Objectives with Free Borders in Problems of Medicine

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***Abstract*— The present paper is devoted to the study of the problem with free boundaries of the Stefan type. An approximate solution of the problem is obtained by the method of equivalent linearization. Using the MatLab matrix-oriented environment, numerical calculations are performed.**

***Keywords*— free boundary; one-dimensional problem; stationary problem**

I. INTRODUCTION

Cryogenic freezing is used in medicine for local irreversible destruction of biological tissue, for which cryoprobes with a flat, cylindrical or hemispherical shape of the cooling surface are used. Cryosurgery is considered to be an accurate and controlled process, although a complete solution to its thermal aspects has not been received up to date. The decisive importance for solving such problems is obtained by mathematical methods of calculation and forecast based on the study of special statements of Stefan's problems related to the problems of mathematical physics. In such problems, the determination of both the temperature field and of its mobile isothermal surfaces is made, the law of motion of which is not known in advance. The proposed paper is devoted to the study of one such boundary-value problem with free boundaries for nonlinear evolution equations arising in the mathematical modeling of problems of cryosurgery.

The exact analytical solution of the stationary problem is obtained in the work, which determines the maximum dimensions of freezing, cryodamage and thermal perturbation that are very important for the surgeon. Using Rothe's method, the approximation of boundary value problem (1)–(3) in the form of a system of boundary-value problems for ordinary differential equations is obtained to determine the approximate value of $u_k\ (x)$ and S_k functions $u\ (x,\ t),\ S\ (t)$ at the points $t = t_k$. Using the Green's function and formulas, the problem is reduced to Volterra's nonlinear integral equations on each time layer. Applying finite-dimensional approximation, the problem is reduced to a system of nonlinear algebraic equations. To determine the approximate solution, the method of equivalent linearization turned out to be especially effective, the ideas of which go back to Leibenson's works in the theory of thermal conductivity and Krylov–Bogolyubov–Mitropolsky in the theory of nonlinear oscillations. The boundary conditions are satisfied. We arrive at the Cauchy problem for determining $x^* = x^*\ (t)$ and $s = s\ (t)$, requiring that the construction used satisfy the differential equation in the sense that the integral residual is zero. As is known, differential equations are widely used for mathematical modeling of processes and phenomena in various fields of science and technology, where in the solution process we face difficulties. This takes a long time and there is a chance of making mistakes, so the idea arose of using MatLab's matrix-oriented mathematical package to study a problem with free boundaries. In this paper, a difference scheme of the problem is constructed. The algorithm for solving the difference problem is implemented using the matrix-oriented environment of MatLab, and also there are graphs constructed using the MatLab environment.

II. FORMULATION OF THE PROBLEM

Determination of the dynamics of the temperature field in cooled and frozen biological tissues is described by solving the following problem with free boundaries: [1, 2, 3]:

$$0<t<t_1:\quad \begin{aligned} & u_{xx}-u_t=u^{\beta}, \quad 0<x<s(t) \\ & u(x,0)=u_0(x), \quad 0<x<s(0), \\ & u_x-Hu=-H\varphi(t), \quad x=0, \\ & u(s(t),t)=0, \quad u_x(s(t),t)=0, \\ & u(0,t_1)=1; \end{aligned} \tag{1}$$

$$t_1<t<t_2:\quad \begin{aligned} & u_{xx}-\frac{1}{a^2}u_t=0, \quad 0<x<x^*(t), \\ & u_{xx}-u_t=u^{\beta}, \quad x^*(t)<x<s(t), \\ & u_x-Hu=-H\varphi(t), \quad x=0, \\ & [u]_{x^*}=0, \quad [u_x]_{x^*}=Px_t^*, \quad u(x^*(t),t)=1, \\ & u(s(t),t)=0, \quad u_x(s(t),t)=0, \\ & u(0,t_2)=u_n; \end{aligned} \tag{2}$$

$$t>t_2:\quad \begin{aligned} & u_{xx}-u_t=u^{\beta}, \quad x^*(t)<x<s(t), \\ & u_x-Hu=-H\varphi(t), \quad x=0, \\ & [u]_{x^{**}}=0, \quad [u_x]_{x^{**}}=P_1x_t^{**}, \quad u(x^{**}(t),t)=1, \\ & u(s((t),t)=0, \quad u_x(s((t),t)=0, \\ & [u]_{x^*}=0, \quad [u_x]_{x^*}=Px_t^*, \quad u(x^*(t),t)=1. \end{aligned} \tag{3}$$

