An Introduction to Ito Diffusion

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1 Definition and first properties

Let $B := \{ B_t, t \geq 0 \}$ be a standard Brownian motion defined on a probability space (Ω,$\mathcal{F}$,$\mathcal{F}_t$,$\mathbb{P}$) with the filtration given by $\mathcal{F}_t = \sigma(B_s$, 0 $\leq$ s $\leq$ t).

Definition 1.1 :Let $a(.) \ : \ \mathbb{R} \to \mathbb{R}$ and $b(.) \ : \ \mathbb{R} \to \mathbb{R}$ be two measurable functions, a Stochastic process $X_t(\omega) : \mathbb{R}_+$ x $\Omega \to \mathbb{R}$ is said to be an Ito Diffusion if X_t is solution of the following Stochastic differential equation :

$$X_t = X_s + \int_s^t a(X_u)\mathrm{d}u + \int_s^t b(X_u)\mathrm{d}B_u \ \ \forall \ \ t \geq s \geq 0 \ \ (SDE)$$

where X_s is a $\mathcal{F}_s$-measurable random variable such that $\mathbb{E}({X_s}^2) < \infty$.

Lemma 1.1 : For any constant $T > 0$ we define the process :
$W := \{ W_t = B_{T+t}$ - B_T , $t \geq 0 \}$, then W is a Brownian motion with respect to the probability measure $\mathbb{P}$.

Proof :
We have, $W_0 = B_T$ - $B_T = 0$, so W_0 is starting at 0.
For almost all $\omega \in \Omega$ the function t $\to B_{T+t}(\omega)$ is the translation function of t $\to B_t(\omega)$ wich is continious thus it is continuous.
$\forall \ t \geq s \geq 0$, W_t - $W_s = B_{T+t}$ - $B_{T+s} \sim \mathcal{N}$(0, T+t - (T+s)) $\approx \mathcal{N}$(0,t-s).
$\forall \ t \geq s \geq 0$, the random variable W_t - W_s is independant from $\mathcal{F}_{T+s}$, since $\mathcal{F}_s \subset \mathcal{F}_{T+s}$, thus W_t - W_s is independant from $\mathcal{F}_s$,
Therefore W_t is a standard Brownian motion.

Lemma 1.2 : Let $f : \mathbb{R}_+$ x $\Omega \to \mathbb{R} \in \mathbb{L}^2_{ad}$ (Ω x [0,T]).
Then, for all $h \geq T \geq 0$ the following egality stands almost surely,

$$\int_T^{T+h} f(t,\omega)\mathrm{d}B_t(\omega) = \int_0^h f(t+T,\omega)\mathrm{d}B_t(\omega)$$

Proof :

Let $\Delta = \{ t_0$=0 ,..., t_{n_T}=T , ..., t_n=T+h $\}$ be a partition of $[0, T+h]$ We first prove for step process and then for $f \in \mathbb{L}_{ad}$ (Ω, L^2 (0,T)).
By Lemma 1.1 since W_t is a Brownian motion we have ,
Let g(t,ω) = $\sum_{i=1}^n \xi_{i-1}(\omega) \ \mathbb{1}_{[t_{i-1},t_i]}$(t) a step process, then,

g(t+T,ω) = $\sum_{i=n_T+1}^n \xi_{i-1}(\omega) \ \mathbb{1}_{[t_{i-1}+T,t_i+T]}$(t)

$$
\begin{aligned}
\int_0^h g(t+T,\omega)\mathrm{d}B_t(\omega) &= \sum_{i=n_T+1}^{n} \xi_{i-1}(\omega)(B_{t_i+T} - B_{t_{i-1}+T}) \\
&= \sum_{i=n_T+1}^{n} \xi_{i-1}(\omega)(W_{t_i} - W_{t_{i-1}}) \\
&= \int_T^{T+h} g(t,\omega)\mathrm{d}W_t(\omega) \\
&= \int_T^{T+h} g(t,\omega)\mathrm{d}B_t(\omega)
\end{aligned}
$$

$f \in \mathbb{L}_{ad}$ $(\Omega,\ L\ ^2$ then there exists a sequence $(f_n)_{n\geq 0}$ of step process such that $I(f_n) \to I(f)$ in $L^2(\Omega)$, then there exists a subsequence $I(f_{\phi(n)})$ converging almost surely to $I(f)$.

$$
\begin{aligned}
\int_T^{T+h} f(t)\mathrm{d}B_t &= \int_T^{T+h} (f(t) - f_{\phi(n)}(t))\mathrm{d}B_t + \int_T^{T+h} f_{\phi(n)}(t)\mathrm{d}B_t \\
&= \int_T^{T+h} f(t)\mathrm{d}B_t - \int_T^{T+h} f_{\phi(n)}(t))\mathrm{d}B_t + \int_0^h f_{\phi(n)}(t+T)\mathrm{d}B_t
\end{aligned}
$$

