A Clausal Tableaux for Modal Logics

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Abstract

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1 Introduction

2 Language

The language of K_n is equivalent to its set of *well-formed formulae*, denoted WFF_{K_n}, which is constructed from a denumerable set of *propositional symbols* $\mathcal{P} = \{p, q, r, \ldots\}$, the negation symbol $\neg$, the disjunction symbol $\vee$ and the modal connectives $\boxed{a}$, that express the notion of necessity, for each index a in a finite, fixed set $\mathcal{A} = \{1, \ldots, n\}, n \in \mathbb{N}$.

Definition 2.1 The set of well-formed formulae, WFF_{K_n}, is the least set such that:

(i) $p \in WFF_{\mathsf{K}_n}$, for all $p \in \mathcal{P}$

(ii) if $\varphi, \psi \in WFF_{\mathsf{K}_n}$, then so are $\neg\varphi, (\varphi \vee \psi)$ and $\boxed{a}\varphi$, for each $a \in \mathcal{A}$

When $n = 1$, we often omit the index in the modal operators, i.e., we just write $\Box\varphi$ and $\Diamond\varphi$, for a formula φ. Other logic operators may be introduced

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as abbreviations, as usual: $\varphi \wedge \psi \stackrel{\text{def}}{=} \neg(\neg\varphi \vee \neg\psi)$ (conjuction), $\varphi \Rightarrow \psi \stackrel{\text{def}}{=} \neg\varphi \vee \psi$ (implication), $\varphi \Leftrightarrow \psi \stackrel{\text{def}}{=} (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ (equivalence), $\Diamond\varphi \stackrel{\text{def}}{=} \neg\boxed{a}\neg\varphi$ (possibility), **false** $\stackrel{\text{def}}{=} \varphi \wedge \neg\varphi$ (*falsum*), **true** $\stackrel{\text{def}}{=} \neg$**false** (*verum*). Parentheses may be ommitted if the reading is not ambiguous.

A *literal* is a propositional symbol $p \in \mathcal{P}$ or its negation $\neg p$. We denote by $\mathcal{L}$ the set of all literals. A *modal literal* is a formula of the form $\boxed{a}l$ o $\Diamond\neg l$, with $l \in \mathcal{L}$ and $a \in \mathcal{A}$.

The maximal number of nesting modal operators in a formula is defined as its *modal depth* and denoted *mdepth*. The maximal number of modal operators in which scope the formula occurs is defined as the *modal level* of that formula, and it is denoted *ml*. For instance, in $\boxed{a}\Diamond p$, $mdepth(p) = 0$ and $ml(p) = 2$.

The semantics of K_n is presented in terms of Kripke structures.

Definition 2.2 A Kripke model for $\mathcal{P}$ and $\mathcal{A}$ is given by the tuple $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$, where W is a non-empty set of possible worlds with a distinguinshed world w_0, the root of $\mathcal{M}$; each R_a, $a \in \mathcal{A}$, is a binary relation on W, and $\pi : W \times \mathcal{P} \longrightarrow \{false, true\}$ is the valuation function that associates to each world $w \in W$ a truth-assignment to propositional symbols.

Satisfiability and validity of a formula in a given world is defined as follows.

Definition 2.3 Let $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ be a Kripke model, $w \in W$ a world, and $\varphi, \psi \in WFF_{\mathsf{K}_n}$. The *satisfiability relation*, denoted by $\langle \mathcal{M}, w \rangle \models \varphi$, between a world w and a formula φ, is inductively defined by:

(i) $\langle \mathcal{M}, w \rangle \models p$ if, and only if, $\pi(w, p) = \mathbf{true}$, for all $p \in \mathcal{P}$;

(ii) $\langle \mathcal{M}, w \rangle \models \neg\varphi$ if, and only if, $\langle \mathcal{M}, w \rangle \not\models \varphi$;

