

AP Calculus AB Notes

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Unit 1. Limits

- A Limit is the intended height of a graph, as x gets very close to a value.
- A Removable discontinuity is a hole discontinuity, whilst a Non-removable discontinuity are jumps and infinite discontinuities
- When solving for limits plug in c for x and always write $\frac{0}{0}$ when the limit after you plug x in is $\frac{0}{0}$

To prove $\lim_{x \rightarrow c} f(x)$ by Squeeze theorem:

1. Prove that: $f(x)$ is between functions $g(x)$ and $h(x)$ this is written as $g(x) \leq f(x) \leq h(x)$
2. Prove that:

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x)$$

3. Write that:

\therefore By Squeeze Theorem $\lim_{x \rightarrow c} f(x) =$ the result of the limits got in the second step



Figure 1: Squeeze Theorem Graph

- When proving a value is in a function by IVT (Intermediate Value Theorem)
 1. Prove that: $f(x)$ is continuous over a range such as $[a, b]$
 2. Write each value $f(a) = c$ and $f(b) = d$
 3. Prove that: $f(a) < \text{value} < f(b)$
 4. Write that \therefore by IVT $\exists k$ s.t. $f(k) = \text{value}$

Unit 2. Derivatives

- Tangent lines are lines that touch a curve at one point
- The definitions of a Derivative are

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} \\f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}\end{aligned}$$

- A derivative of a function $f(x)$ can be written as $f'(x)$, $\frac{d}{dx}f(x)$, or $\frac{df}{dx}$
- If written as an equation ($y = \dots$) then the derivative can be written as y' or $\frac{dy}{dx}$
- To find the tangent lines of a function
 1. Find $f'(x)$
 2. Write in point slope form: $y - y_1 = f'(x_1)(x - x_1)$
- To find the normal lines of a function
 1. Find $f'(x)$ or a tangent line
 2. Write in point slope form: $y - y_1 = -\frac{1}{f'(x_1)}(x - x_1)$
- To find a horizontal tangent line solve for x when $f'(x) = 0$
- A function is differentiable when differentiable at every point
- A function is differentiable at a point when the limit exists
- Derivative Rules

– Derivative of a Constant function: if c is a constant, then $\frac{d}{dx}c = 0$

– Power Rule: $\frac{d}{dx}ax^b = abx^{(b-1)}$

– Product Rule: $\frac{d}{dx}uv = uv' + vu'$

– Quotient Rule: $\frac{d}{dx}\frac{u}{v} = \frac{vu' - uv'}{v^2}$

– Log Rule: $\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$

– Exponential Rule: $\frac{d}{dx}a^x = a^x \ln a$

– Trig Derivatives

$$\begin{array}{ll}\frac{d}{dx} \sin x = \cos x & \frac{d}{dx} \cos x = -\sin x \\ \frac{d}{dx} \sec x = \sec x \tan x & \frac{d}{dx} \csc x = -\csc x \cot x \\ \frac{d}{dx} \tan x = \sec^2 x & \frac{d}{dx} \cot x = -\csc^2 x\end{array}$$

Unit 3. Composite, Inverse, and Implicit Derivatives

- Chain rule says $(f \circ g)'(x) = f'(g(x))g'(x)$
- To find the derivative using implicit differentiation:
 1. Differentiate both sides in Leibniz notation and remember that y is a function of x so you need to apply the chain rule
 2. If necessary distribute into separate terms
 3. Move all terms with $\frac{dy}{dx}$ to one side of the equation and all other terms to the other side
- The derivative of an inverse function is $\frac{1}{f'(f^{-1}(x))}$
- Inverse Trig Derivatives

$$\begin{array}{ll} \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \arctan x = \frac{1}{1+x^2} & \frac{d}{dx} \operatorname{arccot} x = \frac{-1}{1+x^2} \\ \frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}} & \frac{d}{dx} \operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2-1}} \end{array}$$