By taking the limit in the equality, we have ,

$$
\begin{aligned}
\int_T^{T+h} f(t)\mathrm{d}B_t &= \lim_{n\to\infty} \int_T^{T+h} (f(t) - f_{\phi(n)}(t))\mathrm{d}B_t + \lim_{n\to\infty} \int_T^{T+h} f_{\phi(n)}(t)\mathrm{d}B_t \\
&= \lim_{n\to\infty} \int_0^h f_{\phi(n)}(t+T)\mathrm{d}B_t \\
&= \int_0^h f(t+T)\mathrm{d}B_t
\end{aligned}
$$

We now give some recalls on stochastic processes.

By definition, a stochastic process is a collection of random variables $\{X_t, t \geq 0\}$, $\forall\ t \geq 0$, $X_t : (\Omega,\mathcal{A}) \to (E,\mathcal{E})$ is a measurable function taking value in a measurable space $(E,\mathcal{E})$.
For continuous stochastic process we want to deal with the law of the process, therefore we can see the stochastic process as a random variable X taking value in $\mathcal{C}(\mathbb{R}_+,E)$, the space of all continuous functions from $\mathbb{R}_+$ to E.
Note that $\forall\ \omega \in \Omega$, $X(\omega) \in \mathbb{R}^{\mathbb{R}_+}$.

$$
\begin{gathered}
X : (\Omega,\mathcal{A}) \to \mathcal{C}(\mathbb{R}_+,E) \\
\omega \to (t \to X_t(\omega))
\end{gathered}
$$

We define $\forall\ t \geq 0$ the projection π_t as the function,

$$\pi_t : \mathbb{R}^{\mathbb{R}_+} \to \mathbb{R}$$
$$X \to X_t$$

On the space $\mathbb{R}^{\mathbb{R}_+}$, we can build a sigma algebra $\mathcal{B}$ generated by all canonical projections :

$$\mathcal{B} = \sigma(\ \cup_{t\geq 0}\ \pi_t^{-1}\ (B),\ B \in \mathcal{B}(\mathbb{R}))$$

It is the smallest sigma algebra on $\mathbb{R}^{\mathbb{R}_+}$ where all the projections are $\mathcal{B}$-measurable, then we will say that a function $F : \mathbb{R}^{\mathbb{R}_+} \to \mathbb{R}$ is measurable if it is measurable to $\mathcal{B}$.
But we can observe that $\mathcal{C} = \sigma(\ \cap_{t\in J}\ \pi_t^{-1}\ (B),\ B \in \mathcal{B}(\mathbb{R}))$, J finite, is a π-system so that $\mathcal{B} = \sigma(\ \mathcal{C})$. By Dynkin Lemma, the law of the process is uniquely determined by the finite dimensionnal law.
By Kolmogorov Extension Theorem we can build this law such that X be a random variable. In other words,

Proposition 1.1: The law of a stochastic process $X = \{X_t, t \geq 0\}$ is uniquely determined by all the law of vectors $(X_{t_1},...,X_{t_N})$ $\forall\ t_1,t_2,...,t_N \in \mathbb{R}_+$ and $\forall\ N \in \mathbb{N}$.

Therefore, two process $X = \{X_t, t \geq 0\}$ and $Y = \{Y_t, t \geq 0\}$ defined on the same probability space have the same law ($X \stackrel{\mathcal{L}}{=} Y$) if $\forall\ t_1,t_2,...,t_N \in \mathbb{R}_+$ and $\forall\ N \in \mathbb{N}$, $(X_{t_1},...,X_{t_N}) \stackrel{\mathcal{L}}{=} (Y_{t_1},...,Y_{t_N})$ these two vectors have same law.

The following Theorem assure the existance and uniqueness of a solution of a stochastic differentiel equation

Theorem 1.1 : Let a and b be the function given in the Stochastic Differential Equation (SDE), if a and b follow the Lipschitz condition,

$\mid a(x)$ - $a(y) \mid + \mid b(x)$ - $b(y) \mid \leq K \mid x$-$y \mid$ for some $K > 0$ and $\forall\ x,y \in \mathbb{R}$,
Then there exists an unique continous solution for (SDE) in the following sense :
If Y is another solution of (SDE) then $X \stackrel{\mathcal{L}}{=} Y$.

