Stability and Accuracy Assessment of Wave Equation FVM Solution

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Project information:
Course:
Computational Fluid Dynamics
Project No.:
No. 2
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Created:
May 2022

Abstract

In this project we solved wave equation using FVM method and examined its behavior regarding different discretization methods and time schemes. In the beginning an example is solved to describe the whole problem, then the wave equation is spatially discretized and is solved using 2nd order *Runge-Kutta* time scheme. The numerical solution is compared with analytical solution and the error has been calculated, then the order of solution validated the solution. In the end the results are discussed. A *Python* script has been written to solve this problem.

1 An Example

Consider Eq.(1) subjected to the mentioned initial condition:

$$\frac{d\nu}{dt} = -\nu + \sin t \ ; \ \nu(0) = 1 \tag{1}$$

The exact solution of Eq.(1) is:

$$\nu = \nu_h + \nu_p$$

$$\frac{d\nu_h}{dt} = -\nu_h \rightarrow \nu_h = ce^{-t}$$

$$\nu_p = A\sin t + B\cos t$$

Substituting ν_p into Eq.(1):

$$A = \frac{1}{2}$$

$$B = -\frac{1}{2}$$

So, the exact solution is:

$$\nu = \frac{3}{2}e^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t \tag{2}$$

Using *Runge-Kutta* 2nd order as follows:

$$\nu^{(1)} = \nu^m + \frac{\Delta t}{2} f(t^m, \nu^m)$$

$$\nu^{m+1} = \nu^m + \Delta t f(t^m + \frac{\Delta t}{2}, \nu^{(1)}) \tag{3}$$

Also, the maximum stable time step based on the homogeneous Eq.(1) for *Runge-Kutta* 2nd order time scheme :

$$G = \frac{\nu^{m+1}}{\nu^m} = 1 - \Delta t + \frac{\Delta t^2}{2} \tag{4}$$

Applying stability condition $|G| \leq 1$:

$$0 \leq \Delta t \leq 2 \tag{5}$$

So, the maximum stable time step is 2.

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1.1 Example Results

1.1.1 Solution

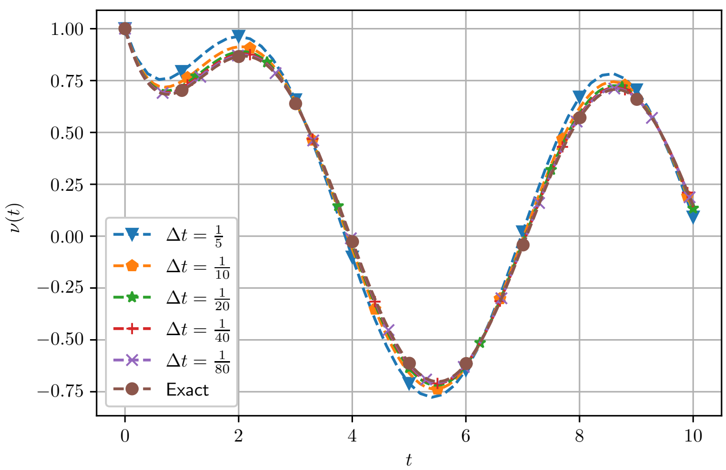

Figure 1: Exact vs Numerical Solution

In Fig.1 the numerical solution is compared with Exact solution in different time steps.