Unit 4. Analytical Applications of differentiation

- A critical point on a graph is when $f' = 0$ but not including an endpoint
- To find the Absolute max and min plug in for both endpoints, and all critical points
- To find local minimums and maximums either graph f' 's sign or find the sign of f''
- To find a slope on a function interval by MVT (Mean Value Theorem):
 1. Prove f is continuous $[a, b]$
 2. Prove f is differentiable (a, b)
 3. Find the average rate of change of the interval $[a, b]$:

$$\frac{f(b) - f(a)}{b - a}$$

4. Write \therefore by MVT $\exists c \in (a, b)$ s.t. $f'(c) =$ the average rate of change

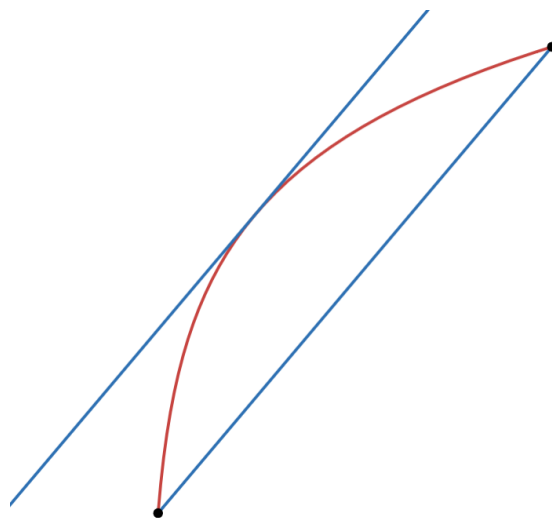













Figure 2: MVT graph

Unit 5. Contextual Applications of Differentiations

- Remember Geometric Equations for volume and surface area for 3D shapes
- And area and perimeter for 2D shapes see Figure 3

Wisconsin Mathematics Formula Reference Sheet

Shape	Formulas for Area (A) and Circumference (C)
Triangle 	$A = \frac{1}{2}bh = \frac{1}{2} \times \text{base} \times \text{height}$
Rectangle 	$A = lw = \text{length} \times \text{width}$
Trapezoid 	$A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2} \times \text{sum of bases} \times \text{height}$
Parallelogram 	$A = bh = \text{base} \times \text{height}$
Circle 	$A = \pi r^2 = \pi \times \text{square of radius}$ $C = 2\pi r = 2 \times \pi \times \text{radius}$
Figure	Formulas for Volume (V) and Surface Area (SA)
Rectangular Prism 	$V = lwh = \text{length} \times \text{width} \times \text{height}$ $SA = 2lw + 2hw + 2lh$ $= 2(\text{length} \times \text{width}) + 2(\text{height} \times \text{width}) + 2(\text{length} \times \text{height})$
General Prisms 	$V = Bh = \text{area of base} \times \text{height}$ $SA = \text{sum of the areas of the faces}$
Right Circular Cylinder 	$V = Bh = \text{area of base} \times \text{height}$ $SA = 2B + Ch = (2 \times \text{area of base}) + (\text{circumference} \times \text{height})$
Right Pyramid 	$V = \frac{1}{3}Bh = \frac{1}{3} \times \text{area of base} \times \text{height}$ $SA = B + \frac{1}{2}Pl$ $= \text{area of base} + (\frac{1}{2} \times \text{perimeter of base} \times \text{slant height})$
Right Circular Cone 	$V = \frac{1}{3}Bh = \frac{1}{3} \times \text{area of base} \times \text{height}$ $SA = B + \frac{1}{2}Cl = \text{area of base} + (\frac{1}{2} \times \text{circumference} \times \text{slant height})$
Sphere 	$V = \frac{4}{3}\pi r^3 = \frac{4}{3} \times \pi \times \text{cube of radius}$ $SA = 4\pi r^2 = 4 \times \pi \times \text{square of radius}$

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Figure 3: Geometric Equations

- To solve for related rates:
 - Draw or look at a drawing of the shape
 - Write what you know
 - Write an equation relating the variables
 - Differentiate both sides
 - Plug in what you know
 - Solve for wanted rate
- Example: The volume of a cube see Figure 4, is increasing at a rate of $20 \text{ cm}^3/\text{sec}$. How fast is the surface area of the cube increasing at the instant when each edge of the cube is 5 cm long?