Remark : By Theorem 1.1 an unique continuous solution X_t is given,

$$X_t = X_s + \int_s^t a(X_u)\mathrm{d}u + \int_s^t b(X_u)\mathrm{d}B_u\ \forall\ t \geq s \geq 0\ (SDE)$$

by the property of a solution, the stochastic process $b(X_u) \in \mathcal{L}(\Omega,L^2(a,b))$ so the Ito integral in (SDE) is well defined and is measurable with respect to $\mathcal{F}_t$, then $\forall$

$t \geq 0$, the random variable X_t is $\mathcal{F}_t$ measurable.

In the other hand $\forall\ t \geq s \geq 0$, B_t - B_s is independant from $\mathcal{F}_s$, but X_t is measurable with respect to the sigma field generated by the random variable B_t - B_s therefore X_t is also independant from $\mathcal{F}_s$.

We now consider the equation (SDE) and Theorem 1.1 assumptions on function a and b so an unique solution exists and we denote it by $\{ X_t^s$, t $\geq 0 \}$ the solution starting at time s.

We have for all t,s ≥ 0,

$$\begin{aligned} X^s_{s+t} &= X_s + \int_s^{s+t} a(X_u)\mathrm{d}u + \int_s^{s+t} b(X_u)\mathrm{d}B_u \\ &= X_s + \int_0^t a(X_{s+v})\mathrm{d}v + \int_0^t b(X_{s+v})\mathrm{d}B_v \end{aligned}$$

the change of variable in the stochastic integral is given by Lemma 1.2.
Thus the process $\{ X^s_{s+t}$, t $\geq 0 \}$ is solution of (SDE) starting at s=0 if we considere $X_0 = X_s = x \in \mathbb{R}$.

By uniqueness of the solution of (SDE) starting at time s=0 the two process $\{ X^s_{s+t}$, t $\geq 0 \}$ and $\{ X^0_t$, t $\geq 0 \}$ have same law.

Thus by Proposition . we have that $\forall\ t_1$,t_2,...,$t_N \in \mathbb{R}_+$ and $\forall\ N \in \mathbb{N}$,
$(X_{t_1}$,...,$X_{t_N}) \overset{\mathcal{L}}{=} (X_{t_1+s}$,...,$X_{t_N+s}) \ \forall\ s \geq 0$.

We say that the process $\{ X^0_t$, t $\geq 0 \}$ is time-homogenuous.

2 Ito Diffusion as a Markov process

Intuitively, a process is a Markov process if the future behaviour only depends of what happened just before and not from the beginning of the trajectory.
For processes $X = \{X_n, n \geq 0\}$ where all the random variables are taking value in a countable space M, the definition of a Markov process can by easily formalize with the conditionnal probability :

$\forall\ x_0\ ,...,x_{n+1} \in M, \forall\ n \in \mathbb{N}$;

$$\mathbb{P}(X_{n+1}{=}x_{n+1} \mid X_n{=}x_n,...,X_0{=}x_0) = \mathbb{P}(X_{n+1}{=}x_{n+1} \mid X_n{=}x_n)$$

Here an Ito Diffusion $X = \{X_t, t \geq 0\}$ describe a trajectory indexed by time and $\forall$ t ≥ 0, the random variable X_t is continuous, hence it is impossible to give the same definition as below.

Definition 2.1 Let $(E,\mathcal{E})$ be a measurable space. We say that an application v defined as :

$$v\,:\,(E,\mathcal{E}) \rightarrow [0,1]$$

is a transition probability if
$\forall\ x \in E,\ A \rightarrow v(x,A)$ is a probability measure on $(E,\mathcal{E})$.
$\forall\ A \in \mathcal{E},\ x \rightarrow v(x,A)$ is $\mathcal{E}$-measurable.

Example For the Ito Diffusion, $v(x,A)\ :\ = \mathbb{E}\ (\mathbb{1}_{X_t \in A} \mid X_s = x)\ \forall\ t \geq 0$ define a transition probability on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$.

We now introduce the notion of conditional law.

Definition 2.2
Let X and Y be two random variables taking values in $(E,\mathcal{E})$. Then the conditionnal law of X knowing Y (denote $\mathcal{L}(X \mid Y)$) is any transition probability v on $(E,\mathcal{E})$ fullfilling :

$$\mathbb{E}(h(X) \mid Y\ ) = \int\ h(y)\ v(Y,dy)\ ,$$

for any $h\,:\,E \rightarrow \mathbb{R}$ measurable and bounded function.

<u>Remark</u> : The conditionnal law of X knowing Y is unique and exists (Theorem admited)
For $h = \mathbb{1}_A,\ A \in \mathcal{E}$, we have $\mathbb{E}(\mathbb{1}_A(X) \mid Y\ ) = \int\ \mathbb{1}_A(y)\ v(Y,dy) = v(Y,A)$.

