MAT110: Differential Calculus
and Co-ordinate Geometry
BRAC University

BRAC UNIVERSITY
Inspiring Excellence

SET: F

Monthly Assignment

*Please write your name and ID on the assignment script. The deadline for submitting the assignment is **20th September**. Solve **all the problems**. You will receive 5 bonus marks for **submitting your assignment in Latex**.*

Any information you need to solve this exam are given in the question.

Watch the videos in this Playlist if you are confused about the assignment process: ***All About Assignments Playlist, Click Here***

*Be creative, use your intuition. Answer the questions by yourself. Cheating and Copying will lead to **50%** deduction from your total marks in the course and a Zero in the assignment. **Total marks is 300**. Each question carries 100 marks. Total marks will be converted to 20. Bonus Question carries 50 marks.*

1. Machine learning is an integral part of todays study of computer science. It is used in popular apps like Samsung Bixby and popular smart speakers like Amazon Echo.

   Sigmoidal functions are frequently used in machine learning, specifically in the testing of artificial neural networks, as a way of understanding the output of a node or "neuron".

   Sigmoid functions are of the form:

   $$\Phi(x) = \frac{1}{1 + \exp(-x)}$$

   (a) Find the Maclaurin series of e^x.

   (b) Taylor expand the Sigmoid function $\Phi(x)$.

   Thus, we have learned the use of Sigmoid functions which is an integral part of modern Machine Learning. [JUST BY TAKING SIMPLE DERIVATIVES AND TAYLOR EXPANDING!]

2. "Differential Equations" constitute an important branch in mathematics. As the name suggests it involves the derivatives of a function with respect to it's variables. Wave equation is a second order partial differential equation that contains description of waves in the field of acoustics, electromagnetics, fluid dynamics etc.

(a) Consider the following equation:

$$\frac{d^2y}{dx^2} + y = 0$$

Show that $y = 2\sin x + 3\cos x$ is a solution of this equation.

(b) Wave equation in one space dimension is as follows:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Show that, the given function $f(x,t) = sin(x - ct)$ is a solution of one dimensional wave equation. [Hint: To find $\frac{\partial f}{\partial x}$, treat t as some constant and differentiate with respect to x. Also, to find $\frac{\partial f}{\partial t}$, treat x as some constant and differentiate with respect to t.]

3. The hyperbolic functions are the complex analogues of the trigonometric functions. The two fundamental hyperbolic functions are cosh x and sinh x, which, as their names suggest, are the hyperbolic equivalents of cosx and sin x. They are defined by the following relations:

$Coshx = \frac{1}{2}\left(e^x + e^{-x}\right)$

$Sinhx = \frac{1}{2}\left(e^x - e^{-x}\right)$

Note: You can answer all questions by using the above two definitions.

(a) Find the taylor expansion of Sinh x and Cosh x.

(b) Find the taylor expansion of Tanhx.

(c) Find the derivative of Cosh x and Sinh x. Is it similar to the trigonometric derivatives?

(d) Find the domain and range of Cosh x and Sinh x. Can you now visualize the graph of the two functions?

(e) Find $Cosh^2x - Sinh^2x$. How does it relate to their trigonometric counterparts? Can you see a relationship?

[BONUS Questions]

4. The classical oscillation is governed by potential $V(x) = \frac{1}{2}kx^2$, where x is the position & k is the spring constant of the oscillation. The force on the oscillation is given by -

$$F(x) = -\frac{dV(x)}{dx} \tag{17}$$

(a) Find an expression for $F(x)$ using the above mentioned formula in equation (1).

According to Newtons law, the force $F(x)$ is given by -

$$F(x) = m\frac{d^2x}{dt^2} \tag{18}$$

Where $\frac{d^2x}{dt^2} = a$ and a stands for acceleration.

(b) Put the expression of F(x) in equation(2) and obtain an expression for $\frac{d^2x}{dt^2}$.

(c) Show that, $x(t) = Ae^{-i\omega t} + Be^{i\omega t}$ will satisfy the expression you got. That is $x(t)$ will satisfy $\frac{d^2x}{dt^2} = -\frac{k}{m}x$. Identify ω in terms of k & m.

Now, we depart from classical oscillation & turn to the problem of Quantum Oscillation. In particular for this problem, we want to look at the thermal behaviour of these oscillations.

To start this analysis, we want to show you the formula for a partition function.It is denoted by Z and it is given by the following formula -

$$Z = \sum_i e^{-\beta E_i}$$

The $\sum$ symbol means a sum and if you expand it, it will look like -

$$Z = e^{\beta E_1} + e^{\beta E_2} + e^{\beta E_3} + ...$$

and so on. Also, β is given by -

$$\beta = \frac{1}{kT}$$

k is a constant and T is the temperature.

Now, the first thing we want to find out about these oscillations, is their average energy. The average energy is given by -

$$\bar{E} = \sum_i E_i P(i)$$

Where, $P(i) = \frac{e^{-\beta E_i}}{Z}$

(d) Show that, $\sum_i E_i P(i) = -(\ln Z)\beta$ [Hint:Focus on the right hand side. There is no need to evaluate the sum]

For the quantum oscillation, the quantum number $'n'$ plays the role of index $'i'$. Now take the partition function for the quantum oscillator -

$$Z_q = \frac{e^{-\beta\ \hbar\omega/2}}{1 - e^{-\beta\hbar w}}$$

Again using the formula -

$$\bar{E_q} = -\ln Z\beta$$

Show that, $\bar{E_q} = \hbar\omega(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1})$

Now, recall $\beta = \frac{1}{kT}$ and we can say that, intuitively it is clear that as the temperature T will go up the oscillator will get more & more excited. Now using what you have learned about limits show that -

$$\lim_{T\to 0}\left[\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right)\right] = kT$$

[Hint: use $\beta = \frac{1}{kT}$]

So, what we can see from here is that, at very very high temperature, quantum behaviour of the oscillator dies out.

Now again, we will use -

$$E_q(T) = \hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right) \qquad \text{Where } \beta = \frac{1}{kT}$$

for the following problem.

Heat capacity is defined by -

$$C(T) = E(t)T......(2)$$

(e) Using equation (1) and (2) find C_q(T) and show that -

$$C_q(T) = 3k\left(\frac{\hbar\omega}{kT}\right)^2 \frac{e^{\frac{\hbar\omega}{kT}}}{(e^{\frac{\hbar\omega}{kT}}-1)^2}$$

(f) Now, when T $\gg \frac{\hbar\omega}{k}$, show that -

$$C_q(T) \approx 3k$$

(g) When T $\ll \frac{\hbar\omega}{k}$, show -

$$C_q(T) \approx 3k\left(\frac{\hbar\omega}{kT}\right)^2 e^{-\frac{\hbar\omega}{kT}}$$

[Hint: Use $e^x = 1 + x + \frac{x^2}{2!} + ...$ expression in both cases]