Bifurcación desde Autovalores Simples

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Resumen en castellano

En esta tesis de fin de grado, exploramos las aplicaciones del análisis funcional no lineal espectral a la teoría de bifurcaciones. Examinamos el Teorema de la Función Implícita sobre espacios de Banach. Además, discutimos el resultado más célebre de la Teoría de Bifurcaciones, el Teorema de Crandall-Rabinowitz, y estudiamos su aplicación a la familia de soluciones de la ecuación $-u'' = \lambda u - au^3$. Finalmente, vemos la aplicación de métodos numéricos a los problemas de bifurcaciones.

Palabras clave

Análisis Funcional No Lineal Espectral; Teoría de Bifurcación, Teorema de Crandall-Rabinowitz; Continuación Numérica

Abstract

In this undergraduate thesis, we explore the applications of spectral non-linear functional analysis to bifurcation theory. We examine the Implicit Function Theorem over Banach spaces. Additionally, we discuss the most celebrated result of the Bifurcation Theory, the Crandall-Rabinowitz Theorem, and study its application to the family of solutions of the equation $-u'' = \lambda u - au^3$. Finally, we see the application of numerical methods to bifurcation problems.

Keywords

Spectral non-linear Functional Analysis; Bifurcation Theory; Crandall-Rabinowitz Theorem; Numerical Continuation

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Chapter 1

Equation $-u'' = \lambda u - au^3$

Our goal is to study the solution of the equation

$$\begin{cases} -u'' = \lambda u - a(x)u^3 & \text{in} \quad (0, L), \\ u(0) = u(L) = 0 \end{cases} \tag{1.1}$$

where $a \in \mathcal{C}[0, L]$ and $\lambda \in \mathbb{R}$ is the *bifurcation parameter*. We will begin to study the structure and properties of the set of non-zero solutions to the nonlinear one-dimensional boundary value problem on the interval (p, q):

$$\begin{cases} -u'' = \lambda u - a(x)u^3 & \text{in} \quad (p, q), \\ u(p) = u(q) = 0 \end{cases} \tag{1.2}$$

where $a \in \mathcal{C}[p, q]$ and $\lambda \in \mathbb{R}$. Using the Fundamental Theorem of Existence and Uniqueness of Solution for initial value problems associated with ordinary differential equations, it follows that the Cauchy problem given by

$$\begin{cases} u' = v \\ v' = -\lambda u + a(x)u^3 \\ u(x_0) = u_0, v(x_0) = v_0 \end{cases} \tag{1.3}$$

admits a unique (global) solution $(u(x), v(x))$ in some interval $(L(x_0), M(x_0))$ where $-\infty \leq L(x_0) < x_0 < M(x_0) \leq \infty$. The following theorem characterizes some properties of the solutions of (1.2).

Theorem 1.0.1. *Assume that $u \neq 0$ is a solution of (1.2). Then,*

(i) $u'(z) \neq 0$ if $u(z) = 0$ for some $z \in [p, q]$, i.e., any zero is simple. And, $u(x)$ changes of sign at z if $z \in (p, q)$

(ii) $u(x)$ admits, at most, finitely many zeroes in (p, q)

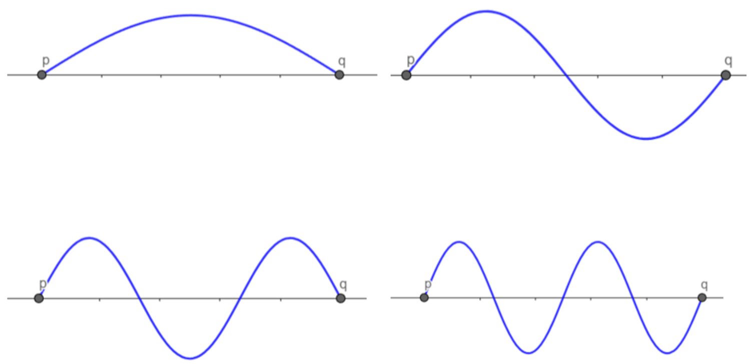


Figure 1.1: Some n-node solutions of (1.1) for $n = 1, \ldots, 4$.

(iii) If u is a positive solution of (1.2), then $u'(p) > 0$, $u'(q) < 0$ and $u(x) > 0$ for all $x \in (p, q)$

Proof. (i) If we assume that there exists a $z \in [p, q]$ such that $u(z) = u'(z) = 0$, then, the Fundamental Theorem of existence and uniqueness of solutions asserts that the unique solution of

$$\begin{cases} -w'' = \lambda w - a(x)w^3 \\ w(z) = 0, w'(z) = 0 \end{cases} \tag{1.4}$$

is the constant 0. Therefore, $u'(z) \neq 0$ if $u(z) = 0$.

(ii) Assume that there exists infinitely many zeros, in particular, let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of zeros in $[p, q]$. By the Arzelà-Ascoli theorem, there exists a subsequence $\{z_{n_m}\}_{m\in\mathbb{N}}$ that converges to $z \in [p, q]$.

Rolle's theorem implies the existence of zeros between z_{n_m} and $z_{n_{m+1}}$ and therefore, there exists a sequence $\tilde{z}_{n_m}$ such that $\lim_{m\to\infty} \tilde{z}_{n_m} = z$. Since solutions are $\mathcal{C}^2[p, q]$, $u(z) = u'(z) = 0$ and by the Fundamental Theorem of existence and uniqueness $u \equiv 0$ which is not possible.

(iii) Owing to (i), if $u(p) = u(q) = 0$ and $u \geq 0$, then $u'(p) > 0$ and $u'(q) < 0$. Moreover, $u(x) > 0$ because otherwise, it would change the sign.

□

The number of zeroes in (p, q) varies depending on the values of λ. Figure 1.1 shows four different types of solutions, categorized by the number of zeros in (p, q). The following proposition establishes a lower bound for λ given the number of interior zeros in a solution u that satisfies (1.2).

Proposition 1.0.2. *If $a > 0$ and (1.2) admits a solution u with $n-1$ interior zeroes in (p,q) for some $n \geq 1$, then*

$$\lambda > \left(\frac{n\pi}{q-p}\right)^2 \tag{1.5}$$

Thus, by the contrapositive, $u \equiv 0$ is the unique solution of equation (1.2) if $\lambda \leq \left(\frac{\pi}{q-p}\right)^2$.

Proof. Firstly, we will show that

$$\lambda > \left(\frac{\pi}{q-p}\right)^2 \tag{1.6}$$

is necessary for the existence of a positive solution u. Multiplying the differential equation (1.2), $-u'' = \lambda u - a(x)u$, by

$$\varphi(x) := \sin\frac{\pi(x-p)}{q-p}, \qquad x \in [p,q] \tag{1.7}$$

and integrating in (p,q) yields

$$-\int_p^q \varphi(x)u''(x)dx = \lambda\int_p^q \varphi(x)u(x)dx - \int_p^q \underbrace{a(x)\varphi(x)u^3(x)}_{>0} \in (p,q)dx \tag{1.8}$$

Therefore,

$$-\int_p^q \varphi(x)u''(x)dx < \lambda\int_p^q \varphi(x)u(x)dx \tag{1.9}$$

On the other hand, integrating the left side by parts twice,

$$\begin{aligned}
\int_p^q \varphi(x)u''(x)dx &= \int_p^q (\varphi(x)u'(x))'dx - \int_p^q \varphi'(x)u'(x)dx \\
&= \cancel{\varphi(q)u'(q)} - \cancel{\varphi(p)u'(p)} - \frac{\pi}{q-p}\int_p^q \cos\frac{\pi(x-p)}{q-p}u'(x)dx \\
&= -\frac{\pi}{q-p}\int_p^q \left(\cos\frac{\pi(x-p)}{q-p}u(x)\right)'dx - \left(\frac{\pi}{q-p}\right)^2\int_p^q \varphi(x)u(x)dx \\
&= -\left(\frac{\pi}{q-p}\right)^2\int_p^q \varphi(x)u(x)dx
\end{aligned}$$

Therefore, this identity combined with (1.9) yields

$$\left(\frac{\pi}{q-p}\right)^2\int_p^q \varphi(x)u(x)dx < \lambda\int_p^q \varphi(x)u(x)dx \tag{1.10}$$

Thus, (1.6) holds because $\int_p^q \varphi(x)u(x)dx > 0$. Now, we proceed to prove the general version, assuming that there exist $n-1$ zeroes in the interior of (p,q), then the minimum distance between zeros is $(q-p)/n$. And thus, applying the first part to this segment of length less or equal to $(q-p)/n$, we obtain (1.5). □

Corollary 1.0.2.1. *Using the proof of Proposition (1.0.2), we can establish a similar result when $a < 0$. Given a solution, u, with $n-1$ interior zeroes in (p,q), it should satisfy*

$$\lambda < \left(\frac{n\pi}{q-p}\right)^2. \tag{1.11}$$

And, equivalently, $u \equiv 0$ is the unique solution of (1.2) if $\lambda \geq \left(\frac{\pi}{q-p}\right)^2$.

1.1 The case where $a(x)$ is a positive constant

We will assume hereafter that $a(x)$ is a positive constant $a > 0$. We will use the phase plane techniques for constructing the solution set of (1.2). Firstly, we notice that the equation

$$-u'' = \lambda u - au^3 \tag{1.12}$$

can be expressed as a conservative first-order system

$$\begin{cases} u' = v, \\ v' = au^3 - \lambda u \\ u(x_0) = u_0, v(x_0) = v_0 \end{cases} \tag{1.13}$$

which is similar to (1.3). For any given $(u_0, v_0) \in \mathbb{R}^2$, let

$$(u(x), v(x)) = (u(x; u_0, v_0), v(x; u_0, v_0))$$

denote the unique solution of (1.13) such that $(u(0), v(0)) = (u_0, v_0)$, and let $I_0 = (L, M)$ denote its maximal interval of definition. By construction, $u(x) = u(x; u_0, v_0)$ is unique. By taking the cross product in (1.13), we obtain

$$v'(x)v(x) = \lambda u(x)u'(x) - au^3(x)u'(x) \quad \text{for all } x \in I_0.$$

Equivalently,

$$\frac{d}{dx}\left(\frac{1}{2}v^2(x) + \frac{\lambda}{2}u^2(x) - \frac{a}{4}u^4(x)\right) = 0 \qquad x \in I_0$$

Thus, using the initial condition

$$\frac{1}{2}v^2(x) + \frac{\lambda}{2}u^2(x) - \frac{a}{4}u^4(x) = \frac{1}{2}v_0^2 + \frac{\lambda}{2}u_0^2 - \frac{a}{4}u_0^4 \qquad x \in I_0 \tag{1.14}$$

We define the function

$$\Phi(\xi) := \frac{\lambda}{2}\xi^2 - \frac{a}{4}\xi^4 \qquad \xi \in \mathbb{R}, \tag{1.15}$$

which can be taken as the associated *potential energy*, because (1.14) can be rewritten in the form

$$\frac{1}{2}v^2(x) + \Phi(u(x)) = \frac{1}{2}v_0^2 + \Phi(u_0) \qquad x \in I_0 \tag{1.16}$$

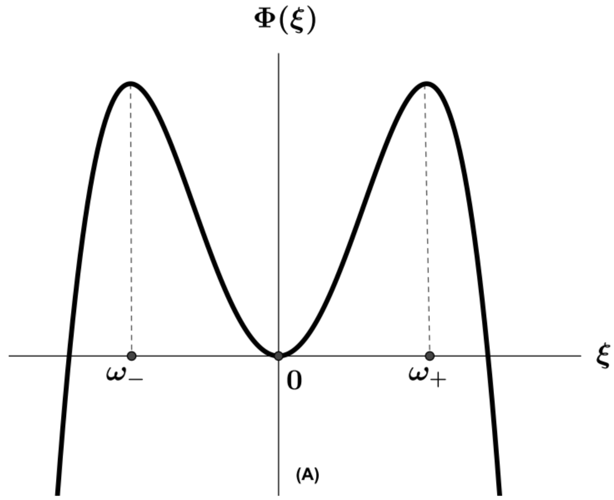


Figure 1.2: Potential energy plot when $\lambda > 0$.

This identity expresses the conservation of the *total energy*, that is the sum of the potential and kinetic energy,

$$\mathcal{E}(U,V) := \frac{1}{2}V^2 + \Phi(U) \tag{1.17}$$

According to (1.16),

$$v(x) = \pm\sqrt{v_0^2 + 2[\Phi(u_0) - \Phi(u(x))]} \qquad x \in I_0. \tag{1.18}$$

Then, the velocity $v(x) = u'(x)$ is determined by the position $u(x)$. When $\lambda > 0$, Φ has three critical points: 0, $\omega_+ := \sqrt{\lambda/a}$, and $\omega_- := -\sqrt{\lambda/a}$ (see Figure 1.2).

Since $\Phi''(\xi) = \lambda - 3a\xi^2$, we have that

$$\Phi''(0) = \lambda > 0 \quad \text{and} \quad \Phi''(\omega_\pm) = \lambda - 3a\frac{\lambda}{a} = -2\lambda < 0 \tag{1.19}$$

Therefore, 0 is a local minimum, and ω_+ and ω_- are local maxima of Φ. Owing to (1.18), the phase portrait of (1.13) looks like Figure 1.3. For instance, if $u_0 \in (\omega_-, \omega_+)$ and

$$\mathcal{E}(u_0,v_0) := \frac{1}{2}v_0^2 + \Phi(u_0) < \Phi(\omega_+), \tag{1.20}$$

then the integral curve is closed and the orbits are periodic. And, since $v(x) = u'(x)$, orbits move clockwise. The set of solutions where $\mathcal{E}(U,V) = \Phi(\omega_+)$ contains the fixed-points $(\omega_-, 0)$ and $(\omega_+, 0)$. 6 arcs pass through these points, and two of them connect both points. Figure 1.3 shows the phase portrait.

