

A Clausal Tableaux for Modal Logics

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Abstract

Keywords:

1 Introduction

2 Language

The Modal Language K_n is equivalent to its set of *well-formed formulae*, denoted WFF_{K_n} , which is constructed from an enumerable set of *propositional symbols* $\mathcal{P} = \{p, q, r, \dots\}$, the negation symbol \neg , the disjunction symbol \vee and the modal connectives \Box_a , that express the notion of necessity, for each index (or agent) a in a finite, fixed set $\mathcal{A} = \{1, \dots, n\}, n \in \mathbb{N}$.

The propositional symbols combined with the logic operators are arranged to form sentences (parentheses can be used to avoid ambiguity also). Therefore, the set of WFF_{K_n} is recursively defined as showed in Definition 2.1.

Definition 2.1 The set of well-formed formulae, WFF_{K_n} , is the least set such that:

- (i) $\mathcal{P} \subset WFF_{K_n}$

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(ii) if $\varphi, \psi \in WFF_{K_n}$, then so are $\neg\varphi, (\varphi \vee \psi)$ and $\boxed{a}\varphi$, for each $a \in \mathcal{A}$

As one might know, other logic operators may be introduced as abbreviation to formulae constructed using the operators defined as essential. In particular, this paper considers the abbreviations: $\varphi \wedge \psi \stackrel{\text{def}}{=} \neg(\neg\varphi \vee \neg\psi)$ (conjunction), $\varphi \Rightarrow \psi \stackrel{\text{def}}{=} \neg\varphi \vee \psi$ (implication), $\varphi \Leftrightarrow \psi \stackrel{\text{def}}{=} (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ (equivalence), $\Diamond\varphi \stackrel{\text{def}}{=} \neg\boxed{a}\neg\varphi$ (possibility), **false** $\stackrel{\text{def}}{=} \varphi \wedge \neg\varphi$ (*falsum*), **true** $\stackrel{\text{def}}{=} \neg\mathbf{false}$ (*verum*). And the precedence of these operators is in the order: $\{\neg, \boxed{a}, \Diamond\}, \{\wedge\}, \{\vee\}, \{\Rightarrow\}, \{\Leftrightarrow\}$.

For simplicity, we often refer to $\boxed{a}\varphi$ as *box* φ and to $\Diamond\varphi$ as *diamond* φ .

Logics that involve n agents in the modal logic, with $n \in \mathbb{N}$, are known as Multi-modal Logics. When $n = 1$, we tend to omit the index in the modal operators, i.e., we just write \Box and \Diamond .

The maximal number of modal operators in a formula is defined as its *modal depth* and denoted $mdepth$. The maximal number of modal operators in which scope the formula occurs is defined as the *modal level* of that formula, and it is denoted ml . For instance, in $\boxed{a}\Diamond p$, $mdepth(p) = 0$ and $ml(p) = 2$.

2.1 Semantics

The semantics of K_n is presented in terms of models based on Kripke structures.

Definition 2.2 A Kripke model for the set of propositional symbols \mathcal{P} and the agents \mathcal{A} is given by the tuple $\mathcal{M} = (W, w_0, R_1, \dots, R_n, \pi)$, where W is a non-empty set of possible worlds with a distinguished world w_0 , the root of \mathcal{M} , each $R_a, a \in \mathcal{A}$, is a binary relation on W , and $\pi : W \times \mathcal{P} \rightarrow \{\mathbf{false}, \mathbf{true}\}$ is the valuation function that associates to each world $w \in W$ a truth-assignment to propositional symbols.

From the definition of a Kripke model, one can define the satisfiability and validity of a formula in K_n .

