Pitch Detection
Frequency Analysis: Zero-crossing Rate and Correlation

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Abstract

This chapter describes two analyses used in audio for detecting the pitch of a signal. The first analysis uses the zero-crossing rate to infer the frequency. The second analysis is based on signal correlation, which is a way to look at similar patterns in signals. Auto-correlation analyzes the similar patterns that occur within a signal and cross-correlation analyzes the similar patterns that occur between two signals. By determining how often cycles repeat in a signal, the pitch can be calculated.

1 Introduction: Frequency Analysis

The frequency information in a signal is an important characteristic for audio signal processing. It is just as fundamental as analyzing the amplitude information in a signal, covered in Chapter 6.7. There are several analyses used for describing a signal's frequency. This chapter presents two approaches which are used specifically for detecting a signal's pitch. The next chapter builds on this material to analyze a signal's frequency, amplitude, and phase using the Discrete Fourier Transform.

2 Pitch Detection

An important frequency characteristic is a signal's pitch. This characteristic is used for a wide range of things in audio from tuning an instrument, to analyzing the intonation in a speech signal. Audio effects, like auto-tune, require first analyzing a signal's pitch before performing pitch shifting (Chapter 15.6).

2.1 Relationship Between Pitch and Frequency

Periodic signals typically contain frequencies which are harmonically related. Listeners perceive the pitch of a signal based on this harmonic relationship. For analyzing a signal's pitch computationally, it is the fundamental frequency which is to be detected.

In many cases, the fundamental frequency is the lowest harmonic contained in a signal. Furthermore, there are many signals where the lowest harmonic is the frequency with the highest amplitude. Although, these two conditions are not always true for all signals. If that is the case, a signal's pitch cannot be detected based on the lowest harmonic or the frequency with the highest amplitude. Instead, the spacing between harmonics can be an indicator of a signal's pitch.

These observations motivate the following two approaches for detecting a signal's pitch: the zero-crossing rate and signal correlation.

3 Zero-crossing Rate

One approach to determine a signal's pitch would be to consider a method for counting the number of cycles that repeat per second. This analysis could also be performed on a shorter time frame (e.g. 1024 samples) and extrapolated to the number of cycles per second, if it is beneficial to know a more time-specific measurement of pitch.

Consider the waveform of the sine wave signal shown in Figure 1. The number of cycles per second can be counted by knowing where the start and end of each cycle occurs. Based on the definition of signal phase (Chapter 7.3), the start of a cycle occurs when the amplitude of the signal switches from having a negative amplitude to having a positive amplitude. Therefore, the start of a cycle is the point in time where the amplitude crosses over the value of zero. This event in time is called a **zero crossing**. The number of zero crossings per second is called the **zero-crossing rate** (ZCR).

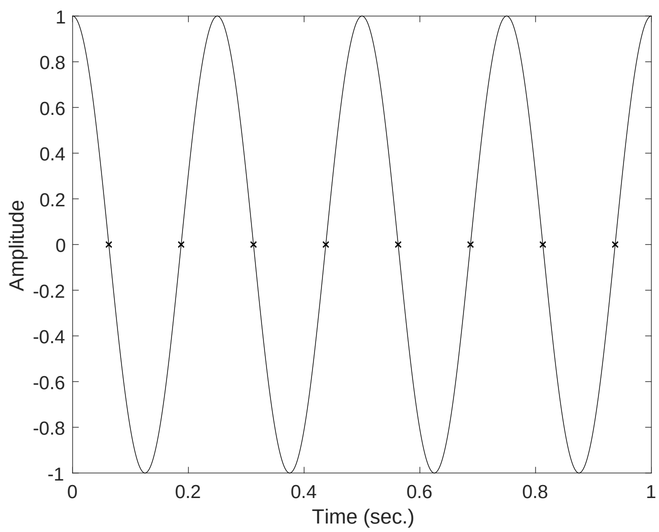

Figure 1: Marking a Signal's Zero Crossings

3.1 Pitch Period and Frequency

During a cycle there are two points in time when the amplitude crosses zero. The first occurs at the phase of 0° and the second occurs at the phase of 180°. Therefore, the frequency in cycles per second can be calculated by dividing the zero-crossing rate by two, $f = \frac{ZCR}{2}$.

3.2 Time-localized analysis with zero-crossings

When analyzing pitch, it is common to determine the pitch at a specific point in time. This is because the pitch of many audio signals varies over time. Using the ZCR of an entire signal, the average overall pitch can be determine. A different approach would be to find the ZCR for short segments of the signal. A segment can also be called a frame or a buffer. It is assumed the pitch does not change during a single segment. This allows for the possibility that the pitch may be different from one segment to the next.

3.3 Limitations of the ZCR

The ZCR is an effective measurement of pitch for some signals, but it does not produce accurate results for all signals. Problems arise for signals which cross zero more than two times a cycle. For a complex waveform with multiple harmonics, it is possible to have three or more zero-crossings per cycle. An example of this is shown in Figure 2. This signal is created by digitally summing a 4 Hz cosine wave and an 8 Hz cosine wave with an additional $\frac{\pi}{2}$ phase offset. Using $\frac{ZCR}{2}$ would estimate a fundamental frequency of 8 Hz, instead of 4 Hz.

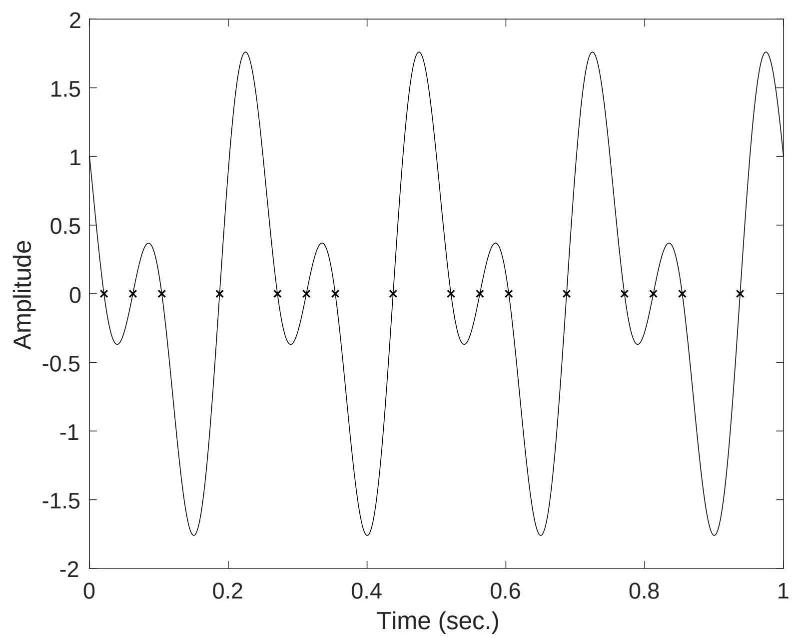

Figure 2: Example of a signal which ZCR cannot be used to determine frequency

3.4 Filtering with the ZCR

One approach to improve the accuracy of estimating pitch uses low-pass filtering to remove higher frequencies prior to the ZCR analysis. This reduces the likelihood of mistaking a higher harmonic for the fundamental frequency. The cut-off frequency of the pre-filter must be carefully selected in order to allow the actual fundamental frequency to pass through, while reducing this amplitude of higher harmonics.

