8. Some Additional Examples

In addition to the Fourier transform and eigenfunction expansions, it is sometimes convenient to have the use of the Laplace transform for solving certain problems in partial differential equations. We will quickly develop a few properties of the Laplace transform and use them in solving some example problems.

Laplace Transform

Definition of the Transform

Starting with a given function of t, $f(t)$, we can define a new function $\hat{f}(s)$ of the variable s. This new function will have several properties which will turn out to be convenient for purposes of solving linear constant coefficient ODE's and PDE's. The definition of $\hat{f}(s)$ is as follows:

Definition Let $f(t)$ be defined for $t \geq 0$ and let the Laplace transform of $f(t)$ be defined by,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \hat{f}(s)$$

For example:

$$f(t) = 1,\ \forall t \geq 0, \qquad L[1] = \int_0^\infty e^{-st}\, dt = \frac{e^{-st}}{-s}\Big|_{t=0}^{t=\infty} = \frac{1}{s} = \hat{f}(s) \quad for\ s > 0$$

$$f(t) = e^{bt},\ \forall t \geq 0, \qquad L[e^{bt}] = \int_0^\infty e^{-(b-s)t}\, dt = \frac{e^{-(b-s)t}}{(s-b)}\Big|_{t=0}^{t=\infty} = \frac{1}{s-b} = \hat{f}(s), \quad for\ s > b.$$

The Laplace transform is defined for all functions of **exponential type**. That is, any function $f(t)$ which is

(a) piecewise continuous = has at most finitely many finite jump discontinuities on any interval of finite length

(b) has exponential growth: for some positive constants M and k

$$|f(t)| \leq M e^{kt} \quad for\ all\ t \geq 0, \quad .$$

Properties of the Laplace Transform

The Laplace transform has the following **general properties**:

1. **Linearity** $\quad L[C_1 f(t) + C_2 g(t)] = C_1 \hat{f}(s) + C_2 \hat{g}(s)$

2. **Homogeneity** $\quad L[f(at)] = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right) \quad for \quad a > 0$

3. **Transform of the Derivative** $\quad L[f'(t)] = s\hat{f}(s) - f(0)$

$$L[f''(t)] = s^2 \hat{f}(s) - sf(0) - f'(0) \quad etc$$

4. **Derivative of the Transform** $L[tf(t)] = -\hat{f}'(s)$

$$L[t^2 f(t)] = (-1)^2 \hat{f}''(s) \quad etc$$

Some Special Transforms

There are some transform pairs that are useful in solving problems involving the heat equation. The derivations are given in an appendix.

$$(S.1) \qquad f(t) = \frac{k}{\sqrt{4\pi t^3}} e^{-k^2/4t}, \ t > 0 \qquad\qquad L[f(t)] = e^{-k\sqrt{s}}, \qquad k > 0$$

$$(S.2) \qquad f(t) = \frac{1}{\sqrt{\pi t}} e^{-k^2/4t}, \ t > 0 \qquad\qquad L[f(t)] = \frac{1}{\sqrt{s}} e^{-k\sqrt{s}}, \qquad k \geq 0$$

$$(S.3) \qquad f(t) = \mathrm{erf}c\left(\frac{k}{2\sqrt{t}}\right), \ t > 0 \qquad\qquad L[f(t)] = \frac{1}{s} e^{-k\sqrt{s}}, \qquad k > 0$$

Here $\qquad \mathrm{erf}c(z) = 1 - \mathrm{erf}(z) \qquad and \qquad \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx.$

Additional Properties of the Transform

Let $f(t)$ be a function of exponential type and suppose that for some $b > 0$,

$$h(t) = \left\{ \begin{array}{ll} 0 & if \ \ 0 < t < b \\ f(t-b) & if \ \ t > b \end{array} \right\}$$

