Numerical Tools for Dynamical Analysis of Chaotic Systems

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***Abstract*— Numerical analysis of chaotic systems is an actual branch of nonlinear science. As the mathematical models become more complex, computational costs of dynamical analysis increase correspondingly. Dynamical map analysis is a valuable tool for determining and classification of oscillation regimes observed in the system, so the efficient algorithms for dynamical map construction are of interest. The paper discusses the performance of dynamical maps plotting algorithms, considering chaotic Hadley system as an example. In this study we compare four different approaches: the calculation of largest Lyapunov exponents, statistical bifurcation analysis, recurrence plots and the commutation analysis method. The proposed algorithm of the performance evaluation consists of two stages. On the first stage, a required simulation time is estimated using the perceptual hash calculation. On the second stage, we examine the performance of the dynamical map plotting algorithms on images with various resolutions. It is shown that the commutation-based algorithm has the best performance among all considered methods. Its implementation in the simulation and analysis software can significantly speed up the calculations when high-resolution maps are needed.**

Keywords*— *dynamical chaos; Hadley system; bifurcation analysis; recurrence analysis; commutation analysis; semi-implicit integration; dynamical map

I. Introduction

Numerical analysis of chaotic systems is one of the most important branches of nonlinear science. With the growing number of chaos applications, dynamical analysis tools are of interest in various research areas [1]. The problem of discovering various phenomena in chaotic processes e.g. hidden attractors or different regimes of oscillations is of great importance. Nowadays, many mathematical models of various nonlinear systems are being developed. Due to the application of advanced identification techniques newly discovered models of chaotic systems became more adequate and, therefore, more complex to simulate. For instance, several modifications of the Lotka–Volterra equations, describing the interaction of predator and prey, were recently suggested [2]. Authors propose three ODEs with seven nonlinearity parameters to describe the dynamics of population instead of two equations with four parameters in the original model.

Thus, the development of high-performance numerical analysis techniques for nonlinear systems is important. The common methods for studying chaotic systems include bifurcation analysis, the Lyapunov spectrum, the Kolmogorov-Sinai and Tsallis entropy estimate, the calculation of system space dimension etc. Many of these approaches require long-term simulation to obtain reliable results. When complexity of the analysis increases, the computational costs also tend to rise significantly. Along with traditional methods, new approaches to the dynamical system study are known in the literature [3, 4]. Some of them, such as fast Lyapunov indicator [4], are applicable only to the systems of a certain type – Hamiltonian, linear or other. However, researchers rarely give the detailed description of the proposed algorithms [5]. The lack of information causes the hard repeatability of the presented results.

In [6] authors presented a new technique of dynamical systems investigation, called the commutation analysis method (CAM). Proposed algorithm is based on the properties of semi-implicit numerical integration methods. Using this approach a new oscillation mode was discovered in the discrete Dequan Li model. It is of interest to compare not only the qualitative results obtained with the use of the CAM, but also the effectiveness of its application to the parametric study of chaotic systems. Since the method is based on estimation of the error between two numerical models, it is expected to obtain reliable results already in a short simulation time interval.

The paper is organized as follows. In Section II, Hadley chaotic system is introduced as a test model and dynamical analysis is described. Section III presents the algorithm for investigation of the efficiency of two-dimensional map construction method. In Section IV, we show some experimental results obtained for the considered analysis techniques through numerical simulations. Finally, some conclusions are given in Section V.

II. Dynamical maps for Hardley chaotic system

As the investigated system, we chose the mathematical model of Hadley's circulation, proposed by Lorenz in 1984 [7]. It consists of three ordinary differential equations (ODE):

$$\begin{aligned} \dot{x} &= -y^2 - z^2 - ax + ab; \\ \dot{y} &= xy - cxz - y + d; \\ \dot{z} &= cxy + xz - z, \end{aligned} \tag{1}$$

