# Computational Algorithm for Optimal Control of an Object with Distributed Parameters in a Nonsmooth Area of Final States

Mikhail Y. Livshits, Aleksey V. Nenashev,
Yuliya E. Pleshivtseva
Chair of control and system analysis of heat-energy and socio-technical complexes
Samara State Technical University
Samara, Russia
usat@samgtu.ru

***Abstract*— The effective computational algorithm is proposed for solving boundary-value problems of time-optimal and maximum accuracy control with a minimax estimation of the deviation of the final trajectory from a given state. The problem is reduced to a nonconvex nonlinear programming problem. An example of the solution of the test optimal control problem for induction heating of a cylindrical billet is given.**

***Keywords— distributed parameters; boundary-value problem; optimality criterion; search procedure; local minimum; global minimum***

## I. INTRODUCTION

Practical effectiveness of methods and algorithms for solving problems of optimal control of technological and production processes is determined by the effectiveness of computational methods and corresponding software.

The determining factor in choosing a computational method is its adequacy to the optimal problem posed. Therefore, an additional and mathematically correct substantiation is required for the widely used numerical methods of approximate solution of optimal problems based on the limitation of an infinite number of relations, for example, in a distributed method of moments and (or) the nonconvex and nonsmooth approximation of quality criterion by convex region [1, 10], especially applied to infinite-dimensional objects with distributed parameters (ODP). Mathematically rigorous analysis shows that this often replaces the formulation of the optimal problem and, in place of the initially posed boundary value problem with fixed ends of the optimal trajectory, the problem with the movable or free right end of the optimal trajectory in the corresponding domain of a finite-dimensional or infinite-dimensional state space is actually solved in the case of ODP.

The main difficulties in implementing computational methods for ODPs are related not only to the necessity of their finite-dimensional approximation in most known cases. There is also a problem of the choice of convenient in computational practice metrics determining the initial and final regions of the corresponding optimal boundary value problems that are adequate to the technology-based estimations of these regions. For some optimization problems the errors of achieving a given set can reach 200% because of incorrect choice of the computational method of optimization problems solving [1, 2, 8, 9].

The work was supported by the Russian Foundation for Basic Research projects No. 17-08-00593 and No. 16-08-00945

## II. FORMULATION OF THE PROBLEM

To simplify the presentation of the proposed numerical method, we will restrict ourselves to the one-dimensional boundary value problem of optimal control of ODP, which does not lead to loss of generality of the algorithm of the computational procedure. It is essential that even with the theoretical fixation of the right end of the optimal ODP' trajectory, the measurement error, the truncation of infinite-dimensional models, the error in numerical implementation and etc., do not allow to solve this problem precisely in practice. Therefore, actually the problem with a movable right end trajectory is solved. However, in this case there is a significant loss of the optimality criterion because of an incorrect substitution of the problem statement. The effectiveness of the solution is determined at the stage of formulation of boundary-value optimal problems and depends in many respects on the formulation of the metric for estimation of the final states set [1, 9].

Let an ODP is described by a linear one-dimensional differential equation of parabolic type

$$\frac{\partial\Theta(l,\varphi)}{\partial\varphi}-\frac{\partial^2\Theta(l,\varphi)}{\partial l^2}-\Gamma l^{-1}\frac{\partial\Theta(l,\varphi)}{\partial l}=F(l,\varphi),\quad l\in(0,1),\ \varphi\in(0,\infty) \tag{1}$$

with boundary conditions:

$$\Theta(l,0)=\Theta_0;\ \frac{\partial\Theta(0,\varphi)}{\partial l}=0;\ \frac{\partial\Theta(1,\varphi)}{\partial l}=q(\varphi) \tag{2}$$

Here $l,\varphi$ are the relative spatial and time (Fourier number) arguments, $\Gamma$ is the parameter of the form ($\Gamma$ = 1 for a cylindrical coordinate system, $\Gamma$ = 0 for a Cartesian coordinate system), $F(l,\varphi)$ and $q(\varphi)$ are the given functions of their

