Numerical Probabilistic Method of Solution Hyperbolic Mass Energy Transfer Equations Using the Wave Equation Solutions

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***Abstract*— Boundary value problems for one-dimensional hyperbolic equations (relaxation models of mass energy transfer) are reduced to integral equations. The kernel of the integral equations are expressed through an evolutionary operator of the wave equation for the original boundary value problem. Built random Markov process and the functionality of it, with the help of which the integral equations are solved by the method Monte-Carlo.**

Keyword— hyperbolic equations of mass transfer; integral equations; random process; exact solutions of the wave equation; Monte Carlo method

When modeling mass-energy transfer processes with allowance for relaxation effects, second-order hyperbolic equations-relaxation (or wave) models-that differ from the classical models (parabolic equations) by the presence of the second time derivative are often used [1, 2]:

$$c''_{tt} = w^2 \cdot c''_{xx} + f(x,t,c,c'_t,c'_x), \qquad (1)$$

where c(*x*,*t*) – is the potential of the transferred physical field; *w* – is the propagation velocity of field perturbations; c''_{tt} and c''_{xx} – second derivatives with respect to time *t* and space *x*.

Probabilistic methods for solving differential equations lack the disadvantages of other numerical methods of solving, including difference ones. On the basis of work on establishing a connection between the theory of probabilities and hyperbolic equations, methods were developed for solving these equations by the Monte Carlo method. In this case, both the solution of the one-dimensional equation in [3] and the multidimensional telegraph equation in [4] used the solutions of the wave equation.

This method is developed in this paper. The method is based on the inversion of a differential operator [5]. From the standpoint of the generalized solution of the abstract evolution equation in functional space, the boundary value problem for (1) is reduced to the Cauchy problem for a system of integral equations, the kernels in which are expressed in terms of the evolution operator for the wave equation with initial and boundary values similary to (1) conditions. The system of integral equations is solved by the Monte-Carlo method, i.e. averaging the multiplicative functional from the constructed random process.

A mixed problem is considered for an abstract evolution equation with respect to the vector- function *V* (*t*) in a functional (Banach) space E:

$$dV(t)/dt = A(t)V(t) + B(t)V(t); V(0) = V_0; V(t) \in Z_t, \ \forall t, \qquad (2)$$

operator $A(t)$ – linear, $B(t)$ – continuous, generally speaking, nonlinear. $V(t) \in Z_t$ – Is an abstract analog of linear non-homogeneous (affine) boundary conditions, i.e. the subspace $Z_t \subset E$ at $\forall\ t$ – is affine and is the result of a parallel translation of the linear (directing) subspace of functions $\tilde{Z}_t \subset E$, satisfying linear homogeneous boundary conditions.

Following the assumptions of the theory, we consider the solution of the integral equation to be a generalized solution of problem (2):

$$V(t) = U(t,0)V_0 + \int_0^t \tilde{U}(t,s)B(s)V(s)ds, \qquad (3)$$

U(*t*,*s*): $Z_s \to Z_t$ – evolutionary operator for the "unperturbed" problem (2), i.e. at *B*(t)*V*(t)≡0. Operator *U*(*t*,*s*) – is affine, continuous, that is, for any $V \in Z_s$, $W \in \tilde{Z}_s$ there is a relation, $U(V+W) = U(V) + \tilde{U}(W)$, where $\tilde{U}(t,s)$: $\tilde{Z}_s \to \tilde{Z}_t$ – is a linear operator, called associated with *U*(*t*,*s*).). It is assumed that the "perturbing" operator *B*(*t*): $Z_t \to \tilde{Z}_t$ does not violate boundary conditions. A vector-valued function *W*(*t*)=*U*(0,*t*) *V*(*t*). The initial conditions have the form: $W_0 = W(0) = V(0) = V_0$. The integral equation (3) is equivalent to the system of equations:

$$W_1(t) = W_0 + \int_0^t \tilde{U}(0,s)W_2(s)ds, W_2(t) = B(t)W_3(t), W_3(t) = U(t,0)W_1(t). \qquad (4)$$

Solution $W_1(t)$ of system (4) determines the solution of problem (2): $V(t) = U(t,0)W_1(t)$. Operator $U(0,t)$ cancels the action $U(t,0)$. Equation (1)) with given initial and boundary conditions reduces to problem (2) with the subsequent transition to system (4), in fact to (3). Consider, for example, a typical boundary-value problem on [0,*X*]:

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$$c|_{t=0} = c_0(x),\ c'_t|_{t=0} = c_{t0}(x),\ x \in [0,X];\ c|_{x=0} = \gamma_1(t),\ c'_x|_{x=X} = \gamma_2(t),\ t \in [0,T] \quad (5)$$

$$c_0 \in C^1[0,X],\ c_{t0} \in C[0,X],\ \gamma_1 \in C^1[0,T],\ \gamma_2 \in C[0,T]$$

and the matching conditions are satisfied. It is assumed that the function f from (1) is continuous by x, is integrable by t. introduce the space E of functions $\varphi(x,i)$, $x \in [0,X]$, i=1,2,3 continuous in on x and such that $\varphi(x,3) = \varphi'_x(x,1)$; E is endowed with the norm. In the resulting functional space E the operators A and $B(t)$ are defined with respect to equation (2):

$$A\varphi(x,1) = \varphi(x,2),\ A\varphi(x,2) = w^2\varphi'_x(x,3) = w^2\varphi''_{xx}(x,1),\ A\varphi(x,3) = \varphi'_x(x,2) \quad (6)$$

$$B(t)\varphi(x,1) = 0,\ B(t)\varphi(x,2) = f(x,t,\varphi(x,1),\varphi(x,2),\varphi(x,3)),\ B(t)\varphi(x,3) = 0.$$

Equation (1) is equivalent to the abstract (2) for the vector function $V(t)(x,i) \in E$, which is related to the solution $c(x,t)$ of equation (1) by the relations:

$$V(t)(x,1) = c(x,t),\ V(t)(x,2) = c'_t(x,t),\ V(t)(x,3) = c'_x(x,t) \quad (7)$$

$$V(0)(x,1) = V_0(x,1) = c_0(x),\ V(0)(x,2) = V_0(x,2) = c_{t0}(x),\ V(0)(x,3) = V_0(x,3) = dc_0(x)/dx.$$

Subspace $Z_t \subset E$ consists of all $\varphi \in E$, that satisfy the following conditions:

$$\varphi(0,1) = \gamma_1(t),\ \varphi(0,2) = d\gamma_1(t)/dt,\ \varphi(X,3) = \gamma_2(t). \quad (8)$$

For the problem (1), (5) in the form (2) all the conditions are satisfied. The solution is understood as the solution (3). $U(t,s)$ – is the evolution operator for the homogeneous wave equation, which can be written out explicitly, using the exact solutions of the wave equation and the derivatives of them. The exact solutions of the inhomogeneous wave equation:

$$\hat{c}''_{tt} = w^2\hat{c}''_{xx} + \hat{f}(x,t), \quad (9)$$

are derived from the integral equation of oscillations:

$$\int_L \left(\frac{\partial \hat{c}}{\partial t}dx + w^2 \frac{\partial \hat{c}}{\partial x}dt \right) + \iint_G \hat{f}(x,t)dxdt = 0, \quad (10)$$

Range G is bounded by the curve L formulas for $\hat{c}$ for $z > T$, $z = t - s$, $T = X/w$ based on $\hat{c}$, corresponding $z \le T$ from the range $\square = \{0 \le z \le T;\ 0 \le x \le X\}$. Points $\square$ do not have more than one reflection of characteristics from boundaries. Consider the case $G \subset \square$. By diagonals $x - wz = 0$ and $x + wz = X$ range $\square$ divided into four zones:

I. $\{0 \le z \le X/2w,\ wz \le x \le X - wz\}$;

II. $\{0 \le z \le X/2w,\ 0 \le x \le wz\} \cup \{X/2w < z \le X/w,\ 0 \le x \le X - wz\}$;

III. $\{0 \le z \le X/2w,\ X - wz < x \le X\} \cup \{X/2w < z \le X/w,\ wz \le x \le X\}$

IV. $\{X/2w < z \le X/w,\ X - wz < x < wz\}$;

Denote the points of zones:

$A^I(x,z) \in I$, $A^{II}(x,z) \in II$, $A^{III}(x,z) \in III$, $A^{IV}(x,z) \in IV$, (11)
$B(x_1,s)$, $B'(-x_1,s)$, $C(x_2,s)$, $C'(x_3,s)$, $D(0,t_1)$, $D'(X,t_2)$, $E'(X,s)$, $x_1 = x - wz$, $x_2 = x + wz$, $x_3 = 2X - x - wz$, $t_1 = z - x/w$, $t_2 = z - (X - x)/w$.

For points $A^I(x,z)$ there are no reflections of characteristics from the boundaries, for $A^{II}(x,z)$ there is a reflection of the characteristic $x - wz = const$ from the boundary x=0, for $A^{III}(x,z)$ there is the reflection $x + wz = \text{const}$ from x=X, for $A^{IV}(x,z)$ of both characteristics from the boundaries.