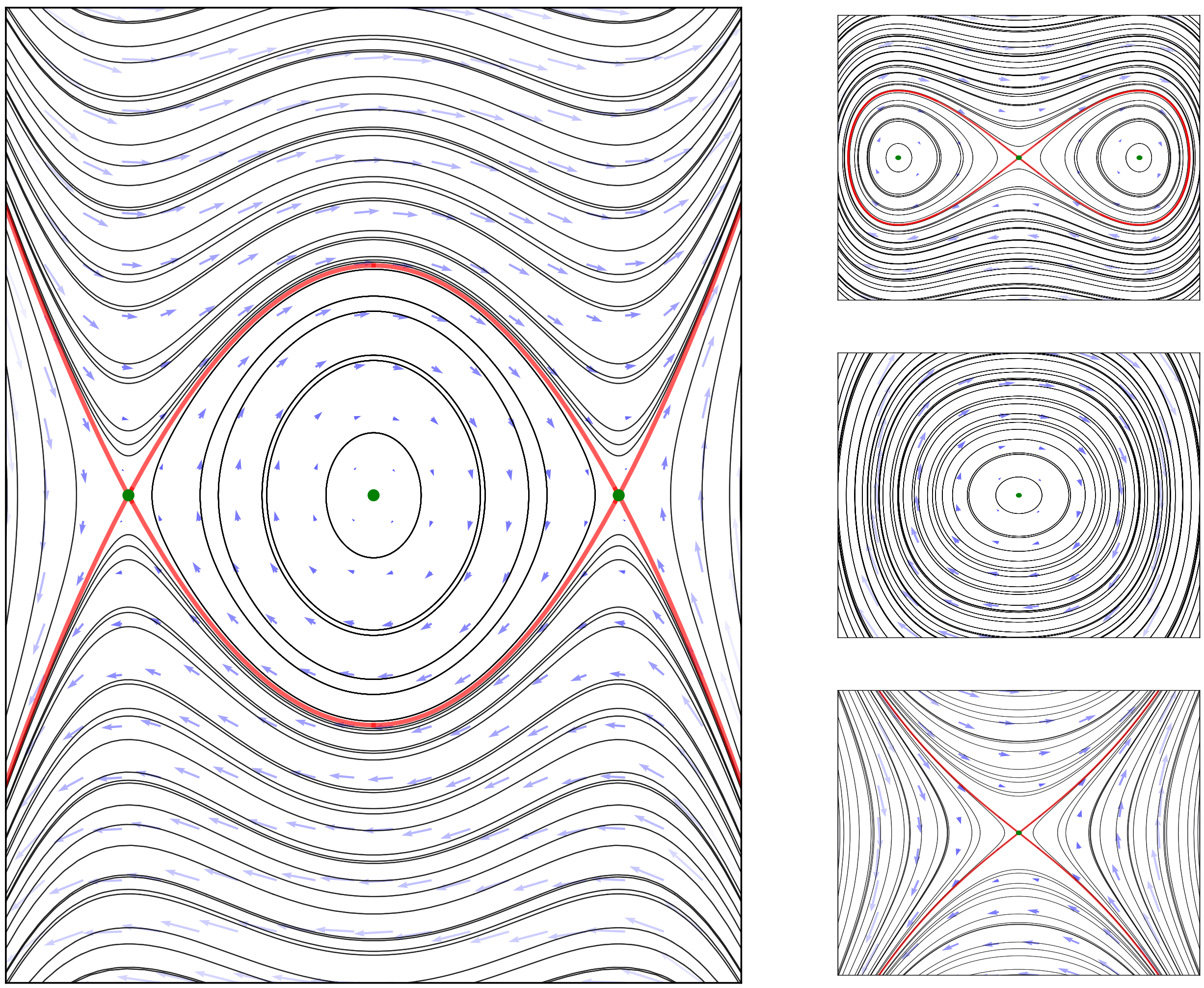


Figure 1.3: Phase portrait given by (1.13). The left image shows the case where $a > 0$ and $\lambda > 0$. On the right, there are three images showing the cases where $a < 0, \lambda < 0$; $a < 0, \lambda > 0$; and $a > 0, \lambda < 0$, respectively.

We denote $v_0^* > 0$ as the critical speed such that $\mathcal{E}(0, v_0^*) = \Phi(\omega_+)$. We are interested in finding positive solutions. We will denote by $u(x) \equiv u(x; 0, v_0)$ the unique solution such that

$$u(0) = 0, \quad u'(0) = v_0. \tag{1.21}$$

To ensure positive solutions of (1.12), we impose $v_0 < v_0^*$ because we already know $u(x; 0, v_0) > 0$ if $v_0 \geq v_0^*$, which does not satisfy the boundary conditions of (1.2). Let T be the minimal time necessary to reach the u-axis, at $(u_0, 0)$, by the periodic orbit. We will consider $v_0 \in (0, v_0^*)$. The graph will look like Figure 1.4. We observe that the periodic orbit satisfies

$$u(0) = 0, \quad u'(0) = v_0, \quad u(T) = u_0, \quad u'(T) = 0, \quad u(2T) = 0, \quad u'(2T) = -v_0. \tag{1.22}$$

Note that any of these positive solutions of the differential equation exists in the interval

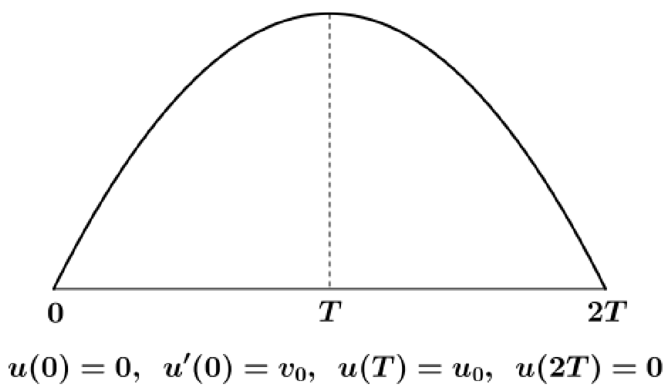


Figure 1.4: The graph of $u(x; 0, v_0)$.

$[0, 2T]$ if and only if $L = 2T$. The next lemma establishes the symmetry of the problem and shows that indeed, $u(2T) = 0$.

Lemma 1.1.1. *Assume that $v_0 \in (0, v_0^*)$, and let $(u(x; 0, v_0), v(x; 0, v_0))$ be the unique solution of (1.13) such that $u(0) = 0$ and $u'(0) = v_0$. Let $T > 0$ be the minimal time such that $v(T) = u'(T) = 0$. Then,*

$$u(x) = u(2T - x) \qquad \forall x \in [0, 2T] \tag{1.23}$$

Therefore, the graph of $u(x)$ on $[T, 2T]$ is a reflection about $x = T$ of its graph on $[0, T]$.

Proof. Consider the auxiliary function $U(x)$ defined by

$$U(x) := \begin{cases} u(x) & \text{if } x \in [0, T], \\ u(2T - x) & \text{if } x \in (T, 2T] \end{cases} \tag{1.24}$$

Then, $U(x)$ is the extension of $u|_{[0,T]}$ to the interval $[0, 2T]$ by reflection about T. By definition,

$$U(0) = u(0) = 0, \quad U(0) = u'(0) = v_0 \tag{1.25}$$

Moreover, the function is continuous

$$\lim_{x\uparrow T} U(x) = \lim_{x\uparrow T} u(x) = u(T), \quad \lim_{x\downarrow T} U(x) = \lim_{x\downarrow T} u(2T - x) = u(T). \tag{1.26}$$

Hence, $U \in \mathcal{C}[0, 2T]$. On the other hand, we can prove that $U(x)$ is differentiable on $[0, 2T]$. Firstly,

$$U'(x) = \begin{cases} u'(x) & \text{if } x \in [0, T), \\ -u'(2T - x) & \text{if } x \in (T, 2T]. \end{cases} \tag{1.27}$$

Consequently,

$$\begin{aligned} \lim_{x\uparrow T} U'(x) = \lim_{x\uparrow T} u'(x) = u'(T) = 0 \\ \lim_{x\downarrow T} U'(x) = \lim_{x\downarrow T} u'(2T - x) = -u'(T) = 0 \end{aligned} \tag{1.28}$$

Therefore, U' is continuous and, hence, $U \in \mathcal{C}^1[0, 2T]$. And, differentiating again

$$U''(x) := 0 \begin{cases} u''(x) & \text{if } x \in [0, T), \\ u''(2T - x) & \text{if } x \in (T, 2T] \end{cases} \tag{1.29}$$

and the limit at T,

$$\begin{aligned} &\lim_{x\uparrow T} U''(x) = \lim_{x\uparrow T} u''(x) = \lim_{x\uparrow T} (-\lambda u(x) + au^3(x)) = -\lambda u_0 + au_0^3 \\ &\lim_{x\downarrow T} U''(x) = \lim_{x\downarrow T} u''(x) = \lim_{x\uparrow T} (-\lambda u(2T - x) + au^3(2T - x)) = -\lambda u_0 + au_0^3 \end{aligned} \tag{1.30}$$

Therefore, $U \in \mathcal{C}^2[0, 2T]$ and it satisfies

$$-U''(x) = \lambda U(x) - aU^3(x) \quad \text{for all } x \in [0, 2T]. \tag{1.31}$$

Since $(U(0), U'(0)) = (0, v_0) = (u(0), u'(0))$, it follows from the unicity of the function that $U(x) = u(x)$ for all $x \in [0, 2T]$ and consequently (1.23) holds. $\square$

Since u solves (1.12) if, and only if, $-u$ does, it can be proven that $2T$ is half of the period of the periodic solution of the differential equation. Moreover, using the same procedure of Lemma 1.1.1, $u(x) = -u(x - 2T)$ for all $x \in (2T, 4T]$. Therefore, the solution of (1.13) such that $u(0) = 0$ and $u'(0) = v_0 < v_0^*$ has period $4T$.

$$\mathcal{E}(u(x), v(x)) = \frac{1}{2} v_0^2 = \Phi(u_0) \qquad \text{for all } x \in \mathbb{R} \tag{1.32}$$

We proceed to determine T as a function of u_0, or v_0. Since $u(x)$ is increasing in $[0,T]$, from 0 up to reach u_0

$$\begin{aligned} T(u_0) \equiv T &= \int_0^T dx = \int_0^T \frac{u'(x)}{v(x)} dx = \int_0^T \frac{u'(x)}{\sqrt{2[\Phi(u_0) - \Phi(u(x))]}} dx \\ &= \int_0^{u_0} \frac{d\xi}{\sqrt{2[\Phi(u_0) - \Phi(\xi)]}} = \int_0^{u_0} \frac{d\xi}{\sqrt{\lambda(u_0^2 - \xi^2) - \frac{a}{2}(u_0^4 - \xi^4)}} \\ &= \int_0^1 \frac{d\theta}{\sqrt{\lambda(1-\theta^2) - \frac{a u_0^2}{2}(1-\theta^4)}} \qquad \text{for all } u_0 \in (0, \omega_+) \end{aligned} \tag{1.33}$$

It is apparent that $T(u_0)$ is increasing with respect to u_0, as well as decreasing with respect to λ. Moreover, the limit of (1.33) when u_0 tends to 0:

$$\lim_{u_0 \to 0} T(u_0) = \int_0^1 \frac{d\theta}{\sqrt{\lambda(1-\theta^2)}} = \int_0^{\pi/2} \frac{\cos t}{\sqrt{\lambda(1-\sin^2 t)}} dt = \int_0^{\pi/2} \frac{1}{\sqrt{\lambda}} dt = \frac{1}{\sqrt{\lambda}} \frac{\pi}{2} \tag{1.34}$$

Similarly, as u_0 tends to ω_+,

$$\begin{aligned} T(\omega_+) &= \int_0^1 \frac{d\theta}{\sqrt{\lambda(1-\theta^2) - \frac{a\omega_+}{2}(1-\theta^4)}} = \int_0^1 \frac{d\theta}{\sqrt{\lambda(1-\theta^2) - \frac{\lambda}{2}(1-\theta^4)}} \\ &= \int_0^1 \frac{d\theta}{\sqrt{(1-\theta^2)[\lambda - \frac{\lambda}{2}(1+\theta^2)]}} = \sqrt{\frac{2}{\lambda}} \int_0^1 \frac{d\theta}{1-\theta^2} \geq \sqrt{\frac{2}{\lambda}} \int_0^1 \frac{d\theta}{1-\theta} = \infty \end{aligned} \tag{1.35}$$

Therefore, the boundary value problem (1.1) admits a positive solution if, and only if, $T(u_0) = \frac{L}{2}$ for some $u_0 \in (0, \omega_+)$, as shown in Figure 1.5. This only happens if $\frac{L}{2} > T(0) = \frac{1}{\sqrt{\lambda}} \frac{\pi}{2}$, or equivalently, $\lambda > \left(\frac{\pi}{L}\right)^2$. This condition is sufficient to prove the existence of such a solution since $T(u_0)$ is an increasing function of u_0, and if there exists, it is unique, i.e., for every $\lambda > (\pi/L)^2$, there exists a unique $u_{0,\lambda}$ such that $T(u_{0,\lambda}) = \frac{L}{2}$. This result is expressed in the following theorem.

Theorem 1.1.2. *The problem (1.1) has a positive solution if and only if*

$$\lambda > \sigma_1 \equiv \left(\frac{\pi}{L}\right)^2 \tag{1.36}$$

Moreover, it is unique in case it exists.

We denote u_λ the unique positive solution of (4.1) when $\lambda > \sigma_1$, and set

$$v_{0,\lambda} := u'_\lambda(0), \qquad u_{0,\lambda} := u_\lambda(L/2) \tag{1.37}$$

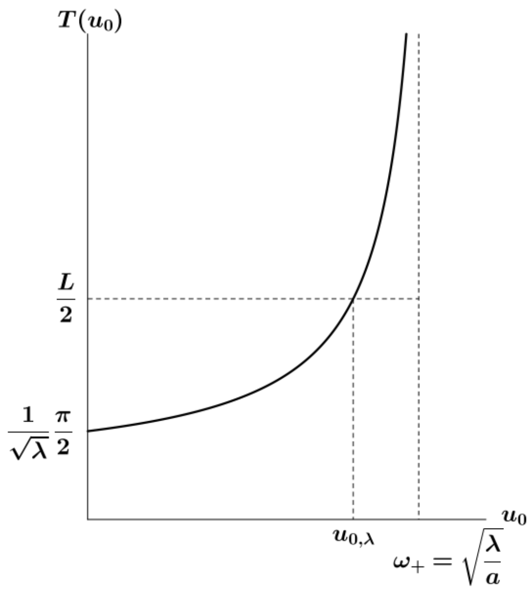


Figure 1.5: The graph of $T(u_0)$.

We notice that the map T defined in (1.33) with respect to the parameter λ varies continuously, therefore, the map defined by $\lambda \to u_{0,\lambda}$ is continuous from (σ_1, ∞) to $(0, \infty)$, by the uniqueness of the positive solution and the map is increasing. Since, $\lambda \to T_\lambda$ is decreasing, the map $\lambda \to u_{0,\lambda}$ is increasing. Furthermore, owing to (1.34),

$$\lim_{\lambda \to \sigma_1} u_{0,\lambda} = \lim_{\lambda \to \sigma_1} \|u_\lambda\|_\infty = 0 \tag{1.38}$$

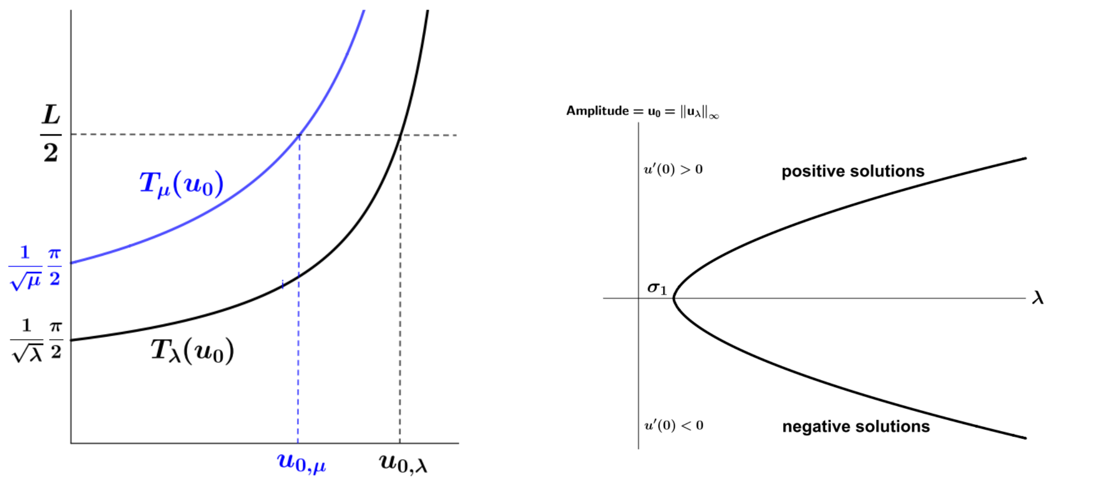
The curve of positive solutions $(\lambda, u_\lambda) \in \mathbb{R} \times \mathcal{C}[0, L]$ bifurcates from $u = 0$ at critical value $\lambda = \sigma_1$.

