Universidade de Brasília
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Combined Proof Methods for Multimodal Logic

Daniella Angelos

Dissertação apresentada como requisito parcial para
qualificação do Mestrado em Informática

Orientadora
Prof.a Dr.a Cláudia Nalon

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Abstract

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Modal Logics

This chapter formally introduces K_n, a *propositional modal logic language*, semantically determined by an account of necessity and possibility [16].

A propositional modal language is the well known propositional language augmented by a collection of *modal operators* [3]. In classical logic, propositions or sentences are evaluated to either true or false, in any model. Propositional logic and predicate logic, for instance, do not allow for any further possibilities. However, in natural language, we often distinguish between various modalities of truth, such as *necessarily* true, *known to be* true, *believed to be* true or yet true *in some future*, for example. Therefore, one may think that classical logics lacks expressivity in this sense.

Modal logics add operators to express one or more of these different modes of truth. Different modalities define different languages. The key concept behind these operators is that they allow us to reason over relations among different contexts or interpretations, an abstraction that here we think as *possible worlds*. The purpose of the modal operators allow is to permit the information that holds at other worlds to be examined — but, crucially, only at worlds visible or accessible from the current one via an accessibility relation [3]. Then, the evaluation of a modal formula depends on the set of possible worlds and the accessibility relations defined for these worlds. It is possible to define several accessibility relations between worlds, and different modal logics are defined by different relations.

The modal language which is the focus of this work is the extension of the classical from the point of view propositional logic plus the unary operators $\boxed{a}$ and $\langle a \rangle$, whose reading are "is necessary by the agent a" and "is possible by the agent a", respectively. This language, known as K_n, is characterized by the schema $\boxed{a}(\varphi \Rightarrow \psi) \Rightarrow (\boxed{a}\varphi \Rightarrow \boxed{a}\psi)$ (axiom K), where a is an index represents an agent in a finite, fixed set, and φ, ψ are well-formed formulae. The addition of other axioms defines different systems of modal logics and it imposes restrictions on the class of models where formulae are valid [5].

Worlds and their accessibility relations define a structure known as a *Kripke model*, a

This is the definition of frame. For models, you also need a valuation function.

structure proposed by Kripke to semantically analyse modal logics [15]. The satisfiability and validity of a formula depend on this structure. For example, given a Kripke model that contains a set of possible worlds, a binary relation of accessibility between worlds and a valuation function that maps in which worlds a proposition symbol holds, we say that a formula $\Box p$ is satisfiable at some world w of this model, if the valuation function establishes that p is true at all worlds accessible from w.

In the following, we will formally define the modal language ~~we will be working with~~. The syntax and semantics of K_n are showed in Sections 2.1 and 5, respectively, and the definitions presented in these two sections follow those in [16]. Section 5?

2.1 Syntax

The language of K_n is equivalent to its set of *well-formed formulae*, denoted by $\mathsf{WFF}_{\mathsf{K}_n}$, which is constructed from a denumerable set of *propositional symbols* or *variables* $\mathcal{P} = \{p, q, r, \ldots\}$, the negation symbol $\neg$, the disjunction symbol $\vee$ and the modal connectives $\boxed{a}$, that express the notion of necessity, for each Text index a in a finite, non-empty fixed set of labels $\mathcal{A} = \{1, \ldots, n\}, n \in \mathbb{N}$.

indexes

Definition 1 The set of well-formed formulae, $\mathsf{WFF}_{\mathsf{K}_n}$, is the least set such that:

1. $p \in \mathsf{WFF}_{\mathsf{K}_n}$, for all $p \in \mathcal{P}$
2. if $\varphi, \psi \in \mathsf{WFF}_{\mathsf{K}_n}$, then so are $\neg\varphi$, $(\varphi \vee \psi)$ and $\boxed{a}\varphi$, for each $a \in \mathcal{A}$

The operator $\langle a \rangle$ is ~~are~~ the dual of $\boxed{a}$, for each $a \in \mathcal{A}$, that is, $\langle a \rangle\varphi$ can be defined as $\neg\boxed{a}\neg\varphi$, with $\varphi \in \mathsf{WFF}_{\mathsf{K}_n}$. Other logical operators are also used as abbreviations. In this work, we consider the usual ones:

- $\varphi \wedge \psi \stackrel{\text{def}}{=} \neg(\neg\varphi \vee \neg\psi)$ (conjuction)
- $\varphi \Rightarrow \psi \stackrel{\text{def}}{=} \neg\varphi \vee \psi$ (implication)
- $\varphi \Leftrightarrow \psi \stackrel{\text{def}}{=} (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ (equivalence)
- $\mathbf{false} \stackrel{\text{def}}{=} \varphi \wedge \neg\varphi$ (*falsum*)
- $\mathbf{true} \stackrel{\text{def}}{=} \neg\mathbf{false}$ (*verum*)

Parentheses may be omitted if the reading is not ambiguous. When $n = 1$, we often omit the index in the modal operators, i.e., we just write $\Box\varphi$ ~~(or 'box' φ)~~ and $\Diamond\varphi$ ~~(or 'diamond' φ)~~, for a well-formed formula φ. You have already given the reading of [] and <>

a literal as a

We define as *literal* a propositional symbol $p \in \mathcal{P}$ or its negation $\neg p$, and denote by $\mathcal{L}$ the set of all literals. A *modal literal* is a formula of the form $\boxed{a}l$ or $\Diamond_a l$, with $l \in \mathcal{L}$ and $a \in \mathcal{A}$.

