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Theory and Methodology

Capacitated inventory problems with fixed order costs: Some optimal policy structure

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Abstract

Almost 40 years ago, H. Scarf (The optimality of (s,S) policies in dynamic inventory problems, in: K. Arrow, S. Karlin, P. Suppes (Eds.), Mathematical Models in the Social Sciences, Stanford University Press, Stanford, 1960) established the optimal (s,S) policy structure for the periodic review inventory problem with fixed ordering costs and no capacity constraint. Since then, the capacitated problem has resisted characterization. In the present paper we partially bridge this gap; using a generalization of Scarf's K-convexity we show that the optimal capacitated policy has an (s,S)-like structure. To do so we divide the parameter space into four regions: In two of these regions the optimal policy is completely specified, while in the other two, it is partially specified. We complement these findings with a computational study. This study suggests that a still simpler optimal policy structure exists. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Managing inventory plays a central role in effective business practice. Firms must balance the costs of ordering and holding inventory with their desire to meet the often unpredictable needs of their customers. To study how inventory decisions should be made, researchers have developed and analyzed an assortment of analytical models. The popular single-period news-vendor was one of the first of these models, with references going back to the 1951 paper by Arrow et al. [1]. The solution to the news-vendor explicitly balances the expected costs of holding inventory with the costs of leaving customers unsatisfied. When extended to multiple periods this model gives rise to so-called base-stock policies, contained in the classic

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paper of Scarf [3] and many references thereafter. These policies specify a single critical parameter which determines the optimal amount of inventory to carry in any period.

While helpful, these models do have shortcomings; the absence of fixed ordering costs and finite ordering/production capacity, for example. Fixed ordering costs arise in manufacturing as the sum of out-of-pocket costs involved in setups plus the opportunity cost of machine down time. In retailing, fixed ordering costs include the cost of processing orders and breaking bulk. When fixed costs are present, a two parameter (s,S) policy is optimal [3]. Under an (s,S) policy, an order to bring the inventory position (inventory on hand plus on order minus backorders) up to S is placed whenever the inventory position drops to or below s. Finite ordering/production capacity is often imposed by restrictions on the availability of raw materials or limits on production/distribution rates. When there is no fixed ordering cost but finite ordering/production capacity, a modified base-stock policy continues to be optimal. In such a situation it is optimal to order up to the critical parameter if there is sufficient capacity in the system, if not, one should order as much as possible. This was first shown by Federgruen and Zipkin [2].

When finite ordering capacity is coupled with a fixed ordering cost the optimal policy structure has proved difficult to isolate. While it might be reasonable to expect a simple (s,S)-like structure, first Wijngaard [6], and more recently Shaoxiang and Lambrecht [5], have published examples where this fails to hold. Shaoxiang and Lambrecht do partially characterize the optimal solution, providing what they call "X–Y bands." These imply, as one would expect, that there is a point below which it is optimal to order full capacity, and one above which it is optimal not to order. This leads us to the issue we address in this paper: Is there a more explicit structure that this problem's optimal solution satisfies? Through the definition of two properties which are similar to the now familiar K-convexity of Scarf, we will show that the answer to this question is yes. Specifically, the solution space can be divided into four regions, as shown in Fig. 1.

In two of the regions, the optimal policy is known – in one you order capacity, C, in the other zero, as in the X–Y bands. In one of the remaining two regions it is optimal for the decision maker to either order nothing, or to bring the inventory at least up to a specified level, s'. In the final region the parameters of the solution dictate one of the two cases hold. In the first case it is optimal to order, again at least up to a specified level. In the second, the optimal policy is to either order the full capacity or nothing.

Before we prove our result we establish the notation for our paper, in Section 2. We then introduce CK-convexity and strong CK-convexity, in Section 3, before presenting our structural result in Section 4. A brief computational study is included in Section 5. We conclude with some further directions for research, as well as speculation as to what the structure of the optimal policy might ultimately prove to be, in Section 6.

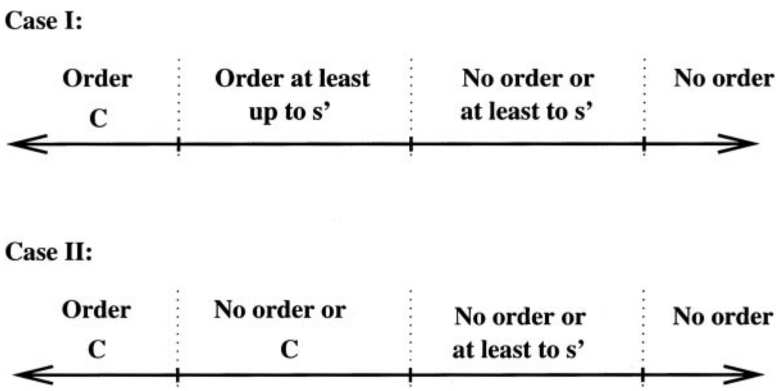

Fig. 1. Optimal policy form – two cases.