The solutions of (4.1) with one zero/node in $(0, L)$ represented in Figure 1.6b. By symmetry of the differential equation, and a is a constant, the 1-node lies in $L/2$ and consists of the positive solution of

$$\begin{cases} -u'' = \lambda u - au^3 & \text{in } (0, \frac{L}{2}), \\ u(0) = u(\frac{L}{2}) = 0 \end{cases} \tag{1.39}$$

and the negative solution

$$\begin{cases} -v'' = \lambda v - av^3 & \text{in } (\frac{L}{2}, L), \\ v(\frac{L}{2}) = u(L) = 0 \end{cases} \tag{1.40}$$



(a) The monotonicity of $u_{0,\lambda}$.

(b) Bifurcation diagram of positive and negative solutions.

Figure 1.6: Prediction and optimization diagram for the use cases described.

because $u'(L/2) = v'(L/2)$,

$$w(x) := \begin{cases} u(x) & \text{if } x \in [0, L/2] \\ v(x) & \text{if } x \in [L/2, L] \end{cases} \tag{1.41}$$

provides a 1-node solution of (1.1) with a unique interior node at $L/2$. Arguing similarly, for n-1 nodes in $(0, L)$ for some $n \geq 2$, they must have their nodes at points

$$z_i := \frac{L}{n} i, \qquad i = 1, \ldots, n-1$$

The symmetries of the problem, from the positive solution of

$$\begin{cases} -u'' = \lambda u a u^3 & \text{in } (0, \frac{L}{n}), \\ u(0) = u(L/n) = 0 \end{cases} \tag{1.42}$$

by odd reflection on each of the subsequent subintervals of $(0, L)$. Figure 1.1 shows the first 4 n-node solutions. Now, this time applying Theorem 1.1.2, to the subinterval of length L/n if $u(x)$ is a nodal solution of (1.1) with $n-1$ interior nodes in $(0, L)$. Therefore, a positive solution exists if, and only if,

$$\lambda > \sigma_n \equiv \left(\frac{\pi}{L/n}\right)^2 = \left(\frac{n\pi}{L}\right)^2, \qquad n \geq 1 \tag{1.43}$$

Thus, the following result holds.

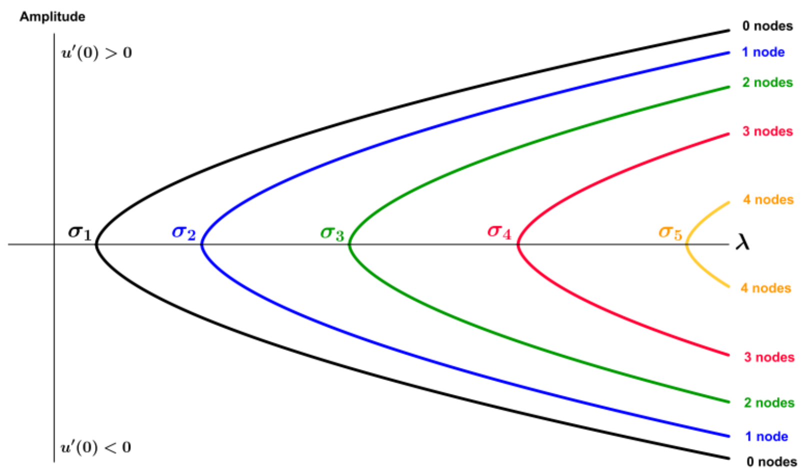


Figure 1.7: Global bifurcation diagram.

Theorem 1.1.3. *For each integer $n \geq 1$ the problem (1.1) possesses a solution with $n-1$ interior nodes in $(0, L)$ and if, and only if $\lambda > \sigma_n$. In such case it has a unique solution such that $u'(0) > 0$, denoted by $u_{n,\lambda}$, whereas $-u_{n,\lambda}$ is the unique solution with $n-1$ interior nodes and such that $u'(0) < 0$.*

Figure 1.7 shows the global bifurcation diagram where the curves

$$\left(\lambda, u_{n,\lambda}(\frac{L}{2n})\right) \qquad \left(\lambda, -u_{n,\lambda}(\frac{L}{2n})\right), \qquad \lambda > \sigma_n, \tag{1.44}$$

Chapter 2

Implicit Function Theorem

There is a handful of problems that arise upon studying the solutions of equations

$$F(u, \lambda) = 0 \tag{2.1}$$

defined in a space, where λ represents a physical parameter of the original system. The conventional approach for finding solutions is to use continuation methods that extend a branch given an initial point (u_0, λ_0) such that $F(u_0, \lambda_0) = 0$. The initial value can be obtained using a numerical approach based on Newton's method. The qualitative properties of the branch of solutions can vary widely depending on the function F being studied. Bifurcation theory mostly studies the points where singularities arise in the set of solutions.

The Newton Method is used to calculate the zeros of an arbitrary function $F(x)$.

$$x_{n+1} := x_n - DF(u_n)^{-1}F(x_n) \qquad n = 0, 1, 2, \ldots \tag{2.2}$$

Under certain conditions on F, the series is convergent, and the convergence rate is quadratic. The Implicit Function Theorem intuitively proves the existence of a local continuation for a neighborhood of (u_0, λ_0) where $F(u_0, \lambda_0) = 0$. Let us use a sufficiently regular function F such that (2.1) can be written as follows

$$0 = D_uF(u_0, \lambda_0)(u - u_0) + D_\lambda F(u_0, \lambda_0)(\lambda - \lambda_0) + o(u - u_0, \lambda - \lambda_0) \qquad \text{as } (u, \lambda) \to (u_0, \lambda_0)$$

Assuming that F is linear, the remainder in the Taylor series vanishes. Moreover, if the differential operator $D_uF(u_0, \lambda_0)$ is invertible, we can solve for u,

$$u = u_0 - D_uF(u_0, \lambda_0)^{-1}D_\lambda F(u_0, \lambda_0)(\lambda - \lambda_0)$$

where $u \equiv u(\lambda)$. A possible way to calculate $u(\lambda)$ is by using an iterative method based on Newton's method (2.2) as follows

$$D_uF(u_0, \lambda)(u_{n+1} - u_n) = -F(u_n, \lambda), \qquad n = 0, 1, 2, \ldots \tag{2.3}$$

Assuming the sequence converges, the limit u_n as $n \to \infty$ satisfies that $F(u, \lambda) = 0$. We notice that (2.3) can be equivalently written as follows

$$u_{n+1} = u_n - D_u F(u_0, \lambda_0)^{-1} F(u_n, \lambda), \qquad n = 0, 1, 2, \dots \tag{2.4}$$

And, taking the limit

$$u(\lambda) = u(\lambda) - D_u F(u_0, \lambda_0)^{-1} F(u, \lambda) \tag{2.5}$$

Defining the function

$$\Phi(u, \lambda) := u - D_u F(u_0, \lambda_0)^{-1} F(u, \lambda),$$

the fixed-points of Φ are the zeros of F. The existence of fixed points is owed to the Contraction Mapping theorem (2.0.1). Let us note that

$$D_u \Phi(u, \lambda) = I - D_u F(u_0, \lambda_0)^{-1} D_u F(u, \lambda) = 0, \tag{2.6}$$

therefore, Φ is a contraction at u uniformly when λ sufficiently closed to λ_0. This pseudo-proof gives intuition into the underlying trappings of the theorem, however, it is not rigorous and we will see in this Chapter the complete statement and proof. Before stating the theorem, we will introduce a few results that will be used in the proof.

Theorem 2.0.1 (Contraction Mapping Theorem)**.** *Also known as the Banach fixed-point theorem, states that given a (X, d) a non-empty complete metric space with a contraction mapping, i.e., Lipschitz constant strictly less than one, $T : X \to X$. Then, T admits a unique fixed-point x^* in X. Furthermore, x^* is the limit of the sequence given by $x_{n+1} := T(x_n)$ where $x_0 \in X$ is an arbitrary element.*

Lemma 2.0.2. *Let E be a Banach space and $L \in \mathcal{L}(E, E)$ such that $\|L\|_{\mathcal{L}(E,E)} \leq \theta < 1$. Then, $I_E - L$ is invertible, and $(I_E - L)^{-1} = \sum_{n=0}^{\infty} L^n$. Furthermore, $\|(I_E - L)^{-1}\|_{\mathcal{L}(E,E)} \leq \frac{1}{1-\theta}$.*

Proof. For every $n \geq 1$, let us define the partial n-sum:

$$S_n := \sum_{j=0}^{n} L^j \tag{2.7}$$

Then for every $n, p \geq 1$ follows

$$\|S_{n+p} - S_n\|_{\mathcal{L}(E,E)} = \left\| \sum_{j=n+1}^{n+p} L^j \right\|_{\mathcal{L}(E,E)} \leq \sum_{j=n+1}^{n+p} \|L\|^j_{\mathcal{L}(E,E)} \leq \sum_{j=n+1}^{n+p} \theta^j = \frac{\theta^{n+1}}{1-\theta} \overset{n\to\infty}{\longrightarrow} 0 \tag{2.8}$$

Thus, $\{S_n\}_{n \geq 1}$ is a Cauchy sequence on $\mathcal{L}(E, E)$, and using that the space is $\mathcal{L}$-complete. Then, the limit exists and is well-defined $S := \lim_{n\to\infty} S_n \in \mathcal{L}(E, E)$.
On the other hand, for every $n \geq 1$ it follows

$$(I_E - L)S_n = \sum_{j=0}^{n} L^j - \sum_{j=0}^{n} L^j L = S_n(I_E - L) = I_E - L^{n+1} \tag{2.9}$$

Moreover, due to the inequality, $\|L^{n+1}\|_{\mathcal{L}(E,E)} \leq \theta^{n+1}$, therefore, $\lim_{n\to\infty} L^{n+1} = 0$. Taking the limit in (2.9)

$$(I_E - L)S = S(I_E - L) = I_E \tag{2.10}$$

In other words, $S = \sum_{n=0}^{\infty} L^n$ is the inverse operator of $(I_E - L)$. Lastly, due to $\|S_n\|_{\mathcal{L}(E,E)} \leq \sum_{j=0}^{n} \theta^j \leq \frac{1}{1-\theta}$, taking the limit $n \to \infty$,

$$\|(I_E - L)^{-1}\|_{\mathcal{L}(E,E)} \leq \frac{1}{1-\theta} \tag{2.11}$$

□

Theorem 2.0.3 (Implicit Function Theorem)**.** *Let E, F, Λ be Banach spaces, $\lambda_0 \in \Lambda, u_0 \in E, \Omega \subset \Lambda \times E$ be an open set such that $(\lambda_0, u_0) \in \Omega$ and $\mathfrak{F} : \Omega \to F$ be a $\mathcal{C}^r(\Omega; F)$ operator for some $r \leq 1$, such that*

1. $\mathfrak{F}(\lambda_0, u_0) = 0$
2. $D_u\mathfrak{F}(\lambda_0, u_0) \in \mathcal{L}(E, F)$ *is topological isomorphism (bijection that is continuous and whose inverse is also continuous)*

Then, there exists $\rho_1 = \rho(\lambda) > 0, \rho_2 = \rho(u_0) > 0$ and a $\mathcal{C}^r$ application

$$\begin{aligned} u : B_{\rho_1}(\lambda_0) &\to B_{\rho_2}(u_0) \\ \lambda &\mapsto u(\lambda) \end{aligned}$$

such that

T1. $u(\lambda_0) = u_0$

T2. $\mathfrak{F}(\lambda, u(\lambda)) = 0$ *for each* $\lambda \in B_{\rho_1}(\lambda_0)$

T3. If $(\lambda, u) \in B_{\rho_1}(\lambda_0) \times B_{\rho_2}(u_0)$ *and* $\mathfrak{F}(\lambda, u) = 0$*, then* $u = u(\lambda)$*.*

Furthermore, the differentials of order less than r of $u(\lambda)$ for each $\lambda \in B_{\rho_1}(\lambda_0)$ can be obtained by implicit differentiation of $\mathfrak{F}(\lambda, u(\lambda)) = 0$.

In particular,

$$D_\lambda u(\lambda_0) = -[D_u\mathfrak{F}(\lambda_0, u_0)]^{-1} D_\lambda \mathfrak{F}(\lambda_0, u_0)$$

Proof. The proof is based on the classic Newton's method for calculating the zeros of a differentiable function. Let us fix $\lambda \sim \lambda_0$. Due to $\mathfrak{F}(\lambda_0, u_0) = 0$, $\mathfrak{F}(\lambda, u_0)$ must be, likewise, close to zero, and henceforth, $u = u_0$ should be a good initialization for the iterative scheme provided by Newton for finding zeros of $\mathfrak{F}(\lambda, \cdot)$ close to u_0. The iterations of the Newton method described in (2.4) are given by

$$u_n = u_{n-1} - [D_u\mathfrak{F}(\lambda_0, u_0)]^{-1}\mathfrak{F}(\lambda, u_{n-1}), \qquad n \geq 1 \tag{2.12}$$

Assuming the iteration converges to $u(\lambda) := \lim_{n\to\infty} u_n$, then

$$u(\lambda) = u(\lambda) - [D_u\mathfrak{F}(\lambda_0, u_0)]^{-1}\mathfrak{F}(\lambda, u(\lambda)) \tag{2.13}$$

which implies that $\mathfrak{F}(\lambda, u(\lambda)) = 0$ because $[D_u\mathfrak{F}(\lambda_0, u_0)]^{-1}$ is linear and injective. Hence, the zeros of $\mathfrak{F}$ are the fixed points of the operator

$$\begin{aligned} \Phi : \Omega \quad & \to E \\ (\lambda, u) \quad & \mapsto u - [D_u\mathfrak{F}(\lambda_0, u_0)]^{-1}\mathfrak{F}(\lambda, u), \qquad (\lambda, u) \in \Omega \end{aligned}$$

We will use the Contraction Mapping theorem to construct $u(\lambda)$ for $\Phi(\lambda, \cdot)$. Firstly, we note that $\Phi \in \mathcal{C}^r(\Omega; F)$ due to being composition of $\mathcal{C}^r$. Furthermore,

$$D_u\Phi(\lambda, u) = I_E - [D_u\mathfrak{F}(\lambda_0, u_0)]^{-1}D_u\mathfrak{F}(\lambda, u)$$

Hence, $D_u\Phi(\lambda_0, u_0) = I_E - I_E = 0$. Due to continuity, for every $\theta \in (0, 1)$, there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\|D_u\Phi(\lambda, u)\|_{\mathcal{L}(E,E)} \le \theta \qquad \forall(\lambda, u) \in \overline{B}_{\rho_1}(\lambda_0) \times \overline{B}_{\rho_2}(u_0) \subset \Omega \tag{2.14}$$

We will prove that ρ_1, ρ_2 could be chosen sufficiently small so that $\Phi(\lambda, \cdot)$ is a uniform contraction at $\lambda \in B_{\rho_1}(\lambda_0)$ such that

$$\Phi(\lambda, u) \in \overline{B}_{\rho_2}(u_0) \qquad \forall u \in \overline{B}_{\rho_2}(u_0) \tag{2.15}$$

