BRAC UNIVERSITY

CSE330

Numerical Methods

Assignment 2

Student Information

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Answer to the question no 1

Given that, $f(x) = \tan x$

here $x_0 = 0$

a) $P_3(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$

here, Taylor coefficients are,

$a_0 = f(x_0) = \tan(0) = 0$

$a_1 = \frac{f'(x_0)}{1!} = \frac{\sec^2(0)}{1} = 1$

$a_2 = \frac{f''(x_0)}{2!} = \frac{2\sec^2(0)\tan(0)}{2!} = 0$

$a_3 = \frac{f'''(x_0)}{3!} = \frac{4\sec^2(0)\tan^2(0) + 2\sec^4(0)}{3!} = \frac{1}{3}$

$P_3(x) = (x - x_0) + \frac{1}{3}(x-x_0)^3$ (Ans)

(b) at $x = \frac{\pi}{4}$

$f(x) = \tan\left(\frac{\pi}{4}\right) = 1$

and, $P_3(x) = 0 + 1.(x-x_0) + 0.(x-x_0)^2 + \frac{1}{3}(x-x_0)^3$

$= \left(\frac{\pi}{4} - 0\right) + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 = 0.94688$

% relative error $= \frac{|f(x) - P_3(x)|}{f(x)} \times 100 \%$

$= \frac{|1 - 0.94688|}{1} \times 100 \%$

$= 5.312 \%$ (Ans)

(c) Lagrange reminder form

$$\Rightarrow \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

here, $n=3$

from, (a), $f^3(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$\therefore f^4(x) = 8\sec^2(x)\tan^3(x) + 16\sec^4(x)\tan(x)$

Given $\xi \in [0, \frac{\pi}{4}]$

for, $\xi = 0$,

$f^4(0) = 8\sec^2(0)\tan^3(0) + 16\sec^4(0)\tan(0)$

$= (8\times1)\times0 + (16\times1)\times0$

$= 0$

and for $\xi = \frac{\pi}{4}$

$f^4(\frac{\pi}{4}) = 8\sec^2(\frac{\pi}{4})\tan^3(\frac{\pi}{4}) + 16\sec^4(\frac{\pi}{4})\tan(\frac{\pi}{4})$

$= 8\times(\sqrt{2})^2\times1 + 16\times(\sqrt{2})^4\times1$

$= 16+64$

$= 80$

here $max = f^4(\frac{\pi}{4})$

$\therefore$ upperbound of truncation error at $x = \frac{\pi}{4}$

$$\Rightarrow \frac{f^4(\frac{\pi}{4})}{4!}\times(x-x_0)^4 \Rightarrow \frac{80}{4!}(\frac{\pi}{4}-0)^4$$

$= 1.26884$ (Ans)

Answer to the question no 2

(a) Given, $f(x) = e^{x} - e^{-x}$

here, $(x_0, x_1, x_2) = (-1, 0, 1)$

the equations are,

$$a_0 + a_1 x_0 + a_2 x_0^2 = f(x_0)$$

$$a_0 + a_1 x_1 + a_2 x_1^2 = f(x_1)$$

$$a_0 + a_1 x_2 + a_2 x_2^2 = f(x_2)$$

here, $f(x_0) = e^{-1} - e^{1} = -2.350$

$f(x_1) = e^{0} - e^{0} = 0$

$f(x_2) = e^{1} - e^{-1} = 2.350$

using Vandermonde Matrix Method

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -2.350 \\ 0 \\ 2.350 \end{bmatrix}}_{b}$$

(b) the determinant of Vandermonde matrix V

$$\Rightarrow 1(1\times 0 - 0\times 1) + 1(1\times 1 - 1\times 0) + 1(1\times 1 - 1\times 0)$$

$$\Rightarrow 0+1+1 = 2$$

(c) we know,

$$A = V^{-1} b$$

$$V^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 1 & 0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 1 & 0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} -2.350 \\ 0 \\ 2.350 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2.35 \\ 0 \end{bmatrix}$$

$\therefore a_0 = 0, \quad a_1 = 2.350, \quad a_2 = 0$ (Ans)

(d) as node = 3, so, $n+1 = 3$ $\therefore$ degree $= 2$

(ii)

Cauchy's theorem,

$$|f(x) - P_n(x)| = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)$$

$f(x) = e^{x} - e^{-x}$

$f'(x) = e^{x} + e^{-x}$

$f^2(x) = e^{x} - e^{-x}$

$f^3(x) = e^{x} + e^{-x}$

Given $\xi \in [-2.1, 2.1]$

for, $\xi = -2.1$

$f^3(-2.1) = e^{-2.1} + e^{2.1} = 8.2886$

for, $\xi = 2.1$

$f^3(2.1) = e^{2.1} + e^{-2.1} = 8.2886$

as both are ~~equally~~ equal we can take any one

$\therefore$ max $\Rightarrow$ $f^3(2.1) = 8.2886$

now,

w_{max};

$w(x) = (x-x_0)(x-x_1)(x-x_2)$

$= (x+1)(x-0)(x-1)$

$= x^3 - x$

now, $w'(x) = 3x^2 - 1 = 0$

$\therefore x^2 = \frac{1}{3}$ $\therefore x = \pm\sqrt{\frac{1}{3}}$

here,

x		$w(x) = x^3 - x$
$\sqrt{\frac{1}{3}}$	$\rightarrow$	-0.3849
$-\sqrt{\frac{1}{3}}$	$\rightarrow$	-0.7698
-2.1	$\rightarrow$	-7.161
2.1	$\rightarrow$	7.161

