A Clausal Tableaux for Modal Logics

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Abstract

Keywords:

1 Introduction

2 Language

The Modal Language K_n is equivalent to its set of *well-formed formulae*, denoted WFF_{K_n}, which is constructed from an enumerable set of *propositional symbols* $\mathcal{P} = \{p, q, r, \ldots\}$, the negation symbol $\neg$, the disjunction symbol $\vee$ and the modal connectives $\boxed{a}$, that express the notion of necessity, for each index (or agent) a in a finite, fixed set $\mathcal{A} = \{1, \ldots, n\}, n \in \mathbb{N}$.

The propositional symbols combined with the logic operators are arranged to form sentences (parentheses can be used to avoid ambiguity). Therefore, the set of WFF_{K_n} is recursively defined as showed in Definition 2.1.

Definition 2.1 The set of well-formed formulae, WFF_{K_n}, is the least set such that:

(i) $\mathcal{P} \subset WFF_{\mathsf{K}_n}$

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(ii) if $\varphi, \psi \in WFF_{\mathsf{K}_n}$, then so are $\neg\varphi, (\varphi \vee \psi)$ and $\boxed{a}\varphi$, for each $a \in \mathcal{A}$

As one might know, other logic operators may be introduced as abbreviation to formulae constructed using the operators defined as essential. In particular, this paper considers the abbreviations: $\varphi \wedge \psi \stackrel{\text{def}}{=} \neg(\neg\varphi \vee \neg\psi)$ (conjuction), $\varphi \Rightarrow \psi \stackrel{\text{def}}{=} \neg\varphi \vee \psi$ (implication), $\varphi \Leftrightarrow \psi \stackrel{\text{def}}{=} (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ (equivalence), $\diamondsuit_a\varphi \stackrel{\text{def}}{=} \neg\boxed{a}\neg\varphi$ (possibility), **false** $\stackrel{\text{def}}{=} \varphi\wedge\neg\varphi$ (*falsum*), **true** $\stackrel{\text{def}}{=} \neg$**false** (*verum*). And the precedence of these operators is in the order: $\{\neg, \boxed{a}, \diamondsuit_a\}$, $\{\wedge\}$, $\{\vee\}$, $\{\Rightarrow\}$, $\{\Leftrightarrow\}$.

For simplicity, we often refer to $\boxed{a}\varphi$ as *box* φ and to $\diamondsuit_a\varphi$ as *diamond* φ.

Logics that involve n agents in the modal logic, with $n \in \mathbb{N}$, are know as Multimodal Logics. When $n = 1$, we tend to omit the index in the modal operators, i.e., we just write $\Box$ and $\Diamond$.

The maximal number of modal operators in a formula is defined as its *modal depth* and denoted *mdepth*. The maximal number of modal operators in which scope the formula occurs is defined as the *modal level* of that formula, and it is denoted *ml*. For instance, in $\boxed{a}\diamondsuit_a p$, $mdepth(p) = 0$ and $ml(p) = 2$.

2.1 Semantics

The semantics of K_n is presented in terms of models based on Kripke structures.

Definition 2.2 A Kripke model for the set of propositional symbols $\mathcal{P}$ and the agents $\mathcal{A}$ is given by the tuple $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$, where W is a non-empty set of possible worlds with a distinguinshed world w_0, the root of $\mathcal{M}$, each $R_a, a \in \mathcal{A}$, is a binary relation on W, and $\pi : W \times \mathcal{P} \longrightarrow \{false, true\}$ is the valuation function that associates to each world $w \in W$ a truth-assignment to propositional symbols.

From the definition of a Kripke model, one can define the satisfiability and validity of a formula in K_n.