2. Notation

We consider the discrete time, finite horizon version of the capacitated inventory problem with stationary stochastic demand and fixed ordering costs. We assume without loss of generality that the leadtime is identically zero. Our result can be generalized to static leadtimes through the use of inventory position in a manner similar to the uncapacitated case, as in [3], for example. The sequence of events is as follows: A period begins, an order is placed and arrives instantaneously, demand is realized and satisfied with on-hand inventory, unsatisfied demand is fully backordered, and holding/penalty costs are assessed.

The notation in this paper follows that of Zheng [7]:

N	length of the horizon
n	period index
x_n	inventory level at the start of period n, before an order is placed
y_n	inventory level after period n order arrives, but before demand is realized
D_n	non-negative demand in period n (we assume $\{D_n:\ 0 \leqslant n \leqslant N\}$ form an iid sequence, with a generic element denoted by D; we further assume $\mathrm{E}[D] < \infty$)
K	non-negative fixed cost for placing an order
C	non-negative fixed capacity for an order
c	non-negative unit cost of ordered goods
p	non-negative unit cost per period of back-ordered goods (we assume $p > c$)
h	non-negative unit cost per period of holding inventory
α	discount factor $(0 \leqslant \alpha \leqslant 1)$
$f_n(x)$	the minimum discounted cost for the n period problem when initial inventory is x (we seek the form of the policy which will return $f_N(x)$ for all $x \in \mathbb{R}$.

If we let $L(y)$ be the expected one-period holding/backorder cost, then

$$L(y) \stackrel{\text{def}}{=} \mathrm{E}[h(y-D)^+ + p(y-D)^-],$$

where for any x, x^+ and x^- denote positive and negative magnitude of x, respectively. For notational simplicity, we define

$$J(y) \stackrel{\text{def}}{=} cy + L(y).$$

As the sum of two convex functions, $J(y)$ itself is convex. Additionally, if we assume that at least one of c or h is strictly positive, then $p > c$ implies $J(y) \to \infty$ as $|y| \to \infty$ (without these assumptions the problem is trivial).

Our problem can then be formulated as follows, where $I\{A\}$ denotes the indicator function of the event A:

$$f_n(x) = -cx + H_n(x), \quad x \in \mathbb{R},\ 0 \leqslant n \leqslant N, \tag{1}$$

$$H_n(x) = \inf_{y \in [x, x+C]} \{KI\{y > x\} + G_n(y)\}, \tag{2}$$

$$G_n(y) = J(y) + \alpha \mathrm{E}[f_{n-1}(y - D_n)]. \tag{3}$$

Also, we assume $f_0(x) \equiv 0$ for all x. This assumption can be relaxed: It is sufficient for $J(y)$ to be a convex function, and $f_0(x)$ to satisfy *strong CK-convexity*, where this property is defined in the next section.

3. *CK*-convexity

To prove our result, we define two analogs of the now familiar *K-convexity* of Scarf [3].

Definition 3.1. Given a non-negative constant K, we call the function $G : \mathbb{R} \to \mathbb{R}$ *K-convex* if for all y, $b > 0,\ z \geqslant 0$,

$$K + G(y+z) \geqslant G(y) + \frac{z}{b}\{G(y) - G(y-b)\}. \tag{4}$$

One way to envision K-convexity is as follows: If one joins the point $(y, G(y))$ with any point on the function behind it, $(y-b, G(y-b))$, and extends this segment another z units forward (so the endpoint lies over the point $y+z$), then this endpoint must lie *below* the point $(y+z, K+G(y+z))$.

We call the first of our analogs *CK-convexity*.

Definition 3.2. Given non-negative constants C and K, we call the function $G : \mathbb{R} \to \mathbb{R}$ *CK-convex* if for all $y,\ b > 0,\ z \in [0, C]$,

$$K + G(y+z) \geqslant G(y) + \frac{z}{b}\{G(y) - G(y-b)\}. \tag{5}$$

CK-convexity can be thought of as a restriction of *K*-convexity, requiring it to hold only at points no more than C units greater than y. Note that the standard K-convexity would correspond to ∞K-convexity in the above definition.

We now extend this definition to what we will call *strong CK-convexity*.