1.1.2 Second Order Solution

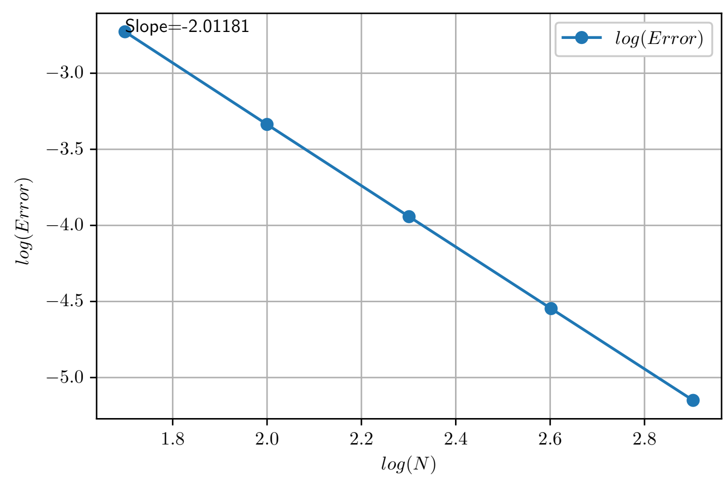

Figure 2: Error

In Fig.2 the Error line slope is found 2, as we expected for our second order solution.

2 Problem Description

The Wave equation to be solved is:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0 \tag{6}$$

Where:

$$u = 2$$

$$0 \leq x \leq 1 \ ; \ 0 \leq t$$

Subjected to boundary and initial conditions:

$$\begin{cases} T(x,0) & = 0 \\ T(0,t) & = \sin(2\pi t) \end{cases} \tag{7}$$

Eq.(6) has an exact solution as following:

$$T(x,t) = \begin{cases} \sin\left(2\pi\left(t - \frac{x}{2}\right)\right) & ; \ x \leq 2t \\ 0 & ; \ x > 2t \end{cases} \tag{8}$$

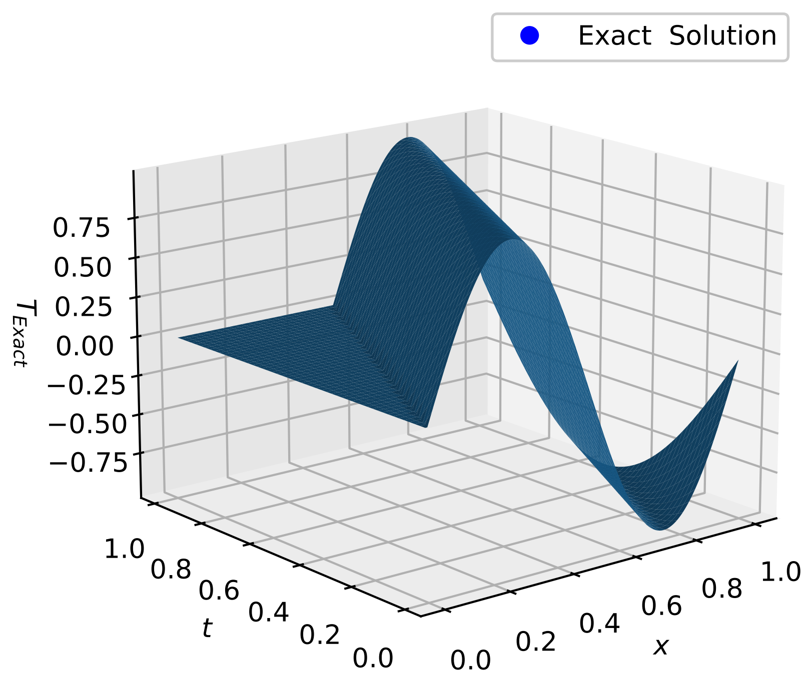

Figure 3: Exact Solution for $0 \leq x \leq 1$ and $0 \leq t \leq 1$

3 Numerical Solution

3.1 Spatial Discretization

Methods used in this project for spatial descretization of the wave equation (Eq.(6)) are 1st and 2nd order upwind and 2nd order central discretization.