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In (1)-(3), the temperature field $u=u(x,t)$ and the boundaries $x^{**}=x^{**}(t)$, $x^*=x^*(t)$, $s=s(t)$ are the desired functions. a, H, P_1, P are known parameters, $U_0(x), \varphi(t)$ are known functions,

$$0 \le \beta < 1,$$
$$a^2 = \underline{\lambda}\bar{c}\bar{\rho} / \bar{\lambda}\underline{c}\underline{\rho},$$
$$H = (\alpha/\underline{\lambda})\sqrt{\bar{\lambda}/W}, \quad W = \tilde{a}W_0, \quad W_0 = c_k m_k,$$
$$\tilde{a} = \left[\bar{T} - \left(\bar{T} - T^*\right)\right]^{1-\beta},$$
$$\varphi = 1 + \underline{\lambda}/\bar{\lambda}u_A(t) - \underline{\lambda}/\bar{\lambda}, \qquad u_A(t) = u_A\left(1 - \exp(-\chi t)\right),$$
$$u_A = \left(\bar{T} - T_A\right)/\left(\bar{T} - T^*\right), \qquad P = P/\bar{c}\bar{\rho}\left(\bar{T} - T^*\right), P = \Lambda\bar{\rho},$$
$$\bar{T} = 36{,}7^0 C, \quad T^{**} = 0 \div -30^0 C,$$
$$T^* = 0 \div -3^0 C, \qquad u_n = \left(\bar{T} - T^{**}\right)/\left(\bar{T} - T^*\right)$$

T_A – applicator temperature; $\lambda, c, \rho, \quad \Lambda$ – thermophysical characteristics of biological tissue; c_k, m_k – heat capacity and blood mass velocity; $\chi > 0$ – the parameter of the output for the given cooling mode; β – nonlinearity parameter; the sign of the line refers to not frozen, but from below to the frozen area of biological tissue; [ ] – means the jump of the function under it.

The stationary problem corresponding to (1)–(3) admits a precise analytic solution. The problem (1)–(3) is reduced to the system of differential equations by Rothe's method. Using the Green's function and formulas for further investigation of the problem, nonlinear integral equations are obtained which, using finite-dimensional approximation, are reduced to a system of nonlinear algebraic equations.

Assuming $x^*=x^*(t)$ and $s=s(t)$ we will seek an approximate solution of problem (1)–(3) in the form:

$$u(x,t) = \begin{cases} 1 + \dfrac{H(\varphi-1)x^*(t)}{1+Hx^*(t)} + \dfrac{H(\varphi-1)x^*(t)}{1+Hx^*(t)}\dfrac{x}{x^*(t)}, & 0 < x < x^*(t), \\ \left(\dfrac{s(t)-x}{s(t)-x^*(t)}\right)^{\frac{2}{1-\beta}}, & x^*(t) < x < s(t). \end{cases} \tag{4}$$

Moreover, the boundary conditions for $x=0$, $x=x^*(t)$, $x=s(t)$ are satisfied automatically. Differential equations and the remaining boundary conditions are satisfied in the sense of general thermal balances for each of the biotissue areas, taking into account the conjugation condition:

$$\begin{aligned} & u_x(x^*+0,t) - Px_t^* - H\left(1 + \frac{H(\varphi-1)x^*(t)}{1+Hx^*(t)} - \varphi\right) - \frac{1}{a^2}\int_0^{x^*} u_t(x,t)dx = 0, \\ & u_x(x^*+0,t) + \int_0^{x^*} u_t(x,t)dx + \int_{x^*}^{s} u^\beta(x,t)dx = 0. \end{aligned} \tag{5}$$