Definition 2.3 Let $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ be a Kripke model for $\mathcal{P}$ and $\mathcal{A}$, and consider $w \in W, p \in \mathcal{P}$ and $\varphi, \psi \in WFF_{\mathsf{K}_n}$. The *satisfiability relation*, denoted by $\langle \mathcal{M}, w\rangle \models \varphi$, among the world w and a formula φ in the model $\mathcal{M}$, is inductively defined by:

(i) $\langle \mathcal{M}, w\rangle \models p$ iff $\pi(w, p) =$ **true**

(ii) $\langle \mathcal{M}, w\rangle \models \neg\varphi$ iff $\langle \mathcal{M}, w\rangle \not\models \varphi$

(iii) $\langle \mathcal{M}, w\rangle \models \varphi \vee \psi$ iff $\langle \mathcal{M}, w\rangle \models \varphi$ or $\langle \mathcal{M}, w\rangle \models \psi$

(iv) $\langle \mathcal{M}, w\rangle \models \boxed{a}\varphi$ iff $\forall\ t \in W$, with $a \in \mathcal{A}$, $(w, t) \in R_a$ implies $\langle \mathcal{M}, t\rangle \models \varphi$

A set of formulae $\mathcal{F} = \{\varphi_1, \ldots, \varphi_r\}, r \in \mathbb{N}$, is satisfiable in a world w if, and only if, each of its formulae is satisfiable in this world, i.e., $\langle \mathcal{M}, w\rangle \models \mathcal{F} \Leftrightarrow \langle \mathcal{M}, w\rangle \models \varphi_1 \wedge \ldots \wedge \varphi_r$.

Definition 2.4 A formula $\varphi \in WFF_{\mathsf{K}_n}$ is said to be *satisfiable* if exists a Kripke model $\mathcal{M}$ such that $\langle \mathcal{M}, w_0\rangle \models \varphi$.

Definition 2.5 A formula $\varphi \in WFF_{\mathsf{K}_n}$ is said to be *valid* if for all Kripke model $\mathcal{M}$, we have that $\langle \mathcal{M}, w_0\rangle \models \varphi$.

When one is considering a set of formulae instead of a single one, both definitions of satisfiability and validity holds similarly to the definition of satisfiability relation. That means you have to check the satisfiability (or validity) of each formula in the set.

The basic satisfiability problem in K_n, then, summarizes to establish if a given set of formulae $\mathcal{F}$ is satisfiable in K_n. To perform this evaluation, there are several proof procedures available, each one designed with a specific purpose to achieve.

3 Clausal Tableaux

Formally, a proof is a finite object constructed according to fixed syntactic rules that refer only to the structure of formulae, not to their intended meaning. The syntactic rules that define proofs are said to specify a *calculus*. A calculus is *sound* for a particular logic if any formula that has a proof must be a valid formula of this logic, and is *complete* for a logic if every valid formula has a proof [1]. Then we are usually interested in sound and complete calculi as they allow us to produce proofs that formulae are or not valid in a specific logic.

Tableaux based proof procedures are frequently (and successfully) used in modal logics because their structure is more obviously related to that notion of possible worlds, mentioned before [1]. There are many varities of tableaux, the kind we present uses formulae in clausal normal form labelled by their modal level. Another feature of tableux calculus is that, in general, proofs are graphically represented by trees where each branch can be thought as a set of formulae.

The calculus proposed by this paper is a tableaux based proof procedure, and it is called a *labelled clausal tableaux calculus* since its rules are formed by formulae in normal clausal form, labelled by their modal level.

3.1 Separated Normal Form with Modal Levels

There is a specific normal form to K_n called *Separated Normal Form with Modal Levels* ($\mathsf{SNF}_{\mathrm{K}}$) which separates the contexts considering the different agents and different modal levels appropriately. The transformation rules and the correction proof of the translation method can be find in [3].

A formula in this clausal form is represented by a set of clauses, which are true in their respectives modal levels. A formula in $\mathsf{SNF}_{\mathrm{K}}$ is of the form: $\bigwedge_i ml : C_i$, where each C_i is a clause and ml is the modal level in which the clause occurs.

Definition 3.1 A *literal* is a propositional symbol $p \in \mathcal{P}$ or its negation $\neg p$. We denote by $\mathcal{L}$ the set of all literals.

Definition 3.2 A *modal literal* is a formula of the form $\boxed{a}\, l$ or its negation $\boxed{a}\neg l$, with $l \in \mathcal{L}$ and $a \in \mathcal{A}$.