Integrating Eq.(6) on CVs we get:

$$\int_{CV} \frac{\partial T}{\partial t} dv = -u \int_{CV} \frac{\partial T}{\partial x} dx$$

Using Gauss's theorem:

$$\frac{\partial T}{\partial t} \Delta x = -u \int_{CS} T.nds$$

So,

$$\frac{\partial T}{\partial t} = -\frac{u}{\Delta x} \left(T_{i+\frac{1}{2}} - T_{i-\frac{1}{2}} \right) \tag{9}$$

There are many ways to find $T_{i+\frac{1}{2}}$ and $T_{i-\frac{1}{2}}$ in Eq.(9). In this project we considered following methods:

A) First Order Upwind

$$\frac{\partial T}{\partial t} = -\frac{u}{\Delta x} (T_i - T_{i-1}) \tag{10}$$

B) Second Order Upwind

$$\frac{\partial T}{\partial t} = -\frac{u}{2\Delta x} (3T_i - 4T_{i-1} + T_{i-2}) \tag{11}$$

C) Second Order Central

$$\frac{\partial T}{\partial t} = -\frac{u}{2\Delta x} (T_{i+1} - T_{i-1}) \tag{12}$$

3.2 Time Schemes

Left side of Eq.s (10,11,12) need to be solved numerically. Among various methods two are discussed in this project 2nd and 4th order *Runge-Kutta*.

A) Second Order *Runge-Kutta*

$$T^{(1)} = T^m + \frac{\Delta t}{2} f(t^m, T^m)$$

$$T^{m+1} = T^m + \Delta t f(t^m + \frac{\Delta t}{2}, T^{(1)}) \tag{13}$$

B) Forth Order *Runge-Kutta*

$$T^{(1)} = T^m + \frac{\Delta t}{4} f(t^m, T^m)$$

$$T^{(2)} = T^m + \frac{\Delta t}{3} f(t^m + \frac{\Delta t}{4}, T^{(1)})$$

$$T^{(3)} = T^m + \frac{\Delta t}{2} f(t^m + \frac{\Delta t}{3}, T^{(2)})$$

$$T^{m+1} = T^m + \Delta t f(t^m + \frac{\Delta t}{2}, T^{(3)}) \tag{14}$$

4 Results and Discussions

4.1 2$^{\text{nd}}$ Order Upwind and Runge-Kutta

Using *second order upwind* spatial descritization combined with *second order Runge-Kutta* time scheme, we expect second order solution as we can see in Fig.4.