**978-1-5386-5824-6/18/$31.00 ©2018 IEEE**

arguments. Let us consider two boundary-value problems of optimal control of the object (1) – (2) with the variable right-end point of the trajectory in infinite non-smooth area of admissible final states $\bar{\Omega}_\Theta = \left\{\Theta(l,\varphi) : \max_{l\in[0,1]}\left|\Theta(l,\varphi_0) - \Theta^*(l)\right| \le \varepsilon\right\}$ for a given $\varepsilon = \varepsilon_э$ or for the least attainable value of error $x_{R+i} = x_{c_2}, x_{R+i+1} = x_{c_1}$ $\varepsilon = \varepsilon_{min}^{(i)}$ for the i-th class of controls in the domain

$$\bar{\Omega}_u = \left\{u(\varphi) : U_{min} \le u(\varphi) \le U_{max}\right\} \tag{3}$$

of admissible controls $u(\varphi)$ for :

- time-optimal control problem

$$j_\varphi^{onm} = \min_{u(\varphi)\in\Omega_u} \varphi_0\Big|_{\Theta\in\bar{\Omega}_\Theta,\ \varepsilon=\varepsilon_э} \tag{4}$$

- maximum accuracy problem

$$j_\varepsilon^{onm} = \min_{u(\varphi)\in\Omega_u} \max_{l\in[0,1]}\left|\Theta(l,\varphi_0) - \Theta^*(l)\right| \tag{5}$$

Here $\Theta^*(l)$ is prescribed final distribution $\Theta(l,\varphi) = \Theta^*(l)$ in the end moment $\varphi = \varphi_0$ of the process, $0 < \varphi_0 < \infty$.

As the control function $u(\varphi)$ we can consider the time component $v(\varphi)$ of the right side of equation (1) in multiplicative form [3]: $F(l,\varphi) = w(l)v(\varphi)$ which has the meaning of intensity of heat sources, if control object (1) – (2) represents a boundary-value problem of heat conduction. There are other options for assigning a control action in the control problems when analyzing the relevant technological or production situation. An effective means of solving these problems is the alternate method of optimization (AMO) [1]. This method has become widespread for solving applied problems of optimal control of various technological processes, but its most effective application is for ODP, which include processes of heat and mass transfer of technological thermal physics (various kinds of heating, chemical-thermal treatment, filtration, etc.). AMO adequately takes into account the incomplete controllability of the process and the non-smoothness of a priori given set $\bar{\Omega}_\Theta$ of final states $\Theta(l,\varphi_0)$ and allows determining the optimal trajectory leading inside this set in the class of realizable controls. A necessary condition for the use of AMO is the parametric representation of control $u(\varphi)$. The parametric representation of the control action allows us to reduce the solution of problems (4) and (5) to the search for the number $i$ and numerical value $\Delta^{(i)} = \left\{\Delta_1^{(i)}, \Delta_2^{(i)}, \ldots\ldots, \Delta_i^{(i)}\right\}$ of the controlling optimized parameters and the least attainable value of error in each i-th subset of controls $\varepsilon = \varepsilon_{min}^{(i)} = \max_{l\in[0,1]}\left|\Theta(l,\varphi_0^{(i)},\Delta^{(i)}) - \Theta^*(l)\right|$ for the uniform approximation of the final profile $\Theta(l,\varphi_0^{(i)},\Delta^{(i)})$ to the desired profile $\Theta^*(l)$ which is prescribed according to the production and technological requirements. In this case And the process time achievable in each parametric i-th control class $\varphi_0 = \varphi_0^{(i)}$ will take the lowest possible value. Unknown control parameters $\Delta^{(i)}$ and the least attainable in i-th control class error $\varepsilon_{min}^{(i)}$ in the optimal problems (4) and (5) are determined by solving a system of transcendental equations developed according to the AMO procedure:

$$\tilde{\Theta}(l,\Delta^{(i)})\Big|_{l=l_{nj}} = (-1)^j\,\vartheta\varepsilon;\ \left.\frac{\partial\Theta(l,\Delta^{(i)})}{\partial l}\right|_{l=l_{эm}} = 0 \tag{6}$$

where $\tilde{\Theta}(l,\varphi_0^{(i)}) = \Theta(l,\varphi_0^{(i)}) - \Theta^*(l)$, $j = \overline{1,R}\,(R{=}i$ *if* $\varepsilon_{min}^{(i-1)} > \varepsilon = \varepsilon_э > \varepsilon_{min}^{(i)}; R = i+1 if\ \varepsilon = \varepsilon_{min}^{(i)}), m = \overline{1,r}$ $l_{nj} \in \left\{l_{nj} : [0, l_{э1}, \ldots l_{эr}, 1], r = R-1, r=R\right\}, 0 \le l_{n1} < l_{n2} < \ldots < l_{nR}, \vartheta = \pm 1$.