Then $h(t)$ is just the function $f(t)$, delayed by the amount b. Then

$$L[h(t)] = \int_0^\infty h(t) e^{-st} dt = \int_b^\infty f(t-b) e^{-st} dt$$

Let $z = t - b$ so that

$$L[h(t)] = \int_0^\infty f(z) e^{-s(z+b)} dz = e^{-bs} \int_0^\infty f(z) e^{-sz} dz = e^{-bs} \hat{f}(s).$$

If we define $\qquad H(t-b) = \left\{ \begin{array}{ll} 0 & if \ \ 0 < t < b \\ 1 & if \ \ t > b \end{array} \right\}$

then

$$h(t) = H(t-b) f(t-b)$$

and we find

5. **Transform of a Delay** $\qquad L[H(t-b)f(t-b)] = e^{-bs}\hat{f}(s), \quad for \quad b > 0.$

A related results is the following

$$L[e^{bt}f(t)] = \int_0^\infty e^{bt}f(t)\,e^{-st}\,dt = \int_0^\infty f(t)\,e^{-(s-b)t}\,dt = \hat{f}(s-b).$$

i.e.,

6. **Delay of a Transform** $\qquad L[e^{bt}f(t)] = \hat{f}(s-b)$

Results 5 and 6 assert that a delay in the function induces an exponential multiplier in the transform and, conversely, a delay in the transform is associated with an exponential multiplier for the function.

A final property of the Laplace transform asserts that

7. **Inverse of a Product** $\qquad L[(f*g)(t)] = \hat{f}(s)\,\hat{g}(s)$

$$where \qquad (f*g)(t) := \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

The product, $(f*g)(t)$, is called the **convolution product** of f and g. Life would be simpler if the inverse Laplace transform of $\hat{f}(s)\,\hat{g}(s)$ was the pointwise product $f(t)\,g(t)$, but it isn't, it is the convolution product. The convolution product has some of the same properties as the pointwise product, namely

$$(f*g)(t) = (g*f)(t) \qquad and \qquad (h*(f*g))(t) = ((h*f)*g)(t).$$

We will not give the proof of the result 7 but will make use of it nevertheless.

Applications to PDE's

1. Consider the IBVP

$$\begin{aligned} &\partial_t u(x,t) = k\,\partial_{xx}u(x,t) && x>0,\ t>0,\\ &u(x,0) = 0, && x>0,\\ &u(0,t) = f(t), \qquad f(0) = 0, && t>0. \end{aligned}$$

Let

$$\hat{U}(x,s) = L[u(x,t)] \qquad \text{Laplace transform in t.}$$

Then

$$\begin{aligned} &s\hat{U}(x,s) - 0 = k\,U''(x,s), \qquad x>0, \qquad U''(x,s) = \frac{d^2}{dx^2}\hat{U}(x,s)\\ &\hat{U}(0,s) = \hat{f}(s). \end{aligned}$$

Solving this ODE in x, we find

$$\hat{U}(x,s) = Ae^{-x\sqrt{s/k}} + Be^{x\sqrt{s/k}} \qquad x>0.$$

We want $\hat{U}(x,s)$ to remain bounded for all positive x, which requires that $B = 0$. Then the boundary condition at x=0 leads to

$$\hat{U}(x,s) = \hat{f}(s)\, e^{-x\sqrt{s/k}}, \qquad x > 0.$$

Then $(S.1)$ together with property 7 of the Laplace transform, gives

$$u(x,t) = f * K(x,\bullet) = \int_0^t \frac{x}{\sqrt{4\pi k(t-\tau)^3}} e^{-x^2/4k(t-\tau)} f(\tau)\, d\tau.$$

as the unique solution of the IBVP. Suppose now that we wish to compute the flux through x=0,