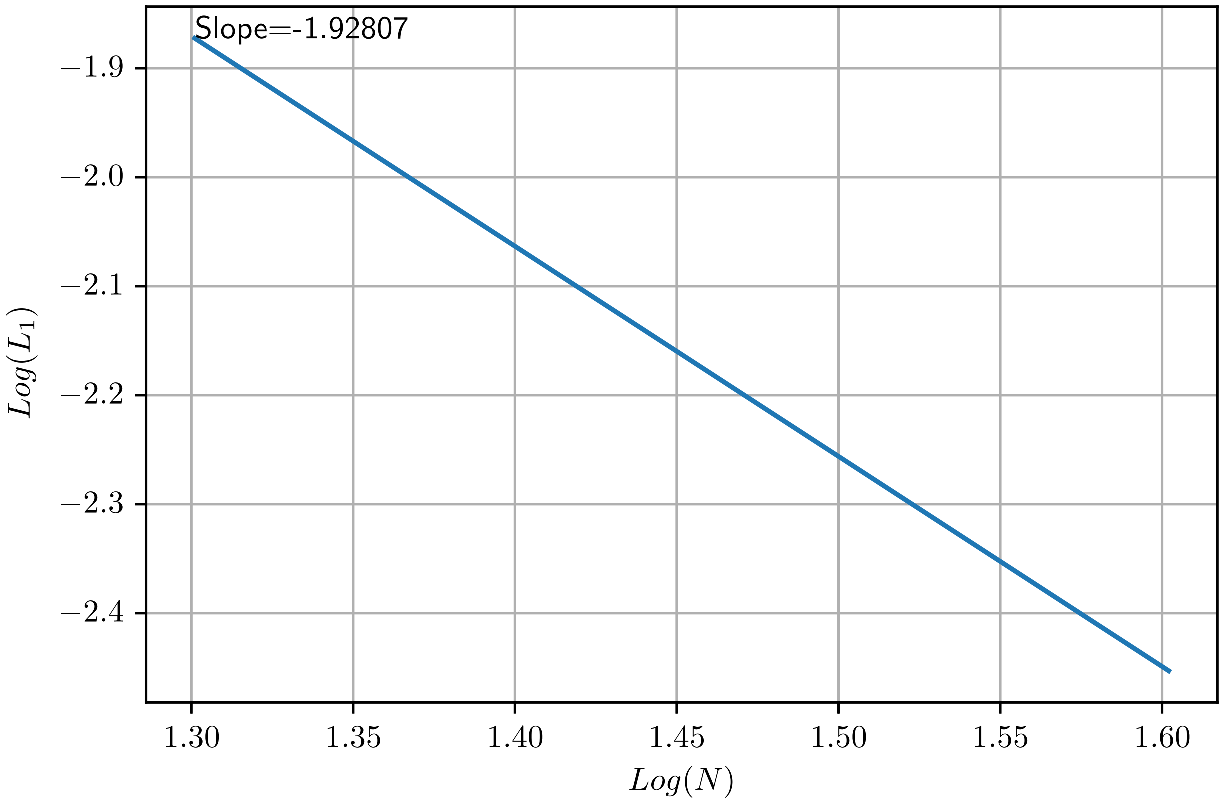

Figure 4: Error plot at $t = 1$ and $CFL = \frac{u\Delta t}{\Delta x} = 0.25$

About the smallest time step or largest Δt we can see in Fig.5 with decreasing of time step the solution will be unstable at some point. According to Fig.5 we can say this smallest time step is about 35 at $t = 0.25$.