Consider for example the boundary-value problem for equation (9) with conditions (5):

$$\hat{c}|_{t=0} = c_0(x),\ \hat{c}'_t|_{t=0} = c_{t0}(x),\ x \in (0,X);\quad \hat{c}|_{x=0} = \gamma_1(t)$$
$$\hat{c}'_x|_{x=X} = \gamma_2(t),\ t>0. \quad (12)$$

The expressions $\hat{c}$, as the expressions obtained from them by differentiation $\hat{c}'_t$ and $\hat{c}'_x$ for for the points of the band $A^{IV}(x,z)$ have the form:

$$\hat{c}(A^{IV}) = [c_0(C') - c_0(B')]/2 + (1/2w)\cdot\int_{B'}^{C'} c_{t0}(y)dy + (1/w)\cdot\int_{C'}^{X} c_{t0}(y)dy +$$
$$+\gamma_1(D) + w\int_{E'}^{D'} \gamma_2(y)dy + (1/w)\left((1/2)\iint_{A^{IV}DB'C'D'} + \iint_{D'C'E'}\right)\hat{f}(x,t)dxdt; \quad (13)$$

$$\hat{c}'_t(A^{IV}) = -(w/2)\cdot[dc_0(B')/dx + dc_0(C')/dx] + (1/2)\cdot[c_{t0}(C') - c_{t0}(B')] +$$
$$+d\gamma_1(D)/dt + w\gamma_2(D') \quad +(1/2)\cdot\left(\int_D^{A^{IV}} + \int_{D'}^{A^{IV}} + \int_{C'}^{D'} - \int_{B'}^{D}\right)\hat{f}(x,t)dt;$$

$$\hat{c}'_x(A^{IV}) = (1/2)\cdot[dc_0(B')/dx - dc_0(C')/dx] + (1/2w)\cdot[c_{t0}(B') + c_{t0}(C')] -$$
$$-(1/w)\cdot d\gamma_1(D)/dt + \gamma_2(D') \quad +(1/2w)\cdot\left(\int_{D'}^{A^{IV}} + \int_{B'}^{D} - \int_{D}^{A^{IV}} + \int_{C'}^{D'}\right)\hat{f}(x,t)dt.$$

Similar formulas obtained for other boundary value problems and for all zones of the range $\square$ were verified on functions satisfying the wave equation [6]. To construct the evolution operator $U(t,s)$ from (2) were used formulas of type (13) at $\hat{f} \equiv 0$ for $0 \le t - s \le X/w$. Let give the result of the action of $U(t,s)$ on function $\varphi(x,i)$, (i=1,2,3) depending on the zones of the range $\square$ taking into account (8). For zone points $A^{IV}(x,z)$:

$$U(t,s)\varphi(x,1) = \frac{\varphi(x_3,1)}{2} - \frac{\varphi(-x_1,1)}{2} + \frac{1}{2w}\int_{-x_1}^{x_3}\varphi(y,2)dy + \frac{1}{w}\int_{x_3}^{X}\varphi(y,2)dy + \gamma(t_1,1) + w\int_s^{t_2}\gamma(y,2)dy,$$
$$U(t,s)\varphi(x,2) = -\frac{w(\varphi(-x_1,3) + \varphi(x_3,3))}{2} + \frac{\varphi(x_3,2)}{2} - \frac{\varphi(-x_1,2)}{2} + \gamma(t_1,3) + w\gamma(t_2,2)$$
$$U(t,s)\varphi(x,3) = \frac{\varphi(-x_1,3) - \varphi(x_3,3)}{2} + \frac{\varphi(-x_1,2)}{2w} + \frac{\varphi(x_3,2)}{2w} - \frac{1}{w}\gamma(t_1,3) + \gamma(t_2,2), \quad (14)$$

where, $\gamma(t,2) = \gamma_2(t)$, $\gamma(t,3) = d\gamma_1(t)/dt$. Function $U(t,s)$ $\varphi(x,i)$ at the point (x,i) expressed as a linear values combina-

tion of functions φ and γ at certain points and integrals of these functions over certain intervals both are united by the concept of an integral in measure. Denoting $\bar{x}$ =(x,i), $\bar{t}$ =(t,i), we obtain $\bar{x} \in \bar{X} = [0, X] \times \{1,2,3\}$, $\bar{t} \in \bar{T} = [0,T] \times \{1,2,3\}$. Formulas of type (14) take the form:

$$U(t,s)\ \varphi(\bar{x}) = \int_{\bar{X}} Q_{t,s}(\bar{x};d\bar{y})\ \varphi(\bar{y}) + \int_{\bar{T}} Q^{\text{гр}}_{t,s}(\bar{x};d\bar{t})\ \gamma(\bar{t}), \quad (15)$$