Definition 3.3. Given non-negative constants C and K, we call the function $G : \mathbb{R} \to \mathbb{R}$ *strong CK-convex* if for all $y,\ 0 \leqslant a < \infty,\ 0 < b < \infty,\ \forall z \in [0, C]$,

$$K + G(y+z) \geqslant G(y) + \frac{z}{b}\{G(y-a) - G(y-a-b)\}. \tag{6}$$

One way to think of strong *CK*-convex functions is as follows: From any point $(y, G(y))$ we extend forward a line segment of length z having as its slope the *supremum* of the slopes of all segments formed by joining points on the function G behind y [joining $(y-a, G(y-a))$ and $(y-a-b, G(y-a-b))$]. The endpoint of this segment must fall below the point $(y+z, K+G(y+z))$, for all $0 \leqslant z \leqslant C$. Setting $a = 0$ shows that any function which is strong *CK*-convex is also *CK*-convex.

Strong *CK*-convex functions (and thus *CK*-convex functions) have many of the same useful properties as *K*-convex functions.

Proposition 3.1. *Using the definition in* (6):
1. *If $G : \mathbb{R} \to \mathbb{R}$ is strong CK-convex, it is also strong DL-convex for any $0 \leqslant D \leqslant C$ and $L \geqslant K$.*
2. *If G is convex, it is strong CK-convex for any non-negative C and K.*
3. *If G_1 is strong CK-convex, and G_2 is strong CL-convex, then for $\alpha, \beta \geqslant 0$, $\alpha G_1 + \alpha G_2$ is strong $C(\alpha K + \beta L)$ convex.*
4. *If G is strong CK-convex and X is a random variable such that* $\mathrm{E}[|G(y-X)|] < \infty$, *then* $\mathrm{E}[G(y-X)]$ *is strong CK-convex.*

Proof.
1. This is immediate.
2. Assume $G(x)$ is a convex function. This implies that for all $0 \leqslant a < \infty$ and $b > 0$,

$$G(y) - G(y-b) \geqslant G(y-a) - G(y-a-b).$$

As any convex function is also K-convex [3], for $K \geqslant 0,\ z \geqslant 0,\ b > 0$ we have

$$\begin{aligned} 0 &\leqslant K + G(y+z) - G(y) - \frac{z}{b}\{G(y) - G(y-b)\} \\ &\leqslant K + G(y+z) - G(y) - \frac{z}{b}\{G(y-a) - G(y-a-b)\}. \end{aligned}$$

3. This is immediate.
4. Using 3, and the Dominated Convergence Theorem, this is immediate. □

4. Optimal policy structure

Given non-negative C and K and a strong CK-convex function $G : \mathbb{R} \to \mathbb{R}$, define:

$$G^* \stackrel{\text{def}}{=} \inf_{\{y \in \mathbb{R}\}} G(y).$$

$$S \stackrel{\text{def}}{=} \inf\{y \in \mathbb{R} \mid G(y) = G^*\}.$$

$$\tilde{G}(x) \stackrel{\text{def}}{=} K + \inf_{\{x \leqslant y \leqslant x+C\}} G(y).$$

$$A(x) \stackrel{\text{def}}{=} \tilde{G}(x) - G(x).$$

$$s \stackrel{\text{def}}{=} \inf\{x \mid A(x) \geqslant 0\}.$$

$$s' \stackrel{\text{def}}{=} \max\{x \leqslant S \mid A(x) \leqslant 0\}.$$

We will assume that $|S|$ is finite; a simple condition such as $\lim_{|y| \to \infty} G(y) = \infty$ guarantees this. Clearly $-\infty \leqslant s \leqslant s' \leqslant S$.

By definition $A(x) < 0$ on $x < s$ and $A(x) > 0$ on $s' < x \leqslant S$. We will shortly show that $A(x) \geqslant 0$ on $x > s'$, and that G is non-increasing up to s'. These properties imply that it is optimal to order C units when $A(x) < 0$ and $x \leqslant s' - C$, and order at least up to s' when $A(x) < 0$, and $s' - C < x \leqslant s'$. For this more detailed analysis of A we need to introduce additional notation:

$$I_+ \stackrel{\text{def}}{=} I\{s' - C > s\}.$$

$$I_- \stackrel{\text{def}}{=} I\{s > s' - C\}.$$

$$G_C(x) \stackrel{\text{def}}{=} K + G(x + C).$$

$$\overline{G}(x) \stackrel{\text{def}}{=} K + \inf_{\{s' \leqslant y \leqslant x+C\}} G(y), \quad s' - C \leqslant x \leqslant s'.$$

We first prove a lemma which is similar to one which appears in [4].