The following definitions are also needed later. The *modal depth* of a formula is recursively defined as follows:

Definition 2 We define $mdepth : \mathsf{WFF}_{\mathsf{K}_n} \longrightarrow \mathbb{N}$ inductively as:

1. $mdepth(p) = 0$
2. $mdepth(\neg\varphi) = mdepth(\varphi)$
3. $mdepth(\varphi \vee \psi) = \max\{mdepth(\varphi), mdepth(\psi)\}$
4. $mdepth(\boxed{a}\varphi) = mdepth(\varphi) + 1$

you are continuing your sentence. It's "with", lowercase.

With $p \in \mathcal{P}$ and $\varphi, \psi \in \mathsf{WFF}_{\mathsf{K}_n}$.

This function represents the maximal number of nesting operators in a formula. For instance, if $\varphi = \boxed{a}\Diamond_a p \vee \Diamond_a q, a \in \mathcal{A}$, then $mdepth(\varphi) = 2$.

The *modal level* of a formula (or a subformula) is given relative to its position in the *annotated syntactic tree*.

Definition 3 Let Σ be the alphabet $\{1, 2, .\}$ and Σ^* the set of all finite sequences over Σ. We define $\tau : \mathsf{WFF}_{\mathsf{K}_n} \times \Sigma^* \times \mathbb{N} \longrightarrow \mathscr{P}(\mathsf{WFF}_{\mathsf{K}_n} \times \Sigma^* \times \mathbb{N})$ as the partial function inductively defined as follows:

1. $\tau(p, \lambda, ml) = \{(p, \lambda, ml)\}$
2. $\tau(\neg\varphi, \lambda, ml) = \{(\neg\varphi, \lambda, ml)\} \cup \tau(\varphi, \lambda.1, ml)$
3. $\tau(\boxed{a}\varphi, \lambda, ml) = \{(\boxed{a}\varphi, \lambda, ml)\} \cup \tau(\varphi, \lambda.1, ml + 1)$
4. $\tau(\varphi \vee \psi, \lambda, ml) = \{(\varphi \vee \psi, \lambda, ml)\} \cup \tau(\varphi, \lambda.1, ml) \cup \tau(\psi, \lambda.2, ml)$

With $p \in \mathcal{P}, \lambda \in \Sigma^*, ml \in \mathbb{N}$ and $\varphi, \psi \in \mathsf{WFF}_{\mathsf{K}_n}$.

The function τ applied to $(\varphi, 1, 0)$ returns the annotated syntactic tree for φ, where each node is uniquely identified by a subformula, its position in the tree (or path order) and its modal level. For instance, p occurs twice in the formula $\boxed{a}\Diamond_a(p \wedge \boxed{a}p)$, at the position 1.1.1, with modal level 2, and again at position 1.1.2.1, with modal level 3.

Definition 4 Let φ be a formula and let $\tau(\varphi, 1, 0)$ be its annotated syntactic tree.

We define *mlevel* : $\mathsf{WFF}_{\mathsf{K}_n} \times \mathsf{WFF}_{\mathsf{K}_n} \times \Sigma^* \longrightarrow \mathbb{N}$, as: if $(\varphi', \lambda, ml) \in \tau(\varphi, 1, 0)$ then $mlevel(\varphi, \varphi', \lambda) = ml$.

This function represents the maximal number of operators in which scope a subformula occurs.

2.2 Semantics

The semantics of K_n is presented in terms of Kripke structures.

Definition 5 A Kripke model for $\mathcal{P}$ and $\mathcal{A} = \{1, \ldots, n\}$ is given by the tuple

$$\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi) \tag{2.1}$$

where W is a non-empty set of possible worlds with a distinguinshed world w_0, the root of $\mathcal{M}$; each R_a, $a \in \mathcal{A}$, is a binary relation on W, that is, $R_a \subseteq W \times W$, and $\pi : W \times \mathcal{P} \longrightarrow \{false, true\}$ is the valuation function that associates to each world $w \in W$ a truth-assignment to propositional symbols.

We write $R_a wv$ to denote that v is accessible from w through the accessibility relation R_a, that is $(w, v) \in R_a$, and $R^*_a wv$, to mean that v is reachable from w through a finite

there

number of steps, that is, it exists a sequence $(w_1, \ldots, w_k)$ of worlds such that $R_a w_i w_{i+1}$, for all $i \leq k$, where $w_1 = w$ and $w_k = v$, with $a \in \mathcal{A}$, $w, v, w_i \in W$ and $i, k \in \mathbb{N}$. Note that R^*_a is the *transitive closure* of R_a, the least transitive set that contains all elements of R_a. In this work, we will also use the *transitive and reflexive closure*, denoted by R^+_a, the least transitive and reflexive set that contains all elements of R_a.

are defined

Satisfiability and *validity* of a formula is defined in terms of the *satisfiability relation*.