Indeed, for every $\lambda \in B_{\rho_1}(\lambda_0)$ and $u, v \in \overline{B}_{\rho_2}(u_0)$,

$$\begin{aligned} \|\Phi(\lambda, u) - \Phi(\lambda, v)\| &= \left\| \int_0^1 \frac{d}{dt}\Phi(\lambda, v + t(u - v))dt \right\| \\ &= \left\| \int_0^1 D_u\Phi(\lambda, v + t(u - v))(u - v)dt \right\| \\ &\le \int_0^1 \|D_u\Phi(\lambda, v + t(u - v))\|_{\mathcal{L}(E,E)}\|u - v\|dt \\ &\le \theta\|u - v\| \end{aligned}$$

Hence, for every $\lambda \in \overline{B}_{\rho_1}(\lambda_0)$, $\Phi(\lambda, \cdot)$ is a contraction with constant θ at $\overline{B}_{\rho_2}(u_0)$. Moreover, owing to $u_0 = \Phi(\lambda_0, u_0)$, whenever $\|\lambda - \lambda_0\| \le \rho_1$ and $\|u - u_0\| \le \rho_2$,

$$\begin{aligned} \|\Phi(\lambda, u) - u_0\| &= \|\Phi(\lambda, u) - \Phi(\lambda_0, u_0)\| \\ &\le \|\Phi(\lambda, u) - \Phi(\lambda, u_0)\| + \|\Phi(\lambda, u_0) - \Phi(\lambda_0, u_0)\| \\ &\le \theta\|u - u_0\| + \|\Phi(\lambda, u_0) - \Phi(\lambda_0, u_0)\| \\ &\le \theta\rho_2 + \|\Phi(\lambda, u_0) - \Phi(\lambda_0, u_0)\| \end{aligned}$$

By continuity of Φ, we can reduce ρ_1 so that $\|\Phi(\lambda, u_0) - \Phi(\lambda_0, u_0)\| \leq (1-\theta)\rho_2$. And, therefore,

$$\|\Phi(\lambda, u) - u_0\| \leq \theta\rho_2 + (1-\theta)\rho_2 = \rho_2$$

which proves (2.15). Using the Contraction Mapping Theorem (2.0.1), for every $\lambda \in \overline{B}_{\rho_1}(\lambda_0)$, there exists a unique $u(\lambda) \in \overline{B}_{\rho_2}(u_0)$ such that $\Phi(\lambda, u(\lambda)) = u(\lambda)$, or equivalently, $\mathfrak{F}(\lambda, u(\lambda)) = 0$. Due to $\Phi(\lambda_0, u_0) = u_0$, there necessarily satisfies $u(\lambda_0) = u_0$, which proves that the application $\lambda \mapsto u(\lambda)$ satisfies T1, T2 and T3. We do still need to prove that the function has the same regularity as $\mathfrak{F}$ and could be obtained by implicit differentiation.

Firstly, let us see that $u(\lambda)$ is locally Lipschitz. For every $\lambda \in B_{\rho_1}(\lambda_0)$ and $h \in \Lambda$ such that $\|h\|_\Lambda$ is sufficiently small,

$$\begin{aligned}\|u(\lambda+h) - u(\lambda)\| &= \|\Phi(\lambda+h, u(\lambda+h)) - \Phi(\lambda, u(\lambda))\| \\ &\leq \|\Phi(\lambda+h, u(\lambda+h)) - \Phi(\lambda+h, u(\lambda))\| + \|\Phi(\lambda+h, u(\lambda)) - \Phi(\lambda, u(\lambda))\| \\ &\leq \theta\|u(\lambda+h) - u(\lambda)\| + \left\|\int_0^1 \frac{d}{dt}\Phi(\lambda+th, u(\lambda))dt\right\|\end{aligned}$$

Hence,

$$\|u(\lambda+h) - u(\lambda)\| \leq \frac{1}{1-\theta}\|h\| \int_0^1 \|D_\lambda\Phi(\lambda+th, u(\lambda))\|dt \tag{2.16}$$

Due to $h \mapsto \|D_\lambda\Phi(\lambda+h, u(\lambda))\|$ being continuous, there exists a constant $C = C(\lambda)$ such that $\|u(\lambda+h) - u(\lambda)\| \leq C(\lambda)\|h\|$ for every $h \in \Lambda$ at a sufficiently small ball centered at 0. Therefore, $u(\lambda)$ is locally Lipschitz at λ, and in particular, is continuous.

We proceed to prove the differentiability. Owing to the construction of $u(\lambda) = \Phi(\lambda, u(\lambda))$ for $\lambda \in \overline{B}_{\rho_1}(\lambda_0)$, in case of being differentiable, it follows

$$Du(\lambda) = D_\lambda\Phi(\lambda, u(\lambda)) + D_u\Phi(\lambda, u(\lambda))Du(\lambda) \tag{2.17}$$

or equivalently,

$$[I_E - D_u\Phi(\lambda, u(\lambda))]Du(\lambda) = D_\lambda\Phi(\lambda, u(\lambda)) \tag{2.18}$$

Henceforth, in case of being differentiable, it should satisfy

$$Du(\lambda) = [I_E - D_u\Phi(\lambda, u(\lambda))]^{-1}D_\lambda\Phi(\lambda, u(\lambda)) \tag{2.19}$$

Note that due to Lemma 2.0.2 and the result obtained in equation 2.14, it follows

$$\|[I_E - D_u\Phi(\lambda, u(\lambda))]^{-1}\|_{\mathcal{L}(E,E)} \leq \frac{1}{1-\theta} \tag{2.20}$$

We will see that operator defined in (2.19) provides $Du(\lambda)$ for each $\lambda \in B_{\rho_1}(\lambda_0)$. And we will prove that

$$\lim_{h\to 0} \frac{u(\lambda+h) - u(\lambda) - [I_E - D_u\Phi(\lambda, u(\lambda))]^{-1}D_\lambda\Phi(\lambda, u(\lambda))h}{\|h\|_\Lambda} = 0 \tag{2.21}$$

Indeed, let us fix $\lambda \in B_{\rho_1}(\lambda_0)$ and $h \in \Lambda$ sufficiently small. We define

$$\Delta_h := \|u(\lambda+h) - u(\lambda) - [I_E - D_u\Phi(\lambda, u(\lambda)]^{-1} D_\lambda \Phi(\lambda, u(\lambda))h\|_E \tag{2.22}$$

Then, due to (2.20),

$$\begin{aligned}
\Delta_h &= \|[I_E - D_u\Phi(\lambda, u(\lambda))]^{-1}\{[I_E - D_u\Phi(\lambda, u(\lambda))][u(\lambda+h) - u(\lambda)] - D_\lambda\Phi(\lambda, u(\lambda))h\}\|_E \\
&\leq \frac{1}{1-\theta}\|u(\lambda+h) - u(\lambda) - D_u\Phi(\lambda, u(\lambda))[u(\lambda+h) - u(\lambda)] - D_\lambda\Phi(\lambda, u(\lambda))h\|_E \\
&= \frac{1}{1-\theta}\|\Phi(\lambda+h, u(\lambda+h)) - \Phi(\lambda, u(\lambda)) - D_u\Phi(\lambda, u(\lambda))[u(\lambda+h) - u(\lambda)] - D_\lambda\Phi(\lambda, u(\lambda))h\|_E \\
&= o(\|h\|_\Lambda + \|u(\lambda+h) - u(\lambda)\|_E)
\end{aligned}$$

due to $u(\lambda+h) = \Phi(\lambda+h, u(\lambda+h))$ and $u(\lambda) = \Phi(\lambda, u(\lambda))$, then, $\Phi \in \mathcal{C}^1$ at $B_{\rho_1}(\lambda_0) \times B_{\rho_2}(u_0)$. Moreover, due to $h \mapsto u(\lambda+h)$ being locally Lipschitz,

$$\Delta_h = o(\|h\|_\Lambda + \|u(\lambda+h) - u(\lambda)\|_E) = o(\|h\|_\Lambda) \tag{2.23}$$

And thus, (2.21) is fulfilled i.e., the application $\lambda \mapsto u(\lambda)$ is differentiable at $B_{\rho_1}(\lambda_0)$ and for every $\lambda \in B_{\rho_1}(\lambda_0)$, and $Du(\lambda)$ is the operator defined in (2.19).

We are now calculating explicitly $Du(\lambda)$ using terms of the operator $\mathfrak{F}$. Using the definition of Φ follows

$$\begin{aligned}
Du(\lambda) &= [I_E - D_u\Phi(\lambda, u(\lambda))]^{-1} D_\lambda\Phi(\lambda, u(\lambda)) \\
&= -\{I_E - I_E + [D_u\mathfrak{F}(\lambda_0, u_0)]^{-1} D_u\mathfrak{F}(\lambda, u(\lambda))\}^{-1}[D_u\mathfrak{F}(\lambda_0, u_0)]^{-1} D_\lambda\mathfrak{F}(\lambda, u(\lambda)) \\
&= -[D_u\mathfrak{F}(\lambda, u(\lambda))]^{-1} D_\lambda\mathfrak{F}(\lambda, u(\lambda))
\end{aligned}$$

Noting that $I_E - D_u\Phi(\lambda, u(\lambda))$ is invertible, then also is $D_u\mathfrak{F}(\lambda, u(\lambda))$ for $\lambda \in B_{\rho_1}(\lambda_0)$. Therefore for every $\lambda \in B_{\rho_1}(\lambda_0)$,

$$D_u\mathfrak{F}(\lambda, u(\lambda))Du(\lambda) + D_\lambda\mathfrak{F}(\lambda, u(\lambda)) = 0 \Leftrightarrow D\mathfrak{F}(\lambda, u(\lambda)) = 0 \tag{2.24}$$

$\square$

Chapter 3

Bifurcation from Simple Eigenvalues. Crandall-Rabinowitz Theorem

Several non-linear problems relevant in practical applications can be expressed as a fixed point equation involving a compact non-linear operator within a real Banach space. In many cases, it is crucial to investigate how the model's behavior changes with variations in a parameter, denoted as λ. In practical applications, λ represents a physical or empirical magnitude of interest, although it could also be a mathematical parameter unrelated to empirical studies. Throughout this chapter, we will assume that $\mathfrak{F} \in \mathcal{C}(\mathbb{R} \times U, V)$, where U, V are real Banach spaces, is of the form

$$\mathfrak{F}(\lambda, u) = \underbrace{\mathfrak{L}(\lambda)u}_{\textit{linear part}} + \underbrace{\mathfrak{R}(\lambda, u)}_{\textit{nonlinear terms}}, \qquad (\lambda, u) \in \mathbb{R} \times U, \tag{3.1}$$

where $\mathfrak{L}$ and $\mathfrak{R}$ must satisfy:

(HL) $\mathfrak{L}(\lambda) \in \mathcal{C}^r(\mathbb{R}, \mathcal{L}(U, V))$ for some $r \geq 2$ and $\mathfrak{L}(\mathbb{R}) \subset \mathrm{Fred}_0(U, V)$.

(HN) $\mathfrak{R} \in \mathcal{C}^r(\mathbb{R} \times U, V)$ satisfies

$$\mathfrak{R}(\lambda, 0) = 0 \quad \text{and} \quad D_u\mathfrak{R}(\lambda, 0) = 0 \text{ for all } \lambda \in \mathbb{R} \tag{3.2}$$

Under these assumptions,

$$\mathfrak{R}(\lambda, u) = \mathfrak{R}(\lambda, 0) + D_u\mathfrak{R}(\lambda, 0)u + o(\|u\|) = o(\|u\|) \text{ as } u \to 0 \tag{3.3}$$

Furthermore, $\mathfrak{F} \in \mathcal{C}^r(\mathbb{R} \times U, V)$,

$$\mathfrak{L}(\lambda) = D_u\mathfrak{F}(\lambda, 0), \quad \lambda \in \mathbb{R} \tag{3.4}$$

and $\mathfrak{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.
In the special case when both $\mathfrak{L}$ and $\mathfrak{R}$ are of class $\mathcal{C}^1$, then

$$\begin{aligned} \mathfrak{L}(\lambda) &= D_u\mathfrak{F}(\lambda, 0) && \text{for all } \lambda \in \mathbb{R} \\ \mathfrak{R}(\lambda, u) &= \mathfrak{F}(\lambda, u) - \mathfrak{L}(\lambda)u && \text{for all } (\lambda, u) \in \mathbb{R} \times U \end{aligned} \tag{3.5}$$

Therefore, $\mathfrak{L}u$ is the *linear* part of $\mathfrak{F}(\lambda, u)$ and $\mathfrak{R}(\lambda, u)$ is the *nonlinear* term of $\mathfrak{F}(\lambda, u)$.

The main goal of bifurcation theory is to get as much information as possible on the local and global structure of the set of zeros of $\mathfrak{F}$ from this simple feature. The most natural strategy to study this problem consists of finding out all possible bifurcation values from $(\lambda, 0)$, which is often referred to as the trivial solution because it is a known solution. Once the bifurcation points from $(\lambda, 0)$ one should study the local and global topological structures of the components bifurcating from them. We will start by defining the concepts that will be used in the following Chapters.

Definition 3.0.1 (Bifurcation point). *Let* $\lambda_0 \in \mathbb{R}$, $(\lambda, 0) \in (\mathbb{R}, U)$ *is said to be a bifurcation point of* $\mathfrak{F}(\lambda, u) = 0$ *from the trivial solution* $(\lambda, 0)$ *if there exists a sequence* $(\lambda_n, u_n) \in \mathfrak{F}^{-1}(0)$, $n \geq 1$, *with* $u_n \neq 0$ *for all* $n \geq 1$ *such that* $\lim_{n\to\infty}(\lambda_n, u_n) = (\lambda_0, 0)$

Definition 3.0.2 (Nonlinear eigenvalue). $\lambda \in \mathbb{R}$ *is said to be a nonlinear eigenvalue of* $\mathfrak{L}$ *if* $(\lambda_0, 0)$ *is bifurcation point of*

$$\mathfrak{L}(\lambda)u + \mathfrak{R}(\lambda, u) = 0 \tag{3.6}$$

for every continuous map $\mathfrak{R} \in \mathcal{C}(\mathbb{R} \times U, U)$ *that is compact, i.e., the operator maps bounded subsets of* $\mathbb{R} \times U$ *to relatively compact subsets of* U.

Theorem 3.0.3 (Open mapping theorem). *Let* U, V *be Banach spaces and* $T \in \mathcal{L}(U, V)$ *be a bijection. Then,* $T^{-1} \in \mathcal{L}(V, U)$. *Thus,* $T \in \mathrm{Iso}(U, V)$.

Definition 3.0.4 (Spectrum of $\mathfrak{L}(\lambda)$). *The spectrum of the family* $\mathfrak{L}(\lambda)$ *is*

$$\Sigma(\mathfrak{L}) := \{\lambda \in \mathbb{R} : \mathfrak{L}(\lambda) \notin \mathrm{Iso}(U)\} = \{\lambda \in \mathbb{R} : \dim N[\mathfrak{L}(\lambda)] \geq 1\} \tag{3.7}$$

Definition 3.0.5 (Fredholm operator). *Let* U, V *be two real Banach spaces and* $L \in \mathcal{L}(U, V)$ *be a linear continuous operator.* L *is said to be a Fredholm operator if*

$$\dim N[L] < +\infty \qquad \operatorname{codim} R[L] < +\infty \tag{3.8}$$

In such case, $R[L]$ *is closed and the index of* L, $\operatorname{ind} L$, *is defined by*

$$\operatorname{ind} L := \dim N[L] - \operatorname{codim} R[L] \tag{3.9}$$

The set of Fredholm operators between U *and* V *is denoted as* $\mathrm{Fred}(U, V)$ *and the operators of index zero is* $\mathrm{Fred}_0(U, V)$.

