بسم اللّه الرّحمن الرّحیم

دانشگاه صنعتی اصفهان ــ دانشکدهٔ مهندسی برق و کامپیوتر
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طراحی الگوریتم‌ها

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حال که دیدیم الگوریتم ***brute-force*** بسیار ناکارآمد عمل می‌کند، سعی می‌کنیم تا از راهبرد برنامه‌ریزی پویا استفاده کنیم:

Suppose that keys Key_i through Key_j are arranged in a tree that minimizes

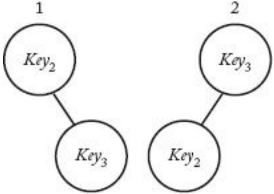
$$\sum_{m=i}^{j} c_m p_m,$$

where c_m is the number of comparisons needed to locate Key_m in the tree. We will call such a tree optimal for those keys and denote the optimal value by $A[i][j]$. Because it takes one comparison to locate a key in a tree containing one key, $A[i][i] = p_i$.

Example: Suppose we have three keys. If

$$p_1 = 0.7, \quad p_2 = 0.2, \quad p_3 = 0.1,$$

then, to determine $A[2][3]$ we must consider following two trees:



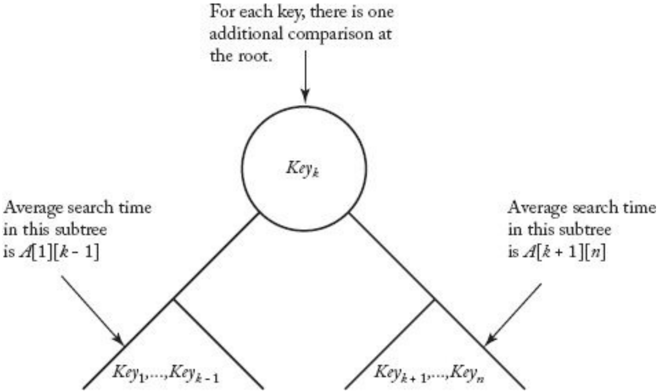
For these two trees we have the following:

1. $1(p_2) + 2(p_3) = 1(0.2) + 2(0.1) = 0.4$
2. $2(p_2) + 1(p_3) = 2(0.2) + 1(0.1) = 0.5$

The first tree is optimal, and $A[2][3] = 0.4$.

Any subtree of an optimal tree must be optimal for the keys in that subtree.☞ *The principle of optimality applies.*

در درخت بهینه یکی از کلیدها (که نمی‌دانیم کدام است) در ریشه قرار می‌گیرد:



For each $m \neq k$ it takes exactly one more comparison (the one at the root) to location Key_m than it does to locate that key in the subtree that contains it. This one comparison adds $1 \times p_m$ to the average search time for Key_m.

$$\underbrace{A[1][k-1]}_{\substack{\text{Average time in}\\ \text{left subtree}}} + \underbrace{p_1 + \cdots + p_{k-1}}_{\substack{\text{Additional time}\\ \text{comparing at root}}} + \underbrace{p_k}_{\substack{\text{Average time}\\ \text{searching for}\\ \text{root}}} + \underbrace{A[k+1][n]}_{\substack{\text{Average time in}\\ \text{right subtree}}} + \underbrace{p_{k+1} + \cdots + p_n}_{\substack{\text{Additional time}\\ \text{comparing at root}}},$$

which equals

$$A[1][k-1] + A[k+1][n] + \sum_{m=1}^{n} p_m.$$

The average search time for the optimal tree is given by

$$A[1][n] = \underset{1 \le k \le n}{minimum}(A[1][k-1] + A[k+1][n]) + \sum_{m=1}^{n} p_m,$$

$A[1][0]$ and $A[n+1][n]$ are defined to be 0. Although the sum of the probabilities in this last expression is clearly 1, we have written it as a sum because we now wish to generalize the result.

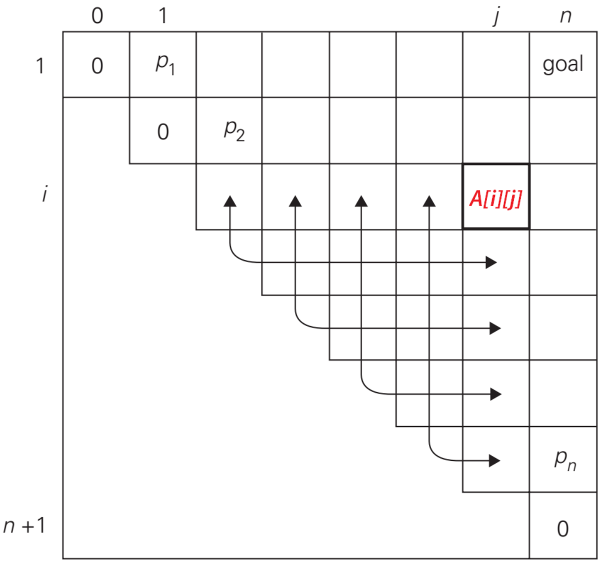
$$A[i][j] = \underset{i \le k \le j}{minimum}(A[i][k-1] + A[k+1][j]) + \sum_{m=i}^{j} p_m \quad i < j$$

$$A[i][i] = p_i$$

$A[i][i-1]$ and $A[j+1][j]$ are defined to be 0.