Definition 2.3 Let $\mathcal{M} = (W, w_0, R_1, \dots, R_n, \pi)$ be a Kripke model for \mathcal{P} and \mathcal{A} , and consider $w \in W, p \in \mathcal{P}$ and $\varphi, \psi \in WFF_{K_n}$. The *satisfiability relation*, denoted by $\langle \mathcal{M}, w \rangle \models \varphi$, among the world w and a formula φ in the model \mathcal{M} , is inductively defined by:

- (i) $\langle \mathcal{M}, w \rangle \models p$ if, and only if, $\pi(w, p) = \mathbf{true}$
- (ii) $\langle \mathcal{M}, w \rangle \models \neg\varphi$ if, and only if, $\langle \mathcal{M}, w \rangle \not\models \varphi$
- (iii) $\langle \mathcal{M}, w \rangle \models \varphi \vee \psi$ if, and only if, $\langle \mathcal{M}, w \rangle \models \varphi$ or $\langle \mathcal{M}, w \rangle \models \psi$
- (iv) $\langle \mathcal{M}, w \rangle \models \boxed{a}\varphi$ if, and only if, $\forall t \in W$, with $a \in \mathcal{A}$, $(w, t) \in R_a$ implies $\langle \mathcal{M}, t \rangle \models \varphi$

A set of formulae $\mathcal{F} = \{\varphi_1, \dots, \varphi_r\}, r \in \mathbb{N}$, is satisfiable in a world w if, and only if, each of its formulae is satisfiable in this world, i.e., $\langle \mathcal{M}, w \rangle \models \mathcal{F} \Leftrightarrow \langle \mathcal{M}, w \rangle \models \varphi_1 \wedge \dots \wedge \varphi_r$.

Definition 2.4 A formula $\varphi \in WFF_{K_n}$ is said to be *satisfiable* if exists a Kripke model \mathcal{M} such that $\langle \mathcal{M}, w_0 \rangle \models \varphi$.

Definition 2.5 A formula $\varphi \in WFF_{K_n}$ is said to be *valid* if for all Kripke model \mathcal{M} , we have that $\langle \mathcal{M}, w_0 \rangle \models \varphi$.

When one is considering a set of formulae instead of a single one, both definitions of satisfiability and validity holds similarly to the definition of satisfiability relation. That means you have to check the satisfiability (or validity) of each formula in the set.

The basic satisfiability problem in K_n , then, summarizes to establish if a given set of formulae \mathcal{F} is satisfiable in K_n . To perform this evaluation, there are several proof procedures available, each one designed with a specific purpose to achieve.

3 Clausal Tableaux

Formally, a proof is a finite object constructed according to fixed syntactic rules that refer only to the structure of formulae, not to their intended meaning. The syntactic rules that define proofs are said to specify a *calculus*. A calculus is *sound* for a particular logic if any formula that has a proof must be a valid formula of this logic, and is *complete* for a logic if every valid formula has a proof [1]. Then we are usually interested in sound and complete calculi as they allow us to produce proofs that formulae are or not valid in a specific logic.

Tableaux based proof procedures are frequently (and successfully) used in modal logics because their structure is more obviously related to that notion of possible worlds, mentioned before [1]. There are many varieties of tableaux, the kind we present uses formulae in clausal normal form labelled by their modal level. Another feature of tableaux calculus is that, in general, proofs are graphically represented by trees where each branch can be thought as a set of formulae.

The definitions presented in this section concerning a tableau calculus (Definition 3.1 to Definition 3.6) are adaptations from [2].

A tableau calculus for a logic L , denoted by \mathcal{C}_L , is a finite set of tableau rules as defined in Definition 3.1.

Definition 3.1 A *tableau rule* σ consists of a numerator N above the line and a finite list of denominators D_1, \dots, D_k , below the line, separated by vertical bars. The numerator and each denominator are a finite set of formulae.

The numerator contains one or more distinguished formulae called *principal formulae*. All formulae in the numerator must be considered while the vertical bars in the denominator have a disjunction meaning. This way, each rule is read downwards as: if the set of formulae forming the numerator is satisfiable in a logic L , so is at least one of the denominators.

Given a \mathcal{C}_L , a *tableau proof*, or just *tableau*, for a formula φ is a tree whose root carries φ , while the other nodes carry finite sets of formulae generated from a parent by the application of one of the rules in \mathcal{C}_L . A *branch* in a tableau represents a path between the tree's root and one of its nodes.

Definition 3.2 Let \mathcal{R} be a set of tableau rules. We say that ψ is obtainable from φ by applications of rules from \mathcal{R} if there exists a tableau for φ which uses only rules from \mathcal{R} and has a branch that carries ψ .