Then we can give a definition for Markov processes.

Definition 2.3

We say that a stochastic process $X = \{X_t, t \geq 0\}$ taking value in $\mathbb{R}$ is a Markov process if,

$$\forall \text{ t} > \text{s} \geq 0,\ \mathcal{L}(X_t \mid X_s...X_0) = \mathcal{L}(X_t \mid X_s)$$

where $X_s...X_0$ denote the stochastic process $\{X_u, u \in [0, s]\}$.
In addition, if $\mathcal{L}(X_t \mid X_s) = \mathcal{L}(X_{t-s} \mid X_0)$ we say that X is a time-homogeneous Markov process.

<u>Remark</u> : If X is a Markov process, then we have the egality almost surely of transition probability $v(X_s..X_0,A) = v(X_s,A)$ and by Remark .

$$\forall \text{ t} \geq \text{s} \geq 0,\ \forall\ A \in \mathcal{B}(\mathbb{R}),\ \mathbb{E}\ (\mathbb{1}_{X_t \in A} \mid \mathcal{F}_s^X\ ) = \mathbb{E}\ (\mathbb{1}_{X_t \in A} \mid X_s\ )$$

where $\mathcal{F}_s^X = \sigma(X_u,\text{u} \leq \text{s})$.

In addition, if $\mathbb{E}\ (\mathbb{1}_{X_t \in A} \mid X_s) = \mathbb{E}\ (\mathbb{1}_{X_{t-s} \in A} \mid X_0)$ we say that the process is time-homogeneous.

The Markov Property

We define the shift operateur $\forall\ h \geq 0$,

$$\theta_h : \underset{(X_t)t \geq 0}{\mathcal{C}([0, \infty[, \mathbb{R})} \underset{\to}{\to} \underset{(X_t)t \geq h}{\mathcal{C}([h, \infty[, \mathbb{R})}$$

This operateur shift a given process starting at time t to the same process starting at time $t + h$.

Theorem 2.1 (The Markov Property)

Let $X = \{X_t, t \geq 0\}$ be an Ito Diffusion (solution of (SDE)) and $F : \mathbb{R}^{\mathbb{R}_+} \to \mathbb{R}$ a measurable or bounded or positive function then X satisfy the Markov property :

$$\forall\ h \geq 0,\ \mathbb{E}(F \text{ o } \theta_h(X) \mid \mathcal{F}_h^X) = u(X_h)$$

where $u(x) = \mathbb{E}^x\ (F(X))\ \forall\ x \in \mathbb{R}$ and $\mathbb{E}^x$ considering the process starting at point x.

<u>Proof :</u>

By Dynkin Lemma and Proposition 1.2 we can suppose X be a finite dimensionnal vector and F=$\mathbb{1}_A$, $A \in \mathcal{C}$, the sigma field generated by a finite number of projections. Let $\{\ t_0$=$0 \leq t_1 \leq ... \leq t_N$=t $\}$ be a partition of $[0,t]$.
X=$(X_{t_0},...,X_{t_n})$

Then $\mathbb{E}(F \text{ o } \theta_h(X) \mid \mathcal{F}_h)$= $\mathbb{E}(\mathbb{1}_A(X_{t_0+h},...,X_{t_n+h}) \mid \mathcal{F}_h)$

$$\begin{aligned}
\mathbb{E}(Fo\theta_h(X)|\mathcal{F}_h) &= \mathbb{E}(\mathbb{1}_A(X_{t_0+h}, ..., X_{t_n+h})|\mathcal{F}_h) \\
&= \mathbb{E}(\mathbb{1}_{X_h \in A_0}\mathbb{1}_{X_{t_1+h} \in A_1}...\mathbb{1}_{X_{t_n+h} \in A_n}|\mathcal{F}_h) \\
&= \mathbb{E}(\mathbb{1}_{X_h \in A_0}\mathbb{1}_{X_{t_1+h} \in A_1}...\mathbb{1}_{X_{t_n+h} \in A_n}) && (1) \\
&= \mathbb{E}(\mathbb{1}_{X_0 \in A_0}\mathbb{1}_{X_{t_1} \in A_1}...\mathbb{1}_{X_{t_n} \in A_n}) && (2)
\end{aligned}$$

(1) we use that X_t is independant from $\mathcal{F}_h$ for $t \geq h$.
(2) is a consequence of the time-homogeneous property for Ito Diffusion

We have proved for $\mathcal{F}_h$ the sigma field generated by B_u , $u \leq t$, but $\mathcal{F}_h^X \subset \mathcal{F}_h$
Then by the tower property,

$$\begin{aligned}
\mathbb{E}(Fo\theta_h(X)|\mathcal{F}_h^X) &= \mathbb{E}(\mathbb{E}(Fo\theta_h(X)|\mathcal{F}_h)|\mathcal{F}_h^X) \\
&= \mathbb{E}(u(X_h)|\mathcal{F}_h^X) \\
&= u(X_h). && (3)
\end{aligned}$$

(3) X_h is $\mathcal{F}_h^X$-measurable.

