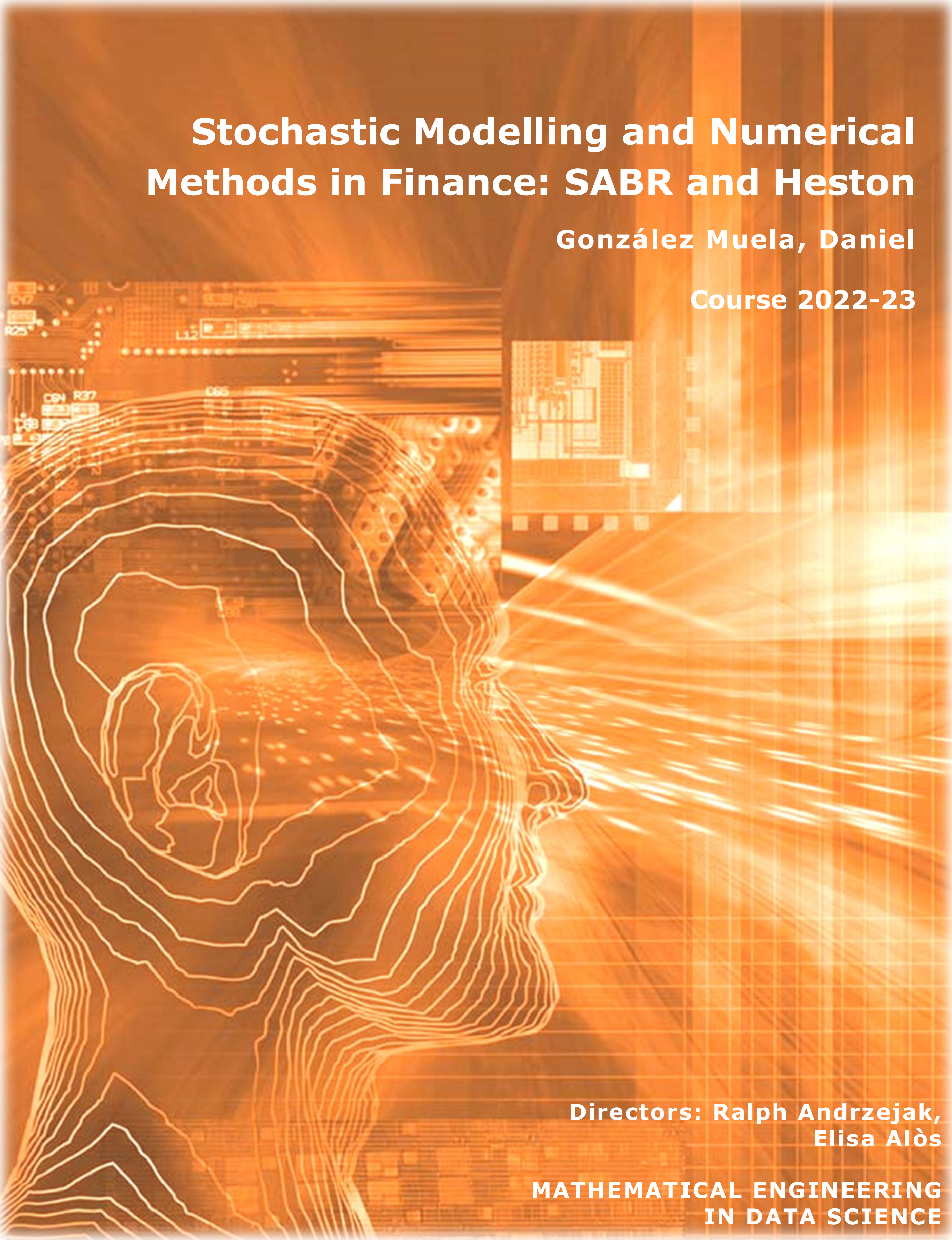

Stochastic Modelling and Numerical Methods in Finance: SABR and Heston

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Abstract

One of the main topics in Financial Mathematics is option pricing. The key for setting a good option price is developing a model that is consistent with the observed market and easy to compute. The Black-Scholes model was the first successful attempt to price options resulting in a well-known closed form solution for vanilla options, but due to its unrealistic assumptions the model is not consistent with the actual market prices and more modern approaches have been developed. One of these approaches lead to stochastic volatility models. However, as more assumptions are removed, the complexity of the models tends to increase, making it very hard to find closed-form solutions and leading to long computational times. On the other hand, many state-of-the-art numerical techniques have appeared in the last years, coming with opportunities to solve the stated problems.

In this project I study two stochastic volatility models, the SABR and the Heston, and I apply computational methods showing how different techniques can be used to improve the computational performance of these models. More specifically, given the model parameters, first I apply variance reduction techniques to compute the price of vanilla options in a faster and more accurate way, and second, with the help of these techniques I simulate the implied volatility surface of both models. Finally, given observed market data of vanilla options, I use minimization techniques to compute the parameters that would fit better the SABR model, which is simpler to calibrate than the Heston model.

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1. Introduction

Nowadays, Financial Mathematics is a very important field used by the majority of banks and insurance companies to provide solutions and services to the society [1]. One of these services is allowing to exchange options at a fair price (consult Appendix A for an introduction to options and the option pricing problem). The key for setting a good option price is developing a model that is: consistent with the observed market (accurate) and easy to compute (simple). The Black-Scholes (BS) model was the first successful attempt to price options resulting in a well-known closed form solution for vanilla options, the simplest type of options (explained in Appendix A). However, due to its unrealistic assumptions the model is not consistent with the actual market prices and more modern approaches have been developed using a similar procedure: taking an unrealistic assumption from the BS model and removing it, or at least adapting it to a more realistic one. The models I focus on this project are stochastic volatility models, which avoid the BS constant volatility assumption, empirically proven to be unrealistic [1].

On the other hand, as more assumptions are removed, the complexity of the models increases substantially, making it very hard to find closed-form solutions or approximations, and leading to very long computational times. In the last years, many results and techniques have appeared on how to increase the computational performance of the models. These state-of-the-art solutions come with lots of opportunities to solve the stated problems and to allow for the use of more complex models.

The goal of this project is to study two stochastic volatility models, the stochastic alpha, beta, rho model (SABR) and the Heston model, and provide concrete examples showing how these techniques can be applied to vanilla options to improve the computational performance of the models and help to provide support on theoretical results. First, I present an overview of the traditional approach to the option pricing problem and its state of art (Section 2). Second, I explain two stochastic volatility models (SABR and Heston) and discuss their properties and contributions to the field (Sections 3 & 4). Then, I use Monte Carlo simulations and variance reduction techniques to price both models (Section 5). Afterwards, I simulate their implied volatility surfaces, analyze their properties, and compare them to the Black Scholes implied volatility surface (Section 6). Finally, I use real market data to calibrate the SABR model (Section 7).

2. How to price a vanilla option? The option pricing problem

Option pricing is a major problem in the financial industry and much research has been done on the subject. In the following, I present an introduction and a review of some of its most famous results for pricing vanilla options. For a more comprehensive explanation see reference [1] and for an introduction to the topic and the basic knowledge to understand this thesis consult Appendix A.

The fair price of an option should be a combination of its expected discounted payoff and its risk (discounted payoff refers to the present value of the payoff, see reference [2] for a full definition). However, there exists the concept of risk neutral probability, under which investors are risk-neutral and the derivative's price is equal to its expected discounted payoff, eliminating thus the risk dependence.