Lemma 4.1. *Under the assumption that $|S|$ is finite*:
1. *G is non-increasing on $(-\infty, s']$ and strictly decreasing on $(-\infty, s)$.*
2. *$A(x) \geqslant 0 \ \forall x > s'$.*
3. *Let*

$$H(x) \stackrel{\text{def}}{=} \inf_{\{x \leqslant y \leqslant x+C\}} \{KI\{y > x\} + G(y)\} = \min\{G(x), \tilde{G}(x)\}.$$

Then

$$H(x) = \begin{cases} G_C(x), & x < \min\{s' - C, s\}, \\ \min\{G(x), G_C(x)\}I_+ + \{\overline{G}(x)\}I_-, & \min\{s' - C, s\} \leqslant x < \max\{s' - C, s\}, \\ \min\{G(x), \overline{G}(x)\}, & \max\{s' - C, s\} \leqslant x \leqslant s', \\ G(x), & s' < x. \end{cases}$$

4. *$H(x)$ is strong CK-convex.*

Proof. 1. (a) Assume $x < x' \leqslant s'$. Since $A(s') \leqslant 0$ we have $G(s') \geqslant \tilde{G}(s') = K + G(s' + z)$ for some $z \in (0, C]$. Set $a = s' - x'$, and $b = x' - x$. By strong CK-convexity

$$G(s') \geqslant K + G(s' + z) \geqslant G(s') + \frac{z}{x' - x}\{G(x') - G(x)\},$$

which immediately implies that

$$G(x) \geqslant G(x') \quad \forall x < x' \leqslant s'.$$

(b) Assume $x < x' < s$. Since $A(x') < 0$, letting $a = 0$ and $b = x' - x$, the same argument leads to

$$G(x) > G(x') \quad \forall x < x' \leqslant s.$$

2. (a) Clearly $A(x) > 0$ in $x \in (s', S]$ by the definition of s'.
(b) If $x > S$, from (6), using $b = x - S,\ a = 0$, we see that

$$\begin{aligned} K + G(x + z) &\geqslant G(x) + \frac{z}{x - S}\{G(x) - G(S)\} \\ &\geqslant G(x) \end{aligned}$$

for all $z \in [0, C]$, as $G(S)$ is minimal. This implies that $\tilde{G}(x) \geqslant G(x)$ and consequently $A(x) \geqslant 0$.

3. We show that $H(x)$ must take the prescribed form.
- $x < \min\{s' - C, s\}$. Then $x < s$ implies that $A(x) < 0$ so $H(x) = \tilde{G}(x)$. On the other hand $x + C < s'$ and part 1 of this lemma implies that $G_C(x) = \tilde{G}(x)$, so $H(x) = G_C(x)$.
- $\min\{s' - C, s\} \leqslant x < \max(s' - C, s)$. There are three cases to consider:
  1. $s' - C = s$. The interval is empty and the property is trivially true.
  2. $s' - C < s$. In this case $I_+ = 0$ and $I_- = 1$. Then $H(x) = \tilde{G}(x)$ on account of $x < s$. Since $s' \leqslant x + C$ we have $H(x) = \tilde{G}(x) = \overline{G}(x)$.

3. $s < s' - C$. In this case $I_+ = 1$ and $I_- = 0$. Here the sign of $A(x)$ is unknown, but $\tilde{G}(x) = G_C(x)$ on account of $x + C < s'$. Consequently $H(x) = \min\{G(x), G_C(x)\}$.

- $\max\{s' - C, s\} \leqslant x \leqslant s'$. Over this interval the sign of $A(x)$ is not determined. However, $\tilde{G}(x) = \overline{G}(x)$ on account of $s' \leqslant x + C$. Consequently, $H(x) = \min\{G(x), \overline{G}(x)\}$.
- $s' < x$. On this interval $A(x) \geqslant 0$, so $H(x) = G(x)$.

4. We will now show that H is strong CK-convex. For simplicity, we will define

$$\Delta \stackrel{\text{def}}{=} K + H(y+z) - H(y) - \frac{z}{b}\{H(y-a) - H(y-a-b)\}.$$

To show that $H(y)$ is strong CK-convex it suffices to show that $\Delta \geqslant 0$. We do so by considering the four different cases for the pair of values $H(y-a-b)$ and $H(y+z)$. In the proof we will replace $H(y)$ and $H(y-a)$ by suitable upper bounds.

Case I. $H(y+z) = G(y+z)$, $H(y-a-b) = G(y-a-b)$. The result follows from the strong CK-convexity of G, by using $H(y) \leqslant G(y)$, $H(y-a) \leqslant G(y-a)$.