Definition 6 Let $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ be a Kripke model, $w \in W$ and $\varphi, \psi \in \mathsf{WFF}_{\mathsf{K}_n}$. The *satisfiability relation*, denoted by $\langle \mathcal{M}, w \rangle \models \varphi$, between a world w and a formula φ, is inductively defined by:

1. $\langle \mathcal{M}, w \rangle \models p$ if, and only if, $\pi(w, p) = true$, for all $p \in \mathcal{P}$;
2. $\langle \mathcal{M}, w \rangle \models \neg\varphi$ if, and only if, $\langle \mathcal{M}, w \rangle \not\models \varphi$;
3. $\langle \mathcal{M}, w \rangle \models \varphi \vee \psi$ if, and only if, $\langle \mathcal{M}, w \rangle \models \varphi$ or $\langle \mathcal{M}, w \rangle \models \psi$;
4. $\langle \mathcal{M}, w \rangle \models \boxed{a}\varphi$ if, and only if, for all $t \in W$, $(w, t) \in R_a$ implies $\langle \mathcal{M}, t \rangle \models \varphi$.

Satisfiability is defined with respect to the root of a model. A formula $\varphi \in \mathsf{WFF}_{\mathsf{K}_n}$ is said to be *satisfiable* if there exists a Kripke model $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ such that $\langle \mathcal{M}, w_0 \rangle \models \varphi$. In this case we simply write $\mathcal{M} \models \varphi$ to mean that $\mathcal{M}$ statisfies φ in at w_0. A formula is said to be *valid* if it is satisfiable in all models. We say that a set $\mathcal{F}$ of formulae is satisfiable if every $\varphi \in \mathcal{F}$ is satisfiable.

this last definition is incomplete. You need to say “there is a model which satisfies all phi in F”.

The satisfiability problem for K_n corresponds to determining the existence of a model in which a formula is satisfied. This problem is proven to be PSPACE-complete [21].

Example 1. Let $\mathcal{M}$ be the model illustrated in Figure 2.1. Take $\mathcal{M} = (W, w_0, R, \pi)$, for $\mathcal{P} = \{p\}$ and $\mathcal{A} = \{1\}$, where

(*i*) $W = \{w_0, w_1, w_2\}$

(*ii*) $R = \{(w_1, w_1), (w_2, w_2), (w_0, w_1), (w_0, w_2)\}$

(*iii*) $\pi(w, p) = \begin{cases} true & \text{if } w = w_0 \\ false & \text{otherwise} \end{cases}$

Note that both p and $\Box\neg p$ are satisfied in $\mathcal{M}$. This is a rather simple example to illustrate that, even though some sentence evaluates to true in the current context, one can see the same sentence occurring with the opposite valuation through an accessibility relation. This kind of reasoning is not possible in classical logic. Other examples of formulae satisfied by this model are: $p \wedge \Diamond\neg p$, $\Box\Box\neg p$ and $\Box\Box\Box\neg p$.

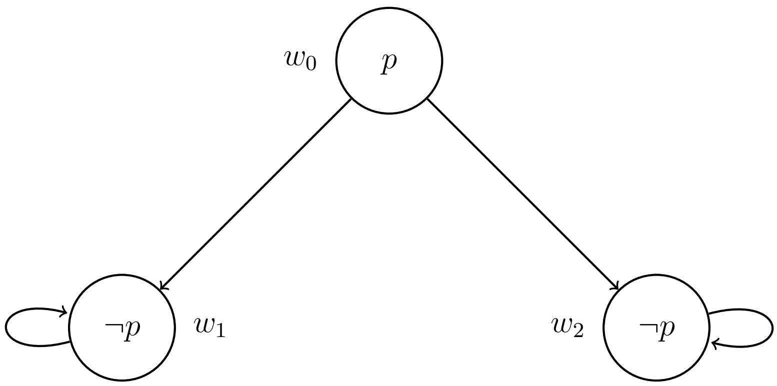
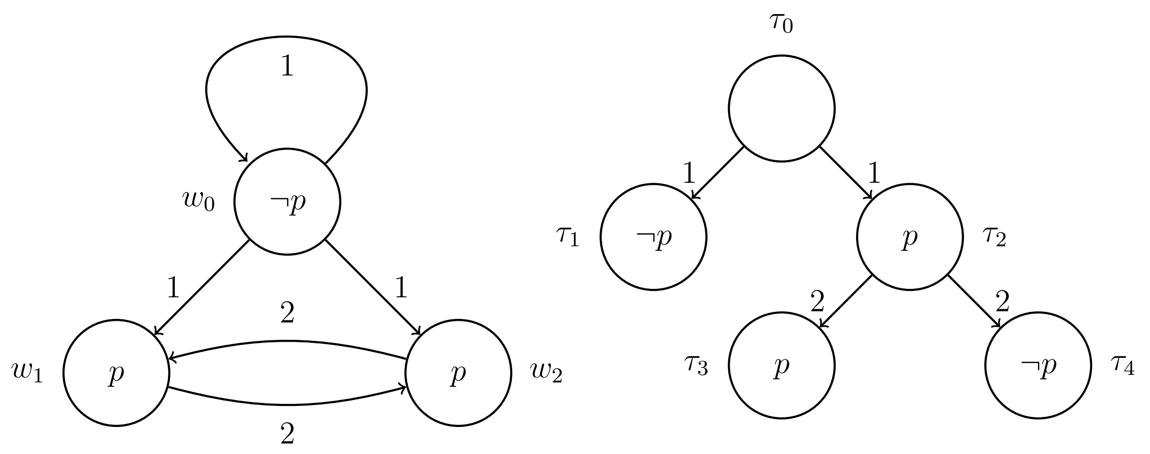


Figure 2.1: Example of a Kripke model for K_n

It is not “similar”. It *is* a tree.