Now let me introduce the fundamental theorem of asset pricing [1]. According to the first fundamental theorem of asset pricing, a market is arbitrage free, if and only if a risk neutral probability exists (a free arbitrage market is a market in which there are no arbitrage opportunities, that is, there is no opportunity of constructing a zero risk strategy with positive benefit [1]). Moreover, due to the second fundamental theorem of asset pricing, this risk neutral probability is unique if and only if the market is complete (markets in which is possible to bet in all the possible future states of the world using existing assets, with no transaction costs) . Vanilla option markets are complete because they can be replicated with a portfolio of the underlying asset and a riskless one, therefore the price of a derivative must equal its expected discounted payoff (for a complete explanation of how to compute the present/discounted value assuming a continuous interest rate, which would correspond to the risk-free rate r, see reference [3]):

$$V(t, S_t) = e^{-rT}\, E[h(S_t)] = e^{-rT}\, E[\max(S_T - K, 0)]$$

Where t is the time in years (unless stated otherwise), S_t the value of stock S at time t, $V(t, S_t)$ the value of an option on the underlying stock S, r the risk-free rate, T the maturity time in years (unless stated otherwise), $h(S_t)$, is the payoff function of a vanilla option on its underlying asset S at time t, $E[h(S_t)]$ is the expected value of $h(S_t)$ across all the realization of S_t, which is assumed to follow a random process and K is the constant strike price of the option $V(t, S_t)$.

Black-Scholes model

The Black-Scholes (BS) model is a model developed in 1973 by Fisher Black, Robert Merton and Myron Scholes [4]. Based on stochastic calculus, Brownian motion and Ito's lemma (more information can be found in [5], [6] and [7]) it was the first successful attempt to price options, with a well-known closed form solution for vanilla options. It has some assumptions:

- No dividends
- Random markets
- No transaction costs
- No arbitrage opportunities and a constant risk-free rate (the theoretical rate of return of an investment with zero risk)
- The returns of the underlying asset are normally distribute and volatility is constant
- The option is a vanilla
- Liquidity: it is possible to buy and sell any amount of asset at any time

From these assumptions, the value of the underlying stock S_t follows a geometric Brownian motion (GBM)

$$dS_t = (rS_t)dt + (\sigma S_t)dB_t$$

Where r is the risk-free rate or drift, σ the volatility, B_t a standard Brownian motion.

Black, Scholes and Merton ended up with the following partial differential equation (PDE) modelling the behavior of the value of a vanilla option $V(t, S_t)$.

$$\frac{\partial V(t,S_t)}{\partial t} + rS_t \frac{\partial V(t,S_t)}{\partial S_t} + \frac{(\sigma S_t)^2}{2} \frac{\partial^2 V(t,S_t)}{\partial {S_t}^2} - rV(t,S_t) = 0$$

Finally, using its similarity to the heat equation, they found the following closed-form solution for the present value of vanilla options, known as the BS formula:

$$V(t,S_t) = e^{-rT}\, E[\max(S_T - K, 0)] = C(\sigma, S_0, r, T, K) = S_0 \Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Where $C(\sigma, S_0, r, T, K)$ is the price for a vanilla call and $\Phi(x)$ is the cumulative distribution function of the standard normal distribution.

To compute the value of a vanilla put $P(\sigma, S_0, r, T, K)$ one could use the call-put parity:

$$C(\sigma, S_0, r, T, K) - P(\sigma, S_0, r, T, K) = S_0 - e^{-rT}K$$

This is a common derivation that begins by assuming the underlying asset model and results in a consistent price, but as mentioned before the objective is to find a good model from the observed prices. More practical derivations, such as the one presented in reference [2], begin by deriving the Black-Scholes PDE from the fact that a vanilla option can be priced as its discounted expectation. Then, the Feynman-Kack formula is used to conclude that in order to get a risk-neutral model that replicates the observed market prices, the underlying asset must follow a GBM.

Implied volatility surface

Given an observed price for a call and all the other parameters for $C(\sigma, S_0, r, T, K)$, implied volatility (IV) is the volatility value σ we should use as input for the Black-Scholes model to get the observed market price of an option with these properties.

The main caveat of the Black-Scholes model is that volatility is assumed to be constant, and therefore its theoretical implied volatility surface, that is plotting implied volatility (IV) versus time to maturity (T) and strike (K), is flat. However, empirical evidence [1] indicates that it is common to observe volatility smiles and skews. When comparing options with the same maturity time but different strike, it is usual to see that at-the-money options ($K = S_0$) the implied volatility is the lowest. Then, as the option is more in-the-money ($S_0 > K$) or out-of-the-money ($S_0 < K$) the implied volatility increases, showing a U-shape in the volatility smile (Figure 1).

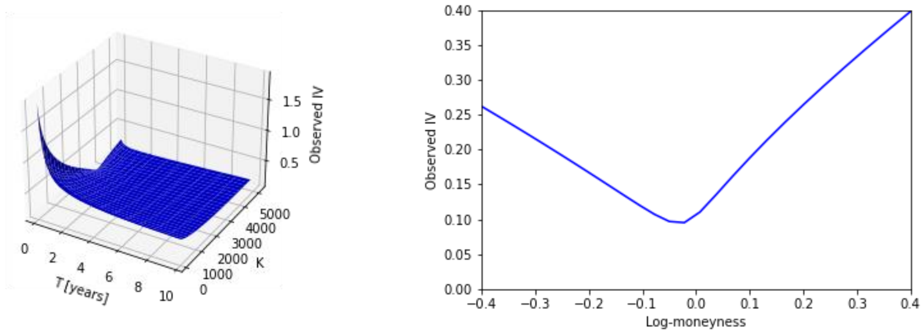


Figure 1: Implied volatility surface (left) and volatility smile (right) observed from real market data courtesy of David García-Lorite (CaixaBank).

Also note that in Figure 1 the data (which can be found in Appendix B) is plotted against T and log-moneyness= $ln(\frac{S_0}{K})$, instead of K. Log-moneyness is zero when an option is at the money and increases/decreases as the option is more in-the-money/out-of-the-money.

To solve the problem of observing implied volatilities which are different from the theoretical ones, models that assumed random volatilities appeared. In local volatility models, volatility $\sigma(S_t, t)$ is assumed to follow a local volatility function, while in stochastic volatility models, σ_t is assumed to follow a diffusion process. The latter ones, like the SABR or the Heston model are the ones I study in this thesis. In practice, both models are imperfect and depending on the context and which model gives better results, experts will work with one model or another [1].

3. SABR

The SABR model [8] is a stochastic volatility model that aims to replicate the volatility smile and describes a single forward F_t. σ_t is the volatility process, and W_t and Z_t are two correlated standard Brownian motions. The remaining parameters are coefficients $0 \leq \beta \leq 1$ (curvature parameter) and $\alpha \geq 0$ (volatility of the volatility).

$$dF_t = \sigma_t F_t^{\beta} dW_t$$

$$d\sigma_t = \alpha \sigma_t dZ_t$$

$W_t$and Z_t have a correlation of ρ.

$$dW_t dZ_t = \rho dt$$

In this model, the risk-free rate r is assumed to be 0. To initialize the model, values for its initial price F_0 and its initial volatility σ_0 are required.

Advantages

Flexibility: captures complex dynamics of asset prices and volatility. Stochastic volatility and skewness can be useful for modeling real-world market conditions [8].

Calibration: It is the main advantage of the SABR model. It has a simple calibration (done in Section 7) and it has fewer parameters than the Heston model.

Disadvantages

Analytical tractability: It has a solution for computing its implied volatility (widely used for model calibration), but there is no closed-form solution for option pricing.

Applicability: it is especially used in the interest rates derivatives market [8] and may not be good for other types of assets.

4. Heston

The Heston model [9] is another stochastic volatility model that describes an asset price S_t (difference between S and F is explained in Appendix A). It follows a process very similar to a GBM, but uses $\sqrt{v_t}$ as the volatility at each step, where v_t is the instantaneous variance described by a Cox-Ingersoll-Ross (CIR) process [10]. Its parameters are k (mean reverting rate), θ (long-term variance) and α (volatility of the volatility). W_t and Z_t are ρ correlated, and it is necessary to define the initial asset price S_0 and initial volatility v_0.

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t$$

$$dv_t = k(\theta - v_t)dt + \alpha\sqrt{v_t} dZ_t$$

For v_t to be strictly positive, the following condition must be satisfied:

$$2k\theta > \alpha^2$$

Advantages

Flexibility: It captures complex dynamics of asset prices and volatility and adds a component of mean reversion. [9]

Analytical tractability: It has an explicit formula for pricing vanilla options. There is no solution either for the implied volatility of European options and numerical methods are often used. [1]

Disadvantages

Calibration: The model has several parameters, and not having a solution for the implied volatility can make it hard and slow to calibrate. [1]

Applicability: it has more parameters, which adds an extra level of complexity compared to the SABR model.

