The Accurate Analytical Method of Calculation of an Anisotropic Photon Crystal for Optical Devices of Information Processing Systems

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***Abstract*— The fundamental solution system in an accurate analytical form for an anisotropic photonic crystal is given and its investigation is carried out. The resulting oscillation in a photonic crystal is represented as the finite sum of equivalent oscillations. The stability conditions for the solutions are presented. It is shown that change in the cell order within a period doesn't affect the structure of solution stability regions.**

Keywords*— *one-dimensional photonic crystal; fundamental solution system; stability conditions of solutions

I. Introduction

For today photonic crystals are widely used in various devices of optics, optoelectronics, microwave technology [1–9]. In particular, photonic crystals are basis elements for Brag filters, optical switching cells, polarization transformer, isolators, etc [1-9]. Therefore theoretical and experimental research of such structures is very important in modern technique. The concept of photonic crystals has been introduced by E. Yablonovich [4] for the first time. Actually photonic crystals are periodic structures. To date, one-dimensional, two-dimensional and three-dimensional crystals are determined depending on the number of periodicity directions. Actually one-dimensional photonic crystals are periodic layered structures in classical electromagnetic models. In quantum models the ones are crystal lattices the properties of which periodically change in one direction only.

To study photonic crystals analytical [5], numerical-analytical [6], and numerical methods [7] are used for today. The transformation matrix method is most general analytical method in electromagnetic theory of photonic crystals [2, 5]. Actually a translation matrix is a fundamental solution system of initial differential equations.

In this paper, the accurate analytical method for calculating and analyzing one-dimensional photonic crystals is presented. Note that this method is applicable in any wavelength domain. The basis for such the assertion is the fact of obtaining the fundamental solution matrix by using the direct identical transformations from the Maxwell equations in the classical model and from the Schrodinger equation in the quantum model. In the general case the system of linear homogeneous differential equations with periodic piecewise-constant coefficients and an arbitrary number of constant parameter intervals in the period is under consideration.

The fundamental solution system is a unimodular 4x4 matrix. This one is presented as the finite sum of the unimodular matrices with the some contribution coefficients. It is shown also that change in an order of cells if a period is invariable doesn't affect the structure of solution stability regions. The sign-functions introduced by the authors [8] are used here to obtain the analytical expression of the fundamental solution matrix.

II. Statement of the Problem

Wave behavior within a one-dimensional photonic crystal with the period T is described by the system of linear homogeneous differential equations with periodic piecewise-constant coefficients ω_1^2, ω_2^2, α_1^2, α_2^2:

$$\begin{aligned} &\frac{\partial U_1}{\partial t} + \omega_1^2 U_1 + \alpha_1^2 U_2 = 0 \\ &\frac{\partial U_2}{\partial t} + \omega_2^2 U_2 + \alpha_2^2 U_1 = 0 \end{aligned} \qquad (1)$$

Our main aim is finding the fundamental solution system of equations (1) in an accurate analytical form for arbitrary piecewise-constant coefficients. Investigation of solution stability conditions is also the purpose of this work.

III. The Fundamental System of a One-Dimensional Photonic Crystal. The Equivalent Modes

First of all note that from the physical point of view a fundamental solution system relates the tangential components of electromagnetic field at two different points of a photonic crystal. For a homogeneous anisotropic medium such a system is already presented in the scientific literature [8, 9]. In that case the solution of the initial differential equation system (1) must be found in an exponential form. For a one-dimensional photonic crystal, a fundamental system can be found as a product of fundamental solutions of intervals with constant parameters.