Example 2. (*Tree-like* model) Consider the formula $\varphi = \boxed{1}(p \Rightarrow \diamondsuit_2 p)$. The Figure 2.2 contains examples of models that satisfy φ, hence, φ is satisfiable. Note that the model from the figure 2.2b has a graphical representation similar to a tree. ~~Finite trees are ubiquitous in computer science, they are used to represent knowledge or data from the most diverse fields, such as linguistics, programming language etc. As trees play such an~~



(a) Kripke model (b) Tree model

Figure 2.2: Models that satisfy φ of Example 2

Connect the next paragraph with the previous one.

~~important role, we will take this opportunity to define them, and next, we will mention some interesting results concerning our modal language.~~ unique node

By a tree $\mathcal{T}$ we mean a relational structure (T, S) where T is a set of nodes and S is
over a binary relation ~~among~~ these nodes. T contains a unique $r_0 \in T$ (called the *root*) such that all other nodes in T are reachable from r_0, that is, for all $t \in T$, with $t \neq r_0$, we have S^*r_0t, besides that, every element of T, distinct from r_0, has a unique S-predecessor, and the transitive and reflexive closure S^+ is acyclic, that is, for all $t \in T$, we have $\neg S^*tt$ [1].

A *tree model* is a Kripke model (W, w_0, R, π), with $\mathcal{A} = \{1\}$, where (W, R) is a tree and w_0 is its root. A *tree-like model* for K_n is a model $(W, w_0, R_1, \ldots, R_n, \pi)$, with $\mathcal{A} = \{1, \ldots, n\}$, such that $(W, \cup_{i \in \mathcal{A}} R_i)$ is a tree, with w_0 as the root.

Let $\mathcal{M} = (W, w_0, R_1, \ldots, R_a, \pi)$ be a tree-like model for K_n. We define the *depth* : $W \longrightarrow \mathbb{N}$ of a world $w \in W$, as the length of the path from w_0 to w through the union of the relations in $\mathcal{M}$. We sometimes say *depth* of $\mathcal{M}$ to mean the largest path from the root to any world in W. It is a tree. There is only one path from the root.

The following theorems are particular cases of the ones presented in [1].

Theorem 2.2.1. *Let $\varphi \in \mathsf{WFF}_{\mathsf{K}_n}$ be a formula and $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ be a model. Then $\mathcal{M} \models \varphi$ if and only if there is a tree-like model $\mathcal{M}'$ such that $\mathcal{M}' \models \varphi$. Moreover, $\mathcal{M}'$ is finite and its depth is bounded by $mdepth(\varphi)$.*

Theorem 2.2.2. *Let $\varphi, \varphi' \in \mathsf{WFF}_{\mathsf{K}_n}$ and $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ be a tree-like model such that $\mathcal{M} \models \varphi$. If $(\varphi', \lambda', ml) \in \tau(\varphi, 1, 0)$ and φ' is satisfied in $\mathcal{M}$, then there is $w \in W$, with $depth(w) = ml$, such that $\langle \mathcal{M}, w \rangle \models \varphi'$. Moreover, the subtree rooted at w has height equals to $mdepth(\varphi')$.*

They are no particular cases. They have been adapted. You need to cite where the proofs can be found.

We don't. Global satisfiability is defined in terms of the satisfiability relation. I needed to add this modality to the paper because I needed to show the relation between global satisfiability and local satisfiability when the labelled language is introduced.

For the *global satisfiability* problem of a modal logic we need to add the universal modality, $\boxed{*}$, to the original modal language [14]. Let K_n^* be the logic obtained by adding $\boxed{*}$ to K_n. A model $\mathcal{M}^*$ for K_n^* is the pair $(\mathcal{M}, R_*)$, where $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ is a tree-like model for K_n and $R_* = W \times W$. The global satisfiability problem is equivalent to the satisfiability problem in the following sense: a formula $\boxed{*}\varphi$ is satisfied at the world $w \in W$, in the model $\mathcal{M}^*$, written $\langle \mathcal{M}^*, w \rangle \models \boxed{*}\varphi$, if, and only if, for all $w' \in W$, we have that $\langle \mathcal{M}^*, w' \rangle \models \varphi$. Therefore, let $\varphi \in \mathsf{WFF}_{\mathsf{K}_n}$ be a formula, we say that φ is globally satisfiable in a model $\mathcal{M}$, denoted $\mathcal{M} \models_G \varphi$, if, and only if, $\mathcal{M}^* \models \boxed{*}\varphi$.

2.3 Proof Systems and Normal Forms

A proof is *not* about satisfiability!

Informally, a proof is an argument that can convince if determined formula is satisfiable or not. Formally, a proof is a finite object constructed according to fixed syntactic rules that refer only to the structure of formulae and not to their intended meaning [11]. The set of syntactic rules that define proofs are said to specify a *proof system* or *calculus*. These rules allow the derivation of formulae from formulae through strict symbol manipulation [9].