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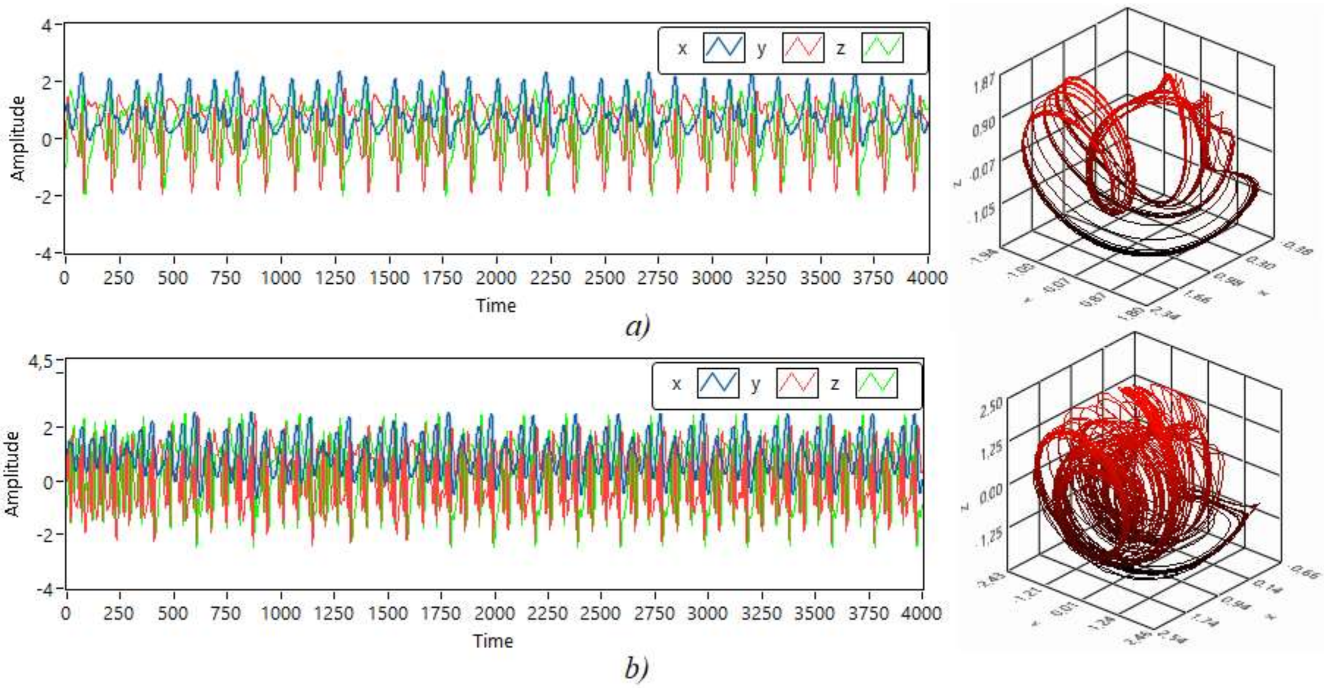

Fig. 1. Time domain and phase portrait of the Hadley system with $a = 0.2$, $d = 1$ and *a*) $b = 2.5$, $c = 10$ *b*) $b = 4.5$, $c = 13$

where $a = 0.2$, $b = 4$, $c = 8$, $d = 1$ are nonlinearity parameters. While changing values of the parameters, the system behavior can vary significantly (Fig. 1). There are various ways to obtain the dynamical map of the chaotic system and due to the properties of numerical calculations, different approaches can give similar but non-identical results (Fig. 2). The most popular technique is to calculate the largest Lyapunov exponents (LLE) and plot them on a two-dimensional map as a colored point (Fig. 2, *a*). The LLE estimates the rate of two close trajectories divergence when they start in the close vicinity to each other. The main shortcoming of this method is the need for long-term simulation. Usually, 10^5 or more iterations of the simulation algorithm are recommended for obtaining reliable results. There are also other techniques of calculating LLE estimation, but they are more computationally costly or are not versatile.

Another technique was proposed in the paper [8] as an evolution of the statistical methods in bifurcation analysis. This method uses kernel density estimation (KDE) to count the number of clusters in Poincare section (Fig. 2,*b*). When calculating KDE, you need to apply the smoothing step. There are several standard ways to estimate this parameter, e.g. the Silverman rule [9]. However, the type of distribution limits the application of this estimation. Silverman's evaluation is better suitable for smoothing the sample of data distributed according to the normal law [9], thus some shortcomings can arise in chaotic applications.

To study systems with multi-wing and multi-scroll attractors, in paper [10] it was proposed to use the recurrence density plot (Fig. 2,*c*). Despite that obtained values are well-correlated with chaotic systems behavior the recurrent nature of the algorithm gives it the N^2 complexity for a diagram size N.

CAM is supposed to be faster technique to construct a dynamical map (Fig. 2,*d*). It is based on the error estimation between two numerical models obtained by semi-implicit discrete operators [6]. Theoretically, this approach can provide correct results even on the short-term simulations. Let us check this by comparing the performance of the described algorithms for constructing the dynamical maps of the system (1).

III. The Performance Comparison Procedure

The proposed comparison procedure consists of two steps. On a first step, we define the simulation time for each considered method. This time should allow constructing the dynamical map with desirable precision. We assume that for the system (1) in the simulation interval $t = 500$ sec, each of the considered approaches make possible the construction of dynamical map which will not change significantly in a case of longer simulation times. For the time t, we construct a dynamical map in 50x50 pixels resolution, which we will take as a reference. We iterate the simulation time down with the step of 20 sec performing the dynamical analysis on each step and comparing the resulting array of data with the reference one. The idea of dynamical maps comparison is based on the calculation of their perceptual hash functions. For the matrix representing the dynamical map, we calculate the average value and use it as a threshold: for numbers above the average, we put one, and otherwise, put zero. Thus, we get a hash of the parametric space map, which can be compared with the hash of reference map through the calculation of Hamming distance (Fig. 3). For simplification, we normalized the obtained values to maximum. Simulation time for each method was chosen by the criterion of similarity of method's map to the reference map. The required level of similarity was set to 90%.

