Bocciu Daniel -243-

Ex 3. $2 \le X+Y \le 8$.

H_i - { telefonul i e defect }, $i = \overline{1,9}$

$P(H_i, H_j) = \frac{1}{\binom{2}{9}} = \frac{1}{\frac{\cancel{9!}\,8\cdot 9}{\cancel{7!}\cdot 2!}} = \frac{1}{36}$. (egal prob. ca tel. i si j sa fie defect.)

$P(X=1, Y=1) = P(H_1, H_2) = \frac{1}{36}$.

$P(X=1, Y=2) = P(H_1, H_3) = \frac{1}{36}$.

$\ldots = \ldots$

$P(X=1, Y=7) = P((H_1, H_8) \cup (H_1, H_9)) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36}$.

$P(X=7, Y=0) = P(H_8, H_9) = \frac{1}{36}$. (dacă nu am găsit unul defect în primele 7, atunci sunt ultimele 2)

X\Y	0	1	2	3	4	5	6	7	
1	0	1/36	1/36	1/36	1/36	1/36	1/36	2/36	8/36
2	0	1/36	1/36	1/36	1/36	1/36	2/36	0	7/36
3	0	1/36	1/36	1/36	1/36	2/36	0	0	6/36
4	0	1/36	1/36	1/36	2/36	0	0	0	5/36
5	0	1/36	1/36	2/36	0	0	0	0	4/36
6	0	1/36	2/36	0	0	0	0	0	3/36
7	1/36	2/36	0	0	0	0	0	0	3/36.
	1/36	8/36	7/36	6/36	5/36	4/36	3/36	2/36	

$\sum x_i y_j = \frac{36}{36} = 1$.

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8/36 & 7/36 & 6/36 & 5/36 & 4/36 & 3/36 & 3/36 \end{pmatrix}$$

$$Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1/36 & 8/36 & 7/36 & 6/36 & 5/36 & 4/36 & 3/36 & 2/36 \end{pmatrix}$$

$$E[X] = \frac{8+14+18+20+20+18+21}{36} = \frac{119}{36} = 3{,}305$$

$$E[X^2] = \frac{8+28+54+80+100+108+147}{36} = \frac{525}{36}$$

$$= 14{,}583$$

$$Var(X) = E[X^2] - E[X]^2 = 14{,}583 - (3{,}305)^2$$

$$= 14{,}583 - 10{,}923 = 3{,}66$$

$$E[Y] = \frac{8+14+18+20+20+18+14}{36} = \frac{112}{36} = 3{,}11$$

$$E[Y^2] = \frac{8+28+54+80+100+108+\cancel{147}\,98}{36} = \frac{476}{36}$$

$$= 13{,}222$$

$$Var(Y) = 13{,}222 - (3{,}11)^2 = 13{,}222 - 9{,}6721 =$$

$$= \cancel{4{,}9109}$$

$$= 3{,}5499.$$

$$Cov(X,Y) = E[XY] - E[X]\,E[Y]$$

$$XY \sim \left(\begin{array}{ccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 10 & 12 & 14 & \\ \frac{36}{36^2} & \frac{64}{36^2} & \frac{112}{36^2} & \frac{96}{36^2} & \frac{129}{36^2} & \frac{64}{36^2} & \frac{132}{36^2} & \frac{40}{36^2} & \frac{70}{36^2} & \frac{36}{36^2} & \frac{56}{36^2} & \frac{102}{36^2} & \frac{35}{36^2} \end{array} \right.$$

$$\left. \begin{array}{ccccccccccccc} 30 & 35 & 36 & 28 & 25 & 20 & 24 & 15 & 18 & 21 & 16 & 42 & 49 \\ \frac{24}{36^2} & \frac{20}{36^2} & \frac{9}{36^2} & \frac{25}{36^2} & \frac{16}{36^2} & \frac{40}{36^2} & \frac{30}{36^2} & \frac{48}{36^2} & \frac{56}{36^2} & \frac{30}{36^2} & \frac{25}{36^2} & \frac{15}{36^2} & \frac{6}{36^2} \end{array} \right)$$