(iii) $\langle \mathcal{M}, w \rangle \models \varphi \vee \psi$ if, and only if, $\langle \mathcal{M}, w \rangle \models \varphi$ or $\langle \mathcal{M}, w \rangle \models \psi$

(iv) $\langle \mathcal{M}, w \rangle \models \boxed{a}\varphi$ if, and only if, for all $t \in W$, $(w, t) \in R_a$ implies $\langle \mathcal{M}, t \rangle \models \varphi$

Satisfiability is defined in terms of the root of a model. A formula $\varphi \in WFF_{\mathsf{K}_n}$ is said to be *satisfiable* if there exists a Kripke model $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ such that $\langle \mathcal{M}, w_0 \rangle \models \varphi$. A formula is said to be *valid* if it is satisfiable in all models.

The local satisfiability problem in K_n corresponds to determining the existence of a model at which a formula is satisfied. The local satisfiability problem for K_n is PSPACE-complete [4].

3 A Clausal Tableaux for K_n

Formally, a proof is a finite object constructed according to fixed set of syntactic rules that refer only to the structure of formulae, not to their intended meaning. The set of syntactic rules that are used to provide are said to specify a *calculus*. A calculus is *sound* for a particular logic if any formula that has a proof is a valid formula of this logic, and is *complete* for a logic if every valid formula has a proof [1]. From these definitions, complete and sound calculi allow us to produce proofs that formulae are valid in a specific logic. As a formula is valid if, and only if, its negation is unsatisfiable, calculi which are constructed for satisfiability checking can also be used to determine whether a formula is valid or not.

Tableaux-based methods are frequently used in modal logics because their structure is more obviously related to the notion of possible worlds [1]. In general, proofs are graphically represented by trees where each branch can be thought as a set of formulae. Here we present a tableaux-based calculus for deciding the satisfiability of formulae in K_n. Unlike many tableaux-based systems, our system is clausal and clauses are labelled by the modal level in which a formula occur. The normal form we use is called *Separated Normal Form with Modal Levels* ($\mathsf{SNF}_{\mathrm{K}}$) which separates the contexts considering the different modal ineceses and different modal levels appropriately. The transformation rules and the correction proof of the translation method can be find in [3].

A formula in this clausal form is represented by a set of clauses, which are true in their respectives modal levels. A formula in $\mathsf{SNF}_{\mathrm{K}}$ is of the form: $\bigwedge_i ml : C_i$, where each C_i is a clause and ml is the modal level in which the clause occurs. A clause in $\mathsf{SNF}_{\mathrm{K}}$ is in one of the following syntactic forms:

- literal clause: $ml : \bigvee_{b=1}^{r} l_b$
- negative a-clause: $ml : l \Rightarrow \Diamond_a m$
- positive a-clause: $ml : l \Rightarrow \boxed{a} m$

where $l_b, l, m \in \mathcal{L}, a \in \mathcal{A}$ and $r, b, ml \in \mathbb{N}$. As conjunctions are associative, commutative, and idempotent, we often refer to a formula into $\mathsf{SNF}_{\mathrm{K}}$ as a set of clauses.

The proposed calculus, denoted by $\mathcal{C}\mathsf{K}_n$, comprehends a set of inference rules to deal with both propositional and modal reasoning. Before presenting the inference rules, we need to define some more notation. Let $\mathcal{C}$ be a set of clauses and $i \in \mathbb{N}$. Consider that γ^i names a literal clause in $\mathcal{C}$ occurring at the modal level i, that is, a clause of the form $i : \bigvee_{b=1}^{r} l_b$; λ^i names a negative a-clause in $\mathcal{C}$, that is, a clause of the form $i : l \Rightarrow \Diamond_a m$, for all $a \in \mathcal{A}$; and θ^i a positive a-clause in $\mathcal{C}$, that is, a clause of the form $i : l \Rightarrow \boxed{a} m$, for all $a \in \mathcal{A}$. We use the capital Greek letters Γ^i, Λ^i and Θ^i to denote the respective sets of clauses. We denote by Δ^i a set of literals and modal literals occurring at the modal level i. Initially, $\Delta^i = \emptyset$. Finally, we denote by Π^i all sets of literals occurring at the modal level i.