Q.E.D.

We have seen so far that $v(X_s,A) := \mathbb{E}(\mathbb{1}_A(X_t) \mid X_s)$ define a probability transition on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, it is the law of X_t knowing X_s.

We pose $P_{s,t}(x,A) := v(X_s,A)\ \forall\ x \in \mathbb{R}$.
Since X_t is time-homogeneous, $\forall\ t \geq s$, $P_{s,t}(x,A)$=$P_{0,t-s}(x,A)$.
So $P_{0,t}(x,A)$ will be denote only by $P_t(x,A)$.

$\forall\ t \geq 0$, P_t is the law of X_t knowning X_0.
And we define the function, $\forall\ x \in \mathbb{R}$, $\forall\ f : \mathbb{R} \to \mathbb{R}$ borel and bounded function,

$$P_t\ f(x) = \int\ f(y)\ P_t(x,dy)$$

Corrolary 2.1 Markov simple
Applying the Markov property with the function $F(X)$=$f(X_t)$ for a given $t \geq 0$, wich is measurable, then we get :

$$\mathbb{E}(f(X_{t+h}) \mid \mathcal{F}_h^X) = P_t\ f(X_h)$$

Definition 2.4 A randam variable $\tau : \Omega \to [0,\infty]$ is said to be a stopping time with respect to the filtration $\{\ \mathcal{F}_t,\ t \geq 0\ \}$ if,

$$\forall\ t \geq 0,\ \{\ \tau \leq t\ \} \in \mathcal{F}_t$$

Definition 2.5 Let $\tau : \Omega \to [0,\infty]$ be a stopping time, $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ a filtred probability space, we define the stopping filtration $\mathcal{F}_\tau$ by

$$\mathcal{F}_\tau = \{ \ A \in \mathcal{F} \ , \ A \cap \{ \ \tau \leq t \ \} \in \mathcal{F}_t \ \forall \ t \ 0 \ \}$$

Proposition 2.1

-$\mathcal{F}_\tau$ is a sigma algebra,
-τ is $\mathcal{F}_\tau$-measurable,
-if s and τ are two stopping time such that $s \leq \tau$ then, $\mathcal{F}_s \subset \mathcal{F}_\tau$.

Theorem 2.2 The Strong Markov property
We have the same result as the Markov property but for a random shift.

Let $X = \{X_t, t \geq 0\}$ be an Ito Diffusion (solution of (SDE)) and $\tau : \Omega \to [0,\infty]$ a stopping time, then almost surely :

$$\mathbb{E}(F \text{ o } \theta_\tau(X) \mid \mathcal{F}_\tau^X) = u(X_\tau) \text{ on the event } \{ \ \tau < \infty \ \}$$

where $u(x) = \mathbb{E}^x \ (F(X)) \ \forall \ x \in \mathbb{R}$ and $\mathbb{E}^x$ considering the process starting at point x.

Proof :

We can decompose the proof assuming first that τ takes value in a countable set. $\tau(\omega) \in \{ \ t_0, \ t_1,...\infty \ \}$.
We want to show that $\mathbb{E}(F \text{ o } \theta_\tau(X) \mid \mathcal{F}_\tau^X) = u(X_\tau)$ on the event $\{ \ \tau < \infty \ \}$ which is equivalent to $\mathbb{E}(F \text{ o } \theta_\tau(X) \ \mathbb{1}_A) = \mathbb{E}(\ u(X_\tau) \ \mathbb{1}_A) \ \forall \ A \in \mathcal{F}_\tau^X$ since X_τ is $\mathcal{F}_\tau^X$-measurable by Proposition 2.1

On $\{ \ \tau < \infty \ \}$, since τ takes countable values,

$$\begin{aligned}
\mathbb{E}(Fo\theta_\tau(X)\mathbb{1}_A) &= \sum_{n=0}^{\infty} \mathbb{E}(Fo\theta_{t_n}(X)\mathbb{1}_{A\cap\{\tau=t_n\}}) && (4)\\
&= \sum_{n=0}^{\infty} \mathbb{E}(Fo\theta_\tau(X)\mathbb{1}_{A\cap\{\tau=t_n\}}) && (5)\\
&= \sum_{n=0}^{\infty} \mathbb{E}(u(X_\tau)\mathbb{1}_{A\cap\{\tau=t_n\}}) && (6)\\
&= \mathbb{E}(u(X_\tau)\mathbb{1}_A) && (7)
\end{aligned}$$

Now for $\tau : \Omega \to [0,\infty]$, since τ is a positive measurable function there exists an increasing sequence of step function τ_n such that $\tau_n \to \tau$ as $n \to \infty$. In addition τ_n is a $\mathcal{F}^X_{\tau_n}$-stopping time.