Definition 3.0.6. *The set of nontrivial solutions of* $\mathfrak{F}(\lambda, u) = 0$ *is defined as*

$$\mathcal{S} := \{(\lambda, u) \in \mathfrak{F}^{-1}(0) : u \neq 0\} \cup \{(\lambda, 0) : \lambda \in \Sigma(\mathfrak{L})\} \tag{3.10}$$

It consists of the solution pairs (λ, u) *with* $u \neq 0$ *plus all possible bifurcation points from* $(\lambda, 0)$.

Let us now define the operator $\mathfrak{L}_j$ for $0 \geq j \geq r$ and $\lambda_0 \in \Sigma(\mathfrak{L})$,

$$\begin{aligned} \mathfrak{L}_j : U &\longrightarrow V \\ u &\longmapsto \mathfrak{L}_j(u) := \frac{1}{j!}\frac{d^j \mathfrak{L}(\lambda_0)}{d\lambda^j}(u) \end{aligned} \tag{3.11}$$

Lemma 3.0.7. *If $(\lambda_0, 0)$ is a bifurcation point of $\mathfrak{F} = 0$ from $u = 0$, then* $\dim N[\mathfrak{L}_0] = n \geq 1$.

Proof. It is an immediate application of the Implicit Function theorem (2.0.3). Indeed, if $\mathfrak{L}_0 \in \mathrm{Iso}(U, V)$, then, the Implicit Function Theorem establishes that $\mathfrak{F}^{-1}$ consits of a curve of class $\mathcal{C}^r$ in a neighborhood of $(\lambda, u) = (\lambda_0, 0)$. Thus, since $\mathfrak{F}(\lambda, 0) = 0$, the equation cannot admit any non-trivial solution in a neighborhood of $(\lambda_0, 0)$. □

Therefore, $\mathfrak{L}_0$ cannot be an isomorphism if $(\lambda_0, 0)$ is a bifurcation point. The next theorem studies the case When $N[\mathfrak{L}_0]$ is one-dimensional, i.e., $N[\mathfrak{L}_0] = \mathrm{span}[\varphi_0]$ for some $\varphi_0 \in U \backslash \{0\}$.

Theorem 3.0.8 (Crandall & Rabinowitz)**.** *Suppose that $\mathfrak{F}(\lambda, u)$ is of class $\mathcal{C}^r$, with $r \geq 2$, and*

$$N[\mathfrak{L}_0] = \mathrm{span}[\varphi_0], \qquad \mathfrak{L}_1(N[\mathfrak{L}_0]) \oplus R[\mathfrak{L}_0] = V \tag{3.12}$$

Let Y be a closed subspace of U such that

$$N[\mathfrak{L}_0] \oplus Y = U \tag{3.13}$$

Then, there exists $\epsilon > 0$ and two maps of class $\mathcal{C}^{r-1}$

$$\lambda : (-\epsilon, \epsilon) \to \mathbb{R}, \qquad y : (-\epsilon, \epsilon) \to Y$$

such that $\lambda(0) = \lambda_0, y(0) = 0$, and, for every $s \in (-\epsilon, \epsilon)$,

$$\mathfrak{F}(\lambda(s), u(s)) = 0$$

where $u(s) := s(\varphi_0 + y(s))$. Moreover, there exists $\rho > 0$ such that if

$$\mathfrak{F}(\lambda, u) = 0 \quad \text{and} \quad (\lambda, u) \in B_{\rho, \mathbb{R}\times U}(\lambda_0, 0),$$

then, either $u = 0$ or $(\lambda, u) = (\lambda(s), u(s))$ for some $s \in (-\epsilon, \epsilon)$. Furthermore, if $\mathfrak{F}$ is of class $\mathcal{C}^\infty$, or real analytic, then so are $\lambda(s)$ and $u(s)$.

Proof. Let the operator $\mathfrak{G} : \mathbb{R}^2 \times Y \to V$ be defined by

$$\mathfrak{G}(s, \lambda, y) := \begin{cases} s^{-1}\mathfrak{F}(\lambda, s(\varphi_0 + y)) & \text{if } s \neq 0 \\ D_u\mathfrak{F}(\lambda, 0)(\varphi_0 + y) & \text{if } s = 0 \end{cases} \tag{3.14}$$

for $(s, \lambda, y) \simeq (0, \lambda_0, 0)$. Note that

$$\lim_{s \to 0} \frac{\mathfrak{F}(\lambda, s(\varphi_0 + y))}{s} = \lim_{s \to 0} \frac{\mathfrak{F}(\lambda, s(\varphi_0 + y)) - \mathfrak{F}(\lambda, 0)}{s} = D_u \mathfrak{F}(\lambda, 0)(\varphi_0 + y).$$

Thus, since $\mathfrak{F}$ is of class $\mathcal{C}^r$, the map $\mathfrak{G}$ is of class $\mathcal{C}^{r-1}$. We know that, by definition,

$$\mathfrak{G}(0, \lambda_0, 0) = D_u \mathfrak{F}(\lambda_0, 0)\varphi_0 = \mathfrak{L}(\lambda_0)\varphi_0 = \mathfrak{L}_0 \varphi_0 = 0 \tag{3.15}$$

and, differentiating with respect to (λ, y) yields,

$$\begin{aligned}
[D_{(\lambda,y)}\mathfrak{G}(0, \lambda_0, 0)](\lambda, y) &= \lim_{h \to 0} \frac{\mathfrak{G}(0, \lambda_0 + h\lambda, hy)}{h} \\
&= \lim_{h \to 0} \frac{D_u \mathfrak{F}(\lambda_0 + h\lambda, 0)(\varphi_0 + hy)}{h} \\
&= \lim_{h \to 0} \frac{\mathfrak{L}(\lambda_0 + h\lambda)(\varphi_0 + hy)}{h} \\
&= [\mathfrak{L}(\lambda_0)](y) + \frac{d}{d\lambda}\mathfrak{L}(\lambda_0)(\lambda)\varphi_0 = \mathfrak{L}_0 y + \lambda \mathfrak{L}_1 \varphi_0
\end{aligned}$$

for all $(\lambda, y) \in \mathbb{R} \times Y$. The linear operator

$$\begin{aligned}
\mathbb{R} \times Y &\longrightarrow V \\
(\lambda, y) &\longmapsto \mathfrak{L}_0 y + \lambda \mathfrak{L}_1 \varphi_0
\end{aligned}$$

is continuous. We can prove that this linear operator is a bijection. Firstly, we will show that it is injective. Assume that $\mathfrak{L}_0 y + \lambda \mathfrak{L}_1 \varphi_0$, then $\lambda \mathfrak{L}_1 \varphi_0 \in R[\mathfrak{L}_0]$. However, the transversality condition (3.12) asserts that $\mathfrak{L}_1 \varphi \cap R[\mathfrak{L}_0] = 0$, and, hence, $\lambda = 0$. Therefore, $\mathfrak{L}_0 y = 0$, so $y \in N[\mathfrak{L}_0]$. Due to (3.13), $y \in Y \cap N[\mathfrak{L}_0] = 0$. Let us prove that it is surjective. Let $v \in V$, we will show that v can be expressed as $v = \alpha \mathfrak{L}_1 \varphi_0 + \mathfrak{L}_0 u$ where $\alpha \in \mathbb{R}$ and $u \in U$. Due to equation (3.13), u can be written as $u = y + \beta \varphi_0$ for some $\beta \in \mathbb{R}$. Therefore,

$$\alpha \mathfrak{L}_1 \varphi_0 + \mathfrak{L}_0 u = \alpha \mathfrak{L}_1 \varphi_0 + \mathfrak{L}_0 (y + \beta \varphi_0) = \alpha \mathfrak{L}_1 \varphi_0 + \mathfrak{L}_0 y. \tag{3.16}$$

The transversality condition (3.12) ensures that we can find some $\lambda, \beta \in \mathbb{R}$ such that $v = \alpha \mathfrak{L}_1 \varphi_0 + \mathfrak{L}_0 y$, and thereby, it is surjective. We have indeed proven the hypothesis of the Open Mapping Theorem (Theorem 3.0.3), hence,

$$D_{(\lambda,y)}\mathfrak{G}(0, \lambda_0, 0) \in \mathrm{Iso}(\mathbb{R} \times Y, V).$$

Therefore, the Implicit Function Theorem assures the existence of two unique maps $\mathcal{C}^{r-1}$, $\lambda : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to Y$, defined in an open neighborhood of $0 \in \mathbb{R}$ such that $\lambda(0) = \lambda_0, y(0) = 0$, and

$$\mathfrak{G}(s, \lambda(s), y(s)) = 0 \qquad s \simeq 0.$$

The fact that $(\lambda(s), u(s))$ is a solution curve of $\mathfrak{F}^{-1}(0)$ follows from the definition of $\mathfrak{G}$, i.e., for $s \neq 0$, $0 = \mathfrak{F}(\lambda(s), s(\varphi_0 + y(s))) = \mathfrak{F}(\lambda(s), u(s))$.

Let us prove the uniqueness assertion. Let $\rho > 0$ and a continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$, such that

$$\mathfrak{F}(\lambda, s\varphi_0 + y) = 0, \qquad y \in Y, \qquad (\lambda, s\varphi_0 + y) \in B_{\rho,\mathbb{R}\times U}(\lambda_0, 0) \tag{3.17}$$

imply that

$$\|y\| + |s||\lambda - \lambda_0| \leq |s|g(|s|). \tag{3.18}$$

Indeed, if $s = 0$, equation (3.18) yields $y = 0$, whereas, if $s \neq 0$, then

$$\|s^{-1}y\| + \|\lambda - \lambda_0\| \leq g(|s|) \tag{3.19}$$

and, by the definition of $\mathfrak{G}$, equation (3.17) implies that $\mathfrak{G}(s, \lambda, s^{-1}y) = 0$. On the other hand, for sufficiently small s, (3.19) follows

$$\|s^{-1}y\| + |\lambda - \lambda_0| \simeq 0$$

due to $g(0) = 0$. Implicit Function Theorem implies the uniqueness of the solution at equation $\mathfrak{G}$, therefore,

$$(\lambda, s^{-1}y) = (\lambda(s), y(s))$$

which completes the proof. We will proof subsequently equation (3.18). It follows

$$\begin{aligned} 0 = \;& \mathfrak{F}(\lambda, s\varphi_0 + y) \\ = \;& \mathfrak{F}(\lambda, s\varphi_0 + y) - \mathfrak{F}(\lambda, s\varphi_0) - D_u\mathfrak{F}(\lambda, s\varphi_0)y \\ & + \mathfrak{F}(\lambda, s\varphi_0) - \mathfrak{F}(\lambda, 0) - sD_u\mathfrak{F}(\lambda, 0)\varphi_0 \\ & + s[D_u\mathfrak{F}(\lambda, 0)\varphi_0 - (\lambda - \lambda_0)D_\lambda D_u\mathfrak{F}(\lambda_0, 0)\varphi_0] \\ & + D_u\mathfrak{F}(\lambda, s\varphi_0)y - D_u\mathfrak{F}(\lambda_0, 0)y \\ & + D_u\mathfrak{F}(\lambda_0, 0)y + s(\lambda - \lambda_0)D_\lambda D_u\mathfrak{F}(\lambda_0, 0)\varphi \end{aligned} \tag{3.20}$$

As $\mathfrak{F}$ is of class $\mathcal{C}^2$, there exists a continuous function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(0) = 0$ and

$$\begin{aligned} \|\mathfrak{F}(\lambda, s\varphi_0 + y) - \mathfrak{F}(\lambda, s\varphi) - D_u\mathfrak{F}(\lambda, s\varphi_0)y\| &\leq \|y\|h(\|y\|), \\ \|\mathfrak{F}(\lambda, s\varphi_0) - \mathfrak{F}(\lambda, 0) - sD_u\mathfrak{F}(\lambda, 0)\varphi_0\| &\leq |s|h(|s|) \\ \|D_u\mathfrak{F}(\lambda, 0)\varphi_0 - (\lambda - \lambda_0)D_\lambda D_u\mathfrak{F}(\lambda_0, 0)\varphi_0\| &\leq |\lambda - \lambda_0|h(|\lambda - \lambda_0|) \end{aligned} \tag{3.21}$$

Thus, taking into account that

$$D_u\mathfrak{F}(\lambda_0, 0) = \mathfrak{L}_0 \quad \text{ and } \quad D_\lambda D_u\mathfrak{F}(\lambda_0, 0) = \mathfrak{L}_1$$

it follows from (3.20) and (3.21) that

$$\begin{aligned} \|\mathfrak{L}_0 y + s(\lambda - \lambda_0)\mathfrak{L}_1\varphi_0\| \leq \;& \|y\|h(\|y\|) \\ & + |s|h(|s|) + |s||\lambda - \lambda_0|h(|\lambda - \lambda_0|) \\ & + \|D_u\mathfrak{F}(\lambda, s\varphi_0) - D_u\mathfrak{F}(\lambda_0, 0)\| \, \|y\|. \end{aligned} \tag{3.22}$$

We already know that the operator

$$\begin{aligned}\mathbb{R}\times Y &\longrightarrow V\\ (\lambda,y)&\longmapsto \mathfrak{L}_0 y+\lambda\mathfrak{L}_1\varphi_0\end{aligned}$$

is an isomorphism. Thus, there is a constant $C>0$, such that

$$\|\mathfrak{L}_0 y+s(\lambda-\lambda_0)\mathfrak{L}_1\varphi_0\|\geq C(\|y\|+|s||\lambda-\lambda_0|)$$

for all $\lambda, s\in\mathbb{R}$ and $y\in Y$. Hence, it follows from (3.22) that

$$\begin{aligned}C(\|y\|+s|s||\lambda-\lambda_0|)\leq \|y\|h(\|y\|)\\ +|s|h(|s|)+|s||\lambda-\lambda_0|h(|\lambda-\lambda_0|)\\ +\|D_u\mathfrak{F}(\lambda,s\varphi_0)-D_u\mathfrak{F}(\lambda_0,0)\|\,\|y\|.\end{aligned}\tag{3.23}$$

Choosing $\rho>0$ so that $(\lambda,s\varphi_0+y)\in B_{\rho,\mathbb{R}\times U}(\lambda_0,0)$ implies

$$h(\|y\|)\leq C/4,\quad h(|\lambda-\lambda_0|)\leq C/2,\quad \|D_u\mathfrak{F}(\lambda,s\varphi_0)-D_u\mathfrak{F}(\lambda_0,0)\|\leq C/4$$

Then, (3.23) provides us with (3.19) for the choice

$$g(|s|)=\frac{2}{C}h(|s|)$$

□

Chapter 4

Application

We will study the general case of the equation presented in Chapter 1.

$$\begin{cases} -u'' = \lambda u - a(x)u^3 & \text{in} \quad (0, L), \\ u(0) = u(L) = 0 \end{cases} \tag{4.1}$$

where $a \in \mathcal{C}[0, L]$ satisfies $a > 0$ and $\lambda \in \mathbb{R}$ is the bifurcation parameter. Firstly, we will solve the eigenvalue problem $-u'' = f$.