$$E[XY] = \frac{13328}{36^2} = 10{,}2839.$$

$$Cov(X,Y) = 10{,}2839 - \cancel{(14{,}583)}\;(3{,}305)\cdot(3{,}11)$$

$$Cov(X,Y) = 10{,}2839 - 10{,}278 \simeq 0{,}005 \simeq 0$$

$$\Rightarrow \rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = 0.$$

c) $X|Y=2 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1/\cancel{36}\,7 & 1/\cancel{36}\,7 & 1/\cancel{36}\,7 & 1/\cancel{36}\,7 & 1/\cancel{36}\,7 & 2/\cancel{36}\,7 & 0 \end{pmatrix}^{8} \quad \Sigma = \frac{7}{36}.$

$$E[X|Y=2] = \frac{1+2+3+4+5+12}{7} = \frac{27}{7} = 3{,}8571$$

$$E[X^2|Y=2] = \frac{1+4+9+16+25+72}{7} = 18{,}1428.$$

$$Var(X|Y=2) = 18{,}1428 - (3{,}8571)^2 =$$

$$= 18{,}1428 - 14{,}877 =$$

$$= 3{,}2658$$

Ex1.

$$X \sim \begin{pmatrix} 4 & 5 \\ 0,33 & 0,67 \end{pmatrix} \quad Y \sim \begin{pmatrix} -4 & 4 \\ p_1 & p_2 \end{pmatrix}, \ p_1, p_2 \in (0,1)$$

a) $P(X=4,\ Y=4) = 0,11$, $E[X|Y=4] = 4,5$

$p(X=4, Y=-4) = 0,33 - 0,11 = 0,22$.

$X \backslash Y$	-4	4	
4	0,22	0,11	0,33
5	$p_1 - 0,22$	$p_2 - 0,11$	0,67
	p_1	p_2	

~~$E[X$~~ $\quad X|Y=4 \sim \begin{pmatrix} 4 & 5 \\ 0,11/p_2 & \frac{p_2 - 0,11}{p_2} \end{pmatrix}$

$$E[X|Y=4] = \frac{0,44}{p_2} + \frac{5p_2 - 0,55}{p_2} = \frac{5p_2 - 0,11}{p_2} = 4,5.$$

$5p_2 - 0,11 = 4,5 p_2 \ \Big| \cdot 2$. $\quad 10p_2 - 0,22 = 9p_2$

$\boxed{p_2 = 0,22}$

$\Rightarrow \boxed{p_1 = 0,78}$

b) $X = 4 \to Y = \begin{cases} -4 \\ 4 \end{cases} \to X+Y = \begin{cases} 0 \\ 8 \end{cases}$

$X = 5 \to Y = \begin{cases} -4 \\ 4 \end{cases} \Rightarrow X+Y = \begin{cases} 1 \\ 9 \end{cases}$.

$$X+Y \sim \begin{pmatrix} 0 & 1 & 8 & 9 \\ 0{,}2574 & 0{,}5226 & 0{,}0726 & 0{,}1474 \end{pmatrix}$$

$$x=4 \rightarrow y=\begin{cases} -4 \\ 4 \end{cases} \rightarrow x-y=\begin{cases} 8 \\ 0 \end{cases}$$

$$x=5 \rightarrow y=\begin{cases} -4 \\ 4 \end{cases} \Rightarrow x-y=\begin{cases} 9 \\ 1 \end{cases}$$

$$X-Y \sim \begin{pmatrix} 0 & 1 & 8 & 9 \\ 0{,}0726 & 0{,}1474 & 0{,}2574 & 0{,}5226 \end{pmatrix}$$

