# The Use of ADAR Method and Theory of Optimal Control for Optimal Control Systems Synthesis

A. A. Kolesnikov[1], A. A. Kuzmenko[2]
Institute of Computer Technology and Information Security, Southern Federal University
ankolesnikov@sfedu.ru[1], aakuzmenko@sfedu.ru[2]

***Abstract*— This report compares the known method of Analytical Design of Aggregated Regulators (ADAR) with method of Analytical Design of Optimal Regulators (ADOR). It is shown that ADAR method has significant advantages associated with simpler procedure of analytical design of nonlinear laws of optimal control, clear physical representation of weighting factors of optimality criteria, and stability of closed-loop optimal systems.**

***Key words— nonlinear control systems; control synthesis; ADOR method; synergetic control theory; ADAR method***

## I. Introduction

Along with Pontryagin's maximum principle, ADOR methods even today forms the most important part of theory and practice of synthesis of optimal control systems. However, it needs to be mentioned that these methods turned out to be applicable mostly for linear objects and quadratic functionals: Letov-Kalman criterion, functional of generalized work (FGW), weighted generalized work criterion (WGWC), etc. With the large scale of even linear objects, the solution to the ADOR problem meets certain difficulties connected with numerical solution of the Ricatti equations [1]. Furthermore, solving the ADOR problem for nonlinear objects has principal difficulties of finding solutions to the nonlinear partial differential equations. That is what accounts for the small number of publications with analytical synthesis (without using the numerical techniques) of laws of control of nonlinear objects via ADOR methods. In existing publications, solution of problem of synthesis of optimal control is preceded by linearization of initial nonlinear mathematical model of the object, or use of predictive linear models and further application of numerical techniques of solving nonlinear partial differential equations. Nevertheless, the interest to the practical application of ADOR methods is still high: thus, the search for «LQR or LQG» in Scopus database only for the 2000-2016 period gives you more than 5000 unique links to the scientific works.

Let's note, that the problems of ADOR mentioned above are worsen by the problem of choice of weighting factors of optimizing functionals, which still did not receive a solution acceptable for designers: the choice of these coefficients is not preliminary determined by clear physical recommendations. If in linear case the quadratic criteria coefficient fit might be somehow organized by using computer to simulate the transient processes in the closed-loop linear system, such approach is mathematically incorrect in nonlinear case. This can be explained by the well-known fact that that for nonlinear systems the principle of superposition is not valid, and their behavior, unlike linear ones, essentially depends on the initial conditions. Considering that, if we will choose some weighting factors of respective criteria, received as the result of computer simulation of nonlinear system and valid at first sight, we can receive practically inoperable nonlinear system, since its behavior may change significantly if given different initial data. Surprisingly, this common fact is effaced in literature sources, despite its influence in the synthesized system behavior is principal. Apparently, the ideology of the classical linear control theory continues to dominate in the minds of many researchers. Currently, the popular field of the optimal control theory is the based on the model predictive control [2–4]. As for the discrete control systems, it is possible to solve many problems of optimal control basing on polyhedral methodology [5].

Modern science has shown, that behavior of nonlinear systems can be most adequately shown not by simulating of its transient process in time, but, primarily, by phase portraits, invariant manifolds, and attractors in their phase space. At the present time, it becomes obvious that we have to move on to new views on the problem of analytical design of regulators, relying on the basic concepts of modern nonlinear dynamics and synergetics: invariant manifolds, attractors, self-organization, asymptotic properties of synthesized systems, etc. [6]. This allows to effectively solve the problem of structure forming and choosing the weighting factors of optimizing functionals that usually have the accompanying nature.

## II. The Problem of Control in ADOR and ADAR Methods

Suppose the object of control is described with vector-matrix differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u}, \tag{1}$$

where $\mathbf{x} = (x_1, ..., x_n)^T$, $\mathbf{u} = (u_1, ..., u_m)^T$ are the vectors of phase coordinates and control; $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x}))^T$ is a vector function; $\mathbf{G}(\mathbf{x}) = (g_{ij}(\mathbf{x}))_{n\times m}$ is a functional matrix.