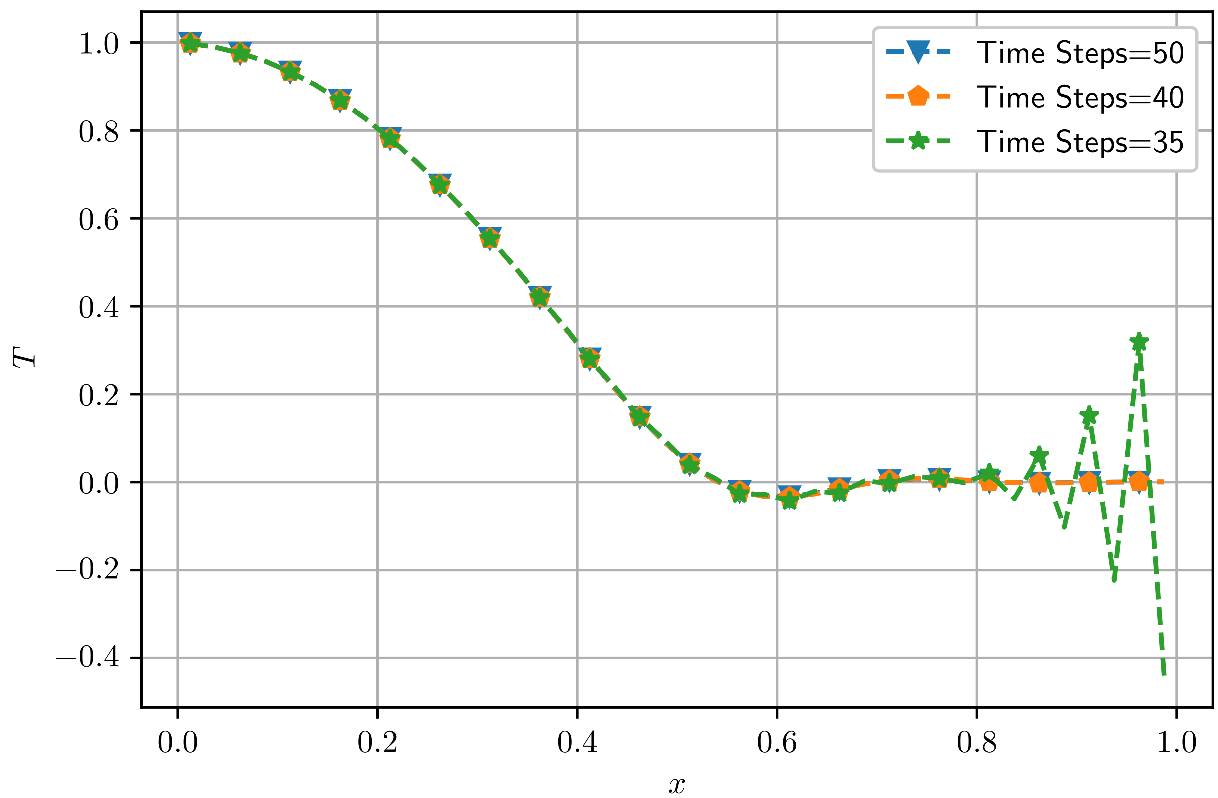

Figure 5: Wave curve at $t = 0.25$

4.2 1st Order Upwind and 2nd Order Runge-Kutta

In Fig.6 first and second order upwind combined with second order Runge-Kutta are compared. According to this figure increasing of the number of Control Volumes will decrease the error but still the second order upwind is the closest one to exact solution in the same number of CVs.

