Residue THM

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$$\frac{\partial^2 \eta}{\partial \xi^2} = \frac{\partial \eta}{\partial \tau} \Rightarrow \eta''(\xi) - s\eta(\xi) = 0.$$

BC: $\frac{\partial \eta}{\partial \xi} = -I^* \frac{\gamma}{1+\gamma}$

$$\frac{\partial \eta}{\partial \xi} = I^* \frac{1}{1+\gamma}.$$

$$\Rightarrow \eta(\xi) = \frac{e^{-\sqrt{s}}\left(e^{\sqrt{s}} + e^{\sqrt{s}} \cdot e^{2\sqrt{s}\xi} + e^{2\sqrt{s}}\gamma + e^{2\sqrt{s}\xi}\gamma\right)}{\left(e^{2\sqrt{s}} - 1\right)(1+\gamma)^2\sqrt{s}}$$

$$= \frac{1 + e^{2\sqrt{s}\xi} + e^{\sqrt{s}}\gamma + e^{2\sqrt{s}\xi} e^{-\sqrt{s}}\gamma}{(1+\gamma)^2\sqrt{s}\left(e^{2\sqrt{s}} - 1\right)}$$

$$= \frac{e^{-\sqrt{s}} + e^{2\sqrt{s}\xi}\left(e^{-\sqrt{s}} + e^{-2\sqrt{s}}\gamma\right) + \gamma}{(1+\gamma)^2\sqrt{s}\left(e^{\sqrt{s}} - e^{-\sqrt{s}}\right)}$$

$$V^*(s) = -\frac{\gamma}{1+\gamma}\eta\Big|_0 + \frac{1+2\gamma}{1+\gamma}\eta\Big|_1 - I^*\frac{\gamma}{(\gamma+1)^2}$$

$$= \frac{1 + 4e^{\sqrt{s}}\gamma + \gamma^2 + e^{2\sqrt{s}}(1+\gamma^2)}{\left(-1+e^{2\sqrt{s}}\right)(1+\gamma)^2\sqrt{s}}\bar{I} - \bar{I}\frac{\gamma}{(\gamma+1)^2}$$

$$= \frac{(1+\gamma^2)\left(e^{2\sqrt{s}}+1\right)}{(1+\gamma)^2\left(e^{2\sqrt{s}}-1\right)\sqrt{s}} + \frac{2 \cdot 2e^{\sqrt{s}}\gamma}{(1+\gamma)^2\left(e^{2\sqrt{s}}-1\right)\sqrt{s}} - \bar{I}\frac{\gamma}{(\gamma+1)^2}$$

$$= \frac{(1+\gamma^2)\bar{I}(s)}{(1+\gamma)^2(\tanh\sqrt{s})\cdot\sqrt{s}} + \frac{2\gamma\bar{I}(s)}{\sqrt{s}(1+\gamma)^2\sinh\sqrt{s}} - \bar{I}\frac{\gamma}{(\gamma+1)^2}$$

poles for $\overset{I}{\bar{I}}$, $\bar{II}$. are the same.

$s = 0$ 2nd order root.

$s = -n^2\pi^2$ simple roots. $n = 1, 2 \cdots$

$$\bar{V}_{elec}(s) = \underbrace{\frac{2\gamma\bar{I}(s)}{(1+\gamma)^2\sqrt{s}\sinh(\sqrt{s})}}_{I} + \frac{(1+\gamma^2)\bar{I}(s)}{(1+\gamma)^2} \times \underbrace{\frac{\cosh(\sqrt{s})}{\sqrt{s}}}_{II} - \frac{\gamma\bar{I}(s)}{(1+\gamma)^2}.$$

Goal: $\mathrm{Res}[e^{st} \mathrm{I}]$, $\mathrm{Res}[e^{st} \mathrm{II}]$.

$$\mathrm{Res}(e^{st}\mathrm{I}, 0) \overset{2nd}{=} \lim_{s\to 0} \frac{d}{ds}\left(s^2 \frac{e^{st}}{\sqrt{s}\sinh\sqrt{s}}\right).$$

$$e^{st} = \left(1 + st + \frac{s^2t^2}{2} \cdots\right).$$

$$= \lim_{s\to 0}\left[\frac{t\cdot s^{3/2} e^{st}}{\sinh(\sqrt{s})} + \frac{3}{2}\frac{\sqrt{s}}{\sinh\sqrt{s}} e^{st} - \frac{1}{2} e^{st} s \frac{1}{\tanh\sqrt{s}} \frac{1}{\sinh\sqrt{s}}\right]$$

$$= 1$$

$$\mathrm{Res}(e^{st}\mathrm{II}, 0) \overset{2nd}{=} \lim_{s\to 0}\frac{d}{ds}\left(s^2 \frac{e^{st}\coth(\sqrt{s})}{\sqrt{s}}\right).$$

$$= \lim_{s\to 0}\left(t s^{3/2} e^{st}/\tanh(\sqrt{s}) + \frac{3}{2}\sqrt{s}\, e^{st}\coth(\sqrt{s}) - \frac{1}{2} s e^{st}/\sinh^2\sqrt{s}\right).$$

$$= 1$$

$$\left.\begin{aligned} \mathrm{Res}(e^{st}\mathrm{I}, -n^2\pi^2) &= \lim_{s\to -n^2\pi^2}(s+n^2\pi^2)\cdot\frac{e^{st}}{\sqrt{s}\sinh\sqrt{s}} = (-1)^n \times 2 e^{-n^2\pi^2 t} \\ \mathrm{Res}(e^{st}\mathrm{II}, -n^2\pi^2) &= \lim_{s\to -n^2\pi^2}(s+n^2\pi^2)\cdot\frac{e^{st}}{\sqrt{s}\tanh\sqrt{s}} = 2e^{-n^2\pi^2 t}. \end{aligned}\right\} \text{simply poles.}$$

$$V_{elec}^{*} = \int_0^{\tau} I(\tau')\left\{1 + 2\sum_{n=1}^{\infty}\left[\frac{2\gamma}{(1+\gamma)^2}(-1)^n + \frac{1+\gamma^2}{(1+\gamma)^2}\right] e^{-n^2\pi^2(\tau-\tau')}\right\} d\tau' - \frac{\gamma I^{*}}{(1+\gamma)^2}$$

For $I(\tau') = I$

$$V_{elec}^{*} = I\left[\tau + 2\sum_{n=1}^{\infty}\left[\frac{2\gamma}{(1+\gamma)^2}(-1)^n + \frac{1+\gamma^2}{(1+\gamma)^2}\right]\left(\frac{1-e^{-n^2\pi^2\tau}}{n^2\pi^2}\right)\right] - \frac{\gamma I^{*}}{(1+\gamma)^2}$$