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The ADOR problem is formulated as follows [7]: find the control law $\mathbf{u}=\mathbf{u}(\mathbf{x})$, which takes the object (1) from any initial state $\mathbf{x}(0)=\mathbf{x}_0$ to the origin of the phase space $\mathbf{x}=0$, ensuring the asymptotic stability of the closed-loop system and affords a minimum to functional

$$J=\int_0^\infty \left(\mathbf{F_0}(\mathbf{x})+\langle \mathbf{u},\mathbf{Du}\rangle\right)dt,$$

where $\mathbf{F_0}(\mathbf{x})$ is a positive-definite function of an $\mathbf{x}$; $\langle , \rangle$ is a dot product of vectors, and $\mathbf{D}=\operatorname{diag}(d_{ij})_{m\times m}$ is a diagonal matrix.

The ADOR problem solving procedure is based on the method of R. Bellman's dynamic programming. Key moments of this procedure are the choice of respective Lyapunov's function and the solving of algebraic or differential equations of Ricatti type.

The problem of Analytical Design of Aggregated Regulators (ADAR) is formulated as follows [6, 7]: it is required to determine such control vector $\mathbf{u}=\mathbf{u}(\mathbf{x})$, which takes the object's representation point (RP) from random initial state (within some acceptable region) first to the $\psi_k(\mathbf{x})=0$ manifolds, and then ensures its motion along them to the specific state (to the origin of coordinates $\mathbf{x}=0$ in particular). Given that, the RP's movement must satisfy the system of basic functional equations of ADAR method:

$$T_k\dot{\psi}_k(t)+\phi_k(\psi_k)=0. \qquad (2)$$

Functions $\phi_k(\psi_k)$ are selected in order to ensure the desired control criteria of RP's movement to the attractor manifolds $\psi_k(\mathbf{x})=0$, besides the asymptotic stability (2). In simplest case, $\phi_k(\psi_k)=\psi_k$. In general case, system's RP moves from one manifold to another until it either reaches the final manifold which is defined by desired object's invariant (1) in case of scalar control, or final manifolds which are defined by desired object's invariants (1) in case of vector control. ADAR problem presented above can also be defined in terms of optimal control: functional equations (2) are the equations of Euler-Lagrange for the following generalized accompanying functional (GAF):

$$J_0=\int_0^\infty \left[\phi_k^2(\psi_k)+T_k^2\dot{\psi}_k^2(t)\right]dt. \qquad (3)$$

Thus, synthesized by ADAR method, $\mathbf{u}=\mathbf{u}(\mathbf{x})$ control vector ensures the GAF's minimum, i.e. is optimal according to the GAF. However, unlike in ADOR method, the functional (3) plays secondary (accompanying) role and it is not directly involved in procedure of analytical synthesis of control vector.

Let's illustrate with particular examples the comparison of ADOR and ADAR methods, which are historically following one another and possessing both a definite analogy and a significant difference in approaches to the essence and procedures of analytical synthesis of control laws.

## III. Comparative Examples of Synthesis of Control Laws by ADOR and ADAR Methods

Let's analyze the problem of synthesis of laws of third-order object optimal control

$$\dot{x}_1(t)=x_2;\quad \dot{x}_2(t)=bf(x_1)+x_3;\quad \dot{x}_3(t)=u. \qquad (4)$$

The work [8] analyzes the problem of optimizing the object control system (4) with $b=0$, i.e

$$\dot{x}_1(t)=x_2;\quad \dot{x}_2(t)=x_3;\quad \dot{x}_3(t)=u. \qquad (5)$$

Equations (5) at a first approximation describe, in particular, the process of aerodynamic braking during ballistic entry of an Earth artificial satellite. The problem is to synthesize an autopilot which is optimal upon criterion

$$J_1=\int_0^\infty \left(q_1x_1^2+q_2x_2^2+q_3x_3^2+u^2\right). \qquad (6)$$

In [8] coefficient values $p_k$ of the autopilot equation are given,

$$u_1=-p_1x_1-p_2x_2-p_3x_3, \qquad (7)$$

which are calculated by numerical solution of the Ricatti equation for different combinations of weighting factors $q_i$ of control criterion (6) and respective transient processes are designed.