Case II. $H(y+z) = G(y+z)$, $H(y-a-b) = \tilde{G}(y-a-b) = K + G(y-a-b')$ for some $b - C \leqslant b' < b$. Then

$$\Delta = K + G(y+z) - H(y) - \frac{z}{b}\{H(y-a) - K - G(y-a-b')\}. \tag{7}$$

(a) Assume first that $H(y-a) \leqslant K + G(y-a-b')$. Then $\Delta \geqslant 0$ on account of $K + G(y+z) - H(y) \geqslant 0$.
(b) Assume now that $H(y-a) > K + G(y-a-b')$. Then from $K + G(y-a) \geqslant H(y-a)$ it follows that

$$G(y-a) - G(y-a-b') \geqslant H(y-a) - K - G(y-a-b') > 0.$$

This coupled with $G(y) \geqslant H(y)$ and $b' < b$ justifies the first two inequalities in

$$\begin{aligned}\Delta &\geqslant K + G(y+z) - G(y) - \frac{z}{b}\{G(y-a) - G(y-a-b')\}\\ &> K + G(y+z) - G(y) - \frac{z}{b'}\{G(y-a) - G(y-a-b')\}\\ &\geqslant 0.\end{aligned}$$

The final inequality follows from the strong CK-convexity of G.

Case III. $H(y+z) = \tilde{G}(y+z) = K + G(y+z+u)$, for some $u \in [0, C]$, $H(y-a-b) = G(y-a-b)$. Since $H(y) \leqslant K + G(y+u)$ and $H(y-a) \leqslant G(y-a)$,

$$\begin{aligned}\Delta &\geqslant K + K + G(y+z+u) - K - G(y+u) - \frac{z}{b}\{G(y-a) - G(y-a-b)\}\\ &= K + G(y'+z) - G(y') - \frac{z}{b}\{G(y'-a') - G(y'-a'-b)\}\\ &\geqslant 0,\end{aligned}$$

where $y' = y + u$ and $a' = a + u$.

Case IV. $H(y+z) = \tilde{G}(y+z) = K + G(y+z+u)$, $H(y-a-b) = \tilde{G}(y-a-b) = K + G(y-a-b+u')$ for some $u, u' \in [0, C]$. Then

$$\Delta = K + \tilde{G}(y+z) - H(y) - \frac{z}{b}\{H(y-a) - \tilde{G}(y-a-b)\}. \tag{8}$$

(a) $H(y-a) \leqslant \tilde{G}(y-a-b)$:
(i) $z + u \leqslant C$: In this case we can reach $y + u + z$ from y, so $H(y) \leqslant K + G(y+z+u) = \tilde{G}(y+z)$ resulting in

$$\Delta \geqslant K + \tilde{G}(y+z) - H(y) \geqslant K > 0.$$

(ii) $z + u > C$: Let $y' = y + C$, $z' = z + u - C \leqslant z$, and $a' = a + C - u' \geqslant 0$. Since $z' \leqslant z$ and $H(y-a) \leqslant \tilde{G}(y-a-b)$, from (8):

$$\begin{aligned}
\Delta &\geqslant K + \tilde{G}(y+z) - H(y) - \frac{z'}{b}\{H(y-a) - \tilde{G}(y-a-b)\} \\
&\geqslant K + G(y+z+u) - G(y+C) - \frac{z'}{b}\{G(y-a+u') - G(y-a-b+u')\} \\
&\geqslant K + G(y'+z') - G(y') - \frac{z'}{b}\{G(y'-a') - G(y'-a'-b)\} \\
&\geqslant 0.
\end{aligned}$$

The second inequality holds as $H(y-a) \leqslant K + G(y-a+u')$, and $H(y) \leqslant K + G(y+C)$.

(b) $H(y-a) > \tilde{G}(y-a-b)$: If $u' \geqslant b$, or $y - a \leqslant y - a - b + u'$, then

$$H(y-a) \leqslant K + G(y-a-b+u') = \tilde{G}(y-a-b) < H(y-a)$$

which is a contradiction. Thus it must be that $y - a - b + u' < y - a$, or $u' < b$.