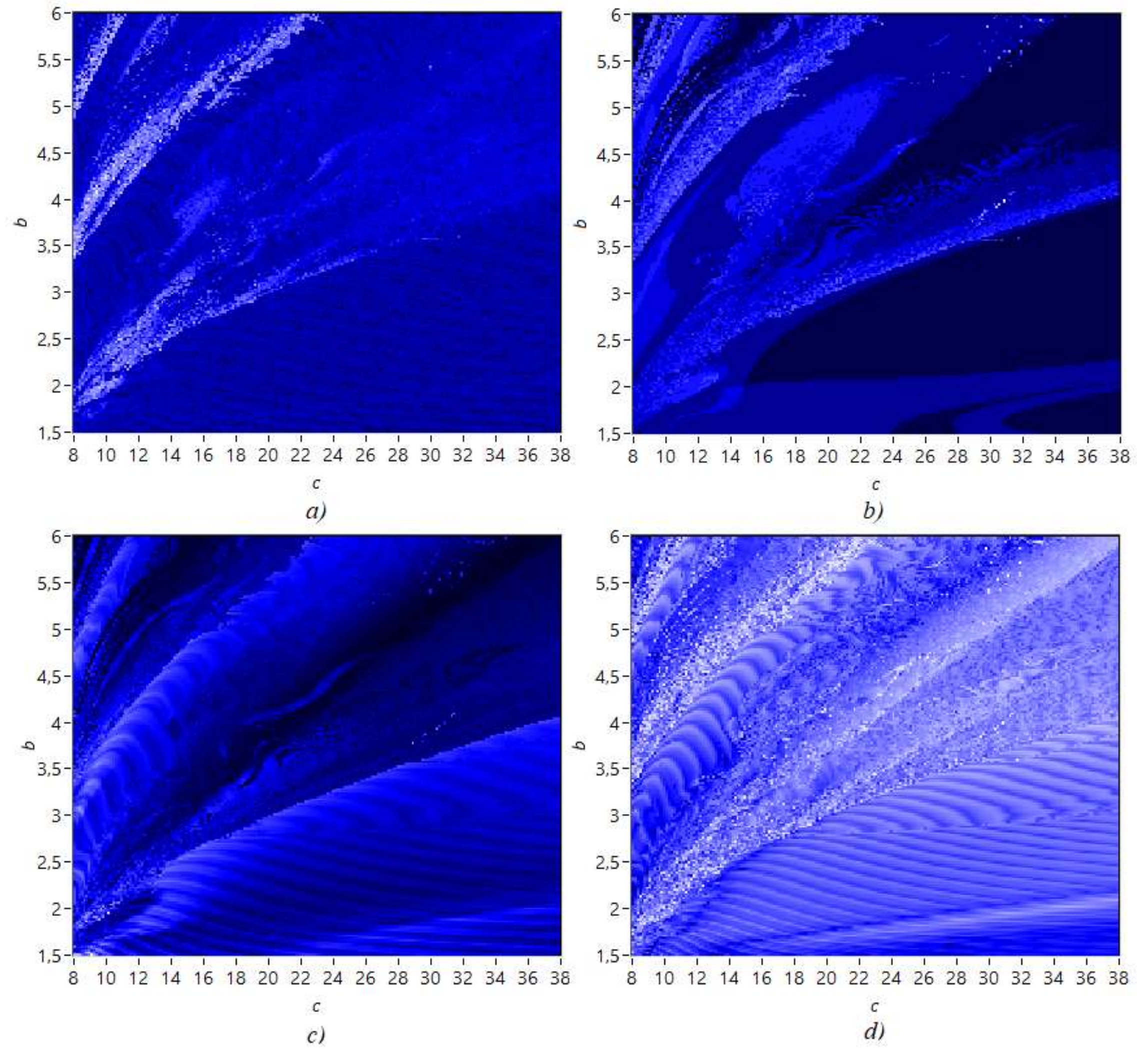

Fig. 2. Dynamical maps of Hadley system obtained by a) LLE method b) KDE technique c) CAM algorithm d) recurrence density technique

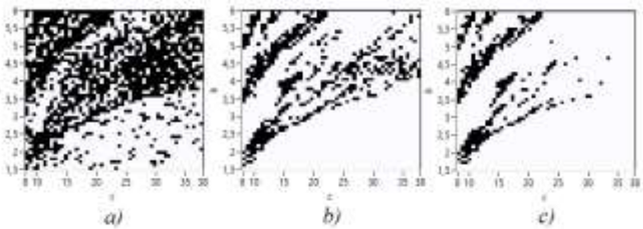

Fig. 3. The dynamical map hash, LLE method, over simulation time *a*) t = 20 sec *b*) t = 200 sec *c*) t = 500 sec (reference)

The second step of performance evaluation algorithm is to measure the time needed to construct a dynamical maps with desirable precision and various resolution. We measured elapsed time by changing the resolution for each dimension from 20 to 200 pixels.

