Nonsmooth Norm Minimization Operators on the Intersection of Linear Manifolds and a Parallelepiped

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***Abstract*— Difference equations with nonsmooth operators are considered for calculating the optimal control on the basis of methods of mathematical programming. The presented results extend the classes of mathematical models for the synthesis of optimal control systems.**

***Keywords*— mathematical programming; control; nonsmooth projectors minimize the Euclidean norm**

I. INTRODUCTION

Approaches to optimizing dynamic systems based on the maximum principle are currently widely used L.S. Pontryagin [14, 15], the theory of optimal control [2, 13, 15], R. Bellman's dynamic programming method [1], mathematical programming methods [4–7, 10].

In this case, an important place in the application of mathematical programming takes a qualitative study of the optimality and stability of closed systems by methods of functional analysis [7, 8, 10].

The results presented in this article expand the classes of mathematical models for the synthesis of optimal control systems.

The models of closed optimal systems with constraints on the coordinates and controls synthesized on the basis of nonsmooth operators for solving problems of mathematical programming, formulated on the basis of nonsmooth optimization operators, are considered [7].

II. MODELS OF CLOSED OPTIMAL SYSTEMS WITH COORDINATE AND CONTROL LIMITATIONS

Models of closed optimal systems with constraints on coordinates and controls are synthesized on the basis of nonsmooth operators for solving problems of mathematical programming and formulated on the basis of nonsmooth optimization operators [7]

In this case, nonsmooth projection operators are formed on the basis of classical orthogonal projectors on linear manifolds and nonsmooth projectors onto parallelepipeds. Solutions are the limit points of fundamental minimizing sequences of relaxation type.

The optimization problem is formulated as: calculate on a nonempty set of space a vector that minimizes the functional in the form of a square of the Euclidean norm under constraints such as equalities and two-sided inequalities for variables. Thus, it is required to generate an operator for calculation.

$$x_* = \arg\min\left\{\varphi = \|X - C\|_2^2 \,\middle|\, AX = b,\ A \in R^{m\times n},\ rang\,A = m,\ X^- \le X \le X^+\right\} =$$
$$= \arg\min\left\{\varphi = \|X - C\|_2^2 \,\middle|\, X \in D = D^0 \cap D^1 \ne \varnothing,\right.$$
$$D^0 = \left\{X \,\middle|\, AX = b\right\} \ne \varnothing,\ D^1 = \left\{X \,\middle|\, X^- \le X \le X^+\right\} \ne \varnothing\Big\} \in R^n. \qquad (1)$$

The problem (1) in the form of minimization of a quadratic functional on the non-empty intersection of a linear manifold and a parallelepiped is often solved by the relaxation-projection method [7]

$$X_{k+1}^0 = P^0\left(X_k^0\right) \in D^0,\ X_{k+1}^1 = P^1\left(X_{k+1}^0\right) \in D^1,\ k = 0,1,2,\ldots,\ X_0^0 = P^0(C). \quad (2)$$

Relations (2) define a recurrence sequence of vectors

$$\begin{array}{lll}
1).\ k=0: & X_1^0 = P^0\left(X_0^0\right) \in D^0, & X_1^1 = P^1\left(X_1^0\right) = X_1^1 \in D^1; \\
 & \ldots\ldots\ldots & \\
s-1).\ k=s-1: & X_s^0 = P^0\left(X_{s-1}^1\right) \in D^0, & X_s^1 = P^1\left(X_s^0\right) = X_s^1 \in D^1; \\
s).\ k=s: & X_{s+1}^0 = P^0\left(X_s^1\right) \in D^0, & X_{s+1}^1 = P^1\left(X_{s+1}^0\right) = X_{s+1}^1 \in D^1; \\
s+1).\ k=s+1: & X_{s+2}^0 = P^0\left(X_{s+1}^1\right) \in D^0, & X_{s+2}^1 = P^1\left(X_{s+1}^0\right) = X_{s+2}^1 \in D^1; \\
 & \ldots\ldots\ldots\ . &
\end{array} \qquad (3)$$

