AP Calculus AB Notes

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1 Limits

- A Limit is the intended height of a graph, as x gets very close to a value.
- A Removable discontinuity is a hole discontinuity, whilst a Non-removable discontinuity are jumps and infinite discontinuities
- When solving for limits plug in c for x and always write $\frac{0}{0}$ when the limit after you plug x in is $\frac{0}{0}$

To prove $\lim_{x\to c} f(x)$ by Squeeze theorem:

1. Prove that: $f(x)$ is between functions $g(x)$ and $h(x)$ this is written as $g(x) \leq f(x) \leq h(x)$
2. Prove that:
$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x)$$
3. Write that:
$$\therefore \text{ By Squeeze Theorem } \lim_{x\to c} f(x) = \text{ the result of the limits got in the second step}$$

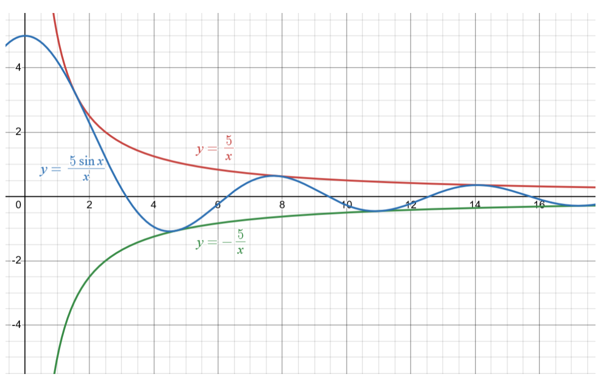


Figure 1: Squeeze Theorem Graph

- When proving a value is in a function by IVT (Intermediate Value Theorem)
  1. Prove that: $f(x)$ is continuous over a range such as $[a, b]$
  2. Write each value $f(a) = c$ and $f(b) = d$
  3. Prove that: $f(a) <$ value $< f(b)$
  4. Write that $\therefore$ by IVT $\exists k$ s.t. $f(k) =$ value

2 Derivatives

- Tangent lines are lines that touch a curve at one point
- The definitions of a Derivative are

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

- A derivative of a function $f(x)$ can be written as $f'(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$, or $\frac{\mathrm{d}f}{\mathrm{d}x}$
- If written as an equation ($y =$...) then the derivative can be written as y' or $\frac{\mathrm{d}y}{\mathrm{d}x}$
- To find the tangent lines of a function
  1. Find f'(x)
  2. Write in point slope form: $y - y_1 = f'(x_1)(x - x_1)$
- To find the normal lines of a function
  1. Find f'(x) or a tangent line
  2. Write in point slope form: $y - y_1 = -\frac{1}{f'(x_1)}(x - x_1)$
- To find a horizontal tangent line solve for x when $f'(x) = 0$
- A function is differentiable when differentiable at every point
- A function is differentiable at a point when the limit exists
- Derivative Rules
  - Derivative of a Constant function: if c is a constant, then $\frac{\mathrm{d}}{\mathrm{d}x} c = 0$
  - Power Rule: $\frac{\mathrm{d}}{\mathrm{d}x} ax^b = abx^{(b-1)}$
  - Product Rule: $\frac{\mathrm{d}}{\mathrm{d}x} uv = uv' + vu'$
  - Quotient Rule: $\frac{\mathrm{d}}{\mathrm{d}x} \frac{u}{v} = \frac{vu' - uv'}{v^2}$
  - Log Rule: $\frac{\mathrm{d}}{\mathrm{d}x} \log_b x = \frac{1}{x \ln b}$
  - Exponential Rule: $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a$
  - Trig Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cot x = \csc^2 x$$

3 Composite, Inverse, and Implicit Derivatives

- Chain rule says $(f \circ g)'(x) = f'(g(x))g'(x)$
- To find the derivative using implicit differentiation:
  1. Differentiate both sides in Leibniz notation and remember that y is a function of x so you need to apply the chain rule
  2. If necessary distribute into separate terms
  3. Move all terms with $\frac{\mathrm{d}y}{\mathrm{d}x}$ to one side of the equation and all other terms to the other side
- The derivative of an inverse function is $\frac{1}{f'(f^{-1}(x))}$
- Inverse Trig Derivatives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x &= \frac{1}{\sqrt{1-x^2}} & \qquad \frac{\mathrm{d}}{\mathrm{d}x}\arccos x &= \frac{-1}{\sqrt{1-x^2}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\arctan x &= \frac{1}{1+x^2} & \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccot} x &= \frac{-1}{1+x^2} \\
\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcsec} x &= \frac{1}{|x|\sqrt{x^2-1}} & \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccsc} x &= \frac{-1}{|x|\sqrt{x^2-1}}
\end{aligned}
$$

