# New numerical method for finding the approximate solutions of coupled fractional order partial differential systems employing operational matrices approach

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***Abstract*—A new numerical method based on the operational matrices of orthogonal polynomial is developed to find the approximate solutions of fractional order partial differential equations (FOPDEs) and its coupled systems having mixed type partial derivative terms. The considered problem is transformed to an algebraic system of equations which are simple in handling by any computational software. Being easily solvable, the associated algebraic system leads to finding the solution of the problem. The accuracy and validity of the proposed method is checked by taking some examples.**

***Index Terms*—Operational matrices of fractional order; Generalized fractional coupled systems; Riemann-Liouville integral; Caputo derivative; Orthogonal Legendre polynomial.**

## I. Introduction

For the last few decades the subject fractional calculus (FC) has gotten considerable attention of the researchers round the globe due to its non-local behaviour. The mathematical modeling of many physical phenomena is based on the idea of FC, see for example [1]–[6]. Therefore due to its vast applications, a number of numerical techniques have been developed to find the approximate solutions of fractional order problems (FOPs), see for example [7]–[9].

For the past couple of years, orthogonal polynomials have been used extensively for obtaining the approximate solutions of FOPs, see for example [10], [11], [17]. These polynomials have ability to reduce FOPs into a system of algebraic equations that indeed makes finding the solution much easier.

Some certain type of fractional differential equations (FDEs) where the non-integer order is left arbitrary, have been solved numerically applying such ideas and types of orthogonal polynomials. Doha et al [13] developed a direct solution technique for linear multi-order FDEs with the aid of a shifted Jacobi tau approximation technique. Hammad and Khan [14] developed the operational matrices of fractional order derivatives and integrals of orthogonal Legendre polynomials to find the approximate solution of the coupled system of FOPDEs. Bhrawy and Zaky [15] developed an accurate spectral collocation method for solving one–and two-dimensional variable-order fractional nonlinear cable equations with the use of shifted Jacobi collocation procedure in conjunction with the shifted Jacobi operational matrix for variable-order fractional derivatives, described in the sense of Caputo. Brawy et al [16] proposed an efficient numerical scheme for time fractional diffusion-wave equations using Jacobi tau spectral procedure together with the Jacobi operational matrix for fractional integrals, described in the Riemann–Liouville sense. Saadatmandi and Dehghan [17] generalized the Legendre operational matrix to the fractional calculus to deal with the numerical solutions of a class of FDEs.

Motivated by the cited research work and vast applications of FDEs, we are encouraged to develop the new operational matrix of orthogonal Legendre polynomials in Caputo sense to study the numerical solutions of FOPDEs having mixed partial derivatives terms. To the best of our knowledge, the operational matrix for mixed partial derivative of Legendre polynomials in the sense of Caputo differential operator is not to be studied yet.

## II. Preliminaries

The unique definition of fractional differential operator is not available in the literature due to its fractional nature: the most commonly used definitions are proposed by Riemann–Liouvill (RL) and Caputo. The RL integral operator and Caputo differential operator are defined as under:

*Definition 2.1:* The RL integral is defined in [12] as:

$$J^{\beta}u_1(t)=\begin{cases}\frac{1}{\Gamma(\beta)}\int_a^t(t-\tau)^{\beta-1}u_1(\tau)d\tau, & \beta>0,\\ u_1(t), & \beta=0.\end{cases}\tag{1}$$

*Definition 2.2:* The Caputo derivative is defined as [12]:

$$\mathcal{D}^{\beta}u_1(t)=\begin{cases}\frac{d^m u_1(t)}{dt^m},\ \beta=m\in\mathbb{N},\\ \frac{1}{\Gamma(m-\beta)}\int_a^t\frac{u_1^{(m)}(t)}{(t-\tau)^{\beta+1-m}}d\tau,\\ m-1<\beta<m\ ,\ m\in\mathbb{N},\end{cases}\tag{2}$$

where $m$ is an integer, $t>0$, and $u_1(t)\in C^m[a,b]$.