5. Simulation and Monte Carlo pricing

As the models become more complex, it is harder to find closed-form solutions for the expected discounted payoff of an option. In consequence, the use of numerical methods and Monte Carlo (MC) simulations become necessary. After defining the parameters of the model, we can simulate future asset movements and compute the expected payoff of an option on that underlying asset. However, using accrued Monte Carlo generations, that is generating thousands of prices in a brute-force way is too slow. Banks need to price options (find a low variance estimator) as fast as possible, and variance reduction techniques are very useful for this purpose.

The simplest way to reduce the variance of an estimator is to generate more simulations, which we want to avoid since this implies a linear increase in the computation time. On the other hand, variance reduction methods compute the simulations in a smarter way that allows reducing the variance of an estimator without having to increase the number of simulations.

In this section, I apply some variance reduction techniques and compare their results to the accrued Monte Carlo simulations. I present a description of the methods explained in reference [12] and contribute by providing a practical application to the SABR and Heston models (there is no explanation on how to apply these methods for these specific models). Some of these techniques like the Conditional Method for the SABR model, have already been applied in some other thesis [11, 12] but to the best of my knowledge there is no other project or source applying any of the other methods. The python code of this section can be found in Notebooks 1 and 2 of Appendix B.

First, I describe the estimator we want to compute and how I do it: as explained in section 2, the fair price of an option should be its expected payoff. Therefore, our estimator is $E[h(F_T)]$ where as before, $h(F_T)$ is the payoff function (section 2) and F_T is the final price simulated by the SABR/Heston model. We also want to compute the variance of the estimator $Var(E[h(F_T)])$. I will simulate L option prices and compute the estimators as the variance and mean of the L prices. For computing each option price, I will generate M sample paths, which is simulating M option payoffs and then taking the average of the M payoffs to get an option price.

Now let me introduce some common notation used for the methods' explanation. Let X be a random variable and $Y(X)$ a function of that random variable. These methods explain how to reduce the variance of θ when we compute $\theta = E[Y(X)]$. In our case $X = F_T$ corresponds to the final price simulated by the SABR/Heston model, and $Y = h(F_T)$ the payoff function for a vanilla option.

5.1 SABR

The simulated SABR model has the following parameters:

$F_0 = 250, \sigma_0 = 0.3, \alpha = 0.2, \beta = 1, p = 0, T = 1, K = 270$

Each of the simulated paths has $n = 1000$ steps, so computationally we can treat each time increment as $dt = \frac{T}{n} = \frac{1}{1000}$. The Brownian motion can be simulated as $dZ_t \sim N(0, dt)$,where $N(\mu, \sigma^2)$ is a normal distribution with mean μ and variance σ^2, and the correlated Brownian motion $dW_t = pdZ_t + \sqrt{1 - p^2}dB_t$, in which $dB_t \sim N(0, dt)$.

<u>Control Variate Monte Carlo</u>

According to reference [12] we can compute an unbiased estimator of θ as $\theta_c = Y + \mathrm{c} * (\mathrm{Z} - \mathrm{E[Z]})$ for any number c and any random variable Z. Clearly the expected values are the same $E[\theta_c] = E[Y] = \theta$. Therefore, choosing an appropriate Z such that $Cov(Y, Z) \neq 0$, E[Z] is known and Z is also obtained during the simulation, we can compute c in a certain way that reduces the variance.

The variance of θ_c:

$$Var(\theta_c) = Var(\theta) + c^2 Var(Z) + 2cCov(Y, Z)$$

Hence, if:

$$c = -\frac{Cov(Y,Z)}{Var(Z)}$$

The variance is minimized and

$$Var(\theta_c) = Var(\theta) - \frac{Cov(Y,Z)^2}{Var(Z)}$$

In this case, I chose $Z = F_T$, which is the last value of the simulated forward price. Z is appropriate since $Cov(Z, Y) = Cov(F_T, h(F_T)) \neq 0$, its expected value is known: similarly to a GBM with $r = 0$ [15], $E[Z] = E[F_T] = F_0 exp(rT) = F_0$. Finally, Z is also obtained during the simulation because F_T is necessary for computing $h(F_t)$.

Antithetic Variates Method

Another way of computing an unbiased estimator of θ (explained in reference [12]), is defining a new estimator $\hat{\theta} = \frac{Y_1+Y_2}{2}$,where both Y_1 and Y_2 are realizations of the original process $Y = h(F_T)$. The variance of this new estimator is $Var(\hat{\theta}) = \frac{Var(Y_1)+Cov(Y_1, Y_2)}{2}$.

The key for applying this method is generating a negatively correlated sample to Y_1 (Y_2) without incurring in extra computations. In the SABR model I was able to simulate negatively correlated prices: to do it, I simulated an accrued price path of F_T computing $d\sigma_t = \alpha\sigma_t dZ_t$ and $dF_t = \sigma_t F_t^\beta dW_t = \sigma_t F_t^\beta p dZ_t + \sqrt{1-p^2}dB_t$. Then I defined a negatively correlated price path $F_{T_{neg}}$, using $d\sigma_{t_{neg}} = \alpha\sigma_t dZ_{t_{neg}}$ and $dF_{t_{neg}} = \sigma_{t_{neg}} F_{t_{neg}}^\beta dW_{t_{neg}}$

Where:

$$dZ_{t_{neg}} = -dZ_t$$

$$dB_{t_{neg}} = -dB_t$$

$$dW_{t_{neg}} = pdZ_{t_{neg}} + \sqrt{1-p^2}dB_{t_{neg}} = -dW_t$$

Once F_T and $F_{T_{neg}}$ are simulated, one can compute Y_1 and Y_2 as $h(Y_{F_T})$ and $h(Y_{F_{t_{neg}}})$, respectively.

Conditional Monte Carlo

The last method explained in reference [12] is the conditional Monte Carlo method. Instead of computing the usual expectation $E[Y]$ we try to condition the expectation on another random variable that allows us to compute the desired estimator with lower variance.

$$V = E[Y|P] = g(P)$$

V is an unbiased estimator of θ:

$$E[V] = E\big[E[Y|P]\big] = E[P] = \theta$$

And its variance:

$$Var(V) \leq Var(Y)$$

Since

$$Var(Y) = E[Var(Y|P)] + Var(E[Y|P])$$

Accordingly, instead of simulating Y and computing its mean (accrued Monte Carlo) one can compute V, that consists of simulating P and then computing $E[Y|P]$ for each P. Then one can compute $E[V]$. The difficulty here is finding a variable Z such that:

1- P can be easily simulated
2- $V = E[Y|P]$ can be computed exactly

In this case it is possible to apply this method using the extended Hull and White formula [16, 17] to compute $E[h(F_T)|Z_t]$. Under the stochastic volatility assumption,

$$\mathrm{E}[h(F_T)|\mathrm{Z_t}] = \mathrm{E}\left[C\left(\sqrt{1-p^2}\mathrm{v_0}, \xi\mathrm{tS_0}, r = 0, T, K\right)|Z_t\right]$$

Where

$$\mathrm{v_t} = \sqrt{\frac{1}{T-t}\int_t^T \sigma_s^2\, ds}$$

$$\xi\mathrm{t} = \exp\left(\rho\int_t^T \sigma_s\, \mathrm{dZ_s} - \frac{1}{2}\rho^2 \mathrm{v_t^2}(\mathrm{T}-\mathrm{t})\right)$$

Consequently,

$$\theta = E[Y] = E[h(F_T)] = E\big[E[h(F_T)|Z_t]\big]$$

This means that we can get our estimator only simulating the Brownian motion Z_t.

Antithetic + Conditional Monte Carlo

Now I combine the Antithetic and Conditional Monte Carlo methods. To do it, I apply the Conditional Monte Carlo, but instead of computing $\theta = E[Y] = E\big[E[h(F_T)|Z_t]\big]$ I define $Y' = \frac{E[h(F_T)|Z_t] + E[h(F_T)|Z_{t_{neg}}]}{2}$ and computed $\theta' = E[Y']$, where, as before, I compute $Z_{t_{neg}}$ using $dZ_{t_{neg}} = -dZ_t$.