Let $y' = y + u, a' = a + u$, and $b' = b - u' > 0$. From $H(y-a) \leqslant K + G(y-a)$ and $H(y) \leqslant K + G(y+u)$ we obtain:

$$\begin{aligned}
\Delta &\geqslant K + G(y+z+u) - G(y+u) - \frac{z}{b}\{G(y+u-a-u) - G(y+u-a-u-b+u')\} \\
&= K + G(y'+z) - G(y') - \frac{z}{b}\{G(y'-a') - G(y'-a'-b')\} \\
&\geqslant K + G(y'+z) - G(y') - \frac{z}{b'}\{G(y'-a') - G(y'-a'-b')\} \\
&\geqslant 0,
\end{aligned}$$

where the second inequality is justified by $G(y'-a') - G(y'-a'-b') > 0$ (as $H(y-a) > \tilde{G}(y-a-b)$) and $b' \leqslant b$. $\square$

Remark. *If $A(x) < 0$ and $x < S \leqslant x + C$, then $H(x) = K + G(S)$.*

Theorem 4.1. *Under the assumptions above, there exists a sequence of numbers $\{s_n, s'_n : n = 0, \ldots, N-1\}$ such that the optimal policy is given by*

$$H_n(x) = \begin{cases} G_{C_n}(x), & x < \min\{s'_n - C, s_n\}, \\ \min\{G_n(x), G_{C_n}(x)\}I_{n+} + \{\overline{G}_n(x)\}I_{n-}, & \min\{s'_n - C, s_n\} \leqslant x < \max\{s'_n - C, s_n\}, \\ \min\{G_n(x), \overline{G}_n(x)\}, & \max\{s'_n - C, s_n\} \leqslant x \leqslant s'_n, \\ G_n(x), & s'_n < x. \end{cases}$$

In the above we define $G_{C_n}(x)$, $\overline{G}_n(x)$, I_{n+}, and I_{n-} as previously (save for the addition of subscripts as necessary).

Proof. In the light of formulae (1)–(3), we must show that $G_n(y)$ is strong CK-convex for all $0 \leqslant n \leqslant N$ in order to apply Lemma 4.1. In addition, we must show that S_n is finite for all considered n. We prove this by induction.

The base case, $n = 0$ is true as $f_0(x) = 0$ for all x, and $G_1(y)$ is CK-convex due to Proposition 3.1. Further, the fact that its value goes to ∞ as $|y| \to \infty$ (inherited from $J(y)$) implies that S_1 is finite.

Now assume the hypothesis is true for $n = k$. We will show it for $k+1$. Under the assumption that $G_k(x)$ is strong CK-convex, we know that $H_k(y)$ has this property also, from Lemma 4.1. Furthermore, as the slope of $H_k(y)$ equals that of $G_k(x)$ as $|y| \to \infty$, $H_k(y)$ has a finite minimizing argument. Therefore $f_k(x)$ is strong CK-convex and has a finite minimizing argument, as $-cx$ is convex and $y \geqslant x$. Therefore from Proposition 3.1 we have that $\mathrm{E}[f_k(y - D_k)]$ is strong CK-convex and has a finite minimizing argument. Coupled with the convexity of $J(y)$, and Proposition 3.1, this implies that $G_{k+1}(y)$ is strong CK-convex and has a finite minimizing argument.

Therefore we apply Lemma 4.1 to arrive at the desired result. □

Corollary 4.1. *If $S_n - s_n \leqslant C$, then under the same assumptions, the optimal policy reduces to*

$$H_n(x) \stackrel{\text{def}}{=} \begin{cases} G_{C_n}(x), & x \leqslant s'_n - C, \\ \overline{G}_n(x), & s'_n - C < x \leqslant S_n - C, \\ K + G_n(S_n), & S_n - C < x \leqslant s'_n, \\ G_n(x), & s'_n < x. \end{cases}$$

Proof. The only interval that requires explanation is $(s_n, s'_n]$. From $A(s_n) \geqslant 0 \geqslant A(s'_n)$ and the fact that we can reach S from both s_n and s'_n we have

$$G_n(s_n) \leqslant K + G_n(S_n) \leqslant G_n(s'_n).$$

But G non-increasing up to s'_n implies that $G(x) = G(s_n) = G(s'_n) = K + G(S_n)$ on the entire interval $(s_n, s'_n]$. □

5. Computational study

We have computed the optimal policy over $n = 52$ periods for factors $h = 1$, $p \in \{9, 19, 99\}$, $K \in \{1, 5, 10, 25, 50\}$, $C \in \{5, 10, 15\}$, discount rates $\alpha \in \{0.9, 0.95, 1\}$, lead times $L \in \{0, 1\}$, and maximal demand $D_{\mathrm{m}} \in \{10, 15\}$. In each case the probability mass function with support on $\{0, 1, \ldots, D_{\mathrm{m}}\}$ was obtain by generating $D_{\mathrm{m}} + 1$ uniform random variables and normalizing them so that the sum was equal to one. We generated 1000 problems for each combination of the above parameters (and thus 540 000 problem instances).