Lemma 4.0.1. *For every $f \in \mathcal{C}[0, L]$, the function*

$$u(x) = \int_0^x (s - x) f(s) ds - \frac{x}{L} \int_0^L (s - L) f(s) dx \tag{4.2}$$

is the unique solution of the linear boundary value problem

$$\begin{cases} -u'' = f & \text{in } [0, L], \\ u(0) = u(L) = 0 \end{cases} \tag{4.3}$$

Proof. Differentiating $u(x)$ with respect to x,

$$\begin{aligned} u'(x) &= xf(x) - \int_0^x f(s) ds - xf(x) - \frac{1}{L} \int_0^L (s - L) f(s) ds \\ &= -\int_0^x f(s) ds - \frac{1}{L} \int_0^L (s - L) f(s) ds \end{aligned}$$

Hence,

$$u''(x) = -f(x) \quad \text{for all } x \in [0, L]$$

Assessing the boundary conditions, by (4.2), it follows that $u(0) = u(L) = 0$. Thus, (4.2) solves (4.3). Actually, it is the unique function of class $\mathcal{C}^2[0, L]$ solving (4.3). Indeed, assuming that two solutions, u, v solves (4.3), then the difference $h := u - v$ satisfies

$$\begin{cases} -h'' = 0 & \text{in } [0, L], \\ h(0) = h(L) = 0 \end{cases} \tag{4.4}$$

So, $h(x) = Ax + B$ where $A, B \in \mathbb{R}$. However, due to the boundary conditions, $h \equiv 0$ and thus, $u \equiv v$ are unique. $\square$

We define the linear operator introduced in the previous Lemma in (4.2) that maps the space of continuous functions to solutions of $u'' = -f$,

$$\mathcal{K} : \mathcal{C}[0, L] \to \mathcal{C}^2[0, L]$$
$$f(x) \mapsto \mathcal{K}f(x) := \int_0^x (s - x)f(s)ds - \frac{x}{L}\int_0^L (s - L)f(s)ds, \quad x \in [0, L],$$

Here, $\mathcal{C}[0, L]$ stands for the real Banach space of continuous functions with the maximum norm. And, $\mathcal{C}^n[0, L]$ stands for the real Banach space of the real functions u of class $\mathcal{C}^n$ endowed with the norm

$$\|u\|_{\mathcal{C}^n[0,L]} := \|u\|_\infty + \sum_{i=1}^{n} \|u^{i)}\|_\infty$$

Subsequently, we define $\mathcal{C}_0^n[0, L]$ that stands for the closed subspace of $\mathcal{C}^n[0, L]$ consisting of all the functions $u \in \mathcal{C}^n[0, L]$ such that $u(0) = u(L) = 0$.

According to Lemma 4.0.1, $\mathcal{K}$ is the resolvent operator of the linear boundary value problem (4.3), which explains why we denote it by $\mathcal{K} = (-D^2)^{-1} = (-\Delta)^{-1}$.

Lemma 4.0.2. *The integral operator $\mathcal{K} : \mathcal{C}[0, L] \to \mathcal{C}_0^2[0, L]$ is a linear continuous operator.*

Proof. $\mathcal{K}$ is a trivially linear operator by definition. To prove the continuity, we will prove that is bounded. Firstly, let $u := \mathcal{K}f$, and derivating once,

$$u'(x) = -\int_0^x f(s)ds - \frac{1}{L}\int_0^L (s - L)f(s)ds \quad \text{for all } x \in [0, L],$$

and $u''(x) = -f(x)$ for all $x \in [0, L]$. Thus,

$$\begin{aligned}
\|u\|_\infty &\leq 2L^2\|f\|_\infty + L^2\|f\|_\infty = 3L^2\|f\|_\infty \\
\|u'\|_\infty &\leq L\|f\|_\infty + L\|f\|_\infty = 2L\|f\|_\infty \\
\|u''\|_\infty &\leq \|f\|_\infty
\end{aligned}$$

Therefore, for every $f \in \mathcal{C}[0, L]$,

$$\|\mathcal{K}f\|_{\mathcal{C}^2[0,L]} \leq (1 + 2L + 3L^2)\|f\|_\infty \tag{4.5}$$

Thus, $\mathcal{K}$ is bounded, and then continuous. $\square$

Lemma 4.0.3. *The following injection*

$$j : \mathcal{C}_0^2[0, L] \hookrightarrow \mathcal{C}_0^1[0, L] \tag{4.6}$$

is a linear compact operator.

Proof. We will show that j is a compact operator by proving that for every bounded sequence, $\{u_n\}_{n\geq 1}$ in $\mathcal{C}^2[0,L]$, then, the sequence $\{j(u_n)\}_n$ contains a converging subsequence. Firstly, owing to u_n being a bounded sequence, there exists a constant $C>0$ such that

$$\|u_n\|_{\mathcal{C}^2} \equiv \|u_n\|_\infty + \|u_n'\|_\infty + \|u_n''\|_\infty \leq C \qquad \text{for all } n \geq 1$$

Since $\|u_n''\|_\infty \leq C$, we find that, for every $x, y \in [0,L]$,

$$|u_n'(x) - u_n'(y)| = |u_n''(\psi_n)||x-y| \leq C|x-y|$$

and hence, the sequence $\{u_n'\}_{n\geq 1}$ is bounded and equi-continuous in $\mathcal{C}[0,L]$. Asoli-Arzelá Theorem guarantees the existence of a function $v \in \mathcal{C}[0,L]$ such that a subsequence, relabeled by n, satisfies

$$\lim_{n\to\infty} \|u_n' - v\|_\infty = 0 \tag{4.7}$$

Using the same argument, since $\|u_n'\|_\infty \leq C$, for every $x, y \in [0,L]$,

$$|u_n(x) - u_n(y)| = |u_n'(z_n)||x-y| \leq C|x-y| \tag{4.8}$$

and hence there exists another $u \in \mathcal{C}[0,L]$ such that

$$\lim_{n\to\infty} \|u_n - u\|_\infty = 0 \tag{4.9}$$

for some relabeled subsequence. On the other hand,

$$u_n(x) = u_n(0) + \int_0^x u_n'(s)ds \qquad \text{for all } n \geq 1, x \in [0,L], \tag{4.10}$$

and taking the limit $n \to \infty$ yields

$$u(x) = u(0) + \int_0^x v(s)ds \tag{4.11}$$

Therefore, $u \in \mathcal{C}^1[0,L]$ and $u' = v$. Therefore,

$$\lim_{n\to\infty} \|u_n - u\|_{\mathcal{C}^1[0,L]} = \lim_{n\to\infty} (\|u_n - u\|_\infty + \|u_n' - v\|_\infty) = 0 \tag{4.12}$$

Then, the injection j is compact. □

As a consequence of Lemma 4.0.2 and 4.0.3, the overloaded operator,

$$\mathcal{K} := j\mathcal{K}|_{\mathcal{C}_0^1[0,L]} : \mathcal{C}_0^1[0,L] \to \mathcal{C}_0^1[0,L], \tag{4.13}$$

is a linear compact operator. It allows us to express the nonlinear problem (4.1) as a fixed point equation in a Real Banach space. Indeed, u is a solution of (4.1) if, and only if,

$$u = \mathcal{K}(\lambda u - au^3). \tag{4.14}$$

Indeed, the nonlinear operator

$$\begin{aligned}\mathfrak{F} : \mathbb{R} \times \mathcal{C}_0^1[0,L] &\to \mathcal{C}_0^1[0,L] \\ (\lambda, u) &\mapsto \mathfrak{F}(\lambda, u) := u - \mathcal{K}(\lambda u - au^3)\end{aligned} \tag{4.15}$$

satisfies the requirement of the Crandall-Rabinowitz Theorem when setting

$$\begin{aligned}\mathfrak{L}(\lambda)u &:= u - \lambda \mathcal{K} u, \\ \mathfrak{R}(\lambda, u) &:= -\mathcal{K}(au^3)\end{aligned} \tag{4.16}$$

By construction, $\mathfrak{F}^{-1}(0)$ provides with the set of solution of (4.1). Since, $\mathfrak{L}(\lambda)$ is a compact perturbation of the identity map in $\mathcal{C}_0^1[0,L]$, by the Fredholm Alternative Theorem [Brezis, 2007], $\mathfrak{L}(\lambda)$ is a Fredholm operator of index zero, which proves the condition (HL) defined in Chapter 3. Let us prove the condition (HN). Firstly, $\mathfrak{R}(\lambda, 0) = -\mathcal{K}(0) = 0$. Secondly, we will prove that the Fréchet derivative of $\mathfrak{R}$ at $(\lambda, 0)$ is 0 using the Gâteaux derivative

$$D_u \mathfrak{R}(\lambda, 0) = \lim_{r \to 0} \frac{\mathfrak{R}(\lambda, ru)) - \cancel{\mathfrak{R}(\lambda, 0)}}{r} = \lim_{r \to 0} \frac{-\mathcal{K}(ar^3u^3)}{r} = \lim_{r \to 0} -ar^2 \mathcal{K}(u^3) = 0 \tag{4.17}$$

Therefore, assumption (HN) and (HL) holds. We will prove hereafter, that we can use the Crandall-Rabinowitz Theorem.

By definition, $\Sigma(\mathfrak{L})$ consists of the set of $\lambda \in \mathbb{R}$ such that there exists $u \in \mathcal{C}_0^1[0,L]$, $u \neq 0$ and

$$0 = u - \lambda \mathcal{K} u. \tag{4.18}$$

Differentiating twice with respect to x,

$$\begin{cases} -u'' = \lambda u & \text{in } [0, L], \\ u(0) = u(L) = 0. \end{cases} \tag{4.19}$$

Thus, $\Sigma(\mathfrak{L})$ consists of the real eigenvalues of (4.19). Assuming that (4.19) admits a solution $u \neq 0$, then, multiplying the differential equation by u yields

$$\lambda u^2 = -uu'' = -(uu')' + (u')^2. \tag{4.20}$$

And integrating in $[0, L]$,

$$\lambda \int_0^L u^2 = -\int_0^L (uu')' + \int_0^L (u')^2 = \int_0^L (u')^2 \geq 0. \tag{4.21}$$

Hence, $\lambda \geq 0$. However, if $\lambda = 0$, then $u(x) = Ax + B$ and necessarily $A, B = 0$ due to the boundary conditions. Therefore, $\lambda > 0$ if $\lambda \in \Sigma(\mathfrak{L})$. The general solution of $-u'' = \lambda u$ is given by

$$u(x) = A \sin(\sqrt{\lambda} x) + B \cos(\sqrt{\lambda} x) \tag{4.22}$$

Owing to $0 = u(0) = B$, we can choose $u(x) = \sin(\sqrt{\lambda}x)$. At $x = L$, $\sin(\sqrt{\lambda}L) = 0 \Leftrightarrow \sqrt{\lambda}L = n\pi$ for some integer $n \geq 1$. Therefore,

$$\Sigma(\mathfrak{L}) = \left\{\sigma_n \equiv \left(\frac{n\pi}{L}\right)^2 : n \geq 1\right\}, \tag{4.23}$$

which reaffirms the result obtained in the first Chapter. Now, in order to apply the Crandall-Rabinowith Theorem at σ_n,

$$N[\mathfrak{L}(\sigma_n)] = \text{span}[\phi_n], \qquad \text{where } \phi_n(x) = \sin\frac{n\pi x}{L}, x \in [0, L]. \tag{4.24}$$

Let $(\lambda_0, \varphi_0) := (\sigma_n, \phi_n)$. And, then,

$$\mathfrak{L}_0 \equiv \mathfrak{L}(\lambda_0) = I - \sigma_n\mathcal{K}, \qquad \mathfrak{L}_1 \equiv \mathfrak{L}'(\lambda_0) = -\mathcal{K}. \tag{4.25}$$

Therefore, $N[\mathfrak{L}_0] = \text{span}[\varphi_0]$. Let us see that the transversality condition holds,

$$\mathfrak{L}_1(N[\mathfrak{L}_0]) \oplus R[\mathfrak{L}_0] = \mathcal{C}_0^1[0, L] \tag{4.26}$$

holds, on the contrary, if $\mathfrak{L}_1\varphi_0 \in R[\mathfrak{L}_0]$. Then, there exists $u \in \mathcal{C}_0^1[0, L]$ such that

$$\mathfrak{L}_1\varphi_0 = -\mathcal{K}\varphi_0 = \mathfrak{L}_0 u = u - \sigma_n\mathcal{K}u \tag{4.27}$$

And differentiating twice with respect to x yields,

$$-\varphi_0 = -u'' - \sigma_n u \tag{4.28}$$

Multiplying by φ_0, integrating in $(0, L)$ and integrating by parts the right hand side yields

$$-\int_0^L \varphi_0^2 = \int_0^L [(-u'' - \sigma_n u)\varphi_0] = \int_0^L [(-\varphi_0'' - \sigma_n\varphi_0)u] = 0 \tag{4.29}$$

And therefore, $\varphi_0 = 0$, which is impossible. Let Y be a closed subset of U such that $U = Y \oplus N[\mathfrak{L}_0]$. We define $Y := N[\mathfrak{L}_0]^\perp = \text{span}[\varphi_0]^\perp = \left\{w \in \mathcal{C}_0^1[0, L] : \int_0^L w(x)\varphi_0(x)dx = 0\right\}$. It can be proven that for this choice of Y, it is a closed subset of U. Indeed, let $w_n \to w$ where $(w_n)_n \in Y$, then $0 = \lim_{n\to\infty}\int_0^L w_n(x)\varphi_0(x)dx = \int_0^L \lim_{n\to\infty} w_n\varphi_0(x)$. Thus, the next result holds due to the Crandall-Rabinowitz Theorem.

Theorem 4.0.4. *For every integer $n \geq 1$, let $(\lambda_0, \varphi_0) := (\sigma_n, \phi_n)$ and consider the closed subspace of $\mathcal{C}_0^1[0, L]$*

$$Y := \left\{w \in \mathcal{C}_0^1[0, L] : \int_0^L w(x)\varphi_0(x)dx = 0\right\}. \tag{4.30}$$

Then, there exist $\nu > 0$ and two analytic maps

$$\lambda : (-\nu, \nu) \to \mathbb{R}, \qquad y : (-\nu, \nu) \to Y, \tag{4.31}$$

such that

1. *$\lambda(0) = \lambda_0, y(0) = 0$*
2. *$\mathfrak{F}(\lambda(s), s(\varphi_0 + y(s))) = 0$ for all $s \in (-\nu, \nu)$*
3. *And besides the trivial solution $(\lambda, 0)$, the solution $(\lambda(s), s(\varphi_0 + y(s))))$, $|s| < \nu$, are the unique zeroes of $\mathfrak{F}$ in a neighborhood of $(\lambda_0, 0)$ in $\mathbb{R} \times \mathcal{C}_0^1[0, L]$*

The nodal behavior of $\varphi_0 = \phi_n$ is the same as the function $u(s) = s(\varphi_0 + y(s))$ for sufficiently small $s \neq 0$ because $y \xrightarrow{s \to 0} 0$ and the zeroes of φ_0 are simple. The number of zeros in y is greater or equal to the number of zeroes of φ_0. We can prove that there cannot be more zeroes. The zeroes of $u(s)$ ought to be near the ones of φ_0, we can prove that the derivative of $u(s)$ does not cancel in such interval for values of s sufficiently small and therefore by Rolle's Theorem has the same number of zeroes.

