The Numerical Soluion Algorithm of the Problem with a Singularity

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Abstract— **In this paper we consider a problem with a singularity, thanks to which it is possible to model a wide class of processes of a different nature. A numerical algorithm suitable for the use of computer computing powers has been constructed for this problem. An example is proposed for illustrating the implementation of the proposed numerical algorithm in the computer algebra system Maxima, based on the Common Lisp language in the application to the problem of anesthesia in open heart surgery.**

Keywords— hyperbolic equation; numerical integration; the geometric graph; the mixed problem; Green's function; modeling anesthesia

I. Introduction

A wide class of the most diverse physical, biological and economic processes can be modeled by an equation of the type (see, for example, [1])

$$u_{xx}(x,t) - q(x)u(x,t) = u_{tt}(x,t) \quad \left(x \in \Gamma, t > 0\right) \qquad (1)$$

where Γ – geometric graph, and the coefficient $q(x)$ is a finite linear combination of δ and δ' functions with supports at the points of Γ

$$q(x) = \sum_i k_i \delta(x - x_i) + \sum_j \tilde{k}_j \delta'(x - \tilde{x}_j) \cdot$$

The list of modeling problems using as an instrument the equation (1) covers: processes in waveguide networks [2], deformations and vibrations of rod grids, deformations of elastic grids and string-rod systems (see, for example, [3, 4, 2]), diffusion in networks (see, for example, [5]), propagation of electric potential in a neuron and neural networks, bifurcation of vortex flows in a fluid, hemodynamics, vibrations of complex molecules, calculation of hydraulic networks.

Particular interest in considering these problems is the prediction of the state of elastic bonds in any future given time interval for any node of the network. The approach in which the solution of equation (1) can be represented in a form suitable for constructing computational algorithms most suitable for the use of computing powers (see, for example, [6]) is seen as the most promising.

II. Basic Basis of the Algorithm of the Numerical Solution of the Problem with Singularity

A mathematical model is considered:

$$(1)\begin{cases} u_{xx}(x,t) = u_{tt}(x,t) \quad (0 < x < \ell,\ t > 0) \\ l_1(u(\cdot,t)) = 0,\ l_2(u(\cdot,t)) = 0 \quad (t > 0) \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \quad (0 \le x \le \ell) \end{cases}, \qquad (2)$$

where $\varphi \in C^2[0;\ell]$, $\psi \in C^1[0;\ell]$, l_1: $l_1 y = y(0)$ or $l_1 y = y'(0)$, $l_2 y = y'(\ell) + k\, y(\ell)$.

Model (2) can be a description of the physical process of oscillation in the vertical plane of the following systems:

Fig. 1. Vibrations of a string with a spring-loaded end

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The system represents a stretched string, one end of which is rigidly fixed (Fig. 1a) or moves freely (without friction) only in the vertical direction (Fig. 1b), and the second is spring-loaded and slides without friction along the (unbending) spokes in the vertical direction. In a state of equilibrium, the string is located in a horizontal position.

The purpose of the problem (2) is to obtain a solution suitable for constructing a computational algorithm with any given accuracy.

Comment. The condition $\psi(x) \equiv 0$ does not narrow the generality of the investigation of problem (2); therefore, such a case will be considered.

In problem (2) we replace the function $\varphi(x)$ by the function $G(x,s)$. Then the solution of problem (2) is the function $g(x,t,s)$. We call it the fundamental solution of problem (2). Obviously, $G(x,s)$ is the Green's function of the boundary value problem

$$\begin{cases} y''(x) = f(x) \\ l_1 y = 0,\ l_2 y = 0 \end{cases}.$$

Statement 1. The fundamental solution of problem (2) is represented in the form of an analogue of the d'Alembert-Euler formula (see, for example, [8]).

