Mathematical Models of the Quality Estimating of the Change Detection Algorithm of the Random Processes

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***Abstract*— Mathematical models have been developed to calculate the average number of delay steps of the detecting changes algorithm of the random processes properties in the case of the decision making statistics constructed on the principle of the discrepancy. The proposed models allow choosing the parameters of the algorithm with the worst-case orientation, which in real-time systems is important for ensuring the change point detection with a delay not exceeding a given value.**

Keywords*— *the moments of the random processes properties changes; the problem of the change point detection; quality measures; decision making statistics; quality estimating

I. INTRODUCTION

The task of detecting the moments of changing the properties of the random processes arises in many practical applications – beginning with such traditional ones as medical and technical diagnostics [1, 2] and ending with relatively new ones – the monitoring activity in the computer networks and analysis of the behavior of the software [3, 4]. Despite the fact that the new methods of processing both the primary signals and the information are actively developing, including natural computing [5, 6], nevertheless, interest in the classical approaches based on the statistical algorithms is remaining generally great [7, 8]. At the same time, the focus is on the tuning (determining the parameters of the algorithms) and on the modifying the detection algorithms, taking into account that a priori information about the processes parameters is often not available, and the modes of the primary statistics accumulations are limited [9–11].

The paper is devoted to the quality estimation of the algorithm constructed on the principle of the discrepancy. Developed mathematical models may be used for algorithm tuning.

Decision making statistic of the change point (CP) detection is defined as [12]:

$$G_k = \frac{\sum_{t=1}^{k}(g_t - 1)}{\sqrt{2k}}, k = \overline{1, N_C}\ , \quad (1)$$

$$g_t = \left\{ \frac{e_t - f(E_t^n, \theta^{(1)})}{\beta^{(1)}} \right\}^2 , \quad (2)$$

where $E_t^n = (e_{t-1}, e_{t-2}, ..., e_{t-n})$ – is the vector of the n previous samples of the stochastic process (SP) e_i, corresponding to the time moment t; $(\theta^{(i)}\beta^{(i)}) = (\theta^{(i)}{}_1, \dots, \theta^{(i)}{}_m, \beta^{(i)})$ – *(1+m)* – dimensional vectors of the parameters of the SP before (*i=1*) and after (*i=2*) CP; $f(E_t^n, \theta^{(1)})$ – is a known function; N_c – the accumulation period of the decision making statistic (the decision making statistic is reset $G_{N_C} = 0$ after performing the comparison operation since the growth rate of the decision making function decreases with the time).

The decision making rules can be represented as follows [12]. Let the hypothesis H^0 corresponds to the case of "change point", and the hypothesis H^1 to its absence, then:

$$\begin{aligned} H^0 &: |G_t| \le h \\ H^1 &: |G_t| > h \end{aligned} \quad (3)$$

Let us consider the case when the SP is represented by the first-order autoregressive model AR(1) (in this case $n = 1$) and then:

$$f(E_t^n, \theta^{(i)}) = (e_{t-1} - \mu^{(i)}), \quad (4)$$

$$(\theta^{(i)}\beta^{(i)}) = (\mu^{(i)}, a^{(i)}, \beta^{(i)}), \quad (5)$$

$$e_t = \mu^{(i)} + a^{(i)}(e_{t-1} - \mu^{(i)}) + \beta^{(i)} z_t\ , \quad (6)$$

where *i=1* – parameter values AR before CP, *i=2* – after CP; z_t – discrete white noise with $\mu=0$ and $\sigma=1$; $\beta^{2(i)} = \sigma^{2(i)}(1 - a^{2(i)})$ [12]. Further it is assumed that $a^{(1)} = a^{(2)} = a$, and the decision making statistic G_k is used to detect CP as "mean changes" and as "variance changes", since in most cases it is necessary to detect precisely these kinds of mismatch errors. The probability of a "false" detection CP by the decision making statistic (1) can be determined by using the algorithm operations representation in a finite Markov graphs form [13].