Therefore, a clause in $\mathsf{SNF}_{\mathrm{K}}$ is in one of the following forms:

- literal clause: $ml : \bigvee_{b=1}^{r} l_b$
- a-negative clause: $ml : l \Rightarrow \Diamond_a m$
- a-positive clause: $ml : l \Rightarrow \boxed{a}\, m$

where $l_b, l, m \in \mathcal{L}, a \in \mathcal{A}$ and $r, b, ml \in \mathbb{N}$.

3.2 Labelled Clausal Tableaux

A clausal tableau calculus for a logic L, denoted by $\mathcal{C}_L$, is a finite set of tableau rules as defined in Definition 3.3.

Definition 3.3 A *tableau rule* σ consists of a numerator N above the line and a finite list of denominators $D_1, \ldots, D_k$, below the line, separated by vertical bars. The numerator and each denominator are a finite set of formulae.

The numerator contains one or more distinguished clauses. All clauses in the numerator must be considered while the vertical bars in the denominator have a disjunction meaning. This way, each rule is read downwards as: if the clauses forming the numerator are satisfiable in a logic L, so is at least one of the denominators.

Given a $\mathcal{C}_L$, a *tableau proof*, or just *tableau*, for a set of clauses C is a set of trees, each one labelled by a modal level, whose nodes carry literals (or literals in the scope of a box or a diamond) generated from a parent by the application of one of the rules in $\mathcal{C}_L$. A *branch* in a tableau represents a path between a tree's root and one of its nodes.

Definition 3.4 Let $\mathcal{R}$ be a set of tableau rules. We say that ψ is obtainable from φ by applications of rules from $\mathcal{R}$ if there exists a tableau for φ which uses only rules from $\mathcal{R}$ and has a branch that carries ψ.

After an application of a tableau rule, it is necessary to verify if the negation of one of the literals recently obtained, already occurs in the same branch. In this case, an inconsistency was inserted and a new node containing $\perp$ is added to this branch. A $\perp$ is also inserted on a branch containing a diamond that can not be satisfied in any branch of the tree at the next modal level.

Definition 3.5 A branch in a tableau proof is *closed* if it contains $\perp$. A tableau is *closed* if every one of its branches at the first modal level is closed. A tableau is *open* if it is not closed.

Definition 3.6 A finite set $\mathcal{F}$ of formulae is $\mathcal{C}_L$*-satisfiable* if every $\mathcal{C}_L$-tableau for $\mathcal{F}$ is open. If there is a closed $\mathcal{C}_L$-tableau for $\mathcal{F}$ then it is $\mathcal{C}_L$*-unsatisfiable.*

Definition 3.7 A tableau calculus $\mathcal{C}_L$ is *sound* if for all finite sets $\mathcal{F}$ of formulae, if $\mathcal{F}$ is L-satisfiable then $\mathcal{F}$ is $\mathcal{C}_L$-satisfiable. It is *complete* if for all finite sets $\mathcal{F}$ of formulae, if $\mathcal{F}$ is $\mathcal{C}_L$-satisfiable then $\mathcal{F}$ is L-satisfiable.

Definition 3.8 Let σ be a rule of $\mathcal{C}_L$. We say that σ is sound with respect to L if for every instance σ' of σ, if the numerator of σ' is L-satisfiable then so is one of the denominators of this instance.

Any tableau calculus for a logic L containing only sound rules, with respect to L, is sound [2].

4 Calculus

The proposed calculus, denoted by $\mathcal{C}_{\mathsf{K}_n}$, comprehends a set of inference rules to deal with both propositional and modal reasoning. Before presenting these rules, it is necessary to define the notation that is used in them, in order to establish the formality needed.