We now present the inference rules that are aplied to theses sets of literals. The inference rules try to build sets of literals for each modal level, starting with $\Pi^i = \{\emptyset\}$ (because $\Delta^i = \emptyset$ initially).

The first inference rule, (PROP), is applied to literal clauses. It takes as premisse the sets of literals already built and expands these sets with literals occurring in clauses in Γ_i, as shown in the conclusion of the rule. The intuition for this rule is that if Δ^i_j and $\gamma^i \in \Gamma^i$ are both satisfiable, then $\Delta^i_j \cup \{l_r\}$ has to be satisfiable for some $1 \leq r \leq t$, therefore, at least one of the conclusions is satisfiable.

The rules (NEG) and (POS) are applied to modal clauses. For (NEG), the premise is applied to every set of literals Δ^i_j already built and to a negative a-clause. As negative a-clauses can be seen as disjunctions, the conclusion is a branching, where the negation of the left-hand side is added to one of the sets and both the left-hand side and the right-hand side is added to the other set. The inference rule (POS) is similar.

In addition to the inference rules for construction of sets, we also have two rules

$$\frac{\Delta^i_j \in \Pi^i \qquad \Gamma^i \cup \{i : \bigvee_{k=1}^{t} l_k\}}{\Delta^i_j \cup \{l_1\} \mid \ldots \mid \Delta^i_j \cup \{l_t\}} \text{ (PROP)}$$

$$\frac{\Delta^i_j \in \Pi^i \qquad \Lambda^i \cup \{i : l \Rightarrow \langle a \rangle m\}}{\Delta^i_j \cup \{\neg l\} \mid \Delta^i_j \cup \{l, \langle a \rangle m\}} \text{ (NEG)} \qquad \frac{\Delta^i_j \in \Pi^i \qquad \Theta^i \cup \{i : l \Rightarrow \boxed{a} m\}}{\Delta^i_j \cup \{\neg l\} \mid \Delta^i_j \cup \{l, \boxed{a} m\}} \text{ (POS)}$$

to eliminate sets that should not be part of the proof:

(ELIM1): Eliminate sets containing both l and $\neg l$ for some l.

(ELIM2): If, at some modal level, we have that every set satisfy $\neg m_0 \vee \neg m_1 \vee \ldots \vee \neg m_r$, eliminate the sets at the previous level that contains $\langle a \rangle m_0$, $\boxed{a} m_1, \ldots, \boxed{a} m_r$.

A tableau proof, or just tableau, that has all the sets at the first modal level eliminated is said do be *closed*. A tableau is *open* if it's not closed. A set of clauses that has an open tableau is said to be $\mathcal{C}_{\mathsf{K}_n}$-satisfiable.

4 Correctness Results

4.1 Soundness

A calculus is sound when it does not prove anything that it shouldn't. A tableau calculus for a logic L containing only sound rules, with respect to L, is sound [2]. A sound rule has a satisfiable conclusion every time its premisses are satisfiable.

Lemma 4.1 *(PROP) Let $\gamma^i = i : \bigvee_{t=1}^{k} l_t$ be a literal clause in SNF_{ml} such that $\gamma^i \in \Gamma^i$, and $\Delta^i_j \in \Pi^i$ a literal set at the i-th modal level. If γ^i and Δ^i_j are both satisfiable in K_n then exists $1 \leq r \leq t$ such that $\Delta^i_j \cup \{l_r\}$ is also satisfiable in K_n.*

Proof Let Π^i be the initial set of literal clauses of the i-th modal level, with $\gamma^i = i : \bigvee_{t=1}^{k} l_t \in \Gamma^i$, and consider $\Delta^i_j \in \Pi^i$ a literal set that, by hypothesis, also belongs to the i-th modal level. Supose that these hypothesis are satisfiable in K_n. By the definition of satisfiability, there is a model $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ and a world $w \in W$ such that:

- $mdepth(w) = i$
- $\langle \mathcal{M}, w \rangle \models \bigvee_{k=1}^{t} l_k$ and
- $\langle \mathcal{M}, w \rangle \models d, \forall\, d \in \Delta^i_j$

To prove that this rule is sound, it is sufficient to proof that, in an instancy of it, at least one of its denominators is satisfiable. In fact, we have that:

- $\langle \mathcal{M}, w \rangle \models \bigvee_{k=1}^{t} l_k \Leftrightarrow \langle \mathcal{M}, w \rangle \models l_1$ ou $\ldots$ ou $\langle \mathcal{M}, w \rangle \models l_t$, by Definition 2.3

This means that exists at least one index r, with $1 \leq r \leq t$, such that $\langle \mathcal{M}, w \rangle \models l_r$. Therefore, $\Delta^i_j \cup \{l_r\}$ is satisfiable in K_n. $\square$

Lemma 4.2 *(NEG) Let $\lambda^i = i : l \Rightarrow \langle a \rangle m$ be an a-negative clause in SNF_{ml} such that $\lambda^i \in \Lambda^i$, and $\Delta^i_j \in \Pi^i$ a literal set at the i-th modal level. If λ^i and Δ^i_j are both* *satisfiable in* K_n *then either* $\Delta^i_j \cup \{\neg l\}$ *or* $\Delta^i_j \cup \{l, \diamondsuit m\}$ *is also satisfiable in* K_n.

Proof Let Λ^i be the initial set of negative modal clauses at the i-th modal level, with $\lambda^i = i : l \Rightarrow \diamondsuit m \in \Lambda^i$, and consider $\Delta^i_j \in \Pi^i$ a literal set that, by hypothesis, also belongs to the i-th modal level. Supose that these hypothesis are all satisfiable in K_n. By the definition of satisfiability, there exists a model $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ and a world $w \in W$ such that:

- $mdepth(w) = i$
- $\langle \mathcal{M}, w \rangle \models l \Rightarrow \diamondsuit m$
- $\langle \mathcal{M}, w \rangle \models d, \forall\, d \in \Delta^i_j$

To prove that this rule is sound, it is sufficient to proof that, in an instancy of it, one of the denominators is satisfiable. In fact, we have that:

- $\langle \mathcal{M}, w \rangle \models l \Rightarrow \diamondsuit m \Leftrightarrow \langle \mathcal{M}, w \rangle \models \neg l$ ou $\langle \mathcal{M}, w \rangle \models \diamondsuit m$, by Definition 2.3

If $\langle \mathcal{M}, w \rangle \models \neg l$ holds then $\Delta^i_j \cup \{\neg l\}$ is satisfiable. Otherwise, $\langle \mathcal{M}, w \rangle \not\models \neg l$ and $\langle \mathcal{M}, w \rangle \models \diamondsuit m$ hold, therefore, $\Delta^i_j \cup \{l, \diamondsuit m\}$ is satisfiable in K_n, by Definition 2.3.□

Lemma 4.3 *(POS) Let* $\theta^i = i : l \Rightarrow \boxed{a} m$ *be an a-positive clause in* SNF_{ml} *such that* $\theta^i \in \Theta^i$, *and* $\Delta^i_j \in \Pi^i$ *a literal set in the i-th modal level. If* θ^i *and* Δ^i_j *are both satisfiable in* K_n *then at least one of* $\Delta^i_j \cup \{\neg l\}$ *or* $\Delta^i_j \cup \{l, \diamondsuit m\}$ *is also satisfiable in* K_n.

Being the rule (POS) quite analog to the rule (NEG), we leave the proof of its soundness omitted.

The proofs for elimination rules use the contrapositive.

Lemma 4.4 *(ELIM1) Let* $\Delta^i_j \in \Pi^i$ *be a literal set at the i-th modal level. If* Δ^i_j *is satisfiable in* K_n *then* Δ^i_j *does not contain both* l *and* $\neg l$ *for any* $l \in \Delta^i_j$.