After an application of a tableau rule, it is necessary to verify if the negation of one of the formulae recently obtained, already occurs in the same branch. In this case, an inconsistency was inserted and a new node containing \perp is added to this branch.

Definition 3.3 A branch in a tableau proof is *closed* if its end node contains only \perp . A tableau is *closed* if every one of its branches is closed. A tableau is *open* if it is not closed.

Definition 3.4 A finite set \mathcal{F} of formulae is \mathcal{C}_L -*satisfiable* if every \mathcal{C}_L -tableau for \mathcal{F} is open. If there is a closed \mathcal{C}_L -tableau for \mathcal{F} then it is \mathcal{C}_L -*unsatisfiable*.

Definition 3.5 A tableau calculus \mathcal{C}_L is *sound* if for all finite sets \mathcal{F} of formulae, if \mathcal{F} is L -satisfiable then \mathcal{F} is \mathcal{C}_L -satisfiable. It is *complete* if for all finite sets \mathcal{F} of formulae, if \mathcal{F} is \mathcal{C}_L -satisfiable then \mathcal{F} is L -satisfiable.

Definition 3.6 Let σ be a rule of \mathcal{C}_L . We say that σ is sound with respect to L if for every instance σ' of σ , if the numerator of σ' is L -satisfiable then so is one of the denominators of this instance.

Any tableau calculus for a logic L containing only rules sound, with respect to L , is sound [2].

The calculus proposed by this paper is a tableaux based proof procedure, and it is called a *labelled clausal tableaux calculus* since its rules are formed by formulae in normal clausal form, labelled by their modal level.

There is a specific normal form to \mathbf{K}_n called *Separated Normal Form with Modal Levels* (\mathbf{SNF}_K) which separates the contexts considering the different agents and different modal levels appropriately. The transformation rules and the correction proof of the translation method can be find in [3].

A formula in this clausal form is represented by a set of clauses, which are true in their respective modal levels. A formula in \mathbf{SNF}_K is of the form: $\bigwedge_i ml : C_i$, where each C_i is a clause and ml is the modal level in which the clause occurs.

Definition 3.7 A *literal* is a propositional symbol $p \in \mathcal{P}$ or its negation $\neg p$. We denote by \mathcal{L} the set of all literals.

Definition 3.8 A *modal literal* is a formula of the form $\boxed{a}l$ or its negation $\boxed{a}\neg l$, with $l \in \mathcal{L}$ and $a \in \mathcal{A}$.

Therefore, a clause in \mathbf{SNF}_K is in one of the following forms:

- literal clause: $ml : \bigvee_{b=1}^r l_b$
- a -negative clause: $ml : l \Rightarrow \Diamond m$
- a -positive clause: $ml : l \Rightarrow \boxed{a}m$

where $l_b, l, m \in \mathcal{L}, a \in \mathcal{A}$ and $r, b, ml \in \mathbb{N}$.

4 Calculus

The proposed calculus, denoted by \mathcal{CK}_n , comprehends a set of inference rules to deal with both propositional and modal reasoning. Before presenting these rules,

it is necessary to define the notation that is used in them, in order to establish the formality needed.

Definition 4.1 Let $\gamma^i, \lambda^i, \theta^i$ be representations of literal, a -negative and a -positive clauses, respectively, as follows:

- (i) $\gamma^i := i : \bigvee_{k=1}^t l_k$
- (ii) $\lambda^i := i : l \Rightarrow \Diamond m$
- (iii) $\theta^i := i : l \Rightarrow \Box m$

Also, let $\Gamma^i, \Lambda^i, \Theta^i$ be the initial set of the respective clauses at the modal level i :

- (i) $\Gamma^i := \{\gamma_1^i, \dots, \gamma_p^i\}, p \in \mathbb{N}$
- (ii) $\Lambda^i := \{\lambda_1^i, \dots, \lambda_q^i\}, q \in \mathbb{N}$
- (iii) $\Theta^i := \{\theta_1^i, \dots, \theta_r^i\}, r \in \mathbb{N}$

Let ι be the maximum modal level of a clause in any of the sets defined above and consider then that the initial set of clauses is represented by:

$$(i) \ C = (\bigcup_{i=1}^{\iota} \Gamma^i) \cup (\bigcup_{i=1}^{\iota} \Lambda^i) \cup (\bigcup_{i=1}^{\iota} \Theta^i)$$

Finally, consider Δ^i a set of literals, either propositional or modal, that belongs to the modal level i with $\Pi^i = \{\Delta_1^i, \dots, \Delta_k^i\}$, for some $k \in \mathbb{N}$. Observe that if $d \in \Delta_j^i$ then $d \in \mathcal{L}$ or d is of form $\Diamond l$ or $\Box l$ (modal literal), for some $l \in \mathcal{L}$.