Because $A[i][j]$ is computed from entries in the ith row but to the left of $A[i][j]$ and from entries in the jth column but beneath $A[i][j]$, we proceed by computing in sequence the values on each diagonal.

عیناً مانند الگوریتمی که برای مسئلهٔ ضرب زنجیره‌ای ماتریس‌ها ارائه دادیم.


0
1
j
n
1
i
n +1
0
p_1
goal
0
p_2
A[i][j]
p_n
0

The array R produced by the algorithm contains the indices of the keys chosen for the root at each step. For example, $R[1][2]$ is the index of the key in the root of an optimal tree containing the first two keys, and $R[2][4]$ is the index of the key in the root of an optimal tree containing the second, third, and fourth keys. After analyzing the algorithm, we will discuss how to build an optimal tree from R.

Optimal Binary Search Tree

***Problem:** Determine an optimal binary search tree for a set of keys, each with a given probability of being the search key.*

***Inputs:** n, the number of keys, and an array of real numbers p indexed from 1 to n, where $p[i]$ is the probability of searching for the ith key.*

***Outputs:** A variable $minavg$, whose value is the average search time for an optimal binary search tree; and a two-dimensional array R from which an optimal tree can be constructed. R has its rows indexed from 1 to $n+1$ and its columns indexed from 0 to n. $R[i][j]$ is the index of the key in the root of an optimal tree containing the ith through the jth keys.*

```
void optsearchtree (int n,
                    const float p[],
                    float& minavg,
                    index R[][])
{
  index i, j, k, diagonal;
  float A[1..n + 1][0..n];

 for (i = 1; i < = n; i++){
    A[i][i - 1] = 0;
    A[i][i] = p[i];
    R[i][i] = i;
    R[i][i - 1] = 0;
 }
 A[n + 1][n] = 0;
 R[n + 1][n] = 0;
 for (diagonal = 1; diagonal < = n - 1; diagonal++)
    for (i = 1; i <= n - diagonal; i++){          // Diagonal-1 is
                                                  // just above the
                                                  // main diagonal.
      j = i + diagonal;
      A[i][j] = minimum(A[i][k - 1] + A[k + 1][j]) + Σ_{m=i}^{j} p_m;
                i≤k≤j
      R[i][j] = a value of k that gave the minimum;
    }
 minavg = A[1][n];
}
```

$$A[i][j] = \min_{i \le k \le j}(A[i][k-1] + A[k+1][j]) + \sum_{m=i}^{j} p_m$$

Every-Case Time Complexity (Optimal Binary Search Tree)

Basic operation: ***The instructions executed for each value of k.***

Input size: ***n, the number of keys.***

تحلیل این الگوریتم بسیار شبیه به آنچه برای الگوریتم ضرب زنجیره ای ماتریسها گفتیم است:

$$\sum_{diagonal=1}^{n-1}(n-diagonal)(diagonal+1)=\frac{n(n-1)(n+4)}{6}\in\Theta(n^3).$$

Recall that R contains the indices of the keys chosen for the root at each step.

Build Optimal Binary Search Tree

Problem: *Build an optimal binary search tree.*

Inputs: *n, the number of keys, an array Key containing the n keys in order, and the array R produced by the previous algorithm. $R[i][j]$ is the index of the key in the root of an optimal tree containing the ith through the jth keys.*

Outputs: *A pointer tree to an optimal binary search tree containing the n keys.*

```
node_pointer tree (index i, j)
{
  index k;
  node_pointer p;

  k = R[i][j];
  if (k == 0)
     return NULL;
  else{
     p = new nodetype;
     p-> key = Key[k];
     p-> left = tree(i, k - 1);
     p-> right = tree(k + 1, j);
     return p;
  }
}
```

Following our convention for recursive algorithms, the parameters n, Key, and R are not inputs to function tree. If the algorithm were implemented by defining n, Key, and R globally, a pointer $root$ to the root of an optimal binary search $tree$ is obtained by calling tree as follows: $root = tree(1, n);$

Example:

Don	Isabelle	Ralph	Wally
Key[1]	Key[2]	Key[3]	Key[4]

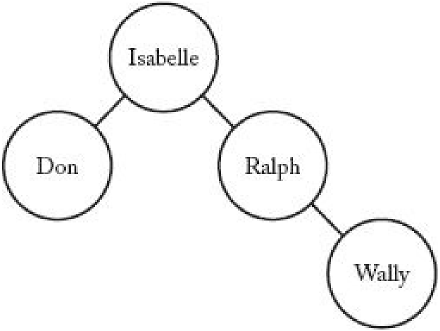
$$p_1 = \frac{3}{8}, \quad p_2 = \frac{3}{8}, \quad p_3 = \frac{1}{8}, \quad p_4 = \frac{1}{8}.$$