Flux at 0 $= -k\partial_x u(0,t).$

Differentiating the integral expression for u does not seem like a pleasant prospect. However, note that

$$L[-k\partial_x u(x,t)] = -k\partial_x \hat{U}(x,s) = \sqrt{ks}\,\hat{f}(s)\, e^{-x\sqrt{s/k}},$$

and

Flux at 0 $= -k\partial_x u(0,t) = L^{-1}\left[\sqrt{ks}\,\hat{f}(s)\right].$

In order to use our inversion formulas, we write

$$\sqrt{ks}\,\hat{f}(s) = \frac{\sqrt{k}}{\sqrt{s}}\, s\hat{f}(s)$$

and, recalling that $f(0) = 0,$ we have $s\hat{f}(s) = L[f'(t)]$. In addition, $(S.2)$ with k=0, gives

$$L^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}} \qquad ,$$

and so,

Flux at 0 $= L^{-1}\left[\dfrac{\sqrt{k}}{\sqrt{s}}\, s\hat{f}(s)\right] = \dfrac{\sqrt{k}}{\sqrt{\pi}} \displaystyle\int_0^t \frac{f'(\tau)}{\sqrt{(t-\tau)}}\, d\tau.$

Note that if $f(t) = At,$ then

Flux at 0 $= \dfrac{\sqrt{k}}{\sqrt{\pi}} \displaystyle\int_0^t \frac{A}{\sqrt{(t-\tau)}}\, d\tau = 2A\sqrt{\frac{kt}{\pi}}\,.$

Problem 1 Show that if $v(x,t)$ solves

$$\begin{aligned} \partial_t v(x,t) &= k\,\partial_{xx} v(x,t) && x > 0,\ t > 0, \\ v(x,0) &= 0, && x > 0, \\ -k\partial_x v(0,t) &= g(t), && t > 0, \end{aligned}$$

then

$$v(0,t) = \int_0^t \frac{g(\tau)}{\sqrt{\pi k(t-\tau)}}\, d\tau, \qquad t > 0,$$

and if $g(t) = B,$ then $\quad v(0,t) = 2B\sqrt{\frac{t}{\pi k}}\,.$

2. Consider the IBVP

$$\begin{aligned} &\partial_{tt}u(x,t) = a^2\,\partial_{xx}u(x,t) \qquad x > 0,\ t > 0,\\ &u(x,0) = \partial_t u(x,0) = 0, \qquad x > 0,\\ &u(0,t) = f(t), \qquad t > 0. \end{aligned}$$

Let

$$\hat{U}(x,s) = L[u(x,t)] \qquad \text{Laplace transform in t.}$$

Then

$$\begin{aligned} &s^2\hat{U}(x,s) - 0 = k\,U''(x,s), \qquad x > 0,\\ &\hat{U}(0,s) = \hat{f}(s). \end{aligned}$$

Solving this ODE in x, we find a general solution of the form,

$$\hat{U}(x,s) = Ae^{\frac{x}{a}s} + Be^{-\frac{x}{a}s},$$

and both

$$\hat{U}(x,s) = \hat{f}(s)\,e^{\frac{x}{a}s} \qquad \hat{U}(x,s) = \hat{f}(s)\,e^{-\frac{x}{a}s}$$

solve the transformed equation and the boundary condition at x=0. To see how to choose the correct solution, recall that for $x > 0,$ property 5 implies that $\hat{f}(s)\,e^{-\frac{x}{a}s}$ is the transform of $f(t)$ delayed by the amount $\frac{x}{a} > 0$. On the other hand, $\hat{f}(s)\,e^{\frac{x}{a}s}$ is the transform of $f(t)$ advanced in time by the amount $\frac{x}{a} > 0$. Another way to say this is to say

$$u(x,t) = L^{-1}\left[\hat{f}(s)\,e^{-\frac{x}{a}s}\right] = f\left(t - \tfrac{x}{a}\right)H\left(t - \tfrac{x}{a}\right),$$

represents the wave form $f(\bullet)$ propagating from L to R into the region $\{x > 0\}$ while

$$u(x,t) = L^{-1}\left[\hat{f}(s)\,e^{\frac{x}{a}s}\right] = f\left(t + \tfrac{x}{a}\right)H\left(t + \tfrac{x}{a}\right)$$

represents the wave form $f(\bullet)$ propagating from R to L out of the region $\{x > 0\}$. Then the solution that is relevant for our problem is the wave that travels from L to R into the region $\{x > 0\}$.