$\cancel{2x^2}$

$$x=4, \; y=\begin{cases} -4 \\ +4 \end{cases} \rightarrow 2x^2+7y^2 - \begin{cases} 9 \cdot 16 \\ 9 \cdot 16 \end{cases}$$

$$x=5 - y-\begin{cases} -4 \\ +4 \end{cases} \rightarrow 2x^2+7y^2 - \begin{cases} 162 \\ 162 \end{cases}$$

$$2X^2+7Y^2 \sim \begin{pmatrix} 144 & 162 \\ 0{,}33 & 0{,}67 \end{pmatrix}$$

$$E[X] = 4 \cdot 0{,}33 + 5 \cdot 0{,}67 = 4{,}67$$

$$E[Y] = -4 \cdot 0{,}78 + 4 \cdot 0{,}22 = -2{,}24$$

$$Var(X) = E[X^2] - E[X]^2$$

$$E[X^2] = 16 \cdot 0{,}33 + 25 \cdot 0{,}67 = 22{,}03$$

$$Var(X) = 22{,}03 - (4{,}67)^2 = 0{,}2211$$

$E[Y^2] = 16 \cdot 0{,}22 + 16 \cdot 0{,}78 = 16.$

$Var(Y) = 16 - (-2{,}24)^2 = 10{,}9823.$

$Var(6X - 4Y + 15) = Var(6X + 4Y) =$

$= 36 Var(X) + 16 Var(Y) =$

$= 36 \cdot 0{,}2211 + 16 \cdot 10{,}9823 =$

$= 7{,}9596 + 175{,}7168 = 183{,}6764.$

$Cov(X,Y) = E[XY] - E[X]E[Y].$

$$XY \sim \begin{pmatrix} -16 & -20 & +20 & 16 \\ 0{,}2574 & 0{,}5226 & 0{,}1474 & 0{,}0726 \end{pmatrix}$$

$E[XY] = -10{,}46.$

$Cov(X,Y) = -10{,}46 - 4{,}67 \cdot (-2{,}24) = 0.$

$\Rightarrow \rho(X,Y) = 0.$

Ex 4

$$f(x) = \frac{x}{25} \cdot e^{\frac{-x^2}{50}} \cdot \mathbb{1}_{\{x \geq 0\}}$$

$$F(t) = \int_0^t \frac{x}{25} \cdot e^{\frac{-x^2}{50}} dx =$$

$$= -e^{\frac{-x^2}{50}} \Big|_0^t = 1 - e^{-\frac{t^2}{50}}$$

$F^{+1}_{(x)} = y \quad | F. \qquad x = F(y).$

$x = 1 - e^{-\frac{y^2}{50}} \Rightarrow e^{-\frac{y^2}{50}} = 1 - x.$

$-\frac{y^2}{50} = \ln(1-x)$

$y^2 = 50 \cdot \ln\left(\frac{1}{1-x}\right)$

$y = 5\sqrt{2\ln\left(\frac{1}{1-x}\right)} = F^{-1}_{(x)}$

$F^{-1}\left(\frac{3}{4}\right) = 5\sqrt{2\cdot\ln 4} \quad , F^{-1}\left(\frac{1}{4}\right) = 5\sqrt{2\ln\frac{4}{3}}$

$Var(X) = E[X^2] - E[X]^2.$

$E[X] = \int_0^\infty x f_{(x)}\, dx = \int_0^\infty x \cdot \frac{x}{25} e^{-\frac{x^2}{50}} dx.$

$= \int_0^\infty (-x)\left(e^{-\frac{x^2}{50}}\right)' dx = \underbrace{-x e^{-\frac{x^2}{50}} \Big|_0^\infty}_{0} + \int_0^\infty x e^{-\frac{x^2}{50}} dx$

$= \int_0^\infty (-25)\left(e^{-\frac{x^2}{50}}\right)' dx = -25 \cdot e^{-\frac{x^2}{50}} \Big|_0^\infty + 25\int_0^\infty e^{-\frac{x^2}{50}} dx.$

$y = \frac{x}{5}.$

$= -25 + 25 \cdot 5 \cdot \underbrace{\int_0^\infty e^{-\frac{y^2}{2}} dy}_{\frac{\sqrt{2\pi}}{2}}.$