Proof Trivial by the definition of satisfiability. □

Lemma 4.5 *(ELIM2) Let* $\Delta^i_k \in \Pi^i$ *be a literal set at the i-th modal level such that* $\{\diamondsuit m_0, \boxed{a} m_1, \ldots, \boxed{a} m_r\} \subseteq \Delta^i_k$ *for some* $m_0, m_1, \ldots, m_r \in \mathcal{L}, r \geq 0$. *If* Δ^i_k *is satisfiable in* K_n *then exists* $\Delta^{i+1}_j \in \Pi^{i+1}$ *such that* $\Delta^{i+1}_j \cup \{m_0, m_1, \ldots, m_r\}$ *is also satisfiable.*

Proof Let's assume that $\Delta^i_k \cup \{\diamondsuit m_0\} \cup \{\boxed{a} m_1, \ldots, \boxed{a} m_r\} \in \Pi^i$ is satisfiable in K_n, i.e., that exists a model $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ and a world $w \in W$, with $mdepth(w) = i$, such that, by Definition 2.3:

- $\langle \mathcal{M}, w \rangle \models \Delta^i_k \cup \{\diamondsuit m_0\} \cup \{\boxed{a} m_1, \ldots, \boxed{a} m_r\}$

And by definition of satisfiability in sets:

(1) $\langle \mathcal{M}, w \rangle \models d \wedge \diamondsuit m_0 \wedge \boxed{a} m_1 \wedge \ldots \wedge \boxed{a} m_r, \forall\, d \in \Delta^i_k$

We have, then, $\langle \mathcal{M}, w \rangle \models \diamondsuit m_0$. Therefore, it exists a world w', where $wR_a w'$ and $mdepth(w') = i + 1$, where $\langle \mathcal{M}, w' \rangle \models m_0$. As $\langle \mathcal{M}, w \rangle \models \boxed{a} m_k$ holds for every $0 \leq k \leq r$, and we already know that $wR_a w'$, then $\langle \mathcal{M}, w' \rangle \models m_k$ also holds, for every $0 \leq k \leq r$, by the definitions of satisfiability of the operators $\diamondsuit$ and $\boxed{a}$.

Supose, by contradiction, that it doesn't exist $\Delta_j^{i+1} \in \Pi^{i+1}$, literal set at $(i+1)$-th modal level, such that $\Delta_j^{i+1} \cup \{m_0, m_1, \ldots, m_r\}$ is satisfiable in $\mathcal{M}$. That means that every set Δ_j^{i+1} satisfies $\neg m_0 \vee \neg m_1 \vee \ldots \vee \neg m_r$. Then, again by definition of satisfiability of $\boxed{a}$, we have that $\langle \mathcal{M}, w'' \rangle \models \boxed{a}(\neg m_0 \vee \neg m_1 \vee \ldots \vee \neg m_r)$ for all w'' with $mdepth(w'') = i$. In particular, because $mdepth(w) = i$, we have that:

(2) $\langle \mathcal{M}, w \rangle \models \boxed{a}(\neg m_0 \vee \neg m_1 \vee \ldots \vee \neg m_r)$

From (1) and (2) and because $\boxed{a}\varphi \wedge \boxed{a}\psi$ is semantically equivalent to $\boxed{a}(\varphi \wedge \psi)$, we obtain that:

(3) $\langle \mathcal{M}, w \rangle \models \boxed{a}((\neg m_0 \vee \neg m_1 \vee \ldots \vee \neg m_r) \wedge m_1 \wedge \ldots \wedge m_r) \wedge \Diamond_a m_0$

By resolution's principle, we have $\langle \mathcal{M}, w \rangle \models \boxed{a}\neg m_0 \wedge \Diamond_a m_0$. Finally, by satisfiability definition of $\Diamond_a$ and $\boxed{a}$, we obtain $\langle \mathcal{M}, w \rangle \models$ **false**, a contradiction. Thus, the suposition of not existence of such Δ_j^{i+1} is untrue. Therefore, the conclusion of the rule is satisfiable. □

4.2 Completeness

A calculus is complete when it proves everything that it should. We show that $\mathcal{C}\mathsf{K}_n$ is complete, by showing that for all sets of clauses C that have an open tableau, that is, for all sets that is $\mathcal{C}\mathsf{K}_n$-satisfiable, C must also be K_n-satisfiable.