In *all* cases we found $s_n = s'_n$ for all $n = 1, 2, \ldots, 52$, so it is optimal to place an order on $x < s_n$ and not to order on $x \geqslant s_n$. Since $s_n = s'_n$ implies that $I_- = 1$, H_n reduces to the following form:

$$H_n(x) \stackrel{\text{def}}{=} \begin{cases} G_{C_n}(x), & x \leqslant s_n - C, \\ \overline{G}_n(x), & s_n - C < x < s_n, \\ G_n(x), & s_n \leqslant x. \end{cases}$$

We found in addition that $S_n - C < s_n$ holds for small to moderate values of K which further reduces H_n to the form

$$H_n(x) \stackrel{\text{def}}{=} \begin{cases} G_{C_n}(x), & x \leqslant s_n - C, \\ \overline{G}_n(x), & s_n - C < x \leqslant S_n - C, \\ K + G_n(S_n), & S_n - C < x < s_n, \\ G_n(x), & s_n \leqslant x. \end{cases}$$

Furthermore, for small values of K, $\overline{G}(x) = G_C(x)$ on $s_n - C < x \leqslant S_n - C$, so the optimal policy reduces to ordering $\min\{C, S_n - x\}$ on $x < s_n$, and not ordering on $x \geqslant s_n$. If we let $y(x)$ be the smallest optimal

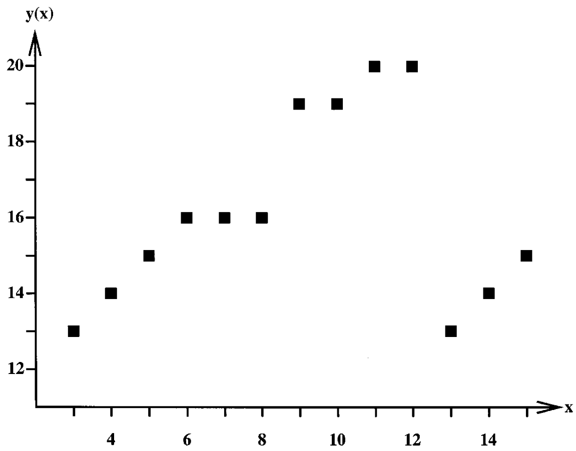

Fig. 2. Optimal order quantities for the example.

inventory after ordering, $y(x) = \min\{x + C, S\}$ on $x < s_n$ and $y(x) = x$ on $x \geqslant s_n$. This policy is known as the modified (s, S) policy [2].

For moderate values of K the ordering policy exhibits a slightly more complicated behavior when $s_n - C < x \leqslant S_n - C$. This is because there may be one or more local minima of G_n in the interval $[s_n, S_n)$. Suppose $s_n \leqslant S_n^1 < S_n^2 < \cdots < S_n^m = S_n$ are m distinct, and successively better local minima of G_n over the interval (s_n, S_n). (By successively better we mean that $G_n(x) > G_n(S_n^k)$ for all $x < S_n^k$.) For each S_n^k let

$$s_n^k = \inf\{y > S_n^k : \ G_n(y) \leqslant G_n(S_n^k)\}.$$

Notice that G_n must be non-increasing over the intervals (s_n^k, S_n^{k+1}) for each $k = 1, \ldots, m-1$. It follows that $y(x) = S_n^k$ over $x \in (S_n^k - C, s_n^k - C)$ and $y(x) = x + C$ over (s_n^k, S_n^{k+1}). The function $y(x)$ jumps from S_n^k to s_n^k at $s_n^k - C$ for each k. This is illustrated in the following example:

Example. For $n = 52, h = 1, b = 9, K = 10, C = 10$, and probability mass function $p_1 = 0.15$, $p_6 = 0.70$, and $p_7 = 0.15$. We found $s_{52} = 13$, $S_{52} = 20$, but there is one local minimum $S_{52}^1 = 16$ between s_{52} and S_{52}. Here $s_{52}^1 = 19$. Consequently $y(x) = x + 10$ on $x \leqslant 5$, $y(x) = 16$ on $x = 6, 7, 8$, $y(x) = 19$ on $x = 9, 10$, followed by $y(x) = 20$ on $x = 11, 12$, and $y(x) = x$ on $x \geqslant 13$. This is shown in Fig. 2.

Finally, for large K we observe that $s_n < S_n - C$, and that $y(x) = x + C$ on $x < s_n$ and $y(x) = x$ on $x \geqslant s_n$. It is possible to show that this simple structure is optimal when the only ordering options are to order C units or nothing at all, provided that $G_n(x + C) - G_n(x)$ is non-decreasing (which holds if G_n is convex).