IV. Experimental results

Fig. 4 represents the dependence between the normalized Hamming distance and the simulation time needed to construct a relevant dynamical map (*b* and *c* parameters) of Hadley system for considered algorithms. We used semi-implicit CD numerical integration methods of accuracy order 2 [11]. One can see, that LLE method is the most time consuming algorithm, while other methods allow obtaining a dynamical map with 90% precision after approximately 30 sec of simulation.

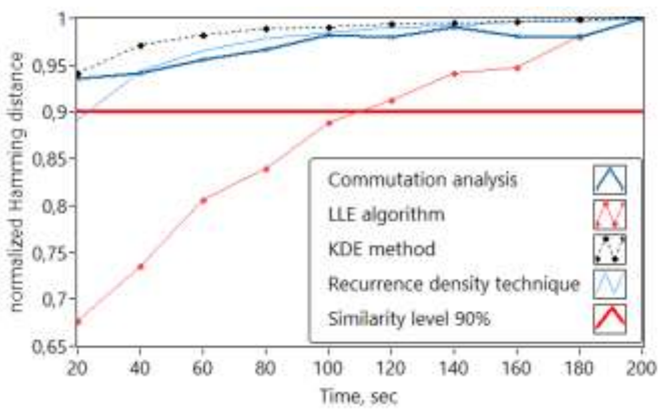

Fig. 4. The dependence between the normalized Hamming distance and the simulation time for the compared algorithms

Table 1 represents found estimated time values for each analysis method.

TABLE I. REQUIRED SYSTEM SIMULATION TIME

Considered algorithm	Required time interval, sec
CAM algorithm	20
LLE algorithm	110
KDE method	20
Recurrence density technique	30

Let us evaluate the performance of considered algorithms. The computational experiments were performed in the environment of the NI LabVIEW 2017 on 4th generation Intel Core i5 processor. One can see from Fig. 5 that recurrence density calculation is the slowest way to construct a parametric map. To obtain 200x200 pixels dynamical map (Fig. 2) it will take 25 times more time, than the CAM algorithm, which is the fastest among considered methods. LLE method appears to be twice slower than the CAM, and KDE algorithm is comparable with the CAM method. However, these results may vary for different systems.

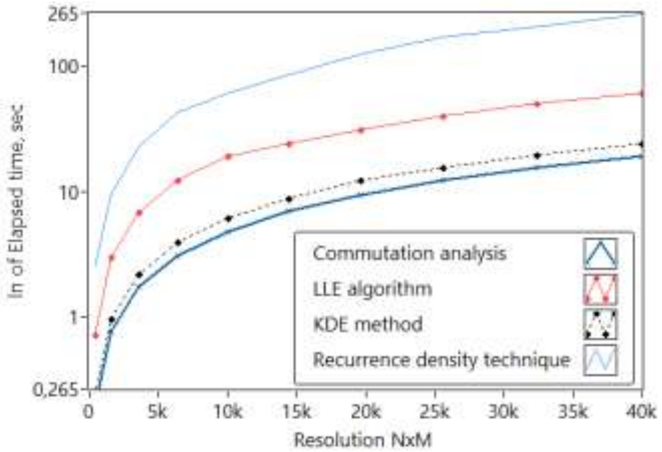

Fig. 5. The performance of dynamical map construction methods

Thus, the most effective technique of calculating parametric maps for system (1) is the commutation analysis method, as it was assumed in the Section III.

V. Conclusion

The main purpose of this study was to compare various techniques of dynamical map construction. A two-stage procedure for experimental comparison of map construction algorithms based on the calculation of a perceptive hash was proposed. Using the Hadley chaotic system as an example, we established that the CAM method is the fastest algorithm. The results of the study are explained by the fact that in order to obtain a reliable dynamical map, an algorithm based on estimating the error between two semi-implicit finite-difference schemes with different commutations does not require a long-term simulation. Being compared with the LLE method, the CAM algorithm treats the trajectories divergence that is already set by different commutations. Thus, there is no need of any analytical procedures such as normalization or the choice of initial divergence value. A qualitative assessment of the chaotic system oscillation modes, which can be distinguished using CAM-based dynamical maps, will be the topic of further research.

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