3. Consider the IBVP

$$\begin{aligned} &\partial_t u(x,t) = K\,\partial_{xx}u(x,t) \qquad 0 < x < 1, \quad t > 0,\\ &u(x,0) = 0, \qquad 0 < x < 1,\\ &u(0,t) = f(t) \qquad \partial_x u(1,t) = 0 \qquad t > 0. \end{aligned}$$

Here, we may use the Laplace transform, or if we prefer, we can use eigenfunction expansion after a suitable modification of the problem. We will solve the problem first by this means. Since the boundary conditions are not homogeneous, the method does not apply directly but if we let

$$v(x,t) = u(x,t) - f(t)$$

then

$$\begin{aligned}
\partial_t v(x,t) &= \partial_t u(x,t) - f'(t) \\
&= K\partial_{xx} u(x,t) - f'(t) \\
&= K\partial_{xx} v(x,t) - f'(t), \\
v(x,0) &= u(x,0) - f(0) = 0, \\
v(0,t) &= u(0,t) - f(t) = 0, \qquad \partial_x v(1,t) = \partial_x u(1,t) = 0.
\end{aligned}$$

Now the method of eigenfunction expansion applies directly to the problem for $v(x,t)$ since the boundary conditions are homogeneous. Since $v(0,t) = \partial_x v(1,t) = 0$, it follows that the eigenfunctions are the eigenfunctions of example 6.3, namely

$$\lambda_n = \left(n - \tfrac{1}{2}\right)^2 \pi^2 = \mu_n^2, \qquad \phi_n(x) = \sin\left(n - \tfrac{1}{2}\right)\pi x = \sin \mu_n x, \qquad n = 1,2,...$$

We write $$v(x,t) = \sum_{n=1}^{\infty} v_n(t)\,\phi_n(x) \qquad \textit{and} \qquad f'(t) = f'(t)\,1 = f'(t)\sum_{n=1}^{\infty} C_n\,\phi_n(x),$$

where

$$C_n = \frac{(1,\phi_n)}{(\phi_n,\phi_n)} = \frac{\int_0^1 \sin\left(n - \frac{1}{2}\right)\pi x dx}{\int_0^1 \sin^2\left(n - \frac{1}{2}\right)\pi x dx} = \frac{4}{(2n-1)\pi} = \frac{2}{\mu_n}.$$

Substituting these expansions into the IBVP for $v(x,t)$, we conclude that

$$v_n'(t) + K\mu_n^2\, v_n(t) = -f'(t)\,C_n, \qquad v_n(0) = 0, \qquad \forall n.$$

and $$v_n(t) = -C_n \int_0^t e^{-K\mu_n^2(t-s)} f'(s)\,ds.$$

For example, if $f(t) = mt$, then

$$v_n(t) = -mC_n \int_0^t e^{-K\mu_n^2(t-s)}\,ds. = \frac{mC_n}{K\mu_n^2}\left(e^{-K\mu_n^2 t} - 1\right)$$

and

$$\begin{aligned}
u(x,t) &= f(t) + \sum_{n=1}^{\infty} \frac{mC_n}{K\mu_n^2}\left(e^{-K\mu_n^2 t} - 1\right)\phi_n(x), \\
&= mt + \frac{2m}{K}\sum_{n=1}^{\infty} \frac{1}{\mu_n^3}\left(e^{-K\mu_n^2 t} - 1\right)\phi_n(x).
\end{aligned}$$