Initially, consider that Π^i carries an empty set Δ_1^i , for all modal levels. We now present the inference rules that are applied to theses sets of literals.

The first inference rule, (PROP), related to literal clauses, has as numerator all literal sets that are already satisfied at modal level i , and considers a clause $\gamma^i = i : \bigvee_{k=1}^t l_k$ of Γ^i that has not been treated. The application of (PROP) for the clause γ^i results in the branching of each set of the numerator to consider every literal in the disjunction. The intuition for this rule is that if Δ_j^i and $\gamma^i \in \Gamma^i$ are both satisfiable, then $\Delta_j^i \cup \{l_r\}$ has to be satisfiable for some $1 \leq r \leq t$, therefore, at least one of the denominators is satisfiable.

The rules (NEG) and (POS) refer to modal clauses. Their meaning are pretty similar, even though they deal with different types of modal clauses, the difference is only the semantics of the operators. Ergo, the following description refers to the rule (NEG), being the one for the rule (POS) analogous.

This rule has as numerator all literal sets Δ_j^i that are already satisfied at modal level i , and considers the a -negative modal clauses $\lambda^i = i : l \Rightarrow \Diamond m$ of Λ^i at the same level. The application of (NEG) results in the duplication of Δ_j^i to consider both sides of the implication, as the implication is, actually, a disjunction between the head of the clause in negative form and the modal literal that occurs as the consequent of it ($\neg l \vee \Diamond m$). By hypothesis, Δ_j^i and λ^i are both satisfiable, so at least one of the denominators, $\Delta_j^i \cup \{\neg l\}$ or $\Delta_j^i \cup \{l, \Diamond m\}$ must also be satisfiable.

The last calculus' rule, (EXP), has as numerator all sets $\Delta_k^i \in \Pi^i$, at the i -th modal level, such that $\{\Diamond m_0, \Box m_1, \dots, \Box m_r\} \subseteq \Delta_k^i$ for some $m_0, m_1, \dots, m_r \in$

$$\begin{array}{c}
\frac{\Delta_j^i \in \Pi^i \quad \Gamma^i \cup \{i : \bigvee_{k=1}^t l_k\}}{\Delta_j^i \cup \{l_1\} \mid \dots \mid \Delta_j^i \cup \{l_t\}} \text{ (PROP)} \\
\\
\frac{\Delta_j^i \in \Pi^i \quad \Lambda^i \cup \{i : l \Rightarrow \Diamond m\}}{\Delta_j^i \cup \{\neg l\} \mid \Delta_j^i \cup \{l, \Diamond m\}} \text{ (NEG)} \quad \frac{\Delta_j^i \in \Pi^i \quad \Theta^i \cup \{i : l \Rightarrow \Box m\}}{\Delta_j^i \cup \{\neg l\} \mid \Delta_j^i \cup \{l, \Box m\}} \text{ (POS)} \\
\\
\frac{\Delta_k^i \cup \{\Diamond m_0\} \cup \{\Box m_1, \dots, \Box m_r\} \in \Pi^i}{\Delta_j^{i+1} \cup \{m_0, m_1, \dots, m_r\}} \text{ for some } \Delta_j^{i+1} \in \Pi^{i+1} \text{ (EXP)}
\end{array}$$

$\mathcal{L}, r \geq 0$. That means that this set contains at least one diamond and an arbitrary number of boxes. This rule expresses that if Δ_k^i is satisfiable then exists at least one set $\Delta_j^{i+1} \in \Pi^{i+1}$ that satisfies $\{m_0, m_1, \dots, m_r\}$, that is, each literal in a diamond must be satisfied at the next modal level along with all the literals in the boxes in the same set.