$= -25 + \frac{125\sqrt{2\pi}}{2}.$

$$E[X^2] = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty (-x^2)\cdot\left(e^{-\frac{x^2}{50}}\right)' dx =$$

$$= \underbrace{-x^2\cdot e^{-\frac{x^2}{50}}\Big|_0^\infty}_{0} + 25\int_0^\infty \left(\frac{-x}{\cancel{25}}\right)'\left(e^{-\frac{x^3}{50}}\right)' dx$$

$$= \underbrace{-25\frac{x}{\cancel{25}}\cdot e^{-\frac{x^2}{50}}\Big|_0^\infty}_{0} + 25^2\int_0^\infty (-1)\left(e^{-\frac{x^2}{50}}\right)' dx.$$

$$= +25^2\cdot(-1)\cdot e^{-\frac{x^2}{50}}\Big|_0^\infty + 25^2\cdot\underbrace{\int_0^\infty e^{-\frac{x^2}{50}}\,dx}_{5\cdot\frac{\sqrt{2\pi}}{2}.}$$

$$= -25^2 + 25^2\cdot 5\cdot\frac{\sqrt{2\pi}}{2}.$$

$\cancel{F^{-1}_{(0,725)}}$ $$Var(X) = \underline{-25^2} + 25^2\cdot 5\frac{\sqrt{2\pi}}{2} - 25^2 - \frac{125\sqrt{2\pi}}{2} + \frac{2\cdot 25^2\cdot 5\cdot\sqrt{2\pi}}{2}$$

$$Var(X) = -2\cdot 25^2 + \frac{\sqrt{2\pi}}{2}\left(25^2\cdot 5 - 125 + 2\cdot 25^2\cdot 5\right)$$

$$\frac{F^{-1}_{(0,75)} - F^{-1}_{(0,25)}}{\sqrt{Var(X)}} = \frac{5\left(\sqrt{2\ln 4} - \sqrt{2\ln\frac{4}{3}}\right)}{\sqrt{-2\cdot 25^2 + \frac{\sqrt{2\pi}}{2}\left(25^2\cdot 5 - 125 + 2\cdot 25^2\cdot 5\right)}}$$

$= - - - - —$

5. $A = \{$ persoana votează cu Florin Cîțu $\}$

$P(A) = 0,07$. Pois (560). , $\lambda = 560$.

$P(A^c) = 0,93$ Not $p = 0,07$.

$X \to$ nr de aleg. a lui Cîțu și Y a lui Orban

a) $P(X = a, Y = b) = P(X = a, Y = b \mid N = a+b) \cdot P(N = a+b)$
(prob. ca a alegători să voteze cu Cîțu)

$\Rightarrow \cancel{P(N = a+b)}$

$= P(X = a \mid N = a+b) \cdot P(Y = b \mid N = a+b)$.

$$P(X = a, Y = b) = C_{b+1}^{a} \, p^a (1-p)^b \cdot e^{-\lambda} \cdot \frac{\lambda^{a+b}}{(a+b)!}$$

$$= \frac{\cancel{(a+b)!}}{a! \cdot b!} \cdot p^a \cdot (1-p)^b \cdot e^{-\lambda} \cdot \frac{\lambda^{a+b}}{\cancel{(a+b)!}} =$$

$$= \frac{e^{-\lambda p} \cdot (\lambda p)^a}{a!} \cdot e^{-\lambda(1-p)} \frac{(\lambda(1-p))}{b!}$$

Repartițiile marginale $P(X = a) = \sum_{b \geq 0} P(X = a, Y = b)$

$$= \sum_{b \geq 0} \frac{e^{-\lambda p} (\lambda p)^a}{a!} \cdot e^{-\lambda(1-p)} \cdot \frac{(\lambda(1-p))^b}{b!} =$$

$$= e^{-\lambda p} \cdot \frac{(\lambda p)^a}{a!} \Rightarrow X \sim \text{Pois}(\lambda p)$$

Analog $Y \sim \text{Pois}(\lambda(1-p))$

b) Analog ca la ⓐ, $P(X=a, Y=b) = P(X=a)\,P(Y=b)$

$\Rightarrow$ X și Y sunt independente.

c) $E[V] = E[X-Y] = E[X] - E[Y]$

Noi știm că X și Y sunt resp. Pois.

$$\left.\begin{array}{l} E[X] = \lambda p \\ E[Y] = \lambda(1-p) \end{array}\right\} \Rightarrow E[X-Y] = \lambda(p-1+p)$$

$E[X-Y] = 2p\lambda = 2 \cdot 0{,}07 \cdot 560 =$

$= 0{,}14 \cdot 560 = 78{,}4$.