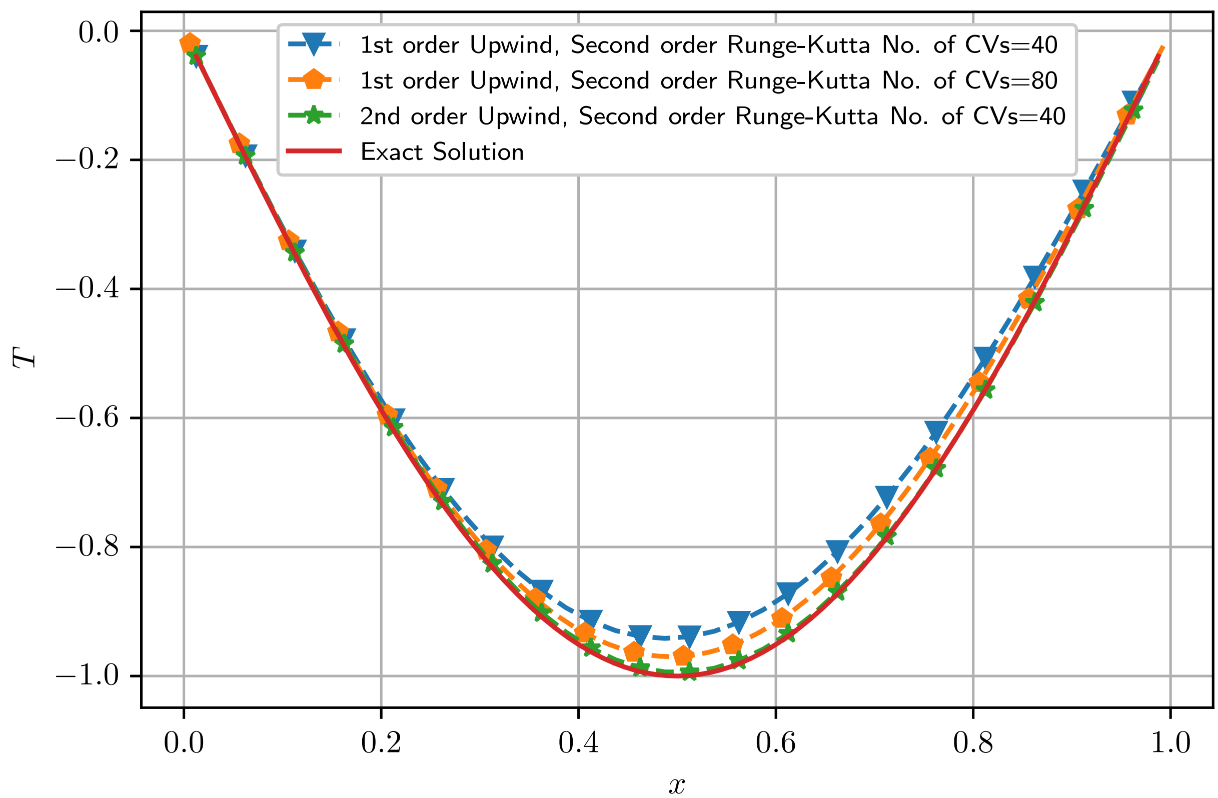

Figure 6: Wave curves at $t = 1$ and $CFL = 0.25$

4.3 2nd Order Central and Runge-Kutta

As we know central discretization is inherently unstable in the following Fig.7 we can see this. Note that we can not get any good results by changing the CFL number.

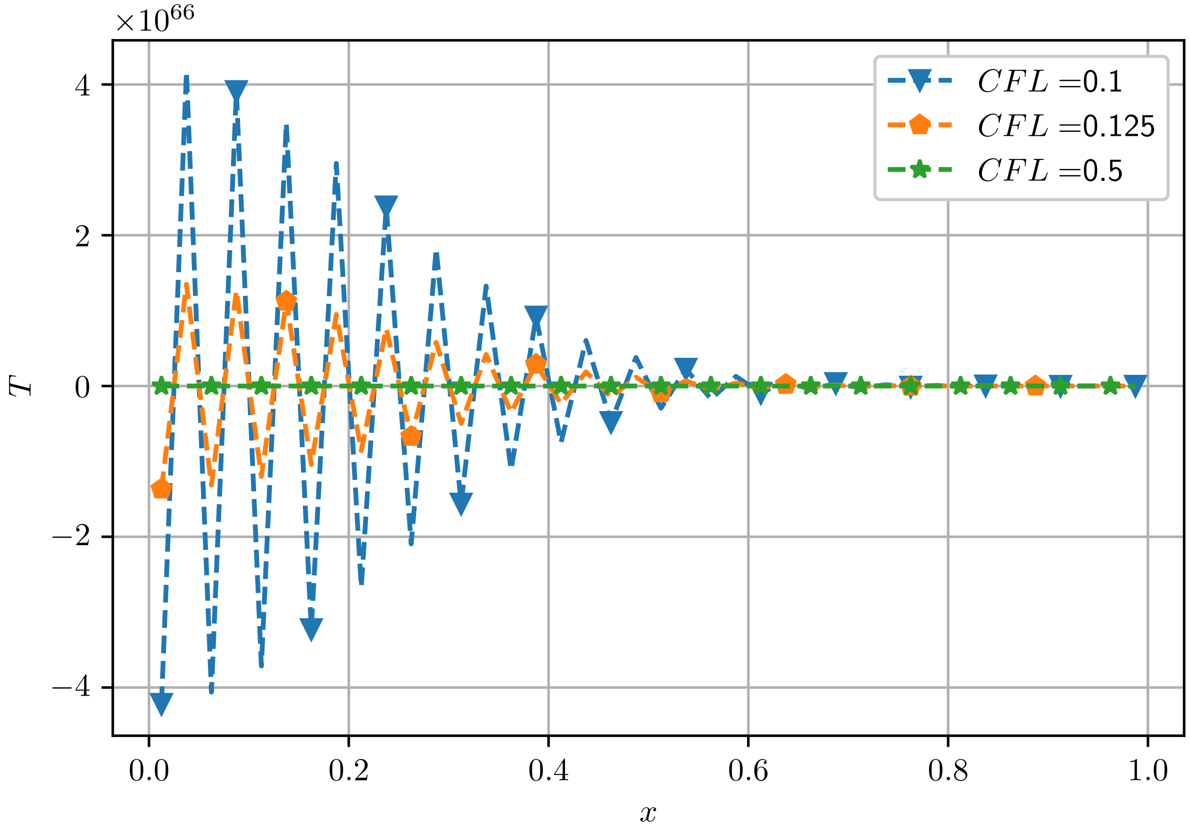

Figure 7: Wave Noisy curves at $t = 1$ and $N = 40$
Lower CFL means smaller time steps