Conditional Monte Carlo + Antithetic + Control variate

Finally, I combine these three methods. I follow the same procedure than for the "Antithetic + Conditional Monte Carlo", but I no longer compute $\theta' = E[Y']$. The new estimator has the same form like the control variate estimator ($\theta_c = Y + \mathrm{c} * (\mathrm{Z} - \mathrm{E[Z]})$) but using Y=$E[Y']$ and $\mathrm{Z} = \frac{\sigma_T^2 + \sigma_{T_{neg}}^2}{2}$ the mean of the two last values of the generated volatilities. Since σ_t follows a GBM [15],

$$\mathrm{E}[\sigma_T] = \sigma_0$$

$$\mathrm{Var}[\sigma_T] = \sigma_0^2 * \left(\mathrm{e}^{\alpha^2 T} - 1\right)$$

Therefore $\mathrm{E}[Z] = \mathrm{E}[\sigma_T^2] = \mathrm{Var}[\sigma_T] + \mathrm{E}[\sigma_T]^2$

Results

In Table 1 I present one of the main results I obtained after applying the variance reduction techniques to the SABR model. To run the SABR simulations (as well as all the other in the thesis), I used my personal laptop, which has an Intel i7, 12 GB of RAM and uses and operative system of 64 bits.

		Accrued			**Antithetic**			**Control variate**		
L	**M**	Time	Price	Std	Time	Price	Std	Time	Price	Std
50	50	27	23.9	8.0	29	21.3	4.0	26	22.3	3.7
50	100	53	22.8	4.8	62	22.5	2.6	53	22.5	2.6
50	200	105	21.9	3.6	115	22.0	2.0	107	22.3	1.9
100	50	55	23.3	7.9	59	22.0	3.8	53	22.1	3.8
100	100	113	23.1	4.6	121	22.0	2.6	105	22.2	2.4
100	200	217	22.5	3.6	238	22.1	2.2	217	22.5	1.7
200	50	105	23.2	6.9	116	21.9	4.2	104	22.0	3.7
200	100	212	22.6	5.2	238	22.1	2.8	218.	22.2	2.6
200	200	434	22.4	3.5	466	22.0	2.0	428	22.4	1.7

		Conditional MC			**Antithetic + Conditional**			**Control variate + Antithetic + Conditional**		
L	**M**	Time	Price	Std	Time	Price	Std	Time	Price	Std
50	50	11	22.03	0.44	12	22.132	0.063	12	22.111	0.032
50	100	23	22.07	0.31	24	22.109	0.041	25	22.120	0.026
50	200	46	22.12	0.21	46	22.111	0.030	46	22.121	0.016
100	50	24	22.06	0.45	23	22.125	0.061	23	22.118	0.034
100	100	48	22.13	0.34	47	22.116	0.045	48	22.113	0.024
100	200	96	22.09	0.24	96	22.111	0.032	96	22.111	0.014
200	50	48	22.11	0.49	48	22.115	0.061	48	22.114	0.034
200	100	96	22.11	0.31	95	22.115	0.042	92	22.118	0.023
200	200	192	22.11	0.21	189	22.118	0.030	195	22.116	0.017

Table 1: Numerical results obtained after applying different variance reduction techniques for pricing the SABR model. Time refers to the computational time in seconds, price to the computed option price estimator and Std is the standard deviation of the L simulated option prices, where each option price is computed as the mean of M option prices. The number of decimals is adjusted to the variability of the estimator: the results slightly vary each time you run the simulations, so I ran the simulations a couple times and took the number of decimals that gave a consistent result. For example if in on simulation I obtained 0.4456 and in another 0.4413, I only noted the first two decimals.

From this table one can see that the standard deviation of the expected price is significatively reduced as the number of prices per option (M) increases regardless of the methods used. However, as the number of simulated options increases (L), the standard deviation does not present any substantial change. Finally, to compare the methods I computed a measure of the estimator precision as $precision = \frac{1}{std}$. Then I used $C = \frac{time}{precision}$ as an indicator of computational cost: how many seconds do we need to gain one unit of precision (Figure 2).

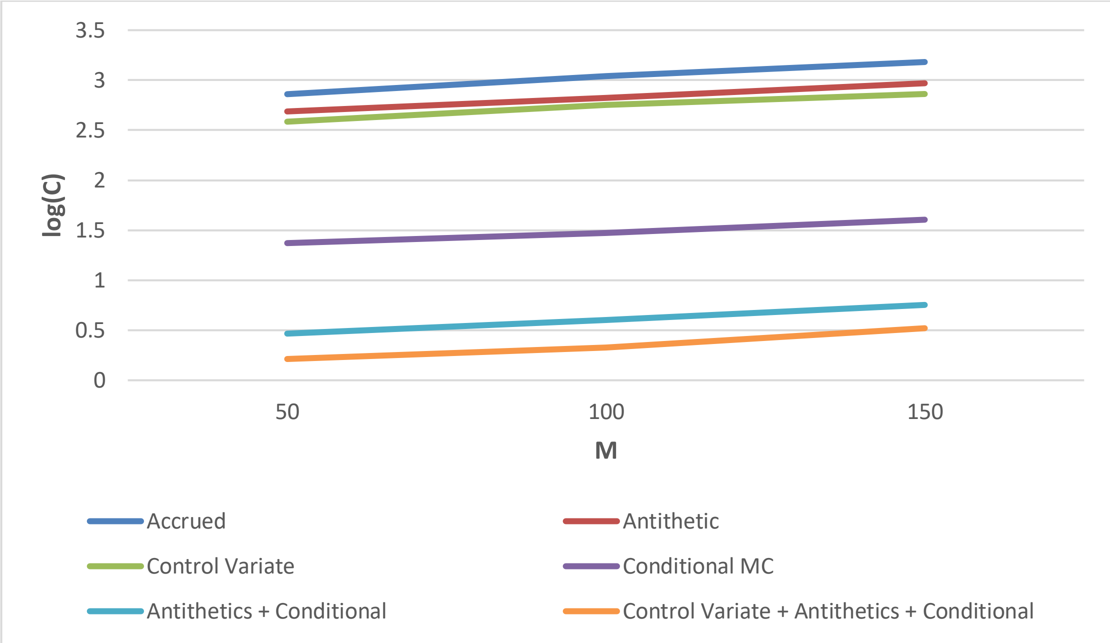


Figure 2: Computational cost C of each of the methods applied to the SABR model as M increases when $L = 200$, expressed in log scale.

Comparing different methods, the Antithetic and Control variate methods had a very similar performance, producing a two-fold improvement of the computational cost of the Accrued simulations. The Conditional MC method is even better (as one would expect from previous work [13, 14]), consistently being around 35 times better than the Accrued method. Finally, the combination of these techniques results in an impressive improvement with respect to any of the other methods, being the “Antithetics + Conditional” 260 times better than the Accrued methods and the “Control Variate + Antithetics + Conditional” 460 times better. For all the methods, every marginal decrease in the standard deviation will have a greater computational cost, as it can be deduced from the continuously increasing lines obtained in Figure 2.

5.2 Heston

The simulated Heston model has parameters:

$S_0 = 250, v_0 = 0.3, r = 0.01, k = 1, \theta = 0.09, \alpha = 0.3, p = -0.5, K = 270, T = 1$

As before, each path will have $n = 1000$ steps. Finally note that even when the volatility process is strictly positive, when you simulate it, you can get negative values. The formula for v_t was $dv_t = k(\theta - v_t)dt + \alpha\sqrt{v_t}dZ_t$, so when $v_t = 0$, dv_t can not be negative. However, computationally, due to the discretization of a continuous process we can obtain a dv that makes v_t become negative, therefore I will use the Milstein scheme [18] to avoid this problem. That scheme consists of considering the second order term in the Taylor expansion for computing dv_t :

$$dv_t = k(\theta - v_t)dt + \alpha\sqrt{v_t}dZ_t + \frac{\alpha^2}{4}(dZ_t^2 - dt)$$

Control Variate Monte Carlo

Recall that from [12] an unbiased estimator of θ is $\theta_c = Y + \mathrm{c}(\mathrm{Z} - \mathrm{E}[\mathrm{Z}])$ for any number c. As I did with the SABR model, letting $Z = S_T$ works well as control variate, where S_T stands for the last value of the simulated stock price. $Cov(Y, Z) = Cov(h(S_T), S_T) \neq 0$ and $E[Z] = E[S_T] = S_0 exp(rT)$.