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A. The fundamental system of a homogeneous medium

The fundamental solution system for a homogeneous anisotropic medium can be written in the kind [8]:

$$\mathbf{L}(T)=\begin{vmatrix}\sum\limits_{k_i=2i-1}^{2i}(-1)^{k_i+1}\eta_{k_i}^{(1,1)}\mathbf{M}_{k_i} & \sum\limits_{k_i=2i-1}^{2i}(-1)^{k_i+1}\eta_{k_i}^{(1,1)}\mathbf{M}_{k_i}\\ \sum\limits_{k_i=2i-1}^{2i}(-1)^{k_i+1}\eta_{k_i}^{(1,1)}\mathbf{M}_{k_i} & \sum\limits_{k_i=2i-1}^{2i}(-1)^{k_i+1}\eta_{k_i}^{(1,1)}\mathbf{M}_{k_i}\end{vmatrix} \quad (2)$$

where

$$\mathbf{M}_{k_i}=\begin{vmatrix}\cos\left(\Omega_{k_i}\Delta t_i\right) & -\frac{j}{\Omega_{k_i}}\sin\left(\Omega_{k_i}\Delta t_i\right)\\ -j\Omega_{k_i}\sin\left(\Omega_{k_i}\Delta t_i\right) & \cos\left(\Omega_{k_i}\Delta t_i\right)\end{vmatrix} \quad (3)$$

$$\eta_{k_i}^{(m,n)}=\frac{1}{\Omega_{2i}-\Omega_{2i-1}}\Big[\left(1-|m-n|\right)\left(\omega_{2i-n+1}^{2}-\Omega_{k_i}\right)+ +\omega_{2i-n+1}^{2}\alpha_{2i-n+1}\Big] \quad (4)$$

Here $\Omega_{2i}=j\lambda_2$, $\Omega_{2i-1}=j\lambda_1$ are the normal modes of the system; $\eta_{2i-1}^{(1,2)}/\eta_{2i-1}^{(1,1)}$, $\eta_{2i}^{(1,2)}/\eta_{2i}^{(1,1)}$ are the distribution coefficients at the eigen-frequencies Ω_{2i} и Ω_{2i-1}; λ_2, λ_1 are eigennumbers of the initial differential equation system (1).

In this work the matrix of the fundamental solutions (2) is written in the form convenient for further transformations.

B. The fundamental system for a one-dimensional photonic crystal

The matrix of the fundamental solutions of a one-dimensional crystal including *N* cells with constant parameters may be found as product of the matrix of cells with constant parameters [10]:

$$\mathbf{L}(T)=\prod_{i=N}^{1}\mathbf{L}\left(\Delta t_i\right) \quad (5)$$

However this matrix kind does not allow us to carry out further analytical investigation and effectively solve inverse problems. Therefore, the authors obtained the system in the form of the finite sum of the 4x4 unimodular matrices. For this, direct analytical transformations are made and two sign functions $f_{q,i}$, $F_{p,i}$, are introduced by authors [8]:

$$\mathbf{L}(T)=\sum_{p=1}^{2^N}(-1)^{\sum_{i=1}^{N}\left[\sum_{k_i=2i-1}^{2i}k_iF_{p,i}\right]}\sqrt{\det\boldsymbol{\eta}_{N,p}} \times\sum_{q=1}^{2^{N-1}}\frac{1}{2^{N-1}}\varsigma_{pq}\mathbf{L}_{pq} \quad (6)$$

where the 4x4 unimodular matrix $\mathbf{L}_{pq}$ is following

$$\mathbf{L}_{pq}=\begin{vmatrix}\frac{\eta_{N,p}^{(1,1)}}{\sqrt{\det\eta_{N,p}}}\mathbf{M} & \frac{\eta_{N,p}^{(1,2)}}{\sqrt{\det\eta_{N,p}}}\mathbf{M}\\ \frac{\eta_{N,p}^{(2,1)}}{\sqrt{\det\eta_{N,p}}}\mathbf{M} & \frac{\eta_{N,p}^{(2,2)}}{\sqrt{\det\eta_{N,p}}}\mathbf{M}\end{vmatrix} \quad (7)$$

where the elements of the matrix **M** are following:

$$M_{11}=\sqrt{\frac{\sum\limits_{k_N=2N-1}^{2N}\left(\Omega_{k_N}F_{p,N}f_{q,N}\right)}{\sum\limits_{k_1=1}^{2}\left(\Omega_{k_1}F_{p,1}\right)}}\times\cos\phi_{pq}= \eta_{pq}^{(1,1)}\cos\varphi_{pq};$$

$$M_{12}=\sqrt{\frac{\sum\limits_{k_1=1}^{2}\left(\Omega_{k_1}F_{p,1}\right)}{\sum\limits_{k_N=2N-1}^{2N}\left(\Omega_{k_N}F_{p,N}f_{q,N}\right)}}\times\cos\phi_{pq}= \eta_{pq}^{(1,2)}\cos\varphi_{pq}; \quad (8)$$

$$M_{21}=-\frac{j}{\sqrt{\sum\limits_{k_1=1}^{2}\left(\Omega_{k_1}F_{p,1}\right)\times\sum\limits_{k_N=2N-1}^{2N}\left(\Omega_{k_N}F_{p,N}f_{q,N}\right)}}\times\sin\phi_{pq}= \eta_{pq}^{(2,1)}\cos\varphi_{pq};$$

$$M_{22}=-j\sqrt{\sum_{k_1=1}^{2}\left(\Omega_{k_1}F_{p,1}\right)\times\sum_{k_N=2N-1}^{2N}\left(\Omega_{k_N}F_{p,N}f_{q,N}\right)}\times\sin\phi_{pq} \eta_{pq}^{(2,2)}\cos\varphi_{pq};$$

The coefficient ς_{pq} is

$$\varsigma_{pq}=\sqrt{\frac{\sum_{k_N=2N-1}^{2N}\left(\Omega_{k_N}F_{p,N}f_{q,N}\right)}{\sum_{k_N=1}^{2}\left(\Omega_{k_1}F_{p,1}\right)}}\times \prod_{i=1}^{N-1}\left(1+\sum_{k_i-2i-1}^{2i}\frac{\Omega_{k_i+1}F_{p,i+1}f_{q,i+1}}{\Omega_{k_i}F_{p,i}f_{q,i}}\right) \quad (9)$$

The elements $\left|\eta_{N,p}^{(m,n)}\right|_1^2$ are determined as the elements of the matrix that is equal to a product of the matrix with the elements (4) of each cell with the constant parameters:

$$\left|\eta_{N,p}^{(m,n)}\right|_1^2=\prod_{i=1}^{N}\left\{\sum_{k_i=2i-1}^{2i}\left[\left|\eta_{k_i}^{(m,n)}\right|_1^2F_{q,i}\right]\right\} \quad (10)$$

The value φ_{pq} is equal to

$$\varphi_{pq}=\sum_{i=1}^{N}\left[\sum_{k_i=2i-1}^{2i}\left(\Omega_{k_i}F_{p,i}f_{q,i}\Delta t_i\right)\right] \quad (11)$$

C. The equivalent mods

Since (6)-(11) the resulting wave can be represented as a superposition of the 2^{2N-1} modes which can be decomposed into the *p* groups of the *q* modes in each according to (6). Introducing the notation

$$\alpha_p = \frac{1}{T}\mathrm{Ln}\sqrt{\det \eta_{N,p}}$$

$$\alpha_{pq} = \frac{1}{T}\mathrm{Ln}\frac{\varsigma_{pq}}{2^{N-1}} \quad (12)$$

we obtain

$$\mathbf{L}(T) = \sum_{p=1}^{2^N} (-1)^{\sum_{i=1}^{N}\left[\sum_{k_i=2i-1}^{2i} k_i F_{p,i}\right]} \times$$

$$e^{\alpha_p t} \sum_{q=1}^{2^{N-1}} \frac{1}{2^{N-1}} e^{\alpha_{pq}} \mathbf{L}_{pq} \quad (13)$$

Let's call α_p as the characteristic exponent of the *p*-th group, and α_{pq} as the characteristic exponent of the *q*-th wave in the *p*-th group. Thus, the resulting oscillation can be represented as the spectrum of the 2^N oscillation groups with 2^{N-1} modes in each group. The φ_{pq} has the physical meaning of the electromagnetic thickness of the *pq* mode. In other words we decomposed the resultant oscillation in a photonic crystal into a finite spectrum for the first time.