If there are many you should say a bit more about other kinds of systems.
on

A proof system is *sound* for a particular logic if any formula that has a proof is a valid formula of the logic, and it is *complete* for a particular logic if any valid formula has a proof [11]. A sound and complete calculus for K_n finds a model for a formula if, and only if, this formula is satisfiable.

Calculi do not find models. They find proofs.
You are mixing satisfiability with validity in those definitions.

There are several kinds of proof systems, and they can be divided in many categories as, for instance, *refutational systems*. To prove a formula using a refutational system, we begin by negating it, then analysing the consequences of doing so, using a kind of tree structure, we verify if the consequences turn out to be impossible, if that is the case, we conclude that the original formula has been proved [11]. The calculus for K_n we use in this work is based in resolution, which is one of the most common examples of refutational systems. This calculus is introduced in Chapter 3, resolution is also briefly introduced in the same chapter. The resolution calculus used is proved to be sound and complete for K_n, it was developed with regard to computer implementations and it uses formulae into a special *normal form*.

do not use consequence. This is a technical term and it does not mean what you intend to say.

TREE?

When citing results, provide references.

translated into

What is the reference for this???

A normal form is an elegant representation of an equivalence class. The equivalence relation in question may determine what kind of normal form is used. The relation considered in proof systems for logic languages relates two formulae if whenever one is satisfiable, the other one also is. Therefore, the transformation rules defined for a specific normal form used in a proof system for a logic must preserve satisfiability, that is, the formula obtained by applying a transformation rule is satisfiable if, and only if, the original one also is. Normal forms can provide constructive proofs of many standard results [10].

What do you mean by constructive proofs? And normal forms do not provide proofs. Calculi do.

This happens because formulae translated into a normal form have a specific, normalized structure, possibly resulting in less operators, which may implicate into a smaller number of rules for a proof system. Hence, a calculus that is planned to be implemented in a computer may take great advantage of normal forms, since the smaller number of rules reduces the chances of implementation errors.

The normal form used in this work is a layered normal form called Separated Normal Form with Modal Levels and it's introduced in Chapter 3, along side with its transformation rules for formulae in K_n. These rules are proved to preserve satisfiability of formulae.

Chapter 3

Modal-Layered Resolution

Resolution appeared in the early 1960's through investigations on performance improvements of refutational procedures based on *Herbrand Theorem*. In particular, Prawitz' studies on such procedures brought back the idea of unification. J. A. Robinson incorporated the concept of unification on a refutation method, creating what was later known as resolution [4].

The standard rule for resolution systems takes two or more premises with literals that are contradictory, and generates a resolvent. Most of these systems work exclusively with clauses in a specific normal form. Resolution systems are refutational systems, that is, to show that a formula φ is valid, $\neg\varphi$ is translated into a normal form. The inference rules are applied until either no new resolvents can be generated or a contradiction is obtained. The contradiction implies that $\neg\varphi$ is unsatisfiable and hence, that φ is valid.

3.1 Clausal Resolution

Clausal resolution was proposed as a proof method for classical logic by Robinson in 1965 [20], and was claimed to be suitable to be performed by computer, as it has only one inference rule that is applied exhaustively.

3.1.1 Modal-Layered Resolution

3.2 Separated Normal Form with Modal Levels

Formulae in K_n can be transformed into a layered normal form called *Separated Normal Form with Modal Levels*, denoted by SNF_{ml}, proposed in [16]. A formula in SNF_{ml} is a conjunction of *clauses* where the modal level in which they occur is emphasized as a label.

We write $ml : \varphi$ to denote that φ occurs at modal level $ml \in \mathbb{N} \cup \{*\}$. By $* : \varphi$ we mean that φ is true at all modal levels. Formally, let $\mathrm{WFF}^{ml}_{\mathsf{K}_n}$ denote the set of formulae with the modal level annotation, $ml : \varphi$, such that $ml \in \mathbb{N} \cup \{*\}$ and $\varphi \in \mathsf{WFF}_{\mathsf{K}_n}$. Let $\mathcal{M}^* = (W, w_0, R_1, \ldots, R_n, R_*, \pi)$ be a tree-like model and take $\varphi \in \mathsf{WFF}_{\mathsf{K}_n}$.

Definition 7 Satisfiability of labelled formulae is given by:

1. $\mathcal{M}^* \models ml : \varphi$ if, and only if, for all worlds $w \in W$ such that $depth(w) = ml$, we have $\langle \mathcal{M}^*, w \rangle \models \varphi$
2. $\mathcal{M}^* \models * : \varphi$ if, and only if, $\mathcal{M}^* \models \boxed{*}\varphi$

Observe that the labels in formulae work as a kind of *weak* universal operator, allowing us to reason about a set of formulae that are all satisfied at a given modal level.

Clauses in SNF_{ml} are defined as follows.

Definition 8 Clauses in SNF_{ml} are in one of the following forms:

1. Literal clause $\quad ml : \bigvee_{b=1}^{r} l_b$
2. Positive a-clause $\quad ml : l' \Rightarrow \boxed{a}\, l$
3. Negative a-clause $\quad ml : l' \Rightarrow \Diamond_a l$

where $r, b \in \mathbb{N}, ml \in \mathbb{N} \cup \{*\}$ and $l, l', l_b \in \mathcal{L}$.

