Parametric Synthesis of PID Controllers with a Filtered Derivative for a Specified Damping Ratio

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***Abstract*— An algorithm for calculating the settings of PID controllers with a filtered derivative for a specified damping ratio analog is considered for the case of low-frequency disturbances acting on the plant. To uniquely determine the derivative gain the formulas are used that was obtained from the condition of suppression of low-frequency disturbances. Thus, the calculation problem is reduced to two parameters optimizing. Using the symbolic computing system Maple made possible to study the features of the curve of a specified damping ratio in the plane of two controller tuning parameters. The efficiency of the method was tested on test models.**

***Keywords*— PID controller; filter on derivative term; damping ratio; Maple**

I. Introduction

The procedure of parametric synthesis of PID controllers is complicated by the presence of three tuning parameters. These are the gain k_p, integration time T_i and differentiation time T_d, or equivalent parameters, $k_1 = k_p$, $k_0 = k_1/T_i$, $k_2 = k_1 \cdot T_d$. Often the coefficient T_d or k_2 is determined by the formulas $T_d = \alpha T_i$ or $k_2 = \alpha k_1^2 / k_0$ [1]. The parameter α is chosen, for example, in the range $\alpha = 0.15...0.25$, and often quite arbitrarily. The selection of optimal parameters (k_1, k_0) is perform from the condition of a minimum of an additional quality index, for example, the integral criterion *IE* [1]. In [2], based on the condition of suppression of the low-frequency disturbances (CSLD), formulas were obtained that make it possible to uniquely determine the coefficient k_2 as a function k_1, k_0 and the parameters of the plant model. In [3], on this basis, an algorithm for parametric synthesis of a PID controller with an ideal differentiator was developed. The synthesis procedure reduces to building a curve of a specified value of damping ratio in the plane of two tuning parameters (k_1, k_0) for different frequency values. The selection of optimal parameters (k_1, k_0) is perform from the condition of a minimum of the integral criterion IE. In [4, 5] the method was used to synthesize PID controllers with ideal differentiator for the specified maximum of the sensitivity function $M_s = \max_{\omega} |S(j\omega)|$ or additional sensitivity function $M_p = \max_{\omega} |T(j\omega)|$ for providing of "stability margin" (robustness). In this paper, based on this method, a control system with a PID controller with a filtered derivative is designed. To provide a stability margin the analog of the damping ratio m is used. This is the "root quality" criterion that specifies the location of the poles of the control system [6].

II. Formulation of the Problem

We consider a control system with one input and one output. In this paper we use the following notation: $G(s)$, $G_c(s)$ - the transfer functions of the plant and controller; x, y, $e = x - y$ - the set point, the process value and the control error; f_1 and f_2 - the disturbance applied to the input and output of the object respectively.

A general view of the plant transfer function

$$G(s) = \frac{1}{s^r}\tilde{G}(s) = \frac{1}{s^r}\frac{b_0 + b_1 s ... + b_m s^m}{a_0 + a_1 s ... + a_n s^n} e^{-s\tau} \qquad (1)$$

where $\tilde{G}(0) = |b_0/a_0| < \infty$; $r = 0, 1$; a_i, b_i constant coefficients, $a_n > 0$, $b_0 > 0$; $m \le n$; τ time delay.

We represent $\tilde{G}(s)$ in the form of a Taylor series with respect to s at the point $s = 0$

$$G(s) = \frac{1}{s^r}\left(\mu_0 + \mu_1 s + ... + \mu_k s^k + ...\right) \qquad (2)$$

where $\mu_k = \frac{1}{k!}\frac{d^k}{ds^k}\tilde{G}(s)\Big|_{s=0}$, $k = 0,1,...$ - the moments of the transfer function $\tilde{G}(s)$.