Alternatively, we may use the Laplace transform to solve this same problem. Let $\hat{u}(x,s)$ denote the Laplace transform of $u(x,t)$. Then,

$$s\,\hat{u}(x,s) - 0 = K\hat{u}''(x,s), \qquad \hat{u}(0,s) = \hat{f}(s), \quad \hat{u}\,'(1,s) = 0.$$

The general solution of the equation may be written as

$$\hat{u}(x,s) = A\exp\left(-x\sqrt{s/K}\right) + B\exp\left(x\sqrt{s/K}\right),$$

and then the boundary conditions lead to

$$A = \frac{\exp\left(\sqrt{s/K}\right)}{\exp\left(-\sqrt{s/K}\right) + \exp\left(\sqrt{s/K}\right)}\hat{f}(s), \qquad B = \frac{\exp\left(-\sqrt{s/K}\right)}{\exp\left(-\sqrt{s/K}\right) + \exp\left(\sqrt{s/K}\right)}\hat{f}(s).$$

Then

$$\hat{u}(x,s) = \frac{\exp\left(-(x-1)\sqrt{s/K}\right)}{\exp\left(-\sqrt{s/K}\right) + \exp\left(\sqrt{s/K}\right)}\hat{f}(s) + \frac{\exp\left(-(1-x)\sqrt{s/K}\right)}{\exp\left(-\sqrt{s/K}\right) + \exp\left(\sqrt{s/K}\right)}\hat{f}(s),$$

$$= \frac{\exp\left(-x\sqrt{s/K}\right)}{\exp\left(-2\sqrt{s/K}\right) + 1}\hat{f}(s) + \frac{\exp\left(-(2-x)\sqrt{s/K}\right)}{\exp\left(-2\sqrt{s/K}\right) + 1}\hat{f}(s),$$

and since the formula for the sum of a geometric series implies that,

$$\frac{1}{\exp\left(-2\sqrt{s/K}\right) + 1} = \sum_{n=0}^{\infty}(-1)^n \exp\left(-2n\sqrt{s/K}\right),$$

this becomes

$$\hat{u}(x,s) = \hat{f}(s)\sum_{n=0}^{\infty}(-1)^n \exp\left(-(x+2n)\sqrt{s/K}\right) + \hat{f}(s)\sum_{n=0}^{\infty}(-1)^n \exp\left((2n+2-x)\sqrt{s/K}\right).$$

It follows from (S.1) that

$$L^{-1}\left[\exp\left(-\frac{x}{\sqrt{K}}\sqrt{s}\right)\right] = \frac{x}{\sqrt{4\pi K t^3}}\exp\left(-\frac{x^2}{4Kt}\right) := G(x,Kt), \qquad t > 0,$$

Then, using property 7, we find

$$u(x,t) = \sum_{n=0}^{\infty}(-1)^n\int_0^t G(x+2n,K(t-\tau))f(\tau)\,d\tau + \sum_{n=0}^{\infty}(-1)^n\int_0^t G(2n+2-x,K(t-\tau))f(\tau)\,d\tau$$

$$= \int_0^t G(x,K(t-\tau))f(\tau)\,d\tau + \int_0^t G(2-x,K(t-\tau))f(\tau)\,d\tau - \int_0^t G(x+2,K(t-\tau))f(\tau)\,d\tau$$

Notice that this representation for the solution looks nothing at all like the eigenfunction expansion obtained previously. However, this problem has a unique solution so the two representations must produce identical results. What can, in fact, be seen is that since each representation involves an infinite series of which only a finite number of terms can actually be computed, each representation produces only an approximate solution. Moreover, it can be shown that the Laplace transform approximation is most accurate at small values of t while the eigenfunction expansion is more accurate as t grows large.

Appendix Some Laplace Transform Formulas

We will derive some Laplace transform formulas which are useful in solving problems involving the heat equation.