We know that for all $n \in \mathbb{N}$, $\mathbb{E}(F \text{ o } \theta_{\tau_n}(X) \mid \mathcal{F}^X_{\tau_n}) = u(X_{\tau_n})$

But since X is a continuous time process then $F \text{ o } \theta_{\tau_n}(X) \to F \text{ o } \theta_{\tau}(X)$ and $u(X_{\tau_n}) \to u(X_\tau)$ as $n \to \infty$

Also remark that $\cap_{n>0} \mathcal{F}^X_{\tau_n} = \mathcal{F}^X_{\tau}$ since the family $(\mathcal{F}^X_{\tau_n})_{n>0}$ are increasing , then we can pass to the limit in $\mathbb{E}(F \text{ o } \theta_{\tau_n}(X) \mid \mathcal{F}^X_{\tau_n})$ however we will not prove this here, this is a consequence of martingales convergence.

Therefore we have the expected result.

Example : The Ornstein-Ulhenbeck process

We give the following stochastic differentiel equation :
dX_t = - $X_t\ dt + \sigma\ dB_t$

To solve this, we apply Ito formula with the function $f(x,t) = x\ e^t$, $X_0 = x$
We have $d\ f(X_t,t) = e^t\ dX_t + X_t\ e^t\ dt = e^t\ (-X_t dt + \sigma dB_t) + X_t\ e^t\ dt = \sigma\ e^t\ dB_t$
$X_t = x\ e^{-t} + \sigma \int_0^t e^{u-t}\ dB_u$
So, $\mathbb{E}((X_{t+s}) \mid \mathcal{F}^X_s) = u(X_s) = X_s\ e^{-t}$

3 Dynkin's formula

The Generator of an Ito Diffusion

Definition 3.1 :

Let X_t be an Ito Diffusion and $x \in \mathbb{R}$, we define the infinitesimal generateur $\mathcal{A}$ as the operateur : $\mathcal{A} : \mathcal{D}_{\mathcal{A}}(x) \to \mathbb{R} : f \in \mathcal{D}_{\mathcal{A}} \to \mathcal{A}(f(x)) = \underset{t\to 0}{lim} \frac{\mathbb{E}^x(f(X_t))-f(x)}{t}$.
Where $\mathcal{D}_{\mathcal{A}}(x) = \{ f : \mathbb{R} \to \mathbb{R}$ measurable ; $\underset{t\to 0}{lim} \frac{\mathbb{E}^x(f(X_t))-f(x)}{t}$ exists.$\}$ and we denote $\mathcal{D}_{\mathcal{A}} = \{ f : \mathbb{R} \to \mathbb{R}$ measurable ; $f \in \mathcal{D}_{\mathcal{A}}(x)\ \forall\ x \in \mathbb{R}\ \}$.
First we want to show the following Lemma

Lemma 3.1 Let Y_t be an Ito process given by $Y_t = x + \int_0^t u(s,\omega)$ ds + $\int_0^t b(s,\omega)$ $\mathrm{dB_s}$ with u_t and b_t bounded process or we differential notation $dY_t = u_t dt + b_t dB_t$, Y_0. For $f \in \mathcal{C}_o^2(\mathbb{R})$ twice derivable and with compact support, and τ a $\mathcal{F}_t$ -integrable and bounded stopping time.
Then we get :

$$\mathbb{E}^x(f(Y_\tau)) = f(x) + \int_0^\tau \{\ f'(Y_s)\ u_s + \tfrac{1}{2}\ f''(Y_s)\ {b_s}^2\ \}\ ds$$

<u>Proof :</u>
For $f \in {C_o}^2(\mathbb{R})$ we apply Ito formula and we get

$$f(Y_\tau) = f(Y_0) + \int_0^\tau \{\ f'(Y_s)\ u_s + \tfrac{1}{2}\ f''(Y_s)\ {b_s}^2\ \}\ ds + \int_0^\tau\ f'(Y_s)\ b_s\ dB_s$$

Taking the expectation,

$$\mathbb{E}^x(f(Y_\tau)) = \mathbb{E}^x(f(Y_0)) + \mathbb{E}^x \int_0^\tau \{f'(Y_s)u_s + \tfrac{1}{2} f''(Y_s){b_s}^2\}ds + \mathbb{E}^x \int_0^\tau f'(Y_s) b_s dB_s$$

We want to see that $\mathbb{E}^x \int_0^\tau f'(Y_s)b_s dB_s = 0$ then the proof will end.