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II. Mathematical Models Of The Average Time Of Delays In Detection Of Change Point

It was shown [12] that $M\,[G_{k+1}]=0$ while CP has not occurred. Let $t=n_0$ is the moment of CP, then taking into account eq.(1)–(6), the estimation of the average time to reach the level h by the decision making statistic mean G_k is the average time delay of the CP detection:

$$M[G_{k+1}] = \frac{\sum_{n_0}^{k} \frac{(\mu^{(2)}-\mu^{(1)})^2(1-a)^2+\beta^{(2)2}-1}{\beta^{(1)2}}}{\sqrt{2k}} =$$

$$= (k-n_0+1)\frac{\frac{(\mu^{(2)}-\mu^{(1)})^2(1-a)^2+\beta^{(2)2}}{\beta^{(1)2}}-1}{\sqrt{2k}} =$$

$$= (k-n_0+1)\frac{\frac{(\mu^{(2)}-\mu^{(1)})^2(1-a)^2}{\sigma^{(1)2}(1-a^2)}+\frac{\sigma^{(2)2}-\sigma^{(1)2}}{\sigma^{(1)2}}}{\sqrt{2k}},$$

and

$$M[G_k] = \left\{u^2+2u+r^2\frac{1-a}{1+a}\right\}\frac{k-n_0+1}{\sqrt{2k}}, \tag{7}$$

where $u=\frac{\sigma_2-\sigma_1}{\sigma_1}$, $r=\mu_2-\mu_1$, and k is determined on the basis of

$$M[G_k]=h, \tag{8}$$

$$N_{\text{detect CP}} = k-n_0+1.$$

It is seen that the detection delay time of the algorithm depends on the time n_0 of the CP appearance (see eq.(7)). The following cases are possible:

1) The detection will occur in the same accumulation period in which the CP occurred, i.e.

$$\exists k:\ M[G_k]=h,\ k=\overline{n_0,N_C};$$

2) The detection will occur at the time moment j in the next accumulation period, i.e.

$$\forall k, M[G_k]<h,\ k=\overline{n_0,N_C}$$

$$\exists\, j: \frac{mj}{\sqrt{2j}}=h,\ j=\overline{1,N_C},$$

$$m=u^2+2u+r^2\frac{1-a}{1+a},\ u=\frac{\sigma_2-\sigma_1}{\sigma_1},\ r=\mu_2-\mu_1, \tag{9}$$

the time moment j is measured from the beginning of the second accumulation period.

3) The detection will never occur (in case of the undetectable CP) if the decision making statistic does not reach the threshold h during the whole accumulation period N_c

$$\forall k, M[G_k]<h,\ k=\overline{n_0,N_C}$$

$$\forall\, j: \frac{mj}{\sqrt{2j}}<h,\ j=\overline{1,N_C},$$

where m, u, r – are calculated by eq.(9).

CP will be detected at the same accumulation period when occurring if the values combination of n_0, h, N_c, r and u will provide the following condition:

$$(N_C-n_0+1)m \ge h\sqrt{2N_C} \tag{10}$$

This condition is more powerful and allows to focus on the worst case. Proceeding from this, we introduce the following notation

$$j^* = \left]2\mathrm{N_c}+1\text{-}\mathrm{h}\frac{\sqrt{2\mathrm{N_C}}}{\mathrm{m}}\right[, \tag{11}$$

where $]X[$ – the nearest whole, not more than X. Then, taking into account eq.(10), we can assume that the detection will occur on the first accumulation period (on the same one on which the CP has arisen), if $1\le n_0\le j^*$, while $j^*+1\le n_0\le N_C$ – on the second accumulation period.

At the same time, $n_0=(j^*+1)$ is the moment when a CP occurs but the decision making statistic does not have enough time to reach the threshold (due to reset after accumulation period N_c), and therefore detection will occur at the second period. From this point of view, this is the most unfavorable moment for the CP appearance, since in this case, the average delay time in detection will be the maximum with all other conditions being equal (increase by t), The waiting time t of the end of the current accumulation period will be shorter if CP moment will be closer to the end of the accumulation period (case 2).

In case 1 – at any moment of CP occurrence – the detection will be at the same period. The later the CP occurs, the longer the delay in detection due to the decrease in the growth rate of the decisive function over time.

The general condition for CP detecting is as follows:

$N_C m \ge h\sqrt{2N_C}$
or $j^*\ge 1$.

The average detection delay time is determined from the ratios, depending on the case under consideration:

1) $\overline{N}_{\text{detect CP}} = j_{\det}-n_0+1$, and $j_{\det}$ is found from the equation:

$$mj_{\det}-h\sqrt{2j_{\det}}+mn_0=0,$$

from where follows

$$j_{\det}=\frac{(h\sqrt{2}+\sqrt{2h^2+4m^2(N_C-1)})^2}{4m^2} \tag{12}$$

2) $\overline{N}_{\text{detect CP}} = N_C - n_0 + j_{обн} + 1$, and $j_{\det}$ is found from the equation:

$$mj_{\det} - h\sqrt{2\,j_{\det}} = 0\ ,$$

from where follows

$$j_{\det} = \frac{2h^2}{m^2}\ . \qquad (13)$$

In view of the fact that the actual time of occurrence of a CP is unknown, it is advisable to take the average number of the detection delay for the worst case CP time occurs.