Let's synthesize the object control laws (5) basing on the ADAR method's ideology in its first simplest version, i.e. using one attractor manifold. To do that, first, let's introduce a linear macrovariable

$$\psi_1=x_3+\beta_1x_1+\beta_2x_2. \qquad (8)$$

Then, applying $\psi_1$ (8) to functional equation (2) with $\psi_k=\psi_1, \varphi_k(\psi_k)=\psi_1, T_k=T_1$, taking into account the object equations (5), we find the control law

$$u_1 = u = -(\beta_1 / T_1)x_1 - \\ -\left(\beta_1 + \beta_2 / T_1\right)x_2 - \left(\beta_2 + 1/T_1\right)x_3. \quad (9)$$

This implies, that control (7) will be equivalent (9) when selecting

$$p_1 = \beta_1 / T_1, \quad p_2 = \beta_1 + \beta_2 / T_1, \quad p_3 = \beta_2 + 1/T_1. \quad (10)$$

Control $u_1$ (9), according to (2), in a time $(4 \div 5)T_1$ takes the system's RP into neighborhood of manifold $\psi_1 = 0$ (8), movement along which is described by the following equations:

$$\dot{x}_{1\psi}(t) = x_{2\psi}, \quad \dot{x}_{2\psi}(t) = -\beta_1 x_{1\psi} - \beta_2 x_{2\psi}.$$

It is obvious, that given the $\beta_1 > 0$, $\beta_2 > 0$, this system is stable, and, therefore, in case of $T_1 > 0$ the closed-loop system (5), (9) is also stable. Depending on chosen values of the $\beta_1$, $\beta_2$ coefficients, it is possible to ensure the desired time and nature of the transient processes in the closed-loop system. The fulfillment of (10) ratios means equivalence of system (5), (9) optimization with quadratic criterion $J_1$ (6) and GAF in form of

$$J_2 = \int_0^\infty \left[\psi_1^2 + T_1^2 \dot{\psi}_1^2(t)\right] dt, \quad (11)$$

where the $\psi_1$ macrovariable is determined by equation (8).

It needs to be underlined, that calculation of coefficients $p_i$ of the control law (7) in case of system optimization with criterion $J_1$ (6) demands the numerical solution of nonlinear Ricatti equation, whereas during optimization according to GAF $J_2$ (11) the simple analytical ratios are used during synthesis of the equivalent control law (9). The advantages of ADAR method enhance even more with increased order of control systems and occurrence of nonlinear components in the model of controlled object.

Now let's analyze the problem of synthesis of optimal system of control of nonlinear object (4), when $b > 0$ and $f(x_1) = \sin x_1$, i.e. the perturbation equations takes the form

$$\dot{x}_1(t) = x_2; \quad \dot{x}_2(t) = b \sin x_1 + x_3; \quad \dot{x}_3(t) = u. \quad (12)$$

The equations (12) describe, in particular, the move of mathematical pendulum at the highest unstable position, therewith $x_1$ is the angle the pendulum swings away from vertical; $x_2$ is the swinging speed; and $x_3$ is the torque on the pendulum. The problem is to stabilize the pendulum by the torque applied to it on the pendulum axis. The mentioned torque is developed by actuating mechanism represented by integrating factor. It is needed to find an equation $u(\mathbf{x})$ at the actuating mechanism's entry, which stabilizes the pendulum in the highest equilibrium position, i.e. $x_1 = 0$, and ensures the asymptotic stability of the system. We take the following criterion of optimal stabilization:

$$J_3 = \int_0^\infty \left[\psi_2^2 + T_2^2 \dot{\psi}_2^2(t)\right] dt, \quad (13)$$

where $\psi_2 = \beta_1 x_1 + \beta_2 x_2 + x_3$. Applying the ADAR method based on the functional equation in form of (2), when $\psi_k = \psi_2$, $\varphi_k(\psi_k) = \psi_2$, $T_k = T_2$, we find the control law:

$$u_2 = u = -(\beta_1 / T_2)x_1 - \beta_2 b \sin x_1 - \\ -\left(\beta_1 + \beta_2 / T_2\right)x_2 - \left(\beta_2 + 1/T_2\right)x_3, \quad (14)$$

affording a minimum to control criterion (13). Let's test the dynamic stability of the synthesized system, which equations of movement along the manifold $\psi_2 = 0$ are taking the form of

$$\dot{x}_{1\psi}(t) = x_{2\psi}, \quad \dot{x}_{2\psi}(t) = b \sin x_{1\psi} - \beta_1 x_{1\psi} - \beta_2 x_{2\psi}.$$

This implies the inequations $\beta_1 > b, \beta_2 > 0, T_2 > 0$, which are the conditions of asymptotic stability of the whole synthesized system.