If the fundamental frequency is limited to a narrow frequency range (e.g. male voice or female voice), then it may be possible to choose an appropriate cut-off frequency *apriori*. Nonetheless, it may not always be possible to know the range of the fundamental frequency without first determining the pitch. Figure 3 shows the two-step process of filtering with the ZCR analysis.

$$x[n] \longrightarrow \boxed{LPF} \longrightarrow \boxed{ZCR(\cdot)} \longrightarrow f$$

Figure 3: Block diagram of pre-filtering the ZCR analysis

Given the limitations of the ZCR analysis for detecting pitch, it is worth considering another approach which is more robust to various types of signals.

4 Signal Correlation Functions

Correlation is a measurement of the similarity in signals. It can be used for detecting repeated patterns in a signal. Therefore, correlation can be used for determining a signal's pitch.

One type of correlation is called: *cross-correlation*. It is a measurement of the similarity between two different signals. The cross-correlation function between signals x and y is written: $r_{x,y}$.

$$r_{x,y}[l] = \sum_{n=1}^{N} x[n] \cdot y[n+l]$$

Another type of correlation is called: *auto-correlation*. It is a measurement of how similar a signal is with itself at different points in time. The auto-correlation function for signal x is written: $r_{x,x}$.

$$r_{x,x}[l] = \sum_{n=1}^{N} x[n] \cdot x[n+l] \tag{1}$$

4.1 Mathematical Background

As a foundation for calculating and programming these two correlation functions, several other related mathematical functions are presented next. These functions provide a review of the mathematical procedures used in the correlation calculations.

4.1.1 RMS Calculation

As described in Chapter 6.7.3, the root-mean-square (RMS) amplitude is a measurement of average signal magnitude. For a signal, $\mathrm{x} = \{x[1], x[2], ..., x[N]\}$, the RMS amplitude is:

$$A_{rms} = \sqrt{\frac{1}{N}\sum_{n=1}^{N}(x[n])^2}$$

By expanding the squared term, an equivalent equation is:

$$A_{rms} = \sqrt{\frac{1}{N}\sum_{n=1}^{N}x[n] \cdot x[n]}$$

4.1.2 Average Power and Signal Energy

In addition to a signal's RMS amplitude, there are two other closely-related functions for defining signal strength: *average power* and *signal energy.*

Recalling a signal's power is related to its squared amplitude ($P = A^2$), a signal's **average power** is related to A_{rms} as:

$$P = \frac{1}{N}\sum_{n=1}^{N}(x[n])^2 = (A_{rms})^2$$

Notice that the average power calculation includes a term to divide by the number of samples, N, in the signal. This is a similar term found in the RMS amplitude and the arithmetic mean calculations. Essentially, this scale factor makes each calculation a measurement of average signal strength *per sample.*

The overall signal strength (i.e. not scaled by the total number of samples) is called: **signal energy**. It can be thought of as the strength of the entire signal. It is defined as:

$$E = \sum_{n=1}^{N}(x[n])^2$$

4.2 Inner Product of Two Signals

The calculation of signal energy is a specific case of an important operation called the **inner product**. Sometimes this operation is also called the **dot product**. For signals (or vectors) $\mathtt{x} = \{x[1], x[2], ..., x[N]\}$ and $\mathtt{y} = \{y[1], y[2], ..., y[N]\}$ the inner product is written:

$$\langle x, y\rangle = \sum_{n=1}^{N} x[n] \cdot y[n]$$

The result is the sum of the element-wise multiplication between the two signals, $\langle x, y\rangle = x[1] \cdot y[1] + x[2] \cdot y[2] + ... + x[N] \cdot y[N]$

4.2.1 Basic Examples of the Inner Product Operation

The following examples provide a basic application of the inner product operation to relatively short signals. First, let $\mathtt{x} = \{2, 3, 4\}$ and $\mathtt{y} = \{5, 6, 7\}$. The inner product of these two signals is:

$$\langle x, y\rangle = 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 = 10 + 18 + 28 = 56$$

Second, consider another example as a step towards using longer audio signals. Let $\mathtt{x} = \{1, 1, 1, -1, -1, -1\}$ and $\mathtt{y} = \{.2, .5, .2, -.2, -.5, -.2\}$. The inner product of these two signals is:

$$\begin{aligned}\langle x, y\rangle &= 1 \cdot .2 + 1 \cdot .5 + 1 \cdot .2 + (-1) \cdot (-.2) + (-1) \cdot (-.5) + (-1) \cdot (-.2) \\ &= .2 \ + \ .5 \ + \ .2 \ + \qquad .2 \qquad + \qquad .5 \qquad + \qquad .2 \qquad = 1.8\end{aligned}$$

Observe that the result of multiplying terms which are both negative produces a result which is positive.

4.2.2 Visualizing Examples of the Inner Product Operation

Consider the following examples of the inner product operation. Suppose signal $\mathtt{x}$ is a sine wave and signal $\mathtt{y}$ is a square wave. As an initial example, these two signals have an identical frequency and starting phase. A plot is shown in Figure 4 of the two waveforms. In this example the segments of the square wave with a positive amplitude correspond with the segments of the sine wave with a positive amplitude. Similarly, the segments with negative amplitude occur for the same samples in both signals.

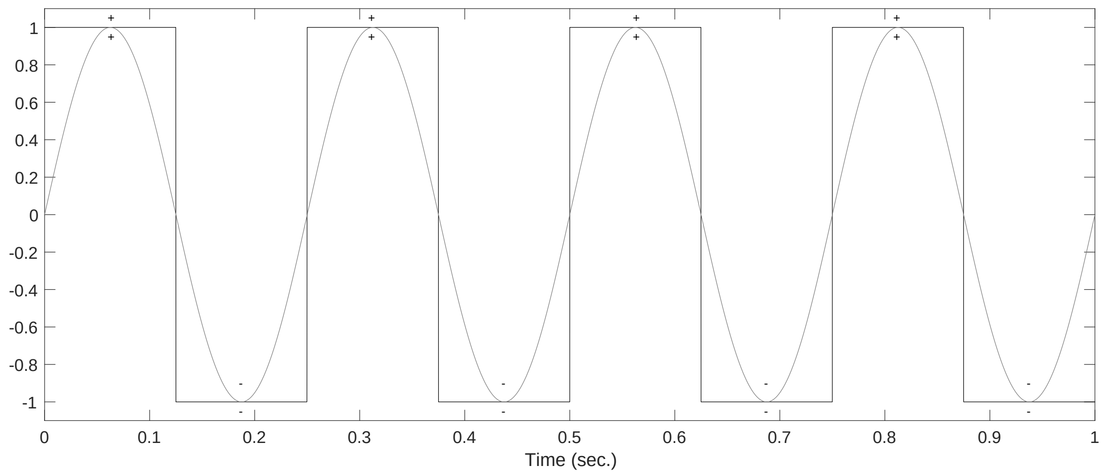

Figure 4: Inner product visual demonstration

For a sample, n, when `x` and `y` are both positive, the result of their multiplication, $x[n] \cdot y[n]$, is positive. When both samples are negative, the result of their multiplication is also positive. Therefore, each element-wise multiplication of the inner product operation for these signals produces a positive number. Finally, the sum of all these positive numbers results in a relatively large positive value for the inner product.