When branching a set of literals results in an inconsistency, i.e., when we generate a set that contains both d and $\neg d$ for some d , this set is eliminated from the proof. Furthermore, if we are not able to satisfy a diamond of a specific set, along with all the boxes in the same set, we also eliminate this set of the proof. A proof that has all the sets at a specific modal level eliminated is called a closed tableau. A closed tableau means that the initial set of clauses C was unsatisfiable while an open tableau gives us enough semantic information to construct a model that witnesses the satisfiability of these clauses.

5 Correctness Results

5.1 Soundness

As mentioned in the Section 3, to prove that \mathcal{CK}_n is sound, we just need to prove that its rules are sound.

Lemma 5.1 (PROP) *Let $\gamma^i \in i : \bigvee_{t=1}^k l_t$ be a literal clause in SNF_{ml} such that $\gamma^i \in \Gamma^i$, and $\Delta_j^i \in \Pi^i$ a literal set at the i -th modal level. If γ^i and Δ_j^i are both satisfiable in \mathcal{K}_n then exists $1 \leq r \leq t$ such that $\Delta_j^i \cup \{l_r\}$ is also satisfiable in \mathcal{K}_n .*

Proof Let Π^i be the initial set of literal clauses of the i -th modal level, with $\gamma^i = i : \bigvee_{t=1}^k l_t \in \Gamma^i$, and consider $\Delta_j^i \in \Pi^i$ a literal set that, by hypothesis, also belongs to the i -th modal level. Suppose that these hypothesis are satisfiable in \mathcal{K}_n . By Definition 2.4, it is known that exists a model $\mathcal{M} = (W, w_0, R_1, \dots, R_n, \pi)$ and a world $w \in W$ such that:

- $mdepth(w) = i$
- $\langle \mathcal{M}, w \rangle \models \bigvee_{k=1}^t l_k$ and
- $\langle \mathcal{M}, w \rangle \models d, \forall d \in \Delta_j^i$

To prove that this rule is sound, it is sufficient to show that, in an instance of it, at least one of its denominators is satisfiable. In fact, we have that:

- $\langle \mathcal{M}, w \rangle \models \bigvee_{k=1}^t l_k \Leftrightarrow \langle \mathcal{M}, w \rangle \models l_1 \text{ ou } \dots \text{ ou } \langle \mathcal{M}, w \rangle \models l_t$, by Definition 2.3

This means that exists at least one index r , with $1 \leq r \leq t$, such that $\langle \mathcal{M}, w \rangle \models l_r$. Therefore, $\Delta_j^i \cup \{l_r\}$ is satisfiable in \mathcal{K}_n . \square

Lemma 5.2 (NEG) *Let $\lambda^i = i : l \Rightarrow \Diamond m$ be an a-negative clause in SNF_{ml} such that $\lambda^i \in \Lambda^i$, and $\Delta_j^i \in \Pi^i$ a literal set in the i -th modal level. If λ^i and Δ_j^i are both satisfiable in \mathcal{K}_n then at least one of $\Delta_j^i \cup \{\neg l\}$ or $\Delta_j^i \cup \{l, \Diamond m\}$ is also satisfiable in \mathcal{K}_n .*

Proof Let Λ^i be the initial set of negative modal clauses at the i -th modal level, with $\lambda^i = i : l \Rightarrow \Diamond m \in \Lambda^i$, and consider $\Delta_j^i \in \Pi^i$ a literal set that, by hypothesis, also belongs to the i -th modal level. Suppose that these hypothesis are all satisfiable in \mathcal{K}_n . By Definition 2.4, it is know that exists a model $\mathcal{M} = (W, w_0, R_1, \dots, R_n, \pi)$ and a world $w \in W$ such that:

- $mdepth(w) = i$
- $\langle \mathcal{M}, w \rangle \models l \Rightarrow \Diamond m$
- $\langle \mathcal{M}, w \rangle \models d, \forall d \in \Delta_j^i$

To prove that this rule is sound, it is sufficient to show that, in an instancy of it, one of the denominators is satisfiable. In fact, we have that:

- $\langle \mathcal{M}, w \rangle \models l \Rightarrow \Diamond m \Leftrightarrow \langle \mathcal{M}, w \rangle \models \neg l \text{ ou } \langle \mathcal{M}, w \rangle \models \Diamond m$, by Definition 2.3

If $\langle \mathcal{M}, w \rangle \models \neg l$ holds then $\Delta_j^i \cup \{\neg l\}$ is satisfiable. Otherwise, $\langle \mathcal{M}, w \rangle \not\models \neg l$ and $\langle \mathcal{M}, w \rangle \models \Diamond m$ hold, therefore, $\Delta_j^i \cup \{l, \Diamond m\}$ is satisfiable in \mathcal{K}_n , by Definition 2.3. \square

Lemma 5.3 (POS) *Let $\theta^i = i : l \Rightarrow \Box m$ be an a-positive clause in SNF_{ml} such that $\theta^i \in \Theta^i$, and $\Delta_j^i \in \Pi^i$ a literal set in the i -th modal level. If θ^i and Δ_j^i are both satisfiable in \mathcal{K}_n then at least one of $\Delta_j^i \cup \{\neg l\}$ or $\Delta_j^i \cup \{l, \Diamond m\}$ is also satisfiable in \mathcal{K}_n .*

Being the rule (POS) quite analog to the rule (NEG), we leave the proof of its soundness omitted.

Lemma 5.4 (EXP) *Let $\Delta_k^i \in \Pi^i$ be a literal set at the i -th modal level such that $\{\Diamond m_0, \Box m_1, \dots, \Box m_r\} \subseteq \Delta_k^i$ for some $m_0, m_1, \dots, m_r \in \mathcal{L}, r \geq 0$. If Δ_k^i is satisfiable in \mathcal{K}_n then exists $\Delta_j^{i+1} \in \Pi^{i+1}$ such that $\Delta_j^{i+1} \cup \{m_0, m_1, \dots, m_r\}$ is also satisfiable.*

Proof Let's assume that $\Delta_k^i \cup \{\Diamond m_0\} \cup \{\Box m_1, \dots, \Box m_r\} \in \Pi^i$ is satisfiable in \mathcal{K}_n , i.e., that exists a model $\mathcal{M} = (W, w_0, R_1, \dots, R_n, \pi)$ and a world $w \in W$, with $mdepth(w) = i$, such that, by Definition 2.4:

- $\langle \mathcal{M}, w \rangle \models \Delta_k^i \cup \{\Diamond m_0\} \cup \{\Box m_1, \dots, \Box m_r\}$

And by definition of satisfiability in sets:

$$(1) \langle \mathcal{M}, w \rangle \models d \wedge \Diamond m_0 \wedge \Box m_1 \wedge \dots \wedge \Box m_r, \forall d \in \Delta_k^i$$

We have, then, $\langle \mathcal{M}, w \rangle \models \Diamond m_0$. Therefore, it exists a world w' , where $wR_a w'$ and $mdepth(w') = i + 1$, where $\langle \mathcal{M}, w' \rangle \models m_0$. As $\langle \mathcal{M}, w \rangle \models \Box m_k$ holds for every

$0 \leq k \leq r$, and we already know that $wR_a w'$, then $\langle \mathcal{M}, w' \rangle \models m_k$ also holds, for every $0 \leq k \leq r$, by the definitions of satisfiability of the operators \Diamond and \Box .

Suppose, by contradiction, that it doesn't exist $\Delta_j^{i+1} \in \Pi^{i+1}$, literal set at $(i+1)$ -th modal level, such that $\Delta_j^{i+1} \cup \{m_0, m_1, \dots, m_r\}$ is satisfiable in \mathcal{M} . That means that every set Δ_j^{i+1} satisfies $\neg m_0 \vee \neg m_1 \vee \dots \vee \neg m_r$. Then, again by definition of satisfiability of \Box , we have that $\langle \mathcal{M}, w'' \rangle \models \Box(\neg m_0 \vee \neg m_1 \vee \dots \vee \neg m_r)$ for all w'' with $mdepth(w'') = i$. In particular, because $mdepth(w) = i$, we have that:

$$(2) \quad \langle \mathcal{M}, w \rangle \models \Box(\neg m_0 \vee \neg m_1 \vee \dots \vee \neg m_r)$$