Definition 4.1 Let γ^i, λ^i e θ^i be representations of literal, a-negative and a-positive clauses, respectively, as follows:

(i) $\gamma^i \stackrel{\text{def}}{=} i : \bigvee_{k=1}^{t} l_k$

(ii) $\lambda^i \stackrel{\text{def}}{=} i : l \Rightarrow \diamondsuit m$

(iii) $\theta^i \stackrel{\text{def}}{=} i : l \Rightarrow \boxed{a} m$

Also, let Γ^i, Λ^i e Θ^i be the initial set of the respective clauses at the i-th modal level:

(i) $\Gamma^i \stackrel{\text{def}}{=} \{\gamma_1^i, \ldots, \gamma_k^i\}$, $k \in \mathbb{N}$

(ii) $\Lambda^i \stackrel{\text{def}}{=} \{\lambda_1^i, \ldots, \lambda_j^i\}$, $j \in \mathbb{N}$

(iii) $\Theta^i \stackrel{\text{def}}{=} \{\theta_1^i, \ldots, \theta_t^i\}$, $t \in \mathbb{N}$

Let ι be the maximum modal level of a clause in any of the sets defined above and consider then that the initial set of clauses is represented by:

(i) $C = (\bigcup_{i=1}^{\iota} \Gamma^i) \cup (\bigcup_{i=1}^{\iota} \Lambda^i) \cup (\bigcup_{i=1}^{\iota} \Theta^i)$

Finally, consider $i : \{d_1, \ldots, d_t\} = \Delta^i$ as a set of literals, which occur either on propositional or modal scope, that belongs to the modal level i with $\Pi^i = \{\Delta_1^i, \ldots, \Delta_k^i\}$, for some $k \in \mathbb{N}$, denoting all the literals sets at the same modal level. Observe that if $d \in \Delta_j^i$ then $d \in \mathcal{L}$ or d is of form $\diamondsuit l$ or $\boxed{a} l$ (modal literal), for some $l \in \mathcal{L}$.

Initiallly, consider that Π^i carries an empty set Δ_1^i, for all modal levels. We now present the inference rules that are aplied to theses sets of literals.

The first inference rule, (PROP), related to literal clauses, has as numerator all literal sets Δ_j^i that are already satisfied at modal level i, and considers a clause $\gamma^i = i : \bigvee_{k=1}^{t} l_k$ of Γ^i that has not been treated. The application of (PROP) for the clause γ^i results in the branching of each set of the numerator to consider every literal in the disjunction. The intuition for this rule is that if Δ_j^i and $\gamma^i \in \Gamma^i$ are both satisfiable, then $\Delta_j^i \cup \{l_r\}$ has to be satisfiable for some $1 \leq r \leq t$, therefore, at least one of the denominators is satisfiable.

The rules (NEG) and (POS) refer to modal clauses. Their meaning are pretty similar, even though they deal with different types of modal clauses, the difference is only the semantics of the operators. Ergo, the following description refers to the rule (NEG), being the one for the rule (POS) analogous.

This rule has as numerator all literal sets Δ_j^i that are already satisfied at modal level i, and considers the a-negative modal clauses $\lambda^i = i : l \Rightarrow \diamondsuit m$ of Λ^i at the

$$\frac{i : \bigvee_{k=1}^{t} l_k \qquad \Delta_j^i \in \Pi^i}{\Delta_j^i \cup \{l_1\} \mid \ldots \mid \Delta_j^i \cup \{l_t\}} \text{ (PROP)}$$

$$\frac{i : l \Rightarrow \Diamond_{a} m \qquad \Delta_j^i \in \Pi^i}{\Delta_j^i \cup \{\neg l\} \mid \Delta_j^i \cup \{l, \Diamond_{a} m\}} \text{ (NEG)} \qquad \frac{i : l \Rightarrow \boxed{a} m \qquad \Delta_j^i \in \Pi^i}{\Delta_j^i \cup \{\neg l\} \mid \Delta_j^i \cup \{l, \boxed{a} m\}} \text{ (POS)}$$

same level. The application of (NEG) results in the duplication of Δ_j^i to consider both sides of the implication, as the implication is, actually, a disjuction between the head of the clause in negative form and the modal literal that occurs as the consequent of it ($\neg l \vee \Diamond_{a} m$). By hypothesis, Δ_j^i and λ^i are both satisfiable, so at least one of the denominators, $\Delta_j^i \cup \{\neg l\}$ or $\Delta_j^i \cup \{l, \Diamond_{a} m\}$ must also be satisfiable.

In addition to the inference rules for constructing a proof by adding literals when branching sets, we also have two rules to eliminate sets that should not be a part of the proof:

(ELIM1): Eliminate sets containing both l and $\neg l$ for some l.