$$g(x,t,s) = \frac{1}{2}\left(\tilde{g}(x+t,s) + \tilde{g}(x-t,s)\right), \tag{3}$$

where $\tilde{g}(y,s)$ is determined respectively:

at $-\ell \le y - 2n\ell < -s$

$$\tilde{g}_1(y,s) = (-\mu)^n\left(\alpha_2(y-2n\ell)+s\right) - (-\mu)^n \sum_{i=1}^{n} \Re_i^n(2ky)e^{-ky}\cdot\left(f_{n,i}(y,s,\alpha_2,s) -\right.$$
$$\left. - f_{n,i}((2n-1)\ell,s,\alpha_2,s)\right) - \sum_{j=1}^{n-1}(-\mu)^j\sum_{i=1}^{j}\Re_i^j(2ky)e^{-ky}\cdot\left(f_{j,i}(2j\ell-s,s,\alpha_2,s) -\right.$$
$$- f_{j,i}((2j-1)\ell,s,\alpha_2,s) + f_{j,i}(2j\ell+s,s,\alpha_1,0) - f_{j,i}(2j\ell-s,s,\alpha_1,0) +$$
$$\left. + f_{j,i}((2j+1)\ell,s,\alpha_2,-s) - f_{j,i}(2j\ell+s,s,\alpha_2,-s)\right),$$

$$\tilde{g}_2(y,s) = (-\mu)^n\alpha_1(y-2n\ell) - (-\mu)^n\sum_{i=1}^{n}\Re_i^n(2ky)e^{-ky}\cdot\left(f_{n,i}(2n\ell-s,s,\alpha_2,s) -\right.$$
$$\left. - f_{n,i}((2n-1)\ell,s,\alpha_2,s) + f_{n,i}(y,s,\alpha_1,0) - f_{n,i}(2n\ell-s,s,\alpha_1,0)\right) -$$
$$-\sum_{j=1}^{n-1}(-\mu)^j\sum_{i=1}^{j}\Re_i^j(2ky)e^{-ky}\left(f_{j,i}(2j\ell-s,s,\alpha_2,s) - f_{j,i}(2j\ell-s,s,\alpha_1,0) -\right.$$
$$- f_{j,i}((2j-1)\ell-s,s,\alpha_2,s) + f_{j,i}((2j+1)\ell,s,\alpha_2,-s) +$$
$$\left. + f_{j,i}(2j\ell+s,s,\alpha_1,0) - f_{j,i}(2j\ell+s,s,\alpha_2,-s)\right),$$

$$\tilde{g}_3(y,s) = (-\mu)^n\alpha_1(y-2n\ell) - (-\mu)^n\sum_{i=1}^{n}\Re_i^n(2ky)e^{-ky}\cdot\left(f_{n,i}(2n\ell-s,s,\alpha_2,s) -\right.$$
$$- f_{n,i}((2n-1)\ell,s,\alpha_2,s) + f_{n,i}(2n\ell+s,s,\alpha_1,0) - f_{n,i}(2n\ell-s,s,\alpha_1,0) +$$
$$\left. + f_{n,i}(y,s,\alpha_2,-s) - f_{n,i}(2n\ell+s,s,\alpha_2,-s)\right) -$$
$$-\sum_{j=1}^{n-1}(-\mu)^j\sum_{i=1}^{j}\Re_i^j(2ky)e^{-ky}\left(f_{j,i}(2j\ell-s,s,\alpha_2,s) - f_{j,i}(2j\ell-s,s,\alpha_1,0) -\right.$$
$$- f_{j,i}((2j-1)\ell-s,s,\alpha_2,s) + f_{j,i}((2j+1)\ell,s,\alpha_2,-s) +$$
$$\left. + f_{j,i}(2j\ell+s,s,\alpha_1,0) - f_{j,i}(2j\ell+s,s,\alpha_2,-s)\right),$$

$$f_{n,i}(y,s,b,\alpha_p) = e^{ky}\left(\frac{b}{k}y^i + \sum_{m=0}^{i-1}(-1)^{m+1}y^{i-m-1}\left\{b\frac{i\cdot\ldots\cdot(i-m)}{k^{m+2}} - (\alpha_p - 2n\ell\cdot b)\cdot\right.\right.$$
$$\left.\left.\cdot\frac{(i-1)\cdot\ldots\cdot(i-m)}{k^{m+1}}\right\}\right),\ (p=1,2),$$

$$\mu = \mp 1,\ \alpha_1 = \frac{1+k(\ell-s)}{1+k\ell},\ \alpha_2 = \frac{2+k(2\ell-s)}{1+k\ell}.$$