Antithetic Variates Method

As before, [12] defines an unbiased estimator of θ as $\hat{\theta} = \frac{Y1+Y_2}{2}$, where Y_1 and Y_2 are negatively correlated realizations of $h(S_T)$. S_t is computed using $dv_t = k(\theta - v_t)dt + \alpha\sqrt{v_t}dZ_t + \frac{\alpha^2}{4}(dZ_t^2 - dt)$ and $dS_t = rS_t dt + \sqrt{v_t}S_t dW_t = rS_t dt + \sqrt{v_t}S_t\left(pdZ_t + \sqrt{1-p^2}dB_t\right)$. On the other hand, I defined $S_{t_{neg}}$ using $dv_{t_{neg}} = k(\theta - v_t)dt + \alpha\sqrt{v_{t_{neg}}}dZ_{t_{neg}} + \frac{\alpha^2}{4}\left(dZ_{t_{neg}}^2 - dt\right)$ and $dS_{t_{neg}} = rS_{t_{neg}}dt + \sqrt{v_{t_{neg}}}S_{t_{neg}}dW_{t_{neg}}$

Where:

$$dZ_{t_{neg}} = -dZ_t$$

$$dB_{t_{neg}} = -dB_t$$

$$dW_{t_{neg}} = pdZ_{t_{neg}} + \sqrt{1-p^2}dB_{t_{neg}} = -dW_t$$

Finally, I computed $Y_1 = h(S_T)$ and $Y_2 = h\left(S_{T_{neg}}\right)$

Conditional Monte Carlo

As I did in the SABR simulation, I used the extended Hull and White formula [16, 17] to compute $E[Y|Z_t]$. Adapting the formula to the Heston model it results in:

$$\mathrm{E}[\mathrm{h}(\mathrm{S_T})|\mathrm{Z_t}] = \mathrm{E}\left[C\left(\sqrt{1-p^2}\mathrm{v}_0, \xi\mathrm{t}\mathrm{S}_0, r=0, T, K\right)|Z_t\right]$$

Where

$$\mathrm{v_t} = \sqrt{\frac{1}{T-t}\int_t^T \mathrm{v_s}\, ds}$$

$$\xi\mathrm{t} = \exp\left(\rho\int_t^T \mathrm{sqrt}(v_s)\,\mathrm{dZ_s} - \frac{1}{2}\rho^2\mathrm{v_t^2}(\mathrm{T}-\mathrm{t})\right)$$

And

$$\theta = E[Y] = E\left[E[h(S_T)|Z_t]\right]$$

Again, we only need to simulate the Brownian motion Z_t.

Conditional Monte Carlo + Antithetic

To combine these methods, I generated two antithetic Brownian motions (Z_t and $Z_{t_{neg}}$ using $dZ_{t_{neg}} = -dZ_t$) and then computed $\theta' = E[Y']$ where $Y' = \frac{E\left[h(F_T)|Z_t\right]+E\left[h(F_T)|Z_{t_{neg}}\right]}{2}$.

Conditional Monte Carlo + Antithetic + Control variate

For this last method, I used the control variate estimator ($\theta_c = Y + c * (Z - E[Z])$) but using $Y = Y'$ and $Z = \frac{v_T^2 + v_{T_{neg}}^2}{2}$. Note that since v_t follows a CIR process [12]:

$$E[v_T] = v_0 * e^{-kT} + \theta(1 - e^{-kT})$$

$$Var[v_T] = v_0 * \frac{\alpha^2}{k}(e^{-kT} - e^{-2kT}) + \frac{\theta\alpha^2}{2k}\left(1 - e^{-kT}\right)^2$$

Therefore $E[Z] = E[v_T^2] = Var[v_T] + E[v_T]^2$

Results

In Table 2 I present the results obtained after applying the variance reduction techniques to the Heston model.

		Accrued			**Control variate**			**Antithetic**		
L	**M**	Time	Price	Std	Time	Price	Std	Time	Price	Std
50	50	23	37.0	9.8	23	38.0	5.2	33	39.1	6.3
50	100	46	39.0	7.4	48	38.8	3.0	67	37.8	4.5
50	200	93	37.8	5.7	94	37.7	2.8	136	38.5	3.0
100	50	44	36.1	10.7	45	37.9	5.3	64	38.9	6.4
100	100	94	37.9	7.2	97	36.8	2.9	134	37.9	4.5
100	200	187	37.7	5.5	186	37.6	2.6	266	37.7	3.2
200	50	94	38.0	9.5	96	37.9	5.1	134	37.1	6.2
200	100	189	37.6	7.2	190	38.2	3.0	272	38.5	3.2
200	200	376	38.1	5.3	380	37.6	2.5	546	37.9	2.9

		Conditional MC			Antithetic + Conditional			Control variate + Antithetic + Conditional		
L	**M**	Time	Price	Std	Time	Price	Std	Time	Price	Std
50	50	12	37.4	3.0	17	37.35	0.90	17	37.55	0.45
50	100	24	36.9	1.8	36	37.70	0.52	35	37.52	0.31
50	200	48	37.7	1.4	71	37.51	0.39	69	37.52	0.23
100	50	24	37.9	2.9	38	37.48	0.88	37	37.50	0.50
100	100	50	37.0	1.9	70	37.50	0.51	71	37.52	0.30
100	200	98	37.6	1.6	146	37.52	0.44	140	37.52	0.23
200	50	47	37.4	3.0	70	37.78	0.86	71	37.56	0.51
200	100	95	37.5	1.7	148	37.52	0.52	144	37.51	0.31
200	200	190	37.3	1.4	289	37.51	0.41	290	37.52	0.23

Table 2: Numerical results obtained after applying different variance reduction techniques for pricing the Heston model. Time refers to the computational time in seconds, price to the computed option price estimator and Std is the standard deviation of the L simulated option prices, where each option price is computed as the mean of M option prices. The number of decimals is obtained as in Table 1.

Exactly as in the SABR model, the standard deviation of the expected price is significatively reduced as M gets larger, but there is no improvement when L is increased. I used the same measure of computational cost C than in Section 5.1 and plotted it in Figure 3.

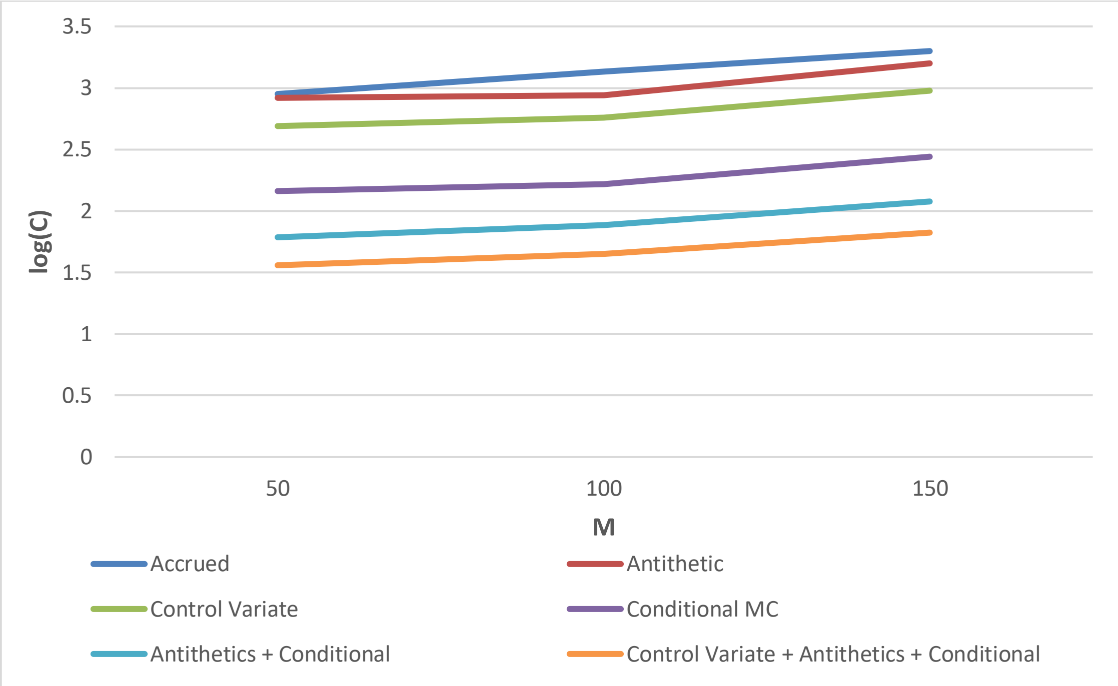


Figure 3: Computational cost C of each of the methods applied to the Heston model as M increases when $L = 200$, expressed in log scale.