Positive and negative a-clauses are together known as *modal a-clauses*, the index a can be omitted if it is clear from the context.

The transformation of a formula $\varphi \in \mathsf{WFF}_{\mathsf{K}_n}$ into SNF_{ml} is achieved by first transforming φ into its *Negation Normal Form*, and then, recursively applying rewriting and renaming [19].

Definition 9 Let $\varphi \in \mathsf{WFF}_{\mathsf{K}_n}$. We say that φ is in Negation Normal Form (NNF) if it contains only the operators $\neg, \vee, \wedge, \boxed{a}$ and $\Diamond_a$. Also, only propositions are allowed in the scope of negations.

Let φ be a formula and t a propositional symbol not occurring in φ. The translation of φ is given by $0 : t \wedge \rho(0 : t \Rightarrow \varphi)$ — for global satisfiability, the translation is given by $* : t \wedge \rho(* : t \Rightarrow \varphi)$ — where ρ is the *translation function* defined below. We refer to clauses of the form $0 : D$, for a disjunction of literals D, as *initial clauses*.

Definition 10 The translation function $\rho : \mathrm{WFF}^{ml}_{\mathsf{K}_n} \longrightarrow \mathrm{WFF}^{ml}_{\mathsf{K}_n}$ is defined as follows:

$$\begin{aligned}
\rho(ml : t \Rightarrow \varphi \wedge \psi) &= \rho(ml : t \Rightarrow \varphi) \wedge \rho(ml : t \Rightarrow \psi)\\
\rho(ml : t \Rightarrow \boxed{a}\varphi) &= (ml : t \Rightarrow \boxed{a}\varphi), \text{ if } \varphi \text{ is a literal}\\
&= (ml : t \Rightarrow \boxed{a}t') \wedge \rho(ml + 1 : t' \Rightarrow \varphi), \text{ otherwise}\\
\rho(ml : t \Rightarrow \diamondsuit\varphi) &= (ml : t \Rightarrow \diamondsuit\varphi), \text{ if } \varphi \text{ is a literal}\\
&= (ml : t \Rightarrow \diamondsuit t') \wedge \rho(ml + 1 : t' \Rightarrow \varphi), \text{ otherwise}\\
\rho(ml : t \Rightarrow \varphi \vee \psi) &= (ml : \neg t \vee \varphi \vee \psi), \text{ if } \psi \text{ is a disjunction of literals}\\
&= \rho(ml : t \Rightarrow \varphi \vee t') \wedge \rho(ml : t' \Rightarrow \psi), \text{ otherwise}
\end{aligned}$$

Where $t, t' \in \mathcal{L}$, $\varphi, \psi \in \mathsf{WFF}_{\mathsf{K}_n}$, $ml \in \mathbb{N} \cup \{*\}$ and $r, b \in \mathbb{N}$.

As the conjunction operator is commutative, associative and idempotent, we will commonly refer to a formula in SNF_{ml} as a set of clauses.

The next lemma, taken from [17], shows that the transformation into SNF_{ml} preserves satisfiability.

Lemma 3.2.1. *Let $\varphi \in \mathsf{WFF}_{\mathsf{K}_n}$ be a formula and let t be a propositional symbol not occurring in φ. Then:*

(i) *φ is satisfiable if, and only if, $0 : t \wedge \rho(0 : t \Rightarrow \varphi)$ is satisfiable;*

(ii) *φ is globally satisfiable if, and only if, $* : t \wedge \rho(* : t \Rightarrow \varphi)$ is satisfiable;*

Example 3.

3.3 Modal-Layered Resolution Calculus for K_n

The motivation for the use of this labelled clausal normal form in a calculus is that inference rules can then be guided by the semantic information given by the labels and applied to smaller sets of clauses, reducing the number of unnecessary inferences, and therefore improving the efficiency of the proof procedure [18].

This calculus comprises a set of inference rules, given in Table 3.1, for dealing with propositional and modal reasoning. In the following, we denote by σ the result of unifying the labels in the premises for each rule. Formally, unification is given by a function

Table 3.1: Inference rules

$$
\text{[LRES]} \quad \begin{array}{rl}
ml_1 : & D \vee l \\
ml_2 : & D' \vee \neg l \\
\hline
\sigma(\{ml_1, ml_2\}) : & D \vee D'
\end{array}
\qquad
\text{[MRES]} \quad \begin{array}{rl}
ml_1 : & l_1 \Rightarrow \boxed{a} l \\
ml_2 : & l_2 \Rightarrow \neg \boxed{a} l \\
\hline
\sigma(\{ml_1, ml_2\}) : & \neg l_1 \vee \neg l_2
\end{array}
$$

$$
\text{[GEN2]} \quad \begin{array}{rl}
ml_1 : & l'_1 \Rightarrow \boxed{a} l_1 \\
ml_2 : & l'_2 \Rightarrow \boxed{a} \neg l_1 \\
ml_3 : & l'_3 \Rightarrow \Diamond_a l_2 \\
\hline
\sigma(\{ml_1, ml_2, ml_3\}) : & \neg l'_1 \vee \neg l'_2 \vee \neg l'_3
\end{array}
$$