4.4 2$^{\text{nd}}$ Order Upwind and 4$^{\text{nd}}$ Order Runge-Kutta

Second order upwind discretization and fourth order Runge-Kutta time scheme will not change the order of solution at all, because our spatial descretization is second order, higher order time schemes will not increase the order of solution, but of course time schemes lower than second order will affect our second order spatial descretization, You can see this in Fig.8 and 9.

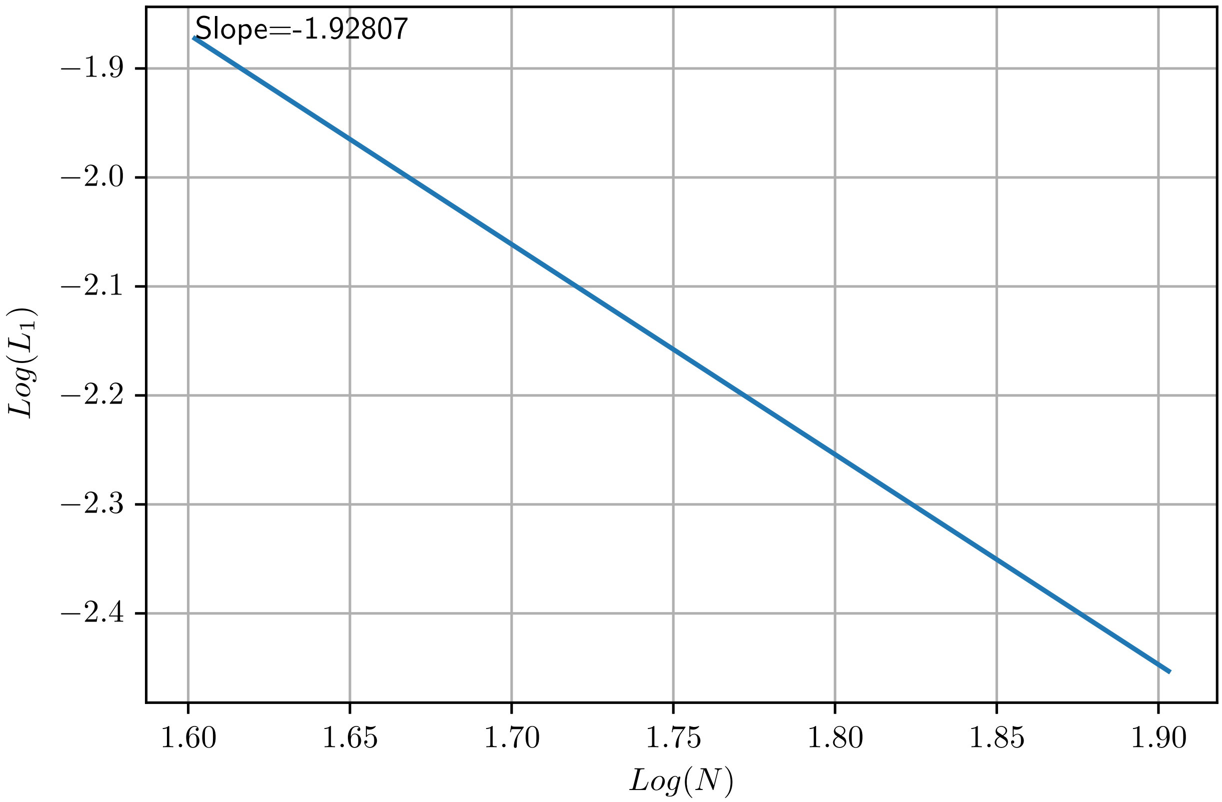

Figure 8: Error for fourth order Runge-Kutta and second order upwind at $t = 1$ and $CFL = 0.25$