Signals with a phase difference

As a related example, consider the same two signal except with a 180° phase difference. A plot is shown in Figure 5 of the two waveforms. In this example the segments of the square wave with a positive amplitude correspond with the segments of the sine wave with a negative amplitude, and vice versa. In either case, each element-wise multiplication of the inner product operation produces a negative number. The sum of all these negative numbers results in a relatively large negative value for the inner product.

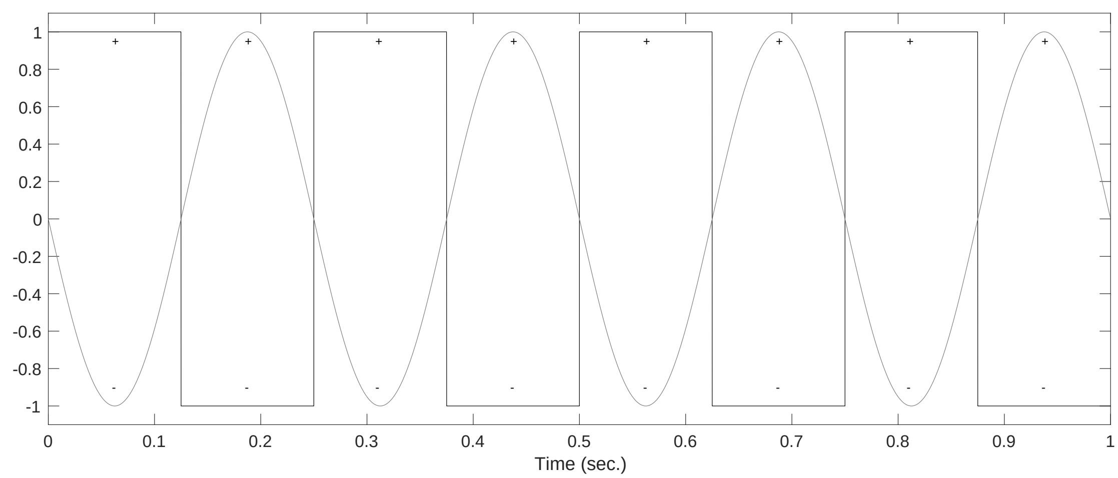

Figure 5: Inner product visual demonstration

5 Auto-correlation Function

Returning now to auto-correlation, it is a measure of how similar a signal is with itself at different points in time. The concept of the measurement is to line up the two signal waveforms and perform the inner product operation. Then, one of the signals if shifted in time and the operation is performed again. As a definition, the auto-correlation function is a comparison of a signal with itself at all possible *time lags* (in integer samples). For a signal, $\mathrm{x} = \{x[1], x[2], ..., x[N]\}$, the auto-correlation function, $r_{x,x}$, at time lag, l, is:

$$r_{x,x}[l] = \sum_{n=1}^{N} x[n] \cdot x[n+l] \qquad (2)$$

In this equation, one auto-correlation value is calculated for each integer time lag over the range of: $l = (-N+1), (-N+2), ..., -1, 0, 1, 2, ..., (N-1)$. When computed in MATLAB, the result is stored in an array with indexing, $r_{x,x}[1], ..., r_{x,x}[2N-1]$, where the index is $r_{x,x}[N+l]$.

5.1 Visual Demonstration

A visual demonstration of the process in shown in Figure 6 using a short signal with $N = 4$. For each value of the auto-correlation function, $r_{x,x}[n]$, the corresponding overlap and lag is shown for signal, $x[n]$. In each calculation, the samples which overlap are multiplied together. Then, all of the products are summed together.

Consider the result for the process of element-wise multiplication followed by addition for different time lags. When $l = 0$ for $r_{x,x}[0]$, all the positive samples in the signal line up with positive samples and all the negative samples line up with negative samples. Therefore, the products from element-wise multiplication are all positive. Summing all the products results in a relatively large number for the auto-correlation function at this time lag.

For another time lag, some of the positive samples line up with positive samples, some of the negative samples line up with negative samples, and *some of the positive samples line up with negative samples*. Therefore, some of the products from element-wise multiplication are positive, but *some are negative*. Summing all the products results in a relatively smaller number for the the auto-correlation function at this time lag.

Also, note the symmetry. A time lag of -3 while produce an identical result as +3. This is true for all time lags, $r_{x,x}[-l] = r_{x,x}[+l]$.