(ELIM2): If, at some modal level, we have that every set satisfy $\neg m_0 \vee \neg m_1 \vee \ldots \vee \neg m_r$, eliminate the sets at the previous level that contains $\Diamond_{a} m_0$, $\boxed{a} m_1, \ldots, \boxed{a} m_r$.

5 Correctness Results

5.1 Soundness

As mentioned in the Section 3, to prove that $\mathcal{C}_{\mathsf{K}_n}$ is sound, we just need to prove that its rules are sound.

Lemma 5.1 *(PROP) Let $\gamma^i \in i : \bigvee_{t=1}^{k} l_t$ be a literal clause in SNF_{ml} such that $\gamma^i \in \Gamma^i$, and $\Delta_j^i \in \Pi^i$ a literal set at the i-th modal level. If γ^i and Δ_j^i are both satisfiable in K_n then exists $1 \leq r \leq t$ such that $\Delta_j^i \cup \{l_r\}$ is also satisfiable in K_n.*

Proof Let Π^i be the initial set of literal clauses of the i-th modal level, with $\gamma^i = i : \bigvee_{t=1}^{k} l_t \in \Gamma^i$, and consider $\Delta_j^i \in \Pi^i$ a literal set that, by hypothesis, also belongs to the i-th modal level. Supose that these hypothesis are satisfiable in K_n. By Definition 2.4, it is know that exists a model $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ and a world $w \in W$ such that:

- $mdepth(w) = i$
- $\langle \mathcal{M}, w \rangle \models \bigvee_{k=1}^{t} l_k$ and
- $\langle \mathcal{M}, w \rangle \models d, \forall\, d \in \Delta_j^i$

To prove that this rule is sound, it is sufficient to show that, in an instancy of it, at least one of its denominators is satisfiable. In fact, we have that:

- $\langle \mathcal{M}, w \rangle \models \bigvee_{k=1}^{t} l_k \Leftrightarrow \langle \mathcal{M}, w \rangle \models l_1$ ou $\ldots$ ou $\langle \mathcal{M}, w \rangle \models l_t$, by Definition 2.3

This means that exists at least one index r, with $1 \leq r \leq t$, such that $\langle \mathcal{M}, w \rangle \models l_r$. Therefore, $\Delta_j^i \cup \{l_r\}$ is satisfiable in K_n. $\square$

Lemma 5.2 *(NEG) Let $\lambda^i = i : l \Rightarrow \diamondsuit m$ be an a-negative clause in SNF_{ml} such that $\lambda^i \in \Lambda^i$, and $\Delta^i_j \in \Pi^i$ a literal set in the i-th modal level. If λ^i and Δ^i_j are both satisfiable in K_n then at least one of $\Delta^i_j \cup \{\neg l\}$ or $\Delta^i_j \cup \{l, \diamondsuit m\}$ is also satisfiable in K_n.*

Proof Let Λ^i be the initial set of negative modal clauses at the i-th modal level, with $\lambda^i = i : l \Rightarrow \diamondsuit m \in \Lambda^i$, and consider $\Delta^i_j \in \Pi^i$ a literal set that, by hypothesis, also belongs to the i-th modal level. Supose that these hypothesis are all satisfiable in K_n. By Definition 2.4, it is know that exists a model $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ and a world $w \in W$ such that:

- $mdepth(w) = i$
- $\langle \mathcal{M}, w\rangle \models l \Rightarrow \diamondsuit m$
- $\langle \mathcal{M}, w\rangle \models d, \forall\ d \in \Delta^i_j$

To prove that this rule is sound, it is sufficient to show that, in an instancy of it, one of the denominators is satisfiable. In fact, we have that:

- $\langle \mathcal{M}, w\rangle \models l \Rightarrow \diamondsuit m \Leftrightarrow \langle \mathcal{M}, w\rangle \models \neg l$ ou $\langle \mathcal{M}, w\rangle \models \diamondsuit m$, by Definition 2.3

