Definition: A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Simply, A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing Fn as some combination of Fi with $i<n$).

Example − Fibonacci series − $F_n=F_{n-1}+F_{n-2}$

, Tower of Hanoi − $F_n=2F_{n-1}+1$

Linear Recurrence Relations

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $C_1, C_2, \ldots, C_k$ are real numbers, and $C_k \neq 0$.

These are some examples of linear recurrence equations –

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1 = a_2 = 1$	Fibonacci number
$F_n = F_{n-1} + F_{n-2}$	$a_1 = 1, a_2 = 3$	Lucas Number
$F_n = F_{n-2} + F_{n-3}$	$a_1 = a_2 = a_3 = 1$	Padovan sequence
$F_n = 2F_{n-1} + F_{n-2}$	$a_1 = 0, a_2 = 1$	Pell number

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant. Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ if and only if

$$\mathbf{r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots\ldots + c_k r^{n-k}}$$

When both sides of this equation are divided by r^{n-k} and the right-hand side is subtracted from the left, we obtain the equation

r^k $\mathbf{- c_1 r^{k-1} - c_2 r^{k-2} - \ldots.. - c_{k-1} r - c_k = 0.}$

Consequently, the sequence {an} with $a_n = r^n$ is a solution if and only if r is a solution of this last equation, which is called the **characteristic equation** of the recurrence relation. The solutions of this equation are called the **characteristic roots** of the recurrence relation.

How to solve linear recurrence relation

Suppose, a two ordered linear recurrence relation is – $F_n = AF_{n-1} + BF_{n-2}$ where A and B are real numbers.

The characteristic equation for the above recurrence relation is −

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots −

Case 1 − If this equation factors as $(x - x_1)(x - x_1) = 0$

and it produces two distinct real roots x_1 and x_2, then $F_n = ax_1^n + bx_2^n$ is the solution. [Here, a and b are constants]

Case 2 − If this equation factors as $(x - x_1)^2 = 0$

and it produces single real root x_1, then $Fn = ax_1^n + bnx_1^n$ is the solution.

Problem 1

Solve the recurrence relation $Fn = 5F_{n-1} - 6F_{n-2}$ Where $F_0 = 1$ and $F_1 = 4$

Solution

The characteristic equation of the recurrence relation is −

$$x^2 - 5x + 6 = 0,$$

So, $(x-3)(x-2)=0$

Hence, the roots are −

$x_1 = 3$ and $x_2 = 2$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is −

$$F_n = ax_1^n + bx_2^n$$

Here, $Fn = a3^n + b2^n$ (As $x_1 = 3$ and $x_2 = 2$)

Therefore,

$1 = F_0 = a3^0 + b2^0 = a + b$

$4 = F_1 = a3^1 + b2^1 = 3a + 2b$

Solving these two equations, we get a=2 and b=−1

Hence, the final solution is −

$$F_n=2.3^n +(-1).2^n=2.3^n-2^n$$

Problem 2

Solve the recurrence relation − $Fn=10F_{n-1}-25F_{n-2}$ where $F_0=3$ and $F_1=17$

Solution

The characteristic equation of the recurrence relation is −

$x^2-10x-25=0$

So $(x-5)^2=0$

Hence, there is single real root $x_1=5$

As there is single real valued root, this is in the form of case 2

Hence, the solution is −

$F_n= ax_1^n + bnx_1^n$

$3=F_0=a.5^0+b.0.5^0=a$

$17=F_1=a.5^1+b.1.5^1=5a+5b$

Solving these two equations, we get a=3

and b=2/5

Hence, the final solution is − $Fn=3.5^n+(2/5).n.2^n$

Problem 3:

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution:

The characteristic equation of the recurrence relation is $r^2 - r - 2 = O$. Its roots are $r = 2$ and $r = -1$. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2(-1)^n,$$

for some constants α_1 and α_2. From the initial conditions, it follows that

$a_0 = 2 = \alpha_1 + \alpha_2$,

$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 . (-1)$.

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$. Hence, the solution to the recurrence relation and initial conditions is the sequence $\{ a_n \}$ with

$$a_n = 3 \cdot 2^n - (-1)^n.$$

Problem 4:

Find the solution to the recurrence relation $a_n = 6a_{n-1} – 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution:
The characteristic polynomial of this recurrence relation is
$r^3 - 6r^2 + 11r - 6$.
The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r – 6 = (r - I)(r - 2)(r - 3)$. Hence, the solutions to this recurrence relation are of the form

$a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives
$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$,
$a_1 = 5 = \alpha_1 + \alpha_2. 2 + \alpha_3. 3$,
$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9$.
When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

Non-Homogeneous Recurrence Relation and Particular Solutions

A recurrence relation is called non-homogeneous if it is in the form

$F_n = AF_{n-1} + BF_{n-2} + f(n)$ where $f(n) \neq 0$

Its associated homogeneous recurrence relation is $F_n = AF_{n-1} + BF_{n-2}$

The solution (a_n) of a non-homogeneous recurrence relation has two parts.

First part is the solution (a_h) of the associated homogeneous recurrence relation and the second part is the particular solution (a_t)

.

$$a_n = a_h + a_t$$

Solution to the first part is done using the procedures discussed in the previous section.

To find the particular solution, we find an appropriate trial solution.

Let $f(n) = cx^n$; let $x^2 = Ax + B$ be the characteristic equation of the associated homogeneous recurrence relation and let x_1 and x_2 be its roots.

- If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
- If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
- If $x = x_1 = x_2$, then $at = An^2x^n$

Example

Let a non-homogeneous recurrence relation be $F_n = AF_{n-1} + BF_{n-2} + f(n)$ with characteristic roots $x_1 = 2$ and $x_2 = 5$. Trial solutions for different possible values of $f(n)$ are as follows –

f(n)	Trial solutions
4	A
5.2^n	$An2^n$
8.5^n	$An5^n$
4^n	$A4^n$
$2n^2+3n+1$	An^2+Bn+C

Problem 5:

Solve the recurrence relation $Fn=3Fn-1+10Fn-2+7*5^n$ where $F0=4$ and $F1=3$

Solution

This is a linear non-homogeneous relation, where the associated homogeneous equation is $Fn=3Fn-1+10Fn-2$ and $f(n)=7*5^n$

The characteristic equation of its associated homogeneous relation is −

$$x^2-3x-10=0$$

Or, $(x-5)(x+2)=0$

Or, $x_1=5$ and $x_2=-2$

Hence $a_h=a.5^n+b.(-2)^n$, where a and b are constants.

Since $f(n)=7.5^n$, i.e. of the form $c.x^n$, a reasonable trial solution of at will be Anx^n

$a_t=Anx^n=An5^n$

After putting the solution in the recurrence relation, we get −

$An5^n=3A(n-1)5^{n-1}+10A(n-2)5^{n-2}+7.5^n$

Dividing both sides by 5^{n-2}, we get

$An5^2=3A(n-1)5+10A(n-2)5^0+7.5^2$

Or, $25An=15An-15A+10An-20A+175$

Or, $35A=175$

Or, $A=5$

So, $Fn=An5^n=5n5^n=n5^{n+1}$

The solution of the recurrence relation can be written as −

$Fn=a_h+at$

$=a.5^n+b.(-2)^n+n5^{n+1}$

Putting values of $F_0=4$ and $F_1=3$, in the above equation, we get $a=-2$ and $b=6$

Hence, the solution is −

$F_n=n5^{n+1}+6.(-2)^n-2.5^n$

For reference:

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