Lemma 1 If $f(t) = L^{-1}\left[\hat{f}(s)\right]$, then $L^{-1}\left[\hat{f}(\sqrt{s}\,)\right] = \dfrac{1}{\sqrt{4\pi t^3}} \int_0^\infty z\, e^{-z^2/4t} f(z)\, dz.$

Proof- if the lemma holds, then

$$\hat{f}(\sqrt{s}\,) = \int_0^\infty e^{-st}\left[\frac{1}{\sqrt{4\pi t^3}} \int_0^\infty z\, e^{-z^2/4t} f(z)\, dz\right] dt$$

$$= \int_0^\infty \frac{1}{\sqrt{\pi}} f(z)\left[\int_0^\infty z\, e^{-z^2/4t}\, \frac{e^{-st}}{\sqrt{4t^3}}\, dt\right] dz.$$

Let $\tau = \dfrac{z}{\sqrt{4t}}$ *then* $d\tau = -\frac{1}{2}\dfrac{z}{\sqrt{4t^3}}\, dt$

$t = \dfrac{z^2}{4\tau^2}$ and as t goes from 0 to ∞, τ goes from ∞ to 0.

Then

$$\hat{f}(\sqrt{s}\,) = \int_0^\infty \frac{1}{\sqrt{\pi}} f(z)\left[\int_0^\infty 2 e^{-z^2 s/4\tau^2} e^{-\tau^2}\, d\tau\right] dz$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty f(z)\left[\frac{\sqrt{\pi}}{2} e^{-z\sqrt{s}}\right] dz = \int_0^\infty f(z) e^{-z\sqrt{s}}\, dz.$$

Here we used the following result,

$$\int_0^\infty e^{-z^2 s/4\tau^2} e^{-\tau^2}\, d\tau = \frac{\sqrt{\pi}}{2} e^{-z\sqrt{s}}. \blacksquare$$

Applications of the Lemma

We have that

for $f(t) = \delta(t-a)$ $\quad \hat{f}(s) = e^{-as}.$

Then, by lemma 1,

$$L^{-1}\left[\hat{f}(s)\right] = L^{-1}\left[e^{-a\sqrt{s}}\right] = \frac{1}{\sqrt{4\pi t^3}}\int_0^\infty z\,e^{-z^2/4t}\delta(z-a)\,dz$$

$$L^{-1}\left[e^{-a\sqrt{s}}\right] = \frac{1}{\sqrt{4\pi t^3}}a\,e^{-a^2/4t} \qquad for\ a \geq 0. \tag{A.1}$$

Now, if we integrate both sides of (A.1) with respect to the parameter a from b to ∞,

$$L^{-1}\left[\int_b^\infty e^{-a\sqrt{s}}\,da\right] = \frac{1}{\sqrt{4\pi t^3}}\int_b^\infty a\,e^{-a^2/4t}da,$$

we get

$$L^{-1}\left[\frac{1}{\sqrt{s}}e^{-b\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}e^{-b^2/4t}, \qquad b \geq 0. \tag{A.2}$$

In particular, for $b = 0$,

$$L^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}.$$

Integrate both sides of (A.2) with respect to the parameter b from a to ∞,

$$L^{-1}\left[\int_a^\infty \frac{1}{\sqrt{s}}e^{-b\sqrt{s}}\,db\right] = \frac{1}{\sqrt{\pi t}}\int_a^\infty e^{-b^2/4t}db.$$

Let $\qquad \xi = \dfrac{b}{\sqrt{4t}} \qquad d\xi = \dfrac{db}{\sqrt{4t}}$

so that $\qquad L^{-1}\left[\int_a^\infty \dfrac{1}{\sqrt{s}}e^{-b\sqrt{s}}\,db\right] = L^{-1}\left[\dfrac{1}{s}e^{-a\sqrt{s}}\right] = \dfrac{2}{\sqrt{\pi}}\int_{\frac{a}{\sqrt{4t}}}^\infty e^{-\xi^2}d\xi.$

If we define $\qquad \text{erfc}(x) = \dfrac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}d\xi,$

then

$$L^{-1}\left[\frac{1}{s}e^{-a\sqrt{s}}\right] = erfc\left(\frac{a}{\sqrt{4t}}\right) \tag{A.3}$$