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## III. Orthogonal Polynomials

This section elaborates the idea of orthogonal Legendre polynomials in two dimensions along-with its orthogonality relationship. The well known Legendre polynomial in one variable is defined as:

$$Q_j(x) = \sum_{l=0}^{j} \frac{(-1)^{j+l}\Gamma(j+k+1)}{\Gamma(j-l+1)(\Gamma(l+1))^2} x^l, \quad j = 1,2,3,.... \tag{3}$$

On the same fashion; the two–dimension Legendre polynomial of order $N$ defined on $[0,\Delta]\times[0,\Delta]$ is written as

$$\begin{aligned} &Q_m(x,y) = Q_j(x)Q_i(y), \\ &m = Nj+i+1,\ j = 0,1,2,...,n,\ i = 0,1,2,...,n. \end{aligned} \tag{4}$$

The orthogonality expression of $Q_m(x,y)$ is as following

$$\begin{aligned} &\int_0^{\Delta}\int_0^{\Delta} Q_b(x)Q_a(y)Q_d(x)Q_c(y)dxdy \\ &= \begin{cases} \frac{1}{(2b+1)(2a+1)} & \text{if } a=c, b=d \\ 0 & \text{Otherwise} \end{cases}. \end{aligned} \tag{5}$$

So, the function $u(x,y)$ defined and square integrable in the region $[0,\Delta]\times[0,\Delta]$ can be expressed in terms of the series of two–dimensional Legendre polynomials as follows

$$u(x,y) \approx \sum_{b=0}^{n}\sum_{a=0}^{n} D_{ba}(Q_b(x))(Q_a(y)), \tag{6}$$

$D_{ba}$ is computed using the following relation

$$D_{ba} = (2b+1)(2a+1)\int_0^{\Delta}\int_0^{\Delta} u(x,y)(Q_b(x))(Q_a(y)dxdy. \tag{7}$$

Equation (6) can also be expressed in the form of vector notation as

$$\begin{aligned} u(x,y) \approx \sum_{m=1}^{N^2} D_m Q_m(x,y) =& \hat{G}_{N^2}^T \hat{\Phi}_{N^2}(x,y), \\ m = Nb+a+1, \quad D_m = D_{ba}.& \end{aligned} \tag{8}$$

## IV. A new Operational matrix of orthogonal Legendre polynomials

In the subject numerical analysis operational matrices have particilar importance and used to solve many classical and fractional differential models. In this section, a new operational matrix of derivatives in Caputo sense is developed capable to solve the fractional order problems involving mixed partial derivative terms.

*Theorem 4.1:* Let $\hat{\Phi}_{N^2}(x,y)$ indicates the column vector of dimensions $N^2\times 1$, defined as

$$\hat{\Phi}_{N^2} = \begin{pmatrix} \hat{\phi}_{11} & \cdots & \hat{\phi}_{1N} & \hat{\phi}_{21} & \cdots & \hat{\phi}_{2N} & \cdots & \hat{\phi}_{NN} \end{pmatrix}^T,$$

Then mixed partial derivative of $\hat{\Phi}_{N^2}(x,y)$ is as follows

$$\frac{\partial^{\tau}}{\partial x^{(\tau/2)}\partial y^{(\tau/2)}}(\hat{\Phi}_{N^2}(x,y)) \simeq \hat{D}^{\tau,x,y}_{N^2\times N^2}\hat{\Phi}_{N^2}(x,y), \tag{9}$$

The operational matrix $\hat{D}^{\tau,x,y}_{N^2\times N^2}$ is given below

$$\hat{D}^{\hat{\tau},x,y}_{N^2\times N^2} = \begin{pmatrix} \daleth_{1,1,l} & \daleth_{1,2,l} & \cdots & \daleth_{1,q,l} & \cdots & \daleth_{1,l,N^2} \\ \daleth_{2,1,l} & \daleth_{2,2,l} & \cdots & \daleth_{2,q,l} & \cdots & \daleth_{2,l,N^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \daleth_{r,1,l} & \daleth_{r,2,l} & \cdots & \daleth_{r,q,l} & \cdots & \daleth_{r,l,N^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \daleth_{N^2,1,l} & \daleth_{N^2,2,l} & \cdots & \daleth_{N^2,q,l} & \cdots & \daleth_{N^2,l,N^2} \end{pmatrix}, \tag{10}$$

and $q = Nj+i+1$, $r = Nb+c+1$, $\daleth_{r,q,l,m} = \varpi_{j,i,b,c,l,m,T}$ for $j,i,b,c = 0,1,2,...,n$,

*Proof 4.1:* To prove the result take the fractional derivative of order $\tau$ of $Q_n(x,y)$ as defined in (4), the following relation is obtained

$$\begin{aligned} \frac{\partial^{\tau}}{\partial x^{(\tau/2)}\partial y^{(\tau/2)}}Q_n(x,y) &= \sum_{l=0}^{b}\Theta_{(b,l,\tau_1,\tau_2,T)} \\ \times\sum_{m=0}^{c}\Theta'_{(c,m,\tau_1,\tau_2,T)}&\mathcal{D}_x^{\tau/2}x^l\mathcal{D}_y^{\tau/2}y^m \end{aligned}$$