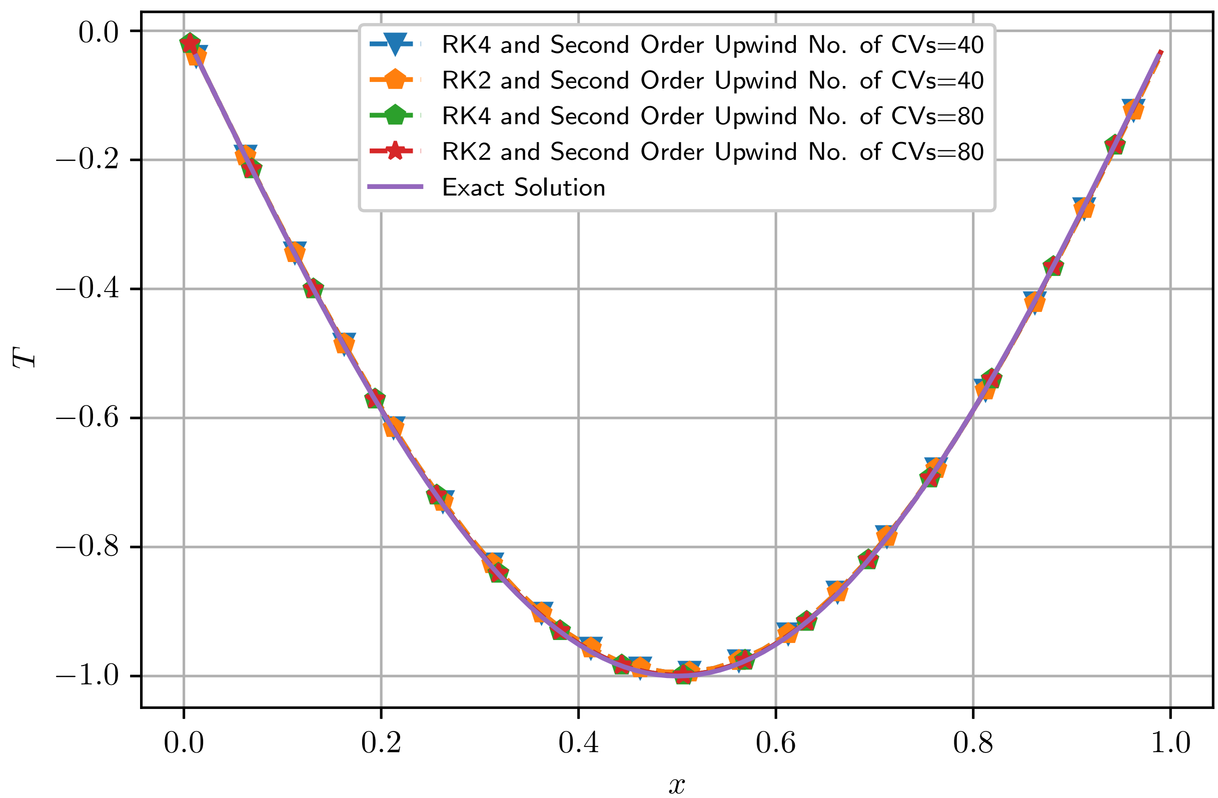

Figure 9: Comparing RK4, Rk2 combined with second order upwind at $t = 1$ and $CFL = 0.25$

5 Conclusion

In this project we examined the behavior of wave equation with different time schemes and spatial descretizations. our spatial discretization was limited to second order in this project so we could not see the actual effect of a fourth order solution on the error. Further examinations can be done by considering higher order spatial descretizations.