First For all integers $n \geq 0$, $\mathbb{E}^x \int_0^{\tau\wedge n} f'(Y_s)b_s dB_s = \mathbb{E}^x \int_0^n \mathbb{1}_{s\leq\tau}\ f'(Y_s)\ b_s\ dB_s$

Since the process $(t,\omega) \to \mathbb{1}_{t\leq\tau(\omega)}\ f'(t)\ b(t,\omega)$ is bounded for all (t,ω) by some constant $K > 0$, then $\mathbb{1}_{t\leq\tau}\ f'(t)\ b(t,\omega) \in \mathbb{L}^2_{ad}$ (Ω x [0,t]) hence $\mathbb{E}^x \int_0^{\tau\wedge n} f'(Y_s)\ b_s\ dB_s = 0$.

By Ito isometry,

$$\mathbb{E}^x\Big(\int_0^{\tau} f'(Y_s)b_s dB_s - \int_0^{\tau\wedge n} f'(Y_s)b_s dB_s)^2 = \mathbb{E}^x\Big(\int_{\tau\wedge n}^{\tau} (f'(Y_s)b_s)^2 ds)$$
$$\leq K^2\mathbb{E}^x(\tau - \tau\wedge n)$$

Since τ is bounded by dominated convergence $\mathbb{E}^x(\ \tau \wedge n) \to \mathbb{E}^x(\ \tau)$ as $n \to \infty$.

Hence, $\int_0^{\tau\wedge n} f'(Y_s)\ b_s\ dB_s \to \int_0^{\tau} f'(Y_s)\ b_s\ dB_s$ in $L^2(\Omega)$ and also in $L^1(\Omega)$.

Then, $\mathbb{E}^x(\ \int_0^{\tau} f'(Y_s)\ b_s\ dB_s\ ) = \underset{n\to\infty}{lim}\ \mathbb{E}^x\ (\int_0^{\tau\wedge n} f'(Y_s)\ b_s\ dB_s\ ) = 0.$

Q.E.D

Lemma 3.2

Let X_t be an Ito Diffusion given by $dX_t{=}u(X_t)dt\ +\ b(X_t)dB_t$, $X_0 = x$, $f \in \mathcal{C}_o^2(\mathbb{R})$, we have $\mathcal{A}(f(x)) = f'(x)\ u(x) + \frac{1}{2}\ f''(x)\ b(x)^2$

Proof :

$$\mathcal{A}(f(x)){=}\ \underset{t\to 0}{lim}\ \frac{\mathbb{E}^x(f(X_t)) - f(x)}{t} = \underset{t\to 0}{lim}\ \frac{1}{t}\ \mathbb{E}^x\ \int_0^t (f'(X_s)\ u(X_s) + \frac{1}{2}\ f''(X_s)\ b(X_t)^2\ )ds$$
$$= \underset{t\to 0}{lim}\ \frac{1}{t}\ \int_0^t \mathbb{E}^x\ g(X_s)\ ds$$

by denoting $g(x) = f'(x)\ u(x) + \frac{1}{2}\ f''(x)\ b(x)^2$.

Since the function g and X_t are both continuous, then $\mathbb{E}^x\ g(X_s)$ is also continuous and the limit is only the derivative at $y{=}0$ of the function $\Phi(y) = \int_0^y \mathbb{E}^x\ g(X_s)\ ds$ and by the fundamental Theorem of analysis $\Phi'(y) = \mathbb{E}^x\ g(X_y)$,
we can conclude $\mathcal{A}(f(x)) = \mathbb{E}^x\ g(X_0) = \mathbb{E}^x\ g(x)$.

Q.E.D

By combining the two last Lemma, we get the Dynkin's formula.