$Var(V) = Var(X-Y) = Var(X) + Var(Y)$

$$\left.\begin{array}{l} Var(X) = \lambda p \\ Var(Y) = \lambda(1-p) \end{array}\right\} \Rightarrow$$

$\Rightarrow Var(V) = \lambda(p+1-p) = \lambda = 560$.

Ex2

1. $E[\log(x)] \boxed{\leq} \log(E[x])$

log(x) este concavă.
folosim inegalitatea lui Jensen.

2. $E[x] \boxed{\geq} \sqrt{E[x]}$.

funcția radical este convexă pt $E[x] \geq 1$
folosim tot ineg. lui Jensen.

3. $E[\sin^2(x)] + E[\cos^2(x)] \boxed{=} 1$

$E[\sin^2(x) + \cos^2(x)] = 1$.

4) $P(x > c) \boxed{\leq} \frac{E[x^3]}{c^3}$

folosind inegalitatea lui Markov, care spune că ~~dacă~~ tv. a > 0, atunci $P(x \geq a) \leq \frac{E[x]}{2}$ $\forall a > 0$.

5) $P(x \leq y) \boxed{?} P(x > y)$

nu cunoaștem cine este x și y.

6). $P(x+y > 10) \boxed{?} P(x>5 \text{ sau } y>5)$

nu știm cine e x și y.
avem cazuri în care se respectă și cazuri în care nu,
$X=20, Y=1$ sau $x=7, y=5$.

7. $E[\min(X,Y)]$ $\boxed{?}$ $\min(E(X), E(Y))$

nu știm ce valori iau X și Y.

8. $E[\frac{X}{Y}]$ $\boxed{\geq}$ $\frac{E[X]}{E[Y]}$.

$$\frac{E[X]}{E[Y]} = \frac{x \cdot P(x)}{y\, P(y)} \neq \frac{x}{y} \cdot \frac{P(x)}{P(y)} \Big| \cdot E \Rightarrow$$

$$\Rightarrow E[\frac{x}{y}] \cdot E\left[\frac{P(x)}{P(y)}\right] = E[\frac{x}{y}] \cdot \frac{P(x)}{P(y)} \geq E[\frac{x}{y}]$$

9. $E[X^2(X^2+1)]$ $\boxed{?}$ $E[X^2(Y^2+1)]$

$$\Updownarrow$$

$E[X^4]$ $\boxed{?}$ $E[X^2Y^2]$ $\Rightarrow E[X^2]$ $\boxed{?}$ $E[Y^2]$

$E[X+Y] = E[X] + E[Y]$

$E[X^4 + X^2] = E[X^2(X^2+1)]$.

$E[X^2(Y^2+1)] = E[X^2Y^2 + Y^2]$.

nu știm.

10. $E[\frac{1}{X}]$ $\boxed{?}$ $\frac{1}{E[X]}$.

$E[X^k]$ = momentul de ordin k, dar $k=1$

$\Rightarrow$ nu se respectă.

<u>Ex6</u> $p = 0{,}26$, avem 5 eșecuri.

k - nr. de succese pt a ajunge la al 5-lea eșec.

~~$P(X=k) = \binom{k+4}{k+4}(1-p)^5 \cdot p^k$ (Binomiala Negativă)~~

trebuie să avem 5 eșecuri $\Rightarrow r = 5$.

$$P(X=k) = \binom{k+4}{4} \cdot (1-p)^5 \cdot p^{k-5} \quad k \in \{r, r+1, \ldots\}.$$

$$E[\cancel{5X+5}\ X] = \sum_{k} k \cdot p^k \cdot (1-p)^5 \cdot C_{k+4}^{4}$$

$$E[X] = (1-p)^5 \cdot \sum k \cdot p^k \cdot \frac{(k+4)!}{4! \cdot k!} =$$

$$= (1-p)^5 \cdot \sum p^k \cdot k \cdot (k+1)(k+2)(k+3)(k+4).$$

$$E[X^2] =$$