$$
\text{[GEN1]} \quad \begin{array}{r@{\,}l}
ml_1 & : l'_1 \Rightarrow \boxed{a} \neg l_1 \\
\vdots & \\
ml_m & : l'_m \Rightarrow \boxed{a} \neg l_m \\
ml_{m+1} & : l' \Rightarrow \Diamond_a \neg l \\
ml_{m+2} & : l_1 \vee \ldots \vee l_m \vee l \\
\hline
ml & : \neg l'_1 \vee \ldots \vee \neg l'_m \vee \neg l'
\end{array}
\qquad
\text{[GEN3]} \quad \begin{array}{r@{\,}l}
ml_1 & : l'_1 \Rightarrow \boxed{a} \neg l_1 \\
\vdots & \\
ml_m & : l'_m \Rightarrow \boxed{a} \neg l_m \\
ml_{m+1} & : l' \Rightarrow \Diamond_a l \\
ml_{m+2} & : l_1 \vee \ldots \vee l_m \\
\hline
ml & : \neg l'_1 \vee \ldots \vee \neg l'_m \vee \neg l'
\end{array}
$$

where $ml = \sigma(\{ml_1, \ldots, ml_{m+1}, ml_{m+2} - 1\})$ where $ml = \sigma(\{ml_1, \ldots, ml_{m+1}, ml_{m+2} - 1\})$

$\sigma : \mathscr{P}(\mathbb{N} \cup \{*\}) \longrightarrow \mathbb{N} \cup \{*\}$, where $\sigma(\{ml, *\}) = ml$ and $\sigma(\{ml\}) = ml$, otherwise, σ is undefined. The following inference rules can only be applied if the unification of their labels is defined (where $* - 1 = *$). Note that for GEN1 and GEN3, if the modal clauses occur at the modal level ml, then the literal clause occurs at the next modal level, $ml + 1$.

The proofs for termination, soundness and completeness of this calculus can be found in [17].

3.4 $\mathrm{K_SP}$

In this section, we present $\mathrm{K_SP}$, the theorem prover presented in [18] for the basic multi-modal logic K_n, which implements a variation of the set of support strategy [22] for the modal resolution-based calculus described in Section 3.3.

Chapter 4

Satisfiability Solvers

The problem of determining whether a formula in classical propositional logic is satisfiable has the historical honor of being the first problem ever shown to be NP-Complete [6]. Great theoretical and practical efforts have been directed in improving the efficiency of solvers for this problem, known as *Boolean Satisfiability Solvers*, or just *SAT solvers*. Despite the worst-case exponential run time of all the algorithms known, satisfiability solvers are increasingly leaving their mark as a general purpose tool in the most diverse areas [13]. In essence, SAT solvers provide a generic combinatorial reasoning and search platform. Beyond that, the source code of many implementations of such solvers is freely available and can be used as a basis for the development of decision procedures for more expressive logics [12].

In the context of SAT solvers for propositional provers, the underlying representational formalism is propositional logic [13]. We are interested in formulae in *Conjunctive Normal Form* (CNF): φ is in CNF if it is a conjunction of *clauses*, where each clause is a disjunction of literals. For example, $\varphi = (p \vee \neg q) \wedge (\neg p \vee r \vee s) \wedge (q \vee r)$ is a CNF formula with four variables and three clauses. We use the symbol $\emptyset$ to denote the *empty clause*, i.e., the clause that contains no literals. A clause with only one literal is referred to as a *unit clause*, and a clause with two literals, as a *binary clause*. When every clause of φ has k literals, we refer to φ as a k-CNF formula.

A propositional formula φ takes value in the set $\{false, true\}$. A *truth assignment* (or just assignment) to a set of variables $\mathcal{P}$, is the valuation function π as defined in Definition 5. As in propositional logic we have a unit set as the set of possible worlds W, we can omit this set from the function signature and write $\pi : \mathcal{P} \longrightarrow \{false, true\}$, for simplicity. A *satisfying assignment* for φ is an assignment π such that φ evaluates to *true* under π. A *partial assignment* for a formula φ is a truth assignment to a subset of the variables in φ. For a partial assignment ρ for a CNF formula φ, $\varphi|_\rho$ denotes the simplified formula obtained by replacing the variables appearing in ρ with their specified values, removing all clauses with at least one *true* literal, and deleting all occurrences of $false$ literals from the remaining clauses [13].

Therefore, the *Boolean Satisfiability Problem* (SAT) can be expressed as: Given a CNF formula φ, does φ have a satisfying assignment? If this is the case, φ is said to be *satisfiable*, otherwise, φ is *unsatisfiable*. One can be interested not only in the answer of this decision problem, but also in finding the actual assignment that satisfies the formula, when it exists. All practical SAT solvers do produce such assignment [7].

4.1 The DPLL Procedure

A *complete* solution method for the SAT problem is one that, given the input formula φ, either produces a satisfying assignment for φ or proves that it is unsatisfiable [13]. One of the most surprising aspects of the relatively recent practical progress of SAT solvers is that the best complete methods remain variants of a process introduced in the early 1960's: the Davis-Putnam-Logemann-Loveland, or DPLL, procedure [8], which performs a backtrack search in the space of partial truth assignments. A key feature of DPLL is efficient pruning of the search space based on falsified clauses. Since its introduction, the main improvements to DPLL have been smart branch selection heuristics, extensions like clause learning and randomized restarts, and well-crafted data structures such as lazy implementations and watched literals for fast unit propagation [13].