It should be noted that the matrix (6) is unimodular since the Ostrogradskii–Liouville theorem [10] as the trace of the coefficient matrix of the initial system of differential equations (1) is equal to zero. From the physical point of view unimodularity of a fundamental solution matrix means the satisfaction of the energy conservation law [11].

IV. The sign-functions

Note that expression (6) and (13) are obtained due to using the sign functions $f_{q,i}$, $F_{p,i}$ [8]:

$$f_{q,i} = sign\left\{\sin\left[\frac{\pi}{2^{N+1-i}}(2q-1)\right]\right\}$$

$$F_{p,i} = \frac{1}{2}\left\langle 1+(-1)^{k_i+1} sign\left\{\sin\left[\frac{\pi}{2^{N+1-i}}(2q-1)\right]\right\}\right\rangle \quad (14)$$

The function $f_{q,i}$ has been described in detail in [12]. Physically $f_{q,i}$ determines the phases with which the interval eigenmodes interact with each other. Mathematically $f_{q,i}$ describes the binary law of the sing change of the phase of eigenmodes in the resulting phase of the equivalent mode.

The detailed description of the function $F_{p,i}$ is presented here for the first time. This function determines the interaction order of the eigenmodes of the constant parameter cells. Thus, for example, the eigenwaves of right-handed polarization of cells with constant parameters interact with each other only or the eigenwaves of left-handed polarization interact with each other only too in the classical model. Right polarization waves cannot interact with left polarization waves. Thus, the function can take two values: either one or zero. The values of the function for three cells with constant parameters are presented in Table 1.

TABLE I. THE VALUES OF THE FUNCTION $F_{p,i}$

p	***i*=1 *k*₁=1**	***i*=2 *k*₂=3**	***i*=3 *k*₃=5**	***i*=1 *k*₁=2**	***i*=2 *k*₂=4**	***i*=3 *k*₃=6**
1	1	1	1	0	0	0
2	1	1	0	0	0	1
3	1	0	1	0	1	0
4	1	0	0	0	1	1
5	0	1	1	1	0	0
6	0	1	0	1	0	1
7	0	0	1	1	1	0
8	0	0	0	1	1	1

Here *p* is the group number, *i* is the number of the cell, k_i with the odd indices describe right polarization modes, and k_i with the even indices are related to the left-polarization modes.

V. The solution stability cjnditions

In accordance with Lyapunov's theory [13] the solutions of the differential equation system will be stable if all eigenvalues of the fundamental solution matrix are less than one or these are equal to one $\lambda_i \le |1|$. The eigenvalues must be found as the roots of the fourth-order characteristic equation

$$\lambda^4 + C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 = 0 \ , \quad (15)$$

where

$$C_0 = \det \mathrm{L}(\Lambda) = 1 \ ;$$

$$C_1 = -\sum_{i=1}^{2}\sum_{j=i+1}^{3}\sum_{k=j+1}^{4}\left(l_{ii}l_{jj}l_{kk} + l_{ij}l_{jk}l_{ki} + l_{ik}l_{ji}l_{kj} - l_{ik}l_{ji}l_{ki} - l_{ij}l_{ji}l_{kk} - l_{ii}l_{jk}l_{kj}\right);$$

$$C_2 = \sum_{i=1}^{3}\sum_{j=i+1}^{4}\left(l_{ii}l_{jj} - l_{ij}l_{ji}\right) \ ; \quad (16)$$

$$C_3 = -tr\ \mathrm{L}(\Lambda) \ .$$

Here l_{ij} are the elements of the fundamental solution matrix, and also $C_1 = C_3$ since the matrix is unimodular.