The Antithetic and Control variate methods had a very similar performance, producing a two-fold and 1.5-fold with respect to the Accrued simulations, respectively. The Conditional MC method keeps producing the same substantial improvement, but this time a bit smaller, around 8 times better than the Accrued method. And, finally, the “Antithetics + Conditional” yield to a 17 times improvement with respect to the Accrued method, and the “Control Variate + Antithetics + Conditional” result in a 30 times increase. One of the main reasons for having such a low improvement compared to the SABR model is using a p different than 0 in the Heston model, which increase the complexity of the conditional method. Therefore, through this work it would not possible to assess if the variance reduction methods work better for one model or another, since it depends on the election of parameters. Also, similarly to the SABR observations, every decrease in the standard deviation leads to a higher computational cost. Overall, the conclusions for the Heston model coincide with the ones of the SABR model. Variance reduction techniques produce a huge increase in computational performance, especially the Conditional Monte Carlo method. Moreover, combining methods it is possible to reach a performance that none of the methods can achieve on its own, resulting in a clear outperformance of the “Control Variate + Antithetics + Conditional” method (a combination of each of the methods).

6. Implied volatility surface

One of the main differences between the SABR and Heston model are they implied volatility surfaces. In this section, given a set of model parameters, I simulate the implied volatility surface of the SABR and Heston models. As I showed in section 2, the implied volatility surface assumed by the Black Scholes model is completely flat, which is completely different from what empirical evidence shows. Stochastic volatility models are the main solution to approach this false assumption. However, when using these models there are no closed-form solutions and the use of computational methods is needed to obtain good approximations of the implied volatility surface. The python code for this section corresponds to Notebook 3 of Appendix B.

Simulating BS IV

Before proceeding with the SABR and Heston models, I will define the approach I follow using as an example the GBM. First of all, I generated a set of option prices: I created a mesh with a wide combination of maturities T (from 0.1 to 4 in steps of 0.3) and strikes K (from 220 to 300 in steps of 10). Then, for every combination of T and K, I computed each option price C_{obs} simulating 15,000 accrued Monte Carlo asset prices following a GBM (Section 2) and taking the mean. After generating the observed option prices, I computed the observed implied volatilities solving for the inverse of the BS formula.

$$\sigma_{implied} = C^{-1}(C_{obs}, S_0, r, T, K)$$

Solving this equation mathematically is hard, so I used the python optimize.brentq function from the scipy library in order to find $0's$ in the function defined as $C_{obs} - C(\sigma_{numerically}, S_0, r, T, K)$, which is 0 only when $\sigma_{numerically} = \sigma_{implied}$. Finally, I plotted every simulated implied volatility ($\sigma_{numerically}$) against T and K, obtaining the simulated implied volatility surface. As a result, the simulated volatility almost identical to the theoretical one (Figure 4).

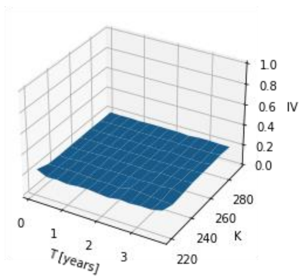


Figure 4: simulated implied volatility surface of a GBM with parameters $S_0 = 250$, $\sigma = 0.2$ and $r = 0.1$

6.1 SABR

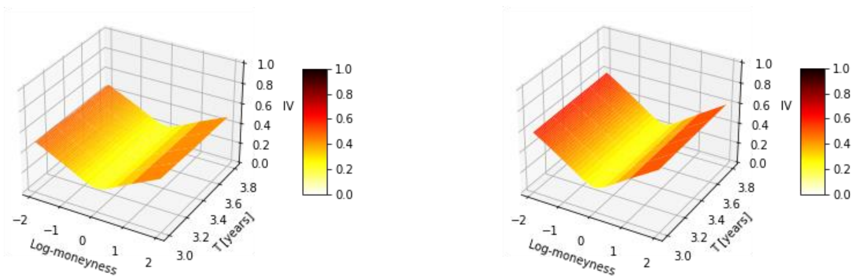
Following the same method as before, I wrote some code that, given the SABR parameters, allow us to simulate the implied volatility surface of a vanilla call (Figure 5). Additionally to the simulated implied volatility surface, I computed the implied volatility surface that we would obtain using the Hagan approximation [19]: given the parameters of the SABR model, we can compute an approximation of the implied volatility. When $B = 1$ we have that

$$IV(T, K) \approx \sigma_0 \left(1 + \left(\frac{\alpha p \sigma_0}{4} + \frac{2 - 3p^2}{24}\alpha^2\right)T\right)\frac{z}{x(z)}$$

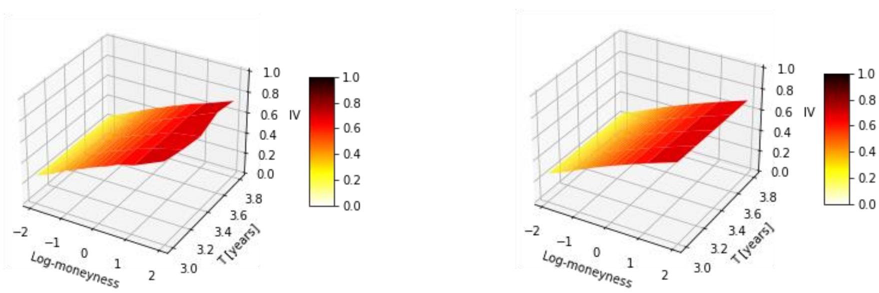
Where

$$z = \frac{\alpha}{\sigma_0}\ln\left(\frac{F_0}{K}\right)$$

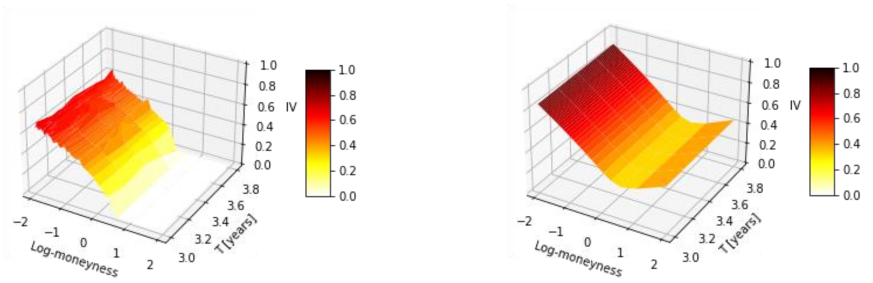
$$x(z) = \ln\left(\frac{\sqrt{1 - 2pz + z^2} + z - p}{1 - p}\right)$$



a) Simulated (left) and Hagan (right) IV surface of SABR model plotted against log-moneyness and T with parameters, $F_0 = 100$, $\sigma_0 = 0.2$, $\alpha = 0.7$, $\beta = 1$ and $p = 0$



b) Simulated (left) and Hagan (right) IV surface of SABR model plotted against log-moneyness and T with parameters, $F_0 = 100$, $\sigma_0 = 0.2$, $\alpha = 0.7$, $\beta = 1$ and $p = 0$