6. Conclusion

In this paper we have established that there does exist a fairly simple structure to the optimal policy for the capacitated periodic review inventory problem with fixed ordering costs. Depending upon whether or not $s' - C \leqslant s$, the optimal policy takes one of the forms as shown in Fig. 3. We consider this result a further step towards establishing the exact form of the optimal policy for this problem. In addition, as the optimal policy is either completely specified or reduced to two actions in all but an interval of length C, this

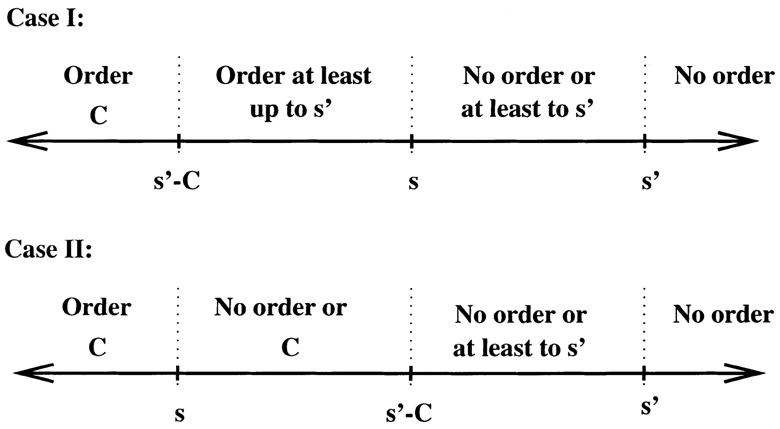

Fig. 3. Optimal policy form – two cases.

structure provides a basis for the calculation of potentially effective and yet simple heuristics. (The case for such heuristics is strengthened by the computational results of Section 5.)

A number of problems still remain. The most vexing is the possibility that under the current structure there could exist a number of intervals between s and s' where it is optimal to start and stop ordering. An optimal policy with a single continuous interval over which ordering is prescribed, as was found for all of the cases tested in Section 5, is much more analytically appealing. A further reduction in the class of strong CK-convex functions is one possible avenue for showing this result. For a single stage problem we can prove that adding the condition $G_n(x + C) - G_n(x)$ non-decreasing reduces the optimal policy to the desired form. Moreover, our computations indicate that this condition is preserved by the DP. Unfortunately, the proof of this has thus far eluded us. It should be mentioned that it is likewise possible, although we believe it unlikely, that such a structure simply does not exist. To show this requires a problem instance in which the optimal policy has multiple disjoint intervals in which ordering is optimal. Our computational study suggests that this is not the case.

Further work also can include proofs of optimality in the infinite horizon discounted and average cost cases, methods for the efficient calculation of s and s', and development of heuristic policies based on the newly established optimal policy structure.

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Corrigendum

Corrigendum to ‘Capacitated inventory problems with fixed order costs: Some optimal policy structure’ [EJOR 126 (2000) 603–613]

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It was recently brought to our attention by Professor Ye Lu, of the City University of Hong Kong, that there is an error in a proof i paper (Gallego & Scheller-Wolf, 2000). The result still holds, and the purpose of this short note is to correct this error.

The problem is Case II of Point 4 in the proof of Lemma 4.1. Specifically, one of the parameters used in the proof, b', could be neg thus rendering an appeal to strong *CK*-convexity invalid. The proof may be corrected as follows:

Proof. Case II: $H(y+z) = G(y+z),\ H(y-a-b) = \tilde{G}(y-a-b) = K + G(y-a-b')$ for some $b - C \leq b' < b$.

Case (i): $b' \geq 0$: In this case the arguments in the paper are correct. We will not duplicate them here.

Case (ii): $b' < 0$: We need to show that:

$$\Delta = K + H(y+z) - H(y) - \frac{z}{b}\{H(y-a) - H(y-a-b)\} \geq 0.$$

$$\begin{aligned}
\Delta &= K + H(y+z) - H(y) - \frac{z}{b}\{H(y-a) - H(y-a-b)\} \\
&= K + G(y+z) - H(y) - \frac{z}{b}\{H(y-a) - K - G(y-a-b')\} \\
&\geq K + G(y+z) - H(y) - \frac{z}{b}\{K + G(y-a-b') - K - G(y-a-b')\} \\
&= K + G(y+z) - H(y) \geq 0.
\end{aligned}$$

The second line substitutes in the assumptions of Case II.

The third line uses the fact that if $b' \leq 0, y-a-b' \geq y-a$, so ordering $-b'$ items is a feasible action at $y-a$. Thus $H(y-a)$ $G(y-a-b')$.

The fourth line simplifies and uses the fact that $H(y) \leq K + G(y+z)$ for any z between zero and C. □

Reference

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