The solution of the sequence of problems of maximum accuracy (5) allows forming a series of inequalities:

$$\varepsilon_{min}^{(1)} > \varepsilon_{min}^{(2)} > \ldots > \varepsilon_{min}^{(\alpha)} = \varepsilon_{inf} \ge 0 \tag{7}$$

The series (7) provides a possibility for choosing the number i of the parameters of the control actions according to the least attainable absolute error $\varepsilon_{min}^{(i)}, i = 1,2,\ldots,\alpha$ when the final profile is reached in each i-th class of optimal control.

## III. COMPUTATIONAL ALGORITHM OF AMO

The basis of the algorithm is a sequential solution of the system of transcendental algebraic equations (6) to determine the terms of the series (7). In this case, each term $\varepsilon_{min}^{(i)}$ of the series (7) is determined in the course of solving the problem (5) by minimizing the discrepancies $f_j(l_{nj},\Delta^{(i)},\varepsilon)$, $f_m(l_{эm},\Delta^{(i)},\varepsilon)$ of the solution of system (6): $\tilde{\Theta}(l,\Delta^{(i)})\Big|_{l=l_{nj}} - (-1)^j\,\vartheta\varepsilon = f_j(l_{nj}, \Delta^{(i)},\varepsilon); \left.\frac{\partial\Theta(l,\Delta^{(i)})}{\partial l}\right|_{l=l_{эm}} = f_m(l_{эm},\Delta^{(i)},\varepsilon)$.

To do this, we introduce an evaluation of the discrepancies in the form of a penalty function:

$$J(l_{nj},\Delta^{(i)},\varepsilon) = J_{big}(l_{nj},\Delta^{(i)},\varepsilon) + J_{small}(l_{nj},\Delta^{(i)},\varepsilon) \tag{8}$$

where $J_{big}(l_{nj},\Delta^{(i)},\varepsilon) = K\left[f_{abs}\right]^2$; $J_{small}(l_{nj},\Delta^{(i)},\varepsilon) = K\sqrt{f_{abs}}$; $f_{abs} = \sum_{j=1}^{R}\left|f_j(l_{nj},\Delta^{(i)},\varepsilon)\right| + \sum_{m=1}^{r}\left|f_m(l_{эm},\Delta^{(i)},\varepsilon)\right|$.

The representation of the evaluation in the form (8) is formulated with the aim of reducing the probability of occurrence of "ravines" because the component $J_{big}(l_{nj},\Delta^{(i)},\varepsilon)$ significantly increases the value $J(l_{nj},\Delta^{(i)},\varepsilon)$, if $f_{abs} > 1$ and decreases, if $f_{abs} < 1$. The component $J_{small}(l_{nj},\Delta^{(i)},\varepsilon)$ acts on the contrary. The scaling factor K provides an increase in the

increment of the evaluation on the segment $f_{abs} \in (\gamma; \delta)$, where the rate of change of the evaluation (8) assumes minimum values, which can lead to the appearance of "ravines" in the specified range. The evaluation (8) is always positive and can be non-convex depending on the optimized process, however, if the system (6) has a solution, the absolute minimum $J_{inf} = J\left(l_{nj}, \Delta^{(i)}, \varepsilon\right) = 0$. The problem of searching for parameters $\Delta^{(i)}$ in each i-th control class is reduced to determination of the coordinate of the absolute minimum $J_{inf} = 0$ of the evaluation (8) defined in the multidimensional area $R_1^c = \left(l_{nj} \in [0, 1], \Delta^{(i)} \in (0, \infty), \varepsilon\right), c_1 = \dim R_1^c = 2(i+1)$ in the case of solving problem (5) and $R_2^c = \left(l_{nj} \in [0, 1], \Delta^{(i)} \in (0, \infty)\right)$, $c_2 = \dim R_2^c = 2i$ in the case of solving problem (4).