From (1) and (2) and because $\Box\varphi \wedge \Box\psi$ is semantically equivalent to $\Box(\varphi \wedge \psi)$, we obtain that:

$$(3) \quad \langle \mathcal{M}, w \rangle \models \Box((\neg m_0 \vee \neg m_1 \vee \dots \vee \neg m_r) \wedge m_1 \wedge \dots \wedge m_r) \wedge \Diamond m_0$$

By resolution's principle, we have $\langle \mathcal{M}, w \rangle \models \Box \neg m_0 \wedge \Diamond m_0$. Finally, by satisfiability definition of \Diamond and \Box , we obtain $\langle \mathcal{M}, w \rangle \models \mathbf{false}$, a contradiction. Thus, the suposition of not existence of such Δ_j^{i+1} is untrue. Therefore, the conclusion of the rule is satisfiable. \square

5.2 Completeness

We show that \mathcal{CK}_n is complete, by showing that for all set of clauses C that has an open tableau, that is, for all set that is \mathcal{CK}_n -satisfiable, C must also be \mathcal{K}_n -satisfiable.

Definition 5.5 We say that a \mathcal{CK}_n tableau proof is *saturated* if all appropriate rule applications have been made. More precisely, an open tableau is saturated if:

- (i) All literal clauses, a -positive and a -negative clauses have been treated. That is, the (PROP), (NEG) and (POS) rules have been applied to all clauses in C , appropriately.
- (ii) In all modal levels i , if there is a set Δ^i that contains a diamond, it must be a set at modal level $i+1$ that satisfies the literal in the diamond and all the literals in boxes that are in Δ^i . That means that the rule (EXP) was applied correctly in all modal levels.

If \mathcal{T} is a saturated \mathcal{CK}_n tableau, we can extract several useful information about it. If, for instance, $i : \bigvee_{k=1}^r l_k \in C$, we know that exists a set of literals Δ^i at modal level i such that $l_k \in \Delta^i$ for some $1 \leq k \leq r$, by item 1. Also by item 1, we know that if $i : l \Rightarrow \Diamond m \in C$, it has to exist a set Δ^i that satisfies either $\neg l$ or $l \wedge \Diamond m$. This last statement is analogous for an a -positive clause. Finally, if, at any modal level i , we have a set Δ^i that contains $\Diamond m_0$ and $\Box m_1, \dots, \Box m_r$, with $r \geq 0$, we will also have a set at modal level $i+1$ that contains m_0, m_1, \dots, m_r , by item 2.

We now specify a systematic construction procedure that, when followed, generates either a closed or a saturated tableau for any set of clauses C .

- (i) Initially, apply the rule (PROP) to all literal clauses at all modal levels. This will be know as the propositional phase. If all sets of literals at some modal level are eliminated then return the closed tableau.
- (ii) Next, apply the rules (NEG) and (POS) to all modal clauses at all modal levels.

This will be know as the modal phase. Again, if some modal level is empty, return the closed tableau.

- (iii) Finally, from the maximum modal level downto the first one do as follows: For each diamond not yet satisfied, search in the next modal level for a set to apply the rule (EXP). If the search fails then eliminate the set holding this diamond. This phase is called the expansion phase. If all sets at some modal level are eliminated, return the closed tableau. Otherwise, return the saturated tableau generated.

There is some flexibility entailed on the procedure described in the sense that it was left open which untreated clause to choose at the first and second phases, and also which unsatisfied diamonds to deal with at the last phase. But it can be shown that, no matter how the choice is made, the process must terminate.

Lemma 5.6 *The systematic construction procedure of a saturated tableau described above terminates for any set of clauses C .*

Proof □

Theorem 5.7 *Let C be a set of clauses in SNF_K . If C is \mathcal{CK}_n -satisfiable then it is K_n -satisfiable.*

Proof Let C be a \mathcal{CK}_n -satisfiable set of clauses. Following the the procedure described above, we are able to generate a saturated tableau for C . □

6 Example

$$\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$$

7 Related Work

References

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