A controller transfer function

$$G_c(s) = k_p\left(1 + \frac{1}{T_i s} + \frac{T_d s}{\gamma T_d s + 1}\right) = k_1 + \frac{k_0}{s} + \frac{k_2 s}{\gamma k_2 s / k_1 + 1} \qquad (3)$$

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where the coefficient γ in industrial systems varies in the bounds of $\gamma = 0.05...0.125$ [1, 2]. From the necessary stability condition for a closed loop system and $b_0 > 0$ in (1) it follows that $k_0 > 0$.

The controllers settings will be determined from the condition of a minimum of the integral criterion IE, subject to a constraint on a given stability margin

$$IE = \min_{k_0,k_1,k_2} I_0,\ m \ge m_s \tag{4}$$

where m is analog of the damping ratio of the system $m = \min_i \left|\frac{\operatorname{Im} s_i}{\operatorname{Re} s_i}\right|$; s_i – the roots of the characteristic polynomial of a closed loop system, m_s – a specified value of m.

To solve the problem, the boundary of the domain $m \ge m_s$ is constructed in the (k_1, k_0) plane and a minimum point of *IE on it is* searched.

The following restrictions on the coefficients of the controller follow from the CSLD criterion, [2, 3].

$$\begin{aligned} &k_0 = \max k_0 \\ &k_2 = \alpha \cdot \frac{k_1^{\,2}}{k_0} + \alpha_1 \cdot \frac{k_1}{k_0} + \frac{\alpha_1^2}{2 \cdot k_0} + \alpha_2 \cdot k_0 + \alpha_3, \end{aligned} \tag{5}$$

where $\alpha = \frac{1}{2}$, $\alpha_1 = \frac{1}{\mu_0}$, $\alpha_2 = \frac{\mu_1^2 - 2 \cdot \mu_0 \cdot \mu_2}{2 \cdot \mu_0^2}$, $\alpha_3 = \frac{\mu_1}{\mu_0^2}$.

The first restriction in (5) corresponds to a minimum *IE* [1].

The formulas for α_i, $i \in 1,3$ depend on the properties of the plant and the point of application of the disturbance. For the disturbance acting at the input of the plant $\alpha_2 = 0$, for an astatic (integrating) plant $\alpha_1 = 0$ and $\alpha_3 = -1/\mu_0$. $\alpha_3 = -1/\mu_0$. If $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and α is a constant, we get the well-known formula $k_2 = \alpha k_1^2 / k_0$. The coefficient α_2 can be a measure of the oscillation of the plant process, and when $\alpha_2 \le 0$ the process is weakly oscillatory [3].

III. Algorithm of the Solution of the Problem

The equation of the domain boundary $m \ge m_{зад}$ (the D-decomposition curve) is found from the characteristic equation of the closed loop system by the substitution $s = -m\omega + j\omega$ [6]

$$\begin{aligned} &\left(k_1 + \frac{k_0}{-m\omega + j\omega} + \frac{k_2\left(-m\omega + j\omega\right)}{\gamma k_2\left(-m\omega + j\omega\right)/k_1 + 1}\right) \times \\ &\times \left(V_1(m,\omega) + jV_2(m,\omega)\right) + 1 = 0 \end{aligned} \tag{6}$$

where $W_y(-m\omega + j\omega) = V_1(m,\omega) + jV_2(m,\omega)$ – the "extended" frequency characteristic of the plant, $j = \sqrt{-1}$.

Here and in what follows we denote m_s as m. Substituting the expression for k_2 from (5) into (6), taking $\alpha = 0.5$, finding the real and imaginary parts, after the transformations we obtain a system of third-order polynomial equations respect to k_1

$$\begin{cases} F_1 = A_3 \cdot k_1^3 + A_2 \cdot k_1^2 + A_1 \cdot k_1 + A_0 = 0 \\ F_2 = B_3 \cdot k_1^3 + B_2 \cdot k_1^2 + B_1 \cdot k_1 + B_0 = 0 \end{cases} \tag{7}$$

where

$A_0 = -\omega \cdot \gamma(2\alpha_2 k_0^2 + 2\alpha_3 k_0 + \alpha_1^2)(\omega \cdot (1 - m^2) + (mV_1 + V_2)k_0)$