The elements of the sequence (3) converge unboundedly, forming a fundamental sequence in space R^n, that converges to the limiting vector $x_* = \lim x_n$, $n \to \infty$, as a solution of problem (1). The superscripts "0" or "1" in (3) indicate the membership of the sequence element in sets D^0 or D^1. Orthogonal projector on a linear manifold D^0 is equal to

$$P^0(C) = P^0 C + P_A b,\ P^0 = E_n - P^\perp,\ P^\perp = P_A A,\ P_A = A^T\left(AA^T\right)^{-1}, \qquad (4.a)$$

and equality (2) gives the orthogonal projection of the vector X^0 onto the parallelepiped

$$P^1\left(x^0\right)=0{,}5\left(\left|x^0-X^-\right|-\left|x^0-X^+\right|+X^-+X^+\right). \qquad (4.b)$$

on the basis of minimizing the individual terms of the quadratic functional $\varphi=\|X-C\|_2^2=\sum_{j=1}^{n}\left(x_j-C_j\right)^2$ on each of the orthogonal subspaces of the space R^n. Here the minimum on the subspaces is achieved with the help of the orthogonal projection (4.b), where $X^0=C$ – the vector of the unconditional minimum of the norm-type functional.

Thus, it follows from equalities (1) – (4) that this relaxation computing scheme generates two sequences X_{k+1}^1 and X_{k+1}^0, the first of which belongs to the set D^1, and the second to the set D^0.

These two sequences asymptotically converge to a nonempty intersection of a linear manifold and a parallelepiped, since they are formed by a set of orthogonal projections onto sets, and the limit element of sequences is a solution of problem (1) on the basis of orthogonal projectors onto linear manifolds and piecewise linear projectors onto a bounding parallelepiped.

III. Algorithm of the solution of the problem

To formulate norm minimization operators on a nonempty intersection of a linear manifold and a parallelepiped, it is required to define an operator for the limit point of the fundamental sequence of the formulated relaxation method. Then problem (1) is solved with the help of a piecewise-linear operator on the basis of the algorithm:

Calculation $p_0=P^0(C),\ p_1=P^1(p_0)$,

where $P^0(C)$ – the projection vector of the parameter of the functional $C\in R^n$ on the linear manifold $Ax=b$.

1. Computation with the help of linear and nonsmooth projection operators specifying a vector $p_2=P^1\left[P^0(p_1)\right]$.
2. Formation of the direction vector in the form of a difference of vectors
$$p=p_2-p_1=P^1\left[P^0\left[P^1\left(P^0(C)\right)\right]\right]-P^1\left(P^0(C)\right).$$
3. Calculation of the limiting vector-solution
$$x_*(\alpha)=p_2+\alpha_* p\in D^0=\left\{x(\alpha_*)\,\middle|\,Ax(\alpha_*)=b\right\},$$

where the parameter $\alpha_*=\left[b-(A_i,p_2)\right](A_i,p)^{-1}\in R$ is such that condition

$$x_*(\alpha_*)=p_2+\alpha_* p\in D^0=\left\{x(\alpha)\right| \quad A_i\,x(\alpha)=b,$$
$$\left.\alpha_*=\left[b-(A_i,p_2)\right](A_i,p)^{-1}\right|,$$

since relations

$$A_i x(\alpha_*)=A_i\left(p_0+\alpha_* p\right)=A_i p_0+\alpha_* A_i p=b,$$

from which follows the value of the parameter

$$\alpha_*=\left[b-(A_i,p_0)\right](A_i,p)^{-1}.$$

The values of the parameter α_* are valid for the matrix

$$A\in R^{1\times n}.$$

For general matrices, which are represented in a vector-line notation:

$$A=\left(A_i\right)_{i=1}^{i=m}\in R^{m\times n},\ rang\ A=m,$$

under the condition of compatibility of equality type constraints in (1), we can use the row vector of this matrix, which gives a joint system of linear equations. It can also be noted that the proposed variant of accounting for restrictions of the type of inequalities that possesses universality makes it possible to perform a separate account of constraints on the groups of components of the solution vector for the corresponding transformation of a linear manifold.

The first form of a piecewise-linear projection operator for a vector $c\in R^n$ on a non-empty compact intersection of a linear manifold and a parallelepiped is defined by the equalities

$$\begin{aligned}x_*(\alpha_*)&=p_2+\alpha_* p=P^1\left[P^0\left[p_1\right]\right]+\alpha_*\left\langle P^1\left[P^0(p_1)\right]-p_1\right\rangle=\\&=P^1\left[P^0\left[P^1\left(P^0(C)\right)\right]\right]+\\&+\alpha_*\left\langle P^1\left[P^0\left[P^1\left(P^0(C)\right)\right]\right]-P^1\left(P^0(C)\right)\right\rangle,\end{aligned} \qquad (5.a)$$

where the vector

$$p_1=P^1(p_0)=P^1\left(P^0(C)\right),$$
$$\alpha_*=\left[b-(A_i,p_2)\right](A_i,p)^{-1},\ (A_i,p)\neq 0,$$
$$p_2=P^1\left[P^0\left[p_1\right]\right],$$

and the scalar product – is not zero.