4 Analytical Applications of differentiation

- A critical point on a graph is when $f' = 0$ but not including an endpoint
- To find the Absolute max and min plug in for both endpoints, and all critical points
- To find local minimums and maximums either graph f''s sign or find the sign of f''
- To find a slope on a function interval by MVT (Mean Value Theorem):
  1. Prove f is continuous $[a, b]$
  2. Prove f is differentiable (a, b)
  3. Find the average rate of change of the interval $[a, b]$:

  $$\frac{f(b) - f(a)}{b - a}$$

  4. Write $\therefore$ by MVT $\exists c \in (a, b)$ s.t. $f'(c) =$ the average rate of change

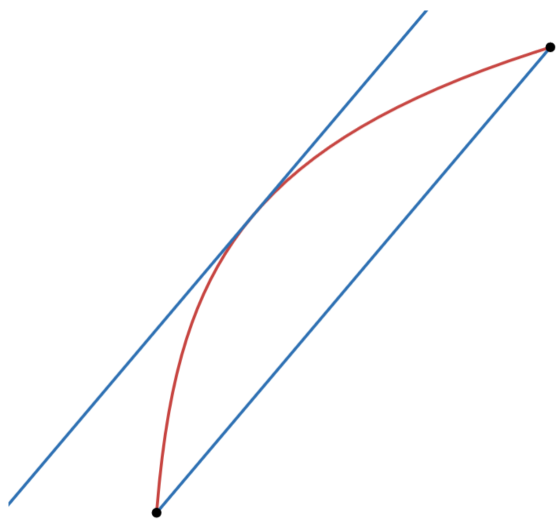
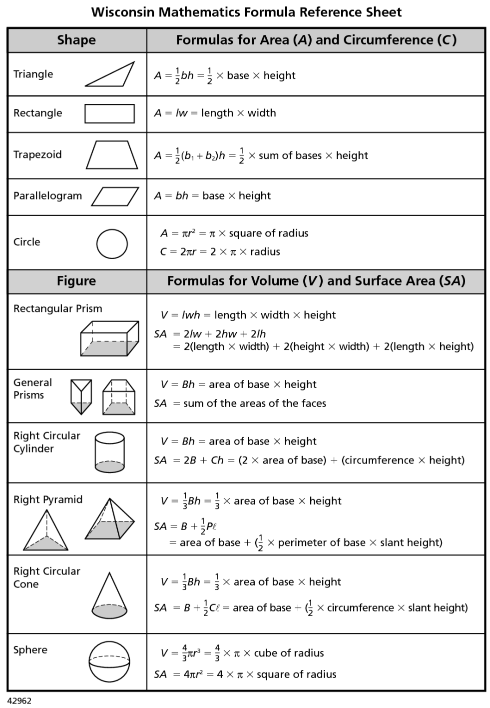


Figure 2: MVT graph

5 Contextual Applications of Differentiations

- Remember Geometric Equations for volume and surface area for 3D shapes
- And area and perimeter for 2D shapes see Figure 3



Wisconsin Mathematics Formula Reference Sheet

Shape	Formulas for Area (A) and Circumference (C)
Triangle	$A = \frac{1}{2}bh = \frac{1}{2} \times$ base $\times$ height
Rectangle	$A = lw =$ length $\times$ width
Trapezoid	$A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2} \times$ sum of bases $\times$ height
Parallelogram	$A = bh =$ base $\times$ height
Circle	$A = \pi r^2 = \pi \times$ square of radius $C = 2\pi r = 2 \times \pi \times$ radius
Figure	**Formulas for Volume (V) and Surface Area (SA)**
Rectangular Prism	$V = lwh =$ length $\times$ width $\times$ height $SA = 2lw + 2hw + 2lh$ $= 2($length $\times$ width$) + 2($height $\times$ width$) + 2($length $\times$ height$)$
General Prisms	$V = Bh =$ area of base $\times$ height $SA =$ sum of the areas of the faces
Right Circular Cylinder	$V = Bh =$ area of base $\times$ height $SA = 2B + Ch = (2 \times$ area of base$) + ($circumference $\times$ height$)$
Right Pyramid	$V = \frac{1}{3}Bh = \frac{1}{3} \times$ area of base $\times$ height $SA = B + \frac{1}{2}P\ell$ $=$ area of base $+ (\frac{1}{2} \times$ perimeter of base $\times$ slant height$)$
Right Circular Cone	$V = \frac{1}{3}Bh = \frac{1}{3} \times$ area of base $\times$ height $SA = B + \frac{1}{2}C\ell =$ area of base $+ (\frac{1}{2} \times$ circumference $\times$ slant height$)$
Sphere	$V = \frac{4}{3}\pi r^3 = \frac{4}{3} \times \pi \times$ cube of radius $SA = 4\pi r^2 = 4 \times \pi \times$ square of radius