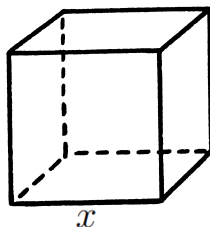


Figure 4: Cube Diagram

1.

$$2. \text{ Know: } \frac{dV}{dt} = 20 \frac{\text{cm}^3}{\text{sec}}$$

$$3. \text{ Find } \frac{dSA}{dt} \text{ when } x = 5$$

$$4. V = x^3$$

$$5. \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$6. 20 = 3(25) \frac{dx}{dt}$$

$$\implies \frac{dx}{dt} = \frac{4}{15} \frac{\text{cm}}{\text{sec}}$$

$$4. SA = 6x^2$$

$$5. \frac{dSA}{dt} = 12x \frac{dx}{dt}$$

$$6. \frac{dSA}{dt} = 12(5)\left(\frac{4}{15}\right) = \boxed{16 \frac{\text{cm}^2}{\text{sec}}}$$

- Example: In Figure 5, a baseball field is a square of side 90 feet. If a runner on second base (II) starts running toward third base (III) at a rate of 20 ft/sec, how fast is his distance from home plate (H) changing when he is 30 ft from third base (III)?

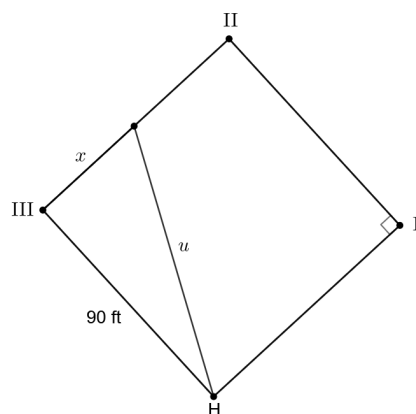


Figure 5: Baseball Square

1.

$$2. \text{ Know } \frac{dx}{dt} = 20 \frac{\text{ft}}{\text{sec}}$$

$$3. \text{ Find } \frac{du}{dt} \text{ when } x = 30$$

$$4. x^2 + 90^2 = u^2$$

$$4.1. u = \sqrt{30^2 + 90^2} = 30\sqrt{10}$$

$$5. 2x \frac{dx}{dt} = 2u \frac{du}{dt}$$

$$6. 30(20) = 30\sqrt{10} \frac{du}{dt}$$

$$\implies \frac{du}{dt} = \boxed{\frac{20}{\sqrt{10}}}$$

- L'Hôpital's Rule:

$$\lim_{x \rightarrow c} \frac{f}{g} = \lim_{x \rightarrow c} \frac{f'}{g'}$$

Unit 6. Estimation and Basic Integration

- To approximate the area under from a to b the curve using n rectangles
- $\Delta x = \frac{b-a}{n}$
- $L_n = \Delta x(f(x_1) + f(x_2) + \dots)$
- Use the anti-derivative to find exact area under the curve

$$\int f(x) dx = F(x) \implies F'(x) = f(x)$$

- For example:

$$\int x^2 dx = \boxed{\frac{x^3}{3} + c}$$

- Another example:

$$\int \frac{1}{x} dx = \boxed{\ln |x| + c}$$

- Fundamental Theorem of Calculus Part 1:
- If f is continuous over $[a, b]$, then the function

$$F(x) = \int_a^x f(t) dt$$

has a derivative at every point x in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

- Thus:

$$\frac{d}{dx} \int_a^{h(x)} f(t) dt = f(h(x)) * h'(x)$$

- For example:

$$\begin{aligned} \frac{d}{dx} \int_2^{x^2+5x} \sin(2t) dt \\ \sin(2(x^2+5x))(2x+5) \end{aligned}$$

- Fundamental Theorem of Calculus Part 2:
- If f is continuous over $[a, b]$ and F is the anti-derivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

- For example:

$$\begin{aligned} \int_{-1}^1 \frac{3}{1+x^2} dx \\ 3 \arctan x \Big|_{-1}^1 \\ 3 [\arctan(1) - \arctan(-1)] \\ 3 \left[\frac{\pi}{4} + \frac{\pi}{4} \right] \\ \boxed{\frac{3\pi}{2}} \end{aligned}$$

- Average Mean Value:
- If they ask for the average they are talking about this rather than the average rate of change

$$\text{Average} = \frac{1}{b-a} \int_a^b f(x) dx$$

- For example: Find the average of the function on the given interval:

$$f(x) = 4x - x^2 \text{ over } [0, 2]$$

$$\frac{1}{2-0} \int_0^2 (4x - x^2) dx$$

$$\frac{1}{2} \left[2x^2 - \frac{x^3}{3} \right]_0^2$$

$$\frac{1}{2} \left[2^3 - \frac{8}{3} \right]$$

$$4 - \frac{4}{3}$$

$$\boxed{\frac{8}{3}}$$

- Definite integrals can be written as the limit of a Riemann sum as the widths of the subintervals approach 0.