Now, we study the nature of these local bifurcations

$$\lambda_1 := \lambda'(0), \quad \lambda_2 := \frac{\lambda''(0)}{2}, \quad y_1 := y'(0), y_2 := \frac{y''(0)}{2}, \tag{4.32}$$

for sufficiently small values of s,

$$\lambda(s) = \lambda_0 + s\lambda_1 + s^2\lambda_2 + \mathcal{O}(s^3), \quad y(s) = sy_1 + s^2 y_2 + \mathcal{O}(s^3). \tag{4.33}$$

By Theorem 4.0.4,

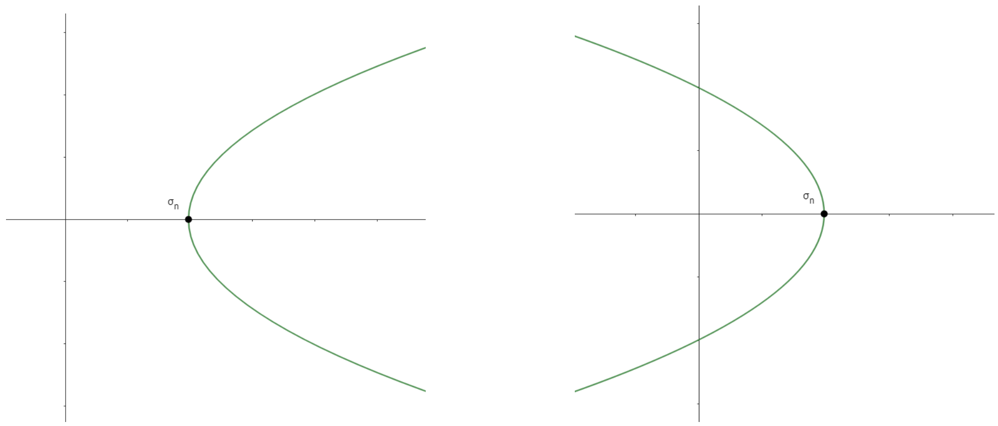
$$\begin{aligned} u'' = u(\lambda - au^2) \Leftrightarrow & -s(\varphi_0 + sy_1 + s^2y_2 + \mathcal{O}(s^3))'' = [\lambda_0 + s\lambda_1 + s^2\lambda_2 + \mathcal{O}(s^3) \\ & - a(x)s^2(\varphi_0 + sy_1 + s^2y_2 + \mathcal{O}(s^2))^2] + (\varphi_0 + sy_1 + s^2y_2 + \mathcal{O}(s^2)) \\ & \Leftrightarrow -s(\varphi_0 + sy_1 + s^2y_2 + \mathcal{O}(s^3))'' = [\lambda_0 + s\lambda_1 + s^2\lambda_2 + \mathcal{O}(s^3) \\ & - a(x)s^2(\varphi_0^2 + \mathcal{O}(s))] + (\varphi_0 + sy_1 + s^2y_2 + \mathcal{O}(s^2)) \end{aligned} \tag{4.34}$$

Particularizing at $s = 0$, yields $-\varphi_0'' = \lambda_0\varphi_0$, which holds true by definition of φ_0. Identifying terms of first and second order in (4.34),

$$\begin{aligned} -y_1'' &= \lambda_0 y_1 + \lambda_1 \varphi_0, \\ -y_2'' &= \lambda_0 y_2 + \lambda_1 y_1 + (\lambda_2 - a\varphi_0^2)\varphi_0. \end{aligned} \tag{4.35}$$

Multiplying first equation by φ_0 and integrating in $(0, L)$ the first equation

$$\begin{aligned} -\int_0^L \varphi_0 y_1'' = \lambda_0 \int_0^L y_1\varphi_0 + \lambda_1 \int_0^L \varphi_0^2 \Leftrightarrow -\int_0^L (\varphi_0\lambda_0 + \varphi_0'')y_1 = \lambda_1 \int_0^L \varphi_0^2 \\ \Leftrightarrow 0 = \lambda_1 \int_0^L \varphi_0^2 \Leftrightarrow \lambda_1 = 0 \end{aligned} \tag{4.36}$$



(a) (b)

Figure 4.1: Local bifurcation diagram when $a > 0$ and $a < 0$, respectively, for every bifurcation point σ_n.

Substituting in the second equation the value in the second equation of (4.34), multiplying the resulting identity by φ_0 and integrating in $(0, L)$ yields

$$-\int_0^L y_2''\varphi_0 = \lambda_0 \int_0^L y_2\varphi_0 + \int_0^L (\lambda_2 - a\varphi_0^2)\varphi_0^2 \Leftrightarrow -\int_0^L y_2(\lambda_0\varphi_0 + \varphi_0'') = \int_0^L (\lambda_2 - a\varphi_0^2)\varphi_0^2$$
$$\Leftrightarrow \lambda_2 = \frac{\int_0^L a(x)\varphi_0^4(x)dx}{\int_0^l \varphi_0^2(x)dx} \tag{4.37}$$

Particularizing when $a > 0$, $\lambda_2 > 0$ provides a supercritical bifurcation, the same as when a was assumed to be a positive constant. Conversely, when $a < 0$, $\lambda_2 < 0$ provides the opposite bifurcation (see Figure 4.1).

Chapter 5

Algorithm: Numerical Continuation

In this chapter, we will explore using numerical methods to approximate the bifurcated branch in problems where the Crandall-Rabinowitz theorem holds. The following algorithm is based on the finite-dimensional version of the theorem; therefore, we should express the infinite-dimensional with this in mind.

Let $A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ and $N : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be a $\mathcal{C}^2$ application such that

$$N(\lambda, 0) = 0, \qquad D_u N(\lambda, 0) = 0$$

for $\lambda \simeq \tilde{\lambda}$ where we assume that $\tilde{\lambda} \in \mathbb{R}$ is a simple eigenvalue of A, and hence, Crandall-Rabinowitz theorem (3.0.8) holds and asserts that $\tilde{\lambda}$, is a bifurcation point for the equation

$$Au = \lambda u + N(\lambda, u) \tag{5.1}$$

Indeed, (5.1) can be written in terms of the linear, $\mathfrak{L}$, and non-linear term, $\mathfrak{R}$ for the equation $\mathfrak{F}(\lambda, u) = \mathfrak{L}(\lambda)u + \mathfrak{R}(\lambda, u)$ by setting

$$\begin{aligned} \mathfrak{L}(\lambda)u &:= \lambda u - Au \\ \mathfrak{R}(\lambda, u) &:= N(\lambda, u) \end{aligned} \tag{5.2}$$

The (HL) condition holds because $\mathfrak{L}(\lambda)$ is a finite-dimensional function, and therefore $\mathfrak{L}(\lambda) \in \mathrm{Fred}_0(U, V) = \mathcal{L}(U, V)$. The (HN) condition holds by definition of $N(\lambda, u)$. Using that $\tilde{\lambda}$ is a simple eigenvalue, $N[\mathfrak{L}_0] = \mathrm{span}[\varphi_i]$ where $\varphi \in \mathbb{R}^m$ is the corresponding eigenvector. Lastly, the transversality condition holds because $\mathfrak{L}_1 = Id$, and therefore, $N[\mathfrak{L}_0] \oplus R[\mathfrak{L}_0] = V$ due to the Rank-Nullity Theorem.

The goal of this chapter is to simulate the branch of non-trivial solutions that stems from $(\lambda, u) = (\tilde{\lambda}, 0)$ in the family of problems described in (5.1). We will start using the inverse power method with the linear problem where $N(\lambda, u) \equiv 0$, and then, we will see the general result for non-linear functions.

5.1 Inverse Power Method for Linear Eigenvalue Problems

Assume that $N(\lambda, u) \equiv 0$, then (5.1) is written as follows

$$Au = \lambda u \tag{5.3}$$

The problem can be written equivalently as $(A - \sigma)u = (\lambda - \sigma I)u$ when $\sigma \neq \lambda$. An iterative method is used based on the following scheme

$$(A - \sigma I)w_k = (\lambda_{k-1} - \sigma)u_{k-1} \tag{5.4}$$

$$u_k = \frac{w_k}{|w_k|} \tag{5.5}$$

$$(\lambda_k - \sigma)|w_k| = \lambda_{k-1} - \sigma \tag{5.6}$$

This scheme yields the following algorithm

$$\begin{cases} w_k = (\lambda_{k-1} - \sigma)(A - \sigma I)^{-1} u_{k-1} \\ u_k = w_k / |w_k|^{-1} \\ \lambda_k = \sigma + \frac{(\lambda_{k-1} - \sigma)}{|w_k|} \end{cases} \tag{5.7}$$

Assuming that λ_k and u_k converges, $\lambda_k \xrightarrow{k \to \infty} \lambda$ and $u_k \xrightarrow{k \to \infty} u$ respectively. Then, owing to (5.6), $|w_k| \xrightarrow{k \to \infty} 1$. Therefore, $w_k \xrightarrow{k \to \infty} u$. Taking the limit as $k \to \infty$ of (5.4), we obtain $(A - \sigma)u = (\lambda - \sigma I)u$, and thus, (λ, u) solves (5.3).

5.2 Inverse Power Method for Non-linear Eigenvalue Problems

[Descloux and Rappaz, 1983] proposed an extension method for solving the non-linear equation (5.1). For a σ relatively close to $\tilde{\lambda}$, let us consider the following variation of equation (5.1),

$$(A - \sigma I)u = (\lambda - \sigma)u + N(\lambda, u) \tag{5.8}$$

Firstly, we consider a normalization of the solution

$$v := R^{-1} u \qquad \text{where} \quad R = |u|$$

Using this notation, equation (5.8) can be rewritten as follows

$$(A - \sigma I)v = (\lambda - \sigma)v + R^{-1} N(\lambda, Rv) \tag{5.9}$$

Thereby, the iterative scheme proposed by Descloux and Rappaz involves two steps: firstly, we fix some initial values and constants: $\sigma, R, v_0, \lambda_0$ and for every $k = 1, 2, \dots$

$$\begin{cases} (A - \sigma I)w_k = (\lambda_{k-1} - \sigma)v_{k-1} + R^{-1}N(\lambda_{k-1}, Rv_{k-1}) \\ v_k = |w_k|^{-1}w_k, \\ u_k = Rv_k, \\ \lambda_k = \sigma + |w_k|^{-1}(\lambda_{k-1} - \sigma) \end{cases} \tag{5.10}$$

In this algorithm, w_k is the solution of a linear system of equations and if λ_k and v_k converge as $k \to \infty$ to $\lambda \neq \sigma$ and v respectively, then w_k converges to w and verifies that $|w| = 1$ and $v = w$. Therefore, u_k converges and (λ, u) solves (5.8).

Let us assume that $N[A - \tilde{\lambda}I] = \text{span}[\phi]$ where $|\phi| = 1$. Then, Desclox and Rappaz introduced the following theorem proving the convergence of this sequence under certain conditions.

Theorem 5.2.1 (Descloux, Rappaz, 1983)**.** *In the neighborhood of* $(\tilde{\lambda}, 0) \in \mathbb{R} \times \mathbb{R}^m$*, there exists a unique branch of non-trivial solutions of*

$$Au = \lambda u + N(\lambda, u) \tag{5.11}$$

that can be parametrized $(\lambda(R), u(R))$*, where* $|R|$ *is sufficiently small and* $\lambda(R) = \tilde{\lambda} + O(R)$ *and* $u(R) = R(\phi + O(R))$*,* $|u(R)| = |R|$*.*

Moreover, there exists $\epsilon > 0$ *such that for every* σ *verifying* $0 < |\tilde{\lambda} - \sigma| < \epsilon$ *exist* $R_0 > 0$ *and* $\delta > 0$ *such that, if* $0 < |R| < R_0$ *and* $|\lambda_0 - \lambda(R)| + |v_0 - R^{-1}u(R)| < \delta$*, then* (λ_k, u_k) *given by algorithm (5.10) converge to* $(\lambda(R), u(R))$ *when* $k \to \infty$*.*

Proof. Let the application

$$\begin{aligned} L : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m &\longrightarrow \mathbb{R}^m \\ (R, \lambda, v) &\longmapsto L(R, \lambda, v) := (A - \sigma I)^{-1}[(\lambda - \sigma)v + R^{-1}N(\lambda, Rv)] \end{aligned} \tag{5.12}$$

which is well-defined when σ is sufficiently closed to $\tilde{\lambda}$ and $\sigma \neq \tilde{\lambda}$. Algorithm (5.10) can be rewritten using L as follows

$$\begin{aligned} v_k &= |L(R, \lambda_{k-1}, v_{k-1})|^{-1}L(R, \lambda_{k-1}, v_{k-1}) \\ \lambda_k &= \sigma + |L(R, \lambda_{k-1}, v_{k-1})|^{-1}(\lambda_{k-1} - \sigma) \end{aligned} \tag{5.13}$$

Now, we define

$$\begin{aligned} S : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m &\longrightarrow \mathbb{R} \times \mathbb{R}^m \\ (R, \lambda, v) &\longmapsto (\sigma + |L(R, \lambda, v)|^{-1}(\lambda - \sigma), |L(R, \lambda, v)|^{-1}L(R, \lambda, v)) \end{aligned}$$

The algorithm (5.10) is rewritten in term of S as

$$(\lambda_k, v_k) := S(R, \lambda_{k-1}, v_{k-1}) \tag{5.14}$$

Using that $\tilde{\lambda}$ is a simple eigenvalue of A, i.e., $A\phi = \tilde{\lambda}\phi$, we know that $(A - \sigma I)\phi = (\tilde{\lambda} - \sigma)\phi$. From which follows that

$$(A - \sigma I)^{-1}(\tilde{\lambda} - \sigma)\phi = \phi$$

Moreover, we know that $R^{-1}N(\lambda, Rv) \xrightarrow{R \to 0} 0$ owing to the initial assumption that $D_u N(\lambda, 0) = 0$. Thereby,

$$L(R, \tilde{\lambda}, \phi) \xrightarrow{R \to 0} (A - \sigma I)^{-1}(\tilde{\lambda} - \sigma)\phi = \phi$$

Due to $|\phi| = 1$,

$$S(R, \tilde{\lambda}, \sigma) \xrightarrow{R \to 0} (\sigma + |\phi|^{-1}(\tilde{\lambda} - \sigma), |\phi|^{-1}\phi) = (\tilde{\lambda}, \phi)$$

On the other hand, the operator $D_{(\lambda,v)}S(0, \tilde{\lambda}, \phi) \in L(\mathbb{R}^{m+1}, \mathbb{R}^{m+1})$ has spectral radius less than 1 whenever σ is sufficiently close to $\tilde{\lambda}$ (see [Descloux and Rappaz, 1983]). Indeed, let us consider,

$$\sigma(A) = \{\lambda_1, \dots, \lambda_{m-1}, \tilde{\lambda}\},$$

we can prove that

$$\rho(D_{(\lambda,v)}S(0, \tilde{\lambda}, \phi)) = \max_{1 \le i \le m-1} \frac{|\tilde{\lambda} - \sigma|}{|\lambda_i - \sigma|} \tag{5.15}$$

Henceforth, for every $\epsilon > 0$ such that, if $0 < |\tilde{\lambda} - \sigma| < \epsilon$, then

$$\rho(D_{(\lambda,v)}S(0, \tilde{\lambda}, \phi)) < 1$$

Considering the function

$$W(R, \lambda, v) := S(R, \lambda, v) - (\lambda, v),$$

it satisfies that

$$W(0, \tilde{\lambda}, \phi) = 0$$

and that

$$D_{(\lambda,v)}W(0, \tilde{\lambda}, \phi) = D_{(\lambda,v)}S(0, \tilde{\lambda}, \phi) - I$$

is an isomorphism over $\mathbb{R}^{m+1}$ due to Lemma 2.0.2. The first part of the proof is due to the Implicit Function theorem. For the second part, let σ such that $0 < |\tilde{\lambda} - \sigma| < \epsilon$, then there exist $R_0 > 0$ such that for every $|R| < R_0$

$$\rho(D_{(\lambda,v)}S(R, \lambda(R), v(R))) \le c \le 1$$

and hence, the iterative scheme

$$S(R, \lambda_{k-1}, v_{k-1}) = (\lambda_k, v_k)$$

converges to a fixed point $(\lambda(R), v(R))$ because the spectral radius is less than 1. In equation (5.10), we notice that

$$\operatorname{sign}(\lambda_k - \sigma) = \operatorname{sign}(\lambda_{k-1} - \sigma),$$

therefore, it is not possible to obtain a solution (λ, u) in (5.1) if

$$\operatorname{sign}(\lambda_0 - \sigma) \neq \operatorname{sign}(\lambda - \sigma)$$

If $\lambda_0 = \sigma$, then $\lambda_k = \sigma$ for $k = 1, 2, \ldots$ and although the scheme may converge, it will not converge to the solution of (5.1). The farthest σ is from $\tilde{\lambda}$, the greater the scheme of convergence. From (5.15) we can derive the speed of the scheme in (5.10) as follows

$$\max_{1 \leq i \leq m-1} \frac{|\tilde{\lambda} - \sigma|}{|\lambda_i - \sigma|} + O(R) \tag{5.16}$$

Although the results of the theorem are local and relative only to the environment of the bifurcation point, numerical implementations show that the iterative method allows the calculation of large chunks of branches of the solution. □

5.3 Numerical Bifurcation of equation $-u'' = \lambda u - a(x)u^3$

Consider the problem in Chapter 1 (1.1). Using the above numerical method, we will calculate the branch of solutions bifurcating from the trivial solution for the following problem

$$\begin{aligned} &- u''(t) = \lambda u(t) - a(t)u^3(t), \qquad 0 \leq t \leq L \\ &u(0) = u(1) = 0 \end{aligned}$$

As shown in Chapter 3, this problem has a unique solution bifurcating from $(\lambda_n, 0)$ where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ as a direct consequence of the Crandall-Rabinowitz Theorem. For simplicity, we will consider the case where $L = 1$ henceforth.