Thus, to solve system (6) it is necessary to solve a nonlinear non-convex mathematical programming problem: $J\left(l_{nj}, \Delta^{(i)}, \varepsilon\right) \to \min_{R_{1,2}^c} J\left(l_{nj}, \Delta^{(i)}, \varepsilon\right)$ under conditions (1), (2), (3). For simplicity let us introduce the notation: $x_1 = l_{n1}, \ldots, x_R = l_{nR}, x_{R+1} = \Delta_1^{(i)}, \ldots, x_{R+i} = \Delta_i^{(i)}, x_{R+i+1} = \varepsilon,$ where $x_{R+i} = x_{c_2}, x_{R+i+1} = x_{c_1}$ then $R_{1,2}^c = \left\{x_1, \ldots, x_{c_{1,2}}\right\}$; $J\left(l_{nj}, \Delta^{(i)}, \varepsilon\right) = J\left(x_1, \ldots, x_{c_{1,2}}\right) = J(X), X = \left(x_1, \ldots, x_{c_{1,2}}\right)$. The procedure for determining the global minimum of function (8) must satisfy the following requirements. It is necessary: to take into account the non-convex and non-linear character of $J(X)$; to provide search in the zone of "ravines"; to provide effective search under conditions of increased dimension of the definition domain $R_{1,2}^c$; to perform the calls of functions under conditions of limited computing resources.

Modern and fairly efficient algorithms of searching for the absolute extremum [1, 6] within the closed domain of multidimensional non-linear non-convex functions are known. The most popular of which are random search methods, for example, the simulated annealing method, genetic algorithms, the interval method of branches and boundaries, and others. All of them are characterized by high computational complexity, and the number of computational operations required for searching represents a power-law dependence on the dimensions number $\dim R_{1,2}^c$ of the definition domain of the optimized function. Such disadvantages are avoided in V.K. Chichinadze's $\Psi$-transformation algorithm [7], which provides the transformation of an optimized multidimensional function (8) into a one-dimensional continuous monotonically decreasing numerically given metric $\Psi(\varsigma)$ of an optimized function whose zero coincides with the absolute extremum $J(X)$. The computational complexity of the $\Psi$-transformation algorithm is not significant. The main consuming of computation time occurs at the stage of data preparation, because at the initial stage of the $\Psi$-transformation algorithm it is necessary to calculate $J(X)$ in S points $X_k \in R_{1,2}^c, k = 1, 2, \ldots, S$ uniformly and regularly distributed in the space $R_{1,2}^c$. The number S of points $X_k$ directly influences on the accuracy of the searching for the extremum by the $\Psi$-transformation method. The minimum $J_{min}$ and maximum $J_{max}$ values are chosen among the calculated values $J(X_k)$, and then the interval $\left[J_{min}; (J_{max} - J_{min}) 2^{-1}\right]$ is divided into N equal parts. Let us determine the estimated levels by the argument $\varsigma$ according to the formula: $\varsigma_\nu = (J_{max} - J_{min}) 2^{-1} - (\nu - 1)\Delta\varsigma$, $\Delta\varsigma = \varsigma_{\nu-1} - \varsigma_\nu$, $\nu = 1, 2, 3, \ldots, N$. Then it is possible to calculate the values $\Psi_\nu = \xi_\nu / S$, where $\xi_\nu = \sum_{k=1}^{S} g_k^\nu$, $g_k^\nu = \begin{cases} 1, & J(X_k) \le \varsigma_\nu \\ 0, & J(X_k) > \varsigma_\nu \end{cases}$.

We will introduce the function $\Psi(\varsigma)$ in an analytical form through approximation on a set $\varsigma_\nu$ of values $\Psi_\nu$, for example, by a polynomial form of the second order:

$$\Psi(\varsigma) = \alpha_0 \varsigma^2 + \alpha_1 \varsigma + \alpha_2 \tag{9}$$

where the coefficients $\alpha_0, \alpha_1, \alpha_2$ are obtained, for example, by the method of least squares. Let us find the smallest root $\varsigma_m = J(X_m)$ of the polynomial equation (9) that is close to the global minimum $J_{inf}$ of function (8). At the next step, we will find a vector $X_m = \left\{x_1^m, x_2^m, \ldots \ldots, x_{c_{1,2}}^m\right\} \in R_{1,2}^c$ whose terms are determined by substituting $\varsigma_m$ into approximating polynomials: $x_t^m(\varsigma) = \beta_0^t \varsigma^2 + \beta_1^t \varsigma + \beta_2^t$, $t = 1, 2, \ldots, c_{1,2}$ introduced similarly to the polynomial (9).