Proceeding, we integrate both sides of (A.3) with respect to the parameter a from b to ∞,

$$L^{-1}\left[\int_b^\infty \frac{1}{s}e^{-a\sqrt{s}}\,da\right] = \int_b^\infty \text{erfc}\left(\frac{a}{\sqrt{4t}}\right)da.$$

We let $\quad \xi = \frac{a}{\sqrt{4t}} \qquad d\xi = \frac{da}{\sqrt{4t}}$

and then,

$$L^{-1}\left[\int_b^\infty \frac{1}{s} e^{-a\sqrt{s}}\, da\right] = L^{-1}\left[\frac{1}{s^{3/2}} e^{-b\sqrt{s}}\right] = \sqrt{4t}\int_{\frac{b}{\sqrt{4t}}}^\infty \text{erfc}(\xi)\, d\xi.$$

That is,

$$L^{-1}\left[\frac{1}{s^{3/2}} e^{-b\sqrt{s}}\right] = \sqrt{4t}\; ierfc\left(\frac{b}{\sqrt{4t}}\right) \tag{A.4}$$

Here

$$\text{ierfc}(x) = \int_x^\infty \text{erfc}(\xi)\, d\xi$$

denotes the so called, iterated complementary error function. It can be shown that

$$\text{ierfc}(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} - x\,\text{erfc}(x).$$

so the iterated complementary error function can be expressed in terms of other functions.

Finally, there are two Laplace transform pairs that are obtainable by elementary means:

$$L^{-1}\left[\frac{1}{s(s+c)}\right] = \frac{1}{c}(1 - e^{-ct})$$

and

$$L^{-1}\left[\frac{e^{-as}}{s(s+c)}\right] = \frac{1}{c}(1 - e^{-c(t-a)})H(t-a) := F_a(t).$$

Then lemma 1 implies

$$L^{-1}\left[\frac{e^{-a\sqrt{s}}}{\sqrt{s}\,(\sqrt{s}+c)}\right] = \frac{1}{\sqrt{4\pi t^3}}\int_0^\infty z\, e^{-z^2/4t} F_a(z)\, dz.$$

$$= \frac{1}{\sqrt{4\pi t^3}}\int_a^\infty z\, e^{-z^2/4t}\frac{1}{c}(1 - e^{-c(z-a)})\, dz.$$

Then integration by parts leads to

$$\int_a^\infty z\, e^{-z^2/4t}\frac{1}{c}(1 - e^{-c(z-a)})\, dz = 2te^{ac}\int_a^\infty e^{-cz}\, e^{-z^2/4t}\, dz$$

so

$$L^{-1}\left[\frac{e^{-a\sqrt{s}}}{\sqrt{s}\,(\sqrt{s}+c)}\right] = \frac{e^{ac}}{\sqrt{\pi t}}\int_a^\infty e^{-cz}\, e^{-z^2/4t}\, dz,$$

$$= \frac{e^{ac}}{\sqrt{\pi t}} \int_a^\infty e^{-(z^2+4zct+4c^2t^2)/4t} e^{-c^2 t} \, dz,$$

$$= \frac{e^{ac+ct^2}}{\sqrt{\pi t}} \int_a^\infty e^{-(z+2ct)^2/4t} \, dz,$$

$$= \frac{2}{\sqrt{\pi}} e^{ac+ct^2} \int_{\frac{a}{\sqrt{4t}}+c\sqrt{t}}^\infty e^{-\xi^2} \, d\xi$$

i.e.,

$$L^{-1}\left[\frac{e^{-a\sqrt{s}}}{\sqrt{s}\,(\sqrt{s}+c)} \right] = e^{ac+ct^2} \operatorname{erfc}\left(\frac{a}{\sqrt{4t}} + c\sqrt{t} \right).$$