Theorem 3.1
Let X_t be an Ito Diffusion, τ an integrable stopping time and $f \in {C_o}^2(\mathbb{R})$

$$\mathbb{E}^x(f(X_\tau)) = f(x) + \mathbb{E}^x(\ \int_0^{\tau} \mathcal{A}(f(X_s))ds)$$

Application

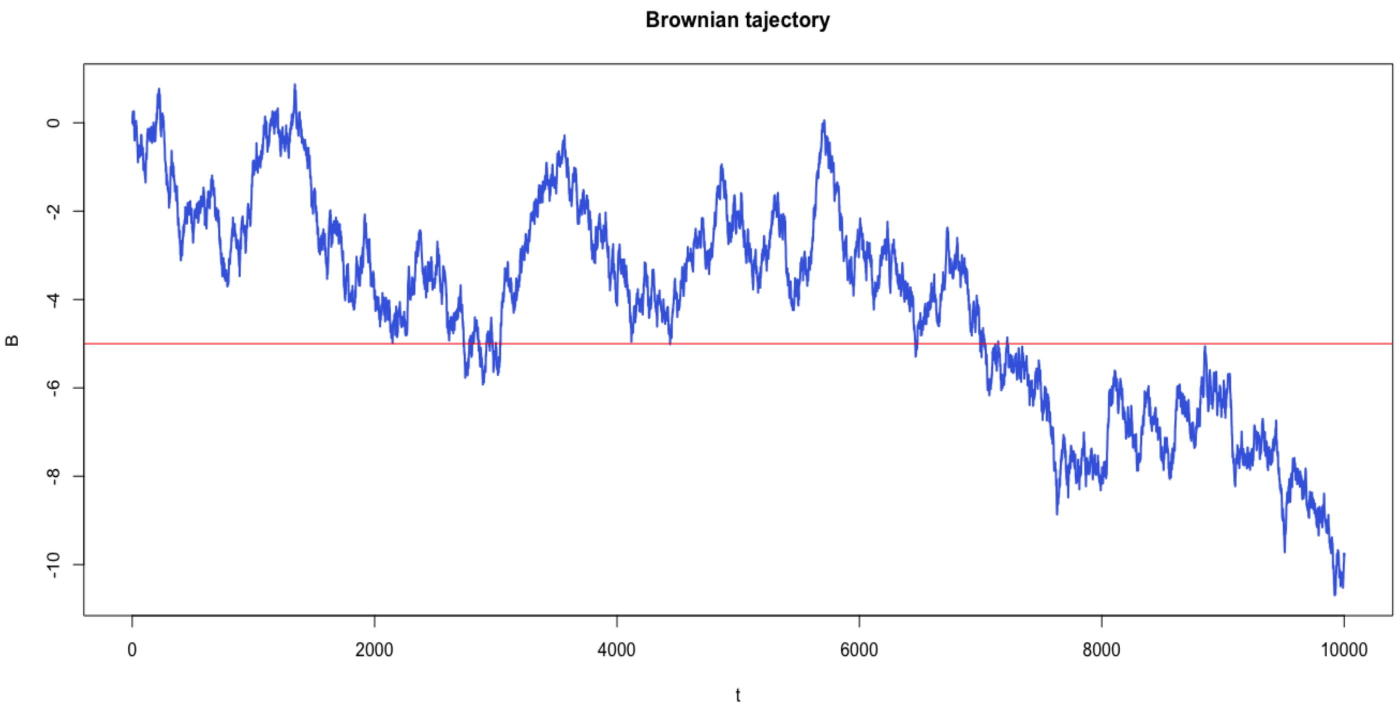
We want to apply the Dynkin's formula for the Ito Diffusion X_t=B_t and for the stopping time $\tau(\omega)$=inf $\{ \ t \geq 0, \ |B_t(\omega)| = R \ \}$, where $R > 0$.

We know that for all $x \in \mathbb{R}$, $\mathbb{P}(\ \exists \ t \geq 0, \ B_t = x) = 1$, so τ is finite almost surely, so $\tau \wedge k \to \tau$ as $k \to \infty$ and by dominated convergence $\mathbb{E}(\tau) < \infty$, thus we can apply the Dynkin's formula to τ and we choose f : [-R,R] $\to \mathbb{R}$, $f(x)$=x^2.

The Generator of B_t is $\mathcal{A}(f(x)) = f'(x) \ u(x) + \frac{1}{2} \ f''(x) \ b(x)^2 = \frac{1}{2} \ f''(x)$.

$\mathbb{E}^0(f(B_\tau)) = f(0) + \mathbb{E}^0(\ \int_0^\tau \ \mathcal{A}(f(B_s))ds) = \mathbb{E}^0(\tau)$.

Since $B_\tau = R$, $\mathbb{E}^0(\tau) = R^2$.



In the graph above, we can see a simulation of Brownian motion starting at 0, and with a partition of [0,100] given by t_i=$\frac{i}{100}$, i=0,...,10 000. The redline is the equation y=-5, we take here R=5 in the previous Application and we can see that the first exit time of [-5,5] is nearly a t_{2500}=25.

Course material : **"Stochastic Differential Equations"**,Chapter 7, by Bernt Øksendal.