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Figure 3: Geometric Equations

- To solve for related rates:
  1. Draw or look at a drawing of the shape
  2. Write what you know
  3. Write an equation relating the variables
  4. Differentiate both sides
  5. Plug in what you know
  6. Solve for wanted rate
- Example: The volume of a cube see Figure 4, is increasing at a rate of 20 cm^3/sec. How fast is the surface area of the cube increasing at the instant when each edge of the cube is 5 cm long?

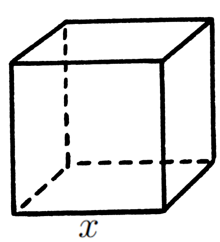

Figure 4: Cube Diagram

1.

2. Know: $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 20\dfrac{\mathrm{cm}^3}{\mathrm{sec}}$
3. Find $\dfrac{\mathrm{d}SA}{\mathrm{d}t}$ when $x = 5$
4. $V = x^3$
5. $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 3x^2\dfrac{\mathrm{d}x}{\mathrm{d}t}$
6. $20 = 3(25)\dfrac{\mathrm{d}x}{\mathrm{d}t}$

   $\implies \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{4}{15}\dfrac{\mathrm{cm}}{\mathrm{sec}}$
4. $SA = 6x^2$
5. $\dfrac{\mathrm{d}SA}{\mathrm{d}t} = 12x\dfrac{\mathrm{d}x}{\mathrm{d}t}$
6. $\dfrac{\mathrm{d}SA}{\mathrm{d}t} = 12(5)(\dfrac{4}{15}) = \boxed{16\dfrac{\mathrm{cm}^2}{\mathrm{sec}}}$

- Example: In Figure 5, a baseball field is a square of side 90 feet. If a runner on second base (II) starts running toward third base (III) at a rate of 20 ft/sec, how fast is his distance from home plate (H) changing when he is 30 ft from third base (III)?

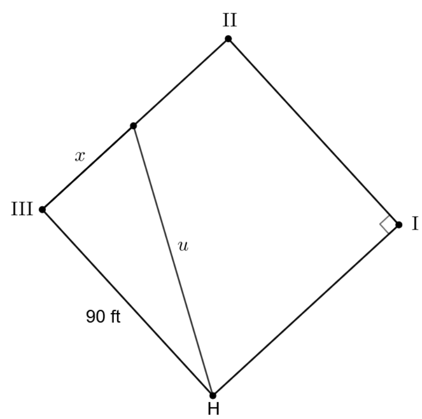

Figure 5: Baseball Square

1.
2. Know $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 20\dfrac{\mathrm{ft}}{\mathrm{sec}}$
3. Find $\dfrac{\mathrm{d}u}{\mathrm{d}t}$ when $x = 30$
4. $x^2 + 90^2 = u^2$

4.1. $u = \sqrt{30^2 + 90^2} = 30\sqrt{10}$

5. $2x\dfrac{\mathrm{d}x}{\mathrm{d}t} = 2u\dfrac{\mathrm{d}u}{\mathrm{d}t}$
6. $30(20) = 30\sqrt{10}\dfrac{\mathrm{d}u}{\mathrm{d}t}$

   $\implies \dfrac{\mathrm{d}u}{\mathrm{d}t} = \boxed{\dfrac{20}{\sqrt{10}}}$

- L'Hôpital's Rule:

$$\lim_{x\to c}\frac{f}{g} = \lim_{x\to c}\frac{f'}{g'}$$

6 Estimation and Basic Integration

- To approximate the area under from a to b the curve using n rectangles
- $\Delta x = \frac{b-a}{n}$
- $L_n = \Delta x(f(x_1) + f(x_2) + ...)$
- Use the anti-derivative to find exact area under the curve

$$\int f(x)\,dx = F(x) \implies F'(x) = f(x)$$

- For example:

$$\int x^2\,dx = \boxed{\frac{x^3}{3} + c}$$

- Another example:

$$\int \frac{1}{x}\,dx = \boxed{\ln|x| + c}$$

- Fundamental Theorem of Calculus Part 1:
- If f is continuous over $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point x in $[a, b]$, and