$w_{max} = 7.161$

$\therefore$ upper bound of interpolation Error,

$$\Rightarrow \frac{f^{n+1}(\xi)}{(n+1)!} \times w_{max}$$

$$= \frac{8.2886}{3!} \times 7.161$$

$$\Rightarrow 9.8924 \text{ (Ans)}$$

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Answer to the question 3

Given,

$$f(x) = e^{x} + e^{-x}$$

and, $(x_0, x_1, x_2) = (-1, 0, 1)$

(a) here, $n+1 = 3 \quad \therefore n = 2$

lagrange method,

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

lagrange basis,

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{(0+1)(0-1)} = \frac{x^2-1}{-1}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{x(x+1)}{2}$$

(Ans)

(c (b) now,

$$f(x_0) = e^{-1} + e^{1} = 3.0861$$

$$f(x_1) = e^{0} + e^{-0} = 2$$

$$f(x_2) = e^{1} + e^{-1} = 3.0861$$

$$P_2(x) = \frac{x(x-1)}{2} \times (3.0861) + (1-x^2) \times 2 + \frac{x(x+1)}{2} \times (3.0861)$$

$$= \frac{x^2 - x}{2} \times 3.0861 + 2 - 2x^2 + \frac{(x^2+x)\,3.0861}{2}$$

$$= \frac{3.0861(x^2 - x) + 4 - 4x^2 + (3.0861)(x^2 + x)}{2}$$

$$= \frac{3.0861(x^2 - x + x^2 + x) + 4 - 4x^2}{2}$$

$$= \frac{6.1722}{2} x^2 - 2x^2 + 2$$

$$= 1.0861 x^2 + 2$$

$$= 2 + 0 + 1.0861 x^2$$

$a_0 = 2$, $a_1 = 0$, $a_2 = 1.0861$

$$f(x) = 1.0861 x^2 + 2$$

$$f(6) = 1.0861 \times 36 + 2$$

$$= 41.096 \text{ (Ans)}$$

$\therefore$ 41.096 is the approximate value for $f(6)$

(c) Percent relative error $= \dfrac{|f(x) - P_2(x)|}{f(x)} \times 100\%$

at $x = 1.5$

$f(x) = e^{1.5} + e^{-1.5} = 4.20$

$P_2(x) = 1.086\, x^2 + 2 = 1.086 \times (1.5)^2 + 2$

$= 4.44$

$\therefore$ % error $= \dfrac{|4.20 - 4.44|}{4.20} \times 100\%$

$= 5.53\%$ (Ans)

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Answer to the question no 4

given, $f(x) = e^{x} - e^{-x}$

nodes are, $(x_0, x_1, x_2) = (-2, 0, 2)$

so, $n+1 = 3$, $\therefore$ degree $= 2$

Newton divided difference method

$P_2(x) = a_0 n_0(x) + a_1 n_1(x) + a_2 n_2(x)$

newton basis:

$n_0(x) = 1$

$n_1(x) = (x - x_0)$

$n_2(x) = (x - x_0)(x - x_1)$

$n_3(x) = (x - x_0)(x - x_1)(x - x_2)$

Co-efficients, a_K

$a_0 = f[x_0]$

$a_1 = f[x_0, x_1]$

$a_2 = f[x_0, x_1, x_2]$

$x_0 \Rightarrow -2 \quad f[x_0] = e^{-2} - e^{2} = \cancel{7.5244}\ -7{,}2537$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$x_1 \Rightarrow 0 \quad f[x_1] = e^{0} - e^{-0} = 0$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

$x_2 \Rightarrow 2 \quad f[x_2] = e^{2} - e^{-2} = \cancel{7.5244}\ 7{,}2537$

$$f[x_0 x_1 x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$a_0 = f[x_0] = \cancel{7.5244}\ -7{,}2537$

$$a_1 = f[x_0 x_1] = \frac{0 \cancel{-7.5244}}{\cancel{2}} \cancel{= -3.7622} \quad \frac{7{,}25}{2} = 3{,}625$$

$$a_2 = f[x_0 x_1 x_2] = \frac{3{,}6268 - \frac{7{,}25}{2}}{4} = \frac{9}{20000} = 4.5 \times 10^{-4}$$

(b) $P_2(x) = -7,2537 \times 1 + \dfrac{7,2537}{2}(x+2) + \dfrac{9}{20000}(x+2)(x-0)$

$= -7,2537 + 3,625\,x + 7,25 + 0,00045\,x^2$

$+ 0,0009\,x$

$= 0 + 3.6259\,x + 0.00045\,x^2$

$\boxed{f(6) = 3.6259 \times 6 + 0.00045 \times 6^2}$

$= 21,7716$

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(c) we know,

$\text{relative error} = \dfrac{|f(x) - P_n(x)|}{f(x)} \times 100\%$

at point $x = 1.5$

$\therefore f(1.5) = e^{1.5} - e^{-1.5} = 4,2585$

$\therefore P_2(1.5) = 3.6259 \times 1.5 + 0.00045 \times (1.5)^2$

$= 5.4398$

$\%\ \text{relative error} = \dfrac{|f(x) - P_2(x)|}{f(x)} \times 100\%$

$= \dfrac{|4,2585 - 5.4398|}{4,2585} \times 100\%$

$= 27.33\%$ Ans