Kernel $Q_{t,s}$ does not take into account the boundaries, kernel $Q^{\text{гр}}_{t,s}$ – allows for. Operator $\tilde{U}(t,s)$, associated with $U(t,s)$, expressed by an integral with kernel $Q_{t,s}$. The solution (3) is realized by the Monte Carlo method through the solution (4) with integrals over the probability measures P {···}:

$$w_1(\bar{x},t) = w_0(\bar{x}) + \int_0^t \mathrm{P}^{\text{вр}}\{t \to ds\}\alpha^{\text{вр}}(t;s)\int_{\bar{X}} \mathrm{P}_{0,s}\{\bar{x} \to d\bar{y}\}\alpha_{0,s}(\bar{x};\bar{y})w_2(\bar{y},s), \quad (16)$$

$$w_2(\bar{x},t) = \{ f(x,t,w_3(x,1,t),w_3(x,2,t),w_3(x,3,t)) \text{ at } \bar{x} = (x,2);\ 0 \text{ at } \bar{x} = (x,1),\ \bar{x} = (x,3)\},$$

$$w_3(\bar{x},t) = \int_{\bar{X}} \mathrm{P}_{t,0}\{\bar{x} \to d\bar{y}\}\alpha_{t,0}(\bar{x};\bar{y})w_1(\bar{y},t) + \int_{\bar{T}} \mathrm{P}^{\text{гр}}_{t,0}\{\bar{x} \to d\bar{t}\}\alpha^{\text{гр}}_{t,0}(\bar{x};\bar{t})\gamma(\bar{t})$$

α – "weight" ("vr" and "gr" – time and boundary). To construct coefficients of the combinations (14) are replaced by positive numbers whose sum =1. assume that the function *f* of (1) is expanded in a series, converging for any a_1, a_2, a_3:

$$f(x,t,a_1,a_2,a_3) = \sum \beta_{\lambda,\mu,\nu}(x,t)p_{\lambda,\mu,\nu}(x,t)a_1^{\lambda}a_2^{\mu}a_3^{\nu};\ \lambda,\mu,\nu \in \{0,1,2,..\},$$

$p_{\lambda,\mu,\nu} \ge 0$ – probability; $\sum p_{\lambda,\mu,\nu} = 1$; $\beta_{\lambda,\mu,\nu}$ – "weight".

$p_{\text{н}}(\bar{x},t)$ and $p_{\text{гр}}(\bar{x},t)$ – are the probabilities of the output of the process modeled in the Monte Carlo method to the initial and boundary conditions, and also the quantities $q_{\text{н}}(\bar{x},t) = 1 - p_{\text{н}}(\bar{x},t)$, $q_{\text{гр}}(\bar{x},t) = 1 - p_{\text{гр}}(\bar{x},t)$. The state of process is a finite set of *Q* points ("particles") Process realization is a sequence of states $\{Q_n\}$, *n*=0,1,2,…, where *n* plays the role of time Simulation begins with the third equation (16). With nonlinear (power-law) dependence of *f* on the arguments c, c'_t, c'_x branching process is modeled. *R* – is the product of all the "weights" corresponding to the transitions of the "particles" in (16) and branching, as well as the values of u corresponding to the disappearance of the "particles" upon the exit of the process to initial conditions or to the boundaries [0,*X*].

Before the algorithm starts, the set *Q* is given from one point $(\bar{x},t) = (x,i,t)$. After a finite number of steps, the algorithm will finish the work; the result is *R*, solution of problem (1), (5) is expected value $\mathrm{E}\{R\}$. If at the beginning in the third equation (16) *i*=1, then c(x,t)= $\mathrm{E}\{R\}$; if *i*=2, then $c'_t(x,t) = \mathrm{E}\{R\}$; if *i*=3, then $c'_x(x,t) = \mathrm{E}\{R\}$. The efficiency of the algorithm depends on the choice of probabilities in (16).

Approbation of the method was carried out [5]. We used equations of the type (1) (with variable coefficients, except the coefficients for the highest derivatives, having constant values, and nonlinear ones) with known analytic (test) solutions, in accordance with which the initial and boundary conditions were given. 10,000 implementations of the simulated process were used. The relative error is less than 5%.

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