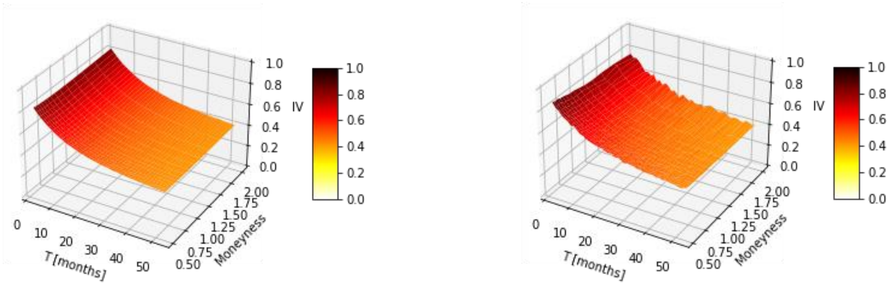
c) Simulated (left) and Hagan (right) IV surface of SABR model plotted against log-moneyness and T with parameters, $F_0 = 100, \sigma_0 = 0.2, \alpha = 0.7, \beta = 1$ and $p = 0$

Figure 5: SABR simulated volatility surfaces (left) vs Hagan IV approximation (right) for different combinations of parameters. I followed the same approach than for the GBM, but the observed option prices C_{obs} are generated using the "Conditional + Control Variate + Antithetic MC" method from section 5.1, with $10{,}000$ simulations per price. T ranges from 3 to 4 in steps of 0.2 and K from 14 to 694 in steps of 20. Following a common practice for this model, I plotted the implied volatility against T and log-moneyness.

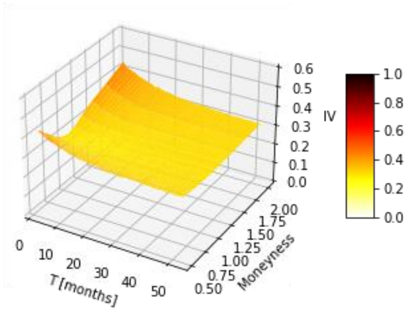
As a result of these simulations, we can see that the simulated surfaces tend to be very close to the Hagan approximation, which verifies the correctness of the surfaces, and now, given any set of parameters one could compute its implied volatility surface. On one hand, this helps to intuitively see how these models could fit the empirical observations (like Figure 1), which would be impossible under the BS model, and on the other hand, practitioners in the industry, like traders for example, can use the implied volatility surface to make better informed decisions when trading. Note that in practice the SABR model is defined for each maturity (as I explain in Section 7) so actually it would allow to replicate the smile for each T, rather than the whole implied volatility surface.

6.2 Heston

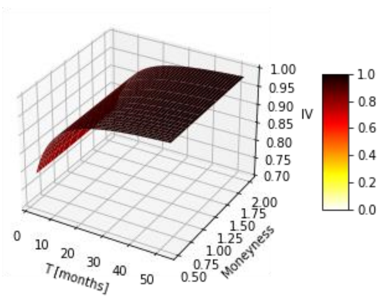
Finally, I simulated some implied volatility surfaces of the Heston model using different parameters. In this case there is no approximation of the IV, so the use of simulation to methods to compute the implied volatility surface is even more important. In Figure 6 you can find some of these simulated surfaces.



a) Simulated IV surface of SABR model plotted against Moneyness and T with parameters $S_0 = 100$, $v_0 = 0.8$, $r = 0$, $k = 2$, $\theta = 0.1$, $\alpha = 0.8$ and $p = 0$ (left) and $p = 0.3$ (right)



b) Simulated IV surface of SABR model plotted against Moneyness and T with parameters $S_0 = 100$, $v_0 = 0.1$, $r = 0$, $k = 2$, $\theta = 0.1$, $\alpha = 1$ and $p = 0$. Note that in this case the z-axis is plotted from 0 to 0.6.



c) Simulated IV surface of SABR model plotted against Moneyness and T with parameters $S_0 = 100$, $v_0 = 0.5$, $r = 0$, $k = 3$, θ =1, $\alpha = 0.3$ and $p = 0$. In this case the z-axis is plotted from 0.7 to 1.

Figure 6: Heston simulated volatility surfaces for different combinations of parameters. Same method than for the GBM, but the observed option prices C_{obs} are generated using the "Conditional + Control Variate + Antithetic MC" method of section 5.2. In this case, T goes from 0.2 to 4.5 in steps of 0.1 and K from 50 to 200 in steps of 10. The number of combinations is larger than in the SABR simulations, so I only simulated 1,000 prices per combination. The common practice for this model is plotting against T and moneyness.

The conclusions for the Heston mode are the same than for the SABR model: plotting implied volatility surfaces gives a visual representation of how the Black Scholes constant volatility assumption can be avoided using stochastic volatility models and provides insights to experts in the industry that use these implied volatility surfaces to make decisions in their day to day.

7. Model Calibration

In previous sections we have seen how given the parameters of a model we can compute option prices (Section 5) and reproduce its implied volatility surface (Section 6), but how can one define the model parameters? Another important topic when developing models for option pricing is model calibration. Given a model and the market observations (set of implied volatilities, maturities, strikes and initial price) the question is how to set the parameters such that the model fits the observed data as accurately as possible. In the last part of this project, I am going to use real market data courtesy of David García-Lorite (CaixaBank) and calibrate the SABR model using as a benchmark actual results obtained by professionals in the industry. One of the strengths of the SABR model against the Heston model is its calibration, that is why I only calibrate the parameters for the SABR. The python code and data used this section are in Notebook 3 of Appendix B.

7.1 SABR

The implied volatility data was originally calibrated in $11/04/2019$ and contains the observed implied volatilities for some options with 18 different maturities, ranging from $17/05/2019$ to $15/12/2028$ and 51 strike prices, from 100 to 5.100. Additionally, for each maturity we also have the calibrated SABR parameters (σ_0, α, p) obtained by the experts, as well as the initial forward price F_0. The parameter B is always assumed to be 1. The common practice for the SABR model calibration is doing it maturity by maturity because the implied volatility smile can vary significantly across options with different maturities. By doing it the model can capture the smile for each maturity, producing much more accurate results than if it was just calibrated overall. To do the calibration, for each maturity I used scipy.minimize to find the parameters (σ_0, α, p) that minimized the squared error between the Hagan implied volatility approximation [19] using this parameters and the actual implied volatility observed in the market. For example, in the case of the first maturity $(17/05/2019)$ the benchmark was $\sigma_0 = 2.40$, $\alpha = 0.11$ and $p = -0.65$. The obtained parameters were $\sigma_0 = 2.45$, $\alpha = 0.10$ and $p - 0.60$, giving a total absolute error of 0.11 between the benchmark and the computed parameters, and a squared error of 0.00017 between the observed implied volatilities and the approximated ones using the Hagan approximation with the calibrated parameters. (Figure 7).

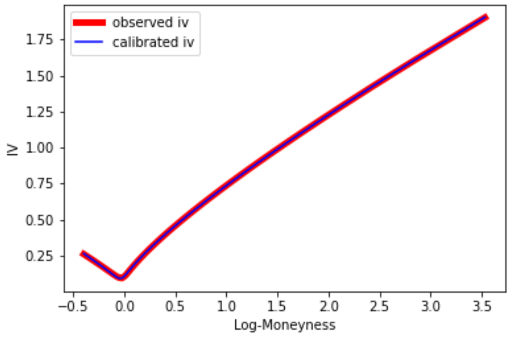


Figure 7: Implied volatility observed in the market (red) vs implied volatility predicted by the calibrated parameters (blue), for a vanilla option with $T = 17/05/2019$

After calibrating for all the maturities, I got a mean absolute error of 0.15 across all the calibrations, and the calibrated implied volatility surface was very close to the observed surface, which means that the results were very close to the benchmark. (Figure 8)

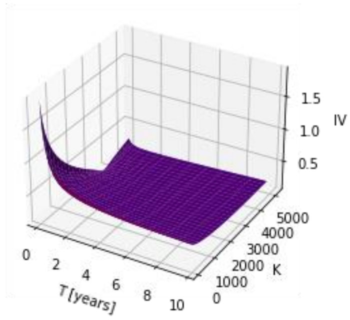


Figure 8: overlapped implied volatility surfaces of the observed data (red) and the Hagan implied volatility predicted by the calibrated parameters (blue)

In summary, the calibrated parameters were very close to the benchmark and produced almost the same implied volatility surface as the observed one. This shows that indeed, the constant volatility assumption originally introduced by the BS models can be avoided using stochastic volatility models like the SABR or Heston. Moreover, now that I showed how to calibrate the SABR model, one could use any observed market data and calibrate it, which would allow to reproduce its implied volatility surface (Section 6) and price vanilla options (Section 5), as well as other type of options.