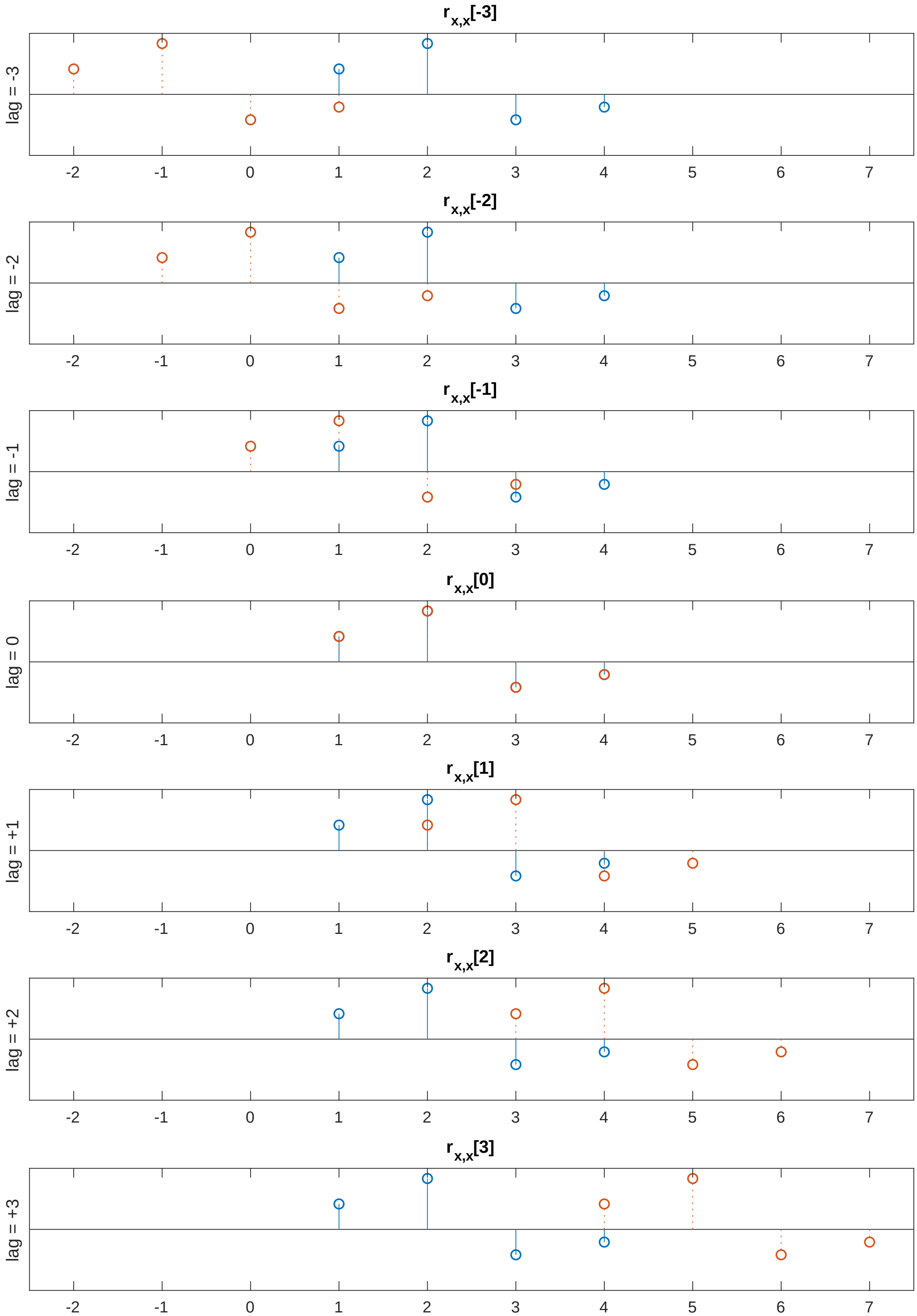

Figure 6: Auto-correlation visual demonstration

5.2 Auto-correlation with Example Signals

Consider the following illustrative examples of calculating the auto-correlation function for various types of signals. Auto-correlation is calculated using the built-in MATLAB function: `xcorr`. This function calculates the auto-correlation function for a signal, `x`, by using the syntax: `xcorr(x,x)`. It returns the output variables, `r` and `lag`, corresponding to the auto-correlation function, $r_{x,x}$, and time lag in samples, l, respectively.

Examples: Create and run the following m-file in MATLAB.

```
% CORRELATIONSINE
% This script plots the auto-correlation function of a
% sine wave signal by using 'xcorr.' The result is plotted
% vs. time lag in units of seconds.
%
% Note: for a sine wave with a frequency of 4 Hz, one cycle
% occurs every 1/4 seconds. In the plot of the
% auto-correlation function, there are local maximas that
% occur every 1/4 of a second. This indicates that sliding
% the sine wave over by this time lag is equal to one
% entire cycle.
%
% See also XCORR

Fs = 48000; Ts = 1/Fs;
f = 4;
t = [0:Ts:1].';
x = sin(2 * pi * f * t); % Test signal

[r,lag] = xcorr(x,x); % Auto-correlation calculation

plot(lag/Fs,r);       % Plot, converting time units to sec.
xlabel('Lag (sec.)');
```

Example 1: Script - `correlationSine.m`

5.2.1 Sine Wave

In Example 1, a sine wave with a frequency of 4 Hz is analyzed. The auto-correlation function is shown in Figure 7. Consider that a sine wave with a frequency of 4 Hz has a period of $\frac{1}{4}$ seconds. Notice, the auto-correlation function has local maxima every $\frac{1}{4}$ seconds. These locations relative to time lag occur when the sine wave signal is shifted by a complete cycle. Therefore, the local maxima are an indicator of the original signal's pitch. In particular, finding the first maximum to the right of the center peak (lag = 0) is a common method for estimating pitch.

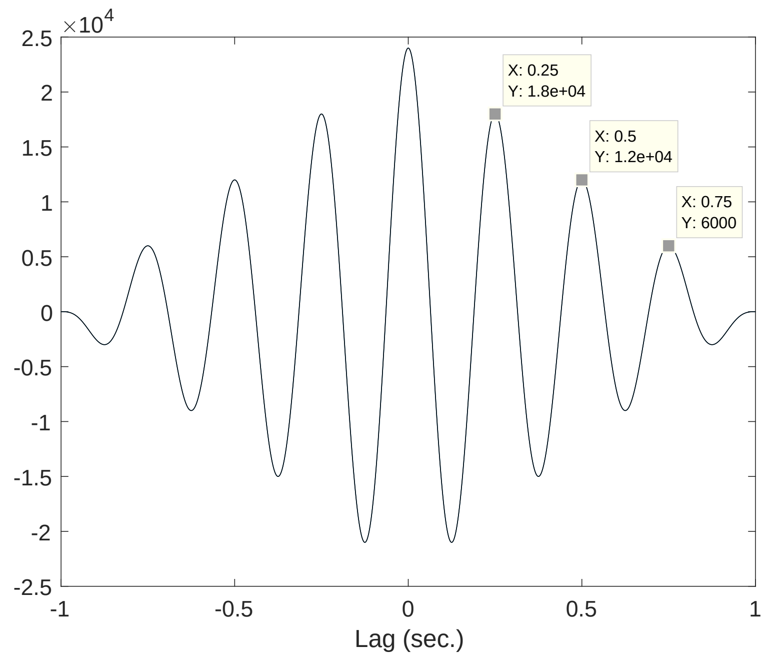

Figure 7: Auto-correlation function of a sine wave

5.2.2 Square Wave

In Figure 8, the auto-correlation function is shown for a square wave signal with a fundamental frequency of 4 Hz. The function has the shape of a damped triangle wave, which is a different shape than the auto-correlation function for the sine wave. One thing similar to the sine wave example is the timing of the local maxima. The auto-correlation function has local maxima every $\frac{1}{4}$ seconds of lag. This illustrates how the auto-correlation function can be used to detect the fundamental frequency of a signal containing multiple harmonics.

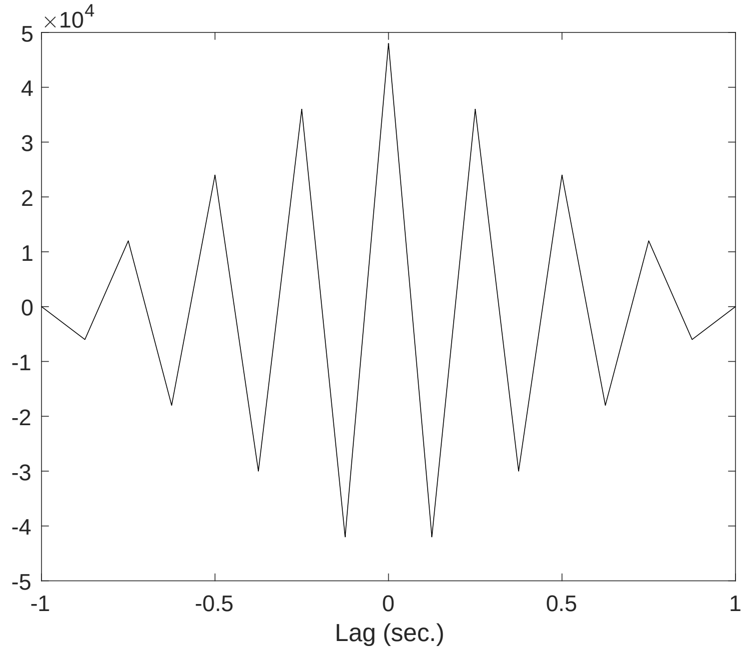

Figure 8: Auto-correlation function of a square wave

5.2.3 White Noise

Although white noise does not have a pitch to detect, calculating the auto-correlation function of white noise demonstrates the result of using this analysis with aperiodic audio signals. An aperiodic signal is defined as signal which does not repeat (i.e. it does not have a period over which it repeats a cycle). Similarly, an aperiodic signal could be defined as a signal which is **uncorrelated** with itself.

