Time Series/Data Processing and Analysis (MATH 587/GEOP 505)

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Notes on Deconvolution

We have seen how to perform convolution of discrete and continuous signals in both the time domain and with the help of the Fourier transform. In this lecture, we'll consider the problem of reversing convolution or deconvolving an input signal, given an output signal and the impulse response of a linear time invariant system.

We begin with the equation

$$d(t) = g(t) * m(t) \tag{1}$$

where $d(t)$ and $g(t)$ are known. Our goal is to solve for the unknown $m(t)$.

Although there's no obvious way to use the convolution integral to solve this equation, the equation becomes much easier to solve in the frequency domain. By the convolution theorem,

$$D(f) = G(f)M(f). \tag{2}$$

Thus

$$M(f) = \frac{D(f)}{G(f)}. \tag{3}$$

Once we have $M(f)$, we can invert the Fourier transform to obtain $m(t)$. Similarly, if we have discrete time signals and

$$d_n = (g_n * m_n)\Delta t \tag{4}$$

then

$$D_k = G_k M_k \Delta t \tag{5}$$

for $k = 0, 1, \ldots, N - 1$. Solving for M_k, we get

$$M_k = \frac{D_k}{G_k \Delta t}. \tag{6}$$

Once we have the vector M we can invert the discrete Fourier transform to obtain m_n. This simple approach to solving the deconvolution problem is called spectral division.

Unfortunately, this method seldom works in practice. The first problem is that denominator in (3) might be zero, at least for some frequencies. In that case, $X(f)$ is undefined, and we can't invert the Fourier transform to obtain $x(t)$. Another way of looking at this is to consider what output the system will produce for sine waves at different frequencies. If the system produces zero output for a sine wave at a particular frequency f_0, then it's clear that we can't solve the deconvolution problem for any input signal that contains a sine wave at frequency f_0 because there's no evidence of this sine wave in the output!

Even if $X(f)$ is non-zero, it may be extremely small for some frequencies. If there is noise in $z(t)$ at such a frequency, then that noise will be greatly amplified when $Z(f)$ is divided by $Y(f)$. Since noise is typically spread out over a broad range of frequencies, this is a very common problem in deconvolution.

There are a number of ways of dealing with this problem. The basic idea is to avoid division by zero by somehow modifying the denominator in (6). This **regularizes** the deconvolution problem. In performing the regularization, we want to do as little as possible to frequencies where the noise is insignificant, while damping out the noise at frequencies where it is larger than the signal. Because the DFT of a real input signal is always Hermitian (i.e. $M_k = M_{N-k}^*$) it is important that we perform the regularization in a way that produces a Hermitian M sequence and a real signal.

For example, we might try

$$M_k = \frac{D_k}{(G_k + \lambda)\Delta t} \tag{7}$$

where λ is a small positive real number. When G_k is much larger than λ, then this will have little effect on M_k. However, when G_k is very small compared to λ, this will effectively zero out the response at frequency k. One problem with this scheme is that if $G_k = -\lambda$, we can still get division by zero. It would obviously be better to work with the absolute value of G_k.

In Tikhonov regularization, we use

$$M_k = \frac{G_k^* D_k}{(G_k^* G_k + \lambda)\Delta t} \tag{8}$$

where λ is a positive real number. This is similar to (7), in that when $|G_k|$ is much larger in magnitude than λ, we get essentially (6). However, when $|G_k|$ is much smaller than λ, M_k is reduced in magnitude. It's not hard to show that if M is obtained by Tikhonov regularization then M will be Hermitian. Furthermore, the denominator in this formula can never be 0.

How do we select the regularization parameter λ? We'll discuss this in considerable detail in the inverse problems course. For now, we'll simply try a range of values to see which regularization parameters produce reasonable values. If λ is too small, the solution will be very noisy at frequencies where

G_k is small, while if λ is too large, all of the frequencies in the solution will be damped down. In most applications the unwanted noise occurs at higher frequencies. Thus Tikhonov regularization typically has the effect of low pass filtering the solution.