Of course, if the parameter $b = 0$, then the equations of objects (5) and (12), as well as control laws (9) and (14) will match. Taking the equations of object (12) into account, the $J_3$ (13) takes form of

$$J_3 = \int_0^\infty \begin{bmatrix} \beta_1^2 x_1^2 + \left(\beta_2^2 + T_2^2 \beta_1^2 - 2\beta_1\right) x_2^2 + \\ + \left(1 + \beta_2^2 T_2^2 - 2\beta_1 T_2^2\right) x_3^2 + \\ + T_2^2 \left(b^2 \sin^2 x_1 + 2b \sin x_1 + u^2\right) \end{bmatrix} dt.$$

This means that for object (5) in case the ratios $q_1 = \beta_1^2 / T_2^2$, $q_2 = \beta_2^2 / T_2^2 + \beta_1^2 - 2\beta_1 / T_2^2$, $q_3 = 1/T_2^2 + \beta_2^2 - 2\beta_1$ are true, $J_1$ (6) and $J_3$ (13) criteria will precisely match, and, therefore the optimal transient processes in the synthesized system will be equivalent.

Fig. 1 and Fig. 2 shows the results of imitating the closed-loop system (12), (14) given the $b = 1$, $\alpha = 2$, $\beta_1 = \beta_2 = 4, T_2 = 1$.

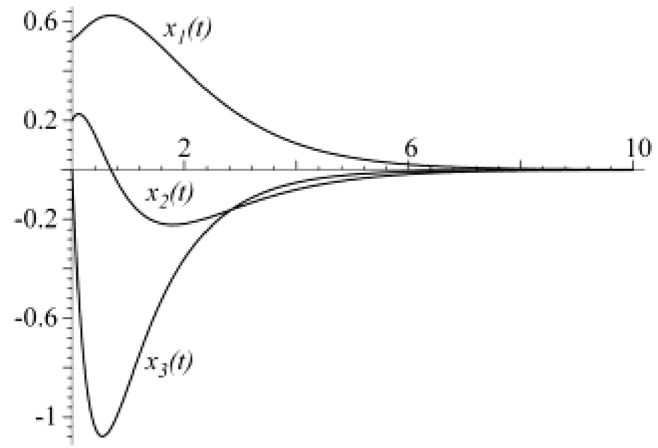


Fig. 1. Graph of change of state variables

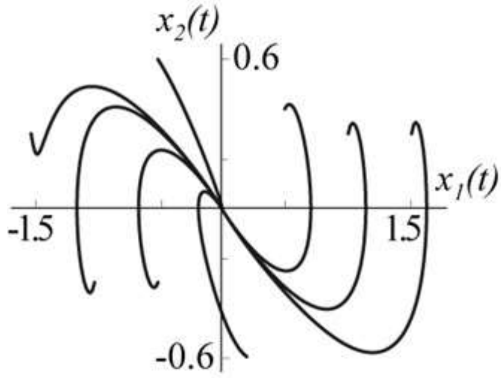


Fig. 2. Phase portrait of the system

## IV. Conclusion

In this work the ADAR and ADOR methods are compared using the particular examples of system synthesis. Both equivalence of these methods and the significant difference in approaches to the analytical synthesis of control laws are shown: unlike in ADOR methods, in ADAR method the optimizing functional is a constructed control criterion, which structure and parameters are defined by the control system designer basing on the physical properties of the object and engineering requirements towards the system.

Clear advantages of ADAR both in relation to procedures of analytical design of control laws for nonlinear objects, supportability and unambiguity of choice of regulator setting parameters, and in relation to ensuring the properties of asymptotic stability of the closed-loop systems are illustrated with examples of synthesis.

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