Values of zero, or close to zero, in the auto-correlation function indicate a signal does not a have a similar, repeated pattern at that time lag. As shown in Figure 9, the values of the auto-correlation function are all near zero for a white noise signal. The only value not close to zero is for a time lag of zero (i.e. no time shift).

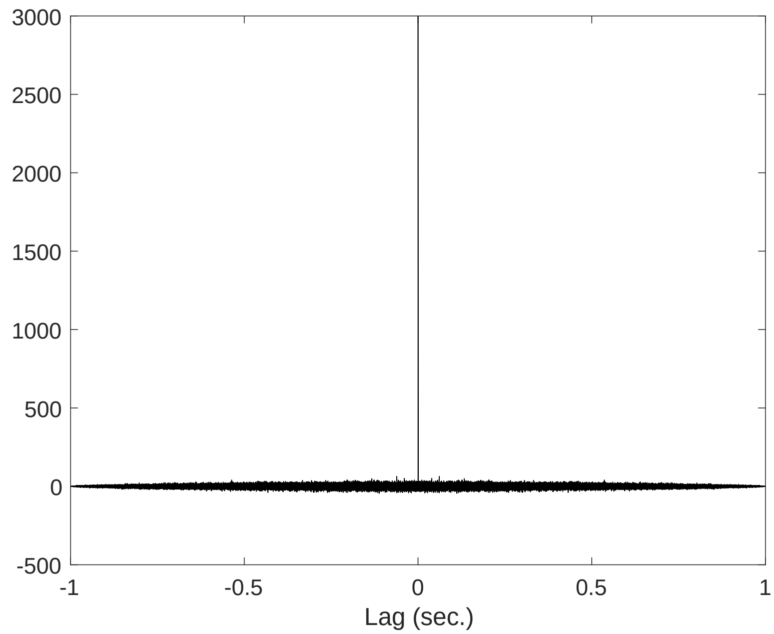

Figure 9: Auto-correlation function of white noise

For white noise, as well as all signals, the value of the auto-correlation function at a time lag of zero, $r_{x,x}[0]$, is equal to the energy, E, of the signal. This is the time lag when the entire signal overlaps with itself in the calculation (shown in Figure 6).

$$r_{x,x}[0] = \sum_{n=1}^{N} x[n] \cdot x[n+0] = \sum_{n=1}^{N} (x[n])^2 = E \tag{3}$$

5.2.4 Pink Noise

As another example for comparison purposes, consider the auto-correlation function of pink noise. As a signal, pink noise has the likelihood of equal energy per octave, such that the range of 20 - 40 Hz has the same energy as 10,000 - 20,000 Hz. The frequency spectrum of pink noise appears to have a decrease in amplitude as frequency increases. One method to create pink noise is to synthesize white noise and then filter it to decrease the amplitude of high frequencies (Chapter 13.9). Therefore, pink noise can be considered as a sequence of random numbers where low frequencies generally contribute more to the signal than high frequencies.

The auto-correlation function of pink noise is shown in Figure 10. There are some similarities between the function from pink noise compared to white noise because they are both aperiodic. Neither function has prominent peaks for time lags other than zero. But, pink noise has a wider range of values for time lags other than zero compared to white noise in which the values are more consistently close to 0.

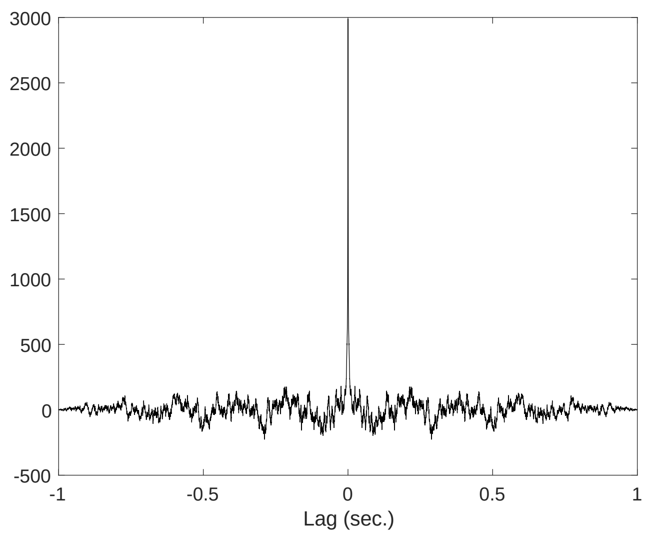

Figure 10: Auto-correlation function of pink noise

5.2.5 Signal and Noise

Finally, consider an example for the auto-correlation function of a signal, $x[n]$ made by digitally summing a periodic signal, $d[n]$, and an aperiodic signal, $v[n]$. The general model for the signal is written: $x[n] = d[n] + v[n]$. This type of signal regularly occurs in audio as a result of having music or voice in the presence of natural background noise. For simplicity, a sine wave signal and white noise are chosen as the signals in this example.

Initially, it may seem relevant to consider how the relative amplitude of $d[n]$ and $v[n]$ impact the auto-correlation function, $r_{x,x}[l]$. In Figure 11, the auto-correlation function is shown for the case when $d[n]$ and $v[n]$ have an equal RMS amplitude for a 0 dB signal-to-noise ratio (SNR). In Figure 12, the auto-correlation function is shown for the case when $d[n]$ has an RMS amplitude 12 dB *less* than $v[n]$.