In the following example, the input signal is $m(t) = te^{-t}$ and the impulse response is $g(t) = e^{-5t}\sin(10t)$. A small amount of noise in the data makes spectral division unstable, but Tikhonov regularization produces very good results.
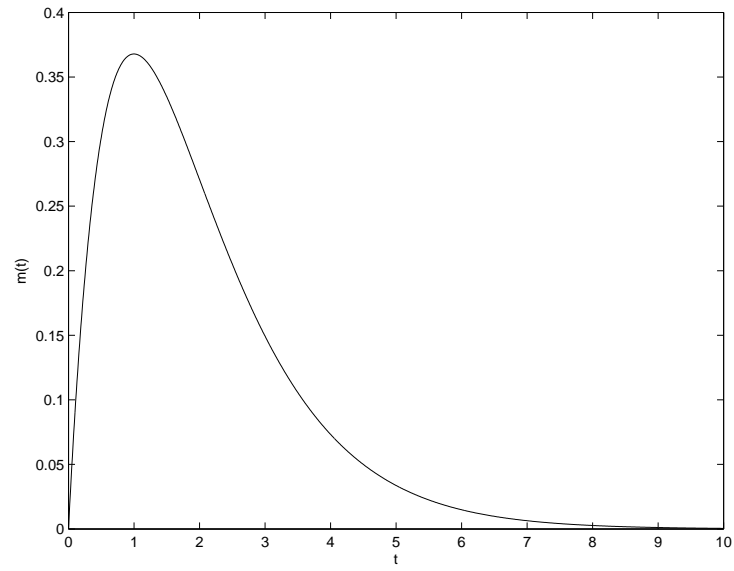
Figure 1: The input signal.
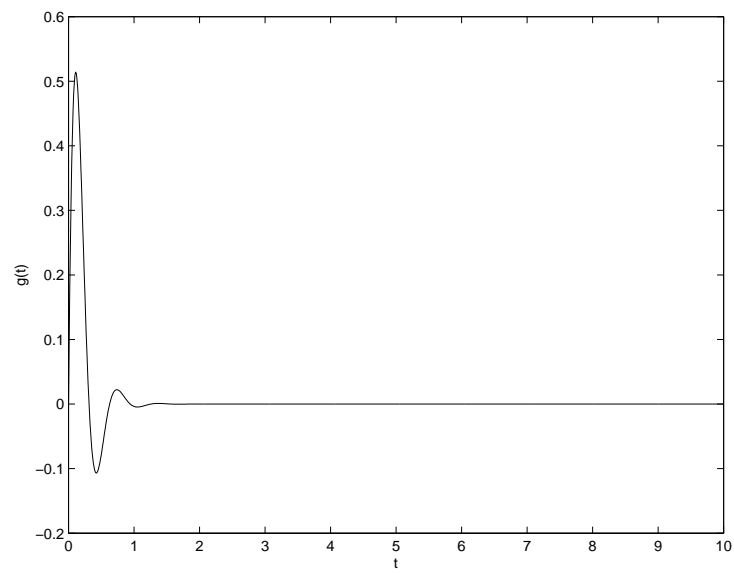
Figure 2: The impulse response.

