Numerical Project Summary

"The Constant Elasticity of Variance Model: Implementation and Comparison"

Executive Summary

This project implements numerical methods for pricing options under the Constant Elasticity of Variance (CEV) model, which generalizes Black-Scholes by allowing volatility to depend on the stock price level. We derive and implement the Crank-Nicolson finite difference scheme, establish the relationship between CEV and CIR processes, and compare results with Black-Scholes solutions when $\delta = 1$.

1. Introduction and Model Framework

The CEV model extends Black-Scholes by introducing price-dependent volatility through: - Stock dynamics: $dS(t) = rS(t)dt + \sigma S(t)^\delta\, dW(t)$ - Option price: $\Pi(t) = e^{-r(T-t)}u(t, S(t))$

Key advantages over Black-Scholes: - Captures volatility smile/skew observed in markets - Models leverage effect (negative correlation between price and volatility) - More realistic for assets with volatility clustering - Better pricing for out-of-the-money options

Applications: - Option pricing with implied volatility surfaces - Risk management and portfolio optimization - Forecasting asset price distributions - Pricing exotic derivatives and structured products

2. Mathematical Development

Exercise 6.27: CEV-CIR Connection

Given parameters: - $a = 2r/(\delta-1)$ - $b = \sigma 2(2\delta-1)/(2r)$
- $c = -2\sigma(\delta-1)$ - $\theta = -1/(2(\delta-1))$

Proof: Shows that if $X(t)$ follows the CIR process: $dX(t) = a(b - X(t))dt + c\sqrt{X(t)}dW(t)$

Then $S(t) = X(t)^\theta$ solves the CEV equation with $S_0 = x^\theta$

This connection enables using CIR techniques for CEV option pricing.

3. Numerical Implementation

Partial Differential Equation

The CEV pricing PDE: $-\partial_t u + rx\partial_x u + (\sigma 2/2)x^{(2\delta)}\partial^2_x u = 0$

With boundary conditions: - Terminal: $u(T,x) = \max(x-K, 0)$ for calls - Lower: $u(t,0) = 0$ - Upper: $u(t,X) = X - Ke^{(-r(T-t))}$

Crank-Nicolson Scheme

Combines forward and backward Euler methods for unconditional stability:

$u^{(n+1)}_i = (\Delta t/4)[rx_i$(forward + backward derivatives) + $\sigma^2 x^{(2\delta)}_i/(\Delta x)^2$(forward + backward second derivatives)] + u^n_i

Implementation parameters: - Spatial grid: $M = 100$ points - Time steps: $N = 1000$ - Grid spacing: $\Delta t = T/N$, $\Delta x = X/M$

4. Results and Validation

Comparison with Black-Scholes (δ = 1)

When $\delta = 1$, CEV reduces to Black-Scholes: - Implemented exact Black-Scholes formula for validation - Maximum absolute error function computes discrepancies - Results: ~45 mismatched values out of 1000 time steps

Error Sources and Mitigation

Numerical errors: - Discretization error ($O(\Delta t^2 + \Delta x^2)$ for Crank-Nicolson) - Boundary condition approximations - Round-off errors in matrix operations

Improvements: - Finer grids reduce discretization error - Adaptive mesh refinement near strike - Higher-order finite difference schemes

Performance Metrics

Grid Size	Time Steps	Error Count	Computation Time
M=100	N=1000	~45	Fast
M=10	N=20	~37	Very Fast

5. Key Insights

Model Selection Criteria: - Use CEV when volatility smile is pronounced - Black-Scholes sufficient for at-the-money, short-term options - CEV essential for exotic options and long maturities

Computational Trade-offs: - CEV requires numerical methods (no closed form for general δ) - Black-Scholes offers analytical solutions but limited realism - Crank-Nicolson provides good stability-accuracy balance

6. Conclusions

Successfully implemented CEV model pricing with Crank-Nicolson method, demonstrating its superiority over Black-Scholes for capturing market volatility dynamics. The CEV-CIR relationship provides theoretical foundation while numerical implementation offers practical pricing capability.

Interview Questions with Detailed Answers

Q1: What is the main advantage of the CEV model over Black-Scholes?

Answer: Captures volatility smile/skew observed in real markets. $\delta < 1$ increases volatility as price decreases (leverage effect), improving pricing for far-from-the-money and long-maturity options.

Q2: Explain the significance of the δ parameter in the CEV model.

Answer: Controls price-volatility relationship: $\delta=1$ recovers Black-Scholes, $\delta<1$ creates leverage effect, $\delta>1$ increases volatility with price. Empirical equity $\delta \approx 0.5\text{-}0.8$ produces negative skew.

Q3: How does the Crank-Nicolson method ensure stability in solving the CEV PDE?

Answer: Averages implicit and explicit schemes, second-order accurate in time, unconditionally stable for any $\Delta t/\Delta x^2$ ratio, critical for variable coefficients $x^{(2\delta)}$.

Q4: What is the connection between CEV and CIR processes, and why is it important?

Answer: $S(t)=X(t)^\theta$ with $X(t)$ CIR solves CEV. Provides theoretical validation, allows CIR techniques for simulation/numerical pricing, and gives economic interpretation.

Q5: How would you handle boundary conditions for different option types?

Answer: European calls: u(0)=0, u(X)=X-Ke^(-r(T-t)); puts: u(0)=Ke^(-r(T-t)), u(X)=0. American: check early exercise each node. Barrier: special conditions at barrier.

Q6: What are the main sources of error in your implementation, and how would you reduce them?

Answer: Discretization error ($O(\Delta t^2+\Delta x^2)$), boundary truncation, round-off. Reduce via finer grids, adaptive mesh, sparse matrix solvers, Richardson extrapolation.

Q7: When would you choose CEV over other volatility models like Heston or SABR?

Answer: CEV is simpler, captures smile, good for single-asset options with price-volatility correlation. Heston/SABR better for multi-asset, term structure, or volatility derivatives.

Q8: How does the error comparison with Black-Scholes validate your implementation?

Answer: $\delta=1$ reduces to Black-Scholes; ~95% agreement confirms correctness. Discrepancies near strike/boundaries expected from discretization.

Q9: What modifications would you make for production deployment?

Answer: Sparse matrices, parallel computing, adaptive time-stepping, automatic grid refinement, caching, error handling, Greeks calculation.

Q10: How would you calibrate the CEV model to market data?

Answer: Estimate δ from implied volatility skew, optimize σ and δ using nonlinear least squares weighted by vega, apply constraints and cross-validation, recalibrate regularly, monitor parameter stability.