8. Conclusions

In conclusion, for solving the option pricing problem we look for models that can price options in a consistent and simple way. One of its most successful approaches, the Black-Scholes model, fails to reproduce some of the implied volatility properties observed in the market. To avoid the constant volatility assumption more modern approaches were developed, leading to stochastic volatility models, like the SABR or Heston. However these models are more complex, and many times is hard to find a closed-form solution, making necessary the use of computational methods. Many state-of-the-art solutions come with opportunities to improve the computational performance of these models, and, in this project I applied some of these techniques and studied its results.

First, given the model parameters I applied some variance reduction techniques for pricing vanilla options. Each of the methods improved the accuracy of the price estimator, reducing its standard deviation. Moreover, the "Conditional + Antithetics + ControlVariate" method (a combination of these three methods), turned out to be the most successful method, improving the computational performance by a factor of 460 for the SABR model and 30 for the Heston. This does not mean that it worked better for the SABR, since the selection of parameters in the Heston model substantially increased the complexity of the variance reduction technique.

Second, given the model parameters, and with help of the "Conditional + Antithetics + ControlVariate" technique, I developed some algorithms that successfully simulated the implied volatility surface of both models (applied to vanilla options). This allows, on one hand to see how these stochastic volatility models could fit the implied volatility surfaces observed in the market, and on the other hand, to help practitioners in the industry to make better informed decisions using these surfaces.

Finally, given some market data of vanilla options, courtesy of David García Lorite (CaixaBank), I designed an algorithm that, minimizing the error between the implied volatility observed in the market and the one predicted by the model, computed the model parameters that better fit the data. I only did it for the SABR model, since one of its main advantages is its simple calibration. Using as a benchmark the results obtained by professionals in the industry, I obtained a mean absolute error of 0.15 across all the calibrations (in stochastic volatility models the calibration is done per maturity), that reproduced an almost identical volatility surface than the observed one.

The next steps in this work could be divided into several paths. On one hand, one could try to improve the current thesis, trying to use other computational methods or directly focusing on implementing the algorithms in a more efficient way. On the other hand, one could follow the same steps of this work and see if these techniques can be adapted to local volatility models. Finally, it also would be possible to use this thesis as a foundation for working with exotic options: in practice, vanilla prices are observed in the market and used as inputs in the model calibration. Then, these calibrated models are used to price exotics. [1]

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Appendix A: Introduction to options and the option pricing problem

Options are the main object of study in this project. To provide the reader with enough background I introduce some of the common terms he/she needs to know.

Like defined in reference [20]: "An asset is anything of value or a resource of value that can be converted into cash. Individuals, companies, and governments own assets. For a company, an asset might generate revenue, or a company might benefit in some way from owning or using the asset". There are three types of assets:

- Tangible assets: These types of assets are physically tangible and have a finite monetary value. Good examples are buildings, machinery or ground. [21]
- Intangible assets: By contrast, intangible assets have a non-physical nature like patents, brands, copyright or intellectual property. [22]
- Financial asset: summarizing the definition of reference [23], financial assets are something in the middle of the other two assets. They may seem intangible since they represent value on for example a dollar bill or a number on a computer screen, but they also represent the claim of ownership of an entity, which could be a public company (stocks) or rights to payments (derivatives or bonds), among others. Different types of financial assets can be distinguished. For our context, derivatives are the most relevant.

Derivatives:

Derivatives are contracts in which, instead of directly exchanging the assets themselves, they allow its holder to trade a previously defined asset (the underlying asset) under certain conditions. The two main types of derivatives are options and forwards:

-Forwards: In a forward contract one participant agrees on buying at a certain time $t = T$ (its maturity time, that is when the contract expires) an asset from the other participant at a forward price F.

For example, imagine that S is the stock price of a certain company and that today ($t = 0$) it is worth 200 € ($S_t = S_0 = 200$ €). In a forward contract with maturity $T = 1\ year$ and forward price $F = 230$ €, in exactly 1 year you would have the obligation of buying the underlying stock S at 230€ regardless of its price at that moment (S_T). If in that year the stock price increased to $S_T = 250$ € you would be lucky and pay 230 € for something that cost 250 € (your profit would be $S_T - F, = 250$ € $- 230$ € $= 20$ €), but if the stock went down to 150 €, by contract you would have to buy this same 150 € stock for 230 € and lose some money (your profit would be $S_T - F = 150\,€ - 230\,€ = -80\,€$). Therefore, the profit function of a forward contract is $S_T - F$.

In these contracts the forward prices F are set in a way such that it is considered fair for both parts, so for any forward contract with maturity T there is a fair F that allows anyone to enter them without having to pay anything in advance. Given the stock price, the forward price can be computed as $F = S_t e^{rT}$, where r is the risk-free rate, the type of interest rate you would get in a zero risk investment.

-Options: give the holder the right (but not the obligation) of buying (in case of calls) or sell (in case of puts) the underlying asset at a certain strike price K. European options, also known as vanilla options, are the most basic type of options: they allow its holder to exercise the option only at maturity time $t = T$. Vanilla options are very similar to forward contracts. The only difference is that forwards obligate both parts to do the transaction while holders of vanilla options can choose doing the transaction or not.

Consider the same stock S from the previous example, a vanilla call that with maturity $T = 1\,year$ and $K = 230\,€$ would allow you to buy this stock in exactly one year for 230 €, regardless of its price at that moment (S_T). Therefore, if $S_T > K$ the owner could obtain an instant payoff of $S_T - K$ €, buying the stock for K € and selling it for S_T €. On the other hand, if $K > S_T$ he/she would have the right to buy the stock for a higher price than its current price, which would be useless, meaning that you would not exercise the option and therefore you would get a payoff of 0 € (in a forward contract you would have to buy it anyways and your payoff would be negative, incurring in losses).

The payoff function for a vanilla option is $h(S_T) = max(S_T - K, 0)$.

Notice that in the case of options I am talking about payoff instead of profit because you are allowed to buy/sell at convenience, so while the price of entering a forward contract is zero, holding option contracts have an associated value $V(S_t, t)$. If they were free you could enter them without any risk of losing money and become rich by only taking profit of the contracts that ensure you some profit. This $V(S_t, t)$ represents its value and is the price you would pay in the market for entering in one of these contracts. Since the value of its underlying asset S_t fluctuates through time, $V(S_t, t)$ also does. Therefore, the profit obtained with a vanilla option can be defined as $h(S_T) - V(S_t, t)$,

Options provide a unique way of hedging risk and the job of many professionals in the field is to determine and exact value for $V(S_t, T)$, also known as the option pricing problem. In practice, the main goal of these professionals is to price options consistently, that is avoiding arbitrage. To achieve it, practitioners' main focus is on finding models that can efficiently replicate the option prices observed in the market $V(S_t, t)$. They begin with the data and end up with the models fitting that data. "Vanilla prices are not computed but observed in the market, and they are the inputs in the calibration of our models. Then, these calibrated models are used to price exotics (another type of options)" [1].

Appendix B: Python Code & Data

GitHub Repository: https://github.com/dani-gonzalez-muela/FinalDegreeThesis

1. SABR variance reduction
2. Heston variance reduction
3. Implied volatility surfaces
4. SABR calibration & Data