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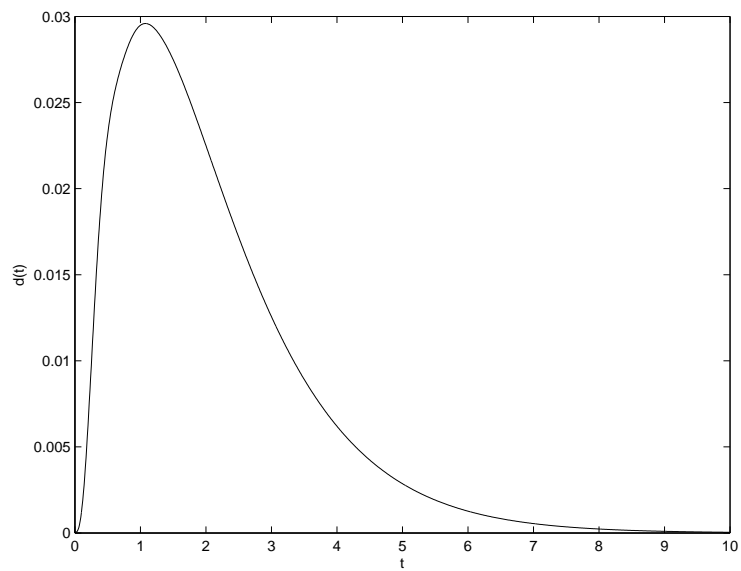
Figure 3: Clean data.
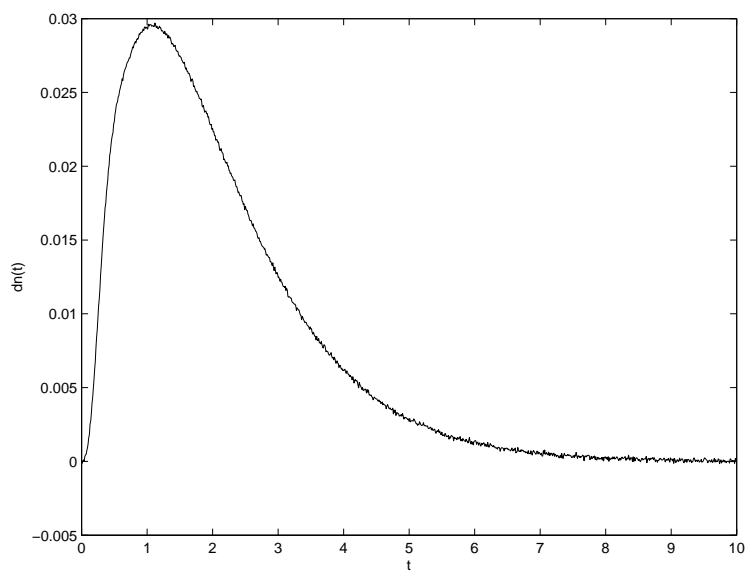
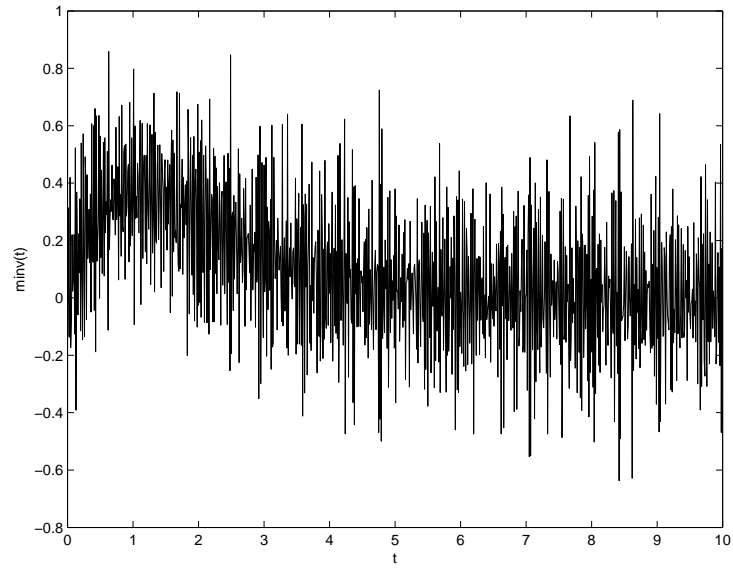Figure 4: Data with a small amount of noise added.