To reduce error of determining $X_m$ and increase the probability of determining a global extremum $J_{inf} = 0$, the obtained results can be used as the initial conditions for a local search algorithm. For example the algorithm of J. Nedler and R. Mead satisfies the requirements to the greatest extent [13]. It is an improved version of the simplex algorithm and does not demand for simplex form, which makes it possible to remove the problem of comparability of dimensions of variables. This algorithm offers a simple and flexible system for resizing simplexes without recalculating all the values of the objective function included in the simplex and provides more faster searching than the classical simplex search algorithm due to using the information about previous iterations. Algorithm parameters are: reflection coefficient $\alpha > 0$; compression ratio $\beta > 0$; coefficient of stretching $\gamma > 0$; search error $\varepsilon_n$. The values of the coefficients are chosen arbitrarily. Typically we can assume: $\alpha = 1$; $\beta = 0.5$; $\gamma = 2$. The value $X_m$ can be used as a reference point, and $c_{1,2} = \dim R_{1,2}^c$ points with coordinates can be selected according to the following expressions:

$$\begin{aligned} X_1 &= \left\{x_1^m + \varepsilon_\varphi x_1^m, x_2^m, x_3^m, \ldots, x_{c_{1,2}}^m\right\} \\ X_2 &= \left\{x_1^m, x_2^m + \varepsilon_\varphi x_2^m, x_3^m, \ldots, x_{c_{1,2}}^m\right\} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ X_{c_{1,2}} &= \left\{x_1^m, x_2^m, x_3^m, \ldots, x_{c_{1,2}}^m + \varepsilon_\varphi x_{c_{1,2}}^m\right\} \end{aligned} \tag{10}$$

which form a simplex in the argument space $R^c_{1,2}$ of the function (8). it is possible to define the values $J(X_z)$, $z=\overline{0,c_{1,2}}$, $X_0 = X_m$ at the points (10). Thus an array of simplex vertexes $V[c_{1,2}+1]$ is obtained where each member of the array is a point $X_z$. Then the following sequence of operations should be performed.

1. The array $V[c_{1,2}+1]$ should be sorted in descending order by $J(V[z])$.

2. Three points $V[0], V[1], V[c_{1,2}]$ from the vertices of the simplex should be chosen. To reduce the value $J_{V[0]}$ the center of gravity of all points should be determined excepting for $V[0]: X_h = \{x_{h1}, x_{h2}, x_{h3}, \ldots, x_{hc_{1,2}}\}$ where $x_{ht} = c_{1,2}^{-1}\sum_{z=1}^{c_{1,2}} x_{zt}$, $t = \overline{1,c_{1,2}}$, it is essential that $J(X_h)$ is not calculated.

3. By reflecting the position of point $V[0]$ with respect to $X_h$ with the reflection coefficient α the point with coordinates $x_{Ht} = (1+\alpha)x_{ht} - \alpha x_{V[0]t}$, $t = 1,2,\ldots,c_{1,2}$ should be obtained.

4. The value $J(X_H)$ should be sequentially compared with $J_{V[0]}$, $J_{V[1]}$, $J_{V[c_{1,2}]}$.

4a. If $J(X_H) < J_{V[c_{1,2}]}$ the simplex is stretched and a new point $X_e$ with coordinates $x_{et} = (1+\gamma)x_{ht} - \gamma x_{Ht}, t = 1,2,\ldots,c_{1,2}$ is got.

If $J(X_e) < J_{V[c_{1,2}]}$, $V[c_{1,2}] = X_e$, then go to step 8.

If $J(X_e) > J_{V[c_{1,2}]}$, $V[0] = X_H$, then go to step 8.

4b. If $J_{V[c_{1,2}]} < J(X_H) < J_{V[1]}$, $V[0] = X_H$, then go to step 8.

4c. If $J_{V[1]} < J(X_H) < J_{V[0]}$, $V[0] = X_H$, then go to step 5.

4d. If $J(X_H) > J_{V[0]}$, then go to step 5.

5. The simplex should be squeezed to the point $X_s$ with coordinates $x_{st} = (1-\beta)x_{ht} + \beta x_{V[0]t}$, $t = 1,2,\ldots,c_{1,2}$.