To determine the presence of the moment of a CP occurrence, which may be detected in the second period of the decision making function accumulation, it is proposed to calculate the value of the parameter j^* by the equation (11). In addition, the value of j^* helps to find the undetectable CP with the predetermined value with the selected parameters of the algorithm.

If these moments exist, then $\overline{N}_{обн.р}$ is computed as in case 2 and $n_0 = (j^*+1)$, and if there are no such moments (i.e. at any moment of CP occurrence it will be detected on the same accumulation period) – as in the case 1 and $n_0 = N_C$. The general analytical equation is as follow:

$$\begin{cases} j^* < 1 \quad \text{- undetectable CP} \\ \overline{\mathrm{N}_{\det\,\mathrm{CP}}} = \dfrac{(h\sqrt{2} + \sqrt{2h^2 + 4m^2(N_C - 1)})^2}{4m^2} + 1 - \mathrm{N_C},\ \text{if}\ \ j^* = N_C \\ \overline{\mathrm{N}_{\det\,\mathrm{CP}}} = N_C - j^* + \dfrac{2h^2}{m^2},\ \text{if}\ \ j^*+1 \le N_C\ . \end{cases} \qquad (14)$$

The case of multiple CPs applied to the considered algorithm is not examined, since detection of properties recovery can be with a significant delay only. Taking into account the worst case, the period of the decision making function accumulation can be taken as the estimation of the average detection delay time of the restoration of the properties of the SP.

III. CONCLUSION

Verification of the reliability of the proposed mathematical models of the evaluation of the CP detection delay was carried out by simulation with the developed simulation model. The dependence of the average CP detection delay from the moment of its appearance for different accumulation periods of the decision making statistic at different levels of "false" detection was investigated and a comparison with the dependencies obtained by the analytical model was made.

The analysis of the simulation experiments showed that the analytical models completely reflect the nature of the investigated dependences. In addition, the developed model (14) provides an estimation for the worst case for CP dealing with of the "change in the mathematical expectation".

However, analysis of the results shows, if at any occurs CP moment it is found in the first period of the decision making function accumulation (see eq.(12)), then the worst moment of the CP occurs in the middle of the period of the decision making function accumulation. Therefore, estimating the average number of steps of detection delay can be reduced and calculated by eq.(14) with $n_0 = \dfrac{N_C}{2}$.

Despite the identical behavior of the analytical and imitation dependencies, the model (14) does not provide a guaranteed estimate of the average number of delay steps of the CP detecting in case of the "variance change", and therefore the dispersion of the decision making function must be taken into account. The dispersion of the decision making statistic (eq.(1)) can be determined from the following expression, obtained by the analogy with the expression of the mathematical expectation (eq.(7)):

$$D[G_k] = \frac{(n_0 - 1) + (k - n_0 + 1)\{2(u+1)^2 r^2 \dfrac{1-a}{1+a} + (1+u)^4\}}{k} \qquad (15)$$

In order to take into account the variance of the decision making function in determining the average number of steps of the CP detecting delay in the case of "variance change" and at the same time not to increase the estimation too much, it is suggested to consider the reaching the threshold h by the lower boundary of the interval instead of eq.(8)

$$\left[M[G_k] - \alpha_1\sigma_G;\ M[G_k] + \alpha_1\sigma_G\right],$$

where α_1 – a coefficient that determines the boundaries of the most decision making function realizations G_k; σ_G – mean-square deviation of the decision making function G_k.

Then, taking into account the reasoning given above in the derivation of the mathematical model of the estimation $\overline{N}_{обн.р}$, when a CP of the form "variance change" is detected, we obtain the following equation for the average number of steps of delay:

$$\begin{cases} j^* < 1 \ \text{-}\ \ \text{undetectable CP} \\ \overline{\mathrm{N}_{\det\,\mathrm{CP}}} = \dfrac{((h+\alpha_1\sigma_G)\sqrt{2} + \sqrt{2(h+\alpha_1\sigma_G)^2 + 4m^2(N_C - 1)})^2}{4m^2} + 1 - \mathrm{N_C},\ \text{if}\ \ j^* = N_C \\ \overline{\mathrm{N}_{\det\,\mathrm{CP}}} = N_C - j^* + \dfrac{2(h+\alpha_1\sigma_G)^2}{m^2},\ \text{if}\ \ j^*+1 \le N_C \end{cases}$$

The simulation experiments was fulfilled in a wide range of conditions and with different variances CP "dispersion change" and with different α_1 values. It was found that a value of α_1 equal 0.1 provides guaranteed estimates of the average number of the CP detecting delay.

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