CC described by the Wiener model [1]

$$\dot{x}(n) = d_1\dot{\xi}_1(n) + d_2\dot{\xi}_1^2(n) + d_3\dot{\xi}_1^3(n),$$

where $d_1 = 1$, $d_2 = 0.2$, $d_3 = 0.1$; $"\cdot"$ is the sign of complexity; $\dot{x}(n)$ is the output signal of non-linear CC at the input signal $\dot{\xi}(n)$, which is the complex envelope of modulated signal; $\dot{\xi}_1(n)$ is the output signal of the linear dynamic part of CC with the following transfer function

$$H(z) = (1,0119 - 0,7589j) + (-0,3796 + 0,5059j) \cdot z^{-1}.$$

8PSK-signal and 4QAM-signal are used as input signals $\dot{\xi}(n)$ of the CC model.

For suppressing non-linear distortions, compensators are synthesized as the following models:

- The Volterra polynomial (VP) [1], [4], [5]

$$\dot{y}(n) = \sum_{i_1=0}^{I_1} \sum_{i_2=0}^{I_2} \ldots \sum_{i_{m/2}=0}^{I_{m/2}} \sum_{i_{m/2+1}=0}^{I_{m/2+1}} \ldots$$
$$\ldots \sum_{i_m=0}^{I_m} \dot{C}_{i_1 i_2 \ldots i_{m/2} i_{(m/2+1)} \ldots i_m} \dot{x}^{i_1}(n)\, \dot{x}^{i_2}(n-1) \ldots$$
$$\ldots \dot{x}^{i_{m/2}}(n - m/2) \left[\dot{x}^*(n)\right]^{i_{m/2+1}} \ldots \left[\dot{x}_m^*(n - m/2)\right]^{i_m},$$

where $\dot{y}(n)$ is the output signal of the compensator model, $I = I_1 + I_2 + \ldots + I_{m/2} + I_{(m/2+1)} + \ldots + I_m$ is the polynomial degree ($I = 3$), $*$ is the sign of complex conjugation, $m/2$ is the memory length ($m = 10$).

- The two-layer perceptron (TLP) [1]–[3]

$$\dot{y}(n) = G\left(c_0 + \sum_{i=1}^{I} c_i\, G\left(\sum_{j=0}^{m} w_{ij} \dot{x}_j(n)\right)\right),$$

where the model input signal is a vector

$$\begin{aligned} &[\, \dot{x}_0(n), \dot{x}_1(n), \dot{x}_2(n), \ldots, \dot{x}_m(n) \,] = \\ &= [\, 1, \dot{x}(n), \dot{x}(n-1), \ldots, \dot{x}(n-(m-1)) \,], \end{aligned} \tag{5}$$

G is a sigmoid function (hyperbolic tangent), I is the number of neurons ($I = 5$), $(m-1)$ is the memory length ($m = 5$).

- The recurrent neural Hammerstein network (RNHN) [1]

$$\dot{y}(n) = \sum_{r_b=0}^{R_b} b_{r_b}\, n\dot{e}t^{(2)}(n - r_b) - \sum_{r_a=1}^{R_a} a_{r_a}\, \dot{y}(n - r_a),$$

where

$$n\dot{e}t^{(2)}(n) = \sum_{k=0}^{I} c_k n\dot{e}t_k^{(1)}(n)\,,\ n\dot{e}t_0^{(1)}(n) = 1\,,$$

$$n\dot{e}t_k^{(1)}(n) = G\left(\dot{u}_k^{(1)}(n)\right),\ \dot{u}_k^{(1)}(n) = \sum_{l=0}^{m} w_{kl}\dot{x}_l(n)\,,\ k = 1,2,...,I\,,$$

$\dot{x}_0(n) = 1$, the input signal of the compensator model is a vector described by (5), G is a sigmoid function (hyperbolic tangent), I is the neurons number ($I = 3$), $(m-1)$ is the memory length ($m = 2$), $R_b = 1$, $R_a = 1$.

The root-mean-square error of non-linear compensation is calculated on the basis of the following equation

$$\varepsilon = \frac{1}{998}\sqrt{\sum_{n=3}^{1000}\left|\dot{y}(n) - \dot{\xi}(n)\right|^2}\,.$$

The root-mean-square error and the number of the compensator model parameters are represented in Table 1.

TABLE I. RESULTS OF COMPENSATORS MODELLING

Model	Error of Modelling and Number of Model Parameters	
	Error	*Number of parameters*
VP	$0.2*10^{-3}$	286
TLP	$0.6*10^{-3}$	36
RNHN	$0.2*10^{-3}$	16

On the basis of represented results it is possible to draw the conclusion that the neural compensator as the recurrent neural Hammerstein network used for cancelling the non-linear signal distortions in CC with the Wiener structure exceeds the two-layer perceptron in the accuracy of the signal processing, as well as the Volterra polynomial in the implementation simplicity.

V. CONCLUSION

The analytical-numerical method used for the analysis of deterministic, non-linear, non-autonomous models with lumped parameters has the following advantages:

- The equations solutions of the model dynamics are described by generalized functions with regular components in the form of the Taylor series (polynomial).
- The existence and uniqueness of the solution of equation (1) and the possibility of obtaining it on the basis of the chosen mathematical approach are proved within the framework of the Taylor series.
- The procedure for definding the parameters of oscillatory regimes in dynamic models has been developed, as well as the regularity and stability of these regimes has been investigated.

Higher bounds, obtained at the discrete instants of the assigned time interval, for the absolute local, relative local and absolute total errors of calculation enable to construct the regions containing the unknown exact solutions of equation (1) and satisfying various estimates of the limit errors of calculation.

Models in the form of neural networks that describe unique mapping between the sets of the input and output signals of dynamic systems are various. Accounting for the example of the synthesis of the nonlinear signal distortion compensator for a power amplifier, it is shown that among different types of neural networks one can find such a network that provides the high accuracy of signal processing and is simpler than polynomial models.

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