Definition 4.6 We say that a $\mathcal{C}\mathsf{K}_n$ tableau proof is *saturated* if all apropriate rule applications have been made. More precisely, an open tableau is saturated if:

(i) All literal clauses, a-positive and a-negative clauses have been treated. That is, the (PROP), (NEG) and (POS) rules have been aplied to all clauses in C, appropriately.

(ii) As a result of applying the rule (ELIM1), there is no set containgin both l and $\neg l$, for any l, at any modal level.

(iii) At all modal levels i, if there is a set Δ^i that contains a diamond, it must be a set at modal level $i+1$ that satisfies the literal in the scope of the diamond and all the literals in the scope of the boxes that are in Δ^i. Otherwise, this set would have been eliminated by the application of (ELIM2).

If $\mathcal{T}$ is a saturated $\mathcal{C}\mathsf{K}_n$ open tableau for C, we can extract a model that satisfies C. This is showed in Theorem 4.8.

We now specify a systematic construction procedure that, when followed, generates either a closed or a saturated tableau for any set of clauses C.

(i) Initially, apply the rule (PROP) to all literal clauses at all modal levels, eliminating the sets where inconsistency is inserted. If all the sets of the first modal level are eliminated, return the closed tableau.

(ii) Next, apply the rules (NEG) and (POS) to all modal clauses at all modal levels, always eliminating the inconsistent sets. Again, if the first modal level is empty, return the closed tableau.

(iii) Finally, from the maximum modal level downto the first one do as follows: For each diamond not yet satisfied, search in the next modal level for a set to satisfy the literals in the scope of this diamond and in the scope of every box belonging to the same set. The search fails when the rule (ELIM2) can be applied to the set holding this diamond. If all sets at the first modal level are eliminated, return the closed tableau. Otherwise, return the saturated tableau generated.

There is some flexibility entailed on the procedure described in the sense that it was left open which untreated clause to choose at the first and second phases, and also which unsatisfied diamonds to deal with at the last phase. But it can be shown that, no matter how the choice is made, the process must terminate.

Lemma 4.7 *The systematic construction procedure of a saturated tableau described above terminates for any set of clauses C.*

Proof □

Theorem 4.8 *Let C be a set of clauses in SNF_K. If C is $\mathcal{C}_{\mathsf{K}_n}$-satisfiable then it is K_n-satisfiable.*

Proof Let C be a $\mathcal{C}_{\mathsf{K}_n}$-satisfiable set of clauses, thus, following the procedure described above, we are able to generate a saturated open tableau $\mathcal{T}$ for C. To proof that C is also satisfiable in K_n, we need to build a model $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ that satisfies C. We do this using the information in $\mathcal{T}$.

If ι is the largest modal level in $\mathcal{T}$, take $W = \Pi^1 \cup \ldots \cup \Pi^\iota$ and $\pi(\Delta^i_j, p) = \mathbf{true}$ if $p \in \Delta^i_j$ and $\pi(\Delta^i_j, p) = \mathbf{false}$, otherwise.

From the systematic procedure, if Δ^{i+1}_k is the return of the search for a set that satisfies the diamond $\diamondsuit$ in Δ^i_j, take $R_a \cup \{(\Delta^i_j, \Delta^{i+1}_k)\}$.

Finally, any set $\Delta^1_j \in \Pi^1$ can be taken as w_0.

Now, to proof that $\mathcal{M}$ satisfies C, □

5 Example

$\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$

6 Related Work

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