	0	1	2	3	4
1	0	$\frac{3}{8}$	$\frac{9}{8}$	$\frac{11}{8}$	$\frac{7}{4}$
2		0	$\frac{3}{8}$	$\frac{5}{8}$	1
3			0	$\frac{1}{8}$	$\frac{3}{8}$
4				0	$\frac{1}{8}$
5					0

A

	0	1	2	3	4
1	0	1	1	2	2
2		0	2	2	2
3			0	3	3
4				0	4
5					0

R


Isabelle
Don
Ralph
Wally

Another example:

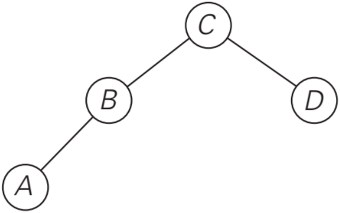
key	A	B	C	D
probability	0.1	0.2	0.4	0.3

main table

	0	1	2	3	4
1	0	0.1	0.4	1.1	1.7
2		0	0.2	0.8	1.4
3			0	0.4	1.0
4				0	0.3
5					0

root table

	0	1	2	3	4
1		1	2	3	3
2			2	3	3
3				3	3
4					4
5					



یک الگوریتم دیگر مبتنی بر راهبرد برنامه‌ریزی پویا برای مسئلهٔ کوله‌پشتی ۱ ـ ۰

The dynamic programming algorithm we designed when we studied the 0-1 Knapsack Problem earlier uses subproblems of the form $F(i, w)$: the subproblem of finding the maximum value of any solution using a subset of the items $1, 2, 3, \ldots, i$ and a knapsack of capacity w.

In our novel approach, we define our subproblems as follows. The subproblem is defined by i and a target value V, and $M(i, V)$ is the smallest knapsack capacity so that one can obtain a solution using a subset of items $\{1, 2, \ldots, i\}$ with value at least V.

We will have a subproblem for all $i = 0, 1, 2, \ldots, n$ and values $V = 0, 1, \ldots, \sum_{j=1}^{i} v_j$.

If v^* denotes $\max_i v_i$, then we see that the largest V can get in a subproblem is $\sum_{j=1}^{n} v_j \leq nv^*$. Thus, assuming the values are integral, there are at most $O(n^2v^*)$ subproblems. None of these subproblems is precisely the original instance of Knapsack, but if we have the values of all subproblems $M(n, V)$ for $V = 0, 1, \ldots, \sum_i v_i$, then the value of the original problem can be obtained easily: it is the largest value V such that $M(n, V) \leq W$.

It is not hard to give a recurrence for solving the subproblem $M(i, V)$. We consider cases depending on whether or not the last item i is included in the optimal solution.

دو حالت داریم: اگر داشته باشیم $V > \sum_{k=1}^{i-1} v_k$ پس قطعاً باید iاُمین آیتم نیز در مجموعهٔ ما قرار گیرد. در نتیجه داریم

$$M(i, V) = w_i + M(i-1, \max(V - v_i, 0)).$$

در غیر اینصورت، یعنی اگر $V \leq \sum_{k=1}^{i-1} v_k$، آنگاه ممکن است iاُمین آیتم در مجموعهٔ ما قرار بگیرد یا نگیرد. پس داریم

$$M(i, V) = \min\left(M(i-1, V), w_i + M(i-1, \max(V - v_i, 0))\right)$$

We can then write down our dynamic programming algorithm. Our algorithm takes $O(n^2 v^*)$ time and correctly computes the optimal values of the subproblems. As was done before, we can trace back through the table M containing the optimal values of the subproblems, to find an optimal solution.

```
Knapsack(n,W,w[],v[]):
    Array M[0,1,...,n][0,1,...,V]
    For i = 0,1,...,n
        M[i,0] = 0
    Endfor
    For i = 1,2,...,n
        For V = 1,2,...,∑_{k=1}^{i} v_k
            If V > ∑_{k=1}^{i-1} v_k then
                M[i,V] = w_i + M[i-1, max(0, V - v_i)]
            Else
                M[i,V] = min(M[i-1,V], w_i + M[i-1, max(0, V - v_i)])
            Endif
        Endfor
    Endfor
    Return the maximum value V such that M[n,V] ≤ W
```

Capacity = 16

item	weight	value
1	2	4
2	5	3
3	10	5
4	5	1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0													
1	0	2	2	2	2									
2	0	2	2	2	2	7	7	7						
3	0	2	2	2	2	7	7	7	12	12	17	17	17	
4	0	2	2	2	2	7	7	7	12	12	17	17	17	22

Therefore, the answer is 9.