Applying Legendre series approximation and convolution theorem of Laplace transformation to the integrands involved in the orthogonality relationships of two dimensional Legendre polynomials and finally taking the inverse Laplace, the following relation is obtained

$$\begin{aligned} &\frac{\partial^{\tau}}{\partial x^{(\tau/2)}\partial y^{(\tau/2)}}Q_b(x)Q_c(y) = \\ &\sum_{j=0}^{n}\sum_{i=0}^{n}\varpi_{j,i,b,a,l,m,\Delta}Q_j(x)Q_i(y). \end{aligned} \tag{11}$$

The required result is obtained by making the use of the notations $q = Nj+i+1$, $r = Nb+c+1$, $\daleth_{r,q,l,m} = \varpi_{j,i,c,b,l,m,T}$ for $i,\ j,\ b,\ c = 0,1,2,3,...,n$.

## V. Application of the operational matrices for FOPDEs and its coupled systems involving mixed partial derivative terms

In this section, computer oriented numerical algorithm to find the approximate solution of FOPDEs and its coupled systems is studied. Consider

$$\begin{aligned} \frac{\partial^{\alpha_1}u_1(x,y)}{\partial x^{\alpha_1}} =& c_1\frac{\partial^{\gamma_1}u_1(x,y)}{\partial y^{\gamma_1}} + c_2\frac{\partial^{\zeta_1}u_1(x,y)}{\partial x^{\zeta_1/2}\partial y^{\zeta_1/2}} \\ &+ G_1(x,y), \end{aligned} \tag{12}$$

corresponding to the following initial condition:

$$u_1^{(j)}(0,y) = h_j(y),\ j = 0,1...,m, \tag{13}$$

The fractional derivatives are approximated by Legendre polynomials of order $N$ in two variables, such that

$$\frac{\partial^{\alpha_1}u_1(x,y)}{\partial x^{\alpha_1}} = \mathbf{X}_{N^2}\hat{\Phi}_{N^2}(x,y). \tag{14}$$

Applying Riemann–Liouville fractional integrals of order $\alpha_1$ on Eq. (14), and making use of Eq. (13), the following expression is obtained

$$u_1(x,y) = \mathbf{X}_{N^2}\mathbf{P}^{(\alpha_1,x)}_{N^2\times N^2}\hat{\Phi}_{N^2}(x,y) + \mathbf{F}^1_{N^2}\hat{\Phi}_{N^2}(x,y). \tag{15}$$

Now applying the operational matrices of derivatives in Caputo sense, the following system of algebraic equations is obtained

$$\begin{aligned}\mathbf{X}_{N^2} =&\mathbf{X}_{N^2}\mathbf{P}^{(\alpha_1,x)}_{N^2\times N^2}(a_1\mathbf{D}^{(\gamma_1,y)}_{N^2\times N^2}+a_2\mathbf{D}^{(\zeta_1,x,y)}_{N^2\times N^2})\\ &+\mathbf{F}^1_{N^2}(a_1\mathbf{D}^{(\gamma_1,y)}_{N^2\times N^2}+a_2\mathbf{D}^{(\zeta_1,x,y)}_{N^2\times N^2})+\mathbf{F}^2_{N^2}.\end{aligned} \tag{16}$$

The unknown matrix $\mathbf{X}_{N^2}$ can be easily computed by solving the above system of algebraic equations with any computational software.

On the same lines the algorithm can easily be extended to the coupled system of FOPDEs, as a result the following system of algebraic equations is obtained

$$\begin{aligned}\begin{pmatrix}\mathbf{X}_{N^2} & \mathbf{Y}_{N^2}\end{pmatrix}-\begin{pmatrix}\mathbf{X}_{N^2} & \mathbf{Y}_{N^2}\end{pmatrix}\widehat{H_1}-\left(\mathbf{F}^1_{N^2}\mathbf{F}^2_{N^2}\right)\widehat{H}\\ -\left(\mathbf{F}^3_{N^2}\mathbf{F}^4_{N^2}\right)=0,\end{aligned} \tag{17}$$