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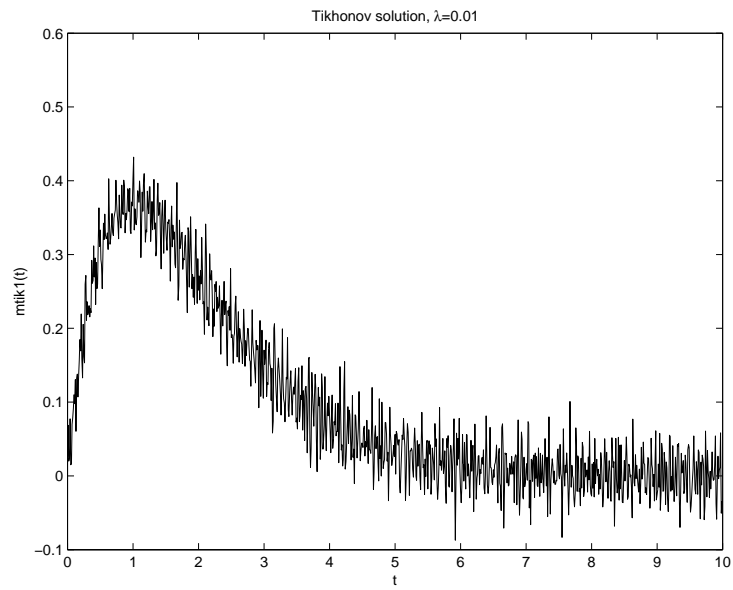Figure 5: Deconvolution by spectral division, no regularization.

Figure 6: Deconvolution with Tikhonov regularization, $\lambda = 0.01$.
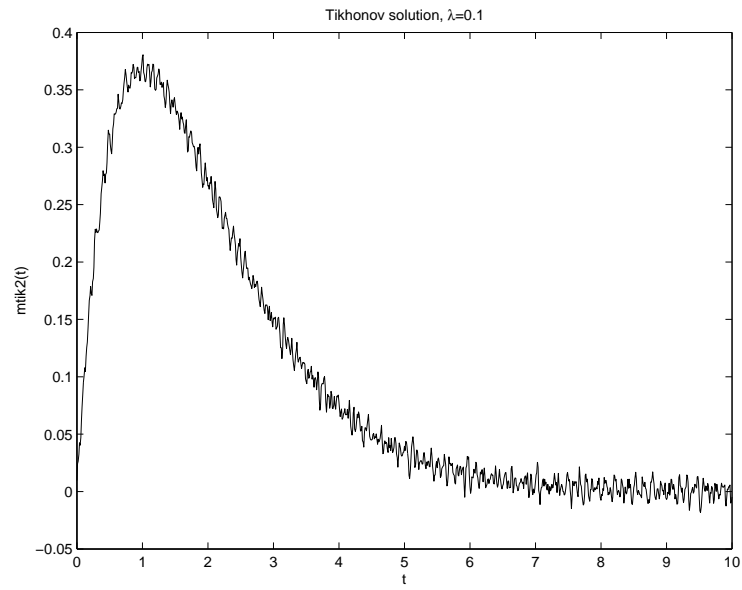
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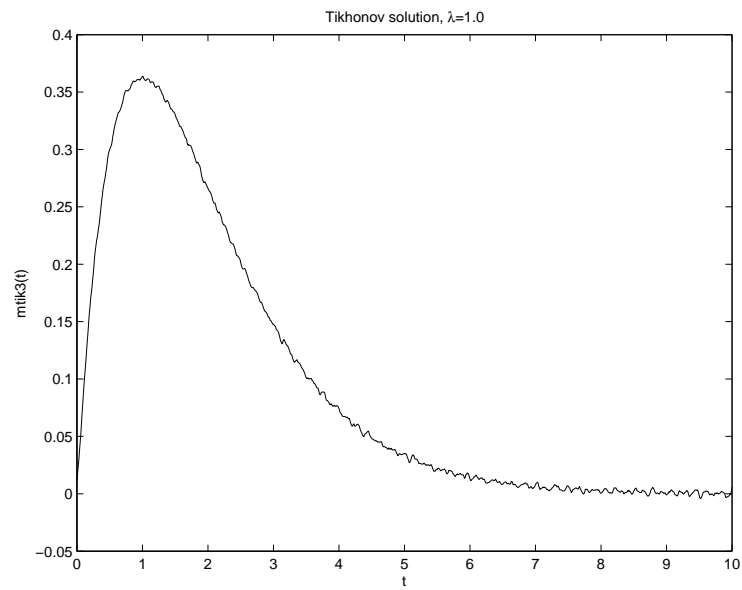Figure 7: Deconvolution with Tikhonov regularization, $\lambda = 0.1$.

Figure 8: Deconvolution with Tikhonov regularization, $\lambda = 1.0$.

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