The second form of the operator, which follows from (5.a), has the form

$$x_*=(1+\alpha_*)P^1\left[P^0\left[P^1\left(P^0(C)\right)\right]\right]-\alpha_* P^1\left(P^0(C)\right). \qquad (5.б)$$

The algorithm corresponding to the operator (5.a) is presented in the form:

Step 1: calculate the vector

$$p_1=P^1(p_0)=P^1\left(P^0(C)\right),$$

where $p_0=P^0(C)$– the projection vector $C\in R^n$ on the manifold $Ax=b$.

Step 2: calculate the vector as a superposition of projectors

$$p_2=P^1\left[P^0\left[p_1\right]\right].$$

Step 3: calculate the direction vector

$$p = p_2 - p_1 = P^1\left[P^0\left[P^1\left(P^0\left(C\right)\right)\right]\right] - P^1\left(P^0\left(C\right)\right).$$

Step 4: calculate the solution vector of the minimization problem

$$x_*\left(\alpha\right) = p_2 + \alpha_* p \in D^0 = \left\{x\left(\alpha_*\right) \left| Ax\left(\alpha_*\right) = b\right.\right\}.$$

where the real parameter $\alpha_* \in R$ is such that the "condition of belonging" to the linear manifold for the solution in the form

$$x_*\left(\alpha_*\right) = p_2 + \alpha_* p \in D^0 =$$

$$= \left\{x\left(\alpha\right)\left| A_i\, x\left(\alpha\right) = b,\ \alpha_* = \left[b - \left(A_i, p_2\right)\right]\left(A_i, p\right)^{-1}\right.\right\} \Rightarrow$$

$$\Rightarrow \alpha_* = \left[b - \left(A_i, p_2\right)\right]\left(A_i, p\right)^{-1},$$

since relations

$$A_i x\left(\alpha_*\right) = A_i\left(p_0 + \alpha_* p\right) = A_i p_0 + \alpha_* A_i p =$$

$$= b \Rightarrow \alpha_* = \left[b - \left(A_i, p_0\right)\right]\left(A_i, p\right)^{-1}.$$

IV. EXAMPLE

Compute the solution vector of a problem of type (1), equal to

$$x_* = \arg\min\left\{\varphi = \left\|X - C\right\|_2^2 \left| AX = b,\ A \in R^{m\times n},\ rang\, A = m,\ X^- \le X \le X^+\right.\right\},$$

where constraints of the type of equality and inequality have the form

$$Ax = \left(1 \mid -1\right)x = b = -1,$$

$$D^1 = \left\{x \;\middle|\; x^- = \begin{pmatrix}0\\1\end{pmatrix} \le \begin{pmatrix}x_1\\x_2\end{pmatrix} \le \begin{pmatrix}5\\4\end{pmatrix} = x^+\right\}.$$

Solution

Step 1: calculation of a vector based on a double projection – on a linear manifold and a parallelepiped

$$p_1 = P^1\left(p_0\right) = P^1\left(P^0(\mathrm{C})\right) = 0.5\left(\left|p_0 - \begin{pmatrix}0\\1\end{pmatrix}\right| - \left|p_0 - \begin{pmatrix}5\\4\end{pmatrix}\right| + \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}5\\4\end{pmatrix}\right) =$$

$$= 0.5\left(\left|\begin{pmatrix}7\\8\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right| - \left|\begin{pmatrix}7\\8\end{pmatrix} - \begin{pmatrix}5\\4\end{pmatrix}\right| + \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}5\\4\end{pmatrix}\right) = \begin{pmatrix}5\\4\end{pmatrix},$$

where the vector $p_0 -$ is the projection of the vector $C \in R^n$, defined in (1) onto the linear manifold

$$p_0 = P^0\left(C\right) = 0{,}5\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \times \begin{pmatrix}c_1\\c_2\end{pmatrix} + P_A b = 0{,}5\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \times \begin{pmatrix}8\\7\end{pmatrix} + \begin{pmatrix}-0{,}5\\0{,}5\end{pmatrix} = \begin{bmatrix}7\\8\end{bmatrix},\ b = -1.$$