$$\Re_i^j(y) = \frac{(2k)^i}{(i-1)!}L_{j-i}^i(y),$$

$L_p^q(y)$ – orthogonal Laguerre polynomials with parameters p and q),

Statement 2. The solution of problem (2) is represented as

$$u(x,t) = -\int_0^l g(x,t,s)\varphi''(s)\,ds \tag{4}$$

The form (4) of the solution of problem (2) allows us to construct a computational algorithm based on the application of (choice of) numerical integration algorithms, which allows us to find the solution of the problem with any given accuracy.

III. Deduction of the Algorithm of the Numerical Solution of the Problem with Singularity

A mathematical model is considered:

$$\begin{cases} u_{yy}(y,t) = u_{tt}(y,t) \quad (0<y<\ell,\ t>0) \\ u_y(0,t) - k_1u(0,t) = 0,\ u_y(\ell,t) + k_2u(\ell,t) = 0 \quad (t>0), \\ u(y,0) = \varphi(y),\ u_t(y,0) = 0 \quad (0 \le y \le \ell) \end{cases} \tag{5}$$

$k_1, k_2 \ge 0$ (the possibility of $k_1 = +\infty$ and/or $k_2 = +\infty$ is not included).

The problem (2) is a special case of the problem (5) under consideration, which we denote by $S(\ell;k_1;k_2;\varphi(y))$ (in case $k_1 = +\infty$ or $k_1 = 0$).

Statement 3. The analog of the computational algorithm of problem (2) is possible for problem (5) in some combination $k_1, k_2 \ge 0$.

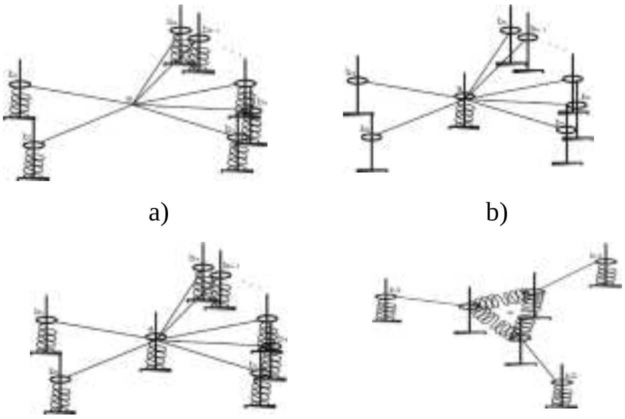
Consider the basic mathematical model of a problem with a singularity:

$$u_{xx}(x,t) - q(x)u(x,t) = u_{tt}(x,t) \quad \left(x \in \Gamma, t>0\right),$$
$$u(x+0\cdot h,t) = 0 \ \left(x \in \partial\Gamma,\ h \in D(x),\ t \ge 0\right) \tag{6}$$

$$u(x,0)=\varphi(x),\ \ u_t(x,0)=0\ \ \left(x\in\overline{\Gamma}\right). \qquad (7)$$

By analogy with (5), we denote it by $B_m\left(\ell;Y;k_1;k_2;\varphi(x)\right)$.

Model (1), (6) - (7) can be a description of the physical process of oscillation in the vertical plane of the following mechanical systems [7] (see Fig.2).



a) b) c) d)

Fig. 2. Stable systems with vertical vibration

Let $(F\varphi)(x)$ and $(G_i\varphi)(x)$ - are operators on Γ:

$$(F\varphi)(x)=\frac{1}{m}\sum_{i=1}^{m}\varphi\left(a+\|x-a\|h_i\right), \quad x\in\overline{\Gamma}$$

$$(G_i\varphi)(x)=\begin{cases}(F\varphi)(x)-\varphi\left(a+\|x-a\|h_i\right) & x\in\overline{\gamma}_1\\ \varphi\left(a+\|x-a\|h_i\right)-(F\varphi)(x) & x\in\overline{\gamma}_i\\ 0 & x\in\overline{\Gamma\setminus(\gamma_1\cup\gamma_i)}\end{cases}$$

Where $h_i=\frac{1}{\|b_i-a\|}(b_i-a)$ – is orientation on Γ.

