CSE330 Assignment 03 Solution

1. A function is given by $f(x) = 2x - e^{-6x}$. Now answer the following:
   (a) (3 marks) Approximate the derivative of $f(x)$ at $x_0 = 0.5$ with step size $h = 0.2$ using the forward difference method up to 5 significant figures.
   (b) (3 marks) Approximate the derivative of $f(x)$ at $x_0 = 0.5$ with step size $h = 0.2$ using the central difference method up to 6 significant figures.
   (c) (4 marks) Calculate the upper bound of truncation error of $f(x)$ at $x_0 = 2$ using $h = 0.1$ in both of the above mentioned methods for the interval $[2.4, 2.7]$.
   (d) (5 marks) Compute $D_{0.5}^{(1)}$ at $x_0 = 0.2$ using Richardson extrapolation method up to 6 significant figures and calculate the truncation error.

① $f(x) = 2x - e^{-6x} \quad \Rightarrow f'(x) = 6e^{-6x} + 2$

a. $$\frac{f(x+h)-f(x)}{h} = \frac{\left[2(x+h) - e^{-6(x+h)}\right] - \left[2x - e^{-6x}\right]}{h}$$
$$= \frac{\left[2(0.7) - e^{-(6\times0.7)}\right] - \left[(2\times0.5) - e^{-(6\times0.5)}\right]}{0.2} = 2.1740$$

b. $$\frac{f(x+h)-f(x-h)}{2h} = \frac{\left[2(x+h) - e^{-6(x+h)}\right] - \left[2(x-h) - e^{-6(x-h)}\right]}{2h}$$
$$= \frac{\left[2(0.7) - e^{-(6\times0.7)}\right] - \left[2(0.3) - e^{-(6\times0.3)}\right]}{2\times0.2} = 2.3758$$

1.c) $f(x) = 2x - e^{-6x}$

max for forward

$f^{(1)}(x) = 2 + 6e^{-6x}$

$f^{(2)}(x) = -36e^{-6x} \xrightarrow{x=2.4} |-2.007 \times 10^{-5}|$

$f^{(2)}(x) = -36e^{-6x} \xrightarrow{x=2.7} |-3.317 \times 10^{-6}|$

$f^{(3)}(x) = 216e^{-6x}$

$x=2.4 \rightarrow 1.204 \times 10^{-4}$

$x=2.7 \rightarrow 1.99 \times 10^{-5}$

1.204×10^{-4} ↓ max value for central

for forward difference,

upper bound error $\leq \left| \frac{f^2(\xi)}{2!} \times (-h) \right|$

$\leq \frac{2.007 \times 10^{-5}}{2} \times 0.1$

(Ans.)

for central difference,

upper bound error $\leq \left| \frac{f^3(\xi)}{3!} \times h^2 \right|$

$\leq \frac{1.204 \times 10^{-4}}{6} \times (0.1)^2$

(Ans.)

d. $D_h^{(1)} = \dfrac{4 D_{h/2} - D_h}{4-1}$ $(x_0 = 0.2)$

$D_h = \dfrac{f(x+h) - f(x-h)}{2h} \Rightarrow D_{h/2} = \dfrac{f(x+h/2) - f(x-h/2)}{2 \times (h/2)}$

$\therefore D_{0.5} = \dfrac{f(0.2+0.5) - f(0.2-0.5)}{2 \times 0.5} \approx 8.0347$

$h = 0.5 \Rightarrow h/2 = 0.25$

$\therefore D_{0.25} = \dfrac{f(0.2+0.25) - f(0.2-0.25)}{2 \times 0.25} \approx 4.5653$

$\therefore D_{0.5}^{(1)} = \dfrac{4 \cdot D_{0.25} - D_{0.5}}{4-1} = 3.4089$

Actual value at $x_0 = 0.2 \Rightarrow f'(0.2) = 3.80716527147$

$\therefore$ truncation error $=$ actual value $- D_{0.5}^{(1)}$

$= 0.39827$

2. During the class, we derived in detail the first order Richardson extrapolated derivative, by using $h \to h/2$,

$$D_h^{(1)} \equiv f'(x_0) - \frac{h^4}{480} f^{(5)}(x_0) + \mathcal{O}(h^6)\ .$$

(a) (4 marks) Using $h \to h/2$, derive the expression for $D_h^{(2)}$ which is the second order Richardson extrapolation.

(b) (5 marks) Now starting from the definition of D_h and using $h \to h/3$, derive the expression for $D_h^{(1)}$.

(c) (3 marks) Now identify the Error Part of the expression found in the previous part, and also find the Error Bound of the expression found in the previous part.

(d) (3 marks) If $f(x) = \ln x$, $x_0 = 1$, $h = 0.1$, find the upper bound of error for $D_h^{(1)}$.

2 a. $D_h^{(1)} = f'(x_0) - \frac{h^4}{480} f^{(5)}(x_0) + O(h^6)$ —— (I)

$\Rightarrow D_{h/2}^{(1)} = f'(x_0) - \frac{1}{4} \frac{f^{(5)}(x_0)}{5!} \frac{h^4}{16} + O(h^6)$ —— (II)

$16 D_{h/2}^{(1)} - D_h^{(1)} = 15 f'(x_0) + O(h^6)$

$\Rightarrow \frac{16 D_{h/2}^{(1)} - D_h^{(1)}}{15} = f'(x) + O(h^6) = D_h^{(2)}$

b. $D_h = f'(x) + \frac{f'''(x)}{3!} h^2 + \frac{f^5(x)}{5!} h^4 + O(h^6)$ — (I)

$\Rightarrow D_{h/3} = f'(x) + \frac{f'''(x)}{3!} \frac{h^2}{9} + \frac{f^5(x)}{5!} \frac{h^4}{81} + O(h^6)$ —— (II)

$9 D_{h/3} - D_h = 8 f'(x) + 0 + \left(\frac{1}{9} - 1\right) \frac{f^5(x)}{5!} h^4 + O(h^6)$

$\Rightarrow \frac{9 D_{h/3} - D_h}{8} = f'(x) - \frac{1}{9} \frac{f^5(x)}{5!} h^4 + O(h^6)$

$= D_h^{(1)}$

c. error part of $D_h^{(1)} = -\frac{1}{9}\frac{f^5(x)}{5!}h^4 + O(h^6)$

error bound of $D_h^{(1)} = -\frac{1}{9}\frac{f^5(x)}{5!}h^4$

d. $f(x) = \ln(x)$; $x_0 = 1$; $h = 0.1$

upper bound of error for $D_h^{(1)} = -\frac{1}{9}\frac{f^5(x)}{5!}h^4$

$f(x) = \ln x$

$f^5(x) = \frac{24}{x^5}$

$= -\frac{1}{9}\frac{24/x^5}{5!}h^4$

$= -\frac{1}{9}\frac{24/1}{5!}(0.1)^4$