Step 2: calculating the vector

$$p_2 = P^1\left[P^0\left[p_1\right]\right] = 0.5\left(\left|\begin{pmatrix}4\\5\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right| - \left|\begin{pmatrix}4\\5\end{pmatrix} - \begin{pmatrix}5\\4\end{pmatrix}\right| + \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}5\\4\end{pmatrix}\right) = \begin{pmatrix}4\\4\end{pmatrix},$$

and the vector $P^0\left[p_1\right]$ has the form

$$P^0\left(p_1\right) = 0{,}5\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \times \begin{pmatrix}p_1^1\\p_2^1\end{pmatrix} + P_A b =$$

$$= 0{,}5\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \times \begin{pmatrix}5\\4\end{pmatrix} + \begin{pmatrix}-0{,}5\\0{,}5\end{pmatrix} = \begin{bmatrix}4\\5\end{bmatrix}.$$

Step 3: Calculation of the "direction vector"

$$p = p_2 - p_1 = P^1\left[P^0\left[P^1\left(P^0\left(C\right)\right)\right]\right] -$$

$$-P^1\left(P^0\left(C\right)\right) = \begin{pmatrix}4\\4\end{pmatrix} - \begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}.$$

Step 4: Calculation of the optimal solution vector

$$x_*\left(\alpha\right) = p_2 + \alpha_* p \in D^0 = \left\{x\left(\alpha_*\right) \left| Ax\left(\alpha_*\right) = b\right.\right\} =$$

$$= \begin{pmatrix}4\\4\end{pmatrix} + 1\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix},$$

where the parameter $\alpha_* \in R$ is defined in (4.a) by the corresponding condition of the "inclusion" type.

This condition can be represented by the following relations

$$x_*\left(\alpha_*\right) = p_2 + \alpha_* p \in D^0 =$$

$$= \left\{x\left(\alpha\right)\left| A_i\, x\left(\alpha\right) = b,\ \alpha_* = \left[b - \left(A_i, p_2\right)\right]\left(A_i, p\right)^{-1}\right.\right\} \Rightarrow$$

$$\Rightarrow \alpha_* = \left[b - \left(A_i, p_2\right)\right]\left(A_i, p\right)^{-1} =$$

$$= \left[-1 - \left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}4\\4\end{pmatrix}\right)\right] \times \left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right)^{-1} = 1,$$

since there are relations for determining the parameter α for which the limit point belongs to one of the hyperplanes of a linear manifold, from which the parameter

$$A_i x\left(\alpha\right) = A_i\left(p_0 + \alpha p\right) = A_i p_0 + \alpha A_i p =$$

$$= b \Rightarrow \alpha = \left[b - \left(A_i, p_0\right)\right]\left(A_i, p\right)^{-1}.$$

The formulated nonsmooth projection operators can be used to solve various problems.

In particular, they can be used in difference operators of a dynamical system with nonsmooth optimization operators for transition states.

This relation is valid, at least for a manifold of dimension $\left(n-1\right)$.

The corresponding difference operators of dynamical optimal systems with constraints on the coordinates and controls and nonsmooth projection control operators can be represented by Cauchy problems

$$x_{k+1} = Hx_k + \gamma F T P^1\left[P^0\left[P^1\left(P^0\left(c\right)\right)\right]\right] +$$
$$+\alpha_*\left(x_k\right)\left\langle P^1\left[P^0\left[P^1\left(P^0\left(c\right)\right)\right]\right] - P^1\left(P^0\left(c\right)\right)\right\rangle,\ x_0 = x^0.$$

In this case, operators can implement local or interval optimal control of dynamic objects.

On the basis of mathematical models of this class, one can investigate stability as the convergence of a countable sequence of closed discrete control systems with optimization operators under constraints on coordinates and control.

These models complement the analytic representations of optimization operators and the corresponding stability conditions of closed control systems for various types of mathematical programming problems, investigated in [4,7, 8, 12, etc.].

The research presented in this article develops within the framework of the scientific and pedagogical school "System Analysis in Design and Management [3, 5, 9, 14].

The presented results extend the classes of mathematical models for the synthesis of optimal control systems.

V. CONCLUSION

Thus, nonsmooth projection operators of admissible and optimal solutions for problems with nonempty compact domains define admissible or optimizing control vectors calculated by projectors with linear and piecewise linear coordinate functions.

Projection operators can also be generalized to minimize linear and quadratic functionals on the basis of projecting "points of unconditional minimum of functionals" to families of intersections of linear manifolds and parallelepipeds.

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