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

- For example: Write the following integral as a Riemann Sum:

$$\int_2^5 (\ln x + \sin x) dx$$

$$\Delta x = \frac{5-2}{n} = \frac{3}{n}$$

$$x_i = 2 + \frac{3i}{n}$$

$$\boxed{\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\ln \left(2 + \frac{3i}{n} \right) + \sin \left(2 + \frac{3i}{n} \right) \right] \frac{3}{n}}$$

- Another example: Write the following Riemann Sum as an integral:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(3 + \frac{5i}{n} \right)^2 \frac{5}{n}$$

$$a = 3$$

$$\frac{b-3}{n} = \frac{5}{n} \implies b = 8$$

$$\boxed{\int_3^8 x^2 dx}$$

Unit 7. Integration and Differential Equations

- u-substitution: make an expression as a function of x to make the integral solvable
- Ways to set u :
 1. Inside Function

$$\int (x+5)\sqrt{x^2+10x+24} \, dx$$

$$\text{let } u = x^2 + 10x + 24$$

$$\frac{du}{dx} = 2x + 10$$

$$2du = x + 5 \, dx$$

$$2 \int \sqrt{u} \, du = 2 \int u^{\frac{1}{2}} \, du$$

$$\frac{4u^{\frac{3}{2}}}{3} + c$$

$$\boxed{\frac{4(x^2 + 10x + 24)^{\frac{3}{2}}}{3} + c}$$

2. Denominator

$$\int \frac{x}{2x^2 - 3} \, dx$$

$$\text{let } u = 2x^2 - 3$$

$$du = 4x \, dx$$

$$\frac{1}{4} \int \frac{1}{u} \, du$$

$$\frac{1}{4} \ln |u| + c$$

$$\boxed{\frac{\ln |2x^2 - 3|}{4}}$$

3. Derivative in Integrand

$$\int \tan^2 x \sec^2 x \, dx$$

$$\text{let } u = \tan x$$

$$du = \sec^2 x \, dx$$

$$\int u^2 \, du$$

$$\frac{u^3}{3} + c$$

$$\boxed{\frac{\tan^3 x}{3} + c}$$

4. Recognize Inverse Trig Function

$$\int \frac{3}{x^2 + 9} \, dx$$

$$\int \frac{3}{9\left(\frac{x^2}{9} + 1\right)} \, dx$$

$$\frac{1}{3} \int \frac{1}{\left(\frac{x}{3}\right)^2 + 1} \, dx$$

$$\text{let } u = \frac{x}{3}$$

$$\frac{1}{3} \int \frac{1}{u^2 + 1} du$$

$$\frac{1}{3} \arctan u + c$$

$$\boxed{\frac{1}{3} \arctan \left(\frac{x}{3} \right) + c}$$

- Definite Integrals: to find the definite integral using u-substitution you need to change the limits of integration

$$\int_a^b f dx = \int_{u(a)}^{u(b)} f du$$

- For Example:

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x}{(1 - \cos x)^2} dx$$

$$\text{let } u = 1 - \cos x$$

$$du = \sin x dx$$

$$u \left(\frac{\pi}{2} \right) = 1$$

$$u \left(\frac{\pi}{3} \right) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\int_{\frac{1}{2}}^1 \frac{1}{u^2} du$$

$$-\frac{1}{u} \Big|_{\frac{1}{2}}^1$$

$$-\ln |u| + c$$

$$\boxed{-\ln |\cos x| + c}$$

- Differential Equations are equations with derivatives in them
- Solvable differential equations take the following form:

$$\frac{dy}{dx} = f(y)g(x)$$

- To solve differential equations in this form:

1. First, separate the variables

$$\frac{1}{f(y)} dy = g(x) dx$$

2. Next, take the anti-derivative of both sides:

$$\int \frac{1}{f(y)} dy = \int g(x) dx$$

3. Then, evaluate and put the +c on the right hand side

$$\int \frac{1}{f(y)} dy = G(x) + c$$

- For Example: find the general solution to the differential equation

$$\frac{dy}{dx} = x^2(1 + y^2)$$

$$\frac{1}{1 + y^2} dy = x^2 dx$$

$$\int \frac{1}{1 + y^2} dy = \int x^2 dx$$

$$\arctan y = \frac{x^3}{3} + c$$

$$y = \tan\left(\frac{x^3}{3} + c\right)$$

- Another Example:

$$\frac{dy}{dx} = \frac{y - 1}{x - 1}$$

$$\int \left(\frac{1}{y - 1}\right) dy = \int \left(\frac{1}{x - 1}\right) dx$$

$$\ln |y - 1| = \ln |x - 1| + c$$

$$|y - 1| = e^{\ln |x - 1| + c}$$

$$|y - 1| = e^{\ln |x - 1|} \cdot e^c$$

$$y - 1 = \pm c |x - 1|$$

$$y = 1 \pm c |x - 1|$$

- To find particular solutions of differential equation you will be given an initial condition
- This initial condition will be a point
- Make sure to check the point is differentiable and continuous
- We can check this by checking the domain of both $\frac{dy}{dx}$ and y
- With this initial condition we will solve for c before solving for y
- Whenever there are multiple solution spaces choose the one the initial condition is in
- For Example: find the particular solution of the following differential equation

$$\frac{dW}{dP} = 2WP \text{ and } W = 3 \text{ when } P = 0$$

$$\int \frac{1}{W} dW = \int 2P dP$$

$$\ln |W| = P^2 + c$$

$$\ln |W| = P^2 + \ln 3$$

$$|W| = 3e^{P^2}$$

$$W = 3e^{P^2}$$

$$\frac{dW}{dP} \in (-\infty, \infty)$$

$$W \in (0, \infty), P \in (-\infty, \infty)$$

- Another Example: find the particular solution

$$\frac{dy}{dx} = (3 - y) \cos x, \quad y(0) = 1$$

$$\int \frac{1}{3 - y} dy = \int \cos x dx$$

$$\text{let } u = 3 - y$$

$$du = -dy$$

$$-\int \frac{1}{u} du = \int \cos x dx$$

$$-\ln |u| = \sin x + c$$

$$-\ln |3 - y| = \sin x + c$$

$$-\ln 2 = \sin 0 + c$$

$$c = -\ln 2$$

$$-\ln |3 - y| = \sin x - \ln 2$$

$$\ln |3 - y| = \ln 2 - \sin x$$

$$|3 - y| = 2e^{-\sin x}$$

$$3 - y = \pm 2e^{-\sin x}$$

$$y = 3 \pm 2e^{-\sin x}$$

$$y = 3 - 2e^{-\sin x}$$

- We can show the general solution to a differential equation via a slope field, see Figure 6

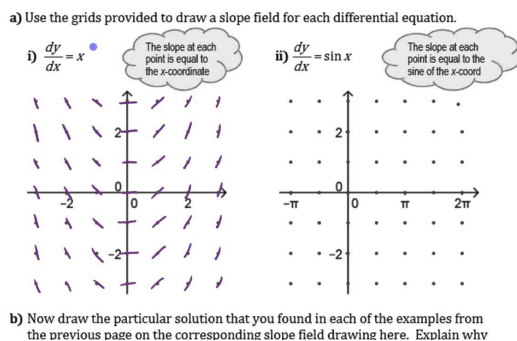


Figure 6: Slope Fields

- Another U-Sub Example:

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$$

$$\text{let } u = e^{2x} + e^{-2x}$$

$$du = 2e^{2x} - 2e^{-2x} dx$$

$$\frac{1}{2} \int \frac{1}{u} du$$

$$\frac{1}{2} \ln |u| + c$$

$$\boxed{\frac{1}{2} \ln (e^{2x} + e^{-2x}) + c}$$