Let us define the following discretization scheme, let $n \in \mathbb{N}, h = 1/N, t_i = ih$ where $0 \leq i \leq N$, $M_h = \{t_i : 0 \leq i \leq N\}$ and let U_h be the set of functions $u : M_h \to \mathbb{R}$ such that $u(0) = u(1) = 0$. We will endow this space with the norm

$$\|u\|_h := \left(h^{-1} \sum_{i=1}^{N} [u(t_i) - u(t_{i-1})]^2 \right)^{1/2} \tag{5.17}$$

We will use the classic Numerov's finite difference method for solving equations of the form

$$\frac{d^2y}{dx^2} = f(x, y)$$

The method gives the following implicit formula

$$y_{n-1} - 2y_n + y_{n-1} = \frac{h^2}{12}(f_{n+1} + 10f_n + f_{n-1}) + \mathcal{O}(h^6) \tag{5.18}$$

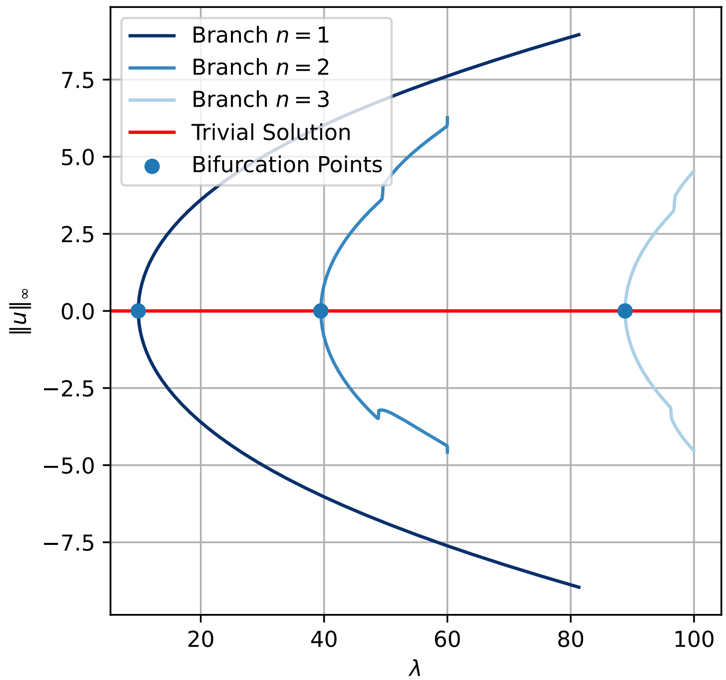


Figure 5.1: Bifurcation diagram when $a > 0$.

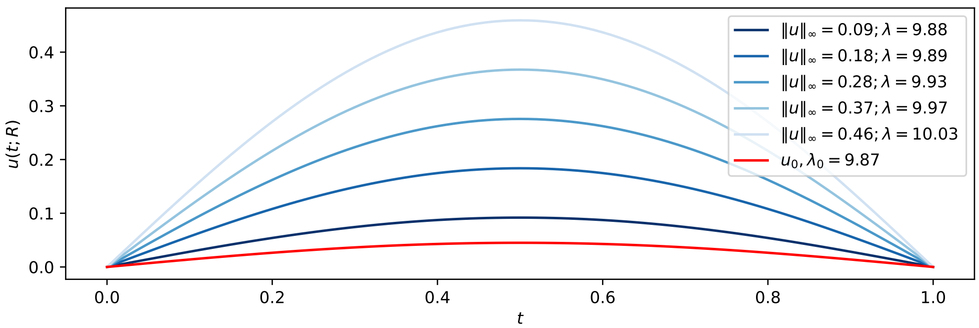
Discretizing (5.10) with $N(\lambda, u) = a(t)u^3$ and $Au = -u''$, and using the Numerov's method, we obtain the following iterative scheme for $k = 1, 2, \ldots$

$$\begin{cases} -\left(1+\frac{h^2\sigma}{12}\right) w_k(t_{i-1}) + \left(2-\frac{10h^2\sigma}{12}\right) w_k(t_i) - \left(1+\frac{h^2\sigma}{12}\right) w_k(t_{i+1}) = \\ \qquad \frac{h^2}{12}[g_{k-1}(t_{i-1}) + 10g_{k-1}(t_i) + g_{k-1}(t_{i+1})] \\ v_k = \|w_k\|_h^{-1}(\lambda_{k-1} - \sigma) \\ u_k = Rv_k \\ \lambda_k = \sigma + \|w_k\|_h^{-1}(\lambda_{k-1} - \sigma) \end{cases} \tag{5.19}$$

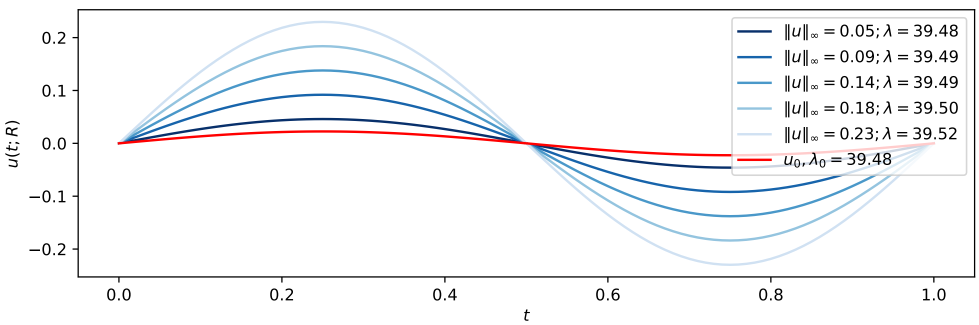
where

$$g_{k-1}(t) := (\lambda_{k-1} - \sigma)v_{k-1}(t) - R^2 v_{k-1}(t) \tag{5.20}$$

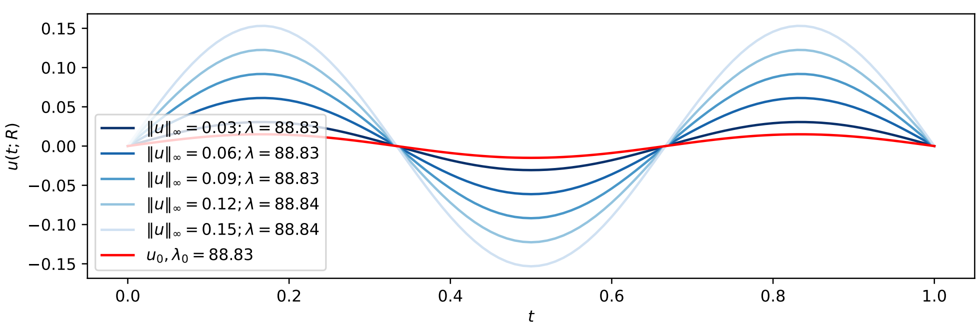
Figure 5.1 and 5.2 show the results of the numerical calculation of the bifurcation diagram for the particular case where $a = 2$. We have continued the first three branches bifurcating from $(\lambda_n, 0) = (n^2\pi^2, 0)$ for $n = 1, 2, 3$. We notice that the bifurcation diagram matches the result obtained in Chapter 1 and Chapter 4 (see Fig. 1.7 and 1.6b). Figure 5.1 shows that the algorithm underperforms far from the bifurcation point.



(a) $n = 1, \lambda_0 = \pi^2 \approx 9.85$.

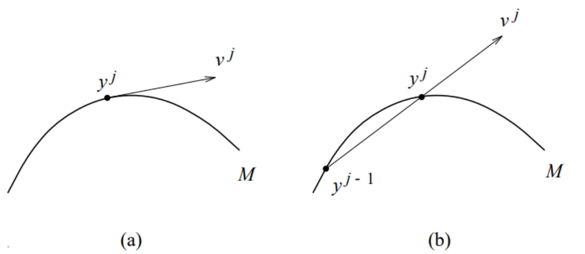


(b) $n = 2, \lambda_0 = (2\pi)^2 \approx 39.43$.

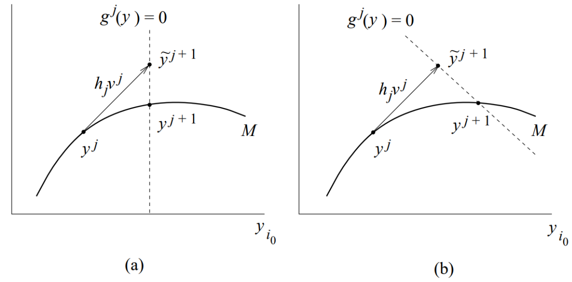


(c) $n = 3, \lambda_0 = (3\pi)^2 \approx 88.73$.

Figure 5.2: Branch continuation for $n = 1, 2, 3$ using the Inverse Power Method.



(a) Tangent and secant predictions.



(b) Natural and pseudo-arclength corrections.

Figure 5.3: Predictor-Corrector scheme (Extracted from [Kuznetsov, 2023]).

5.4 Alternative Methods

We explored the use of alternative algorithms discussed in the literature [Kuznetsov, 2023, Spence and Graham, 1999]. Most algorithms for calculating the branch of solutions are based on a prediction-correction scheme. This algorithm starts with a zero, $(\lambda_0, u_0) \in \mathbb{R} \times U$, of the equation $\mathfrak{F}(\lambda, u) = 0$. Denoting the pair as $y := (\lambda, u)$, the algorithm can be described as follows:

- *Prediction*: Given a zero of the function $\mathfrak{F}(\lambda^j, u^j) = 0$, we calculate an estimate near the given solution on the same branch of solutions. The predicted value $\tilde{y}^{j+1}$ is expressed as
$$\tilde{y}^{j+1} := y^j + h_j v^j \tag{5.21}$$
where h_j is the current *step size*, and $v^j \in \mathbb{R} \times U$ is the normalized tangent vector to the curve of solutions. The most commonly used methods are based on a first-order approximation of the Taylor series expansion of the solution curve (see Figure 5.3a).

- *Correction*: Finds a solution y^{j+1} near the guess $\tilde{y}^{j+1}$ made in the previous step. This process typically relies on Newton-like iterations (see Figure 5.3b). It is generally the most computationally intensive part of the method.

The prediction and correction steps are performed n times to obtain the first n elements of the sequence $(y_n)_n$ on the branch of solutions.

We simulate the branch of solutions using a discretization scheme similar to the one described in the previous section. For this, we employ the Tangent Prediction method along with the Pseudo-arclength Corrector method. Figure 5.4 illustrates the results, showing that the algorithm yields more robust and reliable solutions compared to the Inverse Power Method (see Figure 5.1). Additionally, it is important to note that the solution branches do not intersect. Although this can be proven, such a result lies beyond the scope of this thesis.

The complete code has been uploaded to a public repository at López-Montero [2024][1], allowing the results to be reproduced.

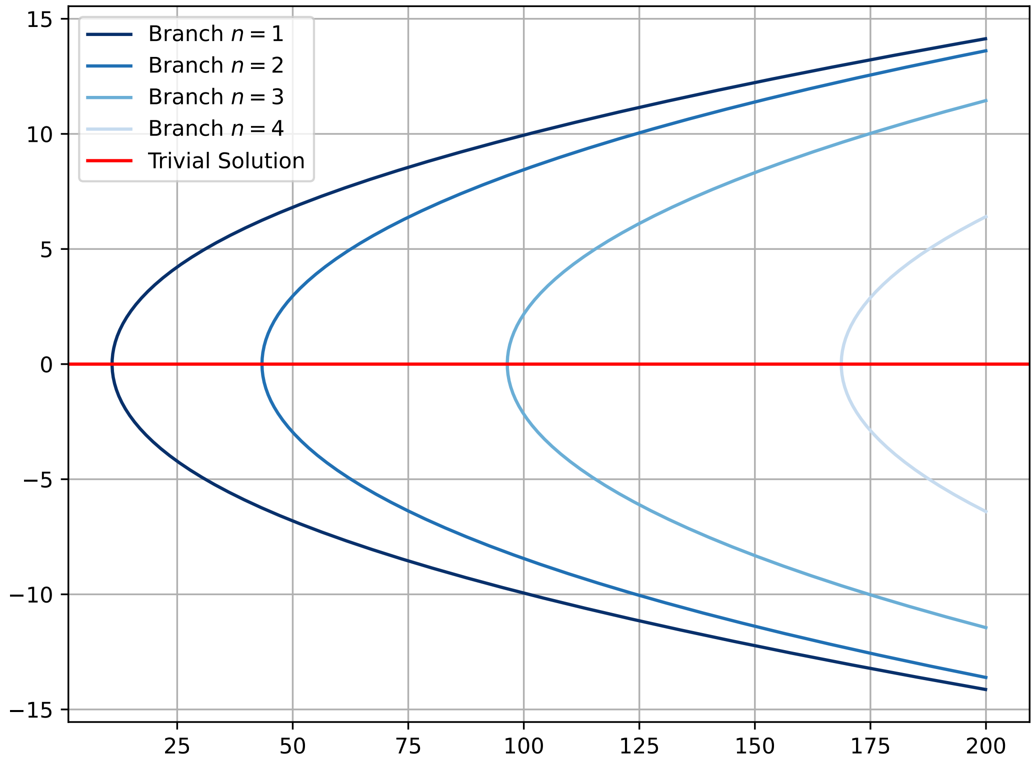


Figure 5.4: Numerically-calculated complete bifurcation diagram.

[1] https://github.com/dani2442/bifurcationjax

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