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,dt = f(x)$$

- Thus:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{h(x)} f(t)\,dt = f(h(x)) * h'(x)$$

- For example:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_2^{x^2+5x} \sin(2t)\,dt$$

$$\sin(2(x^2 + 5x))(2x + 5)$$

- Fundamental Theorem of Calculus Part 2:
- If f is continuous over $[a, b]$ and F is the anti-derivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- For example:

$$\int_{-1}^{1} \frac{3}{1+x^2}\,dx$$

$$3\arctan x \Big|_{-1}^{1}$$

$$3\left[\arctan(1) - \arctan(-1)\right]$$

$$3\left[\frac{\pi}{4} + \frac{\pi}{4}\right]$$

$$\boxed{\frac{3\pi}{2}}$$

- Average Mean Value:
- If they ask for the average they are talking about this rather than the average rate of change

$$\text{Average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

- For example: Find the average of the function on the given interval:

$$f(x) = 4x - x^2 \text{ over } [0,2]$$

$$\frac{1}{2-0}\int_0^2 (4x - x^2)\,dx$$

$$\frac{1}{2}\left[2x^2 - \frac{x^3}{3}\Big|_0^2\right]$$

$$\frac{1}{2}\left[2^3 - \frac{8}{3}\right]$$

$$4 - \frac{4}{3}$$

$$\boxed{\frac{8}{3}}$$

- Definite integrals can be written as the limit of a Riemann sum as the widths of the subintervals approach 0.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

- For example: Write the following integral as a Riemann Sum:

$$\int_2^5 (\ln x + \sin x)\,dx$$

$$\Delta x = \frac{5-2}{n} = \frac{3}{n}$$

$$x_i = 2 + \frac{3i}{n}$$

$$\boxed{\lim_{n\to\infty}\sum_{i=1}^{n}\left[\ln\left(2+\frac{3i}{n}\right) + \sin\left(2+\frac{3i}{n}\right)\right]\frac{3}{n}}$$

- Another example: Write the following Riemann Sum as an integral:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(3+\frac{5i}{n}\right)^2\frac{5}{n}$$

$$a = 3$$

$$\frac{b-3}{n} = \frac{5}{n} \implies b = 8$$

$$\boxed{\int_3^8 x^2\ dx}$$

7 Integration and Differential Equations

- u-substitution: make a expression as a function of x to make the integral solvable
- Ways to set u:

1. Inside Function

$$\int (x+5)\sqrt{x^2+10x+24}\,dx$$

$$\text{let } u = x^2+10x+24$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x+10$$

$$2du = x+5\,dx$$

$$2\int \sqrt{u}\,du = 2\int u^{\frac{1}{2}}\,du$$

$$\frac{4u^{\frac{3}{2}}}{3}+c$$

$$\boxed{\frac{4(x^2+10x+24)^{\frac{3}{2}}}{3}+c}$$

2. Denominator

$$\int \frac{x}{2x^2-3}\,dx$$

$$\text{let } u = 2x^2-3$$

$$du = 4x\,dx$$

$$\frac{1}{4}\int \frac{1}{u}\,du$$

$$\frac{1}{4}\ln|u|+c$$

$$\boxed{\frac{\ln|2x^2-3|}{4}}$$

3. Derivative in Integrand

$$\int \tan^2 x\sec^2 x\,dx$$

$$\text{let } u = \tan x$$

$$du = \sec^2 x\,dx$$

$$\int u^2\,du$$

$$\frac{u^3}{3}+c$$

$$\boxed{\frac{\tan^3 x}{3}+c}$$

4. Recognize Inverse Trig Function

$$\int \frac{3}{x^2+9}\,dx$$

$$\int \frac{3}{9(\frac{x^2}{9}+1)}\,dx$$

$$\frac{1}{3}\int \frac{1}{(\frac{x}{3})^2+1}\,dx$$

$$\text{let } u = \frac{x}{3}$$
$$\frac{1}{3}\int \frac{1}{u^2+1}\,du$$
$$\frac{1}{3}\arctan u + c$$
$$\boxed{\frac{1}{3}\arctan\left(\frac{x}{3}\right) + c}$$

- Definite Integrals: to find the definite integral using u-substitution you need to change the limits of integration

$$\int_a^b f\,dx = \int_{u(a)}^{u(b)} f\,du$$

- For Example:

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x}{(1-\cos x)^2}\,dx$$
$$\text{let } u = 1-\cos x$$
$$du = \sin x\,dx$$
$$u\left(\frac{\pi}{2}\right) = 1$$
$$u\left(\frac{\pi}{3}\right) = 1-\frac{1}{2} = \frac{1}{2}$$
$$\int_{\frac{1}{2}}^{1} \frac{1}{u^2}\,du$$
$$-\frac{1}{u}\bigg|_{\frac{1}{2}}^{1}$$
$$-\ln|u| + c$$
$$\boxed{-\ln|\cos x| + c}$$

- Differential Equations are equations with derivatives in them
- Solvable differential equations take the following form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)g(x)$$

- To solve differential equations in this form:
  1. First, separate the variables
  $$\frac{1}{f(y)}\,dy = g(x)\,dx$$
  2. Next, take the anti-derivative of both sides:
  $$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$
  3. Then, evaluate and put the $+c$ on the right hand side
  $$\int \frac{1}{f(y)}\,dy = G(x) + c$$

- For Example: find the general solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2(1+y^2)$$

$$\frac{1}{1+y^2}\,dy = x^2\,dx$$

$$\int \frac{1}{1+y^2}\,dy = \int x^2\,dx$$

$$\arctan y = \frac{x^3}{3} + c$$

$$y = \tan\left(\frac{x^3}{3} + c\right)$$

- Another Example:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-1}{x-1}$$

$$\int \left(\frac{1}{y-1}\right) dy = \int \left(\frac{1}{x-1}\right) dx$$

$$\ln|y-1| = \ln|x-1| + c$$

$$|y-1| = e^{\ln|x-1|+c}$$

$$|y-1| = e^{\ln|x-1|} \cdot e^c$$

$$y-1 = \pm\, c\,|x-1|$$

$$y = 1 \pm c\,|x-1|$$

- To find particular solutions of differential equation you will be given an initial condition
- This initial condition will be a point
- Make sure to check the point is differentiable and continuous
- We can check this by checking the domain of both $\frac{\mathrm{d}y}{\mathrm{d}x}$ and y
- With this initial condition we will solve for c before solving for y
- Whenever there are multiple solution spaces choose the one the initial condition is in
- For Example: find the particular solution of the following differential equation

$$\frac{\mathrm{d}W}{\mathrm{d}P} = 2WP \text{ and } W = 3 \text{ when } P = 0$$

$$\int \frac{1}{W}\,dW = \int 2P\,dP$$

$$\ln|W| = P^2 + c$$

$$\ln|W| = P^2 + \ln 3$$

$$|W| = 3e^{P^2}$$

$$W = 3e^{P^2}$$

$$\frac{\mathrm{d}W}{\mathrm{d}P} \in (-\infty, \infty)$$

$$W \in (0,\infty), P \in (-\infty,\infty)$$

- Another Example: find the particular solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3-y)\cos x,\ y(0)=1$$
$$\int \frac{1}{3-y}\,dy = \int \cos x\,dx$$
$$\text{let } u = 3-y$$
$$du = -dy$$
$$-\int \frac{1}{u}\,du = \int \cos x\,dx$$
$$-\ln|u| = \sin x + c$$
$$-\ln|3-y| = \sin x + c$$
$$-\ln 2 = \sin 0 + c$$
$$c = -\ln 2$$
$$-\ln|3-y| = \sin x - \ln 2$$
$$\ln|3-y| = \ln 2 - \sin x$$
$$|3-y| = 2e^{-\sin x}$$
$$3-y = \pm 2e^{-\sin x}$$
$$y = 3 \pm 2e^{-\sin x}$$
$$y = 3 - 2e^{-\sin x}$$

- We can show the general solution to a differential equation via a slope field, see Figure 6

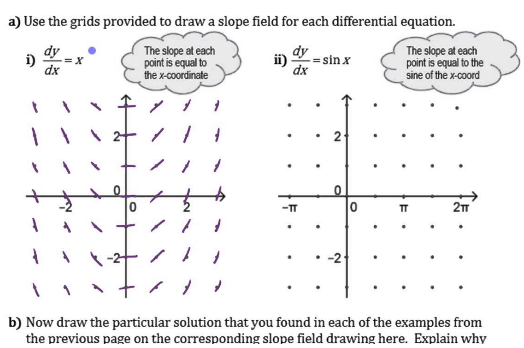

Figure 6: Slope Fields