Numerical Project Summary

"The Asian Option: Pricing Methods and Implementation"

Executive Summary

This project develops and compares numerical methods for pricing Asian options—path-dependent derivatives whose payoffs depend on average asset prices. We implement finite difference schemes (Crank-Nicolson), Monte Carlo simulation with control variates, and derive closed-form solutions for geometric average options, demonstrating their cost advantages over standard European options.

1. Introduction and Motivation

Asian options, introduced in 1987 for crude oil pricing, pay based on average asset prices over time rather than terminal values. This averaging feature provides:

- Protection against price manipulation
- Reduced sensitivity to volatility
- More representative payoffs in volatile markets
- Lower premiums compared to European options

2. Theoretical Framework

Option Types:

- **Arithmetic Average**: Payoff = $(\int_0^T S(t)/T\, dt - K)^+$
- **Geometric Average**: Payoff = $(\exp(1/T \int_0^T \log S(t)dt) - K)^+$

Black-Scholes Adaptation:

- Stock dynamics: $dS = rSdt + \sigma SdW$ under risk-neutral measure
- Closed-form solution exists for geometric average
- Arithmetic average requires numerical methods

Derived Pricing Formulas (Geometric):

- Call: $AC = S_0e^{((a-r)\tau)}\Phi(d_1) - Ke^{(-r\tau)}\Phi(d_2)$
- Put: $AP = Ke^{(-r\tau)}\Phi(-d_2) - S_0e^{((a-r)\tau)}\Phi(-d_1)$
- Where $a = r/2 - \sigma^2/12$

3. Numerical Methods Implementation

Finite Difference Schemes:

PDE Framework:

- Governing equation: $-\partial_t u + (\sigma^2/2)(\gamma(t) - z(t))^2 \partial^2_x u = 0$
- Boundary conditions: $u(0,z) = z^+$, $\lim u(t,z) = 0$

Three Schemes Implemented:

1. **Forward Euler** (explicit): Simple but conditionally stable
2. **Backward Euler** (implicit): Unconditionally stable, requires solving system
3. **Crank-Nicolson**: Second-order accurate, unconditionally stable

Discretization Parameters:

- Grid: 1000×1000 (time × space)
- $\Delta t = T/n$, $\Delta z = 2Z/m$

Monte Carlo Methods:

Standard Approach:

- Simulate paths: $S_{i+1} = S_i + rS_i\Delta t + \sigma S_i\sqrt{\Delta t}\cdot Z$
- Track average: $Q_{i+1} = Q_i + S_i\Delta t$
- Compute payoff: $\max((Q(T)/T - K), 0)$
- Average over 5000+ simulations

Control Variate Enhancement:

- Use S(T) as control (known expectation: $S_0e^{(rT)}$)
- Estimator: $\theta = Z + c(S(T) - E[S(T)])$
- Coefficient: $c = -Cov(Y,Z)/Var(Z)$
- Significantly reduces variance

4. Results and Analysis

Price Behavior:

- Asian calls/puts consistently cheaper than European equivalents
- Lower sensitivity to volatility (flatter price curves)
- Call prices increase with S_0; puts decrease

Volatility Analysis ($\sigma \in [0.1, 0.5]$):

- Asian call premium discount: 30-40% vs European
- Reduced vega (volatility sensitivity)
- More stable pricing in volatile markets

Method Performance:

Method	Execution Time	Accuracy	Characteristics
Finite Difference	~0.1s	High	Deterministic, stable
Monte Carlo	2-5s	Medium	Stochastic, flexible
Control Variate MC	2-5s	High	Reduced variance by 40%

Validation Tests:

- Put-call parity verified: Error < 0.02
- Geometric < Arithmetic average prices confirmed
- Convergence achieved for all methods

5. Key Insights

Practical Advantages:

- Ideal for commodity and FX markets
- Reduced hedging costs
- Natural protection against manipulation
- Suitable for employee compensation plans

Computational Trade-offs:

- Finite differences: Fast, accurate for simple payoffs
- Monte Carlo: Flexible for complex path dependencies
- Control variates: Essential for MC efficiency

6. Code Implementation

Python implementation includes:

- Modular functions for each pricing method
- Visualization tools for sensitivity analysis
- Performance comparison framework
- Automated validation tests

7. Conclusions

The project successfully demonstrates multiple approaches to Asian option pricing, with finite difference methods providing optimal speed-accuracy trade-offs for standard contracts, while Monte Carlo methods offer superior flexibility for exotic variations. Control variates prove essential for practical Monte Carlo implementation.

8. Future Extensions

- American-style Asian options with optimal exercise
- Discrete sampling schedules
- Multi-asset Asian baskets
- Jump-diffusion underlying models
- GPU acceleration for Monte Carlo

Interview Questions with Detailed Answers

Q1: Why are Asian options particularly useful in commodity markets?

Answer: Asian options are especially valuable in commodity markets for three key reasons. First, commodity prices can be highly volatile and subject to short-term manipulation, particularly near contract expiration. By averaging prices over time, Asian options reduce the impact of these temporary price spikes or dips. Second, many commodity transactions involve continuous production or consumption over a period, making the average price more representative of actual business exposure than a single terminal price. Third, the reduced premium (typically 30-40% cheaper than European options) makes hedging more cost-effective for commodity producers and consumers operating on thin margins.

Q2: How does the Crank-Nicolson method improve upon basic finite difference schemes?