Assuming in (5) $u(x,t) \approx \tilde{u}(x,t)$, after computing the derivatives and integrals for the definition of $x^*=x^*(t)$ and $s=s(t)$, we arrive at the Cauchy problem:

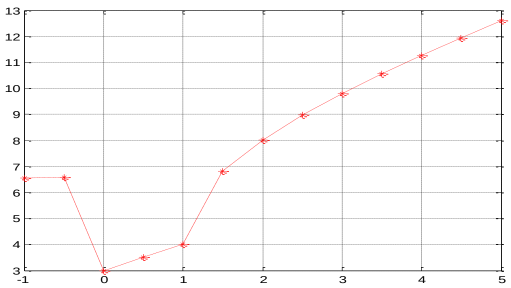
$$\begin{aligned} & \frac{ds}{dt} + \frac{2}{1-\beta}\frac{dx^*}{dt} + \frac{3-\beta}{1+\beta}(s-x^*) - \frac{2(3-\beta)}{(1-\beta)^2}\cdot\frac{1}{s-x^*} = 0, \quad t > t_1, \\ & \frac{d}{dt}\left\{\left[\frac{(\varphi-1)x^*}{2a^2(H^{-1}+x^*)} + P\right]x^*\right\} - \frac{\varphi-1}{H^{-1}+x^*} + \frac{2}{1-\beta}\cdot\frac{1}{s-x^*} = 0, t > t_1 \\ & x^*(t_1) = 0, \quad s(t_1) = s_0. \end{aligned} \tag{6}$$

Replacing the derivatives by finite differences, we obtain a system of nonlinear equations with respect to $x^*=x^*(t)$ and $s=s(t)$ on a given time layer.

Algorithms for obtaining approximate solutions are implemented on a computer using a matrix-oriented MatLab package. Numerical calculations show that the simplest approximate solutions give completely satisfactory results.

III. Realization of Algorithms of the Solution of the Problem in the MatLab System

```
>>b=0.01;
>> h=0.5;
>> u1=-1:0.5:5;
>> ua=-3;
>> f=u1+(1/h).*sqrt((2/(1+b)).*exp((1+b)/2).*log(u1))-ua;
>> plot(u1,f,['R','*','-.']);
>> grid on;
>> s=(2/(1-b)).*sqrt((1+b)/2).*exp((1-b)/2.*log(u1));
>> plot(u1,s,['R','*','.-']);
>> grid on;
0.0225 + 1.4354   0.0160 + 1.0185       0
1.0187     1.4356          0.7547
2.0233     2.2595          2.4730          2.6690
2.8514      3.0226         3.1844
```



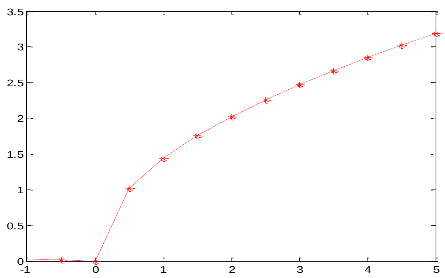


Fig. 1. Dynamics of the temperature field

III. CONCLUSION

The considered one-dimensional problems make possible to calculate a number of typical cases of freezing of biological tissues, to determine with sufficient accuracy the particular and general laws of the cooling and freezing process, to make a series of monograms necessary for use in practical medicine, to outline a further direction of research, and using the MatLab package allows expanding the range of real applications. With the help of MatLab you can save time, analyze and process task data, visualize research results, develop graphic and calculation applications. Application matrix-oriented package MatLab reduces the work of finding solutions, in contrast to the search for solutions in an analytical mathematical way. The package already has built-in functions that separate the interval, which is the scope of the allowed subtask values in the study of the required task. With this package, you can also determine the number of iterations that lead to the solution of the problem.

Further work on this theme involves the application of the results obtained in various fields: the study of mathematical models in the economy, ecology.

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