Algorithm 1, DPLL-recursive(φ, ρ), sketches the basic DPLL procedure on CNF formulae [8]. The main idea is to repeatedly select an unassigned literal l in the input formula and recursively search for a satisfying assignment for $\varphi|_l$ and $\varphi|_{\neg l}$. The step where such an l is chosen is called a *branching step*. Setting l to *true* or $false$ when making a recursive call is referred to as *decision*, and is associated with a *decision level* which equals the recursion depth at that stage. The end of each recursive call, which takes φ back to fewer assigned literals, is called the *backtracking step*.

A partial assignment ρ is maintained during the search and output if the formula turns out to be satisfiable. To increase efficiency, a key procedure in SAT solvers is the *unit propagation* [2], where unit clauses are immediately set to *true* as outlined in Algorithm 1. In most implementations of DPLL, logical consequences are derived with unit propagation. Thus, this procedure is used for identifying variables which must be assigned a specific value. If $\varphi|_\rho$ contains the empty clause, a *conflict* condition is declared, the corresponding clause of φ from which it came is said to be *violated* by ρ, and the algorithm backtracks. The literals whose negation does not appear, called *pure literals*, are also set to *true* as a preprocessing step and, in some implementations, during the simplification process after every branch.

Algorithm 1: DPLL-recursive(φ, ρ)

$(\varphi, \rho) \leftarrow$ `UnitPropagate`(φ, ρ)
if φ *contains the empty clause* **then**
  return UNSAT
end
if φ *has no clauses left* **then**
  Output ρ
  return *SAT*
end
$l \leftarrow$ a literal not assigned by ρ
if *DPLL-recursive*$(\varphi|_l, \rho \cup \{l\}) = SAT$ **then**
  return *SAT*
end
return *DPLL-recursive*$(\varphi|_{\neg l}, \rho \cup \{\neg l\})$

sub `UnitPropagate`(φ, ρ)
  while φ *contains no empty clause but has a unit clause* x **do**
    $\varphi \leftarrow \varphi|_x$
    $\rho \leftarrow \rho \cup \{x\}$
  end
  return (φ, ρ) **return**

Variants of this algorithm form the most widely used family of complete algorithms for the SAT problem. They are frequently implemented in an iterative manner, resulting in significantly reduced memory usage. The efficiency of state-of-the-art SAT solvers relies heavily on various features that have been developed, analysed and tested over the last decade. These include fast unit propagation using watched literals, deterministic and randomized restart strategies, effective clause deletion mechanisms, smart static and dynamic branching heuristics and learning mechanisms. This last one is more discussed in the next section.

4.2 Conflict-Driven Clause Learning

One of the main reasons for the widespread use of SAT in many applications is that solvers based on clause learning are so effective in practice [13]. The main idea is to cache "causes of conflict" as learned clauses, and utilize this information to prune the search in a different part of the search space encountered later. Since their inception in the mid-90s, *Conflict-Driven Clause Learning* (CDCL) SAT solvers have been applied, in many cases with remarkable success, to a number of practical applications [2]. The organization of CDCL SAT solvers is primarily inspired by the DPLL procedure.

In CDCL SAT solvers, each variable x_i is characterized by a number of properties, including the *value*, the *antecedent* and the decision level, denoted respectively by $\nu(x_i) \in \{false, true, u\}, \alpha(x_i) \in \varphi \cup \{nil\}$, and $\delta(x_i) \in \{-1, 0, 1, \ldots, |\mathcal{P}|\}$.

Algorithm 2: CDCL(φ, ν)

4.2.1 Conflict Analysis

Each time the CDCL SAT solver identifies a conflict due to unit propagation, the conflict analysis procedure is invoked. As a result, one or more new clauses are learnt, and a backtracking decision level is computed. This procedure analyses the structure of unit propagation and decides which literals to include in the learnt clause.

The decision levels associated with assigned variables define a partial order of the variables. Starting from a given unsatisfied clause, the conflict analysis procedure visits variables implied at the most recent decision level. HEREEE

Clause learning finds other applications besides the key efficiency improvements to CDCL SAT solvers. One example is *clause reuse*. In a large number of applications, clauses learnt for a given CNF formula can often be reused for related CNF formulae.

Moreover, for unsatisfiable subformulae, the clauses learnt by a CDCL SAT solver encode a resolution refutation of the original formula. Given the way clauses are learnt in this solvers, each learnt clause can be explained by a number of resolution steps, each of which is a trivial resolution step. As a result, the resolution refutation can be obtained from the learnt clauses in linear time and space on the number of learnt clauses.

For unsatisfiable formulae, the resolution refutations obtained from the clauses learnt by a SAT solver serve as certificate for validating the correctness of the SAT solver. Moreover, resolution refutations based on clause learning find practical applications [2].

Besides allowing producing a resolution refutation, learnt clauses also allow identifying a subset of clauses that is also unsatisfiable. For example, a *minimally unsatisfiable subformula* can be derived by iteratively removing a single clause and checking unsatisfiability. Unnecessary clauses are discarded, and eventually a minimally unsatisfiable subformula is obtained.

4.3 Modern CDCL Solvers

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