Answer: The Crank-Nicolson method combines the forward and backward Euler schemes by averaging them, which provides two major advantages. First, it achieves second-order accuracy in time ($O(\Delta t^2)$) compared to first-order accuracy ($O(\Delta t)$) for basic schemes, meaning errors decrease much faster as we refine the time grid. Second, it maintains unconditional stability regardless of the ratio between time and space steps, unlike the forward Euler method which requires $\Delta t \leq \Delta x^2/(2\sigma^2)$ for stability. This allows us to use larger time steps without sacrificing accuracy or causing numerical explosions, significantly improving computational efficiency.

Q3: What challenges arise when pricing arithmetic vs. geometric average options?

Answer: The fundamental challenge is that arithmetic average Asian options don't have closed-form solutions in the Black-Scholes framework, while geometric average options do. For geometric averages, the log-normal property is preserved because the product of log-normal variables remains log-normal, allowing us to derive exact formulas similar to Black-Scholes. However, the sum of log-normal variables (arithmetic average) doesn't follow a known distribution, forcing us to use numerical methods like PDEs or Monte Carlo. Additionally, the arithmetic average is always greater than or equal to the geometric average (AM-GM inequality), making arithmetic options more expensive and requiring more precise numerical methods to capture this premium accurately.

Q4: How does the control variate method reduce Monte Carlo variance?

Answer: The control variate method exploits the correlation between our target variable (the Asian option payoff) and a related variable with known expectation (the terminal stock price S(T)). We adjust our estimate by subtracting c times the error in the control variate: $\theta = Z + c(S(T) - E[S(T)])$. Since S(T) and the average are highly correlated (correlation typically 0.7-0.9), when our simulation overestimates S(T), it likely overestimates the average too. The optimal coefficient $c = -Cov(Y,Z)/Var(Z)$ ensures maximum variance reduction. In our implementation, this reduced variance by approximately 40%, allowing us to achieve the same accuracy with fewer simulations or better accuracy with the same computational budget.

Q5: What boundary conditions are critical for the PDE implementation?

Answer: Three boundary conditions are essential for our PDE implementation. First, the terminal condition $u(0,z) = z^+ = \max(z,0)$ represents the payoff at maturity (noting our time transformation $t = T - t'$). Second, the lower boundary condition $u(t,-Z) = 0$ reflects that deep out-of-the-money options have zero value. Third, the upper boundary condition $u(t,Z) = z$ represents deep in-the-money options behaving like the underlying asset minus strike. These conditions ensure unique, stable solutions and proper option behavior at extremes. The choice of Z (we used 500) must be large enough that further increases don't affect the solution at relevant price levels.

Q6: How would you extend this to handle discrete sampling Asian options?

Answer: Discrete sampling Asian options (where the average is computed at specific dates rather than continuously) require modified approaches. For Monte Carlo, the adaptation is straightforward—simply track prices only at sampling dates. For PDE methods, we'd need to solve a higher-dimensional PDE or use dimension reduction techniques. Tree methods become particularly attractive here: we can build a binomial or trinomial tree and track the running average at each node, though this requires careful management of the state space to avoid exponential growth. Another approach is the moment-matching method, where we approximate the discrete average with a continuous one having adjusted parameters to match the first two moments.

Q7: What are the main trade-offs between finite difference and Monte Carlo methods?

Answer: Finite difference methods excel in speed and accuracy for standard contracts, providing deterministic results in ~0.1 seconds with high precision. They're ideal when we need to price many options with the same underlying but different strikes (the entire surface). However, they struggle with path-dependent features beyond simple averaging and become computationally intensive in higher dimensions. Monte Carlo methods are slower (2-5 seconds) and produce stochastic results requiring many simulations for accuracy. But they naturally handle complex path dependencies, early exercise features, and multi-asset options. They're also easily parallelizable. The choice depends on the specific application: use finite differences for vanilla Asian options in production systems requiring speed, and Monte Carlo for exotic variants or when flexibility is paramount.

Q8: How does volatility smile affect Asian option pricing?

Answer: The volatility smile significantly impacts Asian option pricing, though less severely than for European options. Under constant volatility (our Black-Scholes assumption), we underestimate out-of-the-money option values. In reality, the averaging feature of Asian options means they're exposed to volatility across multiple price levels over time, not just at one strike and maturity. To properly account for the smile, we'd need to use local volatility or stochastic volatility models. The local volatility approach would modify our PDE to include $\sigma(S,t)$, while stochastic volatility would add another dimension to the problem. Interestingly, Asian options are natural "smile traders" since they effectively average across different implied volatilities during the averaging period.

Q9: What practical applications have you seen for Asian options?

Answer: Asian options have numerous real-world applications. In employee compensation, companies use Asian options to reduce the incentive for stock price manipulation around vesting dates and to provide more stable compensation tied to sustained performance. Energy companies use them extensively for hedging since power generation and consumption occur continuously—an electricity producer selling power daily throughout a month is naturally exposed to the average price, not the month-end price. In foreign exchange, corporations with regular international transactions use Asian options to hedge their effective exchange rate over a period. Airlines hedge jet fuel costs with Asian options since they purchase fuel continuously. Investment funds also use them for portfolio insurance where they want protection based on average portfolio value rather than point-in-time values.

Q10: How would you validate your numerical implementation?

Answer: Validation requires multiple approaches to ensure robustness. First, I verify put-call parity holds within numerical tolerance (we achieved error < 0.02). Second, I compare different numerical methods against each other—finite differences and Monte Carlo should converge to the same price. Third, I test against the closed-form solution for geometric Asian options where available. Fourth, I verify limiting cases: as time to maturity approaches zero, the Asian option should converge to a European option; with zero volatility, I can compute the exact deterministic average. Fifth, I check monotonicity properties: prices should increase with spot price for calls, decrease for puts, and both should increase with volatility. Finally, I perform convergence analysis, confirming that refining the grid or increasing simulations improves accuracy at the expected theoretical rate (first or second order depending on the method).