Definition. $K_m^B(\ell;Y;k_1;k_2)$ – class of initial data of the problem $B_m\left(\ell;Y;k_1;k_2;\varphi(x)\right)$, determined by conditions $\varphi(x)\in\widetilde{C}^2(R(\Gamma))$, $\varphi(x)=0$, $\varphi_{hh}^{++}(x)=0$ for $x\in\partial\Gamma$; $\sum_{h\in D(z)}\varphi_h^{+}(z)-k_z\varphi(z)=0$, $\varphi_{hh}^{++}(z)=\varphi_{h_1h_1}^{++}(z)$ for $z\in J(\Gamma)$ and $\forall\ h,h_1\in D(z)$ $\left(D(z)=\{h\in\boldsymbol{R}^n:\|h\|=1,\forall(\varepsilon>0)\forall(a\in\Gamma)[a+\varepsilon h\in\Gamma]\}\right)$,

$$\varphi_h(y+0\cdot h)=\sum_{\eta\in D(y)\setminus\{h\}}\widetilde{k}_y\left(\varphi(y+0\cdot h)-\varphi(y+0\cdot\eta)\right)$$

for $y\in Y$ $\left(Y\subset\{a,b_1,\ldots,b_m\}\right)$.

Statement 4. The solution of the problem $B_m\left(\ell;Y;k_1;k_2;\varphi(x)\right)$ $u^{\varphi}(x,t)$ ($x\in\Gamma$, $t>0$) is representable in the form

$$u^{\varphi}(x,t)=u^{(F\varphi)}(x,t)+\sum_{i=2}^{m}u^{(G_i\varphi)}(x,t),$$

where $u^{(F\varphi)}(x,t)$ – the solution of the problem $B_m\left(\ell;Y;k_1;k_2;(F\varphi)(x)\right)$, a $u^{(G_i\varphi)}(x,t)$ – the solution of the problem $B_m\left(\ell;Y;k_1;k_2;(G_i\varphi)(x)\right)$ ($i=\overline{2,m}$).

Statement 5. Let $\varphi(x)\in K_m^B(\ell;\emptyset;k_1;k_2)$, where pair $(k_1,k_2)=\{(0,k),(k,0),(k,+\infty),(mk,k)\}$ ($k>0$). Then: $u^{(F\varphi)}(x,t)$ – the solution of the problem $B_m\left(\ell;\emptyset;k_1;k_2;(F\varphi)(x)\right)$ exists, and for any $h\in D(a)$ function $u^{(F\varphi)}(a+yh,t)$ is independent of the choice of h and is a solution of the problem $S\left(\ell;\frac{k_1}{m};k_2;(F\varphi)(a+yh)\right);\forall\left(i=\overline{2,m}\right)$ $u^{(G_i\varphi)}(x,t)$ – the solution of the problem $B_m\left(\ell;\emptyset;k_1;k_2;(G_i\varphi)(x)\right)$ exists, and function $u^{(G_i\varphi)}(a+yh_1,t)$ is the solution of the problem $S\left(\ell;+\infty;k_2;(G_i\varphi)(a+yh_1)\right)$ and moreover, $u^{(G_i\varphi)}(a+yh_i,t)=-u^{(G_i\varphi)}(a+yh_1,t)$ and, if $h\notin\{h_1,h_i\}$, then $u^{(G_i\varphi)}(a+yh,t)\equiv 0$.

Statement 6. Let $\varphi(x)\in K_m^B(\ell;\{a\};k_1;k_2)$, where pair a $(k_1,k_2)=\left\{\left(\frac{k}{m},0\right),\left(\frac{k}{m},k\right),\left(\frac{k}{m},+\infty\right)\right\}$ ($k>0$). Then: $u^{(F\varphi)}(x,t)$ – the solution of the problem $B_m\left(\ell;\{a\};k_1;k_2;(F\varphi)(x)\right)$ exists, and for any $h\in D(a)$ function $u^{(F\varphi)}(a+yh,t)$ is independent of the choice of h and is a solution of the problem $S\left(\ell;0;k_2;(F\varphi)(a+yh)\right);\forall\left(i=\overline{2,m}\right)$ $u^{(G_i\varphi)}(x,t)$ – the solution of the problem $B_m\left(\ell;\{a\};k_1;k_2;(G_i\varphi)(x)\right)$ exists, and function $u^{(G_i\varphi)}(a+yh_1,t)$ is a solution of the problem $S\left(\ell;k;k_2;(G_i\varphi)(a+yh_1)\right)$ and moreover if $h\notin\{h_1,h_i\}$, then $u^{(G_i\varphi)}(a+yh,t)\equiv 0$.