If $\langle \mathcal{M}, w\rangle \models \neg l$ holds then $\Delta^i_j \cup \{\neg l\}$ is satisfiable. Otherwise, $\langle \mathcal{M}, w\rangle \not\models \neg l$ and $\langle \mathcal{M}, w\rangle \models \diamondsuit m$ hold, therefore, $\Delta^i_j \cup \{l, \diamondsuit m\}$ is satisfiable in K_n, by Definition 2.3.□

Lemma 5.3 *(POS) Let $\theta^i = i : l \Rightarrow \boxed{a} m$ be an a-positive clause in SNF_{ml} such that $\theta^i \in \Theta^i$, and $\Delta^i_j \in \Pi^i$ a literal set in the i-th modal level. If θ^i and Δ^i_j are both satisfiable in K_n then at least one of $\Delta^i_j \cup \{\neg l\}$ or $\Delta^i_j \cup \{l, \diamondsuit m\}$ is also satisfiable in K_n.*

Being the rule (POS) quite analog to the rule (NEG), we leave the proof of its soundness omitted.

Lemma 5.4 *(EXP) Let $\Delta^i_k \in \Pi^i$ be a literal set at the i-th modal level such that $\{\diamondsuit m_0, \boxed{a} m_1, \ldots, \boxed{a} m_r\} \subseteq \Delta^i_k$ for some $m_0, m_1, \ldots, m_r \in \mathcal{L}, r \geq 0$. If Δ^i_k is satisfiable in K_n then exists $\Delta^{i+1}_j \in \Pi^{i+1}$ such that $\Delta^{i+1}_j \cup \{m_0, m_1, \ldots, m_r\}$ is also satisfiable.*

Proof Let's assume that $\Delta^i_k \cup \{\diamondsuit m_0\} \cup \{\boxed{a} m_1, \ldots, \boxed{a} m_r\} \in \Pi^i$ is satisfiable in K_n, i.e., that exists a model $\mathcal{M} = (W, w_0, R_1, \ldots, R_n, \pi)$ and a world $w \in W$, with $mdepth(w) = i$, such that, by Definition 2.4:

- $\langle \mathcal{M}, w\rangle \models \Delta^i_k \cup \{\diamondsuit m_0\} \cup \{\boxed{a} m_1, \ldots, \boxed{a} m_r\}$

And by definition of satisfiability in sets:

(1) $\langle \mathcal{M}, w\rangle \models d \wedge \diamondsuit m_0 \wedge \boxed{a} m_1 \wedge \ldots \wedge \boxed{a} m_r, \forall\ d \in \Delta^i_k$

We have, then, $\langle \mathcal{M}, w\rangle \models \diamondsuit m_0$. Therefore, it exists a world w', where $wR_a w'$ and $mdepth(w') = i + 1$, where $\langle \mathcal{M}, w'\rangle \models m_0$. As $\langle \mathcal{M}, w\rangle \models \boxed{a} m_k$ holds for every $0 \leq k \leq r$, and we already know that $wR_a w'$, then $\langle \mathcal{M}, w'\rangle \models m_k$ also holds, for every $0 \leq k \leq r$, by the definitions of satisfiability of the operators $\diamondsuit$ and $\boxed{a}$.

Supose, by contradiction, that it doesn't exist $\Delta^{i+1}_j \in \Pi^{i+1}$, literal set at $(i+1)$-th modal level, such that $\Delta^{i+1}_j \cup \{m_0, m_1, \ldots, m_r\}$ is satisfiable in $\mathcal{M}$. That means that every set Δ^{i+1}_j satisfies $\neg m_0 \vee \neg m_1 \vee \ldots \vee \neg m_r$. Then, again by definition of satisfiability of $\boxed{a}$, we have that $\langle \mathcal{M}, w'' \rangle \models \boxed{a}(\neg m_0 \vee \neg m_1 \vee \ldots \vee \neg m_r)$ for all w'' with $mdepth(w'') = i$. In particular, because $mdepth(w) = i$, we have that:

(2) $\langle \mathcal{M}, w \rangle \models \boxed{a}(\neg m_0 \vee \neg m_1 \vee \ldots \vee \neg m_r)$

From (1) and (2) and because $\boxed{a}\varphi \wedge \boxed{a}\psi$ is semantically equivalent to $\boxed{a}(\varphi \wedge \psi)$, we obtain that:

(3) $\langle \mathcal{M}, w \rangle \models \boxed{a}((\neg m_0 \vee \neg m_1 \vee \ldots \vee \neg m_r) \wedge m_1 \wedge \ldots \wedge m_r) \wedge \diamondsuit m_0$

By resolution's principle, we have $\langle \mathcal{M}, w \rangle \models \boxed{a}\neg m_0 \wedge \diamondsuit m_0$. Finally, by satisfiability definition of $\diamondsuit$ and $\boxed{a}$, we obtain $\langle \mathcal{M}, w \rangle \models$ **false**, a contradiction. Thus, the suposition of not existence of such Δ_j^{i+1} is untrue. Therefore, the conclusion of the rule is satisfiable. $\square$

5.2 Completeness

We show that $\mathcal{C}\mathsf{K}_n$ is complete, by showing that for all set of clauses C that has an open tableau, that is, for all set that is $\mathcal{C}\mathsf{K}_n$-satisfiable, C must also be K_n-satisfiable.

Definition 5.5 We say that a $\mathcal{C}\mathsf{K}_n$ tableau proof is *saturated* if all apropriate rule applications have been made. More precisely, an open tableau is saturated if:

(i) All literal clauses, a-positive and a-negative clauses have been treated. That is, the (PROP), (NEG) and (POS) rules have been aplied to all clauses in C, appropriately.

(ii) In all modal levels i, if there is a set Δ^i that contains a diamond, it must be a set at modal level $i+1$ that satisfies the literal in the diamond and all the literals in boxes that are in Δ^i. That means that the rule (EXP) was aplied correctly in all modal levels.

If $\mathcal{T}$ is a saturated $\mathcal{C}\mathsf{K}_n$ tableau, we can extract several useful information about it. If, for instance, $i : \bigvee_{k=1}^{r} l_k \in C$, we know that exists a set of literals Δ^i at modal level i such that $l_k \in \Delta^i$ for some $1 \leq k \leq r$, by item 1. Also by item 1, we know that if $i : l \Rightarrow \diamondsuit m \in C$, it has to exist a set Δ^i that satisfies either $\neg l$ or $l \wedge \diamondsuit m$. This last statement is analogous for an a-positive clause. Finaly, if, at any modal level i, we have a set Δ^i that contains $\diamondsuit m_0$ and $\boxed{a}m_1, \ldots, \boxed{a}m_r$, with $r \geq 0$, we will also have a set at modal level $i+1$ that contains $m_0, m_1, \ldots, m_r$, by item 2.

We now specify a systematic construction procedure that, when followed, generates either a closed or a saturated tableau for any set of clauses C.

(i) Initially, apply the rule (PROP) to all literal clauses at all modal levels. This will be know as the propositional phase. If all sets of literals at some modal level are eliminated then return the closed tableau.

(ii) Next, apply the rules (NEG) and (POS) to all modal clauses at all modal levels. This will be know as the modal phase. Again, if some modal level is empty, return the closed tableau.

(iii) Finally, from the maximum modal level downto the first one do as follows: For each diamond not yet satisfied, search in the next modal level for a set to apply the rule (EXP). If the search fails then eliminate the set holding this diamond.

This phase is called the expansion phase. If all sets at some modal level are eliminated, return the closed tableau. Otherwise, return the saturated tableau generated.

There is some flexibility entailed on the procedure described in the sense that it was left open which untreated clause to choose at the first and second phases, and also which unsatisfied diamonds to deal with at the last phase. But it can be shown that, no matter how the choice is made, the process must terminate.

Lemma 5.6 *The systematic construction procedure of a saturated tableau described above terminates for any set of clauses C.*

Proof $\square$

Theorem 5.7 *Let C be a set of clauses in SNF_K. If C is $\mathcal{C}\mathsf{K}_n$-satisfiable then it is K_n-satisfiable.*

Proof Let C be a $\mathcal{C}\mathsf{K}_n$-satisfiable set of clauses. Following the the procedure described above, we are able to generate a saturated tableau for C. $\square$

6 Example

$\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$

7 Related Work

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