$A_1 = \varphi_1(V_1, V_2, \gamma, m, \omega, \alpha_1, \alpha_2, \alpha_3)$,

$A_2 = -\omega(2+\gamma)(mV_1 + V_2)\,\mathrm{k}_0 +$
$\quad + 2\omega^2\alpha_1(1+\gamma)((\mathrm{m}^2 - 1)\,\mathrm{V}_1 + 2mV_2) + \omega^2\gamma(\mathrm{m}^2 - 1)$

$A_3 = \omega^2(1+\gamma)((m^2 - 1)V_1 + 2mV_2)$

$B_0 = \omega \cdot \gamma(2\alpha_2 k_0^2 + 2\alpha_3 k_0 + \alpha_1^2)(-2\,\mathrm{m}\,\omega + (V_1 - mV_2)k_0)$

$B_1 = \varphi_2(V_1, V_2, \gamma, m, \omega, \alpha_1, \alpha_2, \alpha_3)$

$B_2 = (2+\gamma)\omega(V_1 - mV_2)k_0 +$
$\quad + 2\alpha_1\omega^2(1+\gamma)((m^2 - 1)V_2 - 2mV_1) - 2\gamma m\omega^2$

$B_3 = -\omega^2(1+\gamma)((m^2 - 1)V_2 + 2mV_1)$.

The coefficients A_1 and B_1 are too bulky and are not explicitly given here.

The algorithm of solving the polynomial system (7) is based on the theory of elimination and is detail described in [3, 4, 5]. Applying it to the system (7), we obtain two equations

$$\begin{aligned} &Res = \gamma\omega k_0(2\alpha_2 k_0^2 + 2\alpha_3 k_0 + \alpha_1^2) \times \\ &\quad \times \left(\beta_5 k_0^5 + \beta_4 k_0^4 + ... + \beta_1 k_0 + \beta_0\right) \\ &k_1 = \frac{\eta_4 k_0^4 + \eta_3 k_0^3 + \eta_2 k_0^2 + \eta_1 k_0 + \eta_0}{\lambda_3 k_0^3 + \lambda_2 k_0^2 + \lambda_1 k_0 + \lambda_0} \end{aligned} \tag{8}$$

The coefficients β_i, η_i and λ_i in equations (8), depend on the frequency ω and constant parameters and, because of their complexity, are not given here. The first equation (8) is the resultant of the system (7), the factors k_0 and $P_0 = 2\alpha_2 k_0^2 + 2\alpha_3 k_0 + \alpha_1^2$ determine the special solutions. Substitution of the roots of the equation $P_0 = 0$ into the second equation (8) yields $k_1 = 0$. Varying ω, we define k_0 as positive roots of the polynomial

$$Res_1 = \beta_5 k_0^5 + \beta_4 k_0^4 + ... + \beta_1 k_0 + \beta_0, \qquad (9)$$

the coefficient k_1 is determined from the second equation (8).

Calculations made for various plant model showed that there are no more than three positive solutions. Solutions corresponding to an unstable closed loop system are easily eliminated.

The structure of the "main" D-decomposition curve coincides with the analogous curve for the case when the value k_2 is determined by the formula $k_2 = \alpha k_1^2 / k_0$. This case was studied in detail in [2]. The curve has two main branches converging at one point, at a frequency when the polynomial (9) has a multiple root, Fig.1. At this point the discriminant of the polynomial is zero. The area $m \geq m_{зад}$ is selected according to the hatch rule [7]. At the upper branch points, according to the sign of the Jacobian of the system (7), the right side of the curve is hatched, the bottom branch is hatched on the left side.