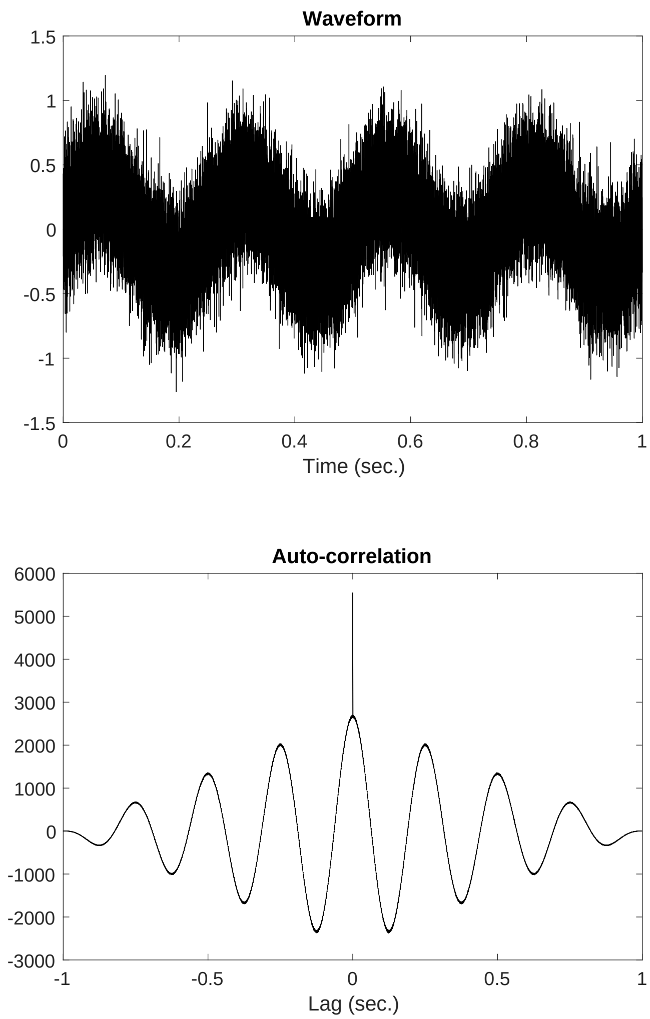

Figure 11: Auto-correlation function of sine wave and noise, 0 dB SNR

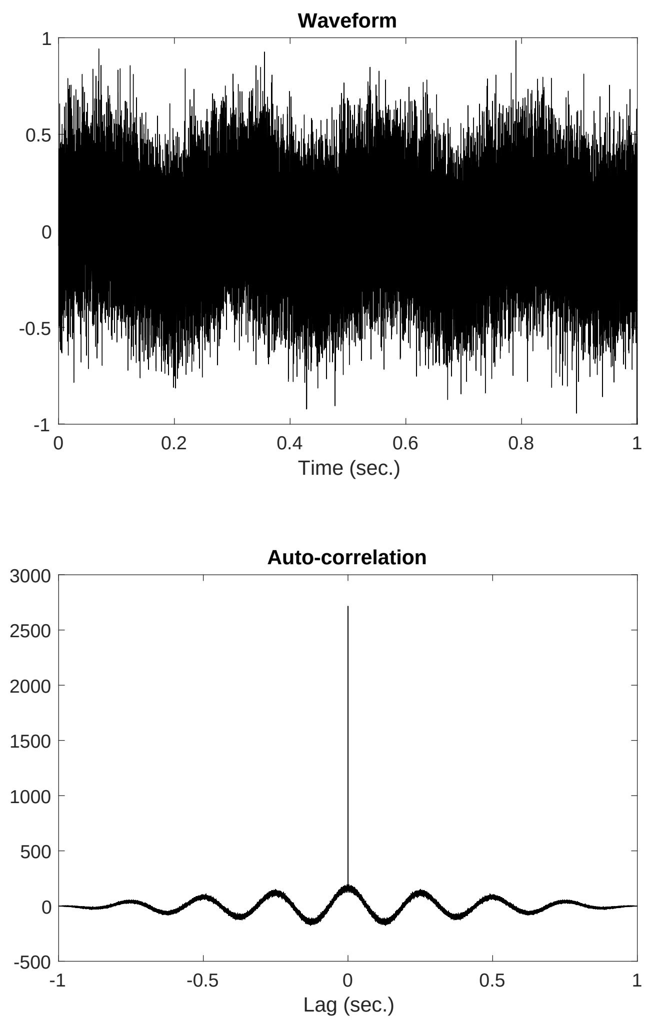

Figure 12: Auto-correlation function of sine wave and noise, -12 dB SNR

5.3 Variations of the Auto-correlation Function

There are several variations of the auto-correlation function which are used for different purposes. Each variation is used to control for some aspect of the auto-correlation function to make it useful for comparing different signals. The unscaled version (Equation 2) of the auto-correlation function is dependent on signal amplitude. With all else being equal, signals with different amplitudes produce different auto-correlation functions. This leads to a *normalized* version of the function to control for amplitude. Similarly, the unscaled version is also dependent on signal length. With all else being equal, signals with different lengths produce different auto-correlation functions. This leads to the *biased* version of the function to control for signal length. Finally, the unscaled version generally shows a dependency on time lag, such that larger time lags have lower values. This leads to the *unbiased* version of the function to control for time lag.

5.3.1 Normalized Auto-correlation

One variation of the function is the normalized auto-correlation. This version of the function ensures the range of values does not exceed a value of one, regardless of the relative signal strength. This is useful for comparing auto-correlation functions to the same relative scale.

Normalization is accomplished by scaling all the values of the function by the maximum value (Section 6.8.1). As shown in Equation 3, the maximum value of the auto-correlation function for any signal occurs at a time lag of zero. This value is equal to the signal energy, E. It can be used to scale all values of auto-correlation function, for the normalized variation.

$$r_{x,x}[l] = \frac{1}{E} \sum_{n=1}^{N} x[n] \cdot x[n+l]$$

5.3.2 Biased Auto-correlation

Another variation of the function is the biased auto-correlation. In some cases, it may be useful to preserve the relationship between signal strength (i.e. amplitude, energy) and the auto-correlation function. Instead, it may be useful to control for the signal length. This prevents the auto-correlation function from having higher values just because a signal is longer. The process of calculating the biased auto-correlation function involves scaling the value for all time lags by the length of the signal, N.

$$r_{x,x}[l] = \frac{1}{N} \sum_{n=1}^{N} x[n] \cdot x[n+l]$$

5.3.3 Unbiased Auto-correlation

The final variation of the function is the unbiased auto-correlation. This type addresses the inherent decrease in values for increased time lag. As shown in Figure 13, the biased auto-correlation has a peak at a time lag of zero. Then it slopes down in either direction for both positive and negative time lags. This tends to de-emphasize those portions of the signal. In other words, the function is inherently biased to a time lag of zero. As an alternative, it may be desired to equally emphasize the values of the function regardless of time lag, leading to the unbiased function shown in Figure 13. This can be accomplished by scaling each value of the function based on its time lag, l:

$$r_{x,x}[l] = \frac{1}{N - |l|} \sum_{n=1}^{N} x[n] \cdot x[n+l]$$

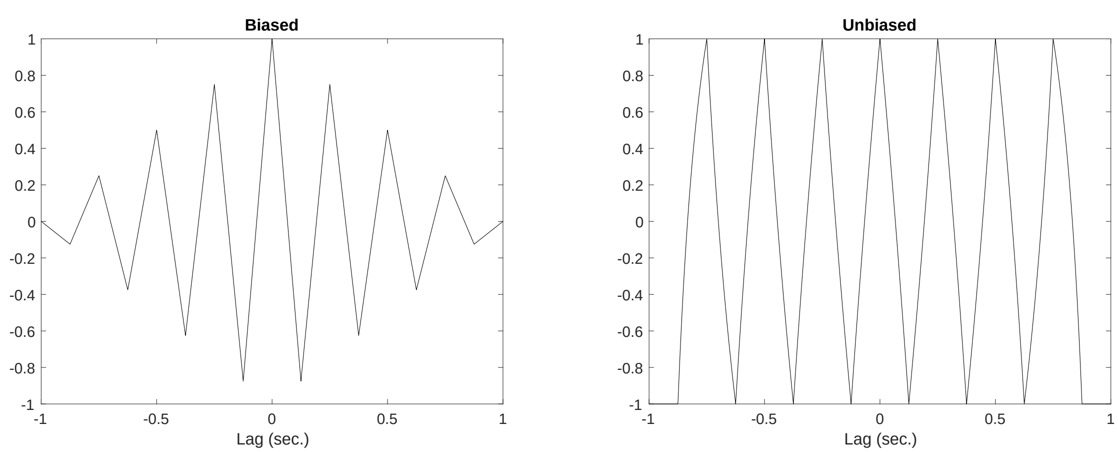

Figure 13: Comparison of Biased and Unbiased Auto-Correlation

6 Cross-correlation

Another approach to using the correlation calculation is to analyze the similarity between two different signals. This is called *cross-correlation*. It is a similar process to auto-correlation except two signals are used. For a signals, $\mathtt{x} = \{x[1], x[2], ..., x[N]\}$ and $\mathtt{y} = \{y[1], y[2], ..., y[N]\}$, with the same length the cross-correlation function, $r_{x,y}$, at time lag, l, is:

$$r_{x,y}[l] = \sum_{n=1}^{N} x[n] \cdot y[n+l]$$

If the two signals have different lengths, the shorter signal can be zero-padded to equal the length of the longer signal.

Unlike auto-correlation, the cross-correlation function may not produce symmetric results over the time lag of zero (i.e. $r_{x,y}[+l] \neq r_{x,y}[-l]$). It is also possible the cross-correlation function $r_{y,x}$ may produce different results than $r_{x,y}$. It can be calculated using the following:

$$r_{y,x}[l] = \sum_{n=1}^{N} y[n] \cdot x[n+l]$$

There is a relationship between the two options for the order of the function. The cross-correlation $r_{y,x}$ produces time-reversed results of $r_{x,y}$ such that $r_{x,y}[+l] = r_{y,x}[-l]$.

6.1 Inner Product Example of Signals with Different Frequencies

Cross-correlation is based on the inner product of two different signals, which may have different frequencies. To demonstrate an important outcome which does not occur with auto-correlation, consider an example with the following two signals. As shown in Figure 14, suppose there is a square wave with a fundamental frequency of 4 Hz and a sine wave with a fundamental frequency of 8 Hz. The sine wave completes two cycles for every one cycle of the square wave.

By performing the inner product operation, a noteworthy result is found. Each positive segment of the square wave corresponds to both a positive and a negative segment of the sine wave. After element-wise multiplication and summation, the result for each positive segment of the square wave is 0. The same is true for the negative segments of the square wave. Therefore, the total inner product for these signals is 0.

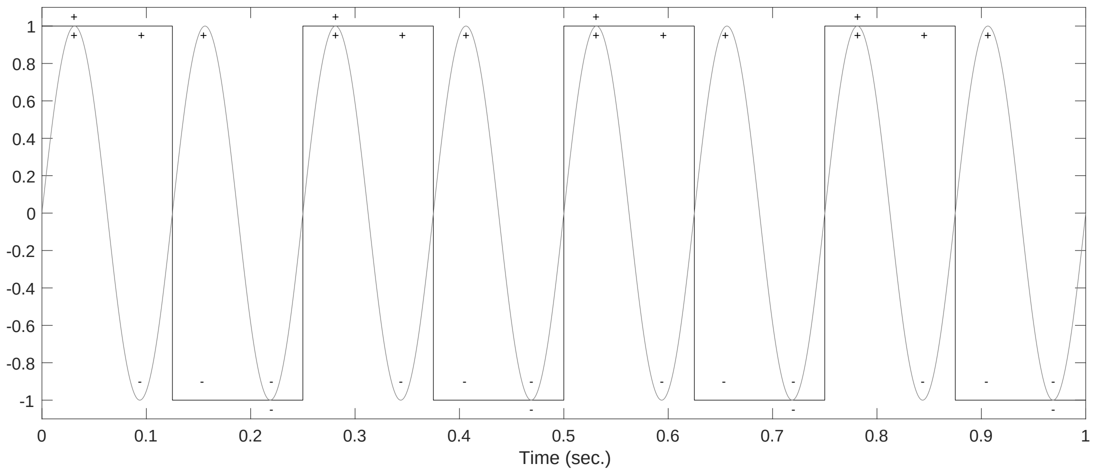

Figure 14: Inner product visual demonstration

6.2 Cross-correlation with Example Signals

The following examples demonstrate the cross-correlation function used with several different signals.

6.2.1 Sine Wave and Square Wave

Consider the cross-correlation function for two signals of different type, namely a sine wave and a square wave. In Figure 15, the cross-correlation function is shown for a square wave and a sine wave with equal fundamental frequencies. The result is a similar result to the auto-correlation function of a sine wave in Figure 7.

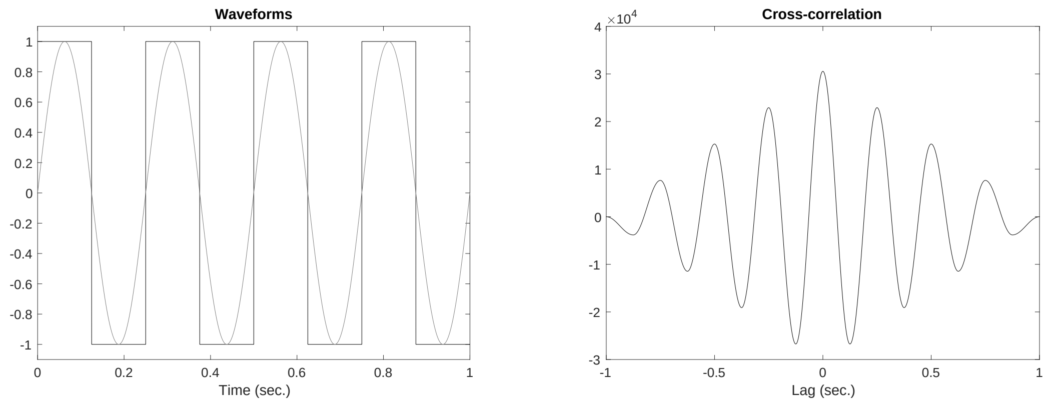

Figure 15: Cross-correlation of square wave and sine wave ($f = 4$)

Next, the cross-correlation function is shown in Figure 16 for a square wave and a sine wave with twice the fundamental frequency of the square wave. The resulting function has a different shape than the example in Figure 15. In particular, the value at a time lag of zero is: $r_{x,y}[0] = 0$. By visual inspection of the waveforms, each positive segment of the square wave has a positive and negative segment of the sine wave. The same is true for each negative segment of the square wave. Through the multiplication and summation operations of the cross-correlation function, these sections cancel each other out to have a value of 0.