Our aim is to express the stability conditions as the functions the elements of the fundamental solution matrix. The equation (15) can be decomposed into two quadratic equations

$$\lambda^2 + b_1\lambda + 1 = 0 \ ;$$

$$\lambda^2 + b_2\lambda + 1 = 0 \quad , \tag{17}$$

the coefficients of (17) satisfy the relations $b_1 + b_2 = C_1$, $2 + b_1 b_2 = C_2$. Having expressed b_1, b_2 through C_1, C_2 we obtain:

$$b_1 = \frac{C_1}{2} + \sqrt{\frac{C_1^2}{4} - (C_2 - 2)};$$

$$b_2 = \frac{C_1}{2} - \sqrt{\frac{C_1^2}{4} - (C_2 - 2)}. \tag{18}$$

Then the eigenvalues of the fundamental solution matrix for a period can be written in the analytical form

$$\lambda_{1,2} = -\frac{C_1}{4} - \sqrt{\frac{C_1^2}{16} - \frac{C_2 - 2}{4}} \pm \sqrt{\frac{1}{4}\left[\frac{C_1}{2} + \sqrt{\frac{C_1^2}{4} - (C_2 - 2)}\right]^2 - 1}$$

$$\lambda_{3,4} = -\frac{C_1}{4} + \sqrt{\frac{C_1^2}{16} - \frac{C_2 - 2}{4}} \pm \sqrt{\frac{1}{4}\left[\frac{C_1}{2} + \sqrt{\frac{C_1^2}{4} - (C_2 - 2)}\right]^2 - 1} \tag{19}$$

Then the system will be stable if the conditions

$$\begin{cases} \left| C_1 + \sqrt{C_1^2 - 4(C_2 - 2)} \right| \le 4 \\ \left| C_1 - \sqrt{C_1^2 - 4(C_2 - 2)} \right| \le 4 \end{cases} \tag{20}$$

are satisfied.

Physically a stability region of the solutions of the system (1) means a wave propagation region (a passband in a crystal), an instability region corresponds to an unpropagation region of a wave (or the stopbands in a crystal).

Investigation of the fundamental solution matrix gives us the following interesting result: changing the order of the cells with constant parameters within a period does not change the structure of the stability regions if a period is invariable. The physical explanation of this phenomenon is based on the fact that a multiple reflection phenomenon is observed in a one-dimensional photonic crystal. And existence of passbands (propagation regions) is determinate by this phenomenon but not Fresnel reflection. Passbands (propagation regions) are corresponded to the system parameters for which the re-reflected waves add up in phase at the end of a period. Changing the order of the cells within a period does not change the phase shift for a period. Consequently the structure of passbands (propagation regions) is invariable.

VI. Conclussion

In this paper the accurate analytical method for investigating one-dimensional photonic crystals is presented. The fundamental solution system of the fourth order differential equations (1) describing wave behavior within a photonic crystal is found. The 4x4 fundamental solution matrix in this case is unimodular in accordance with the Ostrogradsky–Liouville theorem [10,13]. From the physical point of view this fact is a consequence of energy conservation law in the considered electromagnetic systems. Note also that the fundamental system is obtained as the finite sum of unimodular matrices with the corresponding contribution coefficients. Thus the resulting oscillation in the system is represented as the finite spectrum of the so-called equivalent modes. The resulting equivalent modes are divided into the 2^N groups with 2^{N-1} modes in each group. It is found that the law of sign changing of the eigenmode phases of intervals with constant parameters in equivalent modes is binary and it can be described by the sign-function $f_{q,i}$.

Conditions of the solution stability are investigated and the analytical expressions defining the edges of stable solution regions are found. From the physical point of view regions of stable solutions are wave propagation regions (passband) in a photonic crystal under consideration, the instability regions correspond to the unpropagation regions (stopbands).

It is also shown that changing the order of cells within a period does not affect the structure of the solution stability regions.

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