where $\widehat{H_1}=\begin{pmatrix}\mathbf{P}^{(\alpha_1,x)}_{N^2\times N^2}\mathbf{D_1} & \mathbf{P}^{(\alpha_2,x)}_{N^2\times N^2}\mathbf{D_2}\\ \mathbf{P}^{(\alpha_1,x)}_{N^2\times N^2}\mathbf{D_3} & \mathbf{P}^{(\alpha_2,x)}_{N^2\times N^2}\mathbf{D_4}\end{pmatrix}$, $\mathbf{D_1}=(c_1\mathbf{D}^{(\gamma_1,y)}_{N^2\times N^2}+c_4\mathbf{D}^{(\zeta_4,x,y)}_{N^2\times N^2})$,
$\mathbf{D_2}=(c_2^{'}\mathbf{D}^{(\eta_2,x)}_{N^2\times N^2}+c_3^{'}\mathbf{D}^{(\eta_3,y)}_{N^2\times N^2}+c_4^{'}\mathbf{D}^{(\zeta_1,x,y)}_{N^2\times N^2})$,
$\mathbf{D_3}=(c_2\mathbf{D}^{(\gamma_2,x)}_{N^2\times N^2}+c_3\mathbf{D}^{(\gamma_3,y)}_{N^2\times N^2}+c_5\mathbf{D}^{(\zeta_3,x,y)}_{N^2\times N^2})$,
$\mathbf{D_4}=(c_5^{'}\mathbf{D}^{(\zeta_2,x,y)}_{N^2\times N^2}+c_1^{'}\mathbf{D}^{(\eta_1,y)}_{N^2\times N^2})$,
The unknowns $(\mathbf{X}_{N^2}\ \ \mathbf{Y}_{N^2})$ can be easily determined using equation (17).

## VI. Illustrative Examples

In this section, some numerical examples corresponding to classical initial conditions with Caputo fractional derivatives are presented to demonstrate the accuracy and applicability of the proposed method. All the simulations are carried out using 5 Ghz processor. All the results are displayed using plots and tables.

*Example 1:* Consider the following non-homogeneous generalized class of FOPDEs with mixed derivative terms

$$\frac{\partial^{\alpha_1}u_1(x,y)}{\partial x^{\alpha_1}}=c_1\frac{\partial^{\gamma_1}u_1(x,y)}{\partial y^{\gamma_1}}+c_2\frac{\partial^{\zeta_1}u_1(x,y)}{\partial x^{\frac{\zeta_1}{2}}\partial y^{\frac{\zeta_1}{2}}}+G_1(x,y), \tag{18}$$

Corresponding to the conditions of initial type

$$u_1(0,y)=0=u_1^{'}(0,y),u_1^{''}(0,y)=2sin(y).$$

The exact solution $u_1(x,y)$ and the approximate solution is compared at scale level $N=5$, and it is evident that the approximate solution is in a good agreement with the exact solution as shown in Figure 1. It is also examined that for increasing values of $N$, the amount of absolute error decreases significantly (see Figure 2). The behaviour of the approximate solutions at $\gamma_1=1$, $\zeta_1=2$ is also observed for different values of $\alpha_1$, and it is found that, the approximate solutions for different values of $\alpha_1$ approaching to the exact solution as $\alpha_1\to 3$. The results are displayed in Figure 3.

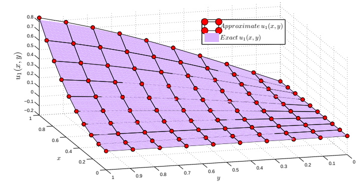

Figure 1. At $(N=5),\alpha_1=3$, the comparison is made among exact and approximate solution of Example (1).

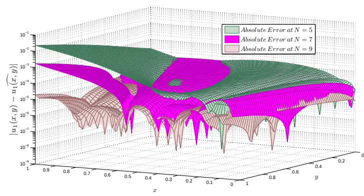

Figure 2. For various values of the scale levels $(N=5,7,9)$, the amount of absolute error in $u_1(x,y)$ of Example (1) is determined at $(\alpha_1=3)$.

*Example 2:* Consider the following non-homogeneous integer-order generalized coupled systems of FOPDEs of the type

$$\begin{aligned}\frac{\partial^{\alpha_1}u_1}{\partial x^{\alpha_1}}=&c_1\frac{\partial u_1}{\partial y}+c_2\frac{\partial u_2}{\partial x}+c_3\frac{\partial u_2}{\partial y}+c_4\frac{\partial^2 u_1}{\partial x\partial y}\\ &+c_5\frac{\partial^2 u_2}{\partial x\partial y}+G_1(x,y),\\ \frac{\partial^{\alpha_2}u_2}{\partial x^{\alpha_2}}=&c_1^{'}\frac{\partial u_2}{\partial y}+c_2^{'}\frac{\partial u_1}{\partial x}+c_3^{'}\frac{\partial u_1}{\partial y}+c_4^{'}\frac{\partial^2 u_1}{\partial x\partial y}\\ &+c_5^{'}\frac{\partial^2 u_2}{\partial x\partial y}+G_2(x,y),\end{aligned} \tag{19}$$

corresponding to the following conditions of initial type

$$u_1^{(j)}(0,y)=0=u_2^{(j)}(0,y),\ j=0,1.$$

The exact solution of the system (19) at $\alpha_1=2=\alpha_2$, and $c_1=2,c_2=2,c_3=3,c_4=5,c_5=6,c_1^{'}=3,c_2^{'}=4,c_3^{'}=$