The last assertions prove the possibility of applying for the problems $B_m\left(\ell;Y;k_1;k_2;\varphi(x)\right)$ of a computational algorithm obtained for problem (2).

IV. Example of Implementation of Algorithm

The proposed numerical algorithm can be used to calculate oscillations in a system of elastically coupled elements [8, 9] Let us consider an example of the implementation of the introduced numerical algorithm on a separate system of anesthetic maintenance in patients undergoing direct myocardial revascularization [10]. The model of the choice of anesthetic management in order to reduce the incidence of complications of a different nature and etiology during the perioperative period can be the string system depicted in Fig. 3

(the central part has an elastic support with a coefficient of rigidity, and the boundaries can freely move in a vertical plane):

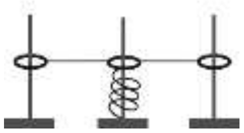


Fig. 3. System for analyzing the operation of the algorithm

In the model, the spring of rigidity k is a means of controlling and regulating the entire system as a whole and reflects the strategy by which the behavior of the system is analyzed. When choosing the strategy of intra- and postoperative analgesia, the spring either stretches or contracts, which determines the prognosis of surgical treatment.

In Fig. 4 shows the initial disturbance in the system.

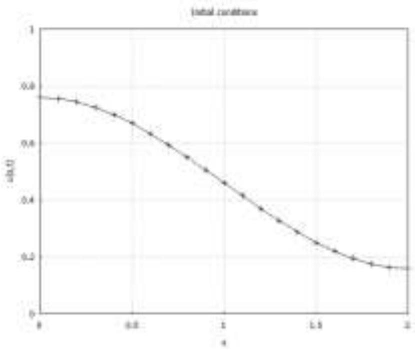
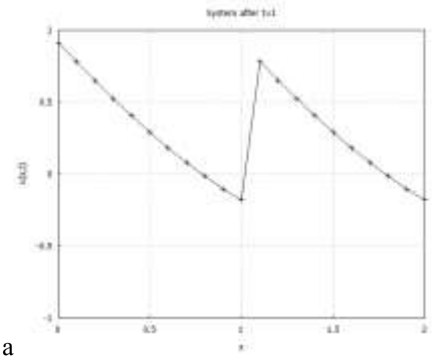
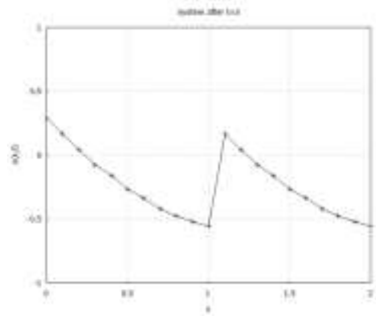


Fig. 4. Initial disturbance

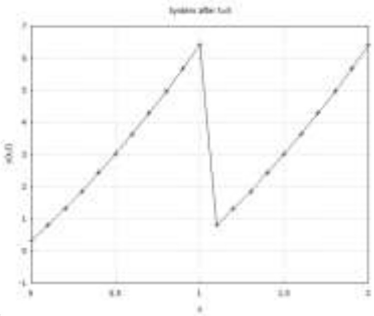
The implementation of the algorithm based on **Statement**s **5-1** was implemented in the computer algebra system Maxima, based on the Common Lisp language [11]. The calculation of the solution in the form of the integral (4) is realized using the quadrature trapezoidal formula, which provides an error of the order of 0.01. The results of such an implementation are shown in Fig. 5.



a



b



c

Fig. 5. The behavior of the system: a – system after t = 1; b – system after t = 3; c – system after t = 5.

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