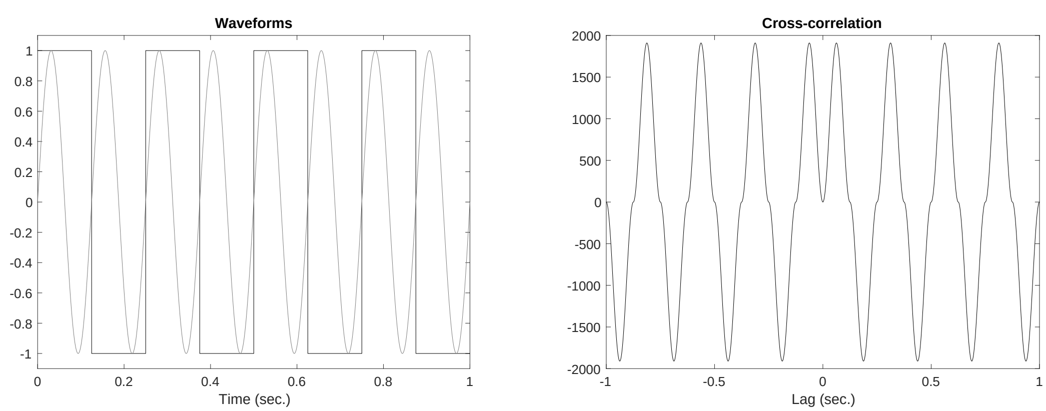

Figure 16: Cross-correlation of square wave ($f = 4$) and sine wave ($f = 8$)

Next, the cross-correlation function is shown in Figure 17 for a square wave and a sine wave with three times the fundamental frequency of the square wave. Although these signals have different fundamental frequencies, the cross-correlation function has a distinct result compared to Figure 16. In this case, the peak of the cross-correlation function occurs at a time lag of zero. By visual inspection of the waveforms, each positive segment of the square wave has two positive segments of the sine wave and only one negative segment. Similarly, each negative segment of the square wave has two negative segments of the sine wave and only one positive segment. Through the multiplication and summation operations of the cross-correlation function, the result is a large positive number.

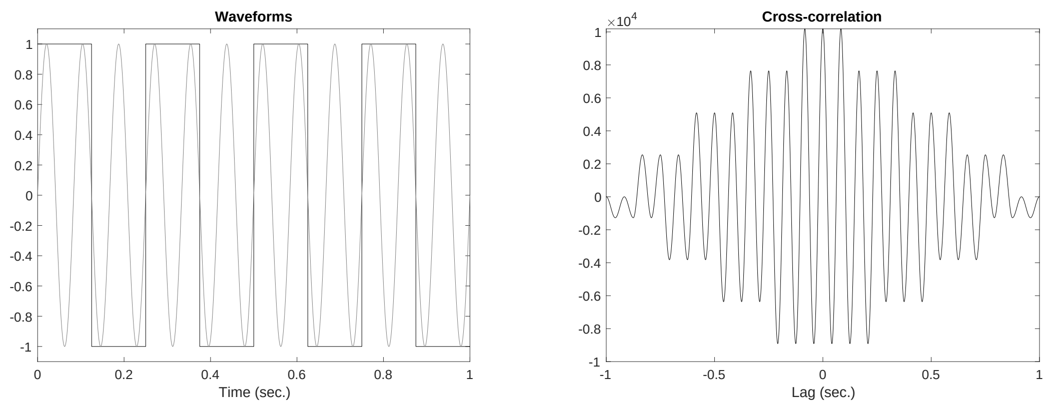

Figure 17: Cross-correlation of square wave ($f = 4$) and sine wave ($f = 12$)

Keep in mind, a square wave is a signal comprised of harmonics at odd multiples of the fundamental frequency. It should be no surprise that there is more similarity between the square wave and a sine wave with a frequency ***equal*** to one of the harmonics, than the square wave and a sine wave with a frequency ***not equal*** to one of the harmonics.

6.2.2 Sine Waves of Different Frequencies

Finally, consider an example for the cross-correlation function using two sine waves of different frequencies. The result is shown in Figure 18. This is another example where the time lag of zero is: $r_{x,y}[0] = 0$. The important observation is that signals of different frequencies show little or no similarity using the operations of the correlation calculation.

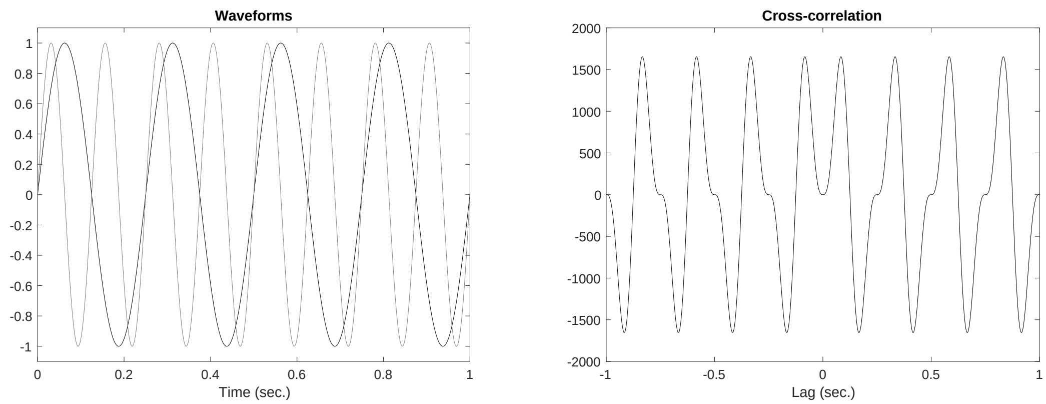

Figure 18: Cross-correlation of two sine waves with different frequencies

6.3 Variations of the Cross-correlation Function

There are similar variations of the cross-correlation function as the auto-correlation function discussed in Section 5.3.

6.3.1 Normalized Cross-correlation

The normalized variation of the cross-correlation function is found by taking into account the energy of the first signal, $E = r_{x,x}[0]$, and the energy of the second signal, $E = r_{y,y}[0]$. The normalization factor is the geometric mean of these two values.

$$r_{x,y}[l] = \frac{1}{\sqrt{r_{x,x}[0] \cdot r_{yy}[0]}} \sum_{n=1}^{N} x[n] \cdot y[n+l]$$

6.3.2 Biased and Unbiased Cross-correlation

The biased and unbiased variations of the cross-correlation function are identical to the auto-correlation versions, with the modification of including a second signal, y.

$$r_{x,y}[l] = \frac{1}{N} \sum_{n=1}^{N} x[n] \cdot y[n+l]$$

$$r_{x,y}[l] = \frac{1}{N-|l|} \sum_{n=1}^{N} x[n] \cdot y[n+l]$$