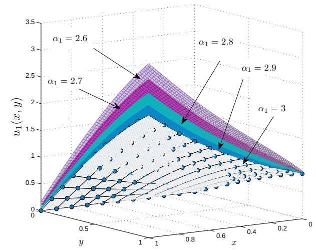


Figure 3. For various values of the fractional parameter $\alpha_1$, the behaviour of the approximate solution of Example (1) is analyzed and observed that as $\alpha_1\to 3$, our numerically approximated solution approaches to our exact solution $u_1(x,y)$ which shows the accuracy of our proposed method.

$2,c_4^{'} = 3,\ c_5^{'} = -1$ is known, and given as under

$$u_1(x,y) = -x^3y^3 + y^4x^4, \quad u_2(x,y) = -y^3x^3 + x^2y^2.$$

For various values of the scale level $N$, the numerically approximated solution of the integer order coupled system (19) is compared with its exact solution $u_1(x,y)$ and $u_2(x,y)$, and examined that our developed numerical scheme yields highly precise results even at low scale level $N$. The results are shown in Figures (4), (5), and (6).

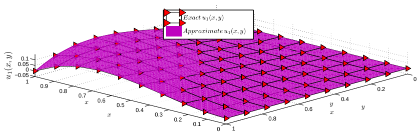


Figure 4. Comparison is made between approximate and exact solution $u_1(x,y)$ by choosing $(N = 5), \alpha_1 = 2$ for the generalized coupled system (19).

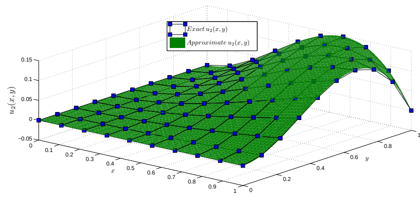


Figure 5. Comparison is made between approximate and exact solution $u_1(x,y)$ by choosing $(N = 5), \alpha_2 = 2$, for the generalized coupled system (19).

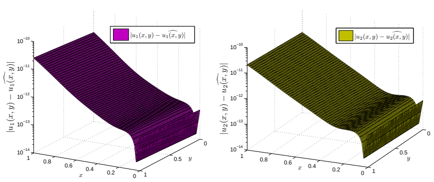

Figure 6. The error analysis is done by choosing scale level $(N = 6)$ at the integer-orders $(\alpha_1 = 2, \alpha_2 = 2)$ of the generalized coupled system (19).

## VII. Conclusion

The new numerical algorithm is developed to get the numerical solutions of FOPDEs. A new operational matrix in the sense of Caputo is constructed able to handle the fractional problems having mixed partial derivative terms. The experimental results show that the results are in a good agreement with the exact solution with a low number of approximating terms.

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Absolute error of coupled system of FOPDEs.

**Table I**

At various scale level $N$ and for different values of $x$ and $y$, the amount of absolute error for Example 2 is computed for the unknown function $u_1(x,y)$.

| $(x,y)$ | $N=5$ | $N=6$ | $N=7$ |
|---|---|---|---|
| $(0.1,0.1)$ | $1.707E-13$ | $1.425E-13$ | $1.567E-13$ |
| $(0.1,0.5)$ | $1.956E-13$ | $1.596E-13$ | $1.235E-13$ |
| $(0.1,0.9)$ | $2.236E-13$ | $1.773E-13$ | $1.756E-12$ |
| $(0.5,0.1)$ | $1.873E-12$ | $6.380E-13$ | $1.857E-13$ |
| $(0.5,0.5)$ | $1.880E-12$ | $7.858E-13$ | $1.648E-12$ |
| $(0.5,0.9)$ | $2.092E-13$ | $9.619E-13$ | $1.829E-12$ |
| $(0.9,0.1)$ | $1.270E-11$ | $9.363E-12$ | $1.472E-11$ |
| $(0.9,0.5)$ | $1.307E-11$ | $1.029E-11$ | $1.497E-13$ |
| $(0.9,0.9)$ | $1.306E-11$ | $1.418E-12$ | $1.652E-13$ |

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