6. If $J(X_s) < J_{V[0]}$, $V[0] = X_s$, then go to step 8.

7. If $J(X_s) > J_{V[0]}$, the simplex should be squeezed to the point $V[c_{1,2}]$ and the coordinates of the remaining points should be recalculated: $V[z]$, $x_t^z = x_{V[c_{1,2}]t} + \left(x_{V[z]t} - x_{V[c_{1,2}]t}\right)2^{-1}, t = \overline{1,c_{1,2}}, z = \overline{0,c_{1,2}-1}$.

8. The length of the edges of the simplex should be calculated. If at least one edge is longer than $\varepsilon_n$, the algorithm should be performed from step 1, otherwise $X_{rez} = V[c_{1,2}]$ and the searching procedure is finished.

## IV. CONCLUSION

The proposed computational procedure has been tested using the known solved problem of the process of induction heating of an infinite cylindrical billet [3], under conditions

$\Theta_0(l) \equiv \Theta_0 = \Theta_a = const, W(\xi,l) = \xi\left(ber'^2(\xi l) + bei'^2(\xi l)\right) / \left/ \left(ber(\xi)ber'(\xi) + bei(\xi)bei'(\xi)\right), \xi = 4, Bi = 0.7, \Theta^*(l) = \Theta^* = 0.5\right.$.

The control parameters $\Delta^{(i)}$ for this problem are the lengths of the constancy intervals of the control $u(\varphi)$ gets value $u = U_{max} = 1$, $U_{min} = -U_{max}$ at these intervals. As a result of the proposed search procedure, parameters of the optimal control accuracy were obtained: $\Delta_1^{(1)} = 0.3475, \Delta_1^{(2)} = 0.3506, \Delta_2^{(2)} = 0.0403$ A well-known solution [3] of the same problem gives the following values: $\Delta_1^{(1)} = 0.349, \Delta_1^{(2)} = 0.35, \Delta_2^{(2)} = 0.04$

Thus, it can be stated that the proposed algorithm has an enough accuracy, meets the requirements and provides an efficient process of searching for optimal control parameters $\Delta^{(i)}$.

## REFERENCES


[1] Rapoport E.Ya. Alternansny metod v prikladnikh zadachakh optimizatsii [An alternative method in applied optimization problems.] Moscowю Science Publ. 2000. 335 p. (in Russian)

[2] Vasilyev F.P. Chislennie metodi resheniya ekstremalnikh zadach. [Numerical methods for solving extremal problems.] Moscow, Science Publ. 1980. 520 p. (in Russian)

[3] Rapoport E.Ya., Pleshivtseva Yu.E. Optimalnoe upravlenie temperaturnymi rezhimami induktsionnogo nagreva. [Optimum control of temperature modes of induction heating.] Moscow, Science Publ. 2012. 309 p. (in Russian)

[4] Samarsky A.A. Vvedenie v chislennye metody. [Introduction to numerical methods.] Moscow, Science Publ. 1982. 271 p. (in Russian)

[5] Chichinadze V.K. Reshenie nelineynykh nevypuklykh zadach optimizatsii. [Solution of nonconvex nonlinear optimization problems.] Moscow, Science Publ. 1983. 256 p. (in Russian)

[6] Livshitc M.Yu., Yakubovich E.A. Optimal Control of Technological Process of Carburization of Automotive Gears. Materials Science Forum. V. 870, 2016. p. 647-653 (in Russian)

[7] Livshits M.Yu. System optimization of processes of heat and mass transfer of technological thermophysics. Sbornik trudov XXIX mezhdunarodnoy konpherentsii pod obshchey redactsiey A.A. Bolshakova. Matematicheskie metody v tekhnike i tekhnologiyakh. [Proceedings of the XXIX International Scientific Conference under the general editorship of A.A. Bolshakov. Mathematical methods in engineering and technology]. P. 11, 2016. Pp. 104-114. (in Russian)

[8] Livshitc M.Yu., Sizikov A.P. Multi-Criteria Optimization of Refinery: EPJ Web of Conferences. Thermophysical Basis of Energy Technologies 2015, Vol. 110, 2016. Article Number 01035. DOI 10.1051/epjconf/201611001035

[9] Nelder J.A., Mead R. A simplex method for function minimization. Computer Journal. 1965. Vol. 7, pp. 308–313.