To limit the frequency range, formulas are obtained for determining the values in which the D-decomposition curve intersects the coordinate axes. Successively substituting into (7) $k_1 = 0$ and $k_0 = 0$ and solving the obtained systems by the elimination method, we obtain for $k_1 = 0$

$$\begin{aligned} &Res_{k1} = \omega(m^2+1)(V_1 + mV_2) = 0, \\ &k_0 = \frac{2m\omega}{V_1 - mV_2}. \end{aligned} \qquad (10)$$

The frequency is found as the point of intersection of the straight line $V_1 + mV_2 = 0$ with the extended frequency characteristic of the plant.

For $k_0 = 0$ we get

$$\begin{aligned} &Res_{k0} = \gamma \cdot \omega \cdot (m^2+1)^2 V_2 = 0, \\ &k_1 = \frac{2\gamma m}{(1+\gamma)(2mV_1 + (1-m^2)V_2}. \end{aligned} \qquad (11)$$

The frequency is found as the point of intersection of the extended frequency characteristic of the plant with the real axis.

As already noted in (5), the condition $k_0 = \max k_0$ corresponds to the minimum of the IE criterion.

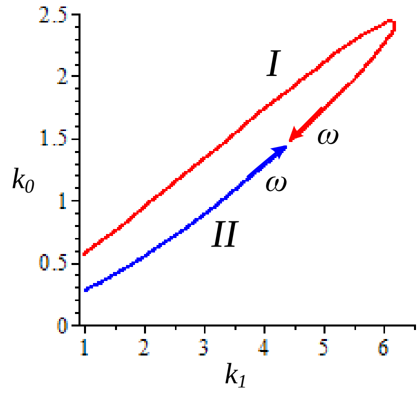

Fig. 1. The D-decomposition for the control system with a PID controller with a filtered derivative

Let's write equations (7) as follows

$$\begin{aligned} F_1 &= F_1(k_1, k_0, \omega) = 0 \\ F_2 &= F_2(k_1, k_0, \omega) = 0 \end{aligned} \qquad (12)$$

At the minimum of *IE* we have $dk_0 / dk_1 = 0$. Consider k_1 and k_0 as the functions of the frequency, we differentiate equations (12) by ω

$$\begin{aligned} &\frac{dF_1}{d\omega} = \frac{\partial F_1}{\partial k_0} \cdot \frac{dk_0}{d\omega} + \frac{\partial F_1}{\partial k_1} \cdot \frac{dk_1}{d\omega} + \frac{dF_1}{d\omega} = 0 \\ &\frac{dF_{12}}{d\omega} = \frac{\partial F_2}{\partial k_0} \cdot \frac{dk_0}{d\omega} + \frac{\partial F_2}{\partial k_1} \cdot \frac{dk_1}{d\omega} + \frac{dF_2}{d\omega} = 0 \end{aligned} \qquad (13)$$

The system (2) is linear with respect to k_1 and k_0, solving it by the Cramer's rule, we obtain

$$\frac{dk_0}{d\omega} = \frac{\Delta_0}{\Delta}, \quad \frac{dk_1}{d\omega} = \frac{\Delta_1}{\Delta}$$

where Δ is the Jacobian of the system, generally not equal to zero; Δ_0, Δ_1 are the corresponding determinants ***for*** the Cramer's rule. From the last equations we find the optimum condition $dk_0 / dk_1 = 0$

$$F_3(k_0, k_1, \omega) = \frac{dk_0}{dk_1} = \frac{\Delta_0}{\Delta_1} = 0 \qquad (14)$$

Equations (12) and (14) determine the optimum controller settings and frequency ω.

IV. Examples of Settings Calculations

The efficiency of the proposed PID design algorithm was tested for $m = 0.3$, $\gamma = 0.125$, on the following plant models:

$$G_1 = \frac{1}{(s+1)^4};\ G_2 = \frac{1}{s\cdot(s+1)^3};\ G_3 = \frac{1-2s}{(s+1)^3};$$
$$G_4 = \frac{1}{(4s-1)\cdot(s+1)^2};\ G_5 = \frac{1}{(s+1)^3}.$$

Models 1, 3 and 5 are stable, model 3 has a right zero, model 2 contains an integrator, and model 4 is unstable. Fig. 1 shows the D-decomposition curve for model 5.

The optimal settings of the PID controller with a filtered derivative k_0 and k_1 were calculated using equations (12) and (14), k_2 using formula (5). The results of the calculations are given in Table I.

There, for comparison, the optimal settings for the ideal PID controller are shown when $\gamma = 0$. In Fig. 2 D-decomposition curves for the control system with PID controllers with ideal (dotted line) and filtered derivative (solid line) for model 2 are shown. Fig. 3 shows the step responses of a closed loop system as reaction on f_1 disturbance (load) for PID controller with ideal and filtered derivative terms and model 2. The roots of the characteristic polynomial of a closed loop system with filtered derivative for this model are equal to

$$s_{1,2} = -0.185225 \pm j0.617416,\ s_{3,4} = -0.317890 \pm j0.234623,$$
$$s_5 = -2.040609,\ s_5 = -5.957241.$$

As can be seen, the pair of complex-conjugate roots with damping ratio m=0.3 is dominant. The other pair has m=1.348. The real roots are far from the imaginary axis. Such a roots distribution is typical for this method.

The results of calculations confirm the efficiency of the proposed algorithm. The advantage of the method is automatic determine of derivative gain K_2.

Note that the calculation program is implemented in the language of the symbolic computing of Maple 14. For example, equations (8) are obtained "manually" and verified with the elimination command Maple 14.

TABLE I. OPTIMUM CONTROLLERS SETTINGS FOR $m = 0.3$, $\gamma = 0.125$

Plant model	PID type	Frequency, ω	k_0	k_1	k_2
G_1	filter	0.936	0.935	2.326	1.917
	ideal	1.584	1.081	2.752	2.510
G_2	filter	0.617	0.131	0.717	0.955
	ideal	1.140	0.166	0.867	1.259
G_3	filter	0.822	0.299	0.816	0.521
	ideal	1.795	0.312	0.872	0.618
G_4	filter	1.071	1.527	6.130	6.615
	ideal	2.960	3.210	10.832	13.056
G_5	filter	1.910	3.860	6.252	3.812
	ideal	4.937	6.931	11.383	8.062

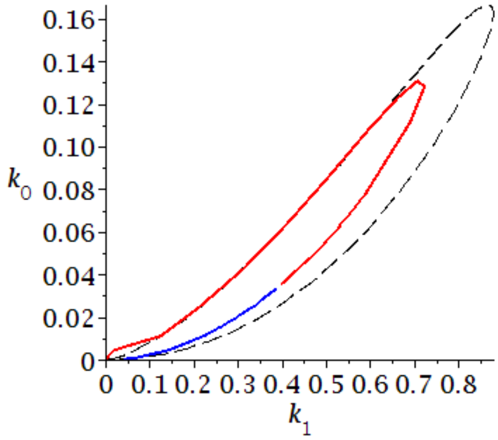

Fig. 2. D-decomposition curves for the control system with PID controllers with an ideal (dotted line) and a filtered derivative (solid line) for model 2

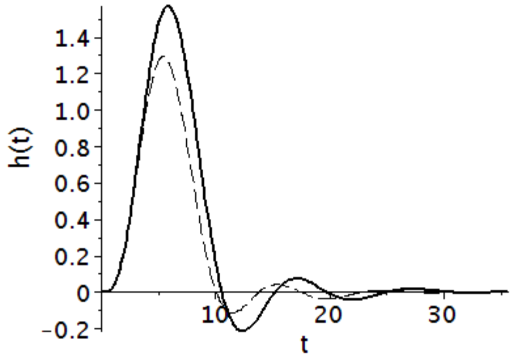

Fig. 3. Step-responses of closed-loop systems with PID